Valuation Algorithms for Structural Models of Financial Networks

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Contents

1 Introduction

Financial networks can be observed in almost all types of markets all over the world. An often cited example is the Japanese economy (see McDonald, 1989; Suzuki, 2002), where the largest stock-listed firms own a relatively large amount of shares of each other. This intertwining of the firm's equity is known as the *Mochiai effect* and arose after the Second World War, see McDonald (1989) for more details. But not only in Northeast Asia such ownership structures can be observed. Other countries in which similar cross-holdings are present are for example Germany (Dorofeenko et al., 2008), Italy (Brioschi et al., 1989) and Norway (Bøhren and Michalsen, 1994).

In a financial network, one party's obligations are the other party's claims, hence a strongly connected network structure bears risks. In prone systems, a firm that gets into financial trouble is a channel of contagion since a potential default of one firm may affect the other firms creditworthiness as well due to interrelations in the firms' balance sheets. Eisenberg and Noe (2001) call this phenomenon *cyclical interdependence* and describe it as follows:

> *"A default by Firm A on its obligations to Firm B may lead B to default on its obligations to C. A default by C may, in turn have a feedback effect on A."*

Even though the publication dates of the articles from the paragraph above are already some years ago, the events in financial markets in recent years showed that financial cross-linkages have become more important than ever. In the world financial crisis that started in 2007, mortgage banks like Fannie Mae and Freddie Mac or the insurance agency AIG got into financial trouble and had to be rescued by the US government to avoid a collapse of the financial system. These bailouts brought to mind that, due to the financial interconnectedness of the companies all over the world, a crisis in one region of the world may trigger financial crisis in other regions. One of many conclusions of the almost breakdown was that modeling the firms' credit risk by ignoring the architecture of the network may lead to dangerous effects. It is therefore necessary to analyze the financial network as a whole. This will help to avoid systemic crashes as the one in 2008 and to keep the associated social costs of a crisis, that can account for up to 300% of the gross domestic product of a country (Boyd et al., 2005), up to a minimum.

Directly associated with the quantification of systemic risks in financial systems is the issue of pricing debt and equity in financial networks since the price of the assets is the crucial information when determining the resilience of a network. This leads to the analysis of credit risk, a field of research that started with the pioneering work of Merton (1974). In the famous *Merton model*, equity and debt value of a firm are interpreted as derivatives of the firm's assets. If the asset value falls below the nominal value of the outstanding debt of a firm, a default event occurs. This approach has lead to many modifications and extensions of the original Merton model. All these models are summarized under the conception *structured models* since they focus on the financial structure of the firms to evaluate their borrowing capacity. For an overview of such models, see for example Bingham and Kiesel (2004) and the references therein. In the works of Crouhy et al. (2000) and Arora et al. (2005) structural models are compared to *reduced-form models* who focus on modeling the firms' default rates.

Though multi-firm models allow correlations of the firms assets utilizing multivariate distributions (cf. Kealhofer and Bohn, 2001) or some works, like Zhou (2001), take potential correlations of asset values in their model into account, a weakness of many structural models is that cross-holdings of the firms in debt and equity are ignored or at least not taken into account on a structural level. Eisenberg and Noe (2001) were the first to structurally model financial systems in which firms can hold each other's financial obligations as assets under the assumption of limited liability. The main difference between Eisenberg and Noe and the standard multi-firm Merton model is that prices at maturity are not trivially determined since the value of one firm's equity or debt may depend on the value of the debt of any other firm in the system which is vividly described by Shin (2008):

> *"The [...] value of my claim against A depends on A's creditworthiness, and so depends on the value of A's claims against B, C, etc. However, B or C may have a claim against me, and so we are back full circle. The task of valuing claims in a financial system thus entails solving for a consistent set of prices [...]."*

This cyclical characteristic described by Shin leads to the challenge of finding a pricing equilibrium for equity and debt of the system members. Spoken mathematically, this is equivalent to determine a fixed point of a certain mapping. In their article, Eisenberg and Noe (2001) gave conditions under which only one equilibrium solution exists at maturity. Other works like Suzuki (2002), Shin (2008) and Elsinger (2009) generalized the Eisenberg and Noe setup by also including cross-holdings in the equities, and by allowing a seniority structure of the liabilities. All these works have in common that they are also faced with the issue of finding a price equilibrium.

While there exists a small but growing amount of research on the existence and the uniqueness of price equilibria in systems with financial interconnectedness, the literature is still somewhat inconsistent. This can, among others, be noticed by the fact that different authors define the underlying financial system on differing ways. In some cases, this bears no conflict since models with less general assumptions can easily be embedded into more general models. For instance, the model of Eisenberg and Noe (2001) is a special case of Shin (2008), since in the former model, only one priority level of debt is considered whereas in the latter model, more than one debt priority is taken into account. In other cases, however, the authors derive contradictory conditions that have to be valid for the existence of a uniquely determinable price equilibrium. As an example may serve the articles of Suzuki (2002) or Gouriéroux et al. (2012) in which the assumptions made on the ownership structure are not compatible with the assumptions of other models such as the one of Eisenberg and Noe (2001). A possible explanation of this plethora of assumptions could be that the authors are sometimes unaware of previous articles or at least do not reference on former publications. Suzuki for example generalizes the model of Eisenberg and Noe (2001), seemingly unbeknown to him. In turn, Suzuki's work stays unmentioned in the article of Gouriéroux et al. (2012) even though they use the same model and come to identical conditions for the uniqueness of a pricing equilibrium.

Another aspect of the lack of uniformity in this research area concerns the procedures that are applied to find the pricing equilibria. In the existing publications, the provided algorithms mainly reflect the individual authors' particular approach to the problem. Beside their results concerning conditions to ensure that the pricing equilibrium is unique, Eisenberg and Noe (2001) also provide a finite numerical algorithm to determine this solution. Other papers (Suzuki, 2002; Elsinger, 2009; Gouriéroux et al., 2012; Fischer, 2014) treating more general models that allow for equity cross-holdings, also present iterative procedures to find the equilibrium. The higher generality, however, comes with the drawback that it cannot be ensured anymore that the equilibrium price vector is reached in a finite number of iteration steps. Comparative studies of the different methods seem to be absent from the existing literature. Furthermore, at present, no numerical algorithm for the setup with cross-holdings of equity and one seniority class of debt (Suzuki, 2002; Elsinger, 2009; Gouriéroux et al., 2012; Fischer, 2014) is known that reaches the exact solution in a finite number of calculation steps.

This work therefore has several objects. In a preliminary step, we attempt to give a definition of the financial system in its most general form in Chapter 2. This model, a system with cross-holdings in both equity and debt with potentially more than one seniority level is referred to as the *standard model* in the remainder of this work. We show that under a regularity condition – the ownership matrices have to possess the *Elsinger Property* – the system has a uniquely defined solution in the form of a payment equilibrium.

Chapter 3 is a survey of the existing articles on this field, not only concerning the historical development of the models but also regarding definitions of financial systems deviating from the standard model in Chapter 2. Possible variations are differing definitions of the exogenous assets, the properties of the ownership matrices that describe the structure of cross-holdings or the definition of the liabilities. We highlight how the differing models can be included into the standard model and, if this is not possible, show the consequences on existence and uniqueness results that follow from the distinct definitions. There are, however, some versions of financial systems that can be embedded into the standard model but then violate the regularity conditions that are required for the uniqueness of a payment equilibrium. This concerns financial systems in which all obligations of the firms are completely owned by other members in the system which contradicts the Elsinger Property. In order to demonstrate that such models still have a unique solution, we need to claim a further condition on the standard model concerning the size of the exogenous assets. Beside a short sketch how to prove this, we also present all further important ideas of proofs for the existence and the uniqueness of a payment equilibrium. In a last subsection of this chapter we briefly discuss some extensions of the standard model, in particular systems in which default costs are included.

The main focus of this work is on the investigation of valuation algorithms to find the unique pricing equilibrium. Based on the standard model in which only one seniority level is allowed, we give in Chapter 4 an overview of already existing iteration procedures, the *Picard* and the *Elsinger Algorithm*, examine their properties and introduce two possible starting vectors for the iteration, a minimum and a maximum possible solution. Moreover, the ideas of Elsinger (2009) and of Eisenberg and Noe (2001) are combined together into a new type of algorithm, called *Hybrid Algorithm* for which we show that it minimizes the number of iteration steps to find a solution. Even if the iteration number is minimized, the drawback of the mentioned methods is that they in general do not reach the solution exactly and, hence, theoretically may need infinitely many iteration steps. This is why we present a new class of finite algorithms that are to able find the equilibrium in a finite number of iteration steps. The innovation of these algorithms, the *Trial-and-Error* and the *Sandwich Algorithm*, is that the *default set*, the number of firms in default for a current iterate, is taken into consideration as an additional information for the search of the solution.

The means of both non-finite and finite algorithms are in Chapter 5 applied to financial systems in which more than one debt priority level is present. It will turn out that the methods also work well for such systems, i.e. the non-finite algorithms will converge to the equilibrium, though potentially infinitely many iteration steps have to be conducted. Trial-and-Error and Sandwich Algorithm can also be generalized for this framework with a slight adaption of the default set to a *default tuple* since in these models in case of a firm's default event, one has to find out in which seniority class the default is exactly located. Nevertheless, we show that the main result of Chapter 4, i.e. that the equilibrium is reached in a finite number of steps, can be retained. At the end of this chapter, we additionally mention another algorithm, called the *Default Structure Algorithm* that is based on the ideas of Elsinger (2009). The approach here is a bit different to the finite algorithms but comes with the advantage that the algorithm has a clear-defined upper bound of maximum iteration steps. The disadvantage is that the finite character of the algorithm gets lost since non-finite methods have to be considered in the substeps of the procedure. We generalize Elsinger's method in more detail than in the original work and additionally present a modification to improve the convergence speed of the procedure.

Except of the Default Structure Algorithm, all mentioned procedures above have in common that they can start from two different directions when searching the payment equilibrium. One possible starting point is the maximum possible solution, another initial iterate consists out of the minimum possible payments. The issue of choosing the optimal one out of the two starting points is addressed in Chapter 6. Based on the error estimates of the Picard Iteration established by Banach (1922), we derive a decision rule that attempts to find the optimal starting point for the Picard Algorithm. Further, we demonstrate that the transition of the decision rule to the Elsinger and the Hybrid Algorithm is not innocuous since for these procedures the required condition necessary for the utilization of the error estimates cannot be guaranteed in general. However, the principle introduced for the Picard Algorithm can be applied even though in order to determine an optimal initial iterate.

Chapter 7 contains a simulation study for a various selection of financial systems with one priority class of debt. The objectives of the study are threefold. In a first part, we assess the goodness of the decision rule from Chapter 6 and find that the number of needed iteration steps is actually minimized when applying the decision rule on all three non-finite algorithms (Picard, Elsinger, Hybrid). However, when taking the runtime of the procedures as an orientation, checking the rule and deciding which initial iterate to take is unfortunately less efficient than using either the upper or the lower starting point each time. Searching for an optimal lag value, a parameter that is needed for the Trial-and-Error Algorithms, is the second part of the study. The results reveal that the smallest possible lag value leads on the one hand to a minimization of the computational effort. On the other hand, the number of additional performed iteration steps, a consequence of small lag values, can be kept acceptable small. The last and most important part entails the investigation of all possible calculation procedures concerning their efficiency. It turns out that the Picard Iteration requires much less computational effort and therefore is the most efficient iteration procedure. Moreover, we find that for financial system of a small size, the Sandwich Algorithm is the most efficient algorithm class and that the higher the number of firms in the system becomes, the more efficient becomes the class of non-finite algorithms.

A summary of the results and an outlook of potential future research topics is given in Chapter 8. Finally, a technical appendix follows with auxiliary results and tables that summarize additional results of the simulation study in Chapter 7

2 Notation and Model Assumptions

2.1 Notation

Vectors are represented by bold small letters (e.g. $\mathbf{x}$), matrices by bold capitals (e.g. $\mathbf{M}$) and a set is written in form of a calligraphic letter (e.g. $\mathcal{I}$). Moreover, a vector $\mathbf{x} \in \mathbb{R}^n$ is always assumed to be a column vector. The components of $\mathbf{x} = (x_1, \dots, x_n)^t$ are written in regular font. Sometimes a vector $\mathbf{x}$ depends on $k \geq 1$ other vectors $\mathbf{y}_1, \dots, \mathbf{y}_k$ which is expressed as $\mathbf{x}(\mathbf{y}_1, \dots, \mathbf{y}_k)$. The i-th element of $\mathbf{x}(\mathbf{y}_1, \dots, \mathbf{y}_k)$ is addressed via $x_i(\mathbf{y}_1, \dots, \mathbf{y}_k)$. By $\mathbf{0}_n$ we denote a (column) vector of length n that contains only zeros and $\mathbf{1}_n$ stands for a vector of the same length with value 1 in every entry. If two vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ are composed to a (column) vector of dimension $2n$, we define for a better readability that

$$(\mathbf{u}, \mathbf{v}) := (\mathbf{u}^t, \mathbf{v}^t)^t \in \mathbb{R}^{2n}, \tag{2.1}$$

which is also possible for k vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ via

$$(\mathbf{u}_1, \dots, \mathbf{u}_k) := ((\mathbf{u}_1)^t, \dots, (\mathbf{u}_k)^t)^t. \tag{2.2}$$

In some cases, $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n \times 2}$ represents a matrix of two vectors that are connected together columnwise. However, no confusion about the dimension of $(\mathbf{u}, \mathbf{v})$ should arise from the context. In this thesis, we will often make use of vectors $\mathbf{R}^k \in \mathbb{R}^{n(m+1)}$ for instance as an iterate of a certain iteration process. Note that this is the only time, a vector is denoted in capitals. For any $k \geq 0$, we interpret $\mathbf{R}^k$ as consisting $m+1$ vectors $\mathbf{r}^{k,m}, \dots, \mathbf{r}^{k,1}, \mathbf{r}^{k,0}$, each of dimension n, which is why we will express $\mathbf{R}^k$ by

$$\mathbf{R}^k := (\mathbf{r}^{k,m}, \mathbf{r}^{k,m-1}, \dots, \mathbf{r}^{k,1}, \mathbf{r}^{k,0}), \tag{2.3}$$

hence the first letter in the superscript on the right hand side of (2.3) always stands for the iterate number $k \geq 0$ and the second letter labels the position of $\mathbf{r}^{k,l}$ in $\mathbf{R}^k$, where $l = 0, 1, \dots, m$.

In general, the entries of a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ are denoted by $M_{ij},\ i, j = 1, \dots, n$. When multiplying a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ with a vector $\mathbf{x} \in \mathbb{R}^n$ of appropriate dimension and adding another vector $\mathbf{y} \in \mathbb{R}^n$, we obtain for the i-th entry of the resulting vector

$$(\mathbf{y} + \mathbf{M}\mathbf{x})_i := y_i + \sum_{j=1}^{n} M_{ij} x_j. \tag{2.4}$$

The symbol $\mathbf{I}_n$ is used for the $(n \times n)$-identity matrix and $\mathbf{0}_{n \times n}$ stands for an $(n \times n)$-matrix with only zero entries. For two matrices $\mathbf{M} \in \mathbb{R}^{n \times n}$ and $\mathbf{N} \in \mathbb{R}^{n \times n}$ we write $\mathbf{M} \geq \mathbf{N}$ if $M_{ij} \geq N_{ij}$ for all $i, j \in \{1, \dots, n\}$ and $\mathbf{M} > \mathbf{N}$ if $M_{ij} > N_{ij}$ for at least one pair (i, j). For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ the definition of $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} > \mathbf{v}$ is analogous to the conventions for matrices. A matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ with $M_{ij} \geq 0$ for all $i, j \in \{1, \dots, n\}$ is said to be *left substochastic* if $\sum_{i=1}^{n} M_{ij} \leq 1$ for all $j \in \{1, \dots, n\}$. The matrix is called *strictly left substochastic* if $\sum_{i=1}^{n} M_{ij} < 1$ for all $j \in \{1, \dots, n\}$ and *fully left stochastic* if $\sum_{i=1}^{n} M_{ij} = 1$ for all $j \in \{1, \dots, n\}$.

For a vector $\mathbf{u} = (u_1, \dots, u_n)^t \in \mathbb{R}^n$, the expression $\operatorname{diag}(\mathbf{u} \leq \mathbf{0}_n)$ stands for an $(n \times n)$-diagonal matrix where the i-th entry on the diagonal is 1 if $u_i \leq 0$ and 0 otherwise, i.e.

$$\operatorname{diag}(\mathbf{u} \leq \mathbf{0}_n) := \begin{cases} 1, & \text{for } i = j \text{ and } u_i \leq 0, \\ 0, & \text{else.} \end{cases} \tag{2.5}$$

The commonly used norm in this thesis is the ℓ^1-norm on $\mathbb{R}^n$ defined as

$$\|\mathbf{x}\| := \|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{for } \mathbf{x} \in \mathbb{R}^n. \tag{2.6}$$

The corresponding norm for a left substochastic matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ is given by

$$\|\mathbf{M}\| := \|\mathbf{M}\|_1 = \max_{\|\mathbf{x}\|=1} \|\mathbf{M}\mathbf{x}\|_1 = \max_j \sum_{i=1}^{n} |M_{ij}| = \max_j \sum_{i=1}^{n} M_{ij} \leq 1, \tag{2.7}$$

meaning that $\|\mathbf{M}\|$ is the maximum of the column sums. One can easily show that $\|\mathbf{M}\mathbf{x}\| \leq \|\mathbf{M}\|\|\mathbf{x}\|$ as well as $\|\mathbf{M} + \mathbf{N}\| \leq \|\mathbf{M}\| + \|\mathbf{N}\|$ for $\mathbf{N} \in \mathbb{R}^{n \times n}$.

Finally, if in a sum the counter at start is higher than in the end, the sum is set to zero, i.e. $\sum_{i=n+1}^{n} x_i := 0$ for $\mathbf{x} \in \mathbb{R}^n$. All operations, such as the minimum, $\min\{\cdot\}$, the maximum, $\max\{\cdot\}$, or the positive part $(\cdot)^+$ are applied element-wise to vectors and matrices. All numbers in the numerical examples during the thesis are rounded to four decimal places, unless otherwise stated.

2.2 Model Assumptions

We consider a system of n financial entities, and denote $\mathcal{N} := \{1, \dots, n\}$. In the following these entities are simply called *firms*. Each firm owns exogenous assets, that are defined in the next step.

Definition 2.1. Let $a_i \geq 0$ denote the market value at maturity of the *exogenous assets* held by firm i. As the name implies, these assets are priced outside the considered system in the sense that the capital structure of the n firms has no influence on the pricing mechanism of such an asset. By $\mathbf{a} = (a_1, \dots, a_n)^t \in (\mathbb{R}_0^+)^n$ we denote the (column) vector of the exogenous assets.

Moreover, we assume that the firms have $m \geq 1$ outstanding liabilities with nominal values at maturity of $\mathbf{d}^1, \dots, \mathbf{d}^m \in \mathbb{R}^n$ meaning that d_j^k is the value of firm j's k-th liability. Suppose that the liabilities are arranged according to their priority in case of a liquidation event. That means that $\mathbf{d}^m$ is the liability with the highest priority, paid first and $\mathbf{d}^1$ has the lowest priority and is paid at last. The number k $(1 \leq k \leq m)$ is often referred to as the *seniority/priority level/class.* To take the interconnectedness of the firms into account, we allow that each firm can own a fraction of the liabilities of the other firms. Beside the liabilities, we also allow for cross-holdings in the equity of the firms. To formalize these possible cross-holdings, we use ownership matrices.

Definition 2.2. The left substochastic matrix $\mathbf{M}^k \in \mathbb{R}^{n \times n}, k = 1, \dots, m$, in which the entry $0 \leq M_{ij}^k \leq 1$ denotes the fraction that firm i owns from the k-th priority level liability of firm j is called *debt ownership matrix* for the k-th priority or seniority level. Since no firm is allowed to hold obligations to itself, we assume $M_{ii}^k = 0$ for all $i \in \mathcal{N}$ and all priority levels $k \in \{1, \dots, m\}$. Moreover, let $\mathbf{M}^0 \in \mathbb{R}^{n \times n}$ be the ownership matrix of the equity which means that firm i owns a fraction of $0 \leq M_{ij}^0 \leq 1$ of firm j's equity.

Note that there is no common convention in literature how to define the diagonal entries of the ownership matrices. In some works (Elsinger, 2009), the diagonal entries of $\mathbf{M}^k$ are for $k = 1, \dots, m$ assumed to be zero and the diagonal entries of $\mathbf{M}^0$ are allowed to be larger than zero. That means it is allowed that firm i holds its own shares but not its own debt. On the other hand there exists articles (Awiszus and Weber, 2015) in which $M_{ii}^k = 0$ for all $k = 0, \dots, m$ has to hold. We refer to the model assumptions in Elsinger (2009) which is why we allow $M_{ii}^0 > 0$ for all $i \in \mathcal{N}$. However, from a mathematical point of view, the restriction that the diagonal entries have to be zero in all ownership matrices is not necessary since all following results in this thesis also hold if the firms own some part of their shares or liabilities.

All available information that was defined above is in the sequel of this work sometimes referred to as the financial system.

Definition 2.3. A *financial system* is given by the tuple $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$ where $\mathbf{M} = (\mathbf{M}^m, \dots, \mathbf{M}^1, \mathbf{M}^0) \in \mathbb{R}^{n \times (n(m+1))}$ contains the ownership matrices for the m seniority levels and the equity ownership matrix and $\mathbf{d} = (\mathbf{d}^m, \dots, \mathbf{d}^1) \in \mathbb{R}^{n \times m}$ represents the nominal values of the outstanding debt for each seniority level.

Associated with the m liability vectors we consider m recovery claim vectors $\mathbf{r}^k \in (\mathbb{R}_0^+)^n, k = 1, \dots, m$. The recovery claim vectors represent the actual payments of the firms at maturity, i.e. in general we have $\mathbf{r}^k \leq \mathbf{d}^k$ since default risk is present. For a given ownership fraction M_{ij}^k, the value of the debt claim that firm i has to firm j in the k-th seniority level is given by $M_{ij}^k r_j^k$. The total value of firm i's debt claims against the other members of the system is $\sum_{j=1}^n M_{ij}^k r_j^k$ for the corresponding seniority class k. Denote by $\mathbf{r}^0 \in \mathbb{R}^n$ the equity values of the n firms. The notation is in accordance with the fact that equity can be interpreted as the liability with the lowest priority level – the residual liability. The total value of claims of firm i against all other firms in the system is therefore given by the i-th entry of

$$\sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k. \tag{2.8}$$

The sum in (2.8) represents the incoming payments on the asset side of the firms' balance sheet stemming from cross-holdings. Since these assets are not exogenously priced such as $\mathbf{a}$, but within the system, we will sometimes use the expression *endogenous assets*.

At this stage, note that in our framework we assume that the entries of $\mathbf{d}^m, \dots, \mathbf{d}^1$ are constant. Since it is assumed that the exogenous assets' prices are given by the constant vector $\mathbf{a}$, the main result of Theorem 2.7 about the uniqueness and the existence of a payment vector also holds if the $\mathbf{d}^k$ $(1 \leq k \leq m)$ depend on $\mathbf{a}$, i.e. if $\mathbf{d}^k = \mathbf{d}^k(\mathbf{a})$. However, for the remainder, we will write $\mathbf{d}^k$ for convenience. This definition of the liability vectors allows the interpretation that the liabilities are simple loans or zero coupon bonds since they are not derivatives that can depend on the other assets within the system. The case of constant liabilities is used in most existing publications in this field, whereas the more general case in which the $\mathbf{d}^k$ can depend on the endogenous assets and on their own recovery values $\mathbf{r}^k$, is also treated in the literature (see Fischer, 2014). An example for such a system is given in Section 3.2.2.

The basic assumption for the model is that the outstanding liabilities of a higher priority level have to be paid completely before any lower ranked liability can be paid back. Equity is treated as the most junior class of liability. This convention is known as the *Absolute Priority Rule*. Hence, the equity value r_i^0 of firm i can only be strictly positive if firm i can fully satisfy all of its obligees. The Absolute Priority Rule immediately leads to the following *liquidation value equations* for the recovery claims and the equities (cf. Fischer, 2014):

$$\mathbf{r}^m = \min\left\{\mathbf{d}^m, \quad \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k\right\} \tag{2.9}$$

$$\mathbf{r}^j = \min\left\{\mathbf{d}^j, \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=j+1}^{m} \mathbf{d}^k\right)^+\right\} \quad (0 < j < m) \tag{2.10}$$

$$\mathbf{r}^0 = \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)^+. \tag{2.11}$$

A solution for the liquidation equations in (2.9) – (2.11) is therefore the fixed point of the mapping $\Phi : (\mathbb{R}_0^+)^{n(m+1)} \to (\mathbb{R}_0^+)^{n(m+1)}$, where

$$\Phi\begin{pmatrix} \mathbf{r}^m \\ \mathbf{r}^{m-1} \\ \vdots \\ \mathbf{r}^1 \\ \mathbf{r}^0 \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}^m, \quad \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k\} \\ \min\{\mathbf{d}^{m-1}, (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \mathbf{d}^m)^+\} \\ \vdots \\ \min\{\mathbf{d}^1, \quad (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=2}^{m} \mathbf{d}^k)^+\} \\ (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k)^+ \end{pmatrix}. \tag{2.12}$$

Note that we consider Φ only on $(\mathbb{R}_0^+)^{n(m+1)}$. This restriction is justifiable since it is guaranteed that every vector $\mathbf{R} = (\mathbf{r}^m, \ldots, \mathbf{r}^0)$ for which the liquidation equations in (2.9) – (2.11) hold is non-negative, see Fischer (2014) for a proof which can easily be adjusted for the case with ownership matrices, for which the Elsinger Property (see Definition 2.5) holds. We will sometimes refer to a fixed point of Φ as a *solution* of the financial system $\mathcal{F}$. A crucial property of the mapping Φ we will make use of is the monotonicity with regard to the recovery claims.

Lemma 2.4. *For $k = 0, \ldots, m$, let the ownership structure of the system be described by left substochastic ownership matrices $\mathbf{M}^k \in \mathbb{R}^{n\times n}$. If for $\mathbf{R}^1 = \left(\mathbf{r}^{1,m}, \ldots, \mathbf{r}^{1,0}\right) \in (\mathbb{R}_0^+)^{n(m+1)}$ and $\mathbf{R}^2 = \left(\mathbf{r}^{2,m}, \ldots, \mathbf{r}^{2,0}\right) \in (\mathbb{R}_0^+)^{n(m+1)}$ we have that $\mathbf{R}^1 \leq \mathbf{R}^2$, then $\Phi(\mathbf{R}^1) \leq \Phi(\mathbf{R}^2)$.*

Proof. Since the $\mathbf{M}^k$ are all ownership matrices, we get that

$$\sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{k,1} \leq \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{k,2}. \tag{2.13}$$

The claim follows immediately from Equation (2.12). □

We are interested in finding the fixed points of Φ, that means the solutions of the financial system $\mathcal{F}$. Without further constraints it is possible that there exist several solutions of $\mathcal{F}$. To ensure that the solution is unique, we have to require a certain condition for the form of the ownership matrices.

Definition 2.5. An ownership matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ possesses the *Elsinger Property* if there exists no subset $\mathcal{J} \subset \mathcal{N}$ such that

$$\sum_{i\in\mathcal{J}} M_{ij} = 1 \quad \text{for all } j \in \mathcal{J}. \tag{2.14}$$

The name of this property is chosen because Elsinger (2009) is, by the best knowledge of the author, the first one to use this assumption in the context of ownership matrices and the valuation under systemic risk. For our model, we demand that the considered ownership matrices have this property.

Assumption 2.6. *The Elsinger Property holds for all ownership matrices* $\mathbf{M}^k$ *with* $0 \leq k \leq m$.

Note that the fact that $\mathbf{M}^k$ having the Elsinger Property is equivalent with the existence of $(\mathbf{I}_n - \mathbf{M}^k)^{-1}$, as shown by Elsinger (2009). Moreover, Assumption 2.6 ensures that there is only one fixed point of Φ, as the following theorem shows. A proof of the Theorem 2.7 is given in the Appendix in Section A, and basically relies on the proof in Hain and Fischer (2015). In a similar form, the proof can also be found in Fischer (2015).

Theorem 2.7. *Under Assumption 2.6, the fixed point of the mapping* Φ *is unique and non-negative for an arbitrary financial system* $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$.

In the sequel, we assume that Assumption 2.6 holds so that $\mathcal{F}$ has only one solution denoted by $\mathbf{R}^* \in (\mathbb{R}_0^+)^{n(m+1)}$, defined as

$$\mathbf{R}^* := \begin{pmatrix} \mathbf{r}^{*,m} \\ \mathbf{r}^{*,m-1} \\ \vdots \\ \mathbf{r}^{*,0} \end{pmatrix} = \Phi \begin{pmatrix} \mathbf{r}^{*,m} \\ \mathbf{r}^{*,m-1} \\ \vdots \\ \mathbf{r}^{*,0} \end{pmatrix} = \Phi(\mathbf{R}^*). \tag{2.15}$$

Remark 2.8. Note that the assumption that $\mathbf{a} \geq \mathbf{0}$ is not necessary for the validity of Theorem 2.7. In the more general case of $\mathbf{a} \in \mathbb{R}^n$ the first n components of the liquidation equations in (2.9) have to be modified to

$$\mathbf{r}^m = \min \left\{ \mathbf{d}^m, \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k \right)^+ \right\} \tag{2.16}$$

and therefore the mapping Φ in (2.12) as well. However, this does not affect the uniqueness of the solution $\mathbf{R}^*$, as shown in Elsinger (2009). See also Section 3.2.1 for more details on the case of negative exogenous asset values.

At the end of this section, we present two assertions that will become useful in the remainder of this work.

Lemma 2.9. *Let* $\mathbf{R}^* = (\mathbf{r}^{*,m}, \ldots, \mathbf{r}^{*,0})$ *the a fixed point of the mapping* Φ. *Then it holds for the equity components that*

$$\mathbf{r}^{*,0} = \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=1}^{m} \mathbf{d}^k \right)^+ = \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=1}^{m} \mathbf{r}^{*,k}. \tag{2.17}$$

Proof. We check Equation (2.17) component-wise for a firm $i \in \mathcal{N}$. The firm i is either in default or solvent under $\mathbf{R}^*$. If it is in default in seniority level k ($k = 1, \ldots, m$), i.e. $r_i^{*,k+1} = d_i^{k+1}$ and $r_i^{*,k} < d_i^k$, it follows for the recovery value in seniority level k that

$$r_i^{*,k} = \min \left\{ d_i^k, \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=k+1}^{m} \mathbf{d}^l \right)_i^+ \right\} = \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=k+1}^{m} \mathbf{d}^l \right)_i. \tag{2.18}$$

Since $r_i^{*,l} = d_i^l$ for $m \geq l > k$ and $r_i^{*,l} = 0$ for $0 \leq l < k$, this yields to

$$\begin{aligned}\left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=1}^{m} \mathbf{r}^{*,l}\right)_i &= \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=k+1}^{m} \mathbf{d}^l - \mathbf{r}^{*,k}\right)_i^+ \\ &= 0 \qquad (2.19) \\ &= \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=1}^{m} \mathbf{d}^l\right)_i^+ .\end{aligned}$$

Aggregating this for all n firms leads to (2.17). For a solvent firm, it holds that $r_i^{*,k} = d_i^k$ for all $1 \leq k \leq m$ and there is nothing to show. □

Lemma 2.10. *Let* $\mathbf{r}^0(\mathbf{r}^m, \ldots, \mathbf{r}^1) \in (\mathbb{R}_0^+)^n$ *be the* equity vector for the corresponding debt payments $\mathbf{r}^m, \ldots, \mathbf{r}^1$ *such that*

$$\mathbf{r}^0(\mathbf{r}^m, \ldots, \mathbf{r}^1) = \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{r}^k + \mathbf{M}^0 \mathbf{r}^0(\mathbf{r}^m, \ldots, \mathbf{r}^1) - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ . \qquad (2.20)$$

Then $\mathbf{r}^0(\mathbf{r}^m, \ldots, \mathbf{r}^1)$ *is in increasing in* $(\mathbf{r}^m, \ldots, \mathbf{r}^1)$.

Proof. See Fischer (2015, Lemma 10.3) for a proof. □

In the remainder of this work, we will use a financial system $\mathcal{F}$ for which the components fulfill the following conditions:

1. Potentially several seniority classes are allowed: $m \geq 1$.
2. Exogenous asset values can only take non-negative values: $\mathbf{a} \in (\mathbb{R}_0^+)^n$.
3. All $m + 1$ ownership matrices possess the Elsinger Property.
4. The liability vectors $\mathbf{d}^m, \ldots, \mathbf{d}^1$ are assumed to be constant.

This model is in this work also referred to as the *standard model* of a financial system. In this thesis, the standard model is considered unless otherwise stated. In Chapter 3, we will discuss in more detail possible modifications of the model components and consequences for existence and uniqueness results arising thereby.

3 Literature Review

In recent years, a substantial growth in research interest in the topic of systemic financial risk could be observed which can be deduced primarily from a continuously increasing number of publications in this research area. Since it is still a relatively young discipline, the works and articles to this subject seem to be a bit inconsistent yet. One indication for this conclusion is the fact that the nomenclature to describe the single components of the model differs from article to article. Take on the one hand the expressions used to describe the members of such a system. For instance, the range of names lasts from "nodes" (Eisenberg and Noe, 2001), "agents" or "investors" (Shin, 2008), "banks" or "institutions" (Gouriéroux et al., 2012) to "organizations" (Elliott et al., 2014) to mention a just few. In a similar manner, this also holds for the other model components like the exogenous assets, see also Section 3.2.1. On the other hand, even the postulated assumptions on the investigated models are not uniformly made in the sense that various regularity conditions are used for the single components like the assets or the ownership matrices. A consequence of this plethora of assumptions are differing results on existence and, in particular, on uniqueness of potential equilibrium solutions of the considered financial systems.

Beside such inconsistencies concerning the naming, it is also observable that some works focus on the same topics without referring to one another. An example is the work of Suzuki (2002) that can be interpreted as some kind of generalization of the model of Eisenberg and Noe (2001). However, no mention of Eisenberg and Noe's ideas can be found in the article. On the other side, it seems that other authors like Shin (2008) and Elsinger (2009), who both develop essential contributions regarding existence and uniqueness results for more general financial systems, were unaware of the insights of Suzuki. Some publications also obtain very similar or even identical results without citing each other. The article of Gouriéroux et al. (2012) uses identical model assumptions as Suzuki (2002) in his article (cf. Table 3.1 for details) yielding the same results about existence and uniqueness. In Demange (2011), a sensitivity analysis of the financial system is conducted similar to the one in Liu and Staum (2010) without mentioning these results. A last example is the work of Ren et al. (2014) which basically contains the same existence and uniqueness results of Demange (2011) but lacks a cross-reference.

With that background in mind, this chapter is an attempt to survey and unify the existing literature on systemic risk. We start with a short chronological documentation of the model development in this area by listing important articles and their results in Section 3.1. A similarity in all works is that the components of the financial system are the same, namely the exogenous assets, the liabilities and the ownership matrices. The specific definition of the components, however, differs from study to study. In Section 3.2 we therefore will for each component compare differing definitions and try to show how they can be unified in a general framework. The main question for financial models of this type is whether there exists a single solution of the financial system in the form of a payment equilibrium and, if so, which conditions have to be fulfilled for this purpose. Based on different existing model assumptions, the authors came to individual conclusions on model assumptions for uniqueness. It will turn out that these conditions can be summarized into several regularity conditions presented in Section 3.3. All existing works can be incorporated into these conditions for which existence and uniqueness results can be stated. Section 3.4 shortly mentions some possible extensions of the standard model.

Note that we do not claim to give a complete overview of all existing articles in this field since this would go far beyond the scope of this work. The primary goal is rather to highlight that the existing model assumptions can be sorted into several regularity conditions and that for each condition a separate uniqueness proof can be given. Our hope is that these efforts will bring some order into the widespread literature and avoid unnecessary effort in further research.

3.1 Evolution of Literature

The work of Eisenberg and Noe (2001) is widely accepted to be the starting point in systemic financial risk and probably one of the most cited papers in this research area. The main result of it is that under some specific conditions – the financial system has to be "regular" – there exists exactly one solution for systems with one seniority and no cross-holdings in equity ($m = 1$, $\mathbf{M}^0 = \mathbf{0}_{n\times n}$). Moreover, the authors present an elegant algorithm ("fictitious default algorithm") that is able to find the solution in no more than n iteration steps. The model defined by Eisenberg and Noe is used in many theoretical and empirical studies to assess the relationship of cross-holdings and contagion in financial networks. For an overview of existing literature on contagion we refer to Staum (2012) or the work of Upper (2011) for a survey of empirical studies on this topic. The mathematical background of such models to assess systemic risks is described in Elsinger et al. (2013).

Only one year after the appearance of the manuscript of Eisenberg and Noe (2001), Suzuki (2002) includes possible cross-holdings in the firms' shares into the model ($\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$). Suzuki's paper is a stand-alone work for two reasons. On the one hand, according to the reference list of the article, the author was not aware of the fact that he is generalizing Eisenberg and Noe's model. His main intention was to generalize the famous Merton-Model by including endogenous assets due to cross-ownership into the firms' balance sheet. On the other hand, the contributions of Suzuki to systemic risk analysis in financial networks have gone unnoticed by the scientific community since no article appearing after 2002 cites Suzuki's work. This is remarkable insofar that some works, like the one of Gouriéroux et al. (2012) rely on the same model assumptions and obtain the same existence and uniqueness results as Suzuki (2002). The first publication that honors Suzuki's insights more than ten years after publication is Fischer (2014). An important development of the model defined by Eisenberg and Noe offers the work of Shin (2008). The author generalizes the liability structure of the financial system in the sense that the debt obligations can be of multiple seniority ($m > 1$, $\mathbf{M}^0 = \mathbf{0}_{n\times n}$). Albeit ignoring cross-holdings in the equities, Shin shows that, under certain assumptions, there exists a unique equilibrium of debt payments.

For two reasons, the next milestone in developing the financial system to a more general structure is the article of Elsinger (2009). First, Elsinger extends Shin's model and includes possible cross-holdings of the firm's shares, i.e. $m > 1$ and $\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$. Second, the model allows for a more general liability structure which means that firms can have obligations to bondholders outside the system. This was not possible in the models of Eisenberg and Noe (2001) and Shin (2008). Together with a new regularity condition on the ownership matrices, the Elsinger Property (cf. Definition 2.5), (sufficient) conditions are derived under which a unique pricing equilibrium exists. Beyond the existence and uniqueness results, the author presents a new algorithm that determines the firm's equity values for a given vector of debt payments, see also Section 4.1.2 for more details.

Table 3.1: Overview of mentioned articles in this chapter with a focus on systemic risk analysis considering cross-holdings in the firms' balance sheet and their model assumptions.

Study	Seniority levels	Equity cross-holdings	Ownership structure	Regularity conditions
Eisenberg and Noe (2001)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC2)
Suzuki (2002)	$m = 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Cifuentes et al. (2005)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC5)
Müller (2006)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	n.s
Shin (2008)	$m > 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC3)
Shin (2009)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC3)
Elsinger (2009)	$m > 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS3)	(RC4)
Liu and Staum (2010)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC2)
Demange (2011)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC4)
Gouriéroux et al. (2012)	$m = 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Gouriéroux et al. (2013)	$m = 2$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Rogers and Veraart (2013)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	n.s.
Fischer (2014)	$m > 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Elliott et al. (2014)	$m = 0$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Ren et al. (2014)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC2)
Acemoglu et al. (2015)	$m = 2$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS2)	(RC2)
Fischer (2015)	$m > 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS1)	(RC1)
Glasserman and Young (2015)	$m = 1$	$\mathbf{M}^0 = \mathbf{0}_{n\times n}$	(OS3)	(RC4)
Awiszus and Weber (2015)	$m = 1$	$\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$	(OS3)	n.s.

Notes: In Elliott et al. (2014), $m = 0$ means that no cross-holdings in debt are taken into account. The Definitions of (OS1) – (OS3) concerning the column sums of the ownership matrices can be found in Section 3.2.3. In Section 3.3.2 the regularity conditions (RC1) – (RC4) are explained in more detail. Note that in the articles of Eisenberg and Noe (2001) and Liu and Staum (2010) the actual regularity condition is a more general one than (RC2) and that (RC2) is only a particular case of it. For convenience, we omit the more general assumption since this does not essentially change the insights, see also the comments after Definition 3.13. The expression "not stated" (n.s.) in the last column of the table includes studies in which either no regularity conditions are stated or where no such conditions are needed since the focus is not on showing the uniqueness of a payment equilibrium. Note that the financial systems in the studies sometimes have additional model assumptions to the ones given in the table. This entails for example the structure of the exogenous assets, where in some articles $\mathbf{a} \geq \mathbf{0}_n$ was demanded and in some works the more general case of $\mathbf{a} \in \mathbb{R}^n$ is allowed.

In later works, some slight modifications of the existing models are presented. As an example, we mention the work of Demange (2011) who utilizes the model of Eisenberg and Noe ($m = 1$, $\mathbf{M}^0 = \mathbf{0}_{n\times n}$) and allows the exogenous asset vector to take also negative values, i.e. $\mathbf{a} \in \mathbf{R}^n$. The article derives conditions that have to be fulfilled for the asset vector $\mathbf{a}$ and the debt ownership matrix $\mathbf{M}^1$ for the existence of a unique solution. Moreover, the author shows how the results of Eisenberg and Noe (2001) can be included in her framework. However, as we will show later in Section 3.2.1, the assumption of negative exogenous asset values is equivalent with the presence of an additional seniority level and, as a consequence, that $\mathbf{a} \geq \mathbf{0}_n$ can be assumed without loss of generality. Another example of a slight modification is the article Gouriéroux et al. (2013) that derive an existence and a uniqueness result for a system with two seniority levels ($m = 2$, $\mathbf{M}^0 \neq \mathbf{0}_{n \times n}$). Their proof relies on different methods than the existing works – they show that the financial system can be converted into a piecewise linear system and show the invertibility of this system. Even though the means to show the uniqueness are innovative, the model used from Gouriéroux et al. can easily be embedded into the work of Elsinger (2009).

In the article of Fischer (2014), a further essential progression of the model is shown concerning the particular form of the debt obligations. In all works before, the liabilities were interpreted as simple zero-coupon bonds with a constant value at maturity. Fischer allows the liabilities to be derivatives that can depend on the exogenous assets as well as on the recovery values, see Section 3.2.2 for more details and examples. Under a regularity assumption on the liabilities, similar to Lipschitz-continuity, there still exists a unique solution of the financial system. A slight drawback of the generalization is that for a unique solution, the condition on the ownership matrices is more strict than the one given in Elsinger (2009). A selection of articles that entail the introduction of default costs are the works of Rogers and Veraart (2013), Elliott et al. (2014) and Glasserman and Young (2015) that are explained in Section 3.4. Table 3.1 gives a short overview of the mentioned articles in this Chapter.

3.2 Differences in Model Assumptions

In its most general form the financial system consists of potentially more than one seniority level, i.e. $m \geq 1$, where cross-holdings of the firms' shares are allowed ($\mathbf{M}^0 \neq \mathbf{0}_{n \times n}$). Contrary to the standard model defined in Section 2.2, we also assume in a first approach that the exogenous assets must not necessarily be positive, that is $\mathbf{a} \in \mathbb{R}^n$. All models that are investigated in the articles of Table 3.1 can be expressed using this framework. In this section, we want to take a closer look at the components of the system, namely the exogenous assets, the ownership matrices and the liabilities and, if present, a potential seniority structure by giving a survey of their differing definitions in the existing articles.

3.2.1 Exogenous Assets

The name "exogenous assets" is not used in all articles, the expressions "fundamental assets" (Shin, 2008), "net worth" or "operating cashflow" (Gouriéroux et al., 2012), "business assets" (Suzuki, 2002), "initial wealth" (Liu and Staum, 2010), "exogenous income" (Elsinger, 2009), "primitive assets" (Elliott et al., 2014) and "net assets" (Rogers and Veraart, 2013) may serve as an incomplete selection of different names for $\mathbf{a}$. Though the assets are called different, their common property is that the pricing mechanism does not depended on the structure or the solvency of the firms in the financial system. An exception from this exogenous approach is for example the work of Cifuentes et al. (2005), where the assets are endogenously modeled as a potential channel of contagion.

In most articles, the assets simply form a vector of size n. However, in some works (see Suzuki, 2002, Fischer, 2014 or Elliott et al., 2014) the asset structure is modeled more general assuming that the firms have access to p different assets that are given in a vector $\mathbf{a} \in \mathbb{R}^p$. In this case, a matrix $\mathbf{M}^{\mathbf{a}} \in (\mathbb{R}_0^+)^{n \times p}$ has to be introduced, where $M_{ij}^{\mathbf{a}}$ denotes the proportion, firm i holds of the j-th asset. The advantage of this more flexible asset structure is a more general modeling of the liabilities, since they can also – under some model assumptions – depend on the values of the exogenous assets, see Fischer (2014) for details and also Example 3.6. The more general representation of the assets, however, does not change the main results about the existence and the uniqueness of solutions of such financial systems. The vector $\mathbf{a}$ in the liquidation equations simply has to be replaced with the product $\mathbf{M^a a}$, which is why we omit the asset matrix $\mathbf{M^a}$ in our model, assume that $p = n$ and set $\mathbf{M^a} = \mathbf{I}_n$ for convenience.

Another property of the assets is that in most articles they are assumed to be non-negative, i.e. $\mathbf{a} \geq \mathbf{0}_n$, whereas in some works (Elsinger, 2009 or Demange, 2011), the asset structure is more generally modeled by $\mathbf{a} \in \mathbb{R}^n$, which entails the particular case that $a_i < 0$ for some $i \in \mathcal{N}$. We want to show in the following that we can assume $\mathbf{a} \geq \mathbf{0}_n$ without loss of generality. For a better comprehensibility, we consider the financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}^1, \mathbf{M}^0, \mathbf{d}^1)$, i.e. we only take one seniority level into account. The forthcoming argumentation can easily be extended to systems with $m > 1$. More precisely, assume in the following that $\mathbf{a} \in \mathbb{R}^n$, where $a_i < 0$ for at least one $i \in \mathcal{N}$. Given the input parameters, a solution of this system is the fixed point of the mapping $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where

$$\Phi \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^0 \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}^1, (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0)^+\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 - \mathbf{d}^1)^+ \end{pmatrix}. \tag{3.1}$$

Clearly, we have to adapt the original mapping in Equation (2.12) and take the positive part in the debt components to avoid negative recovery claim values. However, (3.1) still has a unique fixed-point $\mathbf{R}^*$ if Assumption 2.6 holds (cf. Remark 2.8). Negative values in the vector of the exogenous assets can be interpreted as liabilities of the corresponding firms to debtholders outside the system (cf. Demange, 2011). Note that since the debt ownership matrices must not be fully left stochastic, there are in every seniority level some investors outside the financial system that hold some fractions of the firms' debt. However, in this subsection the expression "outside obligations" only refers to the obligations that are contained in the asset vector $\mathbf{a}$ in the form of negative values. Moreover, these liabilities have a higher priority than the highest priority level in the system. To include these obligations of the firms into a system with nonnegative exogenous assets, we have to add one additional seniority level in which the outer-system obligations of the firms are listed. Hence, the original system $\mathcal{F}$ is modified to a system $\widetilde{\mathcal{F}} = (\tilde{\mathbf{a}}, \widetilde{\mathbf{M}}^2, \widetilde{\mathbf{M}}^1, \widetilde{\mathbf{M}}^0, \tilde{\mathbf{d}}^2, \tilde{\mathbf{d}}^1)$, with

$$\tilde{\mathbf{a}} = \mathbf{a}^+, \; \tilde{\mathbf{d}}^2 = \mathbf{\Lambda}|\mathbf{a}|, \; \tilde{\mathbf{d}}^1 = \mathbf{d}^1, \; \widetilde{\mathbf{M}}^2 = \mathbf{0}_{n\times n}, \; \widetilde{\mathbf{M}}^1 = \mathbf{M}^1 \text{ and } \widetilde{\mathbf{M}}^0 = \mathbf{M}^0, \tag{3.2}$$

where $\mathbf{\Lambda} = \operatorname{diag}(\mathbf{a} < \mathbf{0}_n)$ is the diagonal matrix with the value 1 on the diagonal if the exogenous assets of the corresponding firm are negative. Note that $\mathbf{\Lambda}|\mathbf{a}| = \mathbf{a}^+ - \mathbf{a}$, where $|\mathbf{a}|$ stands for the vector of the element-wise absolute values of the entries in $\mathbf{a}$. The solution of $\widetilde{\mathcal{F}}$ is given as the fixed point of the mapping $\tilde{\Phi} : (\mathbb{R}_0^+)^{3n} \to (\mathbb{R}_0^+)^{3n}$ with

$$\tilde{\Phi} \begin{pmatrix} \tilde{\mathbf{r}}^2 \\ \tilde{\mathbf{r}}^1 \\ \tilde{\mathbf{r}}^0 \end{pmatrix} = \begin{pmatrix} \min\{\tilde{\mathbf{d}}^2, \; \tilde{\mathbf{a}} + \widetilde{\mathbf{M}}^2\tilde{\mathbf{r}}^2 + \widetilde{\mathbf{M}}^1\tilde{\mathbf{r}}^1 + \widetilde{\mathbf{M}}^0\tilde{\mathbf{r}}^0\} \\ \min\{\tilde{\mathbf{d}}^1, (\tilde{\mathbf{a}} + \widetilde{\mathbf{M}}^2\tilde{\mathbf{r}}^2 + \widetilde{\mathbf{M}}^1\tilde{\mathbf{r}}^1 + \widetilde{\mathbf{M}}^0\tilde{\mathbf{r}}^0 - \tilde{\mathbf{d}}^2)^+\} \\ (\tilde{\mathbf{a}} + \widetilde{\mathbf{M}}^2\tilde{\mathbf{r}}^2 + \widetilde{\mathbf{M}}^1\tilde{\mathbf{r}}^1 + \widetilde{\mathbf{M}}^0\tilde{\mathbf{r}}^0 - \tilde{\mathbf{d}}^2 - \tilde{\mathbf{d}}^1)^+ \end{pmatrix}. \tag{3.3}$$

The relationship between the two mappings Φ and $\tilde{\Phi}$ is explained in the following. Beforehand, we take a closer look at the mapping $\tilde{\Phi}$.

Lemma 3.1. *With the definitions above, let*

$$\widetilde{\mathbf{R}}^* = (\tilde{\mathbf{r}}^{*,2}, \tilde{\mathbf{r}}^{*,1}, \tilde{\mathbf{r}}^{*,0}) \in (\mathbb{R}_0^+)^{3n} \tag{3.4}$$

be the fixed point of the mapping $\tilde{\Phi}$. Then, $\widetilde{\mathbf{R}}$ is unique and it holds that

$$\widetilde{\mathbf{R}}^* = \begin{pmatrix} \tilde{\mathbf{r}}^{*,2} \\ \tilde{\mathbf{r}}^{*,1} \\ \tilde{\mathbf{r}}^{*,0} \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{\Lambda}|\mathbf{a}|, \mathbf{a}^+ + \mathbf{M}^1\tilde{\mathbf{r}}^{*,1} + \mathbf{M}^0\tilde{\mathbf{r}}^{*,0}\} \\ \min\{\mathbf{d}^1, (\mathbf{a} + \mathbf{M}^1\tilde{\mathbf{r}}^{*,1} + \mathbf{M}^0\tilde{\mathbf{r}}^{*,0})^+\} \\ (\mathbf{a} + \mathbf{M}^1\tilde{\mathbf{r}}^{*,1} + \mathbf{M}^0\tilde{\mathbf{r}}^{*,0} - \mathbf{d}^1)^+ \end{pmatrix}. \tag{3.5}$$

Proof. The uniqueness of $\widetilde{\mathbf{R}}^*$ follows by that fact that all ownership matrices of $\widetilde{\mathcal{F}}$ obviously have the Elsinger Property. The rest of the claim follows by (3.2) and elementary calculations. □

A direct consequence of the structure of $\widetilde{\mathbf{R}}^*$ in (3.5) is that it is well-defined if only the entries in the recovery values of $\tilde{\mathbf{r}}^{*,1}$ and $\tilde{\mathbf{r}}^{*,0}$ are known. These components can be represented as

$$\begin{pmatrix} \tilde{\mathbf{r}}^{*,1} \\ \tilde{\mathbf{r}}^{*,0} \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}^1, (\mathbf{a} + \mathbf{M}^1\tilde{\mathbf{r}}^{*,1} + \mathbf{M}^0\tilde{\mathbf{r}}^{*,0})^+\} \\ (\mathbf{a} + \mathbf{M}^1\tilde{\mathbf{r}}^{*,1} + \mathbf{M}^0\tilde{\mathbf{r}}^{*,0} - \mathbf{d}^1)^+ \end{pmatrix} = \Phi \begin{pmatrix} \tilde{\mathbf{r}}^{*,1} \\ \tilde{\mathbf{r}}^{*,0} \end{pmatrix}. \tag{3.6}$$

Hence $(\tilde{\mathbf{r}}^{*,1}, \tilde{\mathbf{r}}^{*,0})$ is the fixed point of Φ for $m = 1$. On the other hand, given $(\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ as the fixed point of Φ, this vector can be extended via

$$\overline{\mathbf{R}} := \begin{pmatrix} \min\{\mathbf{\Lambda}|\mathbf{a}|, \mathbf{a}^+ + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\} \\ \min\{\mathbf{d}^1, (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0})^+\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}^1)^+ \end{pmatrix} \tag{3.7}$$

to the (unique) fixed point of $\tilde{\Phi}$ since $\tilde{\Phi}(\overline{\mathbf{R}}) = \overline{\mathbf{R}}$.

We have shown that with the definitions above, the solution $\mathbf{R}^*$ of the system $\mathcal{F}$ is also a solution of the system $\widetilde{\mathcal{F}}$ with the definition in (3.7). Further, if the solution $\widetilde{\mathbf{R}}^*$ of $\widetilde{\mathcal{F}}$ is known, the solution of $\mathcal{F}$ is also known.

Corollary 3.2. *In a financial system $\mathcal{F}(\mathbf{a}, \mathbf{M}^1, \mathbf{M}^0, \mathbf{d}^1)$ it can without loss of generality be assumed that $\mathbf{a} \geq \mathbf{0}_n$.*

Due to Corollary 3.2 we assume for the remainder of the entire thesis that the exogenous assets are non-negative, unless otherwise stated.

Example 3.3. Consider Example 2 from Elsinger (2009) with the system $\mathcal{F} = (\mathbf{a}, \mathbf{M}^1, \mathbf{M}^0, \mathbf{d}^1)$ and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0.75 \\ -9/8 \end{pmatrix}, \quad \mathbf{d}^1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{M}^1 = \begin{pmatrix} 0 & 0.5 & 0.25 \\ 0 & 0 & 0.75 \\ 0 & 0.5 & 0 \end{pmatrix} \tag{3.8}$$

and $\mathbf{M}^0 = \mathbf{0}_{n\times n}$. Using Φ as defined above yields a fixed point of

$$\mathbf{R}^* = \begin{pmatrix} \mathbf{r}^{*,1} \\ \mathbf{r}^{*,0} \end{pmatrix} = (1, 0.75, 0, 0.375, 0, 0)^t, \tag{3.9}$$

i.e. the first firm is solvent and the second and the third firm are in default. Extending the system to $\widetilde{\mathcal{F}} = (\tilde{\mathbf{a}}, \widetilde{\mathbf{M}}^2, \widetilde{\mathbf{M}}^1, \widetilde{\mathbf{M}}^0, \tilde{\mathbf{d}}^2, \tilde{\mathbf{d}}^1)$, where

$$\tilde{\mathbf{a}} = (1, 0.75, 0)^t, \ \tilde{\mathbf{d}}^2 = (0, 0, 9/8)^t \tag{3.10}$$

and the remaining vectors and matrices are defined as in (3.2), yields

$$\widetilde{\mathbf{R}}^* = \begin{pmatrix} \tilde{\mathbf{r}}^{*,2} \\ \tilde{\mathbf{r}}^{*,1} \\ \tilde{\mathbf{r}}^{*,0} \end{pmatrix} = \begin{pmatrix} (0, & 0, & 0.375)^t \\ (1, & 0.75, & 0 \quad)^t \\ (0.375, & 0, & 0 \quad)^t \end{pmatrix}. \tag{3.11}$$

We see that Corollary 3.2 holds, since taking $\tilde{\mathbf{r}}^{*,1}$ and $\tilde{\mathbf{r}}^{*,0}$ together leads to $\mathbf{R}^*$ as in (3.9).

For the considerations above, it was crucial that a new seniority level was installed in which the external obligations were summarized with a higher priority than the already existing seniority classes. In Eisenberg and Noe (2001), the authors mention that assuming $\mathbf{a} \geq \mathbf{0}_n$ without loss of generality can be achieved by adding an additional member, they call it "sink node", to the system. Contrary to the approach above, the external obligations are included in the liabilities of the current highest seniority. This would result in a new financial system $\overline{\mathcal{F}} = (\bar{\mathbf{a}}, \overline{\mathbf{M}}^1, \overline{\mathbf{M}}^0, \bar{\mathbf{d}})$ of size $n+1$ with $m = 1$ and

$$\bar{\mathbf{a}} = \begin{pmatrix} \mathbf{a}^+ \\ 0 \end{pmatrix}, \; \bar{\mathbf{d}} = (\bar{d}_1, \ldots, \bar{d}_n, \bar{d}_{n+1})^t = \begin{pmatrix} \mathbf{d}^1 + \mathbf{\Lambda}|\mathbf{a}| \\ 0 \end{pmatrix}, \; \overline{\mathbf{M}}^0 = \begin{pmatrix} \mathbf{M}^0 & \mathbf{0}_n \\ \mathbf{0}_n^t & 0 \end{pmatrix}. \tag{3.12}$$

The debt ownership matrix $\overline{\mathbf{M}}^1 \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined for $i = 1, \ldots, n$ by

$$\overline{M}_{ji} = \begin{cases} (M_{ji} d_i)/\bar{d}_i, & \text{for } j = 1, \ldots, n, \\ (\mathbf{\Lambda}_{ii}|a_i|)/\bar{d}_i, & \text{for } j = n+1, \end{cases} \tag{3.13}$$

and $\overline{M}_{j,n+1} = 0$ for all $j = 1, \ldots, n+1$, where the matrix $\mathbf{\Lambda}$ is defined as above and $\mathbf{\Lambda}_{ii}$ are the diagonal entries that are 1 if $a_i < 0$ for the i-th firm. In $\overline{\mathcal{F}}$, potential obligations to external debtholders are added up to a new liability vector $\mathbf{d}^1$. As a consequence, the fractions in $\mathbf{M}^1$ in the corresponding columns have to be recalculated. The next example shows that using this approach will not lead to the same results as with the procedure above. This circumstance is also mentioned in a comment in Elsinger (2009).

Example 3.4. We use the same financial system as in Example 3.3. The components of the modified system $\overline{\mathcal{F}}$ are given by

$$\bar{\mathbf{a}} = \begin{pmatrix} 1 \\ 0.75 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\mathbf{d}}^1 = \begin{pmatrix} 1 \\ 2 \\ 2.125 \\ 0 \end{pmatrix}, \quad \overline{\mathbf{M}}^1 = \begin{pmatrix} 0 & 0.5 & 0.1176 & 0 \\ 0 & 0 & 0.3529 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5294 & 0 \end{pmatrix}, \tag{3.14}$$

which leads to a solution

$$\overline{\mathbf{R}} = (1, 0.9107, 0.4554, 0, 0.5089, 0, 0, 0.2411)^t. \tag{3.15}$$

We see that the first three components of the debt recovery values and the first three components of the equities are not identical to (3.9). The addition of a higher seniority class is therefore necessary.

3.2.2 Liability Structure

In most works, the expressions "liabilities", "debt" and "obligations" are used equivalently for the vectors $\mathbf{d}^k$. No matter how they are called, in almost all articles, the liabilities are assumed to be constant, which is for example the case if the liabilities are simple zero-coupon bonds. For practical purposes, this assumption seems to be fairly restrictive, as the debt structure is probably more complicated in many cases.

An exception to this common framework is the article of Fischer (2014) that generalizes the liability structure. In his work, the liabilities can depend on the exogenous assets $\mathbf{a}$ as well as on the recovery values $\mathbf{r}^m, \ldots, \mathbf{r}^0$. As a consequence, it may happen that the liabilities can even depend on their own payoff, as we will see in the Examples 3.6 and 3.7. By the best knowledge of the author, this is the only model that allows for derivatives as liabilities. To take this new aspect into account, the liabilities for the k-th seniority level ($k = 1, \ldots, m$) are defined as functions with $\mathbb{R}^{n(m+2)} \to (\mathbb{R}_0^+)^n$, with

$$\begin{pmatrix} \mathbf{r}^m \\ \vdots \\ \mathbf{r}^0 \\ \mathbf{a} \end{pmatrix} \mapsto \mathbf{d}^k_{\mathbf{r}^m,\ldots,\mathbf{r}^0,\mathbf{a}} = \begin{pmatrix} d_1^k(\mathbf{r}^m, \ldots, \mathbf{r}^0, \mathbf{a}) \\ \vdots \\ d_n^k(\mathbf{r}^m, \ldots, \mathbf{r}^0, \mathbf{a}) \end{pmatrix}. \tag{3.16}$$

The liquidation equations in (2.9) – (2.11) stay the same, the liability vectors for each seniority only have to be replaced by their counterparts defined in (3.16). In a similar manner, the mapping Φ has to be adapted.

Due to this new aspect of potential self-dependency, the question is whether there still exist fixed points of Φ as solutions of the financial system and, if so, whether there are conditions under which a solution is unique. Answers to these questions gives the following theorem.

Theorem 3.5 (Fischer (2014))**.** *Let $\mathcal{F}$ be a financial system for which the liabilities are defined as in* (3.16)*. If $I^{\max} < 1$, where*

$$I^{\max} = \max\{\|\mathbf{M}^m\|, \|\mathbf{M}^{m-1}\|, \ldots, \|\mathbf{M}^0\|\}, \tag{3.17}$$

the following holds:

1. *$\mathcal{F}$ has at least one solution, if the functions $\mathbf{d}^k$ are continuous for $k = 1, \ldots, m$. Further, all solutions are nonnegative.*
2. *The solution of $\mathcal{F}$ is unique if for $k = 1, \ldots, m$ and $i \in \mathcal{N}$ it holds that*

$$d_i^k(\mathbf{r}^m, \ldots, \mathbf{r}^0, \mathbf{a}) = \psi_i^k \left(\sum_{l=0}^{m} \sum_{j=1}^{n} M_{ij}^l r_j^l \right), \tag{3.18}$$

where the functions $\psi_i^k : \mathbb{R} \to \mathbb{R}_0^+$ are monotonically increasing such that for any $\mathbf{y}^1$, $\mathbf{y}^2 \in \mathbb{R}^n$ with $\mathbf{y}^1 \geq \mathbf{y}^2$,

$$\mathbf{y}^1 - \mathbf{y}^2 \geq \sum_{k=1}^{m} \Big(\psi_i^k(y_i^1) - \psi_i^k(y_i^2) \Big)_{i=1,\ldots,n}. \tag{3.19}$$

For a proof of this theorem, we refer to the original paper. The condition in (3.18) can be interpreted as a strong form of Lipschitz continuity for the liabilities. Note that the assumption $I^{\max} < 1$ for the ownership matrices is stronger than the one in Assumption 2.6, where the ownership matrices had to fulfill the Elsinger Property. We will come back to the different model assumptions concerning the properties of the ownership matrices in Section 3.3.

It is also possible to skip the assumption $I^{\max} < 1$ and only assume that the Elsinger Property holds for $\mathbf{M}^0$ but not for the debt ownership matrices. The uniqueness of the solution, however, gets lost in this situation but one can show that under the continuity assumption in (3.18) and the additional assumption that the liability vectors $\mathbf{d}^k$ are bounded for all $k = 1, \ldots, n$, there exists at least a Pareto-dominant solution[1] of the system. A more extensive overview of existence

[1] A solution $\mathbf{R} = (\mathbf{r}^m, \ldots, \mathbf{r}^0)$ is said to be *Pareto-dominant* if there exists no other vector $\widetilde{\mathbf{R}} = (\tilde{\mathbf{r}}^m, \ldots, \tilde{\mathbf{r}}^0)$ such that $\widetilde{\mathbf{R}} \geq \mathbf{R}$ and $\tilde{r}_i^k > r_i^k$ for at least one $i \in \mathcal{N}$ and one $k = 1, \ldots, m$.

and uniqueness results for various financial systems with different assumptions is given in the work of Fischer (2015).

For a better understanding, we give in the following two examples of financial systems in which the liabilities are non-constant. While in the first example, the condition in (3.18) holds and the solution is therefore unique, we will see in the second example that the uniqueness cannot be guaranteed anymore if (3.18) is violated.

Example 3.6 (Fischer (2015))**.** The financial system in this example consists out of $n = 3$ firms and $m = 2$ seniority levels. As described in Section 3.2.1, we assume that the firms have access to $l = 2$ exogenous assets $\mathbf{a} = (a_1, a_2)^t \geq (0, 0)^t$ and that there is an asset matrix $\mathbf{M^a} \in (\mathbb{R}_0^+)^{3\times 2}$ that contains the fractions each firm owns from the corresponding asset. The liability vectors

$$\mathbf{d}^2_{\mathbf{r}^2,\mathbf{r}^1,\mathbf{r}^0,\mathbf{a}} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{d}^1_{\mathbf{r}^2,\mathbf{r}^1,\mathbf{r}^0,\mathbf{a}} = \begin{pmatrix} 300(a_2 - k_1)^+ \\ (0.1r_3^0 + 0.05r_1^1 - k_2)^+ \\ b_3 a_1 \end{pmatrix} \tag{3.20}$$

are for $b_1, b_2, b_3, k_1, k_2 > 0$ derivatives of $\mathbf{a}$ and the vectors of recovery values. The ownership matrices are given by

$$\mathbf{M}^2 = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0.05 & 0.2 & 0 \end{pmatrix}, \; \mathbf{M}^1 = \begin{pmatrix} 0 & \min\{0.9, a_2\} & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

and

$$\mathbf{M}^0 = \begin{pmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.2 \\ 0 & 0.5 & 0 \end{pmatrix}, \; \mathbf{M^a} = \begin{pmatrix} 1000a_2 & 2000 \\ 0 & 5000 \\ 1000 & 0 \end{pmatrix}. \tag{3.22}$$

Note that the ownership matrices here depend on the values of the asset vector $\mathbf{a}$ which bears no problem for the uniqueness of the solution as long as $I^{\max} < 1$ is ensured. For given parameters $b_1, b_2, b_3, k_1, k_2, a_1$ and a_2, all entries of the liability vectors are fixed except of the second entry of $\mathbf{d}^1_{\mathbf{r}^2,\mathbf{r}^1,\mathbf{r}^0,\mathbf{a}}$. To check the validity of (3.18), it suffices therefore to do this only for firm 2. Note that for $x \in \mathbb{R}$, the functions ψ_2^1 and ψ_2^2 are given by

$$\psi_2^1(x) = (0.5x - k_2)^+ \text{ and } \quad \psi_2^2(x) = b_2 \tag{3.23}$$

and it follows directly that the condition in (3.18) is satisfied, see Fischer (2015) for more details.

Example 3.7 (Fischer (2014))**.** Let $n = 2$, $m = 1$ and $\mathbf{M}^0 = \mathbf{0}_{n\times n}$. The further parameters of $\mathcal{F}$ are given by

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \mathbf{M}^1 = \begin{pmatrix} 0 & 0.8 \\ 0.8 & 0 \end{pmatrix} \text{ and } \; \mathbf{d}^1_{\mathbf{r}^1} = \begin{pmatrix} 2 \cdot |r_2^1 - 2| \\ 2 \cdot |r_1^1 - 2| \end{pmatrix}. \tag{3.24}$$

Since the liabilities do not depend on the exogenous assets and on the equity values, we omit the terms $\mathbf{a}$ and $\mathbf{r}^0$. The functions ψ_1^1 and ψ_2^1 to check the conditions in (3.18) are for $x \in \mathbb{R}$ given by

$$\psi_1^1(x) = 2 \cdot |0.8^{-1}x - 2| = \psi_2^1(x) := \psi(x). \tag{3.25}$$

For an example that (3.18) is violated, take the vectors $\mathbf{y} = (4, 4)^t$ and $\mathbf{x} = (1, 1)^t$. It follows that

$$\mathbf{y} - \mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} < \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 2 \cdot |3| - 2 \cdot |-0.75| \\ 2 \cdot |3| - 2 \cdot |-0.75| \end{pmatrix} = \begin{pmatrix} \psi(4) - \psi(1) \\ \psi(4) - \psi(1) \end{pmatrix}, \tag{3.26}$$

but $\mathbf{y} \geq \mathbf{x}$. The solution of $\mathcal{F}$ consequently must not be unique anymore. As noted in Fischer (2014), two possible solutions are given by

$$\mathbf{R}^* = (4/3, 4/3, 11/15, 11/15) \quad \text{and} \quad \mathbf{R}^* = (4, 4, 0.2, 0.2). \tag{3.27}$$

3.2.3 Ownership matrices

The probably most crucial parameter for the properties of a financial system concerning the existence and the uniqueness of a solution is the form of the ownership matrices $\mathbf{M}^k$, $k = 0, \ldots, m$. As we will see in Section 3.3.1, the mildest condition, we can claim on the equity matrix $\mathbf{M}^0$ to ensure the existence of a solution, is the Elsinger Property from Definition 2.5. The priority in this subsection is hence on the debt ownership matrices, in particular we focus on the question whether the obligations of the firms are completely held by other members of the system or not which is expressed by the column sums of the corresponding debt ownership matrix. When surveying the existing articles on this content, we find three different types of debt ownership matrices $\mathbf{M}^k$:

$$\sum_{i=1}^{n} M_{ij}^k < 1 \qquad \text{for all } j \in \mathcal{N} \text{ and all } k = 1, \ldots, m, \tag{OS1}$$

$$\sum_{i=1}^{n} M_{ij}^k = 1 \qquad \text{for all } j \in \mathcal{N} \text{ and all } k = 1, \ldots, m, \tag{OS2}$$

$$\sum_{i=1}^{n} M_{ij}^k \leq 1 \qquad \text{for all } j \in \mathcal{N} \text{ and all } k = 1, \ldots, m. \tag{OS3}$$

The assumption in (OS1) means that for every firm and every seniority there must be at least one debtholder outside the financial system. A drawback of (OS1) is surely that it is a very strict assumption for practical purposes since it is already injured if for only one seniority class and one firm, all debt is held within the system. However, as noted in Gouriéroux et al. (2012), this restriction seems not to be too hard to be fulfilled since in most connected financial systems, it is likely that (OS1) can be ensured. The advantage of the assumption on the other hand, is that it strongly simplifies the argumentation of the proof that $\mathbf{R}^*$ is unique, as we will see in Section 3.3.2.

The counterpart of (OS1) is given in (OS2) where it is demanded that all debt payments have to stay within the system. This assumption is used in most of the studies in Table 3.1. The reason is that the ownership structure is often not directly defined via ownership matrices but by *liability matrices*. In the liability matrix $\mathbf{L}^k \in \mathbb{R}^{n \times n}$ for seniority level k, the entry L_{ij}^k stands for the nominal value of the obligations firm i has to firm j. It follows directly that $d_i^k = \sum_{j=1}^n L_{ij}^k$. Based on $\mathbf{L}^k$, we can define the corresponding ownership matrix $\mathbf{M}^k$ by

$$M_{ji}^k = \begin{cases} \frac{L_{ij}^k}{d_i^k}, & \text{if } d_i^k > 0, \\ 0, & \text{else.} \end{cases} \tag{3.28}$$

Using the definition in (3.28), we immediately see that $\sum_{j=1}^n M_{ji}^k = 1$ for all $i \in \mathcal{N}$ with $d_i^k > 0$ and that all debt ownership matrices are fully left stochastic if $d_i^k > 0$ for all $i \in \mathcal{N}$ and all $k = 1, \ldots, m$. As for (OS1), the assumption in (OS2) also seems to be a bit restrictive if real-life examples of financial networks are considered. The fact that the column sums of $\mathbf{M}^k$ have to be equal to one, might lead to an expansive enlargement of the financial system that has to be taken into account to ensure the validity of (OS2).

While in (OS2), it was not possible that there exist one or more outside debtholders that hold a fraction of each firms' debt, this assumption in generalized in (OS3). When assuming such a structure of the ownership matrices that are based on liability matrices $\mathbf{L}^k$, we suppose that outside obligations can for each firm and each seniority $k = 1, \dots, m$ be summarized in the vector $\tilde{\mathbf{d}}^k \in (\mathbb{R}_0^+)^n$. The nominal obligation of seniority class k is then defined by

$$\mathbf{d}^k = \mathbf{L}^k \mathbf{1}_n + \tilde{\mathbf{d}}^k. \tag{3.29}$$

Replacing d_i^k in the denominator of (3.28) by the one given in (3.29) leads to the corresponding ownership matrix $\mathbf{M}^k$. Note that in this scenario, $\sum_{i=1}^n M_{ij}^k < 1$ is still allowed to hold for some seniority classes if some amount of the firm's debt is held outside the system. The approach in (OS3) is chosen by Elsinger (2009) and both (OS1) and (OS2) can obviously be embedded into this assumption. Clearly, (OS3) is the most flexible assumption on the ownership matrices to model the actual circumstances in a financial network.

Note that the knowledge about the ownership structure of the matrices expressed in (OS1) – (OS3) alone does not give any information whether the recovery vectors are unique or not. In order to make precise statements about the uniqueness of a solution, further regularity conditions are necessary that describe the connections of the single entries in a matrix $\mathbf{M}^k$ in more detail. In Section 3.3.2 these conditions are made concrete. However, the classification in (OS1) – (OS3) will help us with a more convenient structuring of the different regularity conditions.

3.2.4 Seniority Structure

The majority of the models in Table 3.1 treat only one seniority class. In systems with a more elaborate debt structure, which is for example needed if $\mathbf{a} \in \mathbb{R}^n$ is considered (cf. Section 3.2.1), the seniority levels can be arranged in two ways.

Suppose that there are $m > 1$ seniority classes. If from $d_i^k = 0$, $1 < k \leq m$, it follows that $d_i^{k-1} = \ldots = d_i^1 = 0$, we say that the financial system has an *ordered seniority structure*. This is the case in the work of Elsinger (2009). A financial system without this property is said to have an *unordered seniority structure*, that means that there can be seniority classes in which a firm has no obligations, but still it has some at a lower level. Fischer (2014) uses this more general definition of a seniority structure in his work. Aim of this subsection is to show that the solution of an unordered system can be rearranged into the solution of the corresponding ordered system and vice versa.

To this end, note that we can convert a given unordered financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$ into its corresponding ordered financial system $\widetilde{\mathcal{F}}$. This is done by swapping for each firm i the highest seniority level with a zero nominal debt entry, $d_i^k = 0$, with the debt entry of the highest seniority below k that is non-zero, i.e. $d_i^j > 0$, $j < k$, and repeating this procedure until an ordered structure is reached. In the same way the entries of $\mathbf{d}^m, \dots, \mathbf{d}^1$ have been changed, we also have to do so with the corresponding debt ownership matrices. That means if we swap the i-th entry of $\mathbf{d}^k$ with the i-th entry of $\mathbf{d}^j$ $(j < k)$, we also have to swap the i-th column of $\mathbf{M}^k$ with the corresponding column of $\mathbf{M}^j$. We assume further in this context that if a firm i has no obligations in seniority level k, it also follows that the i-th column of $\mathbf{M}^k$ has only zero entries. The transition from an unordered system to an ordered system can be expressed via the two mappings

$$\begin{aligned}\Omega^{\mathbf{M}}(\mathbf{M}^m,\dots,\mathbf{M}^1,\mathbf{M}^0) &\mapsto (\widetilde{\mathbf{M}}^m,\dots,\widetilde{\mathbf{M}}^1,\mathbf{M}^0)\\ \Omega^{\mathbf{d}}(\mathbf{d}^m,\dots,\mathbf{d}^1) &\mapsto (\tilde{\mathbf{d}}^m,\dots,\tilde{\mathbf{d}}^1).\end{aligned} \tag{3.30}$$

If an unordered system $\mathcal{F} = (\mathbf{a},\mathbf{M},\mathbf{d})$ is given, the corresponding ordered system is given by $\widetilde{\mathcal{F}} = (\mathbf{a},\widetilde{\mathbf{M}},\tilde{\mathbf{d}})$, where $\widetilde{\mathbf{M}} = \Omega^{\mathbf{M}}(\mathbf{M}^m,\dots,\mathbf{M}^0)$ and $\tilde{\mathbf{d}} = \Omega^{\mathbf{d}}(\mathbf{d}^m,\dots,\mathbf{d}^1)$. Denote the mapping Φ as in (2.12) with the matrices in $\widetilde{\mathbf{M}}$ and the liabilities in $\tilde{\mathbf{d}}$ by $\widetilde{\Phi}$. To show that the entries of the solution $\mathbf{R}^*$ of an unordered system $\mathcal{F}$ are, except of the order, similar to the entries of the solution of the corresponding ordered system and vice versa, we define for an arbitrary vector $(\mathbf{r}^m,\dots,\mathbf{r}^0) = \mathbf{R} \in (\mathbb{R}_0^+)^{n(m+1)}$ of recovery values the mapping

$$\Omega^{\mathbf{r}}(\mathbf{r}^m,\dots,\mathbf{r}^1,\mathbf{r}^0) \mapsto (\tilde{\mathbf{r}}^m,\dots,\tilde{\mathbf{r}}^1,\mathbf{r}^0), \tag{3.31}$$

that reorders the recovery values in the same manner as $\Omega^{\mathbf{d}}$ reorders the liabilities, where the equity value $\mathbf{r}^0$ remains unchanged. We therefore have to show that $\widetilde{\Phi}(\Omega^{\mathbf{r}}(\mathbf{R}^*)) = \Omega^{\mathbf{r}}(\mathbf{R}^*)$. The next lemma, which obviously holds, will help us with this purpose.

Lemma 3.8. *Let $\mathcal{F} = (\mathbf{a},\mathbf{M},\mathbf{d})$ be an unordered financial system and $\mathbf{R} = (\mathbf{r}^m,\dots,\mathbf{r}^0)$ be an arbitrary vector of recovery values. With the definitions above, it holds that*

$$\sum_{k=1}^{m}\mathbf{M}^k\mathbf{r}^k = \sum_{k=1}^{m}\widetilde{\mathbf{M}}^k\tilde{\mathbf{r}}^k \quad \textit{and} \quad \sum_{k=1}^{m}\mathbf{d}^k = \sum_{k=1}^{m}\tilde{\mathbf{d}}^k. \tag{3.32}$$

For the given fixed point $\mathbf{R}^*$ of Φ, denote $\widetilde{\mathbf{R}} = (\tilde{\mathbf{r}}^m,\dots,\tilde{\mathbf{r}}^0) =: \Omega^{\mathbf{r}}(\mathbf{R}^*)$. Using Lemma 3.8, the last n components of $\widetilde{\Phi}(\widetilde{\mathbf{R}})$ are given by

$$\left(\mathbf{a} + \sum_{k=0}^{m}\widetilde{\mathbf{M}}^k\tilde{\mathbf{r}}^k - \sum_{k=1}^{m}\tilde{\mathbf{d}}^k\right)^+ = \left(\mathbf{a} + \sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^{*,k} - \sum_{k=1}^{m}\mathbf{d}^k\right)^+ = \mathbf{r}^{*,0} = \tilde{\mathbf{r}}^0. \tag{3.33}$$

If the equities are equal for both systems, the recovery values of all solvent firms will be equal to the corresponding nominal amounts, which – except for the order – are the same in both systems. It remains to show the equality for defaulting firms for which we assume that firm $i \in \mathcal{N}$ defaults in seniority level $1 \le k \le m$ under $\mathbf{R}^*$. Since nominal debt amounts of firm i got shifted into higher seniorities (if at all), there exists a seniority level $\tilde{k} \le k$ and $\sum_{l=\tilde{k}+1}^{m}\tilde{d}_i^l = \sum_{l=k+1}^{m} d_i^l$ such that

$$\tilde{r}_i^{\tilde{k}} = \left(\mathbf{a} + \sum_{l=0}^{m}\widetilde{\mathbf{M}}^l\tilde{\mathbf{r}}^l - \sum_{l=\tilde{k}+1}^{m}\tilde{\mathbf{d}}^l\right)_i = \left(\mathbf{a} + \sum_{l=0}^{m}\mathbf{M}^l\mathbf{r}^{*,l} - \sum_{l=k+1}^{m}\mathbf{d}^l\right)_i = r_i^{*,k} < d_i^k = d_i^{\tilde{k}}. \tag{3.34}$$

Consequently, $\widetilde{\Phi}(\Omega^{\mathbf{r}}(\mathbf{R}^*)) = \Omega^{\mathbf{r}}(\mathbf{R}^*)$ and we have shown that solving an unordered system is equivalent to solving the corresponding ordered system.

Example 3.9. Suppose that $n = 5$, $m = 3$, $\mathbf{a} = \mathbf{1}_n$ and

$$\mathbf{d}^3 = (0,2,0,4,0)^t,\ \mathbf{d}^2 = (0,3,1,0,2)^t,\ \mathbf{d}^1 = (3,1,4,2,0)^t \tag{3.35}$$

from which follows that the system is unordered, and assume

$$\mathbf{M}^3 = \begin{pmatrix} 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0 \end{pmatrix}, \ \mathbf{M}^2 = \begin{pmatrix} 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0 & 0.2 \\ 0 & 0.2 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0 & 0.2 & 0.2 & 0 & 0 \end{pmatrix},$$
$$\mathbf{M}^1 = \begin{pmatrix} 0 & 0.2 & 0.2 & 0.2 & 0 \\ 0.2 & 0 & 0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{pmatrix} \text{ and } \ \mathbf{M}^0 = \begin{pmatrix} 0 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0.025 & 0 & 0.025 & 0.025 & 0.025 \\ 0.025 & 0.025 & 0 & 0.025 & 0.025 \\ 0.025 & 0.025 & 0.025 & 0 & 0.025 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0 \end{pmatrix}. \tag{3.36}$$

The solution of this system is given by

$$\mathbf{R}^* = \begin{pmatrix} \mathbf{r}^{*,3} \\ \mathbf{r}^{*,2} \\ \mathbf{r}^{*,1} \\ \mathbf{r}^{*,0} \end{pmatrix} = \begin{pmatrix} (0, & 2, & 0, & 3.4234, & 0 &)^t \\ (0, & 1.4234, & 1, & 0, & 2 &)^t \\ (3, & 0, & 2.4234, & 0, & 0 &)^t \\ (0.4957, & 0, & 0, & 0, & 1.6664)^t \end{pmatrix}. \tag{3.37}$$

The vectors of the nominal liabilities of the corresponding ordered system are given by

$$\tilde{\mathbf{d}}^3 = (3,2,1,4,2)^t, \ \tilde{\mathbf{d}}^2 = (0,3,4,2,0)^t, \ \tilde{\mathbf{d}}^1 = (0,1,0,0,0)^t \tag{3.38}$$

and using the ownership matrices $\widetilde{\mathbf{M}}^k$ by changing the corresponding columns and deleting the column entries in the original matrix, we get

$$\widetilde{\mathbf{R}}^* = \begin{pmatrix} (3, & 2, & 1, & 3.4234, & 2 &)^t \\ (0, & 1.4234, & 2.4234, & 0, & 0 &)^t \\ (0, & 0, & 0, & 0, & 0 &)^t \\ (0.4957, & 0, & 0, & 0, & 1.6664)^t \end{pmatrix} \tag{3.39}$$

as the fixed point of the ordered system and we immediately see that on the one hand the equities are identical and on the other hand that the recovery values are also the same as in $\mathbf{R}^*$ taking the mapping $\Omega^{\mathbf{r}}$ into account.

Remark 3.10. The mappings $\Omega^{\mathbf{M}}$ and $\Omega^{\mathbf{d}}$ transferred the unordered financial system into an ordered system by filling up seniority classes with no debt with the obligations of lower classes. In fact, $\Omega^{\mathbf{M}}$ and $\Omega^{\mathbf{d}}$ can represent any other swapping of debt entries and corresponding entries in the ownership matrices, as long as the priority structure of the liabilities is retained for every firm. This means that if $d_i^k > 0$, $k = 1, \dots, m$, is the nominal non-zero debt entry with highest seniority in the system $\mathcal{F}$ for firm i, it also has to be the nominal non-zero debt entry of highest seniority in the system $\widetilde{\mathcal{F}}$, though the actual seniority levels in both systems must not be identical. For the next highest seniority with a non-zero debt entry, this is analogous, and so on for any other seniority. According to Lemma 3.8 and since nowhere in the preceding argumentation, the detailed function rule of the mappings in (3.30) is used, the solution of the reordered system will stay the same in the sense that $\widetilde{\Phi}(\Omega^{\mathbf{r}}(\mathbf{R}^*)) = \Omega^{\mathbf{r}}(\mathbf{R}^*)$ for any $\Omega^{\mathbf{M}}$ and $\Omega^{\mathbf{d}}$ with the described property. In particular, the transformation of the system $\mathcal{F}$ can be of a reverse order than in an ordered system as described above, i.e. we can apply $\Omega^{\mathbf{M}}$ and $\Omega^{\mathbf{d}}$ such that from $d_i^k > 0$, $1 < k \leq m$, it follows that $d_i^h > 0$ for $1 \leq h < k$. This will become important in Section 5.2.

3.3 Survey of Existence and Uniqueness Results

No matter how the financial system in the different articles is defined exactly, all models have in common that pricing the cashflows at maturity can be interpreted as some kind of fixed point problem and, therefore, are faced with the challenge to find an equilibrium of equity and debt payments as the fixed point of Φ. The main objective in such pricing models is of course the question whether there exists a price at all and, if so, whether it can be ensured that there is exactly one price for the assets or whether there are multiple pricing equilibria. Depending on their model assumptions, the existing works in this field derive different conditions for a unique vector of recovery values and also use differing ideas to show the uniqueness.

In this section we attempt to give a survey of the different means to prove the existence and the uniqueness of a pricing equilibrium. While proving the existence of solutions is very similar in most articles, there are different streams in research to prove that there exists only one solution, mostly due to the different model assumptions. Our aim is to summarize the existing works in the field into several categories of regularity conditions and draft a proof of uniqueness for every category.

3.3.1 A Proof on Existence

The idea of the proof that at least one payment equilibrium exists, is in almost all articles the same. It relies on the fact that, according to the Tarski Fixed Point Theorem (cf. Theorem A.4 in the Appendix), there exists a least and a greatest fixed point for a monotone increasing mapping on a complete lattice. The monotonicity of Φ is shown in Lemma 2.4 and for $m \geq 1$, the $(m+1)$-dimensional lattice is given by

$$[\mathbf{0}_n, \mathbf{d}^m] \times [\mathbf{0}_n, \mathbf{d}^{m-1}] \times \ldots \times [\mathbf{0}_n, \mathbf{d}^0], \tag{3.40}$$

where $\mathbf{d}^0 = \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1)$ is the fixed point of the mapping $\Phi^0 : (\mathbb{R}_0^+)^n \to (\mathbb{R}_0^+)^n$ defined by

$$\Phi^0(\mathbf{r}; \mathbf{d}^m, \ldots, \mathbf{d}^1) = \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k + \mathbf{M}^0 \mathbf{r} - \sum_{k=1}^{m} \mathbf{d}^k\right)^+, \tag{3.41}$$

i.e. $\Phi^0(\mathbf{d}^0; \mathbf{d}^m, \ldots, \mathbf{d}^1) = \mathbf{d}^0$. Alternatively, $\mathbf{d}^0$ can also be defined as

$$\mathbf{d}^0 = (\mathbf{I}_n - \mathbf{M}^0)^{-1} \left(\mathbf{a} + \sum_{k=1}^{m} (\mathbf{M}^k - \mathbf{I}_n) \mathbf{d}^k\right)^+ \tag{3.42}$$

without changing the forthcoming results, see Section 4.1.1 for a justification of this definition. It holds that

$$\mathbf{0}_{n(m+1)} \leq \Phi(\mathbf{0}_{n(m+1)}) \quad \text{and} \quad \begin{pmatrix} \mathbf{d}^m \\ \vdots \\ \mathbf{d}^0 \end{pmatrix} \geq \Phi \begin{pmatrix} \mathbf{d}^m \\ \vdots \\ \mathbf{d}^0 \end{pmatrix}, \tag{3.43}$$

where for $\mathbf{d}^0 = \Phi^0(\mathbf{d}^0; \mathbf{d}^m, \ldots, \mathbf{d}^1)$ this is obviously and for $\mathbf{d}^0$ as in (3.42), this is shown in Proposition 4.2. Therefore, the assumptions of Theorem A.4 are fulfilled. This principle can be applied to all financial systems of the articles in Table 3.1.

To define $\mathbf{d}^0$ as in (3.42), it is necessary, that the matrix $(\mathbf{I}_n - \mathbf{M}^0)$ is invertible. When using $\mathbf{d}^0$ as the fixed point of $\Phi^0(\cdot; \mathbf{d}^m, \ldots, \mathbf{d}^1)$, the invertibility of the matrix is not necessary, but sufficient for the existence of a fixed point Φ^0. In financial systems with $\mathbf{M}^0 = \mathbf{0}_{n\times n}$, at least one fixed point consequently exists without further model constraints, but in a more general model that includes cross-holdings of shares, we have to make the following assumption.

Assumption 3.11. *The equity ownership matrix $\mathbf{M}^0$ fulfills the Elsinger Property given in Definition 2.5. As a consequence, $(\mathbf{I}_n - \mathbf{M}^0)$ is invertible and, hence, $\mathbf{d}^0$ exists no matter which of the two possible definitions is regarded.*

In the remainder of this section, we suppose that Assumption 3.11 is given unless otherwise stated. Recall that this is equivalent with the fact that there is no subset $\mathcal{I} \subset \mathcal{N}$ such that $\sum_{i\in\mathcal{I}} M^0_{ij} = 1$ for all $j \in \mathcal{I}$. The consequences of a violation of Assumption 3.11 are demonstrated in the next example.

Example 3.12. Consider a system with $n = 3$, $m = 1$, $\mathbf{M}^1 = \mathbf{0}_{n\times n}$,

$$\mathbf{a} = (1.5, 0.75, 2)^t, \quad \mathbf{d}^1 = (1, 0.5, 2)^t \quad \text{and} \quad \mathbf{M}^0 = \begin{pmatrix} 0 & 1 & 0.1 \\ 1 & 0 & 0.1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.44}$$

Because of $\mathbf{a} \geq \mathbf{d}^1$, it holds that $\mathbf{r}^{*,1} = \mathbf{d}^1$ and that the equity vector is given by $\mathbf{r}^{*,0} = \mathbf{a} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}^1$. But since $(\mathbf{I}_n - \mathbf{M}^0)^{-1}$ does not exist, we cannot determine $\mathbf{r}^{*,0}$. This also becomes clear when trying to find the solution of $\mathcal{F}$ via the Picard Iteration (cf. Section 4.1.1). For the starting point, we take $\mathbf{R}_{\text{small}}$ because $\mathbf{R}_{\text{great}}$ can not be determined, see Section 4.1.1 for more details. Clearly, $\mathbf{r}^{k,1} = \mathbf{d}^1$ for all iterates $k \geq 0$ but the equity components of the iterates do not converge but become larger in every iteration step. For instance it holds that $\mathbf{r}^{0,0} = (\mathbf{a} - \mathbf{d})^+ = (0.5, 0.25, 0)^t$, $\mathbf{r}^{10,0} = (4.25, 4, 0)^t$ and $\mathbf{r}^{20,0} = (8, 7.75, 0)^t$.

3.3.2 Regularity Conditions for Uniqueness

The Tarski Fixed Point Theorem says that under Assumption 3.11 there is a greatest solution $\overline{\mathbf{R}}$ and a least solution $\underline{\mathbf{R}}$ of the liquidation equations in (2.9) – (2.11). Under some additional conditions on the particular structure of the ownership matrices, it also holds that $\overline{\mathbf{R}} = \underline{\mathbf{R}} = \mathbf{R}^*$ and therefore that there is only one fixed point of Φ. In the existing articles in this field (cf. Table 3.1), several of these conditions have been established to guarantee a unique solution. The listing below is an attempt to sort the conditions into four different categories. Assumption 3.11 about the equity matrix in mind, we focus only on the structure of debt ownership matrices in the following (except of (RC1)).

Definition 3.13. We distinguish between the following *regularity conditions (RC)*.

(RC1) It holds that $\|\mathbf{M}^k\| < 1$ for all $k = 0, \ldots, m$.

(RC2) All debt ownership matrices $\mathbf{M}^k$ $(k = 1, \ldots, m)$ are irreducible.

(RC3) In every seniority there is a firm with no debt outstanding that owns some part of the debt of every indebted firm. Formally, for every $k = 1, \ldots, m$, there exists an $i(k) \in \mathcal{N}$ such that

$$M^k_{i(k),i} > 0 \text{ for all } i \in \mathcal{N}\backslash\{i(k)\} \text{ with } d^k_i > 0 \tag{3.45}$$

and where $d^m_{i(k)} = \ldots = d^1_{i(k)} = 0$.

(RC4) Every debt ownership matrix $\mathbf{M}^k$ $(k = 1, \ldots, m)$ possesses the Elsinger Property (cf. Definition 2.5).

Note that condition (RC1), which implies (OS1), is the only regularity condition that entails the case $k = 0$. This is because $\|\mathbf{M}^0\| < 1$ is a stronger condition than the one in Assumption 3.11 and this condition is needed for a certain version of a proof of uniqueness given in Section 3.3.3. In literature, (RC1) appears in the works of Suzuki (2002)[2], Gouriéroux et al. (2012, 2013) and in Fischer (2014). Though the assumption in (RC2) of irreducibility represents a less mild condition, it appears in the works of Demange (2011), Ren et al. (2014) and Acemoglu et al. (2015). The article in which (RC3) is probably mentioned the first time, is the one of Shin (2008), who denotes this unindebted investor as the so-called "unleveraged investor". Unleveraged investors can, according to Shin (2009), for example be pension funds, insurance companies, foreign central banks or simply households. Note that in financial systems in which (RC3) holds, the unindebted members of the system could also be excluded from the system without changing the solution. Doing so, (RC3) and (RC1) would be equivalent. However, we list (RC3) separately since this assumption is used from Shin (2008) in systems in which (OS2) is demanded. Condition (RC4) is introduced by Elsinger (2009), which is why we refer to the expression Elsinger Property. Later works, like Demange (2011) also use the Elsinger Property for the ownership matrices.

We also found another condition in Eisenberg and Noe (2001) also used in Liu and Staum (2010), where the authors use the expressions "surplus set" and "risk orbit" to define a regularity condition. In order to define this condition exactly, several additional steps are necessary which is why we omit this assumption here and refer to the original work instead. However, a particular case of Eisenberg and Noe's condition is that the debt ownership matrix is irreducible (note that the authors consider a model in which $m = 1$) and, hence, (RC2) must be given.

3.3.3 Sketches of Proofs for Uniqueness

The methods to prove that the fixed point is unique differ in the existing works due to the different assumptions made on the ownership matrices. In combination with the different ways to define the particular form of the matrices concerning their column sums in Section 3.2.3, we can distinguish between three substantial types of proofs to show uniqueness. Note that there also appear other proofs for uniqueness in the mentioned works in Table 3.1. Since we have the objective to unify the regularity conditions, we only want to outline the proofs for the most general situations.

The case if (RC1) holds

Under $\|\mathbf{M}^k\| < 1$ for all $0 \leq k \leq m$, the mapping Φ becomes a strict contraction, and for any $\mathbf{R}^1, \mathbf{R}^2 \in (\mathbb{R}_0^+)^{n(m+1)}$ it holds (Fischer, 2014) that

$$\|\Phi(\mathbf{R}^2) - \Phi(\mathbf{R}^1)\| \leq I^{\max}\|\mathbf{R}^2 - \mathbf{R}^1\|, \tag{3.46}$$

where $0 \leq I^{\max} < 1$ is defined in (3.17). As a consequence, we can apply the Banach Contraction Mapping Theorem (cf. Theorem A.5 in the Appendix) that says that Φ has a unique fixed point.

[2] In his article, Suzuki actually demands that for every ownership matrix ($\mathbf{M}^1$ and $\mathbf{M}^0$), it suffices that there exists only one firm $i \in \mathcal{N}$ for which holds that $\sum_{j=1}^n M_{ji}^k < 1$ for $k = 0, 1$. This is a less strong condition on the ownership matrices than (RC1). However, in his proof for uniqueness a strict contraction mapping argument is used which only holds under (RC1). This is why we assume that Suzuki erroneously defined his regularity condition and actually meant (RC1).

The case if (RC4) holds

This condition implies that (OS2) cannot hold. A proof that the solution must be unique for this regularity condition is given in Elsinger (2009, Theorem 8). Since we already presented a different proof (Theorem 2.7), we only give a short sketch of Elsinger's proof.

The assertion is shown via contraposition. Let $\overline{\mathbf{R}}$ be the greatest and $\underline{\mathbf{R}}$ be the least solution Φ that exist as shown in Section 3.3.1. Subtracting the equity value of $\underline{\mathbf{R}}$ from $\overline{\mathbf{R}}$, Elsinger derives a condition that has to hold if multiple solutions are present in the system. Following his argumentation, if the Elsinger Property in (RC4) holds for all ownership matrices, it is ensured that this condition cannot be fulfilled under any circumstances from which follows that the solution then must be unique.

The case if (OS2) holds

First, check that if (OS2) holds, (RC4) is violated. Further, if all column sums of the debt ownership matrices are equal to one, we cannot conclude from (RC2) that (RC4) holds as well. The same statement is true for (RC3).

The proof for uniqueness in this case slightly differs in the different papers but the idea of the proof is very similar. Before giving a sketch of this proof, we have to make a restriction and demand that $\mathbf{M}^0 = \mathbf{0}_{n\times n}$ since otherwise, the following arguments do not hold anymore. In a first step, it is shown that the equity values are equal for any debt payment vectors that solve the liquidation equations in (2.9) – (2.11), see Demange (2011, Proposition 1) or Eisenberg and Noe (2001, Theorem 1) for a proof. In the second step the uniqueness is shown by contradiction. This is done by assuming that a greatest solution $\overline{\mathbf{R}}$ and a least solution $\underline{\mathbf{R}}$ of Φ exist that differ in at least one component. Summing up the equity values of all firms in the system for $\overline{\mathbf{R}}$ and $\underline{\mathbf{R}}$ and using the fact that $\sum_{i=1}^{n} M_{ij}^k = 1$ for all $j \in \mathcal{N}$ and all $k = 1, \ldots, m$ leads to the result that the equity values must be equal. In case of (RC3), this is a contradiction if the additional assumption $\sum_{i=1}^{n} a_i > 0$ holds, see Shin (2008) for details. If (RC2) holds, we have to assume further that $\sum_{i=1}^{n} a_i > 0$ and that every firm in the system has to be indebted, see Demange (2011, Proposition 2), who shows this for systems with $m = 1$.

The drawback of the approach above is that all proofs do not allow for $\mathbf{M}^0 \neq \mathbf{0}_{n\times n}$. In Elsinger (2009, Theorem 4), an additional assumption for the exogenous asset values is derived under which a unique solution exists even if the Elsinger Property is violated for one of the ownership matrices $\mathbf{M}^k$, $k \geq 0$. This assertion also holds for system in which the equity ownership matrix $\mathbf{M}^0$ is unequal to zero. For a detailed proof, we refer to the original work. We only want to point out that in Elsinger's result it should be excluded that the Elsinger Property is violated for $\mathbf{M}^0$. The reason is given the next example.

Example 3.14. Take the financial system $\mathcal{F}$ from Example 3.12 again, where the Elsinger Property for $\mathbf{M}^0$ is violated since for $\mathcal{I} = \{1, 2\}$ it holds that $\sum_{i\in\mathcal{I}} M_{ij}^0 = 1$ for all $j \in \mathcal{I}$. According to Elsinger (2009, Theorem 4), the solution of $\mathcal{F}$ is in such cases still unique if

$$\sum_{i\in\mathcal{I}} a_i > \sum_{i\in\mathcal{I}} \left(1 - \sum_{j\in\mathcal{I}} M_{ji}^1\right) d_i^1. \tag{3.47}$$

Applying this to the system from Example 3.12, Equation (3.47) boils down to

$$a_1 + a_2 = 2.25 > 1.5 = d_1^1 + d_2^1 = (1 - M_{11}^1 - M_{21}^1)d_1^1 + (1 - M_{12}^1 - M_{22}^1)d_2^1. \tag{3.48}$$

It follows that (3.47) is fulfilled, however, the system has no solution at all.

Summarizing the results of this Section, we have found that regularity condition (RC4) for which a unique solutions exists is the most general one. If (OS2) holds together with (RC2) or (RC3), the additional assumption $\|\mathbf{a}\| > 0$ ensures that the solution is actually unique – at least if no equity cross-holdings are present. All existing models in literature from Table 3.1 can be embedded in these two scenarios. We close this section with two remarks.

Remark 3.15. Another possible approach of finding conditions to ensure a unique solution if the Elsinger Property is injured consists of using the idea of Ren et al. (2014, Corollary 1). Assume that one ownership matrix $\mathbf{M}^k$, $k \geq 0$, does not have the Elsinger Property which means that $(\mathbf{I}_n - \mathbf{M}^k)$ is not invertible. The result in Ren et al. (2014) says that if the recovery value of a firm $i \in \mathcal{N}$ can uniquely and exogenously be determined and if the submatrix of dimension $(n-1) \times (n-1)$ in which i has been deleted has the Elsinger Property, the solution of $\mathcal{F}$ is unique. Depending on the value of k, this leads to two conclusions:

(i) If $k \geq 1$, the firm has to be solvent in seniority level k which is for example ensured if $a_i > \sum_{l=k}^{m} d_i^l$.

(ii) If $k = 0$, the firm has to be in default which can for instance be ensured if $\mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1)$ exists and if $\left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{d}^k + \mathbf{M}^0 \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1)\right)_i < \sum_{k=1}^{m} d_i^k$.

Remark 3.16. In a final remark to terminate this subsection, we want to emphasize that all preceding derived regularity conditions for a unique fixed point of Φ are in general only sufficient but not necessary conditions. Hence, we can easily construct a financial system $\mathcal{F}$ in which all regularity assumptions are violated and in which the solution of $\mathcal{F}$ is unique, even so. For an example for such a system, let $n = 5$, $m = 1$, $\mathbf{M}^0 = \mathbf{0}_{n \times n}$,

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{d}^1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \ \text{and} \ \ \mathbf{M}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}. \tag{3.49}$$

Check that for this system, all regularity conditions in Definition 3.13 are violated, in particular (RC4). If we take the subset $\mathcal{I} = \{2, 3\}$, similar to the approach in Example 3.14, we find that $\sum_{i \in \mathcal{I}} a_i = a_2 + a_3 = 0$ and, hence, the assumption in (3.47) is violated as well. Nevertheless, the fixed point of $\mathcal{F}$ is unique and given by

$$\mathbf{R}^* = (1, 1.5, 1, 3, 2, 3, 0, 1.25, 0, 0.5)^t. \tag{3.50}$$

The firms 1, 3 and 5 are solvent, firm 2 is in default and firm 4 is borderline, i.e. $r_4^{*,1} = d_4$ and $r_4^{*,0} = 0$. Since firm 1 has no endogenous assets in its balance sheet, i.e. $M_{1j}^1 = 0$ for all $j = 1, \ldots, 5$, its recovery value $r_1^{*,1}$ is uniquely defined by $r_1^{*,1} = \min\{d_1^1, a_1\} = \min\{1, 4\} = 1$. Since all intersystem debt payments from this firm flow to firm 2 ($M_{21}^1 = 0.5$), we can interpret this "fixed" income of firm 2 as some kind of exogenous income which in turn allows a unique pricing of the remaining recovery values.

3.4 Extensions of the Model

For the standard model in its most general form, there exists many existence and uniqueness results we tried to survey and to unify in Section 3.3. Beyond that, there are also some extensions of the standard model in order to reflect some practical aspects and issues of financial networks. We briefly want to mention some of these models in this section.

A possible extension embodies the inclusion of default costs. These costs can be modeled by a fixed value of costs a defaulting firm is faced with or by costs that depend on the value of exogenous and endogenous assets. For convenience, we assume a simple model with only one seniority but include cross-holdings in the equities as well. The existing different model extensions can be expressed via the adapted mapping $\Phi_{\mathbf{b}}$ given by

$$\Phi_{\mathbf{b}} \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^0 \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}^1, \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 \ - \mathbf{b}(\mathbf{r}^1)\right)^+\} \\ \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 - \mathbf{b}(\mathbf{r}^1) - \mathbf{d}^1\right)^+ \end{pmatrix}, \tag{3.51}$$

where $\mathbf{b}(\mathbf{r}^1) = (b_1(r_1^1), \ldots, b_n(r_n^1))^t$ is the vector in which the default costs are modeled. If in a liquidation event a fixed value of default costs has to be considered, $\mathbf{b}(\mathbf{r}^1)$ is defined as

$$\mathbf{b}(\mathbf{r}^1) = \operatorname{diag}(\mathbf{r}^1 < \mathbf{d}^1) \cdot \mathbf{c} \tag{3.52}$$

with $\mathbf{c} = (c_1, \ldots, c_n)^t \geq \mathbf{0}_n$ representing the individual default costs. This model is described in the works of Elsinger (2009) and Elliott et al. (2014), who call the default costs *bankruptcy costs* and *failure costs*, respectively. The latter article ignores the effects of debt cross-holdings.

Beside fixed default costs, there is also the possibility to model relative default costs as done by Rogers and Veraart (2013). The idea behind this approach is that if a firm defaults, exogenous and endogenous assets have to be liquidated to service at least partially the debtholders claims. In such a situation it is reasonable to assume that, due to fire sales, the defaulting firm will realize only a fraction of the exogenous assets' price. The same also holds for the endogenous assets, which results in

$$\mathbf{b}(\mathbf{r}^1) = \operatorname{diag}(\mathbf{r}^1 < \mathbf{d}^1) \cdot \left((1-\alpha)\mathbf{a} + (1-\beta)\mathbf{M}^1\mathbf{r}^1 + (1-\gamma)\mathbf{M}^0\mathbf{r}^0\right), \tag{3.53}$$

where $\alpha, \beta, \gamma \in (0, 1]$ denote the realized fractions of the original asset price. The closer α, β and γ are to zero, the higher are the losses of the firm's liquidation. A very similar approach is also chosen in Glasserman and Young (2015). Note that, unlike to the framework of the mentioned articles, we also include cross-holdings in the firms shares in the model.

Other extensions we want to mention are the works of Müller (2006) and Acemoglu et al. (2015). In Müller (2006), credit lines are included into the model. It is based on the idea that if a firm gets into trouble paying off its liquidations, it can raise new credits from non-defaulting banks in the system up to a predefined value, the credit line. Acemoglu et al. (2015) considers a financial system where the exogenous assets are composed out of an amount of cash and the return of a project the firm is involved in. In case of financial troubles, the firm can prematurely liquidate the project to gain further liquidity and service the liabilities. For more mathematical details, we refer to the original works. A last work we want to mention is the one of Awiszus and Weber (2015) that extends the model of Rogers and Veraart (2013) including cross-holdings in the equities as well as the effects of fire sales of the exogenous assets as a potential channel of contagion introduced first by Cifuentes et al. (2005). For more details on this topic, see also the references in Awiszus and Weber (2015).

No matter whether $\mathbf{b}(\mathbf{r}^1)$ is defined as in (3.52) or as in (3.53), all extensions have in common that discontinuities are added to the model. As a result, the uniqueness of a fixed point of $\Phi_{\mathbf{b}}$ gets lost under the usual regularity conditions that are presented in Section 3.3.2, see also

Example 3.17. However, as long as Assumption 3.11 holds, the mapping $\Phi_{\mathbf{b}}$ is bounded from above by $\mathbf{R}_{\text{great}}$ and bounded from below by

$$\mathbf{R}_{\text{small}} = \begin{pmatrix} \min\{\mathbf{d}^1, (\mathbf{a} - \mathbf{b}(\mathbf{0}_n))^+\} \\ (\mathbf{a} - \mathbf{b}(\mathbf{0}_n) - \mathbf{d}^1)^+ \end{pmatrix}. \tag{3.54}$$

This follows immediately by the fact that $\Phi_{\mathbf{b}}(\mathbf{R}_{\text{great}}) \leq \mathbf{R}_{\text{great}}$ and $\Phi_{\mathbf{b}}(\mathbf{R}_{\text{small}}) \geq \mathbf{R}_{\text{small}}$. Together with the monotonicity of $\Phi_{\mathbf{b}}$, we can iteratively apply the mapping to the starting vector $\mathbf{R}_{\text{great}}$ and obtain the greatest fixed point $\mathbf{R}^* = \lim_{k\to\infty} \Phi_{\mathbf{b}}^k(\mathbf{R}_{\text{great}})$ as the limit. $\mathbf{R}^*$ can be interpreted as the "best-case equilibrium" (Elliott et al., 2014) or the Pareto-dominant solution (cf. Section 3.2.2).

According to Rogers and Veraart (2013), $\Phi_{\mathbf{b}}$ is only continuous from above but not from below. As a consequence, starting an iteration with $\mathbf{R}_{\text{small}}$ would lead to a limit that must not necessarily be a fixed point. For an example of this situation, see the original article of Rogers and Veraart (2013, Example 3.3). The problem that possibly multiple fixed points of $\Phi_{\mathbf{b}}$ can exist (no matter how they can be found), is solved by the fact that there exists a greatest fixed point which can be considered as a Pareto-dominant solution. This is very similar to the case of non-constant liabilities that was presented in detailed in Section 3.2.2. The following example demonstrates that the uniqueness gets lost in case of default costs.

Example 3.17. Consider a system of $n = 3$ firms with $m = 1$ and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{d}^1 = \begin{pmatrix} 1.745 \\ 0.75 \\ 1 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 0.5 & 0.25 \\ 0.1 & 0 & 0.5 \\ 0.1 & 0.25 & 0 \end{pmatrix}, \ \mathbf{M}^0 = \begin{pmatrix} 0 & 0.05 & 0.05 \\ 0.05 & 0 & 0.05 \\ 0.05 & 0.05 & 0 \end{pmatrix}. \tag{3.55}$$

The vector of default costs is given by $\mathbf{c} = (0.5, 0.5, 0.5)^t$. Using the Picard Algorithm (see Section 4.1.1 for details) and starting with $\mathbf{R}_{\text{great}}$ leads to a fixed point

$$(1.745, 0.75, 1, 0.0003, 0.9951, 1.4118)^t \tag{3.56}$$

of $\Phi_{\mathbf{b}}$, hence all three firms are solvent. On the other side, when starting with $\mathbf{R}_{\text{small}}$ defined as in (3.54), the Picard Algorithm stops at

$$(1.24, 0.75, 1, 0, 0.9419, 1.3586)^t, \tag{3.57}$$

which is also a fixed point of $\Phi_{\mathbf{b}}$ but now with the difference that the first firm is in default.

4 Valuation Algorithms for Systems with one Seniority Level

We showed in Theorem 2.7 of Chapter 2 that for the standard model, i.e. a financial system $\mathcal{F}$ in which the Elsinger Property holds for all ownership matrices, a unique fixed point $\mathbf{R}^*$ of the mapping Φ exists. In this chapter, our aim is to present possible calculation procedures to obtain this solution. We restrict our considerations to financial systems where the debt payments are all of the same seniority. Because of $m = 1$, we have only one vector that contains the nominal values of the liabilities and omit for convenience the superscript and simply write $\mathbf{d} := \mathbf{d}^1$. The liquidation value equations in (2.9) – (2.11) therefore reduce to

$$\mathbf{r}^1 = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0\} \tag{4.1}$$

$$\mathbf{r}^0 = (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 - \mathbf{d})^+, \tag{4.2}$$

and the mapping Φ in (2.12) becomes $\Phi : (\mathbb{R}_0^+)^{2n} \to (\mathbb{R}_0^+)^{2n}$, where $\mathbf{R} = (\mathbf{r}^1, \mathbf{r}^0) \in (\mathbb{R}_0^+)^{2n}$ and

$$\Phi(\mathbf{R}) = \Phi \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^0 \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 - \mathbf{d})^+ \end{pmatrix}. \tag{4.3}$$

We will sometimes refer to the debt components of $\mathbf{R}$ and mean in such cases the first n components of $\mathbf{R}$ that represent the debt payments of the system. The components $n+1$ to $2n$ of $\mathbf{R}$ are called equity components for the same reasons. This notation is retained during the entire chapter. The reason why we ignore systems with $m > 1$ for now is that the new ideas are at first presented in this more convenient framework for a better comprehensibility. The derived concepts from this chapter will help us in Chapter 5, where the algorithms are generalized for financial systems that allow for a more detailed seniority structure of debt.

In general, we can categorize the developed algorithms into two different classes. The first class, presented in Section 4.1, has the common property that they converge to the solution $\mathbf{R}^*$. However, the procedures must not necessarily reach the fixed point and therefore technically do not deliver an exact solution in a finite number of iteration steps. Of course, due to decimal place restrictions, the convergence is sufficiently exact in the framework of a certain tolerance level but not in a mathematical sense. The second class of algorithms overcomes this problem by, on the one hand, delivering the exact value of $\mathbf{R}^*$ and, on the other hand, by managing this task in a finite number of iteration steps. These algorithms are based on the information whether a firm is in default or not under a current iterate and are outlined in Section 4.2. Most parts of this chapter are based on Hain and Fischer (2015).

4.1 Non-finite Algorithms

Different authors like Eisenberg and Noe (2001), Suzuki (2002) or Elsinger (2009) derived different methods to calculate the solution $\mathbf{R}^*$. We can distinguish between two valuation algorithms that can be used in a financial system with cross-holdings in equity and debt. One algorithm iteratively applies the mapping Φ in (2.12) on a chosen starting vector (Section 4.1.1). A modification of this Picard Iteration is used in the work of Elsinger (2009), where for the determination of the equity component, a more sophisticated subalgorithm is used (Section 4.1.2). Beyond that, in Section 4.1.3, we combine the existing computation techniques together into a new valuation algorithm which we call *Hybrid Algorithm*.

4.1.1 The Picard Algorithm

The most intuitive way to calculate $\mathbf{R}^*$ for the system $\mathcal{F}$ consists of the iterative application of Φ. It will be shown in this section that with an arbitrary starting vector $\mathbf{R}^0 \in (\mathbb{R}_0^+)^{2n}$,

$$\mathbf{R}^* = \lim_{l\to\infty} \Phi^l(\mathbf{R}^0) = \lim_{l\to\infty} \underbrace{\Phi \circ \ldots \circ \Phi}_{l}(\mathbf{R}^0), \tag{4.4}$$

which is commonly known as the *Picard Iteration*. Since $\mathbf{R}^* \geq \mathbf{0}_n$, the range for the starting vector $\mathbf{R}^0$ can be reduced to only non-negative vectors. Beyond that, the search for an optimal starting point can be limited to a specific finite interval as we will show in the next steps. To this end, we introduce the two vectors

$$\mathbf{R}_{\text{great}} := \begin{pmatrix} \mathbf{r}^1_{\text{great}} \\ \mathbf{r}^0_{\text{great}} \end{pmatrix} := \begin{pmatrix} \mathbf{d} \\ (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ \end{pmatrix} \tag{4.5}$$

and

$$\mathbf{R}_{\text{small}} := \begin{pmatrix} \mathbf{r}^1_{\text{small}} \\ \mathbf{r}^0_{\text{small}} \end{pmatrix} := \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a}\} \\ (\mathbf{a} - \mathbf{d})^+ \end{pmatrix} \tag{4.6}$$

The vector $\mathbf{R}_{\text{great}}$ assumes that the debt payments are fully recovered so that in the debt component $\mathbf{r}^1 = \mathbf{d}$. Note that even if $\mathbf{r}^{*,1} = \mathbf{d}$, it must not necessarily hold that $\mathbf{R}_{\text{great}} = \mathbf{R}^*$ (see also Example 6.3). Also note that because of Assumption 2.6, $(\mathbf{I}_n - \mathbf{M}^0)^{-1}$ exists and is non-negative, cf. Lemma A.3 in the Appendix. The second vector $\mathbf{R}_{\text{small}}$ emerges when the liquidation equations (4.1) and (4.2) are applied and the ownership structure of liabilities and equities is completely ignored. In this case the term $\mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0$ that represents the endogenous assets is set to zero. Hence, the firms only have the exogenous assets $\mathbf{a}$ as an income. The starting vector $\mathbf{R}_{\text{small}}$ results from applying the mapping Φ in Equation (2.12) on the vector $\mathbf{0}_{2n}$, i.e.

$$\Phi(\mathbf{0}_{2n}) = \Phi \begin{pmatrix} \mathbf{0}_n \\ \mathbf{0}_n \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a}\} \\ (\mathbf{a} - \mathbf{d})^+ \end{pmatrix} = \mathbf{R}_{\text{small}}. \tag{4.7}$$

Before showing the importance of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ as upper and lower bounds of the solution $\mathbf{R}^*$, we need to introduce the terms *default set* and *default matrix*. For $\mathbf{r}^1 \geq \mathbf{0}_n$ and $\mathbf{r}^0 \geq \mathbf{0}_n$ the set

$$\mathcal{D}(\mathbf{r}^1, \mathbf{r}^0) := \left\{ i \in \mathcal{N} : \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0\right)_i < d_i \right\} \tag{4.8}$$

is called *default set under* $\mathbf{r}^1$ *and* $\mathbf{r}^0$ because – given $\mathbf{r}^1$ and $\mathbf{r}^0$ – the firms in $\mathcal{D}(\mathbf{r}^1, \mathbf{r}^0)$ are not able to fully satisfy their obligations and hence are in default. We say that firm i is in default under $\mathbf{r}^1$ and $\mathbf{r}^0$ if $i \in \mathcal{D}(\mathbf{r}^1, \mathbf{r}^0)$. For $\mathbf{R} = (\mathbf{r}^1, \mathbf{r}^0)$ we will sometimes abbreviate the default set by $\mathcal{D}(\mathbf{R})$. The *default matrix corresponding to* $\mathbf{r}^1$ *and* $\mathbf{r}^0$, $\mathbf{\Lambda}(\mathbf{r}^1, \mathbf{r}^0) \in \mathbb{R}^{n\times n}$, is defined as

$$\mathbf{\Lambda}(\mathbf{r}^1, \mathbf{r}^0) := \text{diag}(\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0 - \mathbf{d} < \mathbf{0}_n) \tag{4.9}$$

and is the diagonal matrix with entry 1 for firms in default under $\mathbf{r}^1$ and $\mathbf{r}^0$ at the corresponding position and with the value 0 for firms not in default. As for default sets, we sometimes write $\mathbf{\Lambda}(\mathbf{R})$ instead of $\mathbf{\Lambda}(\mathbf{r}^1, \mathbf{r}^0)$, if $\mathbf{R} = (\mathbf{r}^1, \mathbf{r}^0)$. With the new notation, we can show the crucial limiting property of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$. In Section 5.1.1, the assertion is showed for systems with a seniority structure ($m > 1$), a different version of the proof can be found in Fischer (2015).

Proposition 4.1. *Let* $\mathbf{R}^* = (\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ *be the fixed point of the mapping* (4.3). *Then* $\mathbf{R}^* \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$.

Proof. Because of (4.7), $\mathbf{R}^* \geq \mathbf{R}_{\text{small}}$, so we only show the validity of the upper bound $\mathbf{R}_{\text{great}}$. Since $\mathbf{R}^*$ is the fixed point of Φ, we can write

$$\Phi \begin{pmatrix} \mathbf{r}^{*,1} \\ \mathbf{r}^{*,0} \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d})^+ \end{pmatrix} = \begin{pmatrix} \mathbf{r}^{*,1} \\ \mathbf{r}^{*,0} \end{pmatrix} = \mathbf{R}^*. \tag{4.10}$$

Obviously, $\mathbf{r}^{*,1} \leq \mathbf{d} = \mathbf{r}^1_{\text{great}}$, hence we reduce our considerations to the equity components of $\mathbf{R}^*$ which, together with $\mathbf{\Lambda}(\mathbf{r}^{*,1}, \mathbf{r}^{*,0}) = \mathbf{\Lambda}^*$, can be represented as

$$\mathbf{r}^{*,0} = (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d})^+ = (\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}). \tag{4.11}$$

Because of $(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{r}^{*,0} = \mathbf{r}^{*,0}$ we can reformulate (4.11) into

$$\begin{aligned} \mathbf{r}^{*,0} &= (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0\mathbf{r}^{*,0} + (\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{d}) \\ &= (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{r}^{*,0} + (\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{d}). \end{aligned} \tag{4.12}$$

Rearranging yields to

$$\mathbf{r}^{*,0} = (\mathbf{I}_n - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))^{-1}(\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{d}). \tag{4.13}$$

Together with Lemma A.6 in the Appendix, this leads to

$$\begin{aligned} \mathbf{r}^{*,0} &= (\mathbf{I}_n - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))^{-1}(\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{d}) \\ &\leq (\mathbf{I}_n - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))^{-1}(\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d}) \\ &\leq (\mathbf{I}_n - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))^{-1}(\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ \\ &\leq (\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{I}_n - \mathbf{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ \\ &\leq (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ \\ &= \mathbf{r}^0_{\text{great}}, \end{aligned} \tag{4.14}$$

from which the assertion follows. □

A direct consequence of Proposition 4.1 is that any iteration procedure that aims to calculate $\mathbf{R}^*$ should make sure that (i) no starting point of the iteration is chosen outside the interval $[\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ and that (ii) every interim result of the procedure is also located in that interval. For these reasons, we present an algorithm that can start either with $\mathbf{R}_{\text{great}}$ or $\mathbf{R}_{\text{small}}$.

Algorithm 1 (Picard Algorithm)**.**

1. *For* $k = 0$, *choose* $\mathbf{R}^0 \in \{\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}\}$ *and* $\varepsilon > 0$.
2. *For* $k \geq 1$, *determine* $\mathbf{R}^k = \Phi(\mathbf{R}^{k-1})$.
3. *If* $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, *stop the algorithm. Else, set* $k = k + 1$ *and proceed with Step 2.*

We will equivalently use the two expressions Picard Iteration and Picard Algorithm for Algorithm 1 in the following. For instance in Suzuki (2002) and Fischer (2014), the Picard Iteration is the algorithm of choice to determine a solution of (4.1) and (4.2). However, no suitable area for the starting vector is proposed in these articles. The question, in which situations $\mathbf{R}_{\text{great}}$ is the more optimal starting vector than $\mathbf{R}_{\text{small}}$ to avoid unnecessary calculation steps is treated in Chapter 6. In the next proposition, it is shown that in case of $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$, the resulting series of iterates will be decreasing, and increasing, if $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$. For these reasons, we will sometimes refer to the Decreasing or the Increasing Picard Algorithm.

Proposition 4.2. *In case of* $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$*, Algorithm 1 generates a sequence of increasing vectors* $\mathbf{R}^k$*, and for* $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ *a sequence of decreasing vectors. For all starting points, the algorithm converges to the solution* $\mathbf{R}^*$*.*

Proof. Let $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$, then

$$\Phi(\mathbf{R}_{\text{small}}) = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{small}} + \mathbf{M}^0\mathbf{r}^0_{\text{small}}\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{small}} + \mathbf{M}^0\mathbf{r}^0_{\text{small}} - \mathbf{d})^+ \end{pmatrix} \geq \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a}\} \\ (\mathbf{a} - \mathbf{d})^+ \end{pmatrix} = \mathbf{R}_{\text{small}}. \tag{4.15}$$

From the monotonicity of Φ (Lemma 2.4), it follows that for all iterates we have $\mathbf{R}^{k+1} \geq \mathbf{R}^k, k \geq 1$. For $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$, first check that because of $\mathbf{r}^0_{\text{great}} = (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+$ and $\mathbf{r}^1_{\text{great}} = \mathbf{d}$,

$$\begin{aligned} &(\mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{great}} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{d})^+ \\ &\quad = \left(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{r}^0_{\text{great}} + \mathbf{r}^0_{\text{great}}\right)^+ \\ &\quad = \left(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} - (\mathbf{I}_n - \mathbf{M}^0)\mathbf{r}^0_{\text{great}} + \mathbf{r}^0_{\text{great}}\right)^+ \\ &\quad = \left(\underbrace{\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} - (\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+}_{\leq \mathbf{0}_n} + \mathbf{r}^0_{\text{great}}\right)^+ \\ &\quad \leq (\mathbf{r}^0_{\text{great}})^+ = \mathbf{r}^0_{\text{great}} \end{aligned} \tag{4.16}$$

and thus

$$\Phi(\mathbf{R}_{\text{great}}) = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{great}} + \mathbf{M}^0\mathbf{r}^1_{\text{great}}\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{great}} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{d})^+ \end{pmatrix} \leq \begin{pmatrix} \mathbf{r}^1_{\text{great}} \\ \mathbf{r}^0_{\text{great}} \end{pmatrix} = \mathbf{R}_{\text{great}}. \tag{4.17}$$

Again it holds, due to the monotonicity of Φ, that $\mathbf{R}^{k+1} \leq \mathbf{R}^k, k \geq 1$. Hence, for any $\mathbf{R} \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ it follows because of $\mathbf{R} \leq \mathbf{R}_{\text{great}}$ that $\Phi(\mathbf{R}) \leq \Phi(\mathbf{R}_{\text{great}}) \leq \mathbf{R}_{\text{great}}$ and with the same argumentation it follows that $\Phi(\mathbf{R}) \geq \Phi(\mathbf{R}_{\text{small}}) \geq \mathbf{R}_{\text{small}}$. This means that any series from the Picard Iteration with a starting point in the interval $[\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ is bounded from above and from below. Since Φ is continuous, it follows that the series must converge to some $\widetilde{\mathbf{R}}$ such that $\Phi(\widetilde{\mathbf{R}}) = \widetilde{\mathbf{R}}$. According to Theorem 2.7, there is only one fixed point, so it must hold that $\widetilde{\mathbf{R}} = \mathbf{R}^*$. □

The result of Proposition 4.2 is not restricted to only non-negative exogenous asset values as the next remark underlines. This will become important in Section 5.2 when algorithms to find $\mathbf{R}^*$ in case of $m > 1$ are investigated.

Remark 4.3. Proposition 4.2 still holds in case of $\mathbf{a} \in \mathbb{R}^n$ (see also Remark 2.8). To avoid negative liabilities, we have to modify the first n components of the mapping Φ according to Equation (2.16), where the positive part of $\mathbf{a}+\mathbf{M}^1\mathbf{r}^1+\mathbf{M}^0\mathbf{r}^0$ is taken. For $\mathbf{a} \in \mathbb{R}^n$, the minimum starting vector then needs to be modified in the first n components to $\mathbf{r}^0_{\text{small}} = \min\{\mathbf{d}, \mathbf{a}^+\}$. Check that Proposition 4.1 still holds under this assumption and as well as Proposition 4.2.

The Picard Iteration – as well as any other iterative algorithm in Section 4.1 – might not reach the solution $\mathbf{R}^*$ in finitely many iteration steps. A financial system in which this problem occurs, can easily be constructed. Everything that is needed, is one firm that is in default and another firm that is solvent in the fixed point. A crucial assumption is that the two firms are directly connected with each other in the sense that the solvent firm owns some part of the debt of the defaulting firm and that the firm in default owns some shares of the solvent firm. The next proposition demonstrates this phenomena.

Proposition 4.4. *Consider a financial system with $n \geq 2$ firms for which either the Increasing or the Decreasing Picard Algorithm is applied to find $\mathbf{R}^*$. Assume that in the fixed point $\mathbf{R}^*$, there exists a firm i_1 which is in default with $d_{i_1} > r^{*,1}_{i_1} > 0$ and that there is another firm i_2 which is solvent in the solution, i.e. $r^{*,0}_{i_2} > 0$ for which in case of the decreasing version further holds that $(\mathbf{r}^0_{\text{great}})_{i_2} > r^{*,0}_{i_2}$. If $M^1_{i_2,i_1} \neq 0$ and $M^0_{i_1,i_2} \neq 0$, the Picard Algorithm will never reach $\mathbf{R}^*$.*

Proof. We show the claim first with the Decreasing Picard Algorithm, i.e. $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. Then, $r^{0,1}_{i_1} = d_{i_1} > r^{*,1}_{i_1}$ since firm i_1 is in default. The next debt iterate of i_1 is given by

$$r^{1,1}_{i_1} = \min\left\{d_{i_1}, \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{0,1}+\mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1}\right\}. \tag{4.18}$$

If $r^{1,1}_{i_1} = d_{i_1}$, then $r^{1,1}_{i_1} > r^{*,1}_{i_1}$ with the same argumentation as above. Else,

$$r^{1,1}_{i_1} = \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{0,1}+\mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1} > \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{*,1}+\mathbf{M}^0\mathbf{r}^{*,0}\right)_{i_1} = r^{*,1}_{i_1}, \tag{4.19}$$

since $r^{0,0}_{i_2} > r^{*,0}_{i_2}$ and $M^0_{i_1,i_2} \neq 0$. For the next equity iterate of i_2, it holds that

$$r^{1,0}_{i_2} = \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{0,1}+\mathbf{M}^0\mathbf{r}^{0,0}-\mathbf{d}\right)_{i_2} > \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{*,1}+\mathbf{M}^0\mathbf{r}^{*,0}-\mathbf{d}\right)_{i_2} = r^{*,0}_{i_2} \tag{4.20}$$

because of $r^{0,1}_{i_1} > r^{*,1}_{i_1}$ and $M^1_{i_2,i_1} \neq 0$. Check that we can omit the $(\cdot)^+$ sign in (4.20) because if i_2 is solvent, it holds that $r^{k,0}_{i_2} > 0$ for every iterate $k \geq 0$. With the same argumentation it follows immediately that $r^{2,1}_{i_1} > r^{*,1}_{i_1}$ and $r^{2,0}_{i_2} > r^{*,0}_{i_2}$. By induction, we see that $r^{k,1}_{i_1} > r^{*,1}_{i_1}$ and $r^{k,0}_{i_2} > r^{*,0}_{i_2}$ for all $k \geq 0$ and therefore $\mathbf{R}^k > \mathbf{R}^*$ for all $k \geq 0$.

Let us now change the direction of the algorithm and set $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$. Note that i_1 is in default from which follows that

$$r^{0,1}_{i_1} = \min\{d_{i_1}, a_{i_1}\} = a_{i_1} < r^{*,1}_{i_1} = \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{*,1}+\mathbf{M}^0\mathbf{r}^{*,0}\right)_{i_1} \tag{4.21}$$

since $r^{*,0}_{i_2} > 0$ and $M^0_{i_1,i_2} \neq 0$. For the equity value of firm i_2 it holds by definition that $r^{0,0}_{i_2} = (a_{i_2} - d_{i_2})^+$. If $a_{i_2} \leq d_{i_2}$, then $r^{0,0}_{i_2} = 0 < r^{*,0}_{i_2}$. Otherwise,

$$r^{0,0}_{i_2} = (\mathbf{a}-\mathbf{d})_{i_2} < \left(\mathbf{a}+\mathbf{M}^1\mathbf{r}^{*,1}+\mathbf{M}^0\mathbf{r}^{*,0}-\mathbf{d}\right)_{i_2} = r^{*,0}_{i_2} \tag{4.22}$$

because of $r_{i_1}^{*,1} > 0$ and $M_{i_2,i_1}^1 \neq 0$. The debt component of the next iterate for i_1 is then given by

$$r_{i_1}^{1,1} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1} < \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\right)_{i_1} = r_{i_1}^{*,1} \tag{4.23}$$

because of $r_{i_2}^{0,0} < r_{i_2}^{*,0}$ and $M_{i_1,i_2}^1 \neq 0$ as well as we immediately find that $r_{i_2}^{1,0} < r_{i_2}^{*,0}$ using the same argumentation as in (4.22). Hence $r_{i_1}^{k,1} < r_{i_1}^{*,1}$ and $r_{i_2}^{k,0} < r_{i_2}^{*,0}$ for all $k \geq 0$ and thus $\mathbf{R}^k < \mathbf{R}^*$ for all $k \geq 0$. □

Example 4.5. Consider a financial system with $n = 5$ firms, where

$$\mathbf{a} = (0.39, 0.38, 0.60, 1.29, 0.46)^t, \qquad \mathbf{d} = (1.62, 1.82, 2.07, 1.14, 0.77)^t \tag{4.24}$$

and

$$\mathbf{M}^1 = \begin{pmatrix} 0 & 0.375 & 0.25 & 0.25 & 0 \\ 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0.375 & 0 & 0 & 0.25 \\ 0.25 & 0 & 0 & 0 & 0.25 \\ 0.25 & 0 & 0.25 & 0.25 & 0 \end{pmatrix}, \; \mathbf{M}^0 = \begin{pmatrix} 0 & 0.05 & 0.067 & 0.05 & 0.05 \\ 0.05 & 0 & 0.067 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.067 & 0 & 0.05 \\ 0.05 & 0.05 & 0.067 & 0.05 & 0 \end{pmatrix}. \tag{4.25}$$

The solution of the system is given by the vector

$$\mathbf{R}^* = (1.62, 1.7342, 1.5316, 1.14, 0.77, 0.1686, 0, 0, 0.7965, 0.8112)^t \tag{4.26}$$

and we see that the second and the third firm are in default, the others are solvent. Since for example the first firm owns some debt of the second firm ($M_{12}^1 = 0.375$) and this firm, on the other hand, holds some shares of the first firm ($M_{21}^0 = 0.05$), the Picard Iteration will theoretically never reach the fixed point $\mathbf{R}^*$. Of course, since $\varepsilon > 0$ is chosen in Algorithm 1, the procedure will stop after some iterations, as Example 4.21 demonstrates.

Remark 4.6. Proposition 4.4 does not claim to be a complete description of situations in which the Picard Iteration does not reach $\mathbf{R}^*$. There are many other possible situations in which the same phenomenon can occur. For instance in case of the Decreasing Picard Algorithm, we can demand that $i_1 \notin \mathcal{D}(\mathbf{R}^0)$ instead of $r_{i_2}^{0,0} > r_{i_2}^{*,0}$. As a consequence, Equation (4.18) becomes $r_{i_1}^{1,1} = d_{i_1} > r_{i_1}^{*,1}$.

From a computational or practical point of view, the fact that the solution can, under some circumstances, not be exactly reached, will result in many needed iteration steps to approach $\mathbf{R}^*$ sufficiently close. This can make the Picard Algorithm somewhat inefficient, as well as the other algorithms that are presented in remainder of this section. The Trial-and-Error Algorithms presented in Section 4.2, however, do not have this drawback since for these procedures it is ensured that they will reach the solution in a finite number of steps.

4.1.2 The Elsinger Algorithm

In Elsinger (2009), an algorithm to find $\mathbf{R}^*$ is presented which differs from the Picard Iteration. This procedure consists of splitting the two components of an iterate $\mathbf{R}^k$, the equity and the debt components, and apply different computation methods to both components in each iteration step. For the equity components, a subalgorithm is applied where the equity payments of the system are determined assuming a fixed amount of debt payments. Denote this vector of debt payments in the following by $\bar{\mathbf{r}}$, hence $\mathbf{0}_n \leq \bar{\mathbf{r}} \leq \mathbf{d}$. Aim of the subalgorithm is to find a fixed point of the mapping $\Phi^0 : (\mathbb{R}_0^+)^n \rightarrow (\mathbb{R}_0^+)^n$ with

$$\Phi^0(\mathbf{r}^0; \bar{\mathbf{r}}) = (\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0 - \mathbf{d})^+. \tag{4.27}$$

This mapping represents the equity components of Φ for given debt payments of $\bar{\mathbf{r}}$. The fixed point of $\Phi^0(\cdot; \bar{\mathbf{r}})$ is denoted by $\mathbf{r}^0(\bar{\mathbf{r}})$, i.e.

$$\Phi^0(\mathbf{r}^0(\bar{\mathbf{r}}); \bar{\mathbf{r}}) = (\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0(\bar{\mathbf{r}}) - \mathbf{d})^+ = \mathbf{r}^0(\bar{\mathbf{r}}). \tag{4.28}$$

As shown in Elsinger (2009, Lemma 4), this fixed point exists and is unique since $\mathbf{M}^0$ has the Elsinger Property.

The following algorithm delivers for given $\bar{\mathbf{r}}$ a series of vectors $\mathbf{w}^k \in \mathbb{R}^n$ that converge to a vector that is the fixed point of (4.27). To explain this in more detail, first define for a given vector $\mathbf{w} \in \mathbb{R}^n$ the set

$$\mathcal{P}(\mathbf{w}) = \{i \in \mathcal{N} : w_i \geq 0\} \tag{4.29}$$

and the matrix

$$\mathbf{G}(\mathbf{w}) = \operatorname{diag}(\mathbf{w} \geq \mathbf{0}_n) \tag{4.30}$$

as the corresponding diagonal matrix. Note that these definitions of $\mathcal{P}(\mathbf{w})$ and $\mathbf{G}(\mathbf{w})$ slightly differ from the original ones in Elsinger (2009), where a strictly larger sign was used in the Equations (4.29) and (4.30). By our definition of default in (4.8), a firm with zero equity value can still be not in default in the sense that all obligations can fully served, but no capital is left for the shareholders. This situation is referred to as *borderline firms* (cf. Section 4.2.2). However, this modification does not change the forthcoming theoretical results.

Algorithm 2.

1. *For* $k = 0$, *set* $\mathbf{w}^0 = \mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} - \mathbf{d}$ *and determine* $\mathcal{P}(\mathbf{w}^0)$ *and* $\mathbf{G}(\mathbf{w}^0)$.
2. *For* $k \geq 1$, *solve* $\Psi_{\mathbf{w}^{k-1}}(\mathbf{w}) = \mathbf{w}$ *where*

$$\Psi_{\mathbf{w}^{k-1}}(\mathbf{w}) = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{k-1})\mathbf{w} \tag{4.31}$$

and denote the solution by $\mathbf{w}^k$, *i.e.* $\Psi_{\mathbf{w}^{k-1}}(\mathbf{w}^k) = \mathbf{w}^k$. *Determine* $\mathcal{P}(\mathbf{w}^k)$ *and* $\mathbf{G}(\mathbf{w}^k)$.

3. *If* $\mathcal{P}(\mathbf{w}^k) = \mathcal{P}(\mathbf{w}^{k-1})$, *stop the algorithm. Else, set* $k = k + 1$ *and proceed with Step 2.*

Before the properties of Algorithm 2 are shown, we give some explanations for a better understanding of its functioning. The starting point is $\mathbf{w}^0$, which is the difference between $\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}}$ and $\mathbf{d}$. The sum represents the firms' incomes on their balance sheet that consists of exogenous and endogenous assets $\mathbf{a}$ and $\mathbf{M}^1\bar{\mathbf{r}}$, respectively. Note that in this step the potential income from equity cross-ownership is ignored since $\mathbf{M}^0$ does not appear. The idea is now as follows: The firms not in $\mathcal{P}(\mathbf{w}^0)$ are not able to fully satisfy their liabilities (assuming debt payments of $\bar{\mathbf{r}}$) and will be in default. On the other hand, the firms that are in $\mathcal{P}(\mathbf{w}^0)$ will be able to satisfy their obligees and can be regarded as solvent (again assuming debt payments of $\bar{\mathbf{r}}$), even though no intersystem payments due to equity cross-ownership are taken into account. As a consequence, the equity payments of the non-defaulting firms are added into the system via the product $\mathbf{M}^0\mathbf{G}(\mathbf{w}^0)\mathbf{w}$. We can interpret the vector $\mathbf{w}^0$, as well as the other iterates $\mathbf{w}^k$, as pseudo equity vectors that give us information about solvent and defaulting firms under the current debt and equity payments. The fact that the entries of $\mathbf{w}^k$ can be negative prevents that they can be naturally interpreted as equity vectors which is why we use the term "pseudo".

The difference compared to the Picard Algorithm is that a linear equation system is solved to achieve a new equity payment vector instead of applying Φ to an iterate $\mathbf{R}^k$. This is because for the fixed point of $\Psi_{\mathbf{w}^{k-1}}$ it holds together with (4.31) that

$$\mathbf{w}^k = (\mathbf{I}_n - \mathbf{M}^0\mathbf{G}(\mathbf{w}^{k-1}))^{-1}\mathbf{w}^0 \tag{4.32}$$

Note that the inverse matrix exists since both $\mathbf{M}^0$ and, hence, $\mathbf{M}^0\mathbf{G}(\mathbf{w}^{k-1})$ have the Elsinger Property.

The vector $\mathbf{w}^1$ can be interpreted as an "updated" version of $\mathbf{w}^0$ since the equity payments of the non-defaulting firms that are in $\mathcal{P}(\mathbf{w}^0)$ are included in $\mathbf{w}^1$. Based on the updated vector $\mathbf{w}^1$, it might appear that some firms that are not in $\mathcal{P}(\mathbf{w}^0)$ have now non-negative entries in $\mathbf{w}^1$. This can be concluded from $\mathbf{w}^1 \geq \mathbf{w}^0$ that we will show later. But these firms are now also able to contribute equity payments to the system. Consequently, the system has to be updated again by determining $\mathbf{w}^2$. The procedure continues until the set of defaulting firms stays the same from one iteration step to the next one.

Proposition 4.7. *Given a fixed vector of debt payments* $\bar{\mathbf{r}} \geq \mathbf{0}_n$*:*

(i) Algorithm 2 generates an increasing sequence of vectors $\mathbf{w}^k$*.*

(ii) Let $1 \leq l \leq n$ *such that*

$$l := \min\{j \in \{0, 1, \ldots, n-1\} : \mathcal{P}(\mathbf{w}^j) = \mathcal{P}(\mathbf{w}^{j+1})\}. \tag{4.33}$$

Then $\mathbf{r}^0(\bar{\mathbf{r}}) = (\mathbf{w}^{l+1})^+$ *is the fixed point of the mapping* $\Phi^0(\cdot; \bar{\mathbf{r}})$*.*

(iii) Let $d_0 = |\mathcal{P}(\mathbf{w}^0)|$ *be the number of firms with a non-negative entry in* $\mathbf{w}^0$*. If* $d_0 \in \{1, \ldots, n\}$*, the fixed point* $\mathbf{r}^0(\bar{\mathbf{r}})$ *is reached after no more than* $n - d_0$ *iteration steps. If* $d_0 = 0$*, no iteration is necessary, since* $\mathbf{r}^0(\bar{\mathbf{r}}) = \mathbf{0}_n$*.*

Proof. (i) This part of the Proposition is shown by Elsinger (2009). We give a different version of the proof. Because of (4.32), the fact that $\mathbf{G}(\mathbf{w}^0)\mathbf{w}^0 \geq \mathbf{0}_n$ and using the series representation of $(\mathbf{I}_n - \mathbf{M}^0\mathbf{G}(\mathbf{w}^0))^{-1}$ as shown in Lemma A.3 of the Appendix we get

$$\begin{aligned}
\mathbf{w}^1 &= (\mathbf{I}_n - \mathbf{M}^0\mathbf{G}(\mathbf{w}^0))^{-1}\mathbf{w}^0 \\
&= (\mathbf{I}_n + \mathbf{M}^0\mathbf{G}(\mathbf{w}^0) + (\mathbf{M}^0\mathbf{G}(\mathbf{w}^0))^2 + \ldots)\mathbf{w}^0 \\
&= \mathbf{w}^0 + \mathbf{M}^0 \underbrace{\mathbf{G}(\mathbf{w}^0)\mathbf{w}^0}_{\geq \mathbf{0}_n} + \mathbf{M}^0\mathbf{G}(\mathbf{w}^0)\mathbf{M}^0 \underbrace{\mathbf{G}(\mathbf{w}^0)\mathbf{w}^0}_{\geq \mathbf{0}_n} + \ldots \\
&\geq \mathbf{w}^0,
\end{aligned} \tag{4.34}$$

which is the induction start. For the induction step we assume $\mathbf{w}^k \geq \mathbf{w}^{k-1}$ and $\mathbf{G}(\mathbf{w}^k) \geq \mathbf{G}(\mathbf{w}^{k-1})$ following from it. We need to show that $\mathbf{w}^{k+1} \geq \mathbf{w}^k$, or, equivalently, $\mathbf{w}^{k+1} = \mathbf{w}^k + \mathbf{e}$ where $\mathbf{e} \geq \mathbf{0}_n$. Since $\mathbf{G}(\mathbf{w}^k)\mathbf{w}^k \geq \mathbf{G}(\mathbf{w}^{k-1})\mathbf{w}^k$ and $\mathbf{w}^k = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{k-1})\mathbf{w}^k$, it follows that

$$\mathbf{u} := \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)\mathbf{w}^k - \mathbf{w}^k = \mathbf{M}^0(\mathbf{G}(\mathbf{w}^k) - \mathbf{G}(\mathbf{w}^{k-1}))\mathbf{w}^k \geq \mathbf{0}_n. \tag{4.35}$$

With this definition we have that

$$\mathbf{w}^k + \mathbf{e} = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)(\mathbf{w}^k + \mathbf{e}) = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)\mathbf{w}^k + \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)\mathbf{e} \tag{4.36}$$

and we can rearrange to

$$\mathbf{e} - \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)\mathbf{e} = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^k)\mathbf{w}^k - \mathbf{w}^k = \mathbf{u} \geq \mathbf{0}_n. \tag{4.37}$$

Solving this for $\mathbf{e}$ leads to

$$\mathbf{e} = (\mathbf{I}_n - \mathbf{M}^0\mathbf{G}(\mathbf{w}^k))^{-1}\mathbf{u} \geq \mathbf{0}_n \tag{4.38}$$

from which follows that $\mathbf{w}^{k+1} \geq \mathbf{w}^k$.

(ii) First, we will show that once a "stable system" has been reached, i.e. for $k \geq 0$ we have $\mathcal{P}(\mathbf{w}^k) = \mathcal{P}(\mathbf{w}^{k+1})$, the sequence $\mathbf{w}^k$ will be constant. Let l be defined as above in (4.33). Note that such an l exists since $\mathbf{w}^k \leq \mathbf{w}^{k+1}$ and therefore $\mathcal{P}(\mathbf{w}^{k+1}) \supseteq \mathcal{P}(\mathbf{w}^k)$ for all $k \geq 0$ as shown above. Due to $\mathbf{G}(\mathbf{w}^l) = \mathbf{G}(\mathbf{w}^{l+1})$, it follows because of

$$\Psi_{\mathbf{w}^l}(\mathbf{w}) = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^l)\mathbf{w} = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{l+1})\mathbf{w} = \Psi_{\mathbf{w}^{l+1}}(\mathbf{w}) \tag{4.39}$$

that the two mappings $\Psi_{\mathbf{w}^l}$ and $\Psi_{\mathbf{w}^{l+1}}$ are the same and consequently $\mathbf{w}^{l+1} = \mathbf{w}^{l+2}$. A direct consequence is that $\mathcal{P}(\mathbf{w}^{l+2}) = \mathcal{P}(\mathbf{w}^{l+1}) = \mathcal{P}(\mathbf{w}^l)$ which implies $\mathbf{G}(\mathbf{w}^{l+2}) = \mathbf{G}(\mathbf{w}^{l+1}) = \mathbf{G}(\mathbf{w}^l)$. By induction, all following vectors will be equal to $\mathbf{w}^{l+1}$.

What remains to be shown out is that the positive part of this iteration vector is the fixed point of the mapping $\Phi^0(\cdot;\bar{\mathbf{r}})$. Since $\mathbf{w}^{l+1}$ is the fixed point of $\Psi_{\mathbf{w}^l}$, it holds that $\mathbf{w}^{l+1} = \mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^l)\mathbf{w}^{l+1}$. This yields to

$$\begin{aligned}\Phi^0((\mathbf{w}^{l+1})^+;\bar{\mathbf{r}}) &= (\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} + \mathbf{M}^0(\mathbf{w}^{l+1})^+ - \mathbf{d})^+ \\ &= (\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{l+1})\mathbf{w}^{l+1} - \mathbf{d})^+ \\ &= (\mathbf{a} + \mathbf{M}^1\bar{\mathbf{r}} + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{l})\mathbf{w}^{l+1} - \mathbf{d})^+ \\ &= (\mathbf{w}^0 + \mathbf{M}^0\mathbf{G}(\mathbf{w}^{l})\mathbf{w}^{l+1})^+ \\ &= (\mathbf{w}^{l+1})^+.\end{aligned} \tag{4.40}$$

(iii) If $d_0 = 0$, then $w_i^0 < 0$ for all $i \in \mathcal{N}$. This means that $\mathbf{G}(\mathbf{w}_0) = \mathbf{0}_{n\times n}$ and that the first iterate $\mathbf{w}^1$ is given by $\mathbf{w}^1 = (\mathbf{I}_n - \mathbf{M}^0\mathbf{G}(\mathbf{w}^0))^{-1}\mathbf{w}^0 = \mathbf{w}^0$ from which follows that $\mathcal{P}(\mathbf{w}^1) = \mathcal{P}(\mathbf{w}^0) = \emptyset$. By definition, the algorithm stops and we obtain $\mathbf{r}^0(\bar{\mathbf{r}}) = (\mathbf{w}^1)^+ = \mathbf{0}_n$ as the fixed point of $\Phi^0(\cdot;\bar{\mathbf{r}})$, as one can easily check. Suppose now that $d_0 \geq 1$. As shown in part (i), the series $\mathbf{w}^k$ increases which means that the firms in $\mathcal{P}(\mathbf{w}^0)$ will maintain their positive entries in every further iteration step. The same statement holds for every firm i with $w_i^k < 0$ and $w_i^{k+1} \geq 0$ for any $k \geq 0$. Because of (ii) this means that the number of iteration steps would certainly be maximal, if in every iteration step the set $\mathcal{P}(\mathbf{w}^k)$ increased by one and if $|\mathcal{P}(\mathbf{w}^{l+1})| = n$. In that case we would therefore have $|\mathcal{P}(\mathbf{w}^{l+1})| - |\mathcal{P}(\mathbf{w}^0)| = n - d_0$ maximal possible iteration steps.

□

Example 4.8. Consider the financial system defined in Example 4.5. To find a solution of this system, we assume in the first step that all firms are able to fully recover their liabilities, i.e. $\bar{\mathbf{r}} = \mathbf{d}$. Under this assumption, we can use Algorithm 2 to find the fixed point of $\Phi^0(\cdot,\mathbf{d})$. Doing so, we find that

$$\mathbf{w}^0 = (0.2550, -0.04, -0.595, 0.7475, 0.8975)^t \quad \text{and} \quad \mathcal{P}(\mathbf{w}^0) = \{1, 4, 5\}.$$

The firms in $\mathcal{P}(\mathbf{w}^0)$ are able to fully satisfy their liabilities, hence we can add their equity payments for the other firms in the system into account which leads to

$$\mathbf{w}^1 = (0.3434, 0.0656, -0.4894, 0.8124, 0.9553)^t \quad \text{and} \quad \mathcal{P}(\mathbf{w}^1) = \{1, 2, 4, 5\}.$$

The additional payments cause that firm number 2 is now also able to fully recover its liabilities, only firm 3 is not able to do so. This also does not change in the next iteration step where we get

$$\mathbf{w}^2 = (0.3471, 0.0661, -0.4856, 0.8161, 0.9590)^t \quad \text{and} \quad \mathcal{P}(\mathbf{w}^2) = \{1, 2, 4, 5\}.$$

So, the third firm stays in default and because of $\mathcal{P}(\mathbf{w}^1) = \mathcal{P}(\mathbf{w}^2)$ the algorithms stops and $(\mathbf{w}^2)^+ = \mathbf{r}^0(\mathbf{d})$ is the searched fixed point of $\Phi^0(\cdot; \mathbf{d})$ as one easily can check.

Using Algorithm 2 to get an equity vector for a given debt payment vector, we can now present the algorithm to calculate the solution $\mathbf{R}^*$. In the sequel, we will make use of the mapping $\Phi^1 : (\mathbb{R}_0^+)^n \to (\mathbb{R}_0^+)^n$ defined by

$$\Phi^1(\mathbf{r}; \bar{\mathbf{r}}^0) = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r} + \mathbf{M}^0\bar{\mathbf{r}}^0\} \tag{4.41}$$

that represents the debt components of Φ for a given equity payment vector $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$.

Algorithm 3 (Elsinger Algorithm)**.** *Set* $\varepsilon > 0$.

1. *For* $k = 0$, *choose* $\mathbf{r}^{0,1} \in \{\mathbf{r}^1_{\text{small}}, \mathbf{r}^1_{\text{great}}\}$ *and determine* $\mathbf{r}^0(\mathbf{r}^{0,1})$ *using Algorithm 2.*
2. *For* $k \geq 1$, *set* $\mathbf{r}^{k,1} = \Phi^1(\mathbf{r}^{k-1,1}; \mathbf{r}^0(\mathbf{r}^{k-1,1}))$ *and calculate* $\mathbf{r}^0(\mathbf{r}^{k,1})$ *by Algorithm 2.*
3. *If* $\left\| \begin{pmatrix} \mathbf{r}^{k-1,1} \\ \mathbf{r}^0(\mathbf{r}^{k-1,1}) \end{pmatrix} - \begin{pmatrix} \mathbf{r}^{k,1} \\ \mathbf{r}^0(\mathbf{r}^{k,1}) \end{pmatrix} \right\| < \varepsilon$, *stop the algorithm. Else, set* $k = k + 1$ *and proceed with Step 2.*

The algorithm starts either assuming that all firms can fully deliver their debt obligations ($\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}} = \mathbf{d}$) or that all firms have only their exogenous assets for paying off their obligations ($\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}} = \min\{\mathbf{d}, \mathbf{a}\}$). With this payment vector, the corresponding equity payments are obtained by using Algorithm 2. In the next step the debt vector has to be adapted to the new equity payments which is done applying Φ^1 to the previous debt vector. The updated debt payment vector is then used for determining a new equity payment vector. This procedure continues until the iterates are sufficiently close to each other. Additional to the original algorithm first presented in Elsinger (2009), Algorithm 3 contains the second possible starting point $\mathbf{r}^1_{\text{small}}$. In Figure 4.1, the schematic workflow of the Elsinger Algorithm is summarized more compactly, we demonstrate the functioning of the Elsinger Algorithm also in Example 4.21. We will show in the next proposition that if $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ is chosen, the vector of debt and equity payments establish an increasing sequence and hence converges to the solution $\mathbf{R}^*$ from below, while for $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$, the series converges from above. Depending on the choice of the initial debt iterate $\mathbf{r}^{0,1}$ in the first step of the procedure, we call the algorithms either *Decreasing Elsinger Algorithm* if $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$ or *Increasing Elsinger Algorithm* in case of $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$.

Proposition 4.9. *The Elsinger Algorithm delivers a series of decreasing vectors if* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$ *and a series of increasing vectors if* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$. *Both series converge to the fixed point of the mapping* Φ *in* (2.12).

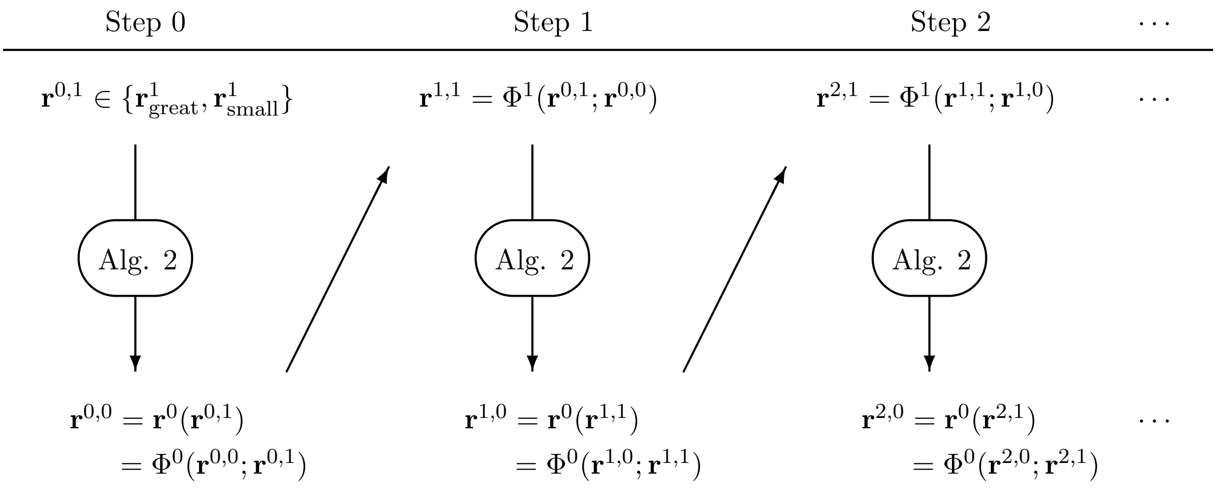


Figure 4.1: Schematic functioning of the Elsinger Algorithm. The oval-shaped symbol means that Algorithm 2 is applied to update the equity vector. The arrow without any further specification means that the function Φ^1 is applied to update the debt vector. In case of $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ the sequence $(\mathbf{r}^k, \mathbf{r}^0(\mathbf{r}^k))$ is increasing, and it is decreasing if $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}} = \mathbf{d}$.

Proof. The decreasing part is shown in Elsinger (2009), we only have to show that the debt iterate in the algorithm therein is identical to $\mathbf{r}^{k,1}$ in Algorithm 3. With our notation, the iterate of the debt component in Elsinger (2009) is defined as

$$\mathbf{r}^{k,1} = \min\{\mathbf{d}, (\mathbf{w}^*(\mathbf{r}^{k-1,1}) + \mathbf{d})^+\}, \tag{4.42}$$

where $\mathbf{w}^*(\mathbf{r}^{k-1,1})$ is the solution of

$$\mathbf{w} = \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1} + \mathbf{M}^0\mathbf{w}^+ - \mathbf{d}. \tag{4.43}$$

However, it follows from (4.40) that for $\bar{\mathbf{r}} = \mathbf{r}^{k-1,1}$,

$$(\mathbf{w}^{l+1})^+ = \Phi^0\left((\mathbf{w}^{l+1})^+; \mathbf{r}^{k-1,1}\right) = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1} + \mathbf{M}^0(\mathbf{w}^{l+1})^+ - \mathbf{d}\right)^+ = \left(\mathbf{w}^*(\mathbf{r}^{k-1,1})\right)^+, \tag{4.44}$$

where $\mathbf{w}^{l+1}$ is the result of Algorithm 2 with the debt payment vector $\mathbf{r}^{k-1,1}$, i.e. $(\mathbf{w}^*(\mathbf{r}^{k-1,1}))^+ = \mathbf{r}^0(\mathbf{r}^{k-1,1})$. Because of (4.43), we have that

$$\mathbf{w}^*(\mathbf{r}^{k-1,1}) = \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1} - \mathbf{d} + \mathbf{M}^0(\mathbf{w}^*(\mathbf{r}^{k-1,1}))^+, \tag{4.45}$$

from which follows with (4.41) and $\mathbf{a} \geq \mathbf{0}_n$ that

$$\begin{aligned}
\mathbf{r}^{k,1} &= \min\{\mathbf{d}, (\mathbf{w}^*(\mathbf{r}^{k-1,1}) + \mathbf{d})^+\} \\
&= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1} + \mathbf{M}^0(\mathbf{w}^*(\mathbf{r}^{k-1,1}))^+\} \\
&= \Phi^1(\mathbf{r}^{k-1,1}; (\mathbf{w}^*(\mathbf{r}^{k-1,1}))^+) \\
&= \Phi^1(\mathbf{r}^{k-1,1}; \mathbf{r}^0(\mathbf{r}^{k-1,1})).
\end{aligned} \tag{4.46}$$

What remains to be shown is that for the starting point $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ the produced series increases and converges to $\mathbf{R}^*$ which is done by induction. For the induction start, check that

$$\mathbf{r}^{0,1} = \min\{\mathbf{d}, \mathbf{a}\} \leq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{0,1})\} = \Phi^1(\mathbf{r}^{0,1}; \mathbf{r}^0(\mathbf{r}^{0,1})) = \mathbf{r}^{1,1}.$$

According to Lemma 2.10, $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$. Hence, $\mathbf{r}^0(\mathbf{r}^{0,1}) \leq \mathbf{r}^0(\mathbf{r}^{1,1})$ which completes the induction start. Assume for the induction step that $\mathbf{r}^{k-1,1} \leq \mathbf{r}^{k,1}$ and consequently $\mathbf{r}^0(\mathbf{r}^{k-1,1}) \leq \mathbf{r}^0(\mathbf{r}^{k,1})$. The next debt iterate emerges as:

$$\begin{aligned}\mathbf{r}^{k+1,1} &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0(\mathbf{r}^0(\mathbf{r}^{k,1}))^+\} \\ &\geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1} + \mathbf{M}^0(\mathbf{r}^0(\mathbf{r}^{k-1,1}))^+\} \\ &= \mathbf{r}^{k,1},\end{aligned} \tag{4.47}$$

from which also follows that $\mathbf{r}^0(\mathbf{r}^{k+1,1}) \geq \mathbf{r}^0(\mathbf{r}^{k,1})$ and, hence, the increasing property of the series. For the convergence, check that $\mathbf{r}^0(\mathbf{r}^{k,1}) \geq \mathbf{0}_n$ and it holds that

$$\mathbf{r}^0(\mathbf{r}^{k,1}) = (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}) - \mathbf{d})^+ \leq \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}) + (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} - \mathbf{d})^+. \tag{4.48}$$

Because of $\mathbf{r}^{k,1} \leq \mathbf{r}^{*,1}$, it follows after some rearrangements that

$$\mathbf{r}^0(\mathbf{r}^{k,1}) \leq (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} - \mathbf{d})^+ \leq (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{d})^+, \tag{4.49}$$

hence the series $\mathbf{r}^0(\mathbf{r}^{k,1})$ is bounded from above as well and therefore converges to some $\mathbf{r}^{*,0}$ from below. The fact that Φ^0 is continuous in $(\mathbf{r}^1, \mathbf{r}^0)$ implies together with $\Phi^0(\mathbf{r}^0(\mathbf{r}^{k,1}); \mathbf{r}^{k,1}) = \mathbf{r}^0(\mathbf{r}^{k,1})$ that $\Phi^0(\mathbf{r}^{*,0}; \mathbf{r}^{*,1}) = \mathbf{r}^{*,0}$. Thus, $(\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ solves (4.2). Similarly, we can argue that because of the continuity of Φ^1, $\Phi^1(\mathbf{r}^{*,1}; \mathbf{r}^{*,0}) = \mathbf{r}^{*,1}$ from which follows that $(\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ also solves (4.1) and therefore must be the fixed point $\mathbf{R}^*$. □

Similar to the Picard Algorithm (cf. Remark 4.3), we can also successfully use the Elsinger Algorithm if negative exogenous asset values are present. A property, we will make use of when applying algorithms to financial systems with $m > 1$ (cf. Section 5.2).

Remark 4.10. The assertions of Proposition 4.9 are also true if $\mathbf{a} \in \mathbb{R}^n$. To see this, first note that for any given debt iterate $\mathbf{r}^{k,1}$, Algorithm 2 still delivers the corresponding equity payments $\mathbf{r}^0(\mathbf{r}^{k,1})$ as the fixed point of $\Phi^0(\cdot; \mathbf{r}^{k,1})$. This follows from the fact that $\mathbf{a} \geq \mathbf{0}$ is not needed in the proof of Proposition 4.7. Moreover, the Elsinger Algorithm obviously delivers a decreasing or an increasing series, depending on the corresponding starting vector. Of course, the mapping Φ^1 has to be adapted to $\Phi^1(\mathbf{r}^1; \mathbf{r}^0) = \min\{\mathbf{d}, (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0)^+\}$, to ensure the proper definition of the debt iterates. For the increasing case in Algorithm 3, we further have to set $\mathbf{r}^1_{\text{small}} = \min\{\mathbf{d}, \mathbf{a}^+\}$ for the starting vector to make sure that the recovery values are positive. However, this does not change the fact that the series staring with $(\mathbf{r}_{\text{small}}, \mathbf{r}^0(\mathbf{r}_{\text{small}}))$ converges from below to $\mathbf{R}^*$.

As described above, the Elsinger Algorithm determines the equity components of the iterates $\mathbf{R}^k$ in a different way than the Picard Iteration. An important consequence of this new approach is that the iterates of the Elsinger Algorithm will for the decreasing version be in every step smaller than the iterates of the Picard Algorithm, as we will show in the next proposition. The same statement holds for the increasing version of the procedure, where the iterates from the Elsinger Algorithm will be greater than the iterates from the Picard Algorithm. Despite the fact that both procedures are difficult to compare concerning their total calculation effort due to different ways to obtain the next equity iterate, only taking the number of needed iterations as a quality criterion into account, we can conclude that the Elsinger Algorithm will not need more iteration steps to reach $\mathbf{R}^*$ sufficiently close than the Picard Iteration, no matter whether the algorithms start from the upper or the lower boundary.

Proposition 4.11. *Let* $\mathbf{R}_{\mathrm{P}}^k = (\mathbf{r}_{\mathrm{P}}^{k,1}, \mathbf{r}_{\mathrm{P}}^{k,0})$ *be the* k*-th iterate of the Picard Algorithm and* $\mathbf{R}_{\mathrm{E}}^k = (\mathbf{r}_{\mathrm{E}}^{k,1}, \mathbf{r}_{\mathrm{E}}^{k,0})$ *the corresponding iterate of the Elsinger Algorithm.*

(i) For any iterate $k \geq 0$ *it holds that* $\mathbf{R}_{\mathrm{P}}^k \geq \mathbf{R}_{\mathrm{E}}^k$ *if* $\mathbf{R}_{\mathrm{P}}^0 = \mathbf{R}_{\text{great}}$ *and* $\mathbf{R}_{\mathrm{E}}^0 = (\mathbf{r}_{\text{great}}^1, \mathbf{r}^0(\mathbf{r}_{\text{great}}^1))$. *In case of* $\mathbf{R}_{\mathrm{P}}^0 = \mathbf{R}_{\text{small}}$ *and* $\mathbf{R}_{\mathrm{E}}^0 = (\mathbf{r}_{\text{small}}^1, \mathbf{r}^0(\mathbf{r}_{\text{small}}^1))$*, we have that* $\mathbf{R}_{\mathrm{P}}^k \leq \mathbf{R}_{\mathrm{E}}^k$ *for every iterate.*

(ii) Let $\mathbf{R}^k$*,* $k \geq 1$*, be an iterate either of the Picard Algorithm with* $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ *or of the Elsinger Algorithm with* $\mathbf{R}^0 = (\mathbf{r}_{\text{great}}^1, \mathbf{r}^0(\mathbf{r}_{\text{great}}^1))$*. Then* $\mathbf{R}_{\mathrm{P}}^{k+1}(\mathbf{R}^k) \geq \mathbf{R}_{\mathrm{E}}^{k+1}(\mathbf{R}^k)$*. If the starting vector is either* $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ *or* $\mathbf{R}^0 = (\mathbf{r}_{\text{small}}^1, \mathbf{r}^0(\mathbf{r}_{\text{small}}^1))$*, it holds that* $\mathbf{R}_{\mathrm{P}}^{k+1}(\mathbf{R}^k) \leq \mathbf{R}_{\mathrm{E}}^{k+1}(\mathbf{R}^k)$*.*

Proof. (i) The assertion is shown by induction. For $k = 0$, suppose that the upper boundary is the starting vector for both algorithms. In Equation (4.49) it was shown that

$$\mathbf{r}_{\mathrm{E}}^{0,0} = \mathbf{r}^0(\mathbf{d}) \leq (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ = \mathbf{r}_{\text{great}}^1 = \mathbf{r}_{\mathrm{P}}^{0,0}. \tag{4.50}$$

Since $\mathbf{r}_{\mathrm{E}}^{0,1} = \mathbf{r}_{\mathrm{P}}^{0,1} = \mathbf{d}$, the induction start is complete. Assume now, that for $k \geq 1$ it holds that $\mathbf{R}_{\mathrm{P}}^k \geq \mathbf{R}_{\mathrm{E}}^k$. From Proposition 4.9, we know that $\mathbf{R}_{\mathrm{E}}^{k+1} \leq \mathbf{R}_{\mathrm{E}}^k$. This leads to

$$\mathbf{r}_{\mathrm{P}}^{k+1,1} = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{P}}^{k,1} + \mathbf{M}^0\mathbf{r}_{\mathrm{P}}^{k,0}\} \geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{E}}^{k,1} + \mathbf{M}^0\mathbf{r}_{\mathrm{E}}^{k,0}\} = \mathbf{r}_{\mathrm{E}}^{k+1,1} \tag{4.51}$$

and

$$\begin{aligned} \mathbf{r}_{\mathrm{P}}^{k+1,0} &= (\mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{P}}^{k,1} + \mathbf{M}^0\mathbf{r}_{\mathrm{P}}^{k,0} - \mathbf{d})^+ \\ &\geq (\mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{E}}^{k,1} + \mathbf{M}^0\mathbf{r}_{\mathrm{E}}^{k,0} - \mathbf{d})^+ \\ &\geq (\mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{E}}^{k+1,1} + \mathbf{M}^0\mathbf{r}_{\mathrm{E}}^{k+1,0} - \mathbf{d})^+ \\ &= \mathbf{r}_{\mathrm{E}}^{k+1,0}. \end{aligned} \tag{4.52}$$

If the starting vector is the lower boundary and the series $\mathbf{R}_{\mathrm{P}}^k$ and $\mathbf{R}_{\mathrm{E}}^k$ are increasing, the argumentation is similar.

(ii) We prove the claim for the decreasing version of the algorithms, the proof for the reverse direction is similar. First, let $\mathbf{R}^k = \mathbf{R}_{\mathrm{P}}^k$. The next iterate of the debt components is equal for both algorithms, i.e. $\mathbf{r}_{\mathrm{P}}^{k+1,1} = \Phi^1(\mathbf{r}^{k,1}; \mathbf{r}^{k,0}) = \mathbf{r}_{\mathrm{E}}^{k+1,1}$. For the equity components, it holds that $\mathbf{r}_{\mathrm{P}}^{k+1,0} = \Phi^0(\mathbf{r}^{k,0}; \mathbf{r}^{k,1})$. The mapping $\Phi^0(\cdot; \mathbf{r}^{k,1})$ has a unique fixed point, that we denote by $\mathbf{r}^0(\mathbf{r}^{k,1})$ and that can be obtained via a Picard Iteration:

$$\lim_{l \to \infty} \left(\Phi^0\right)^l (\mathbf{r}^{k,0}; \mathbf{r}^{k,1}) = \mathbf{r}^0(\mathbf{r}^{k,1}). \tag{4.53}$$

The iterates obviously form a decreasing sequence so that

$$\mathbf{r}_{\mathrm{P}}^{k+1,0} \geq \mathbf{r}^0(\mathbf{r}^{k,1}) \geq \mathbf{r}^0(\mathbf{r}^{k+1,1}) = \mathbf{r}_{\mathrm{E}}^{k+1,0}, \tag{4.54}$$

where the second inequality follows from the fact that $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$ (cf. Lemma 2.10). If the k-th iterate is given by $\mathbf{R}^k = \mathbf{R}_{\mathrm{E}}^k$, the arguments are analogous to the ones above.

□

Even though the iterates of the Elsinger Algorithm are always closer to $\mathbf{R}^*$ than their counterparts of the Picard Algorithm, there as still some situations, in which the Elsinger Algorithm does not reach $\mathbf{R}^*$.

Proposition 4.12. *Consider a financial system with $n \geq 2$ firms for which either the Increasing or the Decreasing Elsinger Algorithm is applied to find $\mathbf{R}^*$. Assume that in the fixed point $\mathbf{R}^*$, there exists a firm i_1 which is in default with $d_{i_1} > r_{i_1}^{*,1} > 0$ and that there is another firm i_2 which is solvent in the solution, i.e. $r_{i_2}^{*,0} > 0$. If $M_{i_2,i_1}^1 \neq 0$ and $M_{i_1,i_2}^0 \neq 0$, the Elsinger Algorithm will never reach $\mathbf{R}^*$.*

Proof. First note that, unlike to Proposition 4.4, we do not have to demand a particular property for $r_{i_2}^{0,0}$. In the decreasing version of the algorithm, $\mathbf{R}^0 = (\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$ and therefore $r_{i_1}^{0,1} > r_{i_1}^{*,1}$. For the equity component of firm i_2, it holds that

$$r_{i_2}^{0,0} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0} - \mathbf{d}\right)_{i_2} > \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}\right)_{i_2} = r_{i_2}^{*,0}, \tag{4.55}$$

since $r_{i_1}^{0,1} > r_{i_1}^{*,1}$ and $M_{i_2,i_1}^1 \neq 0$. The further argumentation that $r_{i_1}^{k,1} > r_{i_1}^{*,1}$ and $r_{i_2}^{k,0} > r_{i_2}^{*,0}$ for $k \geq 1$ is identical to the one in the proof of Proposition 4.4, as well as for the increasing version of the algorithm. □

An example for a system that fulfills the assumptions of Proposition 4.12 is again the system of Example 4.5. Though the stopping criteria will be reached after some iteration steps in the Elsinger Algorithm (cf. Example 4.21), the outcoming vector is not exactly equal to $\mathbf{R}^*$.

4.1.3 A Hybrid Algorithm

To motivate the approach of the next algorithm, we have to compare the functioning of the Elsinger Algorithm and the Picard Algorithm. The major difference between both iterations emerges in the calculation of the equity components. Suppose that we are in iteration step $k \geq 0$ and want to calculate the next iteration of the equity components. We ignore for an instant that both algorithms deliver different iterates and assume that the k-th iterate is given by $\mathbf{R}^k = (\mathbf{r}^{k,1}, \mathbf{r}^{k,0})$. In the Elsinger Algorithm, $\mathbf{r}^{k+1,1}$ is calculated first and then $\mathbf{r}^{k+1,0}$ as the fixed point of $\Phi^0(\cdot; \mathbf{r}^{k+1,1})$ so that it holds that $\mathbf{r}^{k+1,0} = \Phi^0(\mathbf{r}^{k+1,0}; \mathbf{r}^{k+1,1})$. The iterate of the Picard Algorithm, on the other side, can be written as $\mathbf{r}^{k+1,0} = \Phi^0(\mathbf{r}^{k,0}; \mathbf{r}^{k,1})$ from which it becomes clear that the Picard Iteration neither uses the "updated" debt vector $\mathbf{r}^{k+1,1}$, nor does it solve a separate fixed point mapping to obtain $\mathbf{r}^{k+1,0}$.

The determination of the debt components $\mathbf{r}^{k+1,1}$, however, is comparable in both algorithms. Again starting with $\mathbf{R}^k$ we have that $\mathbf{r}^{k+1,1} = \Phi^1(\mathbf{r}^{k,1}; \mathbf{r}^{k,0})$ for both procedures. An obvious extension of the Elsinger Algorithm would be to utilize the principle used for the equity components for the debt components as well. In the article of Eisenberg and Noe (2001), this concept is used for systems with no cross-ownership of equity, i.e. where $\mathbf{M}^0 = \mathbf{0}_{n\times n}$. In this subsection we will generalize the results of this work and it will turn out that combining both ideas, the one of Elsinger (2009) and the one of Eisenberg and Noe (2001), will help to minimize the number needed iteration steps of the global algorithm to find $\mathbf{R}^*$.

To explain this idea in more detail, say that for a debt payment vector $\bar{\mathbf{r}} \in [\mathbf{r}_{\text{small}}^1, \mathbf{r}_{\text{great}}^1]$ we have a corresponding equity vector $\bar{\mathbf{r}}^0 := \mathbf{r}^0(\bar{\mathbf{r}})$, that is, a fixed point of the mapping $\Phi^0(\cdot; \bar{\mathbf{r}})$ in (4.27). In the Elsinger Algorithm, the next debt iterate emerges as $\Phi^1(\bar{\mathbf{r}}; \bar{\mathbf{r}}^0)$. Instead of using this iterate, our aim is now to find the fixed point of $\Phi^1(\cdot;\bar{\mathbf{r}}^0)$ as the new debt iterate. This can be done using the following Algorithm.

Algorithm 4. *Suppose* $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$.

1. *For* $k = 0$, *set* $\mathbf{r}^{0,1} = \bar{\mathbf{r}}$ *and determine* $\mathcal{D}(\mathbf{r}^{0,1}, \bar{\mathbf{r}}^0)$ *and* $\boldsymbol{\Lambda}(\mathbf{r}^{0,1}, \bar{\mathbf{r}}^0)$.
2. *For* $k \geq 1$, *solve* $\Theta_{\mathbf{r}^{k-1,1},\bar{\mathbf{r}}^0}(\mathbf{r}) = \mathbf{r}$ *where*

$$\begin{aligned}\Theta_{\mathbf{r}^{k-1,1},\bar{\mathbf{r}}^0}(\mathbf{r}) = \boldsymbol{\Lambda}(\mathbf{r}^{k-1,1}, \bar{\mathbf{r}}^0)\Big(\mathbf{a} + \mathbf{M}^1\Big(\boldsymbol{\Lambda}(\mathbf{r}^{k-1,1}, \bar{\mathbf{r}}^0)\mathbf{r} + \Big(\mathbf{I}_n - \boldsymbol{\Lambda}(\mathbf{r}^{k-1,1}, \bar{\mathbf{r}}^0)\Big)\mathbf{d}\Big) + \mathbf{M}^0\bar{\mathbf{r}}^0\Big) \\ + \Big(\mathbf{I}_n - \boldsymbol{\Lambda}(\mathbf{r}^{k-1,1}, \bar{\mathbf{r}}^0)\Big)\mathbf{d}\end{aligned} \tag{4.56}$$

3. *Denote the solution by* $\mathbf{r}^{k,1}$, *i.e.* $\Theta_{\mathbf{r}^{k-1,1},\bar{\mathbf{r}}^0}(\mathbf{r}^{k,1}) = \mathbf{r}^{k,1}$ *and determine* $\mathcal{D}(\mathbf{r}^{k,1}, \bar{\mathbf{r}}^0)$ *and* $\boldsymbol{\Lambda}(\mathbf{r}^{k,1}, \bar{\mathbf{r}}^0)$.
4. *If* $\mathcal{D}(\mathbf{r}^{k-1,1}, \bar{\mathbf{r}}^0) = \mathcal{D}(\mathbf{r}^{k,1}, \bar{\mathbf{r}}^0)$, *stop the algorithm. Else, set* $k = k+1$ *and proceed with Step 2.*

The algorithm is identical to the one given in Eisenberg and Noe (2001) with the modification that some additional fixed payments due to equity cross-ownership are included. It solves (4.1) for a fixed amount of equity payments $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$, i.e. is the fixed point of the mapping $\Phi^1(\cdot, \bar{\mathbf{r}}^0)$, as we will show in the next proposition. To see the difference between the calculation of the debt component in the Elsinger Algorithm, assume that an arbitrary debt payment vector $\mathbf{r} \in [\mathbf{0}_n, \mathbf{d}]$ is given and that the corresponding equity payment vector $\mathbf{r}^0(\mathbf{r})$ is given too. The fixed point of the mapping $\Phi^1(\cdot; \mathbf{r}^0(\mathbf{r}))$ can on the one hand be obtained using Algorithm 4 above, but on the other hand, we could also use a Picard Iteration (see Algorithm 5), since it holds that

$$\mathbf{0}_n \leq \Phi^1(\mathbf{r}; \mathbf{r}^0(\mathbf{r})) = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r})\} \leq \mathbf{d}. \tag{4.57}$$

Starting with the vector $\mathbf{r} = \mathbf{r}^{0,1}$, the fixed point is given by

$$\lim_{l\to\infty} (\Phi^1)^l(\mathbf{r}; \mathbf{r}^0(\mathbf{r})). \tag{4.58}$$

In the Elsinger Algorithm, however, the next iterate for the debt component is defined as $\Phi^1(\mathbf{r}; \mathbf{r}^0(\mathbf{r}))$ which is therefore the first iterate of the Picard Iteration in (4.58). Hence, one can say that using in the Elsinger Algorithm, a simple mapping is applied to obtain the next debt iterate, whereas a fixed point problem is solved in Algorithm 4 to get the next debt iterate.

Proposition 4.13. *Let* $\bar{\mathbf{r}}^1 \in [\mathbf{r}^1_{\text{small}}, \mathbf{r}^1_{\text{great}}]$ *be a debt payment vector and* $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$ *a vector of equity payments such that*

$$\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \leq \bar{\mathbf{r}}^1. \tag{4.59}$$

(i) Algorithm 4 generates a well-defined decreasing sequence of vectors $\mathbf{r}^{k,1}$.

(ii) Let $1 \leq l \leq n$ *such that*

$$l := \min\{j \in \{0, 1, \ldots, n-1\} : \mathcal{D}(\mathbf{r}^{j,1}, \bar{\mathbf{r}}^0) = \mathcal{D}(\mathbf{r}^{j+1,1}, \bar{\mathbf{r}}^0)\}. \tag{4.60}$$

Then $\mathbf{r}^{l+1,1}$ *is the fixed point of the mapping* $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$ *defined in* (4.41).

(iii) Let $d_0 = |\mathcal{D}(\bar{\mathbf{r}}^1, \bar{\mathbf{r}}^0)|$ *be the number of firms in default under* $\bar{\mathbf{r}}^1$ *and* $\bar{\mathbf{r}}^0$. *If* $d_0 \in \{1, \ldots, n\}$, *the fixed point* $\mathbf{r}^{l+1,1}$ *is reached after no more than* $n - d_0$ *iteration steps. If* $d_0 = 0$, *no iteration is necessary, since* $\mathbf{r}^{l+1,1} = \mathbf{d}$

Proof. Since the equity vector $\bar{\mathbf{r}}^0$ is considered as fixed we can modify the financial system $\mathcal{F}$ by setting $\tilde{\mathbf{a}} = \mathbf{a} + \mathbf{M}^0\bar{\mathbf{r}}^0$ and $\widetilde{\mathbf{M}}^0 = \mathbf{0}_{n\times n}$. The new system $\tilde{\mathcal{F}} = (\tilde{\mathbf{a}}, \mathbf{M}^1, \widetilde{\mathbf{M}}^0, \mathbf{d})$ is then a system without cross-ownership of equity. Such systems are considered in Eisenberg and Noe (2001), see also Table 3.1.

(i) The proof that the sequence $\mathbf{r}^{k,1}$ decreases is now equivalent to the proof given in Eisenberg and Noe (2001). A needed assumption in the proof therein is that $\bar{\mathbf{r}}^1$ is a so-called super-solution which is given because of $\Phi^1(\bar{\mathbf{r}}^1;\bar{\mathbf{r}}^0) \leq \bar{\mathbf{r}}^1 \leq \mathbf{d}$. What we have to show to complete this part is that the fixed point of the mapping in (4.56) exists and is unique, since their definition of a financial system, differs slightly from ours. Denote by $\mathbf{\Lambda} := \mathbf{\Lambda}(\mathbf{r}^{k,1}, \bar{\mathbf{r}}^0)$ the diagonal matrix in the $(k+1)$-th iteration step. The next iterate $\mathbf{r}^{k+1,1}$, is according to (4.56), given by

$$\begin{aligned}\mathbf{r}^{k+1,1} &= \mathbf{\Lambda}\left(\tilde{\mathbf{a}} + \mathbf{M}^1(\mathbf{\Lambda}\mathbf{r}^{k+1,1} + (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d})\right) + (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d} \\ &= \mathbf{\Lambda}\mathbf{M}^1\mathbf{\Lambda}\mathbf{r}^{k+1,1} + \mathbf{\Lambda}\left(\tilde{\mathbf{a}} + \mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d}\right) + (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d}\end{aligned} \tag{4.61}$$

and rearranging yields to

$$\mathbf{r}^{k+1,1} = \left(\mathbf{I}_n - \mathbf{\Lambda}\mathbf{M}^1\mathbf{\Lambda}\right)^{-1}\left(\mathbf{\Lambda}\left(\tilde{\mathbf{a}} + \mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d}\right) + (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d}\right). \tag{4.62}$$

Note that $\mathbf{M}^1$ has the Elsinger Property and, hence, so does $\mathbf{\Lambda}\mathbf{M}^1\mathbf{\Lambda}$ which means that the inverse of $\mathbf{I}_n - \mathbf{\Lambda}\mathbf{M}^1\mathbf{\Lambda}$ exists. This proves the uniqueness of $\mathbf{r}^{k+1,1}$.

(ii) The argumentation that the sequence converges and becomes constant in the end is analogous to part (ii) of the proof of Proposition 4.7. Since $\mathbf{r}^{k,1}$ is decreasing, we have that $\mathcal{D}(\mathbf{r}^{k,1}, \bar{\mathbf{r}}^0) \subseteq \mathcal{D}(\mathbf{r}^{k+1,1}, \bar{\mathbf{r}}^0)$ that means the number of firms in default increases. If $\mathcal{D}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0) = \mathcal{D}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0)$, then we also have that $\mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0) = \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0)$, from which follows that the mappings $\Theta_{\mathbf{r}^{l,1},\bar{\mathbf{r}}^0}$ and $\Theta_{\mathbf{r}^{l+1,1},\bar{\mathbf{r}}^0}$ have the same fixed point. It must hold then that all consequent iterates are equal.

To show that $\mathbf{r}^{l+1,1}$ is the fixed point of $\Phi^1(\cdot, \bar{\mathbf{r}}^0)$, first check that by definition,

$$\mathbf{r}^{l+1,1} = \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0)\mathbf{r}^{l+1,1} + (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0))\mathbf{d}. \tag{4.63}$$

It then holds that

$$\begin{aligned}\Phi^1(\mathbf{r}^{l+1,1};\bar{\mathbf{r}}^0) &= \min\{\mathbf{d}, \tilde{\mathbf{a}} + \mathbf{M}^1\mathbf{r}^{l+1,1}\} \\ &= (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0))\mathbf{d} + \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0)(\tilde{\mathbf{a}} + \mathbf{M}^1\mathbf{r}^{l+1,1}) \\ &= (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0))\mathbf{d} \\ &\quad + \mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0)\left(\tilde{\mathbf{a}} + \mathbf{M}^1\left(\mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0)\mathbf{r}^{l+1,1} + (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l+1,1}, \bar{\mathbf{r}}^0))\mathbf{d}\right)\right) \\ &= (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0))\mathbf{d} \\ &\quad + \mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0)\left(\tilde{\mathbf{a}} + \mathbf{M}^1\left(\mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0)\mathbf{r}^{l+1,1} + (\mathbf{I}_n - \mathbf{\Lambda}(\mathbf{r}^{l,1}, \bar{\mathbf{r}}^0))\mathbf{d}\right)\right) \\ &= \mathbf{r}^{l+1,1},\end{aligned} \tag{4.64}$$

where the last equality follows from (4.56).

(iii) This part is similar to part (iii) of the proof of Proposition 4.7 with the reverse argumentation. The d_0 firms in default under the starting vector will stay in default since the series decreases. To achieve a maximum theoretical length of the algorithm, exactly one additional default step has to occur in every new iteration step. This results in no more than $n - d_0$ possible iteration steps. If $d_0 = 0$, it follows because of $\mathbf{\Lambda}(\bar{\mathbf{r}}^1, \bar{\mathbf{r}}^0) = \mathbf{0}_{n\times n}$ that $\Theta_{\mathbf{r}^{0,1},\bar{\mathbf{r}}^0}(\mathbf{r}) \equiv \mathbf{d}$ and therefore that $\mathbf{d}$ is the fixed point of $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$.

□

Example 4.14. We reconsider the system in Example 4.8 where we found that $\mathbf{r}^0(\mathbf{d}) = (0.3471, 0.0661, 0, 0.8161, 0.9590)^t$ and $\mathcal{D}(\mathbf{d}, \mathbf{r}^0(\mathbf{d})) = \{3\}$. Since the third firm is in default even assuming maximums payments of equity and debt, the corresponding debt payment has to recalculated using Algorithm 4. First, note that for $\bar{\mathbf{r}}^0 = \mathbf{r}^0(\mathbf{d})$ and $\bar{\mathbf{r}}^1 = \mathbf{d}$ it holds that $\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \leq \mathbf{d} = \bar{\mathbf{r}}^1$ and that the Assumption in (4.59) is fulfilled. We get for the first iterate in Algorithm 4 that

$$\mathbf{r}^{1,1} = (1.62, 1.82, 1.5844, 1.14, 0.77)^t \quad \text{and} \quad \mathcal{D}(\mathbf{r}^{1,1}, \bar{\mathbf{r}}^0) = \{2, 3\}.$$

The default of firm 3 has affected firm 2 that is now also not able to fully satisfy its obligees because of the reduced debt payments. Consequently, the algorithm continues and delivers

$$\mathbf{r}^{2,1} = (1.62, 1.7590, 1.5615, 1.14, 0.77)^t \quad \text{and} \quad \mathcal{D}(\mathbf{r}^{2,1}, \bar{\mathbf{r}}^0) = \{2, 3\}.$$

as the next iterate and default set. Since now, no new firm gets into trouble, the algorithm stops and we have found $\mathbf{r}^{2,1}$ as the fixed point of $\Phi^1(\cdot, \bar{\mathbf{r}}^0)$.

The validity of the inequality in (4.59) is crucial for the monotonicity of the iterates $\mathbf{r}^{k,1}$ produced by Algorithm 4. However, there are situations in which a debt payment vector $\bar{\mathbf{r}} \in [\mathbf{r}^1_{\text{small}}, \mathbf{r}^1_{\text{great}}]$ is given together with an arbitrary vector $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$ and where (4.59) does not hold. Think of an algorithm to find $\mathbf{R}^*$ that starts with $\mathbf{R}_{\text{small}}$. In this case the first debt iterate is $\mathbf{r}^1_{\text{small}} = \min\{\mathbf{d}, \mathbf{a}\}$ and the corresponding equity iterate is $\mathbf{r}^0(\mathbf{r}^1_{\text{small}})$. Applying Φ^1 to these vectors yields to

$$\Phi^1(\mathbf{r}^1_{\text{small}}; \mathbf{r}^0(\mathbf{r}^1_{\text{small}})) = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^1_{\text{small}})\} \geq \min\{\mathbf{d}, \mathbf{a}\} = \mathbf{r}^1_{\text{small}} \tag{4.65}$$

and to a violation of (4.59). Finding the next debt iterate as the fixed point of $\Phi^1(\cdot; \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ and applying Algorithm 4 to do so, can under certain circumstances lead to a non-monotone series, as the following example demonstrates.

Example 4.15. Consider the financial system defined in Example 4.8 again. If the starting vector for the debt iterates is given by $\mathbf{r}^1_{\text{small}}$, we need to find $\mathbf{r}^0(\mathbf{r}^1_{\text{small}})$ that can be determined via Algorithm 2. It holds that

$$(\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}})) = (0.39, 0.38, 0.60, 1.14, 0.46, 0, 0, 0, 0.3746, 0.2412)^t \tag{4.66}$$

as one can check easily as well as $\Phi^1(\mathbf{r}^1_{\text{small}}; \mathbf{r}^0(\mathbf{r}^1_{\text{small}})) \geq \mathbf{r}^1_{\text{small}}$, hence (4.59) does not hold. Using Algorithm 4 even so to obtain the next debt iterate as the fixed point of $\Phi^1(\cdot; \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ would lead to the following series of iterates (the notation of the $\mathbf{r}^{k,1}$ is the notation from Algorithm 4):

$$\mathbf{r}^{0,1} = \min\{\mathbf{d}, \mathbf{a}\} = \begin{pmatrix} 0.39 \\ 0.38 \\ 0.60 \\ 1.14 \\ 0.46 \end{pmatrix}, \ \mathbf{r}^{1,1} = \begin{pmatrix} 1.697 \\ 1.6754 \\ 1.4516 \\ 1.14 \\ 0.77 \end{pmatrix}, \ \mathbf{r}^{2,1} = \begin{pmatrix} 1.62 \\ 1.6542 \\ 1.4436 \\ 1.14 \\ 0.77 \end{pmatrix}, \tag{4.67}$$

where the last iterate $\mathbf{r}^{2,1}$ is the fixed point of $\Phi^1(\cdot, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$. However, the series is not monotone since $\mathbf{r}^{0,1} \leq \mathbf{r}^{1,1} \geq \mathbf{r}^{2,1}$.

This described property makes it difficult to prove the convergence of such a series in general. Nevertheless, given a debt vector $\bar{\mathbf{r}}^1$ and $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$, we can still calculate the fixed point by avoiding Algorithm 4 and use a Picard-type algorithm instead.

Algorithm 5 (Picard Iteration for the Debt Component)**.** *Suppose that* $\bar{\mathbf{r}}^1, \bar{\mathbf{r}}^0 \geq \mathbf{0}_n$ *and* $\varepsilon > 0$.

1. *For* $k = 0$, *set* $\mathbf{r}^{0,1} = \bar{\mathbf{r}}^1$.
2. *For* $k \geq 1$, *determine* $\mathbf{r}^{k,1} = \Phi^1(\mathbf{r}^{k-1,1}; \bar{\mathbf{r}}^0)$.
3. *If* $\|\mathbf{r}^{k-1,1} - \mathbf{r}^{k,1}\| < \varepsilon$, *stop the algorithm. Else, set* $k = k+1$ *and proceed with Step 2.*

Proposition 4.16. *Algorithm 5 delivers a series of decreasing vectors* $\mathbf{r}^{k,1}$ *if* $\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \leq \bar{\mathbf{r}}^1$ *and a series of increasing vectors if* $\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \geq \bar{\mathbf{r}}^1$. *Both series converge to the unique fixed point of* $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$.

Proof. First, note that for fixed equity payments $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$ the mapping Φ^1 has a unique fixed point. To show this, define $\tilde{\mathbf{a}} = \mathbf{a} + \mathbf{M}^0\bar{\mathbf{r}}^0$ and $\widetilde{\mathbf{M}}^0 = \mathbf{0}_{n\times n}$ which together with $\mathbf{d}$ and $\mathbf{M}^1$ represents a new financial system $\widetilde{\mathcal{F}} = (\tilde{\mathbf{a}}, \mathbf{M}^1, \widetilde{\mathbf{M}}^0, \mathbf{d})$ that is a system with no equity cross-holdings. Since both ownership matrices have the Elsinger Property, we can apply Theorem 2.7 from which follows that $\widetilde{\mathcal{F}}$ has a unique solution $\widetilde{\mathbf{R}} = (\tilde{\mathbf{r}}^{*,1}, \tilde{\mathbf{r}}^{*,0})$. From the first liquidation value equation in 4.1, we immediately see that then $\tilde{\mathbf{r}}^{*,1} = \Phi^1(\tilde{\mathbf{r}}^{*,1}, \bar{\mathbf{r}}^0)$ is the searched fixed point of Φ^1.

To show the convergence of the algorithm, note that if $\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \geq \bar{\mathbf{r}}^1 = \mathbf{r}^{0,1}$ for the first iterate, it holds that $\mathbf{r}^{1,1} = \Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \geq \mathbf{r}^{0,1}$. Via induction, it follows that $\mathbf{r}^{k+1,1} \geq \mathbf{r}^{k,1}$ for all $k \geq 1$. Because the monotone series $\mathbf{r}^{k,1}$ is bounded from above by $\mathbf{d}$ and from below by $\mathbf{0}_n$, it must converge to the fixed point. The argumentation is similar if $\Phi^1(\bar{\mathbf{r}}^1; \bar{\mathbf{r}}^0) \leq \bar{\mathbf{r}}^1$. □

The Algorithms 4 and 5 both enable us to calculate a new debt iterate given a fixed equity vector. Together with Algorithm 2 for the equity component, we can now combine both procedures in a common algorithm that searches for the fixed point $\mathbf{R}^*$.

Algorithm 6 (Hybrid Algorithm)**.** *Set* $\varepsilon > 0$.

1. *For* $k = 0$, *choose* $\mathbf{r}^{0,1} \in \{\mathbf{r}^1_{\text{great}}, \mathbf{r}^1_{\text{small}}\}$ *and determine* $\mathbf{r}^0(\mathbf{r}^{0,1})$ *using Algorithm 2.*
2. *For* $k \geq 1$:
   - 2.1 *Determine* $\mathbf{r}^{k,1}$ *using Algorithm 4 if* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$ *or using Algorithm 5 if* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ *in both cases with* $\bar{\mathbf{r}}^0 = \mathbf{r}^0(\mathbf{r}^{k-1,1})$.
   - 2.2 *Determine* $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,1})$ *using Algorithm 2.*
3. *If* $\left\| \begin{pmatrix} \mathbf{r}^{k-1,1} \\ \mathbf{r}^{k-1,0} \end{pmatrix} - \begin{pmatrix} \mathbf{r}^{k,1} \\ \mathbf{r}^{k,0} \end{pmatrix} \right\| < \varepsilon$, *stop the algorithm. Else, set* $k = k+1$ *and proceed with Step 2.*

For given $\mathbf{r}^{k,1}, k \geq 0$, the Hybrid Algorithm determines $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,1})$ as the correct equity value that solves (4.2) and for given $\mathbf{r}^0(\mathbf{r}^{k,1}), k \geq 0$, it determines the correct debt value $\mathbf{r}^{k+1,1}$ that solves (4.1). As such, conditional on the values determined in the previous step, the algorithm calculates an exact solution of either (4.2) or (4.1) in the next iteration step. In Figure 4.2, a schematic illustration of the Hybrid Algorithm is given, which relates the one in Figure 4.1 of the Elsinger Algorithm.

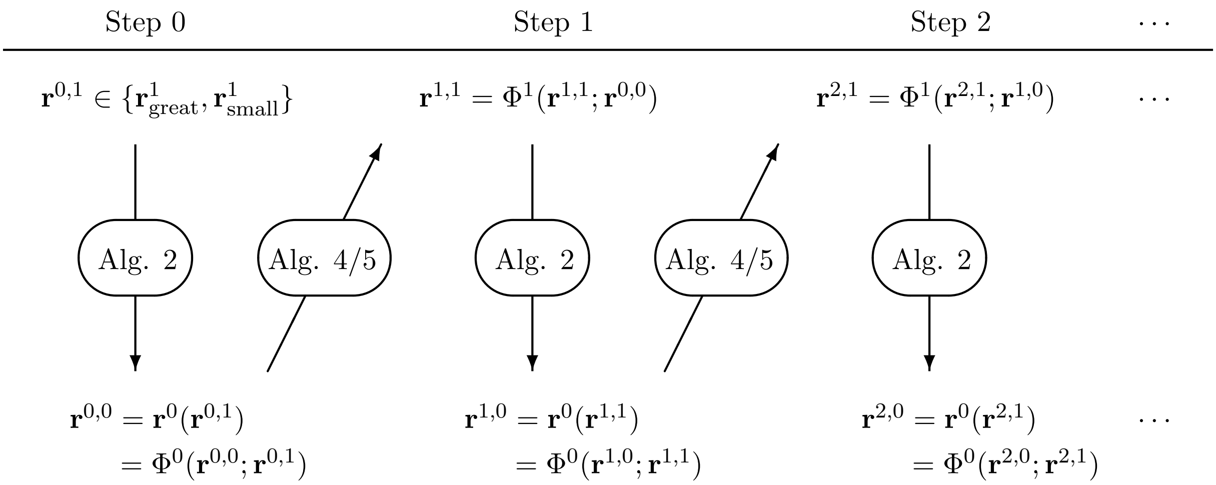


Figure 4.2: Schematic functioning of the Hybrid Algorithm. An oval-shaped symbol means that the corresponding listed algorithm is applied. In case of $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$, Algorithm 4 is used to determine the new debt payment vector which results in a decreasing series of iterates. For $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}} = \min\{\mathbf{d}, \mathbf{a}\}$, Algorithm 5 has to be taken instead. The iterates then form an increasing series. In both cases, the series converges to $\mathbf{R}^*$.

Proposition 4.17. *The Hybrid Algorithm delivers a series of decreasing vectors if* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$ *that converges to the fixed point* $\mathbf{R}^*$. *In case of* $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ *the series is increasing with the same limit.*

Proof. First, suppose that $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$. We will first show by induction that the series decreases. For the induction start note that

$$\mathbf{r}^{1,1} = \Phi^1(\mathbf{r}^{1,1}; \mathbf{r}^0(\mathbf{r}^{0,1})) = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{0,1})\} \le \mathbf{d} = \mathbf{r}^{0,1}. \tag{4.68}$$

According to Lemma 2.10, the equity vector $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$ which yields to $\mathbf{r}^0(\mathbf{r}^{1,1}) \le \mathbf{r}^0(\mathbf{r}^{0,1})$. For the induction step, assume that for $k > 1$ it holds that $\mathbf{r}^{k-1,1} \ge \mathbf{r}^{k,1}$ and consequently $\mathbf{r}^0(\mathbf{r}^{k-1,1}) \ge \mathbf{r}^0(\mathbf{r}^{k,1})$. Since $\mathbf{r}^{k,1} = \Phi^1(\mathbf{r}^{k,1}; \mathbf{r}^0(\mathbf{r}^{k-1,1}))$ and because of

$$\begin{aligned}\Phi^1(\mathbf{r}^{k,1}; \mathbf{r}^0(\mathbf{r}^{k,1})) &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1})\} \\ &\le \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1,1})\} \\ &= \Phi^1(\mathbf{r}^{k,1}; \mathbf{r}^0(\mathbf{r}^{k-1,1})) \\ &= \mathbf{r}^{k,1}\end{aligned} \tag{4.69}$$

the assumption (4.59) is fulfilled. The next iterate $\mathbf{r}^{k+1,1}$ emerges from a decreasing sequence produced by applying Algorithm 4 beginning with $\bar{\mathbf{r}}^1 = \mathbf{r}^{k,1}$. Hence $\mathbf{r}^{k+1,1} \le \mathbf{r}^{k,1}$ and thus $\mathbf{r}^0(\mathbf{r}^{k+1,1}) \le \mathbf{r}^0(\mathbf{r}^{k,1})$. Next step is to show that the series converges to $\mathbf{R}^*$. We have that the two sequences $(\mathbf{r}^{k+1,1}, \mathbf{r}^0(\mathbf{r}^{k,1}))$ and $(\mathbf{r}^{k,1}, \mathbf{r}^0(\mathbf{r}^{k,1}))$ are both decreasing in $(\mathbb{R}_0^+)^{2n}$ and therefore converge to the same limit $(\mathbf{r}^{*,1}, \mathbf{r}^{*,0}) \in (\mathbb{R}_0^+)^{2n}$. Because of the continuity of Φ^1 and Φ^0 it must hold that $\Phi^1(\mathbf{r}^{*,1}; \mathbf{r}^{*,0}) = \mathbf{r}^{*,1}$ and $\Phi^0(\mathbf{r}^{*,0}; \mathbf{r}^{*,1}) = \mathbf{r}^{*,0}$. Thus, $(\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ solves (4.2) and (4.1). The proof for $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$ is similar. □

In a similar manner than for Picard and the Elsinger Algorithm, we make use of the expressions *Decreasing* or *Increasing Hybrid Algorithm* if $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{great}}$ or if $\mathbf{r}^{0,1} = \mathbf{r}^1_{\text{small}}$, respectively.

Remark 4.18. In case of $\mathbf{a} \in \mathbb{R}^n$, the application of Algorithm 4 to obtain the next debt iterate becomes problematic which is also mentioned in Elsinger (2009, Example 2). Assume that for given $\bar{\mathbf{r}}^1 \leq \mathbf{d}$ and $\bar{\mathbf{r}}^0 \geq \mathbf{0}_n$, the fixed point of $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$ has to be calculated. If $\mathbf{a} \in \mathbb{R}^n$, the iterates of the algorithm can be negative, since it is not ensured that the fixed point of the mapping in (4.56) is positive. Suppose that $\tilde{\mathbf{r}}^{k,1}$ is the k-th iterate of Algorithm 4, i.e. it holds that $\Theta_{\mathbf{r}^{k-1},\bar{\mathbf{r}}^0}(\tilde{\mathbf{r}}^{k,1}) = \tilde{\mathbf{r}}^{k,1}$. An intuitive modification of the algorithm would be to define the positive part of $\tilde{\mathbf{r}}^{k,1}$ as the next iterate, viz. $\mathbf{r}^{k,1} = (\tilde{\mathbf{r}}^{k,1})^+$. Doing so, the series of iterates would obviously be well-defined and decreasing as well as it would converge to some “final” iterate $\mathbf{r}^{l,1}$. However, this iterate must not necessarily be the fixed point of $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$, i.e. in general it does not hold $\Phi^1(\mathbf{r}^{l,1}; \bar{\mathbf{r}}^0) = \mathbf{r}^{l,1}$ anymore. For a counterexample, take a look at the system with $n = 5$,

$$\mathbf{M}^0 = \begin{cases} 0.0125 & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases} \tag{4.70}$$

and

$$\mathbf{a} = \begin{pmatrix} 0.31 \\ -0.33 \\ 2.98 \\ -4.55 \\ -2.10 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 1.38 \\ 1.02 \\ 0.48 \\ 0.29 \\ 0 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 1/8 & 1/6 & 1/4 & 1/4 \\ 1/8 & 0 & 0 & 1/4 & 0 \\ 1/8 & 1/8 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/6 & 0 & 1/4 \\ 1/8 & 1/8 & 1/6 & 0 & 0 \end{pmatrix}. \tag{4.71}$$

For $\bar{\mathbf{r}}^1 = \mathbf{d}$ it holds that $\bar{\mathbf{r}}^0 = \mathbf{r}^0(\bar{\mathbf{r}}^1) = (0, 0, 2.8, 0, 0)^t$. We want to find the fixed point of $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$ as the next debt iterate. This fixed point is given by $\mathbf{r} = (0.425, 0, 0.48, 0, 0)^t$, i.e. the third firm is considered to be solvent, the first firm is in default with a positive recovery value and the remaining firms are in default with no recovery payments at all for the firms within the system. Using the modified version of the algorithm above would lead to the “final” iterate $\mathbf{r}^l = (0, 0, 0.48, 0, 0)^t$, which is not the fixed point of $\Phi^1(\cdot; \bar{\mathbf{r}}^0)$.

A consequence from the insights of Remark 4.18 is that for financial systems with $\mathbf{a} \in \mathbb{R}^n$, we have to use the Picard-type Algorithm 5 for the calculation of the next debt iterate in Algorithm 6. The reason is that Proposition 4.16 holds also for negative values of the exogenous assets. Note that we have to use the modified version $\Phi^1(\mathbf{r}; \mathbf{r}^0) = \min\{\mathbf{d}, (\mathbf{a} + \mathbf{M}^1\mathbf{r} + \mathbf{M}^0\mathbf{r}^0)^+\}$ of the mapping. This will become important in Chapter 5 when more than one seniority level is treated.

In Proposition 4.11, we have shown that when using the Elsinger Algorithm, the iterates will always be nearer to the solution $\mathbf{R}^*$ than the corresponding iterates of the Picard Algorithm. This leads to the conclusion that the iteration number is minimized for the Elsinger Algorithm. The next Proposition shows the same when comparing the Elsinger and the Hybrid Algorithm and it will become clear that the Hybrid Algorithm will need in no situation more iteration steps to reach $\mathbf{R}^*$ than the Elsinger Algorithm.

Proposition 4.19. *As in Proposition 4.11, we denote the iterates of the two algorithms with subscripts, where E stands for the Elsinger and H for the Hybrid Algorithm.*

(i) For any iterate $k \geq 1$ it holds that $\mathbf{R}^k_{\text{E}} \geq \mathbf{R}^k_{\text{H}}$ if $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$ and $\mathbf{R}^k_{\text{E}} \leq \mathbf{R}^k_{\text{H}}$ when $(\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ is the starting vector of both algorithms.

(ii) Let $\mathbf{R}^k = (\mathbf{r}^{k,1}, \mathbf{r}^{k,0})$, $k \geq 0$, be an iterate either of the Elsinger Algorithm or of the Hybrid Algorithm that started with $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$. Then $\mathbf{R}^{k+1}_{\text{E}}(\mathbf{r}^{k,1}) \geq \mathbf{R}^{k+1}_{\text{H}}(\mathbf{r}^{k,1})$ for

the next iterates which were calculated with either the Elsinger or the Hybrid Algorithm starting from $\mathbf{R}^k$. *If* $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$, *it holds that* $\mathbf{R}^{k+1}_{\text{E}}(\mathbf{r}^{k,1}) \leq \mathbf{R}^{k+1}_{\text{H}}(\mathbf{r}^{k,1})$.

Proof. (i) Let $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$. From Proposition 4.17 we know that $\mathbf{r}^{1,1}_{\text{H}} \leq \mathbf{r}^{0,1}_{\text{H}} = \mathbf{d}$ which yields to

$$\mathbf{r}^{1,1}_{\text{E}} = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\} \geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\} = \mathbf{r}^{1,1}_{\text{H}}. \tag{4.72}$$

Further, since $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$ (cf. Lemma 2.10), $\mathbf{r}^{1,0}_{\text{E}} = \mathbf{r}^0(\mathbf{r}^{1,1}_{\text{E}}) \geq \mathbf{r}^0(\mathbf{r}^{1,1}_{\text{H}}) = \mathbf{r}^{1,0}_{\text{H}}$, which completes the induction start. For the induction step, assume that it holds for $k > 1$ that $\mathbf{R}^{k-1}_{\text{E}} \geq \mathbf{R}^{k-1}_{\text{H}}$. Because of Proposition 4.17, $\mathbf{r}^{k-1,1}_{\text{H}} \geq \mathbf{r}^{k,1}_{\text{H}}$ and thus

$$\begin{aligned} \mathbf{r}^{k,1}_{\text{E}} &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1}_{\text{E}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{E}})\} \\ &\geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{H}})\} \\ &\geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{H}})\} \\ &= \mathbf{r}^{k,1}_{\text{H}}, \end{aligned} \tag{4.73}$$

where we again used the fact that $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$ from which follows that $\mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{E}}) \geq \mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{H}})$ and also $\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{E}}) \geq \mathbf{r}^0(\mathbf{r}^{k,1}_{\text{H}})$. The proof when $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ is completely analogous.

(ii) Let $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$ and $\mathbf{R}^k = \mathbf{R}^k_{\text{E}}$. Note that because of $\mathbf{r}^{k,1}_{\text{E}} \leq \mathbf{r}^{k-1,1}_{\text{E}}$ it holds that

$$\begin{aligned} \mathbf{r}^{k+1,1}_{\text{E}} &= \Phi^1(\mathbf{r}^{k,1}_{\text{E}}; \mathbf{r}^0(\mathbf{r}^{k,1}_{\text{E}})) \\ &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1}_{\text{E}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{E}})\} \\ &\leq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1,1}_{\text{E}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1,1}_{\text{E}})\} \\ &= \mathbf{r}^{k,1}_{\text{E}}. \end{aligned} \tag{4.74}$$

Therefore, the assumption in (4.59) is fulfilled which ensures that $\mathbf{r}^{k+1,1}_{\text{H}} \leq \mathbf{r}^{k,1}_{\text{E}}$. For the next iterate it follows that

$$\begin{aligned} \mathbf{r}^{k+1,1}_{\text{E}} &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1}_{\text{E}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{E}})\} \\ &\geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k+1,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{E}})\} \\ &= \mathbf{r}^{k+1,1}_{\text{H}}, \end{aligned} \tag{4.75}$$

which in turn implies $\mathbf{r}^{k+1,0}_{\text{E}} \geq \mathbf{r}^{k+1,0}_{\text{H}}$. On the other hand, starting with $\mathbf{R}^k = \mathbf{R}^k_{\text{H}}$ yields because of $\mathbf{r}^{k+1,1}_{\text{H}} \leq \mathbf{r}^{k,1}_{\text{H}}$ to

$$\begin{aligned} \mathbf{r}^{k+1,1}_{\text{E}} &= \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{H}})\} \\ &\geq \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k+1,1}_{\text{H}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k,1}_{\text{H}})\} \\ &= \mathbf{r}^{k+1,1}_{\text{H}}. \end{aligned} \tag{4.76}$$

If follows from these results that $\mathbf{r}^{k+1,0}_{\text{E}} \geq \mathbf{r}^{k+1,0}_{\text{H}}$. A similar argumentation together with Proposition 4.16 delivers the proof in case of $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$.

□

Of course, within an iteration step of the Hybrid Algorithm, potentially many linear equation systems have to be solved since for the Decreasing Hybrid Algorithm, Algorithm 4 is applied to obtain the debt iterates which can result in higher computational costs. But if we ignore for a moment this circumstance, it follows from Proposition 4.19 that the convergence speed of the Hybrid Algorithm is higher than the one of the Elsinger Algorithm.

Another similarity is that also for the Hybrid Algorithm, it cannot be ensured that the fixed point is reached exactly. A possible structure of a financial system with this property is given in the next Proposition. The proof is very similar to the proof of Proposition 4.12.

Proposition 4.20. *Under the assumptions of Proposition 4.12, the Hybrid Algorithm will not exactly reach the fixed point* $\mathbf{R}^*$*, no matter which starting point is used for the iteration.*

We close this section with an example comparison of the three introduced algorithms.

Example 4.21. To demonstrate the different functionings of the algorithms described above, we calculate the solution $\mathbf{R}^*$ of the financial system of Example 4.5 using the Picard, the Elsinger and the Hybrid Algorithm. In all three cases, we use both the increasing and the decreasing version of the procedure. With a tolerance level of $\varepsilon = 10^{-6}$, we determined the first iterates that are shown in Table 4.1. The Increasing Picard Algorithm needs more iterations than its decreasing counterpart (21 vs. 18 steps) which is also the case for the Increasing Elsinger Algorithm that needs 17 steps for the increasing version compared to 14 iteration steps in the decreasing case. Using the Hybrid Algorithm, in both directions 7 iteration steps are required to reach the stopping criteria.

The fact that the number of iterations is smallest for the Hybrid Algorithm, does not necessarily mean that the Hybrid Algorithm is also the most efficient one. When for example using the Decreasing Hybrid Algorithm, in every iteration, a new debt iterate and, based on it, a new equity iterate have to be determined using the Algorithms 4 and 2. Running these algorithms means solving several linear equation systems which is an expensive task from a computational point of view. In the mentioned case of the Decreasing Hybrid Algorithm for instance, 17 such linear equation systems have to be solved (in Section 4.2, such a step will be defined as a *calculation step*). This phenomena also occurs when using the Elsinger Algorithm to find $\mathbf{R}^*$. For the purpose of quantifying these different computation methods in order to make them comparable, we will investigate the computational efficiency of the algorithms in the framework of a simulation study in Chapter 7.

Also note that values of the iterates in Table 4.1 are in line with the insights of the Propositions 4.19 and 4.11: The iterates of the Picard Algorithm are always larger (smaller) than the iterates of the Elsinger Algorithm in the decreasing (increasing) version. This relation also holds for the Elsinger and the Hybrid Algorithm.

Table 4.1: Iterates of the Algorithms 1, 3 and 6 for the financial system defined in Example 4.5. The header "Increasing" means that the lower bound was used as the starting point, i.e. $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ for the Picard and $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ for the Elsinger and the Hybrid Algorithm. Analogously, "Decreasing" stands for the decreasing versions of the algorithms, where the initial iterate is given by $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ or $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$.

Type		Increasing						Decreasing		
		$\mathbf{R}^0$	$\mathbf{R}^1$	$\mathbf{R}^2$	…	$\mathbf{R}^*$	…	$\mathbf{R}^2$	$\mathbf{R}^1$	$\mathbf{R}^0$
Picard		0.39	0.9750	1.3090		1.62		1.62	1.62	1.62
		0.38	1.0350	1.3471		1.7342		1.7669	1.82	1.82
	$\mathbf{r}^{k,1}$	0.6	0.8650	1.2103	…	1.5316	…	1.5861	1.5881	2.07
		1.14	1.14	1.14		1.14		1.14	1.14	1.14
		0.46	0.77	0.77		0.77		0.77	0.77	0.77
		0	0	0		0.1686		0.2277	0.3577	0.3577
		0	0	0		0		0	0.0749	0.1149
	$\mathbf{r}^{k,0}$	0	0	0	…	0	…	0	0	0.1131
		0.15	0.3625	0.5978		0.7965		0.8173	0.8268	0.8268
		0	0.2300	0.4531		0.8112		0.8400	0.9625	0.9625
Elsinger		0.39	1.1065	1.5755	…	1.62	…	1.62	1.62	1.62
		0.38	1.1670	1.5940	…	1.7342	…	1.7507	1.82	1.82
	$\mathbf{r}^{k,1}$	0.6	0.9760	1.4393	…	1.5316	…	1.5671	1.5844	2.07
		1.14	1.14	1.14	…	1.14	…	1.14	1.14	1.14
		0.46	0.77	0.77	…	0.77	…	0.77	0.77	0.77
		0	0	0.0902	…	0.1686	…	0.1842	0.2149	0.3471
		0	0	0	…	0	…	0	0	0.0661
	$\mathbf{r}^{k,0}$	0	0	0	…	0	…	0	0	0
		0.3746	0.6455	0.7795	…	0.7965	…	0.7978	0.7996	0.8161
		0.2412	0.5279	0.7722	…	0.8112	…	0.8209	0.8268	0.9590
Hybrid		0.39	1.62	1.62	…	1.62	…	1.62	1.62	1.62
		0.38	1.6542	1.7286	…	1.7342	…	1.7361	1.7590	1.82
	$\mathbf{r}^{k,1}$	0.6	1.4436	1.5254	…	1.5316	…	1.5337	1.5615	2.07
		1.14	1.14	1.14	…	1.14	…	1.14	1.14	1.14
		0.46	0.77	0.77	…	0.77	…	0.77	0.77	0.77
		0	0.1152	0.1648	…	0.1686	…	0.1699	0.1859	0.3471
		0	0	0	…	0	…	0	0	0.0661
	$\mathbf{r}^{k,0}$	0	0	0	…	0	…	0	0	0
		0.3746	0.7926	0.7962	…	0.7965	…	0.7966	0.7978	0.8161
		0.2412	0.7863	0.8094	…	0.8112	…	0.8117	0.8196	0.9590

4.2 Finite Algorithms

The Algorithms in the previous section all had the drawback that it could not be ensured that the solution $\mathbf{R}^*$ was reached exactly. As we have seen in the Propositions 4.4, 4.12 and 4.20, there are potentially infinitely many iteration steps needed to reach the fixed point using the corresponding algorithm. In this section we will present two ways in which such non-finite solution algorithms can be turned into procedures that reach the solution in finitely many steps. The common principle of these methods is to include the information which firms are in default under a current iterate $\mathbf{R}^k$. It turns out that this modification helps to overcome the disadvantage of potentially infinitely many iteration steps. To guarantee that the forthcoming procedures are well-defined, we have to drop the Elsinger Property and demand a stricter property for the ownership matrices (see also the Sections 3.2.3 and 3.3.2 for more details).

Assumption 4.22. *For debt and equity ownership matrix it holds that* $\|\mathbf{M}^1\| < 1$ *and* $\|\mathbf{M}^0\| < 1$.

For the remainder of this section we suppose that Assumption 4.22 holds. Note that Assumption 4.22 implies Assumption 2.6, but not the other way round. The financial system therefore still has a unique solution.

Definition 4.23. Let $\mathbf{R} \in (\mathbb{R}_0^+)^{2n}$ be an arbitrary vector with corresponding default set $\mathcal{D}(\mathbf{R})$ and default matrix $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{R})$. The *pseudo solution* $\widehat{\mathbf{R}} \in \mathbb{R}^{2n}$ *of* (4.1) *and* (4.2) *that belongs to* $\mathcal{D}(\mathbf{R})$ is defined by

$$\widehat{\mathbf{R}} = \begin{pmatrix} (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d} + \mathbf{\Lambda}\mathbf{x} \\ (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{x} \end{pmatrix}, \tag{4.77}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the solution of the linear equation system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \mathbf{I}_n - \big(\mathbf{M}^1\mathbf{\Lambda} + \mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda})\big) \in \mathbb{R}^{n\times n} \tag{4.78}$$

and

$$\mathbf{b} = \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d} - (\mathbf{I}_n - \mathbf{\Lambda})\mathbf{d} \in \mathbb{R}^n. \tag{4.79}$$

To motivate the definition of a pseudo solution, assume that it was known for each firm whether it was in default under the solution of (4.1) and (4.2) or not. Denote by $\mathcal{D}^* \subseteq \mathcal{N}$ the set of firms that are in default under $\mathbf{R}^*$, i.e.

$$\mathcal{D}^* = \mathcal{D}(\mathbf{r}^{*,1}, \mathbf{r}^{*,0}) = \big\{i \in \mathcal{N} : \big(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\big)_i < d_i\big\} \tag{4.80}$$

and let $\mathbf{\Lambda}^* = \mathbf{\Lambda}(\mathbf{r}^{*,1}, \mathbf{r}^{*,0})$ be the corresponding default matrix. We assume that the set was known even though this information is not available *a priori*. However, if we had this information, no iteration procedure would be needed to find the fixed point $\mathbf{R}^*$. We only had to compute the pseudo solution that belongs to $\mathcal{D}^*$, as shown in Proposition 4.24.

The reason why we have to restrict the following considerations to ownership matrices with a matrix norm smaller one is because we have to guarantee that $\mathbf{x}$ from Definition 4.23 is uniquely defined. This can only be ensured if $\|\mathbf{M}^1\| < 1$ and $\|\mathbf{M}^0\| < 1$ since then $\|\mathbf{M}^1\mathbf{\Lambda} + \mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda})\| < 1$ for any $\mathbf{\Lambda}$ as well, which in turn implies that $\mathbf{A}$ in (4.78) is invertible. If the ownership matrices $\mathbf{M}^1$ and $\mathbf{M}^0$ have the Elsinger Property, the invertibility of $\mathbf{A}$ is not always given. For a simple counterexample, let $n = 3$, $\mathcal{D}(\mathbf{R}) = \{2, 3\}$ for an arbitrary $\mathbf{R} \in (\mathbb{R}_0^+)^{2n}$ and

$$\mathbf{M}^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{M}^0 = \begin{pmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.81}$$

Both matrices do have the Elsinger Property, but the matrix $\mathbf{M}^1\boldsymbol{\Lambda}+\mathbf{M}^0(\mathbf{I}_n-\boldsymbol{\Lambda})$, with $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\mathbf{R})$ is given by

$$\mathbf{M}^1\boldsymbol{\Lambda} + \mathbf{M}^0(\mathbf{I}_n - \boldsymbol{\Lambda}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.82}$$

that does not have the Elsinger Property. Consequently, $\mathbf{A} = \mathbf{I}_n - \left(\mathbf{M}^1\boldsymbol{\Lambda} + \mathbf{M}^0(\mathbf{I}_n - \boldsymbol{\Lambda})\right)$ is not invertible.

Proposition 4.24. *The pseudo solution that belongs to $\mathcal{D}^*$ is the solution $\mathbf{R}^*$ of the financial system $\mathcal{F}(\mathbf{a}, \mathbf{M}^1, \mathbf{M}^0, \mathbf{d})$, i.e.*

$$\mathbf{R}^* = \begin{pmatrix} (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d} + \boldsymbol{\Lambda}^*\mathbf{x} \\ (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{x} \end{pmatrix}, \tag{4.83}$$

where $\boldsymbol{\Lambda}^*$ is the default matrix belonging to $\mathcal{D}^*$ and $\mathbf{x}$ is the solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ defined in (4.78) *and* (4.79).

Proof. According to the liquidation value equations in (4.1) and (4.2), the vectors $\mathbf{r}^{*,1}$ and $\mathbf{r}^{*,0}$ are given as

$$r_i^{*,1} = \begin{cases} d_i, & \text{if } i \notin \mathcal{D}^*, \\ \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\right)_i, & \text{if } i \in \mathcal{D}^* \end{cases} \tag{4.84}$$

and

$$r_i^{*,0} = \begin{cases} \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}\right)_i, & \text{if } i \notin \mathcal{D}^*, \\ 0, & \text{if } i \in \mathcal{D}^*. \end{cases} \tag{4.85}$$

In matrix notation this means in particular that $(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,1} = (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d}$ and $\boldsymbol{\Lambda}^*\mathbf{r}^{*,0} = \mathbf{0}_n$ and thus

$$\mathbf{R}^* = \begin{pmatrix} (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d} + \boldsymbol{\Lambda}^*\mathbf{r}^{*,1} \\ (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} \end{pmatrix}. \tag{4.86}$$

For the firms in default we only have to calculate the debt payments and for the firms not in default we have to determine the equity values. The solution $\mathbf{R}^*$ does hence contain only n unknown values and we only have to consider the two subsystems

$$\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} = \boldsymbol{\Lambda}^*\mathbf{a} + \boldsymbol{\Lambda}^*\mathbf{M}^1\mathbf{r}^{*,1} + \boldsymbol{\Lambda}^*\mathbf{M}^0\mathbf{r}^{*,0} \tag{4.87}$$

and

$$(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} = (\mathbf{I}_n - \boldsymbol{\Lambda}^*)(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}). \tag{4.88}$$

We can add the two equations and write the system more compact as:

$$\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} = \mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d}. \tag{4.89}$$

Because of $(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} = \mathbf{r}^{*,0}$ we get

$$\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} = \mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} - (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d}, \tag{4.90}$$

which leads after some rearrangements to

$$\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} - \mathbf{M}^1\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} - \mathbf{M}^0(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0} = \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,1} - (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d} \tag{4.91}$$

that is equivalent to

$$\left(\mathbf{I}_n - \left(\mathbf{M}^1\boldsymbol{\Lambda}^* + \mathbf{M}^0(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\right)\right)\left(\boldsymbol{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0}\right) = \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d} - (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{d}, \tag{4.92}$$

since $\boldsymbol{\Lambda}^*(\mathbf{I}_n - \boldsymbol{\Lambda}^*) = \mathbf{0}_{n\times n}$. Setting $\mathbf{x} = \boldsymbol{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \boldsymbol{\Lambda}^*)\mathbf{r}^{*,0}$ and with the notation of Definition 4.23, the equation system becomes $\mathbf{A}\mathbf{x} = \mathbf{b}$. □

Example 4.25. We keep the system as defined in Example 4.5. Under the solution vector $\mathbf{R}^*$, firm 2 and 3 are in default, i.e. $\mathcal{D}^* = \{2, 3\}$. If we assume that this information was available, the equation system $\mathbf{Ax} = \mathbf{b}$ is given by

$$\begin{pmatrix} 1 & -0.375 & -0.25 & -0.05 & -0.05 \\ -0.05 & 1 & -0.25 & -0.05 & -0.05 \\ -0.05 & -0.375 & 1 & -0.05 & -0.05 \\ -0.05 & 0 & 0 & 1 & -0.05 \\ -0.05 & 0 & -0.25 & -0.05 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -0.9450 \\ 1.2625 \\ 0.7925 \\ 0.7475 \\ 0.3800 \end{pmatrix}.$$

The solution of this system is given by

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^t = (0.1686, 1.7342, 1.5316, 0.7965, 0.8112)^t.$$

If we set $x_1 = r_1^{*,0}, x_2 = r_2^{*,1}, x_3 = r_3^{*,1}, x_4 = r_4^{*,0}$ and $x_5 = r_5^{*,0}$, we get together with $r_1^{*,1} = d_1, r_4^{*,1} = d_4, r_5^{*,1} = d_5$ and $r_2^{*,0} = r_3^{*,0} = 0$ the solution $\mathbf{R}^*$ of the system.

The main challenge in this solution approach is of course that the final default set $\mathcal{D}^*$ is unknown. Algorithms that follow this idea to find $\mathbf{R}^*$, consequently have to find $\mathcal{D}^*$ in a fast way. A naive strategy could be to check all possible default scenarios of the financial system, calculate the pseudo solution for the corresponding default set and check whether it actually is the fixed point of Φ. However, there are 2^n possible scenarios that would have to be checked, which could be cumbersome for large n. If $m > 1$, the number of possible scenarios is $(m+1)^n$ Therefore, more efficient algorithms are needed that require less computation to find $\mathcal{D}^*$. Some possible algorithms are presented in the next subsections .

4.2.1 Decreasing Trial-and-Error Algorithms

The three Algorithms 1, 3 and 6 from Section 4.1 can start with a vector $\mathbf{R}^0$ that is the upper boundary of the solution vector $\mathbf{R}^*$. The procedures in this subsection have in common that they also start with this upper boundary and calculate a corresponding default set. For every following iterate, the corresponding default set is determined as well. To avoid that for every default set it is checked whether it actually is $\mathcal{D}^*$, the algorithm will identify potential default sets to reduce the computational effort. If it turns out that the potential default set is $\mathcal{D}^*$, the algorithm stops. Otherwise, the procedure continues until a new potential default set is found that has to be checked again, and so on. Due to these characteristics, we name this type of algorithm *Trial-and-Error Algorithm.* The general procedure of algorithms of this type is similar.

Algorithm 7 (Decreasing Trial-and-Error Algorithm)**.** *Set $l \geq 2$ and $p = 0$.*

1. *Choose either the Picard (Algorithm 1) or the Elsinger (Algorithm 3) or the Hybrid Algorithm (Algorithm 6) which is used in the following to generate the next iterate.*
2. *If in Step 1 the Picard Algorithm is chosen, set $d = -1$, $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ and determine $\mathcal{D}(\mathbf{R}^0)$. Else, set $d = 0$, $\mathbf{R}^0 = \begin{pmatrix} \mathbf{d} \\ \mathbf{r}^0(\mathbf{d}) \end{pmatrix}$ and determine $\mathcal{D}(\mathbf{R}^0)$.*
3. *If $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$, set $\mathbf{R}^* = \begin{pmatrix} (\mathbf{I}_n - \mathbf{M}^1)^{-1}\mathbf{a} \\ \mathbf{0}_n \end{pmatrix}$ and stop the algorithm.*
4. *If the Elsinger or the Hybrid Algorithm is chosen in Step 1 and if $\mathcal{D}(\mathbf{R}^0) = \emptyset$, set $\mathbf{R}^* = \mathbf{R}^0$ and stop the algorithm.*

5. *Else, calculate for $k > p$ the iterates $\mathbf{R}^k$ starting with $\mathbf{R}^p$ using the algorithm chosen in Step 1 and the corresponding default sets $\mathcal{D}(\mathbf{R}^k)$ until $k = q$ with*

$$q = \min\{m > p : \mathcal{D}(\mathbf{R}^{m-l+1}) = \ldots = \mathcal{D}(\mathbf{R}^m) \text{ and } |\mathcal{D}(\mathbf{R}^m)| > d\} \tag{4.93}$$

is reached. Determine the pseudo solution belonging to $\mathcal{D}(\mathbf{R}^q)$ and denote it by $\widehat{\mathbf{R}}^q$.

6. *If $\Phi(\widehat{\mathbf{R}}^q) = \widehat{\mathbf{R}}^q$, stop the algorithm. Else, set $d = |\mathcal{D}(\mathbf{R}^q)|$ and $p = q$ and proceed with Step 5.*

The Algorithms 1, 3 and 6 in their decreasing versions produce decreasing sequences of iterates and thus increasing sequences of default sets, i.e. $\mathcal{D}(\mathbf{R}^k) \subseteq \mathcal{D}(\mathbf{R}^{k+1})$ for $k \geq 0$. Algorithm 7 means that one iterates and checks whether the default set has not changed compared to the previous default set. If the default set stays the same for the next l consecutive iterations, this is an indication that the actual default set $\mathcal{D}^*$ might have been reached. To check this, the pseudo solution is calculated and it is checked whether it solves (4.1) and (4.2). Provided that this is not the case, one iterates again until a larger default sets stays identical for $l-1$ consecutive times, and the described procedure can be repeated. In the event of a solution is reached, the procedure stops. Due to its described property, we call l the *lag value*.

In the special case of $l = 2$ this means that the pseudo solution is calculated if the default set stays the same from one iteration step to another. Obviously, choosing a higher lag value inspires more confidence in the potential default set since the longer the default set stays unchanged, the higher is the chance that it is the actual default set. Depending on the choice of the algorithm to calculate the next iterates in Step 1 of the Decreasing Trial-and-Error Algorithm, we obtain three different versions of Algorithm 7:

(i) The *Decreasing Trial-and-Error Picard Algorithm* with $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ and where the iterates are given by $\mathbf{R}^k = \Phi(\mathbf{R}^{k-1})$.

(ii) The *Decreasing Trial-and-Error Elsinger Algorithm* with $\mathbf{R}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$, where $\mathbf{r}^0(\mathbf{r}^1_{\text{great}})$ is obtained via Algorithm 2 and the next iterates are obtained using Algorithm 3.

(iii) The *Decreasing Trial-and-Error Hybrid Algorithm* with the same starting vector as in (ii) and where the next iterates are obtained using Algorithm 6.

The particular cases when $\mathcal{D}(\mathbf{R}^0) \in \{\emptyset, \mathcal{N}\}$ in the steps 3 and 4, deserve a separate mention since in such situations, no iteration is necessary and the solution $\mathbf{R}^*$ can be given explicitly under some circumstances. The justification of this phenomena is given in the following proposition.

Proposition 4.26. *For the Decreasing Trial-and-Error Hybrid Algorithm the following holds:*

(i) If $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$, then $\mathbf{R}^* = \begin{pmatrix} (\mathbf{I}_n - \mathbf{M}^1)^{-1}\mathbf{a} \\ \mathbf{0}_n \end{pmatrix}$, no matter which version of the algorithm is taken.

(ii) If $\mathcal{D}(\mathbf{R}^0) = \emptyset$ and either the Decreasing Trial-and-Error Elsinger Algorithm or the Decreasing Trial-and-Error Hybrid Algorithm is used, $\mathbf{R}^0 = \mathbf{R}^*$.

Proof. (i) First, assume that the Picard Algorithm is chosen in Step 1 of Algorithm 7. Because of $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$, it must hold that $\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} < \mathbf{d}$ and also $\mathbf{a} + \mathbf{M}^1\mathbf{d} < \mathbf{d}$. A consequence is that $\mathbf{r}^0_{\text{great}} = (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d})^+ = \mathbf{0}_n$. From Proposition 4.1 it follows that $\mathbf{r}^{*,0} = \mathbf{0}_n$. For $\mathbf{r}^{*,0} = \mathbf{0}_n$, Equation (4.1) is now solved by $\mathbf{r}^{*,1} = (\mathbf{I}_n - \mathbf{M}^1)^{-1}\mathbf{a}$, where Lemma A.3 proves that $(\mathbf{I}_n - \mathbf{M}^1)^{-1}$ exists. If the Elsinger or the Hybrid Algorithm is chosen in Step 1, we have that $\mathbf{a}+\mathbf{M}^1\mathbf{d}+\mathbf{M}^0\mathbf{r}^0(\mathbf{d}) < \mathbf{d}$. It follows that $\mathbf{r}^0(\mathbf{d}) = \mathbf{0}_n$ since $\mathbf{r}^{*,1} \leq \mathbf{d}$ and the fact that $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$ implies that $\mathbf{0}_n = \mathbf{r}^0(\mathbf{d}) \geq \mathbf{r}^0(\mathbf{r}^{*,1}) = \mathbf{r}^{*,0} = \mathbf{0}_n$ The solution of Equation (4.1) is therefore the same as in the Picard case.

(ii) Now, $\mathbf{R}^0 = (\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$ and since $\mathcal{D}(\mathbf{R}^0) = \emptyset$, it holds that $\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) \geq \mathbf{d}$. This leads to

$$\Phi \begin{pmatrix} \mathbf{d} \\ \mathbf{r}^0(\mathbf{d}) \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d})^+ \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{r}^0(\mathbf{d}) \end{pmatrix}, \tag{4.94}$$

which proves the claim.

□

Remark 4.27. (i) Even if in case of $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$ for the Hybrid Algorithm, the procedure is not stopped immediately, the algorithm would deliver the solution $\mathbf{R}^*$ straight in the next iteration step. To see this, note that the next debt iterate is the fixed point of the mapping $\mathbf{r} = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\}$. Because of $\mathbf{r}^0(\mathbf{d}) = \mathbf{0}_n$ and $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$, the mapping reduces to $\mathbf{r} = \mathbf{a}+\mathbf{M}^1\mathbf{r}$ which obviously has the fixed point $\mathbf{r}^{*,1} = (\mathbf{I}_n - \mathbf{M}^1)^{-1}\mathbf{a}$. Moreover, note that because of $\mathbf{r}^0(\mathbf{d}) = \mathbf{0}_n$, the Picard and the Elsinger version of Algorithm 7 would deliver the same iterates $\mathbf{R}^k$ (if the procedure would not be stopped immediately).

(ii) For the Decreasing Trial-and-Error Picard Algorithm, we cannot conclude that $\mathbf{R}^0 = \mathbf{R}_{\text{great}} = \mathbf{R}^*$ if $\mathcal{D}(\mathbf{R}^0) = \emptyset$. A counterexample is given for the financial system with $n = 3$ that is defined in Example 6.3 in Section 6.1. For the starting vector it holds that

$$\mathbf{R}_{\text{great}} = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0_{\text{great}}) = (1.745, 0.75, 1, 0.121, 1.0015, 1.4181)^t \tag{4.95}$$

and therefore $\mathcal{D}(\mathbf{R}_{\text{great}}) = \emptyset$. However,

$$\mathbf{R}^* = (\mathbf{d}^1, \mathbf{r}^{*,0}) = (1.745, 0.75, 1, 0.0003, 0.9951, 1.4118)^t, \tag{4.96}$$

hence $\mathbf{R}_{\text{great}} \neq \mathbf{R}^*$.

Proposition 4.28. *Algorithm 7 reaches the solution $\mathbf{R}^*$ of (4.1) and (4.2) in a finite number of iteration steps.*

Proof. By definition of $\mathcal{D}(\mathbf{R}^k)$ in (4.8) and since $\mathbf{R}^k$ converges to $\mathbf{R}^*$ from above for any of the three algorithms 1, 3 and 6, there exists a $k^0 \geq 0$ such that $\mathcal{D}(\mathbf{R}^k) = \mathcal{D}(\mathbf{R}^*) = \mathcal{D}^*$ for all $k \geq k^0$. □

A special property of the Decreasing Trial-and-Error Hybrid Algorithm is that it is possible to give a maximum number of steps that are needed to reach the first potential default set. To this end, we distinguish between the expressions *iteration step* and *calculation step*. An iteration step comprises the determination of a new iterate $\mathbf{R}^{k+1}$ based on an iterate $\mathbf{R}^k, k \geq 0$. A calculation step is defined as the calculation of either the fixed point of the mapping Θ in (4.56) of the debt components or of the fixed point of Ψ for the equity components in (4.31). Consequently, each calculation step consists of solving a linear equation system and each iteration step contains potentially several calculation steps to find the equity components $\mathbf{r}^{k+1,1}$ and the debt components $\mathbf{r}^0(\mathbf{r}^{k+1,1})$. Recall that, as shown in the Propositions 4.7 (iii) and 4.13 (iii), there are not more than $n - 1$ calculation steps needed to find the debt or equity iterate no matter at which point of the iteration you are.

Proposition 4.29. *The Decreasing Trial-and-Error Hybrid Algorithm reaches the first potential default set after no more than $(n-1)(l-1)$ iteration steps that consist of no more than $(n-1)(2l-1)$ calculation steps.*

Before we can proof Proposition 4.29, we need to mention the following Lemma.

Lemma 4.30. *Let in the Decreasing Trial-and-Error Hybrid Algorithm the k-th iterate ($k \geq 1$) be given by $\mathbf{R}^k$ and denote by d_k the number of firms in default under $\mathbf{R}^k$, i.e. $d_k = |\mathcal{D}(\mathbf{R}^k)|$. If it holds that $|\mathcal{D}(\mathbf{R}^{k+1})| = |\mathcal{D}(\mathbf{R}^k)| + x$ with $0 \leq x < n - d_k$, then $2+x$ calculation steps are needed to determine $\mathbf{R}^{k+1}$.*

Proof. First, assume $x = 0$. To determine the next iterate $\mathbf{R}^{k+1}$, the debt component $\mathbf{r}^{k+1,1}$ has to be calculated first via Algorithm 4. The algorithm starts with the default set $\mathcal{D}(\mathbf{r}^{k,1}, \mathbf{r}^0(\mathbf{r}^{k,1}))$. Since no new default occurs from the k-th to the $(k+1)$-th iteration step, this means that also in Algorithm 4 no new default can occur. Consequently, the algorithm has to stop after the first iteration which means that there is only one calculation step. For the calculation of the equity component with Algorithm 2, the argumentation is analogous which leads to two calculation steps in total. Assume now that $x = 1$, i.e. that one new firm defaults from iteration step k to $k+1$. This new default can either occur in Algorithm 4 or in Algorithm 2. If it occurs when calculating the next debt iterate, there must be two iteration steps since if there were only one, no new default would occur. For the equity component, only one iteration step is needed which summarizes up to $3 = 2 + x$ calculation steps. If the default occurs in the equity calculation step, the number of calculation steps is the same. Suppose now that $x > 1$. In this case, the defaults can potentially split up arbitrarily and occur in the debt or in the equity calculation step. However, the argument stays the same. □

Proof of **Proposition 4.29**.
According to Proposition 4.26, only $\mathcal{D}(\mathbf{R}^0) \notin \{\emptyset, \mathcal{N}\}$ has to be considered, since otherwise no iteration would be necessary to find $\mathbf{R}^*$. Depending on the choice of the lag value, the number of firms in default can stay constant from one iterate to another. The number of iteration and calculation steps certainly becomes maximal if $|\mathcal{D}(\mathbf{R}^0)| = 1$ and the number of defaults each time increases by one after $l-1$ consecutive iterates to ensure that the lag value is fully "exploited". Note that if the set of defaults does not change for l consecutive iterates, the algorithm would have reached the first potential default set by definition. To determine a new iterate without a change of the default set, needs two calculation steps, as shown in Lemma 4.30. If the number of defaults increases by one, there are three calculation steps necessary (cf. Lemma 4.30). For a better understanding, we can split the complete iteration procedure into parts and count the iteration and calculation steps of each part:

- Iterates $\mathbf{R}^k$ with $|\mathcal{D}(\mathbf{R}^k)| = 1$:
  These iterates are the first $l-1$ iterates $\mathbf{R}^0, \mathbf{R}^1, \ldots, \mathbf{R}^{l-2}$. To find $\mathbf{R}^0$, only one calculation step is needed since the debt component is already given by $\mathbf{d}$. For the determination of the remaining $l-2$ iterates, two calculation steps are necessary, respectively.
- Iterates $\mathbf{R}^k$ with $|\mathcal{D}(\mathbf{R}^k)| = h$, where $2 \leq h \leq n-1$:
  For every value of h, there are $l-1$ iterates, whereas for the first iterate three and for the remaining $l-2$ iterates, two calculation steps are necessary, respectively.

If the default set consists of $h = n-1$ firms for $l-1$ consecutive iterates, the default set of the next iterate can either consist also of $n-1$ firms or it can contain all n firms. In the latter case, no further iteration is needed since it must hold that $\mathcal{D}^* = \mathcal{N}$. In the former case, the algorithm stops for "Trial-and-Error" by definition. In both cases, 2 calculation steps are needed. Excluding the iterate $\mathbf{R}^0$ as the zeroth iterate from our counting, there are clearly $(n-1)(l-1)$ iteration steps for the algorithm. Summing up all calculation steps leads to:

$$1 + 2(l-2) + (n-2)\big(3 + 2(l-2)\big) + 2 = (n-1)\big(3 + 2(l-2)\big) = (n-1)(2l-1). \tag{4.97}$$

□

4.2.2 Increasing Trial-and-Error Algorithms

In contrast to the decreasing algorithms presented in the subsection above, it is of course also possible to use an algorithm with the reverse direction, i.e. in which the series of produced iterates is increasing and in which the default sets are decreasing. The general form is very similar to Algorithm 7.

Algorithm 8 (Increasing Trial-and-Error Algorithm)**.** *Set $l \geq 2$, $d = n+1$ and $p = 0$.*

1. *Select between the Picard, the Elsinger and the Hybrid Algorithm as the algorithm of choice. For the Picard Algorithm, set $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ and for the other ones, set $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$. Moreover, determine $\mathcal{D}(\mathbf{R}^0)$.*
2. *If $\mathcal{D}(\mathbf{R}^0) = \emptyset$, set $\mathbf{R}^* = \binom{\mathbf{d}}{\mathbf{r}^0(\mathbf{d})}$ and stop the algorithm.*
3. *Else, calculate for $k > p$ the iterates $\mathbf{R}^k$ starting with $\mathbf{R}^p$ using the algorithm chosen in Step 1 and the corresponding default sets $\mathcal{D}(\mathbf{R}^k)$ until $k = q$ with*

$$q = \min\{m > p : \mathcal{D}(\mathbf{R}^{m-l+1}) = \ldots = \mathcal{D}(\mathbf{R}^m) \text{ and } |\mathcal{D}(\mathbf{R}^m)| < d\} \tag{4.98}$$

   is reached. Determine the pseudo solution belonging to $\mathcal{D}(\mathbf{R}^q)$ and denote it by $\widehat{\mathbf{R}}^q$.

4. *If $\Phi(\widehat{\mathbf{R}}^q) = \widehat{\mathbf{R}}^q$, stop the algorithm. Else, set $d = |\mathcal{D}(\mathbf{R}^q)|$ and $p = q$ and proceed with Step 3.*

The functioning of Algorithm 8 is similar to the Decreasing Trial-and-Error Algorithms with the difference that the resulting sequence of default sets is obviously decreasing. As in Section 4.2.1, the way of choosing the calculation method to determine the next iterate, allows three different modifications:

(i) The *Increasing Trial-and-Error Picard Algorithm* with $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ and $\mathbf{R}^k = \Phi(\mathbf{R}^{k-1})$.

(ii) The *Increasing Trial-and-Error Elsinger Algorithm* with $\mathbf{R}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$, where $\mathbf{r}^0(\mathbf{r}^1_{\text{small}})$ is obtained via Algorithm 2 and the next iterates are obtained using Algorithm 3.

(iii) The *Increasing Trial-and-Error Hybrid Algorithm* with the same starting vector as in (ii) and where the next iterates are obtained using Algorithm 6. Note that for the next debt iterate, Algorithm 5 is used instead of Algorithm 4 in the decreasing version.

The justification of the stopping criteria in Step 2 of Algorithm 8 is as follows. Suppose that the Picard version of the algorithm is chosen and that $\mathcal{D}(\mathbf{R}^0) = \emptyset$, which means that $\mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{small}} + \mathbf{M}^0\mathbf{r}^0_{\text{small}} > \mathbf{d}$. Since $\mathbf{r}^1_{\text{small}} \leq \mathbf{d}$ and $\mathbf{r}^0_{\text{small}} \leq \mathbf{r}^0(\mathbf{d})$, it also holds that $\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) > \mathbf{d}$ and $\mathbf{r}^0(\mathbf{d}) > \mathbf{0}_n$ following from this. With Equation (4.94), we see that $\mathbf{R}^* = (\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$. Also note that, in contrast to Algorithm 7, there is no stopping criteria in case of $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$. The reason is that in this case, no general statement about the structure of the solution $\mathbf{R}^*$ can be made, no matter which version of the algorithm is used. In particular, from $\mathcal{D}(\mathbf{R}_{\text{small}}) = \mathcal{N}$ it does not follow in general that $\mathcal{D}(\mathbf{R}^*) = \mathcal{N}$, as the next example demonstrates.

Example 4.31. We take the financial system of Example 4.5 but replace the vector of the nominal values of the liabilities with $\mathbf{d} = (1.6, 1.5, 1.5, 1.8, 1.2)^t$ which alters the fixed point $\mathbf{R}^*$ to

$$\mathbf{R}^* = (1.6, 1.5, 1.5, 1.8, 1.2, 0.2423, 0.4589, 0.0369, 0.2542, 0.5328)^t \tag{4.99}$$

and we see that all five firms are solvent in the solution, i.e. $\mathcal{D}(\mathbf{R}^*) = \emptyset$. However, it holds that

$$\mathbf{a} + \mathbf{M}^1\mathbf{r}^1_{\text{small}} + \mathbf{M}^0\mathbf{r}^0_{\text{small}} = (1.005, 1.065, 0.8575, 1.5025, 1.03)^t < \mathbf{d} \tag{4.100}$$

and thus $\mathcal{D}(\mathbf{R}_{\text{small}}) = \mathcal{N}$.

The reason why we distinguish between decreasing and increasing Trial-and-Error Algorithms is that Algorithm 7 will always find the correct default set $\mathcal{D}^*$ and this in a finite number of iteration steps. For the Increasing Trial-and-Error Algorithms such a statement is not possible in general since there are some situations in which the default sets do not converge to $\mathcal{D}^*$, no matter which lag value is chosen. Situations in which this “anomaly” occurs are always financial systems that contain a so-called *borderline firm*. The expression borderline is taken from Liu and Staum (2010) and denotes a firm $i \in \mathcal{N}$ in a financial system with fixed point $\mathbf{R}^*$ for which it holds that $r_i^{*,1} = d_i$ and $r_i^{*,0} = 0$. In other words, borderline firms are just able to fully cover their liabilities, but have no remaining capital left in their balance sheet that can be furnished to their shareholders. By definition of a default set in (4.8), a borderline firm i is not in default since

$$0 = r_i^{*,0} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}\right)_i \tag{4.101}$$

and therefore $i \notin \mathcal{D}(\mathbf{R}^*)$ However, when using an Increasing Trial-and-Error Algorithm it can happen for such a borderline firm i that $i \in \mathcal{D}(\mathbf{R}^k)$ for every iterate $\mathbf{R}^k$, $k \geq 0$. This means that the actual default set $\mathcal{D}^*$ will never be identified by the algorithm, as shown in Example 4.34.

To show that in such situations, the fixed point $\mathbf{R}^*$ can still be determined via the calculation of the pseudo solution, assume that the set $\mathcal{B} \subset \mathcal{N}$ contains an arbitrary selection of borderline firms. The common set of defaulting firms and the selected borderline firms is denoted by $\widetilde{\mathcal{D}}$, i.e. $\widetilde{\mathcal{D}} = \mathcal{D}^* \cup \mathcal{B}$. The corresponding “default” matrices are given by $\widetilde{\mathbf{\Lambda}} = \mathbf{\Lambda}(\widetilde{\mathcal{D}})$ and $\mathbf{\Lambda}^* = \mathbf{\Lambda}(\mathcal{D}^*)$, respectively. Following this notation, $\widetilde{\mathbf{A}}$ and $\mathbf{A}^*$ define the matrices from (4.78) with the corresponding default matrix, and $\widetilde{\mathbf{b}}$ and $\mathbf{b}^*$ are defined analogously. Moreover, we define $\widetilde{\mathbf{\Lambda}}_{\mathcal{B}} = \widetilde{\mathbf{\Lambda}} - \mathbf{\Lambda}^*$ as the diagonal matrix that indicates only the selected borderline firms.

Lemma 4.32. *The vector* $\mathbf{x}^* = \mathbf{\Lambda}^*\mathbf{r}^{*,1} + (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{r}^{*,0}$ *solves the equation system* $\mathbf{A}^*\mathbf{x} = \mathbf{b}^*$ *if and only if* $\widetilde{\mathbf{x}} = \mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}$ *is the solution of* $\widetilde{\mathbf{A}}\mathbf{x} = \widetilde{\mathbf{b}}$.

Proof. Without loss of generality, we assume that the first n_1 firms of the system are solvent, that the next $n_2 - n_1$ firms are the selected borderline cases and that the remaining firms are in default under $\mathbf{R}^*$. This means that

$$\mathcal{N} = \{1, \ldots, n_1\} \cup \mathcal{B} \cup \mathcal{D}^* = \{1, \ldots, n_1\} \cup \{n_1 + 1, \ldots, n_2\} \cup \{n_2 + 1, \ldots, n\}. \tag{4.102}$$

It follows from Proposition 4.24 that

$$\begin{aligned} \mathbf{x}^* &= (x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_2}, x_{n_2+1}, \ldots, x_n)^t \\ &= (r_1^{*,0}, \ldots, r_{n_1}^{*,0}, 0, \ldots, 0, r_{n_2+1}^{*,1}, \ldots, r_n^{*,1})^t. \end{aligned} \tag{4.103}$$

Further, note that

$$\mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{x}^* = \mathbf{M}^0(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{x}^* = \mathbf{M}^0(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})(\mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}) \tag{4.104}$$

and

$$\mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} - \mathbf{M}^1\widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d} = \mathbf{M}^1(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{d} \tag{4.105}$$

and that

$$\mathbf{M}^1\mathbf{\Lambda}^*\mathbf{x}^* + \mathbf{M}^1\widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d} = \mathbf{M}^1\widetilde{\mathbf{\Lambda}}(\mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}) \tag{4.106}$$

because of the structure of $\mathbf{x}^*$. By (4.78) and (4.79), $\mathbf{x}^*$ solves $\mathbf{A}^*\mathbf{x} = \mathbf{b}^*$ if and only if

$$\begin{aligned}\mathbf{x}^* &= \mathbf{b}^* + (\mathbf{M}^1\mathbf{\Lambda}^* + \mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))\mathbf{x}^* \\ &= \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} + (\mathbf{M}^1\mathbf{\Lambda}^* + \mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))\mathbf{x}^* \\ &= \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} - \mathbf{M}^1\widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d} - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} + (\mathbf{M}^1\mathbf{\Lambda}^* + \mathbf{M}^0(\mathbf{I}_n - \mathbf{\Lambda}^*))\mathbf{x}^* + \mathbf{M}^1\widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d} \\ &= \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{d} - (\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} + (\mathbf{M}^1\widetilde{\mathbf{\Lambda}} + \mathbf{M}^0(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}}))(\mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}).\end{aligned} \tag{4.107}$$

Since $(\mathbf{I}_n - \mathbf{\Lambda}^*)\mathbf{d} = (\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{d} + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}$, we can add $\widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}$ on both sides of the equation and obtain

$$\mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d} = \mathbf{a} + \mathbf{M}^1(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{d} - (\mathbf{I}_n - \widetilde{\mathbf{\Lambda}})\mathbf{d} + (\mathbf{M}^1\widetilde{\mathbf{\Lambda}} + \mathbf{M}^0(\mathbf{I}_n - \widetilde{\mathbf{\Lambda}}))(\mathbf{x}^* + \widetilde{\mathbf{\Lambda}}_{\mathcal{B}}\mathbf{d}), \tag{4.108}$$

Therefore $\widetilde{\mathbf{x}} = (r_1^{*,0}, \ldots, r_{n_1}^{*,0}, d_{n_1+1}, \ldots, d_{n_2}, r_{n_2+1}^{*,1}, \ldots, r_n^{*,1})^t$ is the solution of $\widetilde{\mathbf{A}}\mathbf{x} = \widetilde{\mathbf{b}}$ if and only if $\mathbf{x}^*$ solves $\mathbf{A}^*\mathbf{x} = \mathbf{b}^*$. □

The pseudo solution belonging to $\mathcal{D}^*$ is the solution $\mathbf{R}^*$ of the system. A direct consequence of Lemma 4.32 is that the pseudo solution that belongs to $\widetilde{\mathcal{D}}$ is also equal to $\mathbf{R}^*$. Similar to the proof of Proposition 4.28, we can argue that the set $\widetilde{\mathcal{D}}$ will be reached by the Increasing Trial-and-Error Algorithms in a finite number of steps. Note that this statement holds in particular for the Increasing Hybrid Trial-and-Error Algorithm, where Algorithm 5 is used to calculate the next debt iterate. Even though a Picard-typed procedure is used in this auxiliary algorithm, we can conclude together with Proposition 4.19 and $\varepsilon > 0$ that the number of iterations will still be finite. We summarize the findings in the next proposition.

Proposition 4.33. *Algorithm 8 reaches the solution* $\mathbf{R}^*$ *of* (4.1) *and* (4.2) *in a finite number of iteration steps.*

For a better understanding of Proposition 4.33, the next example contains such a situation of a borderline firm.

Example 4.34. We define a financial system that consist of $n = 3$ firms with

$$\mathbf{a} = \begin{pmatrix} 5 \\ 2 \\ 4.4 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 0.25 & 0.1 \\ 0.5 & 0 & 0.25 \\ 0.25 & 0.5 & 0 \end{pmatrix}, \ \mathbf{M}^0 = \begin{pmatrix} 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0 \end{pmatrix}. \tag{4.109}$$

The solution of the system is given by

$$\mathbf{R}^* = (116/15, 7, 1, 0, 0, 53/6)^t = (7.7333, 7, 1, 0, 0, 8.8333)^t, \tag{4.110}$$

so it follows that the first firm is in default, the second is borderline and the third firm is solvent, and that $\mathcal{D}(\mathbf{R}^*) = \{1\}$. Running the Increasing Trial-and-Error Algorithm will lead to $\mathcal{D}(\mathbf{R}^0) = \{1, 2\}$, no matter which version is used, as one can easily check. This default set will remain the same for all other iterates, i.e. $\mathcal{D}(\mathbf{R}^k) = \{1, 2\}$ for $k \geq 1$. However, the pseudo solution that belongs to the default set $\{1, 2\}$ delivers the fixed point $\mathbf{R}^*$.

The next lemma attempts to describe in more general form how the structure of a financial system can look like such that situations as described in Example 4.34 can occur. However, there are many other possible structures of financial systems in which the same phenomena is also present.

Lemma 4.35. *Consider a financial system with $n \geq 3$ firms for which the Increasing Trial-and-Error Algorithm is applied to find $\mathbf{R}^*$. Assume that in the fixed point $\mathbf{R}^*$, firm i_1 is borderline, firm i_2 is in default and firm i_3 is solvent. Further, suppose that $i_1 \in \mathcal{D}(\mathbf{R}^0)$.*

(i) If $r^{0,1}_{i_2} < r^{*,1}_{i_2}$, $M^1_{i_1,i_2} \neq 0$ and $M^1_{i_2,i_1} \neq 0$ and if either the Increasing Trial-and-Error Picard Algorithm or the Increasing Trial-and-Error Elsinger Algorithm is used, then $i_1 \in \mathcal{D}(\mathbf{R}^k)$ for all $k \geq 0$.

(ii) If $\mathbf{r}^0(\mathbf{r}^{0,1})_{i_3} = r^{0,0}_{i_3} > 0$, $M^0_{i_1,i_3} \neq 0$ and $M^1_{i_3,i_1} \neq 0$ then $i_1 \in \mathcal{D}(\mathbf{R}^k)$ for all $k \geq 0$ for the Increasing Trial-and-Error Hybrid Algorithm.

Proof. (i) We will show the claim assuming that the Increasing Trial-and-Error Picard Algorithm is used. For the first iterate of the debt components, it holds for the firm i_1 that

$$\begin{aligned} r^{1,1}_{i_1} &= \min\left\{d_{i_1}, \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1}\right\} \\ &= \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1} < d_{i_1} = r^{*,1}_{i_1}, \end{aligned} \tag{4.111}$$

because of $i_1 \in \mathcal{D}(\mathbf{R}^0)$. Since $M^1_{i_2,i_1} \neq 0$, $r^{0,1}_{i_1} \leq r^{1,1}_{i_1}$ and $r^{*,1}_{i_2} < d_{i_2}$ it follows for the first debt iterate of firm i_2 that

$$\begin{aligned} r^{1,1}_{i_2} &= \min\left\{d_{i_2}, \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_2}\right\} \\ &= \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_2} < r^{*,1}_{i_2}. \end{aligned} \tag{4.112}$$

Therefore, because of $M^1_{i_1,i_2} \neq 0$,

$$r^{2,1}_{i_1} = \min\left\{d_{i_1}, (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^{1,0})_{i_1}\right\} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^{1,0}\right)_{i_1} \tag{4.113}$$

from which follows that $i_1 \in \mathcal{D}(\mathbf{R}^1)$ as well as $r^{2,1}_{i_1} < d_{i_1} = r^{*,1}_{i_1}$. This in turn implies that $r^{2,1}_{i_2} < r^{*,1}_{i_2}$. The inductive continuation of these arguments proves the claim. Note that for the Increasing Trial-and-Error Elsinger Algorithm the argumentation does not change since the debt component is calculated in the same way as in the Picard version of the algorithm.

(ii) It holds that $r^{0,1}_{i_1} = \min\{d_{i_1}, a_{i_1}\}$. Since $i_1 \in \mathcal{D}(\mathbf{R}^0)$ by assumption, it follows that $r^{0,1}_{i_1} = a_{i_1} < d_{i_1}$. Together with $M^1_{i_3,i_1} \neq 0$, we have that

$$0 < \mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,1})_{i_3} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{0,1} + \mathbf{M}^0\mathbf{r}^{0,0} - \mathbf{d}\right)_{i_3} < r^{*,0}_{i_3} \tag{4.114}$$

which in turn implies because of $M^0_{i_1,i_3} \neq 0$ that

$$r^{1,1}_{i_1} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^{0,0}\right)_{i_1} < d_{i_1} = r^{*,1}_{i_1}. \tag{4.115}$$

This implies

$$r^{1,0}_{i_3} = \left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^{1,0} - \mathbf{d}\right)_{i_3} < r^{*,0}_{i_3}, \tag{4.116}$$

due to $M^1_{i_3,i_1} \neq 0$ and because of $M^0_{i_1,i_3} \neq 0$, it follows that

$$\left(\mathbf{a} + \mathbf{M}^1\mathbf{r}^{1,1} + \mathbf{M}^0\mathbf{r}^{1,0}\right)_{i_1} < d_{i_1} = r^{*,1}_{i_1} \tag{4.117}$$

and, hence, $i_1 \in \mathcal{D}(\mathbf{R}^1)$. Using the same argumentation leads to $r^{2,1}_{i_1} < d_{i_1} = r^{*,1}_{i_1}$ and to $r^{2,0}_{i_3} < r^{*,0}_{i_3}$ from which in turn follows that $i_1 \in \mathcal{D}(\mathbf{R}^2)$. An inductive use of this principle shows the assertion.

□

4.2.3 Sandwich Algorithms

A disadvantage of the Trial-and-Error Algorithms was that when a potential default set is reached, the only way to find out whether this default set is actually $\mathcal{D}^*$, is to calculate the corresponding pseudo solution and check whether it is a fixed point of (4.3). The choice of a high lag value can of course increase the chance that $\mathcal{D}^*$ is reached at the first trial, but there is no certainty.

Another way to find $\mathcal{D}^*$ is to start an iteration simultaneously with the largest and smallest possible solution and use one of the Algorithms 1, 3 or 6 to obtain the next iterate. For $k \geq 0$ denote by $\overline{\mathbf{R}}^k$ the k-th iterate of the series that emerges when starting the algorithm with the maximum and by $\underline{\mathbf{R}}^k$ its counterpart when starting with the minimum possible solution. Depending which algorithm is chosen, the starting vector can either be $\overline{\mathbf{R}}^0 = \mathbf{R}_{\text{great}}$ (Picard Iteration) or $\overline{\mathbf{R}}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$ (Elsinger and Hybrid Algorithm). Analogously, we have $\underline{\mathbf{R}}^0 = \mathbf{R}_{\text{small}}$ or $\underline{\mathbf{R}}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$. By the Propositions 4.2, 4.9, 4.17 and by Equation (4.8), the iterative use of one of the mentioned algorithms entails that the default sets approach one another, i.e. for $k \geq 0$,

$$\mathcal{D}(\underline{\mathbf{R}}^k) \supseteq \mathcal{D}(\underline{\mathbf{R}}^{k+1}) \supseteq \mathcal{D}^* \supseteq \mathcal{D}(\overline{\mathbf{R}}^{k+1}) \supseteq \mathcal{D}(\overline{\mathbf{R}}^k). \tag{4.118}$$

Let

$$l = \min\{k \geq 0 : \mathcal{D}(\underline{\mathbf{R}}^k) = \mathcal{D}(\overline{\mathbf{R}}^k)\} \tag{4.119}$$

be the first iteration step in which the default set for both starting vectors is the same. Then we must have that $\mathcal{D}(\overline{\mathbf{R}}^l) = \mathcal{D}^*$ and, by Proposition 4.24, determining the pseudo solution belonging to $\mathcal{D}^*$ leads to $\mathbf{R}^*$. Because of its characteristics we call this algorithm the *Sandwich Algorithm*.

Algorithm 9 (Sandwich Algorithm)**.**

1. *Determine* $\overline{\mathbf{R}}^0$ *and* $\underline{\mathbf{R}}^0$ *as well as their corresponding default sets* $\mathcal{D}(\overline{\mathbf{R}}^0)$ *and* $\mathcal{D}(\underline{\mathbf{R}}^0)$.
2. *For* $k \geq 1$*, calculate the iterates* $\overline{\mathbf{R}}^k$ *and* $\underline{\mathbf{R}}^k$ *using one of the Algorithms 1, 3 or 6 and the corresponding default sets* $\mathcal{D}(\overline{\mathbf{R}}^k)$ *and* $\mathcal{D}(\underline{\mathbf{R}}^k)$.
3. *If* $\mathcal{D}(\overline{\mathbf{R}}^k) = \mathcal{D}(\underline{\mathbf{R}}^k)$*, stop the algorithm, set* $\mathcal{D}^* = \mathcal{D}(\overline{\mathbf{R}}^k)$ *and calculate the pseudo solution that belongs to* $\mathcal{D}^*$ *following Definition 4.23. Else, set* $k = k + 1$ *and go back to Step 2.*

As for the Trial-and-Error Algorithms in the sections above, the Sandwich Algorithm results in different versions:

(i) The *Sandwich Picard Algorithm* with $\overline{\mathbf{R}}^0 = \mathbf{R}_{\text{great}}$ and $\underline{\mathbf{R}}^0 = \mathbf{R}_{\text{small}}$ and the use of Algorithm 1 in Step 2.

(ii) The *Sandwich Elsinger Algorithm* with $\overline{\mathbf{R}}^0 = (\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$ and $\underline{\mathbf{R}}^0 = (\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ and the use of Algorithm 3 in Step 2.

(iii) The *Sandwich Hybrid Algorithm* with the same starting points as the Sandwich Elsinger Algorithm and the iterative use of Algorithm 6 in Step 2.

Recall the insights from Section 4.2.2, where it was shown that, under some circumstances, it may happen that the series of default sets $\mathcal{D}(\underline{\mathbf{R}}^k)$ will never converge to the actual default set $\mathcal{D}^*$. Situations in which this problem occurs always contain at least one firm that is on borderline in the solution $\mathbf{R}^*$. As a result of this behavior, the Sandwich Algorithm may not converge in the sense that the default sets $\mathcal{D}(\overline{\mathbf{R}}^k)$ and $\mathcal{D}(\underline{\mathbf{R}}^k)$ will never become identical. The system from Example 4.34 is an example in which the Sandwich Algorithm fails. In all three versions of the algorithm, $\mathcal{D}(\overline{\mathbf{R}}^0) = \{1\}$ when starting from the upper boundary and this default set stays constant for every further iterate $\mathcal{D}(\overline{\mathbf{R}}^k), k \geq 1$. As shown in Example 4.34, it holds that $\mathcal{D}(\underline{\mathbf{R}}^0) = \{1, 2\}$ for all $k \geq 0$. The two default sets will hence never become identical and so the Sandwich Algorithm would never come to an end.

However, if we consider a stochastic setting and assume a distribution for the vector $\mathbf{a}$ of the exogenous assets' prices which has a density with respect to the Lebesgue measure on $(\mathbb{R}_0^+)^n$, then situations in which the convergence cannot be assured occur only with probability zero as the next Proposition shows. Note that this assumption is fulfilled in the usual n-firm Merton models where the individual a_i are log-normally distributed.

Proposition 4.36. *The Sandwich Algorithm generates a sequence of decreasing default sets $\mathcal{D}(\underline{\mathbf{R}}^k)$ and a sequence of increasing default sets $\mathcal{D}(\overline{\mathbf{R}}^k)$ that reach the default set $\mathcal{D}^*$ of the solution $\mathbf{R}^*$ almost surely after finitely many steps. Thus, it reaches the solution $\mathbf{R}^*$ of* (4.3) *almost surely after finitely many steps.*

Proof. The increasing and decreasing property of the default sets follows directly from the Propositions 4.2, 4.9 and 4.17. The two series of default sets of the algorithm both converge in finitely many iteration steps to $\mathcal{D}^*$ if there is no firm in the financial system that is borderline. Lemma A.7 in the Appendix shows that the probability for borderline firms in $\mathbf{R}^*$ is zero from which follows the almost sure convergence. ☐

By its nature, the Sandwich Algorithm converges to $\mathcal{D}^*$ from both directions which doubles the computation and makes the algorithm somewhat inefficient from a computational point of view. On the other hand, the algorithm computes an exact solution in finitely many iteration steps without wasting time on "Trial-and-Error". In contrast to the Trial-and-Error Algorithms, the drawback of the Sandwich Algorithm is that the convergence of the procedure cannot be ensured when borderline firms are present in the system. To overcome this problem, we recommend for practical purposes to apply the idea of a lag value in the Sandwich Algorithm as well.

Algorithm 10 (Modified Sandwich Algorithm)**.** *Set $l \geq 2$.*

1. *Determine $\overline{\mathbf{R}}^0$ and $\underline{\mathbf{R}}^0$ as well as their corresponding default sets $\mathcal{D}(\overline{\mathbf{R}}^0)$ and $\mathcal{D}(\underline{\mathbf{R}}^0)$.*
2. *For $k \geq 1$, calculate the iterates $\overline{\mathbf{R}}^k$ and $\underline{\mathbf{R}}^k$ using one of the Algorithms 1, 3 or 6 and their corresponding default sets $\mathcal{D}(\overline{\mathbf{R}}^k)$ and $\mathcal{D}(\underline{\mathbf{R}}^k)$.*
3. *If $\mathcal{D}(\overline{\mathbf{R}}^k) = \mathcal{D}(\underline{\mathbf{R}}^k)$, stop the algorithm, set $\mathcal{D}^* = \mathcal{D}(\overline{\mathbf{R}}^k)$ and calculate the pseudo solution that belongs to $\mathcal{D}^*$ following Definition 4.23. Else, if $k \geq l$ and*

$$|\mathcal{D}(\underline{\mathbf{R}}^k)| - |\mathcal{D}(\overline{\mathbf{R}}^k)| = \ldots = |\mathcal{D}(\underline{\mathbf{R}}^{k-l+1})| - |\mathcal{D}(\overline{\mathbf{R}}^{k-l+1})|, \tag{4.120}$$

calculate the pseudo solution belonging to $\mathcal{D}(\overline{\mathbf{R}}^k)$ *and stop the algorithm if it solves the Equations* (4.1) *and* (4.2)*. Else, set* $k = k + 1$ *and go back to Step 2.*

The modification consists of interrupting the algorithm if the default sets $\mathcal{D}(\overline{\mathbf{R}}^k)$ and $\mathcal{D}(\underline{\mathbf{R}}^k)$ for both iteration directions are not identical but stay constant for l consecutive times. If l is chosen large enough (e.g. $l \geq 5$) and (4.120) holds, this is a strong indication that at least one firm in the system is borderline and that the convergence of both series is not given. In this situation, a check whether the default set has already been reached is suitable.

Table 4.2: Default sets $\mathcal{D}(\mathbf{R}^k)$ of the corresponding iterates of the Picard, Elsinger and Hybrid Algorithm for both directions of the financial system given in Example 4.5. The default sets serve as the decision criteria whether to stop the Trial-and-Error Algorithms and the Sandwich algorithms.

Type	Iteration Step k	$\mathcal{D}(\mathbf{R}^k)$ (Increasing)		$\mathcal{D}(\mathbf{R}^k)$ (Decreasing)
Picard	0	$\{1,2,3\}$	$\supset$	$\{3\}$
	1	$\{1,2,3\}$	$\supset$	$\{2,3\}$
	2	$\{1,2,3\}$	$\supset$	$\{2,3\}$
	3	$\{2,3\}$	$=$	$\{2,3\}$
Elsinger	0	$\{1,2,3\}$	$\supset$	$\{3\}$
	1	$\{1,2,3\}$	$\supset$	$\{2,3\}$
	2	$\{2,3\}$	$=$	$\{2,3\}$
Hybrid	0	$\{1,2,3\}$	$\supset$	$\{3\}$
	1	$\{2,3\}$	$=$	$\{2,3\}$

Example 4.37. The fact that the number of needed iterations to find $\mathbf{R}^*$ tends to be much shorter for the Trial-and-Error and the Sandwich Algorithms can once again be demonstrated using the system from Example 4.5. Suppose that we choose the Trial-and-Error Picard Algorithm that starts with $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ and a lag value of $l = 2$. The series of default sets that belongs to the corresponding iterates is given by (cf. Table 4.2)

$$\mathcal{D}(\mathbf{R}^0) = \{3\} \supset \mathcal{D}(\mathbf{R}^1) = \{2,3\} = \mathcal{D}(\mathbf{R}^2) = \{2,3\}, \tag{4.121}$$

from which follows that the algorithms stops the first time after two iteration steps. Since $\mathcal{D}(\mathbf{R}^2) = \mathcal{D}^*$, the pseudo solution belonging to $\mathcal{D}(\mathbf{R}^2)$ is identical to $\mathbf{R}^*$ and the algorithm stops. Note that in case of $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$, the first potential default set is $\{1,2,3\}$, as can be seen in the left-hand part of Table 4.2. The pseudo solution belonging to this default set will not solve the mapping Φ and therefore the algorithm continues and stops the next time after the fourth iteration step (not shown in Table 4.2). Running the Decreasing Trial-and-Error Elsinger Algorithm, also leads to two iteration steps as well as when the Decreasing Trial-and-Error Hybrid Algorithm is used.

Table 4.2 also demonstrates the functioning of the Sandwich Picard Algorithm. The two default sets for every iteration direction become identical in the third iteration step, so the Algorithm stops. The Elsinger and the Hybrid version of the Sandwich Algorithm stops after two steps and one step, respectively.

Recall that concluding from the results of Table 4.2 that the Hybrid Algorithm is more efficient than the Elsinger Algorithm which in turn is more efficient than the Picard Algorithm (no matter whether the Trial-and-Error or the Sandwich algorithm is considered), is not entirely correct since once again, the hidden calculation steps in the Elsinger and Hybrid version of the algorithms are not considered here. Section 7.5 will deliver more precise answers to these questions in the therein performed simulation study.

Remark 4.38. We close this section with a short comment about financial systems with a general exogenous asset structure, i.e. systems in which $\mathbf{a} \in \mathbb{R}^n$. Using non-finite algorithms to detect $\mathbf{R}^*$, we showed in the Remarks 4.3, 4.10 and 4.18 that the algorithms still work if the modified mapping $\Phi^1(\mathbf{r}^1; \mathbf{r}^0) = \min\{\mathbf{d}, (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0)^+\}$ is applied. For the finite algorithms in this section, such a generalization is, however, not possible. Take the financial system of Example 4.5 again and modify the vector of exogenous assets to

$$\mathbf{a} = (0.39, 0.38, -0.6, 1.29, -0.46)^t, \tag{4.122}$$

i.e. we multiply the asset value of the third and the fifth firm by -1 and remain the other entries unchanged. The fixed point $\mathbf{R}^*$ of this new systems is now

$$\mathbf{R}^* = (1.0655, 0.9820, 0, 1.14, 0.1136, 0, 0, 0, 0.4448, 0)^t \tag{4.123}$$

and the default set changes to $= \mathcal{D}(\mathbf{R}^*) = \{1, 2, 3, 5\}$. Calculating a pseudo solution that belongs to $\mathcal{D}(\mathbf{R}^*)$ leads to

$$\widehat{\mathbf{R}} = (0.9575, 0.8671, -0.2497, 1.14, 0.0217, 0, 0, 0, 0.3948, 0)^t \tag{4.124}$$

and hence $\widehat{\mathbf{R}} \neq \mathbf{R}^*$. The crucial difference for $\mathbf{a} \in \mathbb{R}^n$ is that it might happen that the value a_i is for some firms that negative, that these firms will not be able to service any of their debt payments and have $r_i^{*,1} = 0$. In our example, this is the case for the third firm. The non-finite algorithms can deal with this circumstance by considering the modified version of Φ^1. When calculating the pseudo solution, this is not possible. This insight will become important in Section 5.2 when systems with a seniority structure are considered.

5 Valuation Algorithms for Systems with a Seniority tructure

While in Chapter 4, the valuation algorithms dealt with only one seniority level, we want to cover in the current chapter calculation procedures that are able to find the solution of financial systems with a seniority structure. As a consequence, a financial system $\mathcal{F}$ is now represented by $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$, where $\mathbf{M} = (\mathbf{M}^m, \ldots, \mathbf{M}^0) \in (\mathbb{R}_0^+)^{n \times n(m+1)}$ and $\mathbf{d} = (\mathbf{d}^m, \ldots, \mathbf{d}^1) \in (\mathbb{R}_0^+)^{n \times m}$. By analogy to the structure of Chapter 4, we can in a system with $m > 1$ also distinguish between non-finite and finite algorithms. The non-finite algorithms (Section 5.1) are straightforward extensions of the algorithms defined in Section 4.1 for systems with a general seniority structure. All non-finite algorithms have the drawback that they might not reach the solution exactly and therefore need potentially many iteration steps to get sufficiently close to $\mathbf{R}^*$. In Section 5.2, we will generalize the ideas of Section 4.2, where a default set was used to develop a finite iteration procedure that reaches $\mathbf{R}^*$ exactly. Moreover, another method is presented in Section 5.3 that pursues a different approach to find $\mathbf{R}^*$. This procedure which we will call Default Structure Algorithm is mentioned first in Elsinger (2009). We pick up his ideas and demonstrate the method in more detail.

5.1 Non-finite Algorithms

The Picard, the Elsinger and the Hybrid Algorithm from Section 4.1 are generalized in this section to financial systems with $m > 1$.

5.1.1 The Picard Algorithm

In Section 4.1.1, the Picard Algorithm is presented in detail for $m = 1$. The principle of this procedure was the iterative application of the mapping Φ in (4.3). Now, we consider the extended version of the mapping defined in (2.12). However, the idea of iteratively applying Φ remains unchanged. As we adapted the mapping for a seniority structure, we also have to do so for the two possible starting vectors $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$, following Fischer (2015). The natural extension of $\mathbf{R}_{\text{great}}$ is then given as

$$\mathbf{R}_{\text{great}} := \begin{pmatrix} \mathbf{r}_{\text{great}}^m \\ \vdots \\ \mathbf{r}_{\text{great}}^0 \end{pmatrix}, \tag{5.1}$$

where the first nm components contain the face values of the m liabilities, i.e. $\mathbf{r}_{\text{great}}^k = \mathbf{d}^k, k = 1, \ldots, m$, and where

$$\mathbf{r}_{\text{great}}^0 := (\mathbf{I}_n - \mathbf{M}^0)^{-1} \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k \right)^+ \tag{5.2}$$

represents the equity components. The second starting vector, $\mathbf{R}_{\text{small}}$, becomes to

$$\mathbf{R}_{\text{small}} := \begin{pmatrix} \mathbf{r}^m_{\text{small}} \\ \vdots \\ \mathbf{r}^0_{\text{small}} \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned} \mathbf{r}^m_{\text{small}} &= \min\left\{\mathbf{d}^m, \quad \mathbf{a}\right\}, \\ \mathbf{r}^j_{\text{small}} &= \min\left\{\mathbf{d}^j, \left(\mathbf{a} - \sum_{k=j+1}^{m} \mathbf{d}^k\right)^+\right\} \qquad (0 < j < m), \\ \mathbf{r}^0_{\text{small}} &= \qquad \left(\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k\right)^+. \end{aligned} \tag{5.4}$$

Similar to the case of $m = 1$, it also obviously holds in this situation that $\Phi(\mathbf{0}_{n(m+1)}) = \mathbf{R}_{\text{small}}$. Moreover, we also need to adapt the definitions for the default set and the default matrix in (4.8) and (4.9), respectively. For a given vector $\mathbf{R} = (\mathbf{r}^m, \ldots, \mathbf{r}^0) \in (\mathbb{R}^+_0)^{n(m+1)}$ of recovery values, the *default set under* $\mathbf{R}$ is defined as

$$\mathcal{D}(\mathbf{R}) = \mathcal{D}(\mathbf{r}^m, \ldots, \mathbf{r}^0) = \left\{ i \in \mathcal{N} : \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)_i < 0 \right\}. \tag{5.5}$$

A firm $i \in \mathcal{D}(\mathbf{R})$ is said to be *in default* without further specification, since it is only known that for given payments $\mathbf{R}$, there is at least one seniority level – the lowest level – the firm cannot fully service. However, only knowing that $i \in \mathcal{D}(\mathbf{R})$ does not deliver any detailed information in which seniority the transition from full payment to partial or no payment occurs. This is why we introduce another mode of speaking, where a closer look at the default structure of the firms under a given payment vector is taken to find the solution $\mathbf{R}^*$. We say that a firm is *in default in the l-th seniority level* ($1 \leq l \leq m$) *under* $\mathbf{R}$, if

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=l+2}^{m} \mathbf{d}^k\right)_i \geq d_i^{l+1} \quad \text{and} \quad \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=l+1}^{m} \mathbf{d}^k\right)_i < d_i^l. \tag{5.6}$$

Such a firm is therefore able to fully satisfy the obligees in seniority classes higher than l, and for all debt payments in a level at or below l, no capital is left for the debtholders (and shareholders). The *default matrix corresponding to* $\mathbf{R}$, $\mathbf{\Lambda}(\mathbf{R}) \in \mathbb{R}^{n \times n}$, is defined as

$$\mathbf{\Lambda}(\mathbf{R}) = \mathbf{\Lambda}(\mathbf{r}^m, \ldots, \mathbf{r}^0) = \operatorname{diag}\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k < \mathbf{0}_n\right) \tag{5.7}$$

and is the diagonal matrix with entry 1 for firms in default under $\mathbf{R}$ and with the value 0 for firms not in default.

In Proposition 4.1 it is shown that for $m = 1$, the solution $\mathbf{R}^*$ lies in the interval $[\mathbf{R}_{\text{great}}, \mathbf{R}_{\text{small}}]$. This assertion obviously holds for $m > 1$ as well, for the proof that $\mathbf{R}_{\text{great}} \geq \mathbf{R}^*$, we have to replace $\mathbf{M}^1\mathbf{d}$ by $\sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k$ and $\mathbf{d}$ by $\sum_{k=1}^{m} \mathbf{d}^k$. The remaining argumentation stays unchanged, in particular, it clearly holds that $\Phi(\mathbf{R}_{\text{small}}) \geq \mathbf{R}_{\text{small}}$. Moreover, the claim of Proposition 4.2 also holds for $m > 1$, which becomes immediately clear when checking that the Equations (4.15) and (4.17) also hold for more than one seniority. As a consequence, the Picard Algorithm can be used without further adaption to situations with $m > 1$, which is why we do not repeat the pseudo-code and refer to Algorithm 1 in Section 4.1.1 instead. This also implies that the Picard Algorithm generates a decreasing sequence when $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ and an increasing sequence if $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$. Let us conclude this subsection with an example for the Picard Algorithm with a financial system that we will also make use of in the subsequent parts of this chapter.

Example 5.1. We consider a system with $n = 5$ firms and $m = 3$ seniority classes. The vector of the exogenous assets and the liability vectors are given by

$$\mathbf{a} = \begin{pmatrix} 9 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{d}^3 = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 5 \\ 1 \end{pmatrix}, \ \mathbf{d}^2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 5 \end{pmatrix}, \ \mathbf{d}^1 = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 5 \end{pmatrix}, \tag{5.8}$$

the ownership matrices are defined as

$$\mathbf{M}^3 = \begin{pmatrix} 0 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 1/4 & 1/6 & 0 & 0 & 1/2 \\ 0 & 0 & 1/6 & 0 & 0 \\ 1/4 & 1/6 & 1/6 & 0 & 0 \end{pmatrix}, \ \mathbf{M}^2 = \begin{pmatrix} 0 & 1/6 & 1/8 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/4 \\ 0 & 1/6 & 1/8 & 0 & 0 \\ 1/2 & 1/6 & 1/8 & 0 & 0 \end{pmatrix} \tag{5.9}$$

and

$$\mathbf{M}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 1/6 & 0 & 0 & 1/2 & 1/2 \\ 1/6 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{M}^0 = \begin{pmatrix} 0 & 1/40 & 0 & 0 & 1/20 \\ 1/40 & 0 & 0 & 0 & 0 \\ 1/40 & 1/40 & 0 & 1/20 & 0 \\ 0 & 0 & 1/20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

The fixed point $\mathbf{R}^*$ of Φ is given by

$$\mathbf{R}^* = \begin{pmatrix} \mathbf{r}^{*,3} \\ \mathbf{r}^{*,2} \\ \mathbf{r}^{*,1} \\ \mathbf{r}^{*,0} \end{pmatrix} = \begin{pmatrix} (3, & 2, & 4, & 3.7917, & 1 &)^t \\ (2, & 3, & 1, & 0, & 3.875)^t \\ (3, & 1, & 1.1596, & 0, & 0 &)^t \\ (2.6669, & 1.636, & 0, & 0, & 0 &)^t \end{pmatrix}. \tag{5.11}$$

We see that the first two firms are both solvent since their equity values are positive. Firm 3 can fully satisfy its obligees in the two highest seniorities and defaults in seniority level 1, whereas firm 5 already defaults in the second seniority level. The fourth firm is not even able to fully deliver the highest seniority level nor any other seniorities and therefore is in default in level 3. The first iterates $\mathbf{R}^k$ when starting the Picard Algorithm either with $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ or $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ are listed in Table 5.1. For a tolerance level of $\varepsilon = 10^{-6}$ both directions stop the procedure when iterate $\mathbf{R}^{10}$ is reached.

Remark 5.2. Proposition 4.4 demonstrates that the Picard Algorithm for $m = 1$ will under some circumstances never reach $\mathbf{R}^*$. The crucial assumption needed for the proof was that there is a "circular cashflow" from a defaulting and a solvent firm, where the defaulting firm must own some shares of the solvent firm and the solvent firm must own some fraction of the debt from the defaulting firm. Another assumption was that the initial equity iterate $\mathbf{R}^0$ for the solvent firm is larger than the equity value in $\mathbf{R}^*$. This principle can obviously be applied to systems with $m > 1$ as well. Since there are now several seniorities, we have to adapt the first part of the assumption. The solvent firm must therefore own a fraction of the defaulting firms' debt in exactly that seniority, which the defaulting firm is not fully able to service. If these assumptions are fulfilled, the ideas of the proof of Proposition 4.4 can easily be applied. Note that in Example 5.1 we have that firm 2 is solvent and that firm 3 defaults in the lowest seniority. Since $M^0_{32} = 1/40$ and $M^1_{23} = 1/2$ and $r^{*,0}_2 = 1.6360 < 3.9431 = (\mathbf{r}^0_{\text{great}})_2$, the mentioned assumptions are fulfilled. The Picard Algorithm will therefore never reach $\mathbf{R}^*$.

Table 5.1: Iterates of the Picard Algorithm for the financial system in Example 5.1.

	$\mathbf{R}^0 = \mathbf{R}_{\text{small}}$						$\mathbf{R}^0 = \mathbf{R}_{\text{great}}$		
	$\mathbf{R}^0$	$\mathbf{R}^1$	$\mathbf{R}^2$	...	$\mathbf{R}^*$	...	$\mathbf{R}^2$	$\mathbf{R}^1$	$\mathbf{R}^0$
	3	3	3		3		3	3	3
	2	2	2		2		2	2	2
$\mathbf{r}^{k,3}$	3	4	4	...	4	...	4	4	4
	2	3.3333	3.7917		3.7917		3.9420	3.9420	5
	1	1	1		1		1	1	1
	2	2	2		2		2	2	2
	2	3	3		3		3	3	3
$\mathbf{r}^{k,2}$	0	1	1	...	1	...	1	1	1
	0	0	0		0		0	0	4
	0	3.4167	3.8750		3.8750		3.8750	3.8750	5
	3	3	3		3		3	3	3
	0	0.0250	1		1		1	1	1
$\mathbf{r}^{k,1}$	0	0.1083	0.9917	...	1.1596	...	1.2188	4	4
	0	0	0		0		0	0	2
	0	0	0		0		0	0	5
	1	2.1667	2.6250		2.6659		2.7236	2.7236	2.7236
	0	0	0.7542		1.6360		3.1329	3.9431	3.9431
$\mathbf{r}^{k,0}$	0	0	0	...	0	...	0	3.0075	3.0075
	0	0	0		0		0	0	0.1504
	0	0	0		0		0	0	0

5.1.2 The Elsinger Algorithm

The difference in the functioning of the Picard and the Elsinger Algorithm in Section 4.1 is that for the Elsinger Algorithm, the equity iterate is calculated as the fixed point of the mapping $\Phi^0(\cdot; \bar{\mathbf{r}})$ in (4.27) for a given debt payment vector $\bar{\mathbf{r}} \leq \mathbf{d}$ instead of applying Φ^0 on the former equity and debt iterates as this is the case for the Picard Algorithm. This principle can directly be transformed to systems with $m > 1$ as we will show in the following. First, we need to adapt the mapping for the equity components for more than one seniority. Let $\mathbf{r}^m \leq \mathbf{d}^m, \ldots, \mathbf{r}^1 \leq \mathbf{d}^1$

be some arbitrary recovery vectors. Then the mapping $\Phi^0 : (\mathbb{R}_0^+)^n \to (\mathbb{R}_0^+)^n$ is defined as

$$\Phi^0(\mathbf{r};\mathbf{r}^m,\ldots,\mathbf{r}^1) = \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k\mathbf{r}^k + \mathbf{M}^0\mathbf{r} - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ . \tag{5.12}$$

We have shown in Section 4.1.2 that $\Phi^0(\cdot;\mathbf{r})$ has for given debt payments $\mathbf{r} \leq \mathbf{d}$ a unique fixed point. This obviously holds for the mapping in (5.12) as well, since the $\mathbf{r}^m,\ldots,\mathbf{r}^1$ are considered as fixed values. Denote this fixed point by $\mathbf{r}^0(\mathbf{r}^m,\ldots,\mathbf{r}^1)$, that means

$$\Phi^0(\mathbf{r}^0(\mathbf{r}^m,\ldots,\mathbf{r}^1);\mathbf{r}^m,\ldots,\mathbf{r}^1) = \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^m,\ldots,\mathbf{r}^1) - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ . \tag{5.13}$$

This fixed point represents for given debt payments $\mathbf{r}^m,\ldots,\mathbf{r}^1$ the corresponding equity payments of the system.

To obtain $\mathbf{r}^0(\mathbf{r}^m,\ldots,\mathbf{r}^1)$, we use a modified version of Algorithm 2, where

$$\mathbf{w}^0 = \mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k\mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k \tag{5.14}$$

is set as the starting iterate for given debt payments $\mathbf{r}^m,\ldots,\mathbf{r}^1$. The remaining steps in Algorithm 2 stay unchanged. Clearly, Proposition 4.7 is also valid in this situation, i.e. the algorithm converges in a finite number of iteration steps to $\mathbf{r}^0(\mathbf{r}^m,\ldots,\mathbf{r}^1)$. In order to formulate the Elsinger Algorithm for more than one seniority, we need to introduce for each seniority class j $(1 \leq j \leq m)$ the mapping $\Phi^j : (\mathbb{R}_0^+)^n \to (\mathbb{R}_0^+)^n$ with

$$\Phi^j(\mathbf{r};\mathbf{r}^m,\ldots,\mathbf{r}^{j+1},\mathbf{r}^{j-1},\ldots,\mathbf{r}^0) = \min\left\{\mathbf{d}^j, \left(\mathbf{a} + \sum_{\substack{l=0\\ l\neq j}}^{m} \mathbf{M}^l\mathbf{r}^l + \mathbf{M}^j\mathbf{r} - \sum_{l=j+1}^{m} \mathbf{d}^l\right)^+\right\}, \tag{5.15}$$

where the $\mathbf{r}^m,\ldots,\mathbf{r}^{j+1},\mathbf{r}^{j-1},\ldots,\mathbf{r}^0$ are considered as fixed debt payments for the other seniorities and the equity.

Algorithm 11 (Elsinger Algorithm $(m > 1)$)**.** *Set $\varepsilon > 0$.*

1. *For $k = 0$, choose*

$$\left(\mathbf{r}^{0,m},\ldots,\mathbf{r}^{0,1}\right) \in \left\{(\mathbf{r}^m_{\text{great}},\ldots,\mathbf{r}^1_{\text{great}}),(\mathbf{r}^m_{\text{small}},\ldots,\mathbf{r}^1_{\text{small}})\right\} \tag{5.16}$$

   and determine $\mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,m},\ldots,\mathbf{r}^{0,1})$ using Algorithm 2 with the modification in (5.14)*. Denote the iterate $\mathbf{R}^0$ by $\mathbf{R}^0 = (\mathbf{r}^{0,m},\ldots,\mathbf{r}^{0,0})$.*

2. *For $k \geq 1$, set for $1 \leq j \leq m$,*

$$\mathbf{r}^{k,j} = \Phi^j(\mathbf{r}^{k-1,j};\mathbf{r}^{k-1,m},\ldots,\mathbf{r}^{k-1,j+1},\mathbf{r}^{k-1,j-1},\ldots,\mathbf{r}^{k-1,0}), \tag{5.17}$$

   calculate $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,m},\ldots,\mathbf{r}^{k,1})$ and set $\mathbf{R}^k = (\mathbf{r}^{k,m},\ldots,\mathbf{r}^{k,0})$.

3. *If $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, stop the algorithm. Else, set $k = k+1$ and proceed with Step 2.*

Similar to the version without a seniority structure, Algorithm 11 can start either with an iterate that assumes maximum debt payments or with an iterate in which the debt payments are set to a minimum level. Depending on this choice in (5.16), the corresponding equity vector is determined which completes the specification of the zeroth iterate. To obtain the next iterate, the mapping Φ is applied to the debt vectors, as indicated in (5.17). Using the new debt payments, the corresponding equity vector is determined with Algorithm 2. The differences between Algorithm 1 and 11 are therefore the differing ways to calculate the equity components. Figure 5.1 attempts to demonstrate the functioning of the Elsinger Algorithm in an idealized scheme.

In Proposition 4.9, we showed that for $m = 1$ the Elsinger Algorithm generates a series of decreasing or increasing iterates that converges to the fixed point $\mathbf{R}^*$. A crucial part of the proof was that the equity component $\mathbf{r}^0(\mathbf{r})$ is increasing in $\mathbf{r}$, which also holds for the case of $m > 1$, i.e. $\mathbf{r}^0(\mathbf{r}^m, \dots, \mathbf{r}^1)$ is increasing in $\mathbf{r}^m, \dots, \mathbf{r}^1$ (see Lemma 2.10). That means, starting with $(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1}) = (\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})$ will deliver a decreasing series and starting with $(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1}) = (\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})$ will result in an increasing series of iterates. Both series are bounded from above and from below and therefore have to converge to $\mathbf{R}^*$. Additionally, it is also possible to generalize the assertion of Proposition 4.11 to the current situation by obviously extending the argumentation used therein to systems with $m > 1$. Hence, we can conclude that the Elsinger Algorithm for $m > 1$ will always be closer to the fixed point $\mathbf{R}^*$ than the Picard Algorithm.

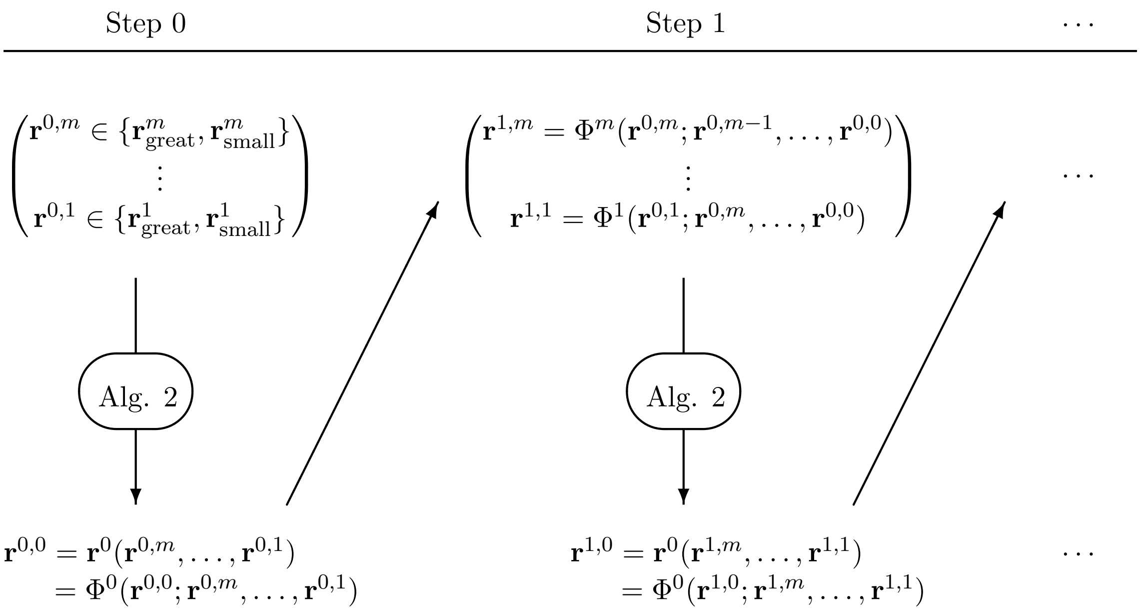


Figure 5.1: Schematic description of Algorithm 11 beginning with the initial iteration step in the left part of the figure. Note that, according to (5.16), the choice of $\mathbf{r}^{0,k} \in \{\mathbf{r}^k_{\text{great}}, \mathbf{r}^k_{\text{small}}\}$ has to be made consistent for every seniority. The arrows indicate the different calculation steps of the procedure. If an arrow is interrupted by an oval-shaped symbol, the listed auxiliary algorithm in the oval is performed to obtain the next iterate. An arrow without any further specification means that the mappings Φ^j are applied that are defined in (5.15).

Table 5.2: Iterates of the Elsinger Algorithm for the financial system in Example 5.1.

	$\mathbf{r}^{0,k} = \mathbf{r}^k_{\text{small}}$ $(k = 1, 2, 3)$					$\mathbf{r}^{0,k} = \mathbf{r}^k_{\text{great}}$ $(k = 1, 2, 3)$				
	$\mathbf{R}^0$	$\mathbf{R}^1$	$\mathbf{R}^2$	...	$\mathbf{R}^*$	...	$\mathbf{R}^2$	$\mathbf{R}^1$	$\mathbf{R}^0$
	3	3	3		3		3	3	3
	2	2	2		2		2	2	2
$\mathbf{r}^{k,3}$	3	4	4	...	4	...	4	4	4
	2	3.3333	3.7917		3.7917		3.7917	3.9417	5
	1	1	1		1		1	1	1
	2	2	2		2		2	2	2
	2	3	3		3		3	3	3
$\mathbf{r}^{k,2}$	0	1	1	...	1	...	1	1	1
	0	0	0		0		0	0	4
	0	3.4167	3.8750		3.8750		3.8750	3.8750	5
	3	3	3		3		3	3	3
	0	0.0542	1		1		1	1	1
$\mathbf{r}^{k,1}$	0	0.1375	1.0231	...	1.1596	...	1.1980	4	4
	0	0	0		0		0	0	2
	0	0	0		0		0	0	5
	2.1667	2.6445	2.6642		2.6659		2.6664	2.7033	2.7236
	0	0.7807	1.5678		1.6360		1.6552	3.1322	3.9431
$\mathbf{r}^{k,0}$	0	0	0	...	0	...	0	0	3
	0	0	0		0		0	0	0
	0	0	0		0		0	0	0

Example 5.3. Take the financial system from Example 5.1. Setting $(\mathbf{r}^{0,3}, \mathbf{r}^{0,2}, \mathbf{r}^{0,1}) = (\mathbf{r}^3_{\text{great}}, \mathbf{r}^2_{\text{great}}, \mathbf{r}^2_{\text{great}})$, it holds for the equity component, that

$$\mathbf{r}^0(\mathbf{r}^{0,3}, \mathbf{r}^{0,2}, \mathbf{r}^{0,1}) = (2.7236, 3.9431, 3, 0, 0)^t. \tag{5.18}$$

Hence, firm 4 is already considered to be in default, which was not the case for the Picard Algorithm (see Table 5.1). The first and some consequent iterates when starting with the other starting vector are shown in Table 5.2. Note that the pure number of iteration steps is 6 for both staring vectors which is smaller than for the Picard Algorithm. The tolerance level was with $\varepsilon = 10^{-6}$ the same than in Example 5.1.

As already mentioned after the introduction of the Elsinger Algorithm just before Proposition 4.11, judging the computational efficiency of the Picard and the Elsinger Algorithm based on the number of needed iterations only, is not entirely accurate. The reason is already mentioned in Section 4.1.2: The calculation of the equity iterate in the Elsinger Algorithm can be more expensive since, under circumstances, many linear equations systems have to be solved which is not the case for the Picard Algorithm. For instance in Example 5.3, 7 linear equation systems have to be solved in total for each direction of the algorithm. This is why we have to be very careful when only considering the iteration steps as the crucial parameter to compare both algorithms. In fact, the runtime for both procedures to find $\mathbf{R}^*$ seems to be the more comparative measure for a comparison.

In Remark 5.2, we mentioned that the Picard Algorithm does not always reach $\mathbf{R}^*$. This statement also holds for the Elsinger Algorithm; in particular we can generalize the insights of Proposition 4.12 in a straightforward way.

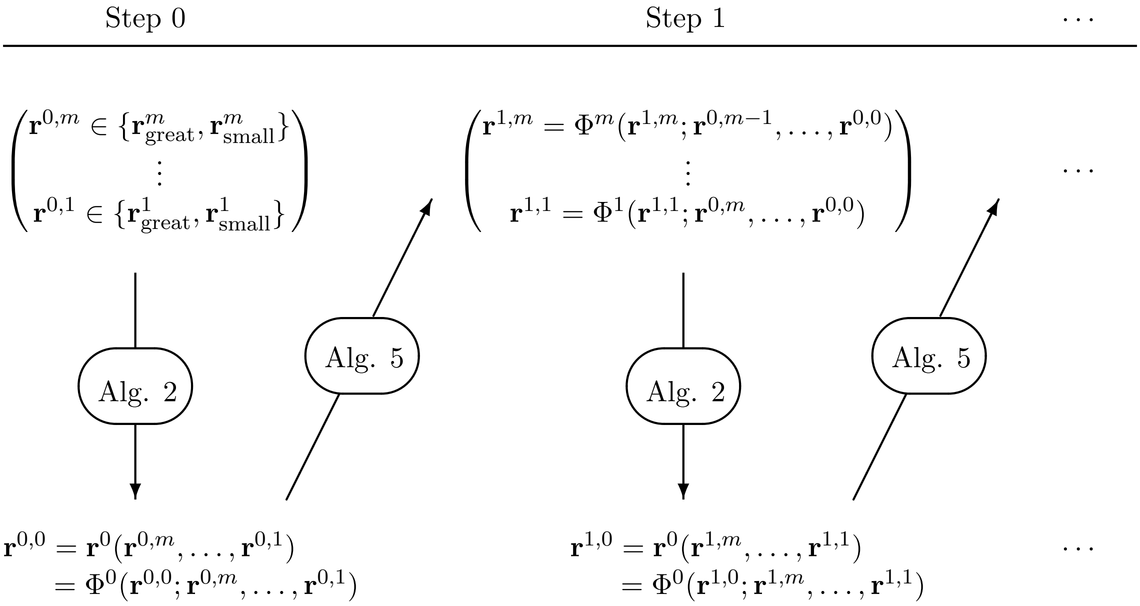


Figure 5.2: Schematic description of Algorithm 12 beginning with the initial iteration step in the left part of the figure. According to (5.19), the choice of $\mathbf{r}^{0,k} \in \{\mathbf{r}^k_{\text{great}}, \mathbf{r}^k_{\text{small}}\}$ has to be made consistent for every seniority. The arrows indicate the different calculation steps of the procedure. To obtain the next iteration in the algorithm, the corresponding auxiliary algorithm listed in the oval-shaped symbol needs to be performed. Note that for the debt components, Algorithm 5 has to be conducted for every debt seniority separately.

5.1.3 A Hybrid Algorithm

As we extended the Picard and the Elsinger Algorithms on systems with a seniority structure, we can also do so for the Hybrid Algorithm defined in Section 4.1.3. The principle is the same as for systems with $m = 1$. In the Elsinger Algorithm, the next debt iterate in seniority level j ($1 \leq j \leq m$) was the application of the mapping Φ^j defined as in (5.15). For the Hybrid Algorithm, we calculate the fixed point of Φ^j for each seniority level j. To obtain this fixed point, Algorithm 5 should be chosen which uses a Picard-type iteration to reach the fixed point. This is necessary since the positive part $(\cdot)^+$ in the right part of Equation (5.15) prevents the usage of Algorithm 4, see also the comments in Remark 4.10. The procedure is summarized in the next algorithm, the schematic functioning of the algorithm is visualized in Figure 5.2.

Algorithm 12 (Hybrid Algorithm ($m > 1$))**.** *Set* $\varepsilon > 0$.

1. *For* $k = 0$*, choose*

$$\left(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1}\right) \in \left\{(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}}), (\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})\right\} \tag{5.19}$$

and determine $\mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1})$ *using Algorithm 2 with the modification in* (5.14)*. Denote the iterate* $\mathbf{R}^0$ *by* $\mathbf{R}^0 = (\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,0})$.

2. *For $k \geq 1$, determine for every $1 \leq j \leq m$ the fixed point $\mathbf{r}^{k,j}$ of the mapping in* (5.15) *using Algorithm 5, where for Φ^j the fixed vectors $\mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,j+1}, \mathbf{r}^{k-1,j-1}, \ldots, \mathbf{r}^{k-1,0}$ are used from the preceding iterate. Calculate $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,1})$ using Algorithm 2 and set $\mathbf{R}^k = (\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,0})$.*
3. *If $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, stop the algorithm. Else, set $k = k + 1$ and proceed with Step 2.*

It is obvious that the assertions of Proposition 4.17 can be transferred to Algorithm 12 as well. This means that the procedure generates a series of increasing or decreasing iterates, depending on the choice of the starting vector in (5.19). Both series will converge to the fixed point $\mathbf{R}^*$. The same transformation is also possible for Proposition 4.19, i.e. the iterates of Algorithm 12 will in every iteration step be closer to $\mathbf{R}^*$ than the iterates of the Elsinger Algorithm in Algorithm 11, where we once again ignore the effects of different calculation methods (applying a map vs. solving a linear equation system). Finally, the Proposition 4.20 can obviously also be generalized to the current situation. Hence, the procedure must also not necessarily exactly reach the fixed point.

Before concluding the subsection with an example, we want to present a slight adaption of the Hybrid Algorithm. The idea of this modification is to include the newest available information about the debt payments. Suppose we are in the k-th iteration step ($k \geq 0$) in Algorithm 12. The debt iterate of seniority level 1, $\mathbf{r}^{k,1}$, is calculated as the fixed point of the mapping Φ^1, where $\mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,2}, \mathbf{r}^{k-1,0}$ are considered as fixed. The vector $\mathbf{r}^{k,1}$ is closer to actual debt payments $\mathbf{r}^{*,1}$ than the preceding iterate $\mathbf{r}^{k-1,1}$. To get the next iterate for seniority class 2, we have to solve the mapping Φ^2, where in Algorithm 12 the vector $\mathbf{r}^{k-1,1}$ was taken for the fixed debt payments of the lowest seniority level. But instead of $\mathbf{r}^{k-1,1}$, we can also take the new iterate $\mathbf{r}^{k,1}$ for the calculation and consider this vector a the "updated" debt payments for seniority class 1. This results in a new iterate $\mathbf{r}^{k,2}$ that can now also be used, together with $\mathbf{r}^{k,1}$ for the calculation of the next highest seniority level 3, and so on. What we obtain is a series of iterates in which every debt iterate contains the updated informations of all debt payments of lower seniorities. For this reason, we call this modified version of the Hybrid Algorithm, the *Updated Hybrid Algorithm*.

Algorithm 13 (Updated Hybrid Algorithm)**.** *Set $\varepsilon > 0$.*

1. *For $k = 0$, choose*

$$\left(\mathbf{r}^{0,m}, \ldots, \mathbf{r}^{0,1}\right) \in \left\{(\mathbf{r}^m_{\text{great}}, \ldots, \mathbf{r}^1_{\text{great}}), (\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}})\right\} \tag{5.20}$$

and determine $\mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,m}, \ldots, \mathbf{r}^{0,1})$ using Algorithm 2 with the modification in (5.14)*. Denote the iterate $\mathbf{R}^0$ by $\mathbf{R}^0 = (\mathbf{r}^{0,m}, \ldots, \mathbf{r}^{0,0})$.*

2. *For $k \geq 1$, determine the next debt iterates $\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,1}$ using the following scheme:*
   - $\mathbf{r}^{k,1}$ *is the fixed point of* $\Phi^1(\mathbf{r}; \mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,2}, \mathbf{r}^{k-1,0})$.
   - $\mathbf{r}^{k,2}$ *is the fixed point of* $\Phi^2(\mathbf{r}; \mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,3}, \mathbf{r}^{k,1}, \mathbf{r}^{k-1,0})$.
   - $\mathbf{r}^{k,3}$ *is the fixed point of* $\Phi^3(\mathbf{r}; \mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,4}, \mathbf{r}^{k,2}, \mathbf{r}^{k,1}, \mathbf{r}^{k-1,0})$.
   - ...
   - $\mathbf{r}^{k,m-1}$ *is the fixed point of* $\Phi^{m-1}(\mathbf{r}; \mathbf{r}^{k-1,m}, \mathbf{r}^{k,m-2}, \ldots, \mathbf{r}^{k,1}, \mathbf{r}^{k-1,0})$.
   - $\mathbf{r}^{k,m}$ *is the fixed point of* $\Phi^m(\mathbf{r}; \mathbf{r}^{k,m-1}, \ldots, \mathbf{r}^{k,1}, \mathbf{r}^{k-1,0})$.

   Calculate $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,1})$ and set $\mathbf{R}^k = (\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,0})$.

3. If $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$*, stop the algorithm. Else, set* $k = k+1$ *and proceed with Step 2.*

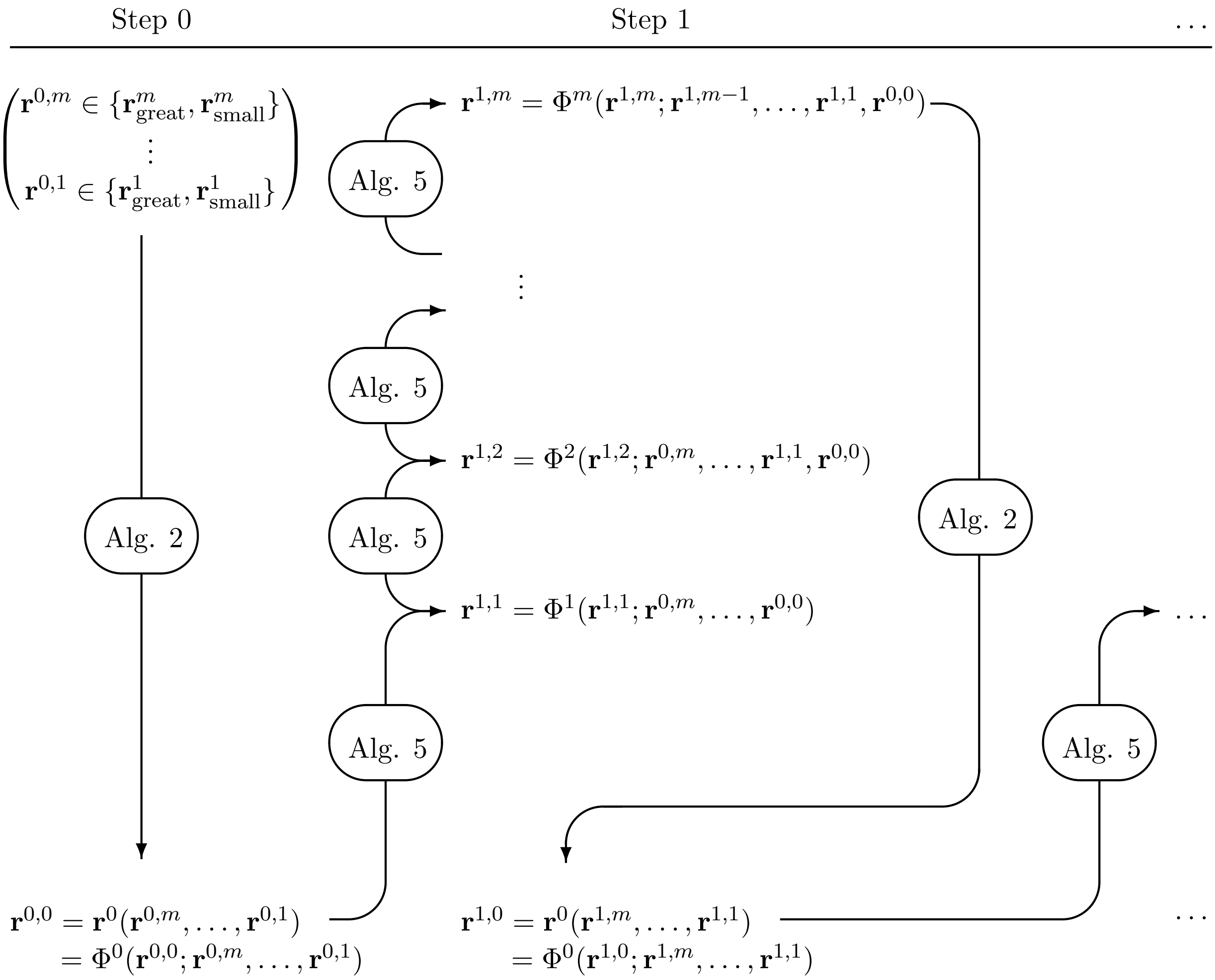


Figure 5.3: Schematic description of Algorithm 13 beginning with the initial iteration step in the left part of the figure. According to (5.20), the choice of $\mathbf{r}^{0,k} \in \{\mathbf{r}^k_{\text{great}}, \mathbf{r}^k_{\text{small}}\}$ has to be made consistent for every seniority. The arrows indicate the different calculation steps of the procedure. To obtain the next iteration in the algorithm, the corresponding auxiliary algorithm listed in the oval-shaped symbol needs to be performed.

It is clear that the Updated Hybrid Algorithm will generate a sequence of decreasing or increasing series of iterates depending on the starting vector. Moreover, this sequence converges to $\mathbf{R}^*$, using a similar argumentation as for the Hybrid Algorithm. The improvement of the Updated Hybrid Algorithm lies in the fact that its iterates are for the decreasing version smaller than or equal to the corresponding iterates of the Hybrid Algorithm. If the increasing version is considered, the iterates of the updated version are greater than or equal to those of the ordinary Hybrid Algorithm. To realize this, suppose we use the decreasing version of both algorithms and compare its iterates. Denote the iterate of the Hybrid Algorithm with the subscript H and the iterates of the updated version with UH. By construction, it holds that $\mathbf{R}^0_{\text{H}} = \mathbf{R}^0_{\text{UH}}$ and also

$\mathbf{r}_{\mathrm{H}}^{1,1} = \mathbf{r}_{\mathrm{UH}}^{1,1}$ for the first debt iterate of the lowest seniority. For the Updated Hybrid Algorithm, it follows for the debt iterate of the next seniority class that

$$\mathbf{r}_{\mathrm{UH}}^{1,2} = \Phi^2(\mathbf{r}_{\mathrm{UH}}^{1,2}; \mathbf{r}^{0,m}, \ldots, \mathbf{r}^{0,3}, \mathbf{r}_{\mathrm{UH}}^{1,1}, \mathbf{r}^{0,0}). \tag{5.21}$$

The corresponding iterate of the Hybrid Algorithm is given by

$$\mathbf{r}_{\mathrm{H}}^{1,2} = \Phi^2(\mathbf{r}_{\mathrm{H}}^{1,2}; \mathbf{r}^{0,m}, \ldots, \mathbf{r}^{0,3}, \mathbf{r}^{0,1}, \mathbf{r}^{0,0}), \tag{5.22}$$

from which it follows, because of $\mathbf{r}_{\mathrm{UH}}^{1,1} \leq \mathbf{r}^{0,1} = \mathbf{d}^1$, that $\mathbf{r}_{\mathrm{UH}}^{1,2} \leq \mathbf{r}_{\mathrm{H}}^{1,2}$. This argumentation can inductively be applied to all debt payments of higher seniority levels and for the next iteration steps as well. The case of the increasing version, is similar. Therefore, taking only the number of iteration steps into account, the Updated Hybrid Algorithm converges to $\mathbf{R}^*$ even faster than the Hybrid Algorithm. However, as Example 5.5 shows, there exists circumstances under which this effect can be almost negligible.

Remark 5.4. (i) The Updated Hybrid Algorithm started with calculating the next debt iterate of the lowest seniority level. Note that it is also possible to start with the highest seniority and determine the next debt iterate as the fixed point of Φ^m. The new iterate can then be used when the debt iterate of seniority level $m-1$ is calculated and so on. This would result in a reverse direction of the updating process. Since the iterates would still be decreasing or increasing depending on which version is used, this modification of the algorithm would still converge to $\mathbf{R}^*$ as well. In fact, any order to work through the seniorities can be chosen.

(ii) The idea processed in the Updated Hybrid Algorithm to calculate the debt iterates similar to Step 2 in Algorithm 13 can also be used for the Picard and the Elsinger Algorithm. Suppose we are in the $(k-1)$-th iteration step. The next debt iterate $\mathbf{r}^{k,j}$ for seniority level j is for these algorithms not the fixed point of Φ^j but the result of a single application of the mapping.

Example 5.5. Applying Algorithm 12 with both starting vectors to the financial system defined in Example 5.1, delivers the iterates shown in the upper part of Table 5.3. With the usual tolerance level $\varepsilon = 10^{-6}$, both the decreasing and the increasing version need 6 iteration steps. Similar to the Elsinger Algorithm in Example 5.3, 7 linear equation systems had to be solved in total for both versions until the stopping criteria is reached. Comparing the iterates with the ones of the Elsinger Algorithm in Table 5.2, we observe that they are always larger or equal for the increasing and smaller or equal for the decreasing version.

The iterates in the lower part of Table 5.3 are the ones that are generated with the Updated Hybrid Algorithm – once again for both directions. We can see that the effect of using the "updated" values for the calculation of the next debt iterates has only a slight effect on the actual iterates $\mathbf{R}^k$. This becomes also visible when counting the number of iterates and linear equation systems that have to be solved to reach the final iterate. Both for the Increasing and the Decreasing Updated Hybrid Algorithm these number are identical to their counterparts of the ordinary Hybrid Algorithm.

Table 5.3: Iterates of the Hybrid and the Updated Hybrid Algorithm for the financial system in Example 5.1.

Type		$\mathbf{r}^{0,k} = \mathbf{r}^k_{\text{small}}$ $(k = 1, 2, 3)$						$\mathbf{r}^{0,k} = \mathbf{r}^k_{\text{great}}$ $(k = 1, 2, 3)$		
		$\mathbf{R}^0$	$\mathbf{R}^1$	$\mathbf{R}^2$	…	$\mathbf{R}^*$	…	$\mathbf{R}^2$	$\mathbf{R}^1$	$\mathbf{R}^0$
Hybrid		3	3	3		3		3	3	3
		2	2	2		2		2	2	2
	$\mathbf{r}^{k,3}$	3	4	4	…	4	…	4	4	4
		2	3.5000	3.7917		3.7917		3.7917	3.9417	5
		1	1	1		1		1	1	1
		2	2	2		2		2	2	2
		2	3	3		3		3	3	3
	$\mathbf{r}^{k,2}$	0	1	1	…	1	…	1	1	1
		0	0	0		0		0	0	4
		0	3.7083	3.8750		3.8750		3.8750	3.8750	5
		3	3	3		3		3	3	3
		0	0.1229	1		1		1	1	1
	$\mathbf{r}^{k,1}$	0	0.1375	1.1001	…	1.1596	…	1.1916	3.5000	4
		0	0	0		0		0	0	2
		0	0	0		0		0	0	5
		2.1667	2.6484	2.6652		2.6659		2.6663	2.6971	2.7236
		0	0.9370	1.6062		1.6360		1.6520	2.8820	3.9431
	$\mathbf{r}^{k,0}$	0	0	0	…	0	…	0	0	3
		0	0	0		0		0	0	0
		0	0	0		0		0	0	0
Updated Hybrid		3	3	3		3		3	3	3
		2	2	2		2		2	2	2
	$\mathbf{r}^{k,3}$	3	4	4	…	4	…	4	4	4
		2	3.7917	3.7917		3.7917		3.7917	3.9417	5
		1	1	1		1		1	1	1
		2	2	2		2		2	2	2
		2	3	3		3		3	3	3
	$\mathbf{r}^{k,2}$	0	1	1	…	1	…	1	1	1
		0	0	0		0		0	0	4
		0	3.7083	3.8750		3.8750		3.8750	3.8750	5
		3	3	3		3		3	3	3
		0	0.1229	1		1		1	1	1
	$\mathbf{r}^{k,1}$	0	0.1375	1.1038	…	1.1596	…	1.1916	3.5000	4
		0	0	0		0		0	0	2
		0	0	0		0		0	0	5
		2.1667	2.6521	2.6652		2.6659		2.6663	2.6971	2.7236
		0	1.0830	1.6081		1.6360		1.6520	2.8820	3.9431
	$\mathbf{r}^{k,0}$	0	0	0	…	0	…	0	0	3
		0	0	0		0		0	0	0
		0	0	0		0		0	0	0

5.2 Finite Algorithms

A common property of the algorithms in Section 5.2 is that they ignore any information about the default structure inherent in the iterate, that means the information whether a firm is able to fully service the debt payments of a certain seniority level or not given a payment vector $\mathbf{R}^k$. In Chapter 4 we developed finite valuation algorithms based on the information that is included in the default set $\mathcal{D}(\mathbf{R}^k)$ of an iterate $\mathbf{R}^k$. While for systems with $m = 1$, there were only the two possibilities that a firm is either in default or solvent, we have to become more detail in case of two or more seniority classes.

More precisely, we have to identify for every defaulting firm $i \in \mathcal{N}$ exactly the seniority level at which the firm gets into trouble, i.e. the highest seniority class $c_i \in \{1, \dots, m\}$ for which holds that $r_i^{*,c_i} < d_i^{c_i}$, or $c_i = 0$ if there is no such class. The debt payments of the seniority classes k with $m \geq k > c_i$ are fully honored and the payments of all classes k, where $1 \leq k < c_i$, are not serviced at all. The particular case of $c_i = 0$ means that the firm is able to pay all of its obligations and therefore is solvent. We can summarize the values c_i in a tuple $\mathbf{c} = (c_1, \dots, c_n) \in \{0, \dots, m\}^n$ which is also referred to as the *default tuple* $\mathbf{c}$. Aim of the algorithms in this section, is to find the tuple $\mathbf{c}^* = (c_1^*, \dots, c_n^*)$ of seniority levels, that contains for each firm the lowest seniority for which the obligees are – at least partially – satisfied under $\mathbf{R}^*$. Before we start with some definitions, we have to make two assumptions on the financial system.

Assumption 5.6. *For $k = 0, \dots, m$, $\|\mathbf{M}^k\| < 1$, i.e. all ownership matrices in the system are strictly left substochastic matrices.*

Assumption 5.6 is necessary to ensure the invertibility of certain ownership matrices, see also the comments before Proposition 5.10.

Assumption 5.7. *For every seniority level $1 \leq k \leq m$ and every $i \in \mathcal{N}$ it follows from $d_i^k > 0$ that $d_i^h > 0$ for $1 \leq h < k$.*

Assumption 5.7 says that if a firm has positive nominal liabilities in seniority level k, it will also have positive nominal liabilities in every lower seniority. The reason for this approach is that it simplifies the argumentation in Section 5.2.1 when borderline firms are taken into account. However, this conditions is not too strict since we have shown in Section 3.2.4 (see also Remark 3.10) that a seniority structure as given in Assumption 5.7 can be assumed without loss of generality.

Definition 5.8. For a given financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$ with $m > 1$, let $\mathbf{c} = (c_1, \dots, c_n) \in \{0, \dots, m\}^n$ be an arbitrary tuple of seniority levels. The vector $\mathbf{a}^{\mathbf{c}} = (a_1^{\mathbf{c}}, \dots, a_n^{\mathbf{c}})$ is defined componentwise via

$$a_i^{\mathbf{c}} = a_i + \sum_{j=1}^{n} \sum_{k=c_j+1}^{m} M_{ij}^k d_j^k - \sum_{k=c_i+1}^{m} d_i^k. \tag{5.23}$$

Moreover, let

$$\mathbf{d}^{\mathbf{c}} = (d_1^{\mathbf{c}}, \dots, d_n^{\mathbf{c}})^t \tag{5.24}$$

be the vector in which for every firm i the nominal value of the liabilities of the c_i-th seniority level is listed. Note that for a better readability, the superscript in $d_i^{\mathbf{c}}$ is the tuple $\mathbf{c}$ but the required seniority to define $d_i^{\mathbf{c}}$ is only contained in the i-th entry c_i. In case of $c_i = 0$, set $d_i^{\mathbf{c}} := 0$. The corresponding ownership matrix $\mathbf{M}^{\mathbf{c}}$ is defined as

$$\mathbf{M}^{\mathbf{c}} = \begin{pmatrix} M_{11}^{c_1} & M_{12}^{c_2} & \dots & M_{1n}^{c_n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1}^{c_1} & M_{n2}^{c_2} & \dots & M_{nn}^{c_n} \end{pmatrix}, \tag{5.25}$$

i.e. the i-th column of $\mathbf{M}^{\mathbf{c}}$ is equal to the i-th column of the ownership matrix of seniority class c_i.

The value $a_i^{\mathbf{c}}$ can be interpreted as the sum of the exogenous and all endogenous assets of firm i reduced by the debt payments of firm i assuming secure payments based on $\mathbf{c}$ but no payments at or below the seniorities $\mathbf{c}$, including no equity payments. Note that $\mathbf{a}^{\mathbf{c}}$ is in general not positive. The expressions from Definition 5.8 are needed to defined a pseudo solution similar to Definition 4.23 for the case of $m = 1$. Before doing this, define for a given default tuple $\mathbf{c}$ and $k = 0, \dots, m$ the diagonal matrices

$$\mathbf{\Lambda}^{k,\mathbf{c}} = \operatorname{diag}(\mathbf{c} = (k, \dots, k)) \tag{5.26}$$

that identifies for each firm i whether c_i equals k.

Definition 5.9. Let $\mathbf{c} = (c_1, \dots, c_n) \in \{0, \dots, m\}^n$ be an arbitrary default tuple. The *pseudo solution* $\widehat{\mathbf{R}}^{\mathbf{c}} \in \mathbb{R}_0^{n(m+1)}$ *of* (2.9) – (2.11) *that belongs to* $\mathbf{c}$ is defined by $\widehat{\mathbf{R}}^{\mathbf{c}} = (\hat{\mathbf{r}}^{m,\mathbf{c}}, \dots, \hat{\mathbf{r}}^{0,\mathbf{c}})$, with $\hat{\mathbf{r}}^{k,\mathbf{c}} = (\hat{r}_i^{k,\mathbf{c}})_{i=1,\dots,n}$ for $1 \le k \le m$, where

$$\hat{\mathbf{r}}^{k,\mathbf{c}} = \mathbf{\Lambda}^{k,\mathbf{c}}\mathbf{x} + \left(\sum_{l=0}^{k-1} \mathbf{\Lambda}^{l,\mathbf{c}}\right)\mathbf{d}^k \tag{5.27}$$

for $1 \le k \le m$ and $\hat{\mathbf{r}}^{0,\mathbf{c}} = \mathbf{\Lambda}^{0,\mathbf{c}}\mathbf{x}$ and $\mathbf{x} \in \mathbb{R}^n$ is given by

$$\mathbf{x} = (\mathbf{I}_n - \mathbf{M}^{\mathbf{c}})^{-1}\mathbf{a}^{\mathbf{c}}. \tag{5.28}$$

The reason to define the pseudo solution as in Definition 5.9 is given in the next Proposition. Before, note that for the invertibility of the matrix $(\mathbf{I}_n - \mathbf{M}^{\mathbf{c}})$ we have to claim that Assumption 5.6 holds, since if only the Elsinger Property is valid for the ownership matrices, the inverse matrix might not exist in general, see also Section 4.2.

Proposition 5.10. *Let* $\mathbf{c}^* = (c_1^*, \dots, c_n^*)$ *be the tuple that denotes the seniority classes in which every firm first defaults under* $\mathbf{R}^*$ *or the value 0 for solvent firms under* $\mathbf{R}^*$ *and let* $\widehat{\mathbf{R}}^{\mathbf{c}^*}$ *be the corresponding pseudo solution. Then it holds that* $\widehat{\mathbf{R}}^{\mathbf{c}^*} = \mathbf{R}^*$*.*

Proof. For a default tuple $\mathbf{c}^* = (c_1^*, \dots, c_n^*)$ of a solution it holds that

$$r_i^{*,c_i^*} = a_i + \sum_{j=1}^{n} \sum_{k=c_j^*+1}^{m} M_{ij}^k d_j^k + \sum_{j=1}^{n} M_{ij}^{\mathbf{c}^*} r_j^{*,c_j^*} - \sum_{k=c_i^*+1}^{m} d_i^k \ge 0, \tag{5.29}$$

see Fischer (2015) for more details. Set $\mathbf{x}^* := \left(r_i^{*,c_i^*}\right)_{i=1,\dots,n}$ and $\mathbf{a}^{\mathbf{c}^*} := \left(a_i^{\mathbf{c}^*}\right)_{i=1,\dots,n}$, then (5.29) becomes in matrix notation to

$$\mathbf{x}^* = \mathbf{a}^{\mathbf{c}^*} + \mathbf{M}^{\mathbf{c}^*}\mathbf{x}^*. \tag{5.30}$$

Hence, $\mathbf{x}^* = (\mathbf{I}_n - \mathbf{M}^{\mathbf{c}^*})^{-1}\mathbf{a}^{\mathbf{c}^*}$. Using Equation (5.27), it follows directly that $\widehat{\mathbf{R}}^{\mathbf{c}^*} = \mathbf{R}^*$. □

Note that in Proposition 4.24 it was shown that for $m = 1$ there exists a maximum number of 2^n possible solutions for the system. If $m > 1$, Proposition 5.10 says that the maximum number of possible solutions is $(m+1)^n$.

Example 5.11. We take up the financial system that is defined in Example 5.1 and from which we know that $\mathbf{c}^* = (0, 0, 1, 3, 2)$. This leads to

$$\mathbf{a}^{\mathbf{c}^*} = \begin{pmatrix} 2.6250 \\ -1.8750 \\ 0.0833 \\ 3.7917 \\ 3.8750 \end{pmatrix} \quad \text{and} \quad \mathbf{M}^{\mathbf{c}^*} = \begin{pmatrix} 0 & 1/40 & 0 & 0 & 0 \\ 1/40 & 0 & 0.5 & 0.5 & 0.25 \\ 1/40 & 1/40 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.31}$$

The linear equation system $(\mathbf{I}_n - \mathbf{M}^{\mathbf{c}^*})\mathbf{x} = \mathbf{a}^{\mathbf{c}^*}$ is then given by

$$\begin{pmatrix} 1 & -0.025 & 0 & 0 & 0 \\ -0.025 & 1 & -0.5 & -0.5 & -0.25 \\ -0.025 & -0.025 & 1 & 0 & -0.25 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2.6250 \\ -1.8750 \\ 0.0833 \\ 3.7917 \\ 3.8750 \end{pmatrix}.$$

The solution of this system is

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^t = (2.6659, 1.6360, 1.1596, 3.7917, 3.8750)^t.$$

Following (5.27), we set $x_1 = r_1^{*,0}, x_2 = r_1^{*,0}, x_3 = r_3^{*,1}, x_4 = r_4^{*,3}$ and $x_5 = r_5^{*,2}$. Completing $\widehat{\mathbf{R}}^{\mathbf{c}^*}$ via Equation (5.27) leads to the solution $\mathbf{R}^*$ of the system.

5.2.1 Trial-and-Error Algorithms

A Trial-and-Error approach to find the default tuple $\mathbf{c}^*$ for such systems, consists of determining for every iterate $\mathbf{R}^k$ the corresponding default tuple $\mathbf{c}^k = (c_1^k, \ldots, c_n^k)$, where each entry of $\mathbf{c}^k$ is defined by

$$c_i^k = \begin{cases} m, & \text{if } r_i^{k,m} < d_i^m \\ 0, & \text{if } r_i^{k,0} > 0 \text{ or } d_i^m = \ldots = d_i^1 = 0 \\ \max\{1 \leq h < m : r_i^{k,h} < d_i^h \text{ and } r_i^{k,h+1} = d_i^{h+1}\}, & \text{else.} \end{cases} \tag{5.32}$$

The value c_i^k is in case of a firm in default the highest seniority level in which the firm gets into financial difficulties. If the firm is solvent under $\mathbf{R}^k$, c_i^k is set to 0. Depending on the choice of the lag value $l \geq 2$, one would calculate iterates $\mathbf{R}^k$ and corresponding tuples $\mathbf{c}^k$ until the stopping criteria $\mathbf{c}^k = \mathbf{c}^{k+1} = \ldots = \mathbf{c}^{k+l-1}$ is reached. To test whether $\mathbf{c}^k = \mathbf{c}^*$, we would calculate the pseudo solution $\widehat{\mathbf{R}}^{\mathbf{c}^k}$ that belongs to $\mathbf{c}^k$ as a (possible) solution of the system $\mathcal{F}$. In case of $\Phi(\widehat{\mathbf{R}}^{\mathbf{c}^k}) = \widehat{\mathbf{R}}^{\mathbf{c}^k}$, the algorithm stops.

Algorithm 14 (Decreasing Trial-and-Error Algorithm $(m > 1)$)**.** *Set* $l \geq 2$, $p = 0$ *and* $\mathbf{c} = (-1, \ldots, -1)$.

1. *Choose either the Picard (Algorithm 1), the Elsinger (Algorithm 11) or the Hybrid Algorithm (Algorithm 12) which is used in the following to generate the next iterate.*

2. *If in Step 1 the Picard Algorithm is chosen, set* $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$*, if the Elsinger or the Hybrid Algorithm is chosen, set* $\mathbf{R}^0 = (\mathbf{d}^m, \ldots, \mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1))$*. In all cases, calculate* $\mathbf{c}^0$ *according to* (5.32).
3. *If the Elsinger or the Hybrid Algorithm is chosen in Step 1 and* $\mathcal{D}(\mathbf{R}^0) = \emptyset$*, set* $\mathbf{R}^* = \mathbf{R}^0$ *and stop the algorithm.*
4. *Else, calculate for* $k > p$ *the iterates* $\mathbf{R}^k$ *starting with* $\mathbf{R}^p$ *using the algorithm chosen in Step 1 and calculate the corresponding tuples* $\mathbf{c}^k$ *defined in* (5.32) *until* $k = q$ *with*

$$q = \min\{m > p : \mathbf{c}^{m-l+1} = \ldots = \mathbf{c}^m \text{ and } \mathbf{c}^m \neq \mathbf{c}\} \tag{5.33}$$

   is reached. Determine the pseudo solution $\widehat{\mathbf{R}}^{\mathbf{c}^q}$ *that belongs to* $\mathbf{c}^q$*.*
5. *If* $\Phi(\widehat{\mathbf{R}}^{\mathbf{c}^q}) = \widehat{\mathbf{R}}^{\mathbf{c}^q}$*, stop the algorithm. Else, set* $\mathbf{c} = \mathbf{c}^q$ *and* $p = q$ *and proceed with Step 4.*

The Decreasing Trial-and-Error Algorithm obviously generates a series of increasing tuples $\mathbf{c}^k$. In Algorithm 7, the series of default sets $\mathcal{D}(\mathbf{R}^k)$ was increasing as well. Note that the particular case $\mathcal{D}(\mathbf{R}^0) = \emptyset$ in Step 3 of the Algorithm is equivalent to

$$\mathbf{a} + \sum_{k=1}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{d}^k + \mathbf{M}^0 \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1) \geq \mathbf{0}_n \tag{5.34}$$

from which directly follows that $\mathbf{R}^* = (\mathbf{d}^m, \ldots, \mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1))$ using a similar argumentation than in the proof of Proposition 4.26 (i). Check that the fact that $\mathcal{D}(\mathbf{R}^0) = \emptyset$ is not a sufficient condition for $\mathbf{R}^* = (\mathbf{d}^m, \ldots, \mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1))$ if $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. Note that $\mathcal{D}(\mathbf{R}^0) = \mathcal{N}$ is no suitable stopping criteria. This is because in this situation we only know that $\mathbf{r}^{*,0} = \mathbf{0}_n$ but we get no detailed information in which seniority level the firm exactly defaults.

Similar to the Increasing Trial-and-Error Algorithm for $m = 1$ (Algorithm 7), it is of course possible to obtain an increasing series of iterates and a decreasing series of tuples $\mathbf{c}^k$ when the initial iterate is the minimum possible starting vector.

Algorithm 15 (Increasing Trial-and-Error Algorithm ($m > 1$))**.** *Set* $l \geq 2$*,* $p = 0$ *and* $\mathbf{c} = (-1, \ldots, -1)$*.*

1. *Choose either the Picard (Algorithm 1), the Elsinger (Algorithm 11) or the Hybrid Algorithm (Algorithm 12) which is used in the following to generate the next iterate.*
2. *If in Step 1 the Picard Algorithm is chosen, set* $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$*, if the Elsinger or the Hybrid Algorithm is chosen, set* $\mathbf{R}^0 = (\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}}))$*. In all cases, calculate* $\mathbf{c}^0$ *according to* (5.32).
3. *If* $\mathcal{D}(\mathbf{R}^0) = \emptyset$*, set* $\mathbf{R}^* = (\mathbf{d}^m, \ldots, \mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1))$ *and stop the algorithm.*
4. *Else, calculate for* $k > p$ *the iterates* $\mathbf{R}^k$ *starting with* $\mathbf{R}^p$ *using the algorithm chosen in Step 1 and calculate the corresponding tuples* $\mathbf{c}^k$ *defined in* (5.32) *until* $k = q$ *with*

$$q = \min\{m > p : \mathbf{c}^{m-l+1} = \ldots = \mathbf{c}^m \text{ and } \mathbf{c}^m \neq \mathbf{c}\} \tag{5.35}$$

   is reached. Determine the pseudo solution $\widehat{\mathbf{R}}^{\mathbf{c}^q}$ *that belongs to* $\mathbf{c}^q$*.*
5. *If* $\Phi(\widehat{\mathbf{R}}^{\mathbf{c}^q}) = \widehat{\mathbf{R}}^{\mathbf{c}^q}$*, stop the algorithm. Else, set* $\mathbf{c} = \mathbf{c}^q$ *and* $p = q$ *and proceed with Step 4.*

In analogy to the comments in Section 4.2.2, we also have to address the issue of borderline firms. Recall that for $m = 1$, borderline firms are defined as firms for which holds in the fixed point $\mathbf{R}^*$ that $r_i^{*,1} = d_i$ and $r_i^{*,0} = 0$. Such firms are able to fully cover their liabilities but have no remaining capital for shareholders. By definition (cf. Equation (5.5)), these firms were not considered as being in default in $\mathbf{R}^*$. The problem for borderline firms was that the Increasing Trial-and-Error Algorithm does not manage to reach the actual default set $\mathcal{D}^*$ in general. We showed in Lemma 4.32 that the algorithm still works in such situations and actually finds the solution of the system. For the purpose of showing this for Algorithm 15 too, we introduce the notation that a firm $i \in \mathcal{N}$ is *borderline in seniority level* k ($k = 1, \dots, m$), if $d_i^k > 0$,

$$r_i^{*,k} = d_i^k \quad \text{and} \quad r_i^{*,h} = 0 \quad \text{for all } 0 \leq h < k. \tag{5.36}$$

Note that for $m > 1$ a firm can be borderline and still be in default, if $k \geq 2$ in (5.36). Following the definition of $\mathbf{c}^*$ at the beginning of Section 5.2 and because of Assumption 5.7, it must hold for a borderline firm in seniority level k that $c_i^* = k - 1$, since the debt payments in seniority class k are fully covered and level $k - 1$ is the first level where the firm gets into trouble. If $c_i^* = 0$, the firm is solvent by definition but $r_i^{*,0} = 0$.

Since Algorithm 14 generates an increasing series of default tuples $\mathbf{c}^k$, the tuple $\mathbf{c}^*$ will be reached at some stage of the procedure using a similar argument as in the proof of Proposition 4.28, for the case of $m = 1$. For Algorithm 15, however, we obtain a decreasing series of default tuples that might not reach $\mathbf{c}^*$ if borderline firms are present. To show that this is no problem and that the algorithm still works, assume that there is a set $\mathcal{S} \subset \mathcal{N}$ of borderline firms, where it does not matter in which seniority level exactly the firms are borderline. The Increasing Trial-and-Error Algorithm will finally reach to a default tuple $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_n)$ for which holds that

$$\bar{c}_i = c_i^* + 1 \text{ for all } i \in \mathcal{B} \text{ and } \bar{c}_i = c_i^* \text{ for all } i \notin \mathcal{B} \text{ for some } \mathcal{B} \subset \mathcal{S}. \tag{5.37}$$

We need to show that the pseudo solution $\widehat{\mathbf{R}}^{\bar{\mathbf{c}}}$ that belongs to $\bar{\mathbf{c}}$ is equal to $\mathbf{R}^*$. To this end, set $\mathbf{x}^* := (\mathbf{I}_n - \mathbf{M}^{\mathbf{c}^*})^{-1}\mathbf{a}^{\mathbf{c}^*}$ and $\tilde{\mathbf{x}} := \mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}}$, where $\mathbf{\Lambda}_{\mathcal{B}}$ denotes the diagonal matrix with value 1 for all borderline firms in $\mathcal{B}$. Check that

$$\mathbf{a}^{\mathbf{c}^*} = \mathbf{a}^{\bar{\mathbf{c}}} + \mathbf{M}^{\bar{\mathbf{c}}}\mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} - \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} = \mathbf{a}^{\bar{\mathbf{c}}} + (\mathbf{M}^{\bar{\mathbf{c}}} - \mathbf{I}_n)\mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} \tag{5.38}$$

and that $\mathbf{M}^{\mathbf{c}^*}\mathbf{x}^* = \mathbf{M}^{\bar{\mathbf{c}}}\mathbf{x}^*$, since $x_i^* = 0$ for all $i \in \mathcal{B}$. This yields

$$\mathbf{M}^{\mathbf{c}^*}\mathbf{x}^* + \mathbf{M}^{\bar{\mathbf{c}}}\mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} = \mathbf{M}^{\bar{\mathbf{c}}}(\mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}}), \tag{5.39}$$

which implies the following equivalences

$$\begin{aligned}
\mathbf{x}^* = (\mathbf{I}_n - \mathbf{M}^{\mathbf{c}^*})^{-1}\mathbf{a}^{\mathbf{c}^*} &\iff (\mathbf{I}_n - \mathbf{M}^{\mathbf{c}^*})\mathbf{x}^* = \mathbf{a}^{\mathbf{c}^*} \\
&\iff \mathbf{x}^* - \mathbf{M}^{\mathbf{c}^*}\mathbf{x}^* = \mathbf{a}^{\bar{\mathbf{c}}} + (\mathbf{M}^{\bar{\mathbf{c}}} - \mathbf{I}_n)\mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} \\
&\iff \mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} - (\mathbf{M}^{\mathbf{c}^*}\mathbf{x}^* + \mathbf{M}^{\bar{\mathbf{c}}}\mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}}) = \mathbf{a}^{\bar{\mathbf{c}}} \\
&\iff \mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}} - (\mathbf{M}^{\bar{\mathbf{c}}}(\mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}})) = \mathbf{a}^{\bar{\mathbf{c}}} \\
&\iff (\mathbf{I}_n - \mathbf{M}^{\bar{\mathbf{c}}})(\mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}}) = \mathbf{a}^{\bar{\mathbf{c}}} \\
&\iff (\mathbf{I}_n - \mathbf{M}^{\bar{\mathbf{c}}})\tilde{\mathbf{x}} = \mathbf{a}^{\bar{\mathbf{c}}}.
\end{aligned} \tag{5.40}$$

As a consequence, the pseudo solutions that belong to $\mathbf{c}^*$ and $\bar{\mathbf{c}}$ can for $1 \leq k \leq m$ be represented by

$$\hat{\mathbf{r}}^{k,\mathbf{c}^*} = \mathbf{\Lambda}^{k,\mathbf{c}^*}\mathbf{x}^* + \left(\sum_{l=0}^{k-1} \mathbf{\Lambda}^{l,\mathbf{c}^*}\right)\mathbf{d}^k \tag{5.41}$$

and

$$\hat{\mathbf{r}}^{k,\bar{\mathbf{c}}} = \mathbf{\Lambda}^{k,\bar{\mathbf{c}}}(\mathbf{x}^* + \mathbf{\Lambda}_{\mathcal{B}}\mathbf{d}^{\bar{\mathbf{c}}}) + \left(\sum_{l=0}^{k-1} \mathbf{\Lambda}^{l,\bar{\mathbf{c}}}\right)\mathbf{d}^k. \tag{5.42}$$

In (5.41), the i-th component is for $1 \le k \le m$ given by

$$\hat{r}_i^{k,\mathbf{c}^*} = \begin{cases} d_i^k, & \text{if } k > c_i^*, \\ x_i^*, & \text{if } k = c_i^*, \\ 0, & \text{if } k < c_i^*, \end{cases} \tag{5.43}$$

and in (5.42), the i-th component has for $1 \le k \le m$ the following form

$$\hat{r}_i^{k,\bar{\mathbf{c}}} = \begin{cases} d_i^k, & \text{if } k > \bar{c}_i, \\ x_i^*, & \text{if } k = \bar{c}_i \text{ and } i \notin \mathcal{B}, \\ x_i^* + d_i^k, & \text{if } k = \bar{c}_i \text{ and } i \in \mathcal{B}, \\ 0, & \text{if } k < \bar{c}_i. \end{cases} \tag{5.44}$$

For firms i with $i \notin \mathcal{B}$, it holds that $c_i^* = \bar{c}_i$ and therefore $\hat{r}_i^{k,\bar{\mathbf{c}}} = \hat{r}_i^{k,\mathbf{c}^*}$ for all $1 \le k \le m$ using (5.43) and (5.44). Let now $i \in \mathcal{B}$, hence $\bar{c}_i = c_i^* + 1$ (cf. Assumption 5.7) and $x_i^* = 0$. To show that also in this case, the pseudo solutions of $\bar{\mathbf{c}}$ and $\mathbf{c}^*$ are identical, we distinct between four cases and apply the Equations (5.43) and (5.44) each time.

(i) Let $k < c_i^*$ from which follows that $k < \bar{c}_i$ which leads to $\hat{r}_i^{k,\mathbf{c}^*} = \hat{r}_i^{k,\bar{\mathbf{c}}} = 0$.

(ii) Let $k = c_i^*$ from which follows that $\bar{c}_i = k + 1 > k$. Then $\hat{r}_i^{k,\mathbf{c}^*} = x_i^* = 0 = \hat{r}_i^{k,\bar{\mathbf{c}}}$.

(iii) Let $k - 1 = c_i^*$ from which follows that $\bar{c}_i = k$. This yields $\hat{r}_i^{k,\mathbf{c}^*} = d_i^k = x_i^* + d_i^k = \hat{r}_i^{k,\bar{\mathbf{c}}}$.

(iv) Finally, let $k - 1 > c_i^*$ and thus $\bar{c}_i < k$. Then $\hat{r}_i^{k,\mathbf{c}^*} = d_i^k = \hat{r}_i^{k,\bar{\mathbf{c}}}$.

Hence, $\hat{r}_i^{k,\bar{\mathbf{c}}} = \hat{r}_i^{k,\mathbf{c}^*}$ for all $i \in \mathcal{N}$ and all $1 \le k \le m$.

This solves the problem of the Increasing Trial-and-Error Algorithm in the presence of borderline firms. The default tuple $\bar{\mathbf{c}}$ will also lead to the fixed point $\mathbf{R}^*$. The insights from above are summarized in the following Proposition.

Proposition 5.12. *The Decreasing Trial-and-Error Algorithm (Algorithm 14) generates a series of decreasing iterates* $\mathbf{R}^k$ *and a series of increasing default tuples* $\mathbf{c}^k$*, whereas the iterates of the Increasing Trial-and-Error Algorithm (Algorithm 15) form an increasing sequence and the corresponding default tuples a decreasing sequence. Both default tuples of the algorithms converge in a finite number of iteration steps to a final tuple whose corresponding pseudo solution is the solution* $\mathbf{R}^*$ *of the financial system.*

Following the notation in Section 4.2, we denote the Trial-and-Error Algorithms depending on the choice of the algorithm to obtain the iterates, i.e. the procedures are called *Decreasing Trial-and-Error Picard/Elsinger/Hybrid Algorithm* and analogously for the increasing version.

5.2.2 Sandwich Algorithm

To complete the generalization of the finite algorithms from Section 4.2, we have to extend the Sandwich Algorithm to $m > 1$ as well. Starting with both, the minimum and the maximum possible debt payment vector, determining the corresponding default tuple $\mathbf{c}^k$ each time and stopping the procedure when the tuples for both directions are identical, leads to the Sandwich Algorithm for $m > 1$. The notation $\overline{\mathbf{R}}^k$ and $\underline{\mathbf{R}}^k$ is based on the one from Algorithm 9.

Algorithm 16 (Sandwich Algorithm ($m > 1$))**.**

1. *Determine* $\overline{\mathbf{R}}^0$ *and* $\underline{\mathbf{R}}^0$ *as well as their corresponding default tuples* $\overline{\mathbf{c}}^0$ *and* $\underline{\mathbf{c}}^0$ *following* (5.32)*.*
2. *For* $k \geq 1$*, calculate the iterates* $\overline{\mathbf{R}}^k$ *and* $\underline{\mathbf{R}}^k$ *using one of the Algorithms 1, 11 or 12 and the corresponding default tuples* $\overline{\mathbf{c}}^k$ *and* $\underline{\mathbf{c}}^k$*.*
3. *If* $\overline{\mathbf{c}}^k = \underline{\mathbf{c}}^k$*, stop the algorithm and calculate the pseudo solution* $\widehat{\mathbf{R}}^{\overline{\mathbf{c}}^k}$ *that belongs to* $\overline{\mathbf{c}}^k$*. Else, set* $k = k + 1$ *and go back to Step 2.*

The procedure is self-explaining and it is also clear that the tuples $\overline{\mathbf{c}}^k$ and $\underline{\mathbf{c}}^k$ approach one another and will approach to $\mathbf{c}^*$ from two sides. If in Step 2 the Picard/Elsinger/Hybrid Algorithm is chosen for the iteration, we call the algorithm the *Sandwich Picard/Elsinger/Hybrid Algorithm*. Note that if there are borderline firms present in the financial system, the convergence of the default tuples can not be ensured. Assuming that the vector $\mathbf{a}$ follows some probability distribution which has a density with respect to the Lebesgue measure on $(\mathbb{R}_0^+)^n$, we can at least show an almost sure convergence. The proof of the following proposition is the direct extension of the proof of Proposition 4.36 for systems with $m = 1$.

Proposition 5.13. *The Sandwich Algorithm generates a sequence of increasing default tuples* $\overline{\mathbf{c}}^k$ *and a sequence of decreasing default tuples* $\underline{\mathbf{c}}^k$ *that reach* $\mathbf{c}^*$ *almost surely after finitely many steps. Thus, it reaches the solution* $\mathbf{R}^*$ *almost surely after finitely many iteration steps.*

To avoid situations in which the default tuples in Algorithm 16 do not converge, a modified version of the Sandwich Algorithm should be chosen for practical purposes. Algorithm 10 describes the principle for $m = 1$, an extension to systems with $m > 1$ is straightforward.

Example 5.14. Once again, we make use of the financial system in Example 5.1 to demonstrate the functioning of the Trial-and-Error and the Sandwich Algorithms. The Decreasing Trial-and-Error-Picard Algorithm leads to the following default tuples:

$$\mathbf{c}^0 = (0,0,0,0,0), \quad \mathbf{c}^1 = (0,0,0,3,2), \quad \mathbf{c}^2 = (0,0,1,3,2), \quad \mathbf{c}^3 = (0,0,1,3,2). \tag{5.45}$$

Hence, after two iteration steps, the target $\mathbf{c}^* = (0,0,1,3,2)$ is reached. No matter which lag value l is chosen, the first time the iteration stops and $\mathbf{c}^q$ with q defined as in (5.35) is calculated, it holds that $\widehat{\mathbf{R}}^{\mathbf{c}^q} = \mathbf{R}^*$. Using the increasing version of the algorithm, we obtain the default tuples

$$\mathbf{c}^0 = (0,2,3,3,2), \quad \mathbf{c}^1 = (0,1,1,3,2), \quad \mathbf{c}^2 = (0,0,1,3,2), \quad \mathbf{c}^3 = (0,0,1,3,2), \tag{5.46}$$

hence the same statements from above also holds for the increasing version. Clearly, the Sandwich Picard Algorithm would stop after iteration step 2. For the Elsinger versions of the algorithms above, the default tuples are for both directions of the algorithm identical as for the Picard Algorithm. Using the Increasing Trial-and-Error Hybrid Algorithm leads to the same default tuples as in (5.46). For the decreasing version of the Hybrid Algorithm, we get

$$\mathbf{c}^0 = (0,0,0,0,0), \quad \mathbf{c}^1 = (0,0,1,3,2), \quad \mathbf{c}^2 = (0,0,1,3,2). \tag{5.47}$$

The Decreasing Trial-and-Error Algorithm reaches $\mathbf{c}^*$ therefore already after only one iteration step.

A disadvantage of any Trial-and-Error Algorithm is the fact that the algorithm can also stop erroneously and deliver a set of tuples $\mathbf{c}^k$ that is unequal to $\mathbf{c}^*$. In such cases, the pseudo solution $\widehat{\mathbf{R}}^{\mathbf{c}^k}$ is determined unnecessarily. The drawback of a Sandwich approach is that the computational effort is doubled since both directions are considered which results in an unnecessary use of computer capacity. It could be more efficient if a procedure would deliver the final tuple $\mathbf{c}^*$ without a Trial-and-Error aspect, i.e. that there is a clear stopping criteria that indicates that $\mathbf{c}^*$ is actually reached. Moreover this should be achieved without double computational work, i.e. $\mathbf{c}^*$ should be reached from only one direction and not from two as done in the Sandwich Algorithm. The algorithm in the next section is designed to overcome this problem by approaching $\mathbf{c}^*$ from only one direction and, additionally, will come up with a clearly defined stopping criterion when $\mathbf{c}^*$ is actually found.

5.3 Default Structure Algorithm

This section presents a procedure that does not focus on generating new iterates $\mathbf{R}^k$, but focuses on calculating new tuples $\mathbf{c}^k$ of seniority levels in every iteration step. Beginning with an initial tuple $\mathbf{c}^0 = (c_1^0, \dots, c_n^0)$ of seniority levels, $\mathbf{c}^0$ is "updated" in a stepwise approach until a final tuple is reached. The k-th iteration step of the algorithm therefore entails the calculation steps to get from $\mathbf{c}^k$ to $\mathbf{c}^{k+1}$. This is achieved by solving a financial subsystem with one seniority, which is a modification of the initial financial system $\mathcal{F}$ with $m > 1$. We will see in the following that the procedure results in an increasing series of tuples. Unlike the algorithms in Section 5.2.1, this new algorithm will make sure that the default tuple will definitely change from one iteration step to another. If it does not, the final default tuple is reached. A consequence of this property is that the procedure will reach a solution of the system $\mathcal{F}$ in a finite number of iteration steps as will be shown below in Propositions 5.15.

The idea of this approach is initially mentioned in the article of Elsinger (2009), where the principle of the algorithm is demonstrated and a short sketch of the proof that it actually works is given. We work out the ideas of Elsinger and give a detailed overview of the procedure's functioning and also present a detailed proof that the algorithm will find $\mathbf{R}^*$ in a finite number of iteration steps. Moreover, we extend the procedure and add an improved version of the Algorithm at the end of the section. Note that the Assumptions 5.6 and 5.7 from Section 5.2 are required to hold in this section as well.

Before explaining the algorithm in more detail, we have to introduce some new notation. For given $\mathbf{c} \in \{1, \dots, m\}^n$, the vectors $\mathbf{a}^{\mathbf{c}}$ and $\mathbf{d}^{\mathbf{c}}$ form together with $\mathbf{M}^{\mathbf{c}}$ from Definition 5.8 and the equity ownership matrix $\mathbf{M}^0$ a new financial system denoted as $\mathcal{F}^{\mathbf{c}} = (\mathbf{a}^{\mathbf{c}}, \mathbf{M}^{\mathbf{c}}, \mathbf{M}^0, \mathbf{d}^{\mathbf{c}})$ with $m = 1$. The solution of $\mathcal{F}^{\mathbf{c}}$ is the fixed point of the mapping

$$\Phi \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}^{\mathbf{c}}, (\mathbf{a}^{\mathbf{c}} + \mathbf{M}^{\mathbf{c}}\mathbf{r} + \mathbf{M}^0\mathbf{s})^+\} \\ (\mathbf{a}^{\mathbf{c}} + \mathbf{M}^{\mathbf{c}}\mathbf{r} + \mathbf{M}^0\mathbf{s} - \mathbf{d}^{\mathbf{c}})^+ \end{pmatrix}, \tag{5.48}$$

where we modified the debt component as in (2.16) to ensure non-negative recovery values. Though $\mathbf{a}^{\mathbf{c}}$ is not assumed to be positive for every firm anymore, the fixed point of the mapping in (5.48) is still unique (cf. Remark 2.8). Denote this fixed-point by

$$\begin{pmatrix} \mathbf{r}^{\mathbf{c}} \\ \mathbf{s}^{\mathbf{c}} \end{pmatrix} = \begin{pmatrix} (r_1^{\mathbf{c}}, \dots, r_n^{\mathbf{c}})^t \\ (s_1^{\mathbf{c}}, \dots, s_n^{\mathbf{c}})^t \end{pmatrix} = \Phi \begin{pmatrix} \mathbf{r}^{\mathbf{c}} \\ \mathbf{s}^{\mathbf{c}} \end{pmatrix}. \tag{5.49}$$

The vector $\mathbf{r}^{\mathbf{c}}$ contains the residual values of the seniority levels given in $\mathbf{c}$ assuming that for each firm, seniorities higher than c_i are fully delivered and seniorities lower than c_i are not delivered at all. $\mathbf{s^c}$ contains the corresponding equity values of the firms. A slight difference in the structure of the default tuple is now that the value zero is excluded, i.e. $\mathbf{c} \in \{1,\dots,m\}^n$. If for a firm $i \in \mathcal{N}$ it holds that $c_i = 1$, the firm can then be in default in the first seniority class or be solvent. The reason for this modification is that we do not need to distinguish between the fact that a firm defaults in the lowest seniority or is solvent since this information will be included in the fixed point $(\mathbf{r^c}, \mathbf{s^c})$ of $\mathcal{F}^{\mathbf{c}}$. This means that a firm with $c_i = 1$ is solvent if $r_i^{\mathbf{c}} = d_i^{\mathbf{c}}$ and hence $s_i^{\mathbf{c}} \geq 0$ or is in default in seniority level 1, if $r_i^{\mathbf{c}} < d_i^{\mathbf{c}}$ and $s_i^{\mathbf{c}} = 0$ following from it.

The fixed point of $\mathcal{F}^{\mathbf{c}}$ is still of dimension $2n$. However, we are interested in finding the fixed point $\mathbf{R}^*$ of the system $\mathcal{F}$ that is of dimension $n(m+1)$. To extract it from the fixed point $(\mathbf{r^c}, \mathbf{s^c})$ of $\mathcal{F}^{\mathbf{c}}$ to the system $\mathcal{F}$, we set

$$\mathbf{R^c} = (\mathbf{r}^{m,\mathbf{c}}, \mathbf{r}^{m-1,\mathbf{c}}, \dots, \mathbf{r}^{0,\mathbf{c}}) \in (\mathbb{R}_0^+)^{n(m+1)}, \tag{5.50}$$

and define $\mathbf{R^c}$ for every firm $i \in \mathcal{N}$ componentwise via

$$r_i^{k,\mathbf{c}} = \begin{cases} d_i^k, & \text{for } k > c_i, \\ r_i^{\mathbf{c}}, & \text{for } k = c_i, \\ 0, & \text{for } 0 < k < c_i, \\ s_i^{\mathbf{c}}, & \text{for } k = 0. \end{cases} \tag{5.51}$$

Additionally, define the matrix $\underline{\mathbf{\Lambda}}^k$ by

$$\underline{\mathbf{\Lambda}}^k = \underline{\mathbf{\Lambda}}^k(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k}) = \operatorname{diag}\left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k} < \mathbf{0}_n\right), \tag{5.52}$$

where $\mathbf{c}^k$ is an arbitrary tuple with $\mathbf{c}^k \in \{1,\dots,m\}^n$ and $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$ is the fixed point of the mapping in (5.48). The matrix $\underline{\mathbf{\Lambda}}^k$ identifies the firms i that are in default in a higher seniority level than c_i given payments as in (5.51) under $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$. For a firm i that is in default in seniority level c_i, it will hold because of (5.48) that $\underline{\mathbf{\Lambda}}_{ii}^k = 0$. Moreover, for firms with $r_i^{\mathbf{c}^k} = d_i^{\mathbf{c}^k}$, i.e. firms that are able to fully satisfy their obligees in seniority level c_i, it also holds that $\underline{\mathbf{\Lambda}}_{ii}^k = 0$. With the definitions from above in mind, we are now able to formulate the algorithm that is able to find $\mathbf{R}^*$.

Algorithm 17 (Default Structure Algorithm)**.** *Denote the initial set of seniority classes by* $\mathbf{c}^0 = (c_1^0, \dots, c_n^0)$*, where*

$$c_i^0 = \begin{cases} 1, & \text{if } d_i^k = 0 \text{ for all } k = 1,\dots,m \\ \min\left\{k \in \{1,\dots,m\} : d_i^k > 0\right\}, & \text{else.} \end{cases} \tag{5.53}$$

1. *For* $k \geq 0$*, specify the financial system* $\mathcal{F}^{\mathbf{c}^k} = (\mathbf{a}^{\mathbf{c}^k}, \mathbf{M}^{\mathbf{c}^k}, \mathbf{M}^0, \mathbf{d}^{\mathbf{c}^k})$ *and determine its (unique) solution using one of the algorithms defined in Section 4.2.*
2. *Denote the solution of the system by* $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$ *and calculate the matrix* $\underline{\mathbf{\Lambda}}^k$ *following* (5.52).
3. *If* $\underline{\mathbf{\Lambda}}^k = \mathbf{0}_{n\times n}$*, determine* $\mathbf{R}^{\mathbf{c}^k}$ *according to* (5.51) *and stop the algorithm. Else, update the set of seniority classes by calculating*

$$\mathbf{c}^{k+1} = \mathbf{c}^k + \mathbf{1}_n^t \underline{\mathbf{\Lambda}}^k, \tag{5.54}$$

set $k = k+1$ *and proceed with Step 1.*

In Step 1 of Algorithm 17, the solution of a financial system $\mathcal{F}^{\mathbf{c}^k}$ has to be determined. Under Assumption 5.6, this is a system with $m = 1$ that has a unique fixed point $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$. The fact that the asset vector $\mathbf{a}^{\mathbf{c}^k}$ can contain negative entries has no influence on the uniqueness of the solution of $\mathcal{F}^{\mathbf{c}^k}$, see also Remark 2.8. When actually determining the solution $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$, the Trial-and-Error and the Sandwich Algorithms from Section 4.2 should not be used since negative entries in $\mathbf{a}^{\mathbf{c}^k}$ prevents the usage of the mentioned procedures, see also the comments in Remark 4.38. Instead, we recommend either the Picard, the Elsinger or the Hybrid Algorithm presented in Section 4.2 since they can handle with negative assets values as noted in the Remarks 4.3, 4.10 and 4.18. Both versions, i.e. the decreasing and the increasing version of the algorithms can be applied to find $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$.

Algorithm 17 starts with the tuple $\mathbf{c}^0$ that contains the lowest seniority level in which each firm has non-negative liabilities. Assuming that every firm is able to fully deliver its obligees for higher seniorities than the one in $\mathbf{c}^0$, the solution of $\mathcal{F}^{\mathbf{c}^0}$ is determined. Consider the firms $i \in \mathcal{N}$ with $\underline{\mathbf{\Lambda}}^0_{ii} = 1$. Clearly, these firms are in default under the solution $\mathbf{R}^*$. Moreover, these firms are not able to cover any debt payments of their lowest seniority class. As a consequence, first, the intersystem debt payments stemming from this seniority to the other firms, that are contained in $\mathbf{a}^{\mathbf{c}^0}$, have to be erased in the next asset vector $\mathbf{a}^{\mathbf{c}^1}$. Second, the next lowest seniority debt payments that are still subtracted in $\mathbf{a}^{\mathbf{c}^0}$, have to be ignored in $\mathbf{a}^{\mathbf{c}^1}$ and turn into the next liability values in $\mathbf{d}^{\mathbf{c}^1}$. This results in a new financial system $\mathcal{F}^{\mathbf{c}^1}$ whose solution is calculated now. It might happen now that (i) firms with $\underline{\mathbf{\Lambda}}^0_{ii} = 1$ are also not able to pay off any of their debt in the next lowest seniority class or that (ii) firms with $\underline{\mathbf{\Lambda}}^0_{ii} = 0$ receive lower intersystem payments due to the defaults in the former iteration step and now also are not able to repay any debt payments of their seniority class in $\mathbf{c}^1$. If one of these two cases (or both) are present, it will hold that $\underline{\mathbf{\Lambda}}^1_{ii} = 1$ for some $i \in \mathcal{N}$. The procedure of adapting the asset values to $\mathbf{a}^{\mathbf{c}^2}$ and the liabilities to $\mathbf{d}^{\mathbf{c}^2}$ then continues.

These steps are repeated until $\underline{\mathbf{\Lambda}}^k = \mathbf{0}_{n\times n}$ for some $k \geq 0$. That means that for every firm, exactly the seniority level in which the corresponding firm will default is identified and contained in $\mathbf{c}^k$. If $c^k_i = 1$, this means that the firm will either be solvent, if $r^{\mathbf{c}^k}_i = d^1_i$, or that the firm is in default in the first seniority level, if $r^{\mathbf{c}^k}_i < d^1_i$. For all firms i with $c^k_i > 1$, it will hold by construction of the algorithm that $s^{\mathbf{c}^k}_i = 0$, as we will show in the next Proposition. Due to the fact that Algorithm 17 does not calculate new iterates $\mathbf{R}^k$, but rather takes into account which firm defaults in which seniority class, we call this algorithm *Default Structure Algorithm.*

Proposition 5.15. *(i) Algorithm 17 generates a series of increasing tuples* $\mathbf{c}^k$*. Further, the algorithm is well-defined in the sense that once a firm is identified to be in default at a certain seniority level, it will remain being in default at that level in all further iteration steps.*

(ii) Let $l \in \mathbb{N}_0$ *such that*

$$l = \min\{k \in \mathbb{N}_0 : \underline{\mathbf{\Lambda}}^k = \mathbf{0}_{n\times n}\}. \tag{5.55}$$

Then it holds for $\mathbf{R}^{\mathbf{c}^l}$ *defined as in* (5.51) *that* $\Phi(\mathbf{R}^{\mathbf{c}^l}) = \mathbf{R}^{\mathbf{c}^l}$*, i.e.* $\mathbf{R}^{\mathbf{c}^l} = \mathbf{R}^*$*.*

(iii) The tuple $\mathbf{c}^l$ *is reached after no more than* $n(m-1)$ *iteration steps.*

Proof. (i) The fact that the tuple $\mathbf{c}^k$ increases follows immediately from (5.54). Assume that we are in the k-th iteration step ($k \geq 0$) and that we have determined the solution $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$ of $\mathcal{F}^{\mathbf{c}^k}$. To show that the algorithm is well-defined, we have to prove that for firms with $\underline{\mathbf{\Lambda}}^k_{ii} = 1$, it follows that $r^{\mathbf{c}^{k+1}}_i < d^{\mathbf{c}^{k+1}}_i$, i.e. firms that are identified to be in default in a seniority level higher that c_i^k, have to stay in default under the next financial subsystem $\mathcal{F}^{\mathbf{c}^{k+1}}$. To this end, we define a potential starting vector for an iteration procedure to find $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$, show that it is an upper bound of $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$ and demonstrate the property of well-definition of the algorithm with the help of this starting vector. Denote by

$$\Phi^{\mathbf{c}^{k+1}}\begin{pmatrix}\mathbf{r}\\ \mathbf{s}\end{pmatrix} := \begin{pmatrix}\min\{\mathbf{d}^{\mathbf{c}^{k+1}}, (\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\mathbf{r} + \mathbf{M}^0\mathbf{s})^+\}\\ (\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\mathbf{r} + \mathbf{M}^0\mathbf{s} - \mathbf{d}^{\mathbf{c}^{k+1}})^+\end{pmatrix} \tag{5.56}$$

the mapping whose fixed point is the solution of $\mathcal{F}^{\mathbf{c}^{k+1}}$ and the two mappings

$$\Phi^{1,\mathbf{c}^{k+1}}(\mathbf{r};\mathbf{s}) = \min\{\mathbf{d}^{\mathbf{c}^{k+1}}, (\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\mathbf{r} + \mathbf{M}^0\mathbf{s})^+\} \tag{5.57}$$

and

$$\Phi^{0,\mathbf{c}^{k+1}}(\mathbf{s};\mathbf{r}) = (\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\mathbf{r} + \mathbf{M}^0\mathbf{s} - \mathbf{d}^{\mathbf{c}^{k+1}})^+ \tag{5.58}$$

that represent the debt and the equity components of $\Phi^{\mathbf{c}^{k+1}}$, where for $\Phi^{1,\mathbf{c}^{k+1}}$, the vector $\mathbf{s} \geq \mathbf{0}_n$ is considered as fixed and for $\Phi^{0,\mathbf{c}^{k+1}}$, $\mathbf{r} \geq \mathbf{0}_n$ is assumed to be fixed. The starting vector to find $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$ is given by $(\tilde{\mathbf{r}}, \mathbf{r}^0(\tilde{\mathbf{r}}))$, where

$$\tilde{\mathbf{r}} = (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{r}^{\mathbf{c}^k} + \underline{\mathbf{\Lambda}}^k\mathbf{d}^{\mathbf{c}^{k+1}} \tag{5.59}$$

and $\mathbf{r}^0(\tilde{\mathbf{r}})$ is the fixed point of $\Phi^{0,\mathbf{c}^{k+1}}(\cdot;\tilde{\mathbf{r}})$ that can for example be obtained using Algorithm 2, i.e.

$$\Phi^{0,\mathbf{c}^{k+1}}(\mathbf{r}^0(\tilde{\mathbf{r}});\tilde{\mathbf{r}}) = (\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\tilde{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}}) - \mathbf{d}^{\mathbf{c}^{k+1}})^+ = \mathbf{r}^0(\tilde{\mathbf{r}}). \tag{5.60}$$

Note that this fixed point is unique also for negative entries in the asset vector $\mathbf{a}^{\mathbf{c}^{k+1}}$ (cf. Remark 4.10). Check that $(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{d}^{\mathbf{c}^{k+1}} = (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{d}^{\mathbf{c}^k}$ and

$$\mathbf{a}^{\mathbf{c}^{k+1}} = \mathbf{a}^{\mathbf{c}^k} - (\mathbf{M}^{\mathbf{c}^{k+1}} - \mathbf{I}_n)\underline{\mathbf{\Lambda}}^k\mathbf{d}^{\mathbf{c}^{k+1}} \tag{5.61}$$

and, because of $\mathbf{M}^{\mathbf{c}^{k+1}}(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k) = \mathbf{M}^{\mathbf{c}^k}(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)$ and $(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\tilde{\mathbf{r}} = (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{r}^{\mathbf{c}^k} = \mathbf{r}^{\mathbf{c}^k}$, that

$$\mathbf{M}^{\mathbf{c}^{k+1}}(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\tilde{\mathbf{r}} = \mathbf{M}^{\mathbf{c}^k}(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{r}^{\mathbf{c}^k} = \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k}. \tag{5.62}$$

Consequently, $\mathbf{r}^0(\tilde{\mathbf{r}})$ can be expressed as

$$\begin{aligned}\mathbf{r}^0(\tilde{\mathbf{r}}) &= \left(\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\tilde{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}}) - \mathbf{d}^{\mathbf{c}^{k+1}}\right)^+\\ &= \left(\mathbf{a}^{\mathbf{c}^k} - (\mathbf{M}^{\mathbf{c}^{k+1}} - \mathbf{I}_n)\underline{\mathbf{\Lambda}}^k\mathbf{d}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\tilde{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}}) - \mathbf{d}^{\mathbf{c}^{k+1}}\right)^+\\ &= \left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^{k+1}}(\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}}) - (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{d}^{\mathbf{c}^{k+1}}\right)^+\\ &= \left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}}) - (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{d}^{\mathbf{c}^k}\right)^+.\end{aligned} \tag{5.63}$$

Check that $\mathbf{s}^{\mathbf{c}^k}$ can be interpreted as the fixed-point of the mapping $\Phi^{0,\mathbf{c}^k}(\cdot;\mathbf{r}^{\mathbf{c}^k})$, meaning

$$\begin{aligned}\mathbf{s}^{\mathbf{c}^k} &= \left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k} - \mathbf{d}^{\mathbf{c}^k}\right)^+\\ &= \left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k} - (\mathbf{I}_n - \underline{\mathbf{\Lambda}}^k)\mathbf{d}^{\mathbf{c}^k}\right)^+,\end{aligned} \tag{5.64}$$

where the last equality follows from the fact that for firms with $\underline{\mathbf{\Lambda}}^k_{ii} = 1$ it holds by definition that the sum $\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k}$ is negative in the corresponding components which means that we can also omit the nominal values of the liabilities $\mathbf{d}^{\mathbf{c}^k}$. Hence, the Equations in (5.63) and (5.64) are identical from which follows that $\mathbf{s}^{\mathbf{c}^k} = \mathbf{r}^0(\tilde{\mathbf{r}})$. Moreover,

$$\begin{aligned}\Phi^{1,\mathbf{c}^{k+1}}(\tilde{\mathbf{r}};\mathbf{r}^0(\tilde{\mathbf{r}})) &= \min\left\{\mathbf{d}^{\mathbf{c}^{k+1}}, \left(\mathbf{a}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\tilde{\mathbf{r}} + \mathbf{M}^0\mathbf{r}^0(\tilde{\mathbf{r}})\right)^+\right\} \\ &= \min\left\{\mathbf{d}^{\mathbf{c}^{k+1}}, \left(\mathbf{a}^{\mathbf{c}^k} - (\mathbf{M}^{\mathbf{c}^{k+1}} - \mathbf{I}_n)\underline{\mathbf{\Lambda}}^k\mathbf{d}^{\mathbf{c}^{k+1}} + \mathbf{M}^{\mathbf{c}^{k+1}}\tilde{\mathbf{r}} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k}\right)^+\right\} \\ &= \min\left\{\mathbf{d}^{\mathbf{c}^{k+1}}, \left(\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k} + \underline{\mathbf{\Lambda}}^k\mathbf{d}^{\mathbf{c}^{k+1}}\right)^+\right\}.\end{aligned} \tag{5.65}$$

From (5.65), it becomes clear that for firms with $\underline{\mathbf{\Lambda}}^k_{ii} = 0$, and, thus, $d_i^{\mathbf{c}^{k+1}} = d_i^{\mathbf{c}^k}$ it holds that $\Phi^{1,\mathbf{c}^{k+1}}(\tilde{\mathbf{r}};\mathbf{r}^0(\tilde{\mathbf{r}}))_i = r_i^{\mathbf{c}^k} = \tilde{r}_i$. For firms with $\underline{\mathbf{\Lambda}}^k_{ii} = 1$, we see that $\Phi^{1,\mathbf{c}^{k+1}}(\tilde{\mathbf{r}};\mathbf{r}^0(\tilde{\mathbf{r}}))_i < d_i^{\mathbf{c}_i^{k+1}} = \tilde{r}_i$ since the sum $\mathbf{a}^{\mathbf{c}^k} + \mathbf{M}^{\mathbf{c}^k}\mathbf{r}^{\mathbf{c}^k} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^k}$ is negative for these firms. We therefore have shown that

$$\Phi^{\mathbf{c}^{k+1}}\begin{pmatrix}\tilde{\mathbf{r}}\\ \mathbf{r}^0(\tilde{\mathbf{r}})\end{pmatrix} = \begin{pmatrix}\Phi^{1,\mathbf{c}^{k+1}}(\tilde{\mathbf{r}};\mathbf{r}^0(\tilde{\mathbf{r}}))\\ \Phi^{0,\mathbf{c}^{k+1}}(\mathbf{r}^0(\tilde{\mathbf{r}});\tilde{\mathbf{r}})\end{pmatrix} \leq \begin{pmatrix}\tilde{\mathbf{r}}\\ \mathbf{r}^0(\tilde{\mathbf{r}})\end{pmatrix}. \tag{5.66}$$

Equation (5.66) reveals that the Picard Algorithm with starting vector $(\tilde{\mathbf{r}}, \mathbf{r}^0(\tilde{\mathbf{r}}))$ will lead, due to the monotonicity of $\Phi^{\mathbf{c}^{k+1}}$, to a decreasing series of iterates that converges to the fixed point $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$ using a similar argument than in the proof of Proposition 4.2. Note that for the Elsinger Algorithm, the first iterate is in this case identical to the first iterate of the Picard Algorithm. Together with Lemma 2.10 we can argue the same way than in the proof of Proposition 4.9 that the resulting series converges to $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$. If the Hybrid Algorithm is used, the first debt iterate is given by the fixed point of the mapping $\Phi^{1,\mathbf{c}^{k+1}}(\cdot;\mathbf{r}^0(\tilde{\mathbf{r}}))$ which, because of (5.66), must be smaller or equal than $\tilde{\mathbf{r}}$. The first equity iterate is then using Lemma 2.10 smaller or equal to $\mathbf{r}^0(\tilde{\mathbf{r}})$ and the convergence to $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$ follows in an analogous way. At this stage note once again that the fact that negative entries in the asset vector $\mathbf{a}^{\mathbf{c}^{k+1}}$ can be present does not influence the results. This is because Picard, Elsinger and Hybrid Algorithm can deal with negative asset vectors and still find the unique fixed point. Hence, for any of the three Algorithms (Picard, Elsinger and Hybrid), the vector $(\tilde{\mathbf{r}}, \mathbf{r}^0(\tilde{\mathbf{r}}))$ can be used as an initial iterate for the decreasing version of the algorithms to find $(\mathbf{r}^{\mathbf{c}^{k+1}}, \mathbf{s}^{\mathbf{c}^{k+1}})$. Consequently, it must hold that $\tilde{\mathbf{r}} \geq \mathbf{r}^{\mathbf{c}^{k+1}}$, where a strict inequality holds for firms with $\underline{\mathbf{\Lambda}}^k_{ii} = 1$ as shown in (5.65). Thus, $r_i^{\mathbf{c}^{k+1}} < d_i^{\mathbf{c}^{k+1}}$ and therefore $s_i^{\mathbf{c}^{k+1}} = 0$. Firms that are identified to be in default in an arbitrary iteration step will stay in default in all further iteration steps.

(ii) If $\underline{\mathbf{\Lambda}}^l = \mathbf{0}_{n\times n}$, it follows by definition that

$$\mathbf{a}^{\mathbf{c}^l} + \mathbf{M}^{\mathbf{c}^l}\mathbf{r}^{\mathbf{c}^l} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^l} \geq \mathbf{0}_n. \tag{5.67}$$

A direct consequence from this is that $\mathbf{c}^{l+1} = \mathbf{c}^l$ and $\underline{\mathbf{\Lambda}}^{l+1} = \mathbf{0}_{n\times n}$ as well, hence the iteration stops at this point. We show that $\Phi(\mathbf{R}^{\mathbf{c}^l}) = \mathbf{R}^{\mathbf{c}^l}$ by checking the equality firmwise. Doing so, we take a closer look at the i-th component of $\mathbf{r}^{\mathbf{c}^l}$ for which, together with (5.61) and (5.51), it holds that

$$
\begin{aligned}
r_i^{\mathbf{c}^l} &= \min\left\{ d_i^{\mathbf{c}^l}, a_i^{\mathbf{c}^l} + \sum_{j=1}^{n} M_{ij}^{c_j^l} r_j^{\mathbf{c}^l} + \sum_{j=1}^{n} M_{ij}^{0} s_j^{\mathbf{c}^l} \right\} \\
&= \min\left\{ d_i^{\mathbf{c}^l}, a_i + \sum_{j=1}^{n} \sum_{k=c_j^l+1}^{m} M_{ij}^{k} d_j^{k} - \sum_{k=c_i^l+1}^{m} d_i^k + \sum_{j=1}^{n} M_{ij}^{c_j^l} r_j^{\mathbf{c}^l} + \sum_{j=1}^{n} M_{ij}^{0} s_j^{\mathbf{c}^l} \right\} \\
&= \min\Bigg\{ d_i^{\mathbf{c}^l}, \underbrace{a_i + \sum_{j=1}^{n}\sum_{k=0}^{m} M_{ij}^{k} r_j^{k,\mathbf{c}^l} - \sum_{k=c_i^l+1}^{m} d_i^k}_{\geq 0 \text{ because of } (5.67)} \Bigg\}.
\end{aligned}
\tag{5.68}
$$

It follows that

$$
a_i + \sum_{j=1}^{n}\sum_{k=0}^{m} M_{ij}^{k} r_j^{k,\mathbf{c}^l} - \sum_{k=c_i^l+2}^{m} d_i^k \geq d_i^{c_i^l+1}. \tag{5.69}
$$

According to (5.51), we set $r_i^{c_i^l+1,\mathbf{c}^l} = d_i^{c_i^l+1}$ and we immediately see that this fulfills the fixed point equation in (2.12). For higher seniority levels than $c_i^l + 1$, the argumentation is the same. Let $c_i^l > 1$. By definition, it follows that $s_i^{\mathbf{c}^l} = 0$ which means that

$$
a_i + \sum_{j=1}^{n}\sum_{k=0}^{m} M_{ij}^{k} r_j^{k,\mathbf{c}^l} - \sum_{k=c_i^l}^{m} d_i^k \leq 0. \tag{5.70}
$$

Since (5.51) means that for lower seniority levels than c_i^l, we set $r_i^{c_i^l-1,\mathbf{c}^l} = \ldots = r_i^{1,\mathbf{c}^l} = r_i^{0,\mathbf{c}^l} = 0$, this is also in line with the fixed point property. If $c_i^l = 1$, we can distinguish the two cases $s_i^1 = 0$ and $s_i^1 > 0$. In the former case, setting by definition $r_i^{0,\mathbf{c}^l} = 0$, $r_i^{1,\mathbf{c}^l} = r_i^{c_i^l}$ and $r_i^{k,\mathbf{c}^l} = d_i^k$ for all seniority levels $k > 1$, the fixed point property of $\mathbf{R}^{\mathbf{c}^l}$ becomes obvious, as well as for the latter case where $r_i^{k,\mathbf{c}^l} = d_i^k$ for all $k \geq 1$ and $r_i^{0,\mathbf{c}^l} = s_i^1$. Summing up the results for all firms leads to $\Phi(\mathbf{R}^{\mathbf{c}^l}) = \mathbf{R}^{\mathbf{c}^l}$ for such a tuple $\mathbf{c}^l$.

(iii) The smallest possible tuple $\mathbf{c}^{\min} = (1, \ldots, 1)$ leads to $\mathbf{a}^{\mathbf{c}^{\min}} = \mathbf{a} + \sum_{k=2}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=2} \mathbf{d}^k$ and the largest possible tuple is $\mathbf{c}^{\max} = (m, \ldots, m)$ for which holds that $\mathbf{a}^{\mathbf{c}^{\max}} = \mathbf{a}$. The number of iteration steps would become maximal if one started with $\mathbf{c}^{\min}$, and the value one was added in a non-maximal component to the tuple in every iteration step of the algorithm. This would yield exactly $n(m-1)$ iteration steps.

□

Depending on the choice of the algorithm in Step 1 of Algorithm 17, we obtain three versions, the *Default Structure Picard/Elsinger/Hybrid Algorithm.* The Default Structure Algorithm itself converges in a finite number of iteration steps to the final tuple $\mathbf{c}^l$ with $\mathbf{R}^{\mathbf{c}^l} = \mathbf{R}^*$. However, recall that the Picard, Elsinger or Hybrid Algorithm that are used in Step 1 might under some circumstances not reach the fixed point $(\mathbf{r}^{\mathbf{c}^k}, \mathbf{s}^{\mathbf{c}^k})$, see the Propositions 4.4, 4.12 and 4.20 and the subsequent Examples for more details.

After having presented the Default Structure Algorithm as an alternative procedure to find $\mathbf{R}^*$, we should try to point out the similarities and also the differences between this approach and the Trial-and-Error Algorithms of Section 5.2.1. Both types of algorithms do have a Trial-and-Error aspect since they both check a fixed point criteria of their iterates. For the Trial-and-Error Algorithm, the fixed point criteria is directly verified in Step 5 of both versions of the algorithm when the equation $\Phi(\widehat{\mathbf{R}}^{\mathbf{c}^q}) = \widehat{\mathbf{R}}^{\mathbf{c}^q}$ is checked for validity, where $\widehat{\mathbf{R}}^{\mathbf{c}^q}$ is the pseudo solution of the potential default tuple $\mathbf{c}^q$. The Trial-and-Error interpretation for the Default Structure Algorithm is as follows. In every iteration step of the procedure, it is checked whether the sum

$$\mathbf{a}^{\mathbf{c}} + \mathbf{M}^{\mathbf{c}}\mathbf{r}^{\mathbf{c}} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}} \tag{5.71}$$

is non-negative, where $\mathbf{c}$ is the default tuple of the corresponding step. If there are still some firms with a negative entry in (5.71), the final tuple to determine $\mathbf{R}^*$ is not found yet and at least one further iteration step has to be performed. The first time (5.71) is non-negative in all components, the procedure stops.

Next to this similarity, there is also the obvious difference that the Default Structure Algorithm has a clear stopping criteria, namely if $\underline{\mathbf{\Lambda}}^k = \mathbf{0}_{n\times n}$ for the first time in the procedure. The Trial-and-Error Algorithms do not have this property. However, the advantage of having this definite stopping point, comes with the drawback that for every default tuple $\mathbf{c}^k$, an additional financial system $\mathcal{F}^{\mathbf{c}^k}$ has to be solved. This might be accompanied with an increased computational effort of this procedure compared to the Trial-and-Error Algorithms.

Example 5.16. Let us find the solution of the financial system of Example 5.1 using the Default Structure Picard Algorithm. For the subalgorithm to find the fixed point of $\mathcal{F}^{\mathbf{c}^k}$, we applied the Decreasing Picard Algorithm (Algorithm 1) each time. The starting tuple $\mathbf{c}^0$ is given by $\mathbf{c}^0 = (1,1,1,1,1)$. Therefore, $\mathbf{d}^{\mathbf{c}^0} = \mathbf{d}^1$, $\mathbf{M}^{\mathbf{c}^0} = \mathbf{M}^1$ and

$$\mathbf{a}^{\mathbf{c}^0} = \mathbf{a} + \mathbf{M}^3\mathbf{d}^3 + \mathbf{M}^2\mathbf{d}^2 - \mathbf{d}^3 - \mathbf{d}^2 = (5.6250, 2.8750, 2.8333, -5.7083, -1.6250)^t. \tag{5.72}$$

The solution of the corresponding system $\mathcal{F}^{\mathbf{c}^0}$ is given by

$$\begin{pmatrix} \mathbf{r}^{\mathbf{c}^0} \\ \mathbf{s}^{\mathbf{c}^0} \end{pmatrix} = \begin{pmatrix} (3, & 1, & 3.4935, & 0, & 0)^t \\ (2.7172, & 3.6897, & 0, & 0, & 0)^t \end{pmatrix}. \tag{5.73}$$

Because of

$$\mathbf{a}^{\mathbf{c}^0} + \mathbf{M}^{\mathbf{c}^0}\mathbf{r}^{\mathbf{c}^0} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^0} = (5.7172, 4.6897, 3.4935, -5.2083, -1.1250)^t, \tag{5.74}$$

we find that firm 4 and 5 are not able to pay off any of the debt of the lowest seniority, so they will default for sure in a seniority class higher than one. The tuple becomes $\mathbf{c}^1 = (1,1,1,2,2)$ and the procedure repeats. This time,

$$\mathbf{a}^{\mathbf{c}^1} + \mathbf{M}^{\mathbf{c}^1}\mathbf{r}^{\mathbf{c}^1} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^1} = (5.6812, 3.2484, 1.1753, -1.2083, 3.8750)^t \tag{5.75}$$

and it becomes clear that firm 4 will also not be able deliver any debt payments in seniority level two. With $\mathbf{c}^2 = (1,1,1,3,2)$ in the next iterate, we obtain

$$\mathbf{a}^{\mathbf{c}^2} + \mathbf{M}^{\mathbf{c}^2}\mathbf{r}^{\mathbf{c}^2} + \mathbf{M}^0\mathbf{s}^{\mathbf{c}^2} = (5.6659, 2.6360, 1.1596, 3.7917, 3.8750)^t, \tag{5.76}$$

which lets the algorithm stop after two iteration steps. One can easily check that $\mathbf{R}^{\mathbf{c}^2} = \mathbf{R}^*$.

The example demonstrates that the total number of iteration steps is much smaller than for the Picard and the Elsinger Algorithms in the Examples 5.1 and 5.3. However, this number should not be overestimated for a premature statement that the Default Structure Algorithm is the computational more efficient procedure. This is because of the additional calculation effort that is necessary in every iteration step to solve the system $\mathcal{F}^{\mathbf{c}^k}$. Moreover, the only procedures that can be used for determining the solution of $\mathcal{F}^{\mathbf{c}^k}$ are the non-finite algorithms of Section 4.1. This means that the Default Structure Algorithm obviously has only a non-finite character as well.

Example 5.16 also reveals some weakness of the Default Structure Algorithm. The initial default tuple $\mathbf{c}^0$ can under circumstances be chosen too small which would result in unnecessary iteration steps. To see this, assume maximal debt payments given in the vector $\mathbf{R}_{\text{great}} = (\mathbf{r}^3_{\text{great}}, \ldots, \mathbf{r}^0_{\text{great}})$ defined as in (5.1). Then we can calculate

$$\mathbf{a} + \sum_{k=0}^{3} \mathbf{M}^k \mathbf{r}^k_{\text{great}} - \mathbf{d}^3 - \mathbf{d}^2 = (5.625, 3.0112, 3.5301, -5.6331, -1.6250)^t. \tag{5.77}$$

The sign of the entries in this sum reveals an information whether the firm is able to recover any payments the lowest seniority level. If the entry of the corresponding firm is negative, all capital will be expended for the two highest seniority classes which means that the firm has no capital left that can be furnished to the creditors in class 1 even if maximum payments as given in $\mathbf{R}_{\text{great}}$ are assumed. Consequently, the firm is not able to do so for all other vectors $\mathbf{R} \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ too. Therefore, using $c^0_i = 1$ in Algorithm 17 seems not suitable for the last two firms, since it must hold in the final tuple $\mathbf{c}^l$ that $c^l_i > 1$ for these firms. It hence would be appropriate to raise the entry of the firms 4 and 5 by 1. The principle can be continued by calculating the sum again and ignore the liabilities in $\mathbf{d}^2$ this time:

$$\mathbf{a} + \sum_{k=0}^{3} \mathbf{M}^k \mathbf{r}^k_{\text{great}} - \mathbf{d}^3 = (7.625, 6.0112, 4.5301, -1.6331, 3.3750)^t. \tag{5.78}$$

We see now that firm 5 has a positive value, so $c^0_5 = 2$ is a logical choice. Firm 4, on the other hand, still has a negative entry which means that it will also not be able to cover any payments of seniority level 2. Thus, we can raise the value in $\mathbf{c}^0$ for this firm to $c^0_4 = 3$. Putting the new values together leads to the modified default tuple $\mathbf{c}^0 = (1, 1, 1, 3, 2)$. But this means that there would be no further iteration step other than the initial step which essentially reduces the calculation effort in this example.

To put this in a general framework, we define the entries c^0_i of the initial default tuple $\mathbf{c}^0$ by

$$\begin{aligned} c^0_i = \Bigg\{ k \in \{2, \ldots, m\} : & \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} - \sum_{l=k}^{m} \mathbf{d}^l\right)_i < 0 \\ & \text{and} \quad \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} - \sum_{l=k+1}^{m} \mathbf{d}^l\right)_i \geq 0 \Bigg\}. \end{aligned} \tag{5.79}$$

If $(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} - \sum_{l=1}^{m} \mathbf{d}^l)_i > 0$, set $c^0_i = 1$. Firms without any capital for a particular seniority class assuming the maximum payment vector given by $\mathbf{R}_{\text{great}}$, will also not be able to pay off any capital in this class for any other vector $\mathbf{R}^k$ of recovery values. That means we have to identify the seniority level k in which a firm $i \in \mathcal{N}$ runs out of capital for the first time. Any seniorities smaller than k can definitely not be serviced since the firm has no capital left even for level k and even when maximal payments are assumed. The class k is therefore a logical starting point for the search of the final tuple $\mathbf{c}^l$. Since c_i^0 in (5.79) is obviously always larger or equal to c_i^0 in (5.53), we can save unnecessary iteration steps. For these reasons we call Algorithm 17 that starts with $\mathbf{c}^0$ defined as in (5.79) the *Smart Default Structure Algorithm*.

6 Optimizing Non-finite Algorithms

In Chapter 4 we have demonstrated that the iteration procedures of the algorithms to find $\mathbf{R}^*$ can start from two directions resulting either in a decreasing or an increasing series of iterates $\mathbf{R}^k$. The number of needed iterations to reach the fixed point $\mathbf{R}^*$ differs in general for both algorithm directions. If financial systems are highly indebted and therefore contain many firms in default, starting with the minimum possible solution seems to be preferable than starting with the maximum possible payment vector. On the other side, in sound systems, i.e. in systems with only a few number of defaults or even no defaulting firms at all, the number of iteration steps is minimized if the maximum possible solution is taken as the initial iterate. An analytical approach to solve this decision problem is presented in this chapter.

For the Picard Algorithm the two starting vectors are $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$. We will show in Section 6.1 that by calculating the a priori and the initial error (see Lemma 6.1), a fairly reliable rule can be derived to decide which of the two starting points is the optimal one in the sense that the number of needed iterations are minimized. This decision rule can be applied *a priori* that means only the information given in the financial system $\mathcal{F}$ is needed to judge whether $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ should be used as the zeroth iterate. Using the Elsinger or the Hybrid Algorithm, we have to decide between the starting vectors $(\mathbf{d}^m, \ldots, \mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1))$ and $(\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}}))$. It will turn out in Section 6.2 that the assumptions that are necessary to develop a decision rule for the Picard Algorithm are not fulfilled in general for all versions of the Elsinger and the Hybrid Algorithm. However, we still can apply the ideas of Section 6.1 which also results in a rule to decide between one of the two starting vectors.

Before we present the findings of this chapter, we have to skip the Elsinger Property of the ownership matrices of the standard model (cf. Section 2.2) and demand that Assumption 5.6 holds in the remainder of this chapter, i.e. $\|\mathbf{M}^k\| < 1$ for all $k = 0, \ldots, m$. Assumption 5.6 in particular ensures that the solution $\mathbf{R}^*$ of the system $\mathcal{F}$ is still unique.

6.1 The Picard Algorithm

An essential property of the mapping Φ in (2.12) is that it is a *strict contraction* on $(\mathbb{R}_0^+)^{n(m+1)}$, i.e. for two arbitrary vectors $\mathbf{R}^1, \mathbf{R}^2 \in (\mathbb{R}_0^+)^{n(m+1)}$ it holds that

$$\|\Phi(\mathbf{R}^1) - \Phi(\mathbf{R}^2)\| \leq I^{\max}\|\mathbf{R}^1 - \mathbf{R}^2\|, \tag{6.1}$$

with $I^{\max} \in [0, 1)$ defined as in (3.17). For a proof of Equation (6.1), see Lemma 4.1 in Fischer (2014). Note that Assumption 5.6 is crucial for Φ to be a strict contraction. As first shown in the work of Banach (1922), strict contractions do have a unique fixed point. In his work, Banach also showed that the fixed point can be reached via the iterative usage of the mapping Φ on a given starting vector $\mathbf{R}^0$ which we picked up in the Picard Algorithm in Section 4.1.1. The article also contains informations about the rate of convergence of the iteration based on an initial iteration shown in the next lemma and which will serve as the main result for our following considerations.

Lemma 6.1. *Let* $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n(m+1)}$ *be the fixed point iteration described in* (2.12), $\mathbf{R}^0 \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ *the initial iterate to find* $\mathbf{R}^*$ *and* $\mathbf{R}^k, k \geq 1$ *be the corresponding iterates based on* $\mathbf{R}^0$*, i.e.* $\mathbf{R}^k = \Phi^k(\mathbf{R}^0)$*. Then the following holds:*

$$\|\mathbf{R}^* - \mathbf{R}^k\| \leq \frac{(I^{\max})^k}{1 - I^{\max}} \|\Phi(\mathbf{R}^0) - \mathbf{R}^0\| = \frac{(I^{\max})^k}{1 - I^{\max}} \|\mathbf{R}^1 - \mathbf{R}^0\|, \tag{6.2}$$

and

$$\|\mathbf{R}^* - \mathbf{R}^k\| \leq (I^{\max})^k \|\mathbf{R}^* - \mathbf{R}^0\|. \tag{6.3}$$

The inequality in (6.2) *is often called* a priori error *of* Φ *(Allen and Isaacson, 1998). Moreover, we label* $\|\mathbf{R}^* - \mathbf{R}^k\|$ *as the* k-th iteration error *and* $\|\mathbf{R}^* - \mathbf{R}^0\|$ *as the* initial error*.*

Proof. (i) Because of Equation (6.1) we get for the k-th iteration of an arbitrary starting vector $\mathbf{R}^0$:

$$\begin{aligned}
\|\mathbf{R}^{k+1} - \mathbf{R}^k\| &= \|\Phi(\mathbf{R}^k) - \Phi(\mathbf{R}^{k-1})\| \\
&\leq I^{\max} \|\mathbf{R}^k - \mathbf{R}^{k-1}\| \\
&= I^{\max} \|\Phi(\mathbf{R}^{k-1}) - \Phi(\mathbf{R}^{k-2})\| \\
&\leq (I^{\max})^2 \|\mathbf{R}^{k-1} - \mathbf{R}^{k-2}\| \\
&\leq \cdots \leq (I^{\max})^k \|\mathbf{R}^1 - \mathbf{R}^0\|.
\end{aligned} \tag{6.4}$$

Let now l and k be two iteration steps with $l > k$. Using the triangle inequality and the results above, it holds that

$$\begin{aligned}
\|\mathbf{R}^l - \mathbf{R}^k\| &\leq \|\mathbf{R}^l - \mathbf{R}^{l-1}\| + \|\mathbf{R}^{l-1} - \mathbf{R}^{l-2}\| + \ldots + \|\mathbf{R}^{k+1} - \mathbf{R}^k\| \\
&\leq (I^{\max})^{l-1} \|\mathbf{R}^1 - \mathbf{R}^0\| + (I^{\max})^{l-2} \|\mathbf{R}^1 - \mathbf{R}^0\| \\
&\qquad + \ldots + (I^{\max})^k \|\mathbf{R}^1 - \mathbf{R}^0\| \\
&= \Big((I^{\max})^{l-1} + (I^{\max})^{l-2} + \ldots + (I^{\max})^k \Big) \|\mathbf{R}^1 - \mathbf{R}^0\| \\
&\leq \Big((I^{\max})^k + (I^{\max})^{k+1} + \ldots \Big) \|\mathbf{R}^1 - \mathbf{R}^0\| \\
&= \frac{(I^{\max})^k}{1 - I^{\max}} \|\mathbf{R}^1 - \mathbf{R}^0\|,
\end{aligned} \tag{6.5}$$

where the last equality follows the formula of a geometric series that starts at point k. As the last line does not depend on l, we can let l go to infinity which leads to $\|\mathbf{R}^* - \mathbf{R}^k\|$ on the left hand side and proves part (i).

(ii) For this part, we use the fact that in the fixed point, $\Phi(\mathbf{R}^*) = \mathbf{R}^*$. Together with Equation (6.1) we get:

$$\begin{aligned}
\|\mathbf{R}^* - \mathbf{R}^k\| &= \|\Phi(\mathbf{R}^*) - \Phi(\mathbf{R}^{k-1})\| \\
&\leq I^{\max} \|\mathbf{R}^* - \mathbf{R}^{k-1}\| \\
&= I^{\max} \|\Phi(\mathbf{R}^*) - \Phi(\mathbf{R}^{k-2})\| \\
&\leq (I^{\max})^2 \|\mathbf{R}^* - \mathbf{R}^{k-2}\| \\
&\leq \cdots \leq (I^{\max})^k \|\mathbf{R}^* - \mathbf{R}^0\|.
\end{aligned} \tag{6.6}$$

□

For the purpose of minimizing the computational effort, our aim is to find a starting vector $\mathbf{R}^0$ that fast converges to the searched fixed point $\mathbf{R}^*$. In other words, we have to find $\mathbf{R}^0$ such that the corresponding k-th iterate is near $\mathbf{R}^*$, i.e. $\|\mathbf{R}^* - \Phi^k(\mathbf{R}^0)\| = \|\mathbf{R}^* - \mathbf{R}^k\|$ has to become small. The question that arises now is whether $\mathbf{R}^0$ can be chosen in an optimal way such that the k-th iteration error is minimal. Of course, the fixed point $\mathbf{R}^*$ solves this problem since $\|\Phi(\mathbf{R}^*) - \mathbf{R}^*\| = \|\mathbf{R}^* - \mathbf{R}^*\| = 0$. But without any knowledge of $\mathbf{R}^*$ and without any other knowledge than the input parameters given by the financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$, can $\mathbf{R}^0$ be chosen optimally such that the computational effort is kept to a minimum?

The findings of Lemma 6.1 suggest that there are two ways to do so. We can choose $\mathbf{R}^0$ such that the a priori error is minimized from which it can be concluded that the bound (6.2) of the k-th iteration error is minimal. Another approach is to take a starting vector that minimizes $\|\mathbf{R}^* - \mathbf{R}^0\|$. Obviously, we have in this case the problem that there is no a priori information available about the fixed point $\mathbf{R}^*$ which makes the estimation a difficult task. In the subsequent parts of this section, we will see that it depends on the structure of the financial system to decide, which of the two estimates in (6.2) and (6.3) delivers more precise error estimate.

In any case, the search for an optimal starting point can be limited from the space $(\mathbb{R}_0^+)^{n(m+1)}$ to the interval $[\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$, where $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ are defined in the Equations (5.1) and (5.3) in Section 5.1.1. Clearly, $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ demand no other informations than the *a priori* information contained in the financial system $\mathcal{F}$ and therefore can be determined without additional calculus.

6.1.1 Properties of $\mathrm{R}_{\text{great}}$ and $\mathrm{R}_{\text{small}}$

The justification for the choice of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ was given in Proposition 4.1, where it was shown that for the fixed point $\mathbf{R}^*$ is must hold that $\mathbf{R}^* \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$. The proof was given for systems with only one seniority , but can easily be extended to systems with $m > 1$, see the comments in Section 5.1.1. In this subsection we list some useful properties of the two starting vectors that are needed for minimizing the iteration error.

Proposition 6.2. *The starting vector $\mathbf{R}_{\text{great}}$ is an upper bound of the actual solution $\mathbf{R}^*$, i.e. we have that $\mathbf{R}^* \leq \mathbf{R}_{\text{great}}$. Equality holds if and only if*

$$\left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)_i \geq 0 \quad \textit{for all } i \in \mathcal{N}. \tag{6.7}$$

Proof. The part that $\mathbf{R}^* \leq \mathbf{R}_{\text{great}}$ is shown in Proposition 4.1. For the second claim, assume first that (6.7) holds from which directly follows that

$$\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=j+1}^{m} \mathbf{d}^k \geq \mathbf{d}^j \quad \text{for all } j \geq 1. \tag{6.8}$$

This implies $\mathbf{r}^{*,j} = \mathbf{d}^j$ for all $j = 1, \ldots, m$. The equity components of $\mathbf{R}^*$ can then be written as

$$\begin{aligned} \mathbf{r}^{*,0} &= \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k + \mathbf{M}^0 \mathbf{r}^{*,0} - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ \\ &= \mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k + \mathbf{M}^0 \mathbf{r}^{*,0} - \sum_{k=1}^{m} \mathbf{d}^k, \end{aligned} \tag{6.9}$$

where in the last line we used (6.8) and the fact that $\mathbf{r}^{*,0} \geq \mathbf{0}_n$. But this means that $\mathbf{r}^{*,0} = (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \sum_{k=1}^m \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^m \mathbf{d}^k)$ and, hence, $\mathbf{R}^* = \mathbf{R}_{\text{great}}$.

Assume now that $\mathbf{R}^* = \mathbf{R}_{\text{great}}$ which means that

$$\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=j+1}^{m} \mathbf{d}^k \geq \mathbf{d}^j \quad \text{for all } j \geq 1 \tag{6.10}$$

and that $\mathbf{r}^{*,j} = \mathbf{d}^j$ for all $j = 1, \ldots, m$. Thus, all firms are by definition solvent which means that

$$\mathbf{r}^{*,0} = \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ = \mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k + \mathbf{M}^0 \mathbf{r}^{*,0} - \sum_{k=1}^{m} \mathbf{d}^k. \tag{6.11}$$

Rearranging yields to

$$\mathbf{r}^{*,0} = (\mathbf{I}_n - \mathbf{M}^0)^{-1} \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k\right). \tag{6.12}$$

Because of $\mathbf{r}^{*,0} = \mathbf{r}^0_{\text{great}}$ and since $(\mathbf{I}_n - \mathbf{M}^0)^{-1}$ has full rank (cf. Lemma A.3), it must hold that

$$\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k = \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)^+ \tag{6.13}$$

and therefore (6.7). □

The next examples demonstrates that even in a solvent system, i.e. in a system in which all n firm are able to satisfy their m obligations completely, $\mathbf{R}_{\text{great}}$ is not necessary the solution of the system.

Example 6.3. Consider a system of $n = 3$ firms with $m = 1$ and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{d}^1 = \begin{pmatrix} 1.745 \\ 0.75 \\ 1 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 0.5 & 0.25 \\ 0.1 & 0 & 0.5 \\ 0.1 & 0.25 & 0 \end{pmatrix}, \ \mathbf{M}^0 = \begin{pmatrix} 0 & 0.05 & 0.05 \\ 0.05 & 0 & 0.05 \\ 0.05 & 0.05 & 0 \end{pmatrix}. \tag{6.14}$$

Check that the fixed point of this system is

$$\mathbf{R}^* = \left(\mathbf{r}^{*,1}, \mathbf{r}^{*,0}\right) = (1.745, 0.75, 1, 0.0003, 0.9951, 1.4118)^t, \tag{6.15}$$

i.e. $\mathbf{r}^{*,1} = \mathbf{d}^1$ and that

$$\mathbf{r}^0_{\text{great}} = (0.1210, 1.0015, 1.4181)^t > \mathbf{r}^{*,0}. \tag{6.16}$$

Therefore, the system is solvent, but still $\mathbf{r}^{*,0} < \mathbf{r}^0_{\text{great}}$. We see that because of $\mathbf{a} + \mathbf{M}^1\mathbf{d}^1 - \mathbf{d}^1 = (-0.12, 0.9245, 1.362)^t$, (6.7) is violated which is why equality of $\mathbf{r}^{*,0}$ and $\mathbf{r}^0_{\text{great}}$ does not hold.

Remark 6.4. The result in Example 6.3 might lead to the conclusion that instead of $\mathbf{r}^0_{\text{great}}$, the optimal starting vector should rather be defined as

$$\tilde{\mathbf{r}}^0_{\text{great}} = \left((\mathbf{I}_n - \mathbf{M}^0)^{-1} \left(\mathbf{a} + \sum_{k=1}^{m} \mathbf{M}^k \mathbf{d}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)\right)^+. \tag{6.17}$$

In Example 6.3, the equity vector of the solution is exactly given by $\tilde{\mathbf{r}}^0_{\text{great}} = \mathbf{r}^{*,0}$. However, this is not plausible because $\tilde{\mathbf{r}}^0_{\text{great}}$ must not be an upper bound of $\mathbf{r}^{*,0}$ in general. To see this, we retain the system of (6.14) but modify the liability vector and the debt ownership matrix to

$$\mathbf{d}^1 = \begin{pmatrix} 2.245 \\ 1.25 \\ 1.5 \end{pmatrix} \quad \text{and} \quad \mathbf{M}^1 = \begin{pmatrix} 0 & 0.05 & 0.025 \\ 0.01 & 0 & 0.05 \\ 0.01 & 0.025 & 0 \end{pmatrix}, \tag{6.18}$$

i.e. we add the value 0.5 on each entry in the original liability vector and multiply each entry of $\mathbf{M}^1$ in (6.14) by 0.1. The fixed point now becomes to $\mathbf{R}^* = (1.1201, 1.1132, 1.5, 0, 0, 0.5390)^t$. Moreover, we have

$$\mathbf{r}^0_{\text{great}} = \begin{pmatrix} 0.0293 \\ 0.0293 \\ 0.5566 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{r}}^0_{\text{great}} = \begin{pmatrix} 0 \\ 0 \\ 0.4880 \end{pmatrix} \tag{6.19}$$

and therefore $\tilde{\mathbf{r}}^0_{\text{great}} < \mathbf{r}^{*,0}$ for the third firm.

A result of Proposition 4.2 was that $\Phi(\mathbf{R}_{\text{great}}) \leq \mathbf{R}_{\text{great}}$ and $\Phi(\mathbf{R}_{\text{small}}) \geq \mathbf{R}_{\text{small}}$. This was shown for systems with $m = 1$ but the generalization for systems with $m > 1$ is straightforward. From the monotonicity of Φ (cf. Lemma 2.4), it follows that when starting the Picard Algorithm with $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$, we get a decreasing sequence of

$$\mathbf{R}_{\text{great}} \geq \Phi(\mathbf{R}_{\text{great}}) \geq \ldots \geq \Phi^k(\mathbf{R}_{\text{great}}) \geq \ldots \geq \mathbf{R}^* \tag{6.20}$$

and if $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$, the algorithm generates an increasing sequence of

$$\mathbf{R}_{\text{small}} \leq \Phi(\mathbf{R}_{\text{small}}) \leq \ldots \leq \Phi^k(\mathbf{R}_{\text{small}}) \leq \ldots \leq \mathbf{R}^*. \tag{6.21}$$

In the literature, this iteration procedure is also called *Kleene chain*, (cf. Fischer, 2015). The following example attempts to make the described iteration procedures more clear.

Example 6.5. Consider a system with $n = 100$ firms and no seniority structure, i.e. $m = 1$, to keep the example as simple as possible. The entries of $\mathbf{a}$ and $\mathbf{d}^1 = \mathbf{d}$ are equal for all firms with $\mathbf{a} = (5, \ldots, 5)^t \in \mathbb{R}^{100}$ and $\mathbf{d} = (50, \ldots, 50)^t \in \mathbb{R}^{100}$. For the equity ownership matrix $\mathbf{M}^0$ we set a constant ownership fraction of $M^0_{ij} = M^0_{ji} = 0.1/99 \approx 0.0010$ for $i \neq j$ which leads to $\|\mathbf{M}^0\| = 0.1$. The ownership matrix $\mathbf{M}^1$ is also symmetric such that $M^1_{ij} = M^1_{ji}$ for all $1 \leq i, j \leq n$ and $i \neq j$. In this example we consider three different scenarios for $\mathbf{M}^1$:

(i) Let $M^1_{ij} = 0.95/99 = 0.0096$ which leads to $\|\mathbf{M}^1\| = 0.95 = I^{\max}$. In this scenario the degree of cross-ownership is very high, i.e. almost all debt payments will stay within the system. In Chapter 7, the extend of cross-holdings measured by the norm of $\mathbf{M}^1$ will be called the *integration level*. As a consequence, the firms are solvent even though the nominal debt values are much higher than the exogenous assets. We have that $r^{*,1}_i = 50 = d_i$ and $r^{*,0}_i = 2.7778$ for all firms.

(ii) Let $M^1_{ij} = 0.7/99 = 0.0071$ which leads to $\|\mathbf{M}^1\| = 0.7 = I^{\max}$. This is a scenario where a medium degree of cross-ownership is present. However, due to the high burden of debt, the firms will be in default so that in $\mathbf{R}^*$, we have that $r^{*,1}_i = 16.6667 < d_i$ which leads to a recovery rate of $16.6667/50 = 33.33\%$.

(iii) Let $M^1_{ij} = 0.15/99 = 0.0015$ which leads to $\|\mathbf{M}^1\| = 0.15 = I^{\max}$ and a system with a low degree of cross-ownership. Only 15% of each firm's debt is held within in the system. The remaining payments belong to debtholders outside the system. In this case, the system defaults as well with $r^{*,1}_i = 5.8824 < d_i$ and a recovery rate of $5.8824/50 = 11.76\%$.

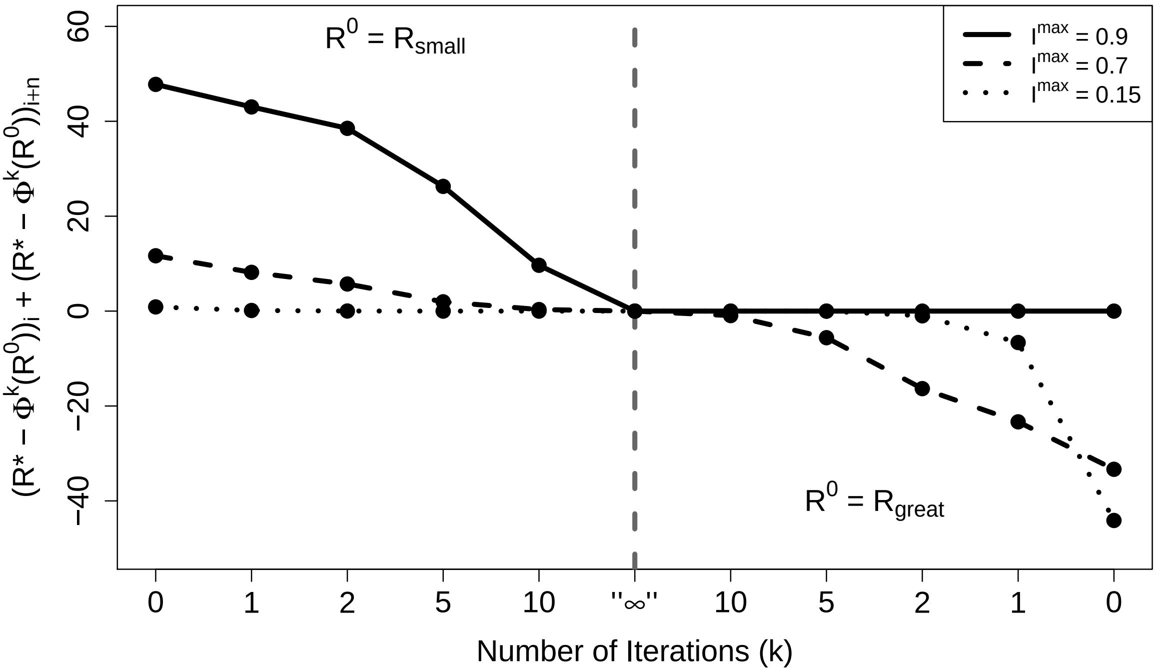


Figure 6.1: Iteration numbers plotted against the sum in (6.22) for the different scenarios described in Example 6.5. In the left part of the figure where the sum is positive, the starting vector was $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ and in the right part the starting vector was $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. Consequently the sum is negative for this starting vector (cf. Equation (6.20)). The somewhat sloppy notation "∞" on the x-axis denotes the number of iterations needed to reach the fixed point $\mathbf{R}^*$ sufficiently close for the tolerance level of $\varepsilon = 10^{-6}$ that differed for each of the considered financial systems.

To demonstrate the functioning of the Kleene chain, we consider for an arbitrary $i \in \mathcal{N}$ in every iteration step k the sum

$$\Big(\mathbf{R}^* - \Phi^k(\mathbf{R}^0)\Big)_i + \Big(\mathbf{R}^* - \Phi^k(\mathbf{R}^0)\Big)_{i+n} = (r_i^{*,1} - r_i^{k,1}) + (r_i^{*,0} - r_i^{k,0}). \tag{6.22}$$

Hence, we take only one of the n firms into account which in this case is reasonable, since because of the structure of $\mathbf{a}$, $\mathbf{d}$ and the ownership matrices, all firms will have the same entries for $\mathbf{R}^*$ and $\mathbf{R}^k$. Note that we do not calculate the norm of the difference and allow the sum in (6.22) to positive and negative for a better illustration of the two ways to approach to the fixed point $\mathbf{R}^*$ depending on the starting vector. Beginning with both starting vectors $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$, the sum in Equation (6.22) is calculated for $k = 0, 1, 2, 5, 10$ where $\Phi^0(\mathbf{R}^0) = \mathbf{R}^0$. The results are shown in Figure 6.1.

Clearly, the degree of ownership influences the speed of convergence. We see that in case of $I^{\max} = 0.95$ the system is solvent with $\mathbf{R}^* = \mathbf{R}_{\text{great}}$. Hence, the sum in (6.22) is zero for $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. On the other hand, the difference for $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ is relatively high for the first iterations but decreases with increasing k. Based on a tolerance level of $\varepsilon = 10^{-6}$, there are 23 iterations needed to reach $\mathbf{R}^*$ sufficiently close. While for $I^{\max} = 0.95$ it is clear that $\mathbf{R}_{\text{great}}$ is the better choice for the starting vector, we see for $I^{\max} = 0.7$ that the difference of the sum in (6.22) for both starting vectors becomes smaller for $k = 0$. In this case, the choice between $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ becomes more difficult. We will see later that the choice of $\mathbf{R}_{\text{small}}$ is preferable in this scenario which is also underlined by the fact that 57 iterations are needed when starting with $\mathbf{R}_{\text{small}}$ whereas 60 iterations are needed in case of starting with $\mathbf{R}_{\text{great}}$. In case of a low degree of cross-ownership ($I^{\max} = 0.15$), the firms have only a low recovery rate so that the choice of $\mathbf{R}_{\text{small}}$ seems to be preferable. This is also visible in Figure 6.1, where the sum in (6.22) on an absolute level is much closer to zero for small values of k when starting with $\mathbf{R}_{\text{small}}$ than the corresponding sum for the starting vector $\mathbf{R}_{\text{great}}$. Note that 11 iteration steps are needed when starting with $\mathbf{R}_{\text{small}}$ and 13 when starting with $\mathbf{R}_{\text{great}}$.

The findings of this example already provide a first insight to the theoretical results in the following. For solvent systems or systems in which the degree of indebtedness is high, the preferable starting vector is $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. The higher the liabilities compared to the exogenous assets get or the lower the degree of cross-ownership becomes, the higher the chances that defaults appear with the consequence that the choice of $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$ is better.

After having presented some properties of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$, we can come back to our initial problem of estimating the iteration error given in the Equations (6.2) and (6.3).

Lemma 6.6. *Let* $\mathbf{R} = \left(\mathbf{r}^m, \ldots, \mathbf{r}^0\right) \in \mathbb{R}^{n(m+1)}$ *with* $\mathbf{R} \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ *be an arbitrary vector such that either* $\Phi(\mathbf{R}) \leq \mathbf{R}$ *or* $\Phi(\mathbf{R}) \geq \mathbf{R}$. *Then:*

$$\|\Phi(\mathbf{R}) - \mathbf{R}\| = \left\| \mathbf{a} + \sum_{k=0}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k \right\|. \tag{6.23}$$

Proof. First, write the left hand side of (6.23) in a more extended form for a better understanding of the forthcoming arguments:

$$\|\Phi(\mathbf{R}) - \mathbf{R}\| = \left\| \begin{pmatrix} \min\{\mathbf{d}^m, & \mathbf{a} + \sum_{k=0}^m \mathbf{M}^k \cdot \mathbf{r}^k\} & - \mathbf{r}^m \\ \min\{\mathbf{d}^{m-1}, & (\mathbf{a} + \sum_{k=0}^m \mathbf{M}^k \cdot \mathbf{r}^k - \mathbf{d}^m)^+\} & - \mathbf{r}^{m-1} \\ \vdots & & \\ \min\{\mathbf{d}^1, & (\mathbf{a} + \sum_{k=0}^m \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=2}^m \mathbf{d}^k)^+\} & - \mathbf{r}^1 \\ & (\mathbf{a} + \sum_{k=0}^m \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=1}^m \mathbf{d}^k)^+ & - \mathbf{r}^0 \end{pmatrix} \right\|. \tag{6.24}$$

To prove Equation (6.23), we have to check its validity component-wise for each firm. Following (6.24), the norm for the i-th firm is given by

$$\begin{aligned} &\sum_{l=1}^{m} \left| \min\left\{ d_i^l, \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=l+1}^{m} \mathbf{d}^k \right)_i^+ \right\} - r_i^l \right| \\ &\qquad + \left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k \right)_i^+ - r_i^0 \right|. \end{aligned} \tag{6.25}$$

Suppose for the first part of the proof that

$$\Phi(\mathbf{R}) \leq \mathbf{R}. \tag{6.26}$$

To show the claim, we have to consider three cases, where we set $d_i^0 := 0$ in the following.

(i) Let

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k\right)_i \leq d_i^m. \tag{6.27}$$

The sum in (6.25) reduces to

$$\underbrace{\left|\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k\right)_i - r_i^m\right|}_{\leq 0 \text{ because of (6.26)}} + \sum_{k=0}^{m-1} |\underbrace{-r_i^k}_{\leq 0}| = \left|\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k\right|_i \tag{6.28}$$

since in case of equal signs, we can write the sum of the absolute values as the absolute value of the sum.

(ii) Let for $m > j \geq 0$,

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+1}^{m} \mathbf{d}^k\right)_i \leq d_i^j \text{ and } \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+2}^{m} \mathbf{d}^k\right)_i > d_i^{j+1}. \tag{6.29}$$

Then

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=l+1}^{m} \mathbf{d}^k\right)_i \geq d_i^l \quad \text{for all } l \geq j+1 \tag{6.30}$$

and

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=l+1}^{m} \mathbf{d}^k\right)_i \leq 0 \quad \text{for all } l \leq j-1. \tag{6.31}$$

It follows because of (6.26) that

$$\begin{aligned} &\sum_{k=j+1}^{m} |\underbrace{d_i^k - r_i^k}_{=0}| + \Bigg|\underbrace{\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+1}^{m} \mathbf{d}^k\right)_i - r_i^j}_{\leq 0}\Bigg| + \sum_{k=0}^{j-1} |\underbrace{-r_i^k}_{\leq 0}| \\ &= \left|\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k\right|_i \end{aligned} \tag{6.32}$$

with the same argumentation as above.

(iii) Let now

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)_i > 0. \tag{6.33}$$

Together with (6.26), the formula in (6.25) can be written as

$$\sum_{k=1}^{m} |\underbrace{d_i^k - r_i^k}_{=0}| + \Bigg|\underbrace{\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k\right)_i - r_i^0}_{\leq 0}\Bigg| = \left|\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k\right|_i. \tag{6.34}$$

Applying this to all n firms leads to (6.23).

For the second part, if $\Phi(\mathbf{R}) \geq \mathbf{R}$, the argumentation is very similar. Note that in this case, the sign in the norms change to larger or equal. We only demonstrate this for (6.28) that becomes to

$$\underbrace{\left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k \right)_i - r_i^m \right|}_{\geq 0} + \sum_{k=0}^{m-1} |\underbrace{-r_i^k}_{=0}| = \left| \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k \right|_i , \tag{6.35}$$

where $r_i^k = 0$ for $k = 0, \dots, m-1$ follows from $\Phi(\mathbf{R}) \geq \mathbf{R}$. The remaining cases are analogous. □

Lemma 6.6 says that we can explicitly calculate the norm $\|\mathbf{R}^{k+1} - \mathbf{R}^k\|$, since when starting with either $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ or with $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$, the assumptions $\Phi(\mathbf{R}) \leq \mathbf{R}$ or $\Phi(\mathbf{R}) \geq \mathbf{R}$ are fulfilled, respectively. This immediately leads to an a priori error for $\mathbf{R}_{\text{great}}$ and for $\mathbf{R}_{\text{small}}$.

Corollary 6.7. *It holds that:*

(i) $\|\mathbf{R}_{\text{small}}\| = \|\mathbf{a}\|$,

(ii) $\|\Phi(\mathbf{R}_{\text{small}}) - \mathbf{R}_{\text{small}}\| = \| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}_{\text{small}}^k \|$,

(iii) $\|\Phi(\mathbf{R}_{\text{great}}) - \mathbf{R}_{\text{great}}\| = \|\mathbf{a} + \sum_{k=0}^{m} (\mathbf{M}^k - \mathbf{I}_n) \mathbf{r}_{\text{great}}^k \|$.

Proof. The equation in (iii) is a simple application of Lemma 6.6. For (i) we further used the fact that $\mathbf{R}_{\text{small}} = \Phi(\mathbf{0}_{n(m+1)})$. For the equation in (ii), observe that

$$\sum_{k=0}^{m} \mathbf{r}_{\text{small}}^k = \min\{\mathbf{d}^m, \mathbf{a}\} + \sum_{k=1}^{m-1} \min\left\{ \mathbf{d}^k, \left(\mathbf{a} - \sum_{j=k+1}^{m} \mathbf{d}^j \right)^+ \right\} + \left(\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k \right)^+ = \mathbf{a} \tag{6.36}$$

which is easy to check component-wise (cf. Fischer, 2014, Lemma A5). Together with (6.23), we get

$$\begin{aligned} \|\Phi(\mathbf{R}_{\text{small}}) - \mathbf{R}_{\text{small}}\| &= \left\| \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}_{\text{small}}^k - \sum_{k=0}^{m} \mathbf{r}_{\text{small}}^k \right\| \\ &= \left\| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}_{\text{small}}^k \right\| . \end{aligned} \tag{6.37}$$

□

Remark 6.8. Note that a direct consequence of Lemma 6.6 is that in the fixed point $\mathbf{R}^*$ it holds that

$$\|\Phi(\mathbf{R}^*) - \mathbf{R}^*\| = \left\| \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=0}^{m} \mathbf{r}^{*,k} \right\| = 0 \tag{6.38}$$

from which follows that

$$\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} = \sum_{k=0}^{m} \mathbf{r}^{*,k}. \tag{6.39}$$

These are the balance sheet equations which summarize assets (left) and liabilities (right).

Another consequence of Lemma 6.6 is that the a priori error of $\mathbf{R}_{\text{great}}$ is independent from the ownership structure of the equity described in $\mathbf{M}^0$. To see this, we use the equation in part (iii) of Corollary 6.7 and the definition of $\mathbf{r}^0_{\text{great}}$ in (5.2) to get

$$\begin{aligned}
\|\Phi(\mathbf{R}_{\text{great}}) - \mathbf{R}_{\text{great}}\| &= \left\| \mathbf{a} + \sum_{k=0}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} \right\| \\
&= \left\| \mathbf{a} + \sum_{k=1}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} - (\mathbf{I}_n - \mathbf{M}^0)\mathbf{r}^0_{\text{great}} \right\| \qquad (6.40) \\
&= \left\| \mathbf{a} + \sum_{k=1}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} - \left(\mathbf{a} + \sum_{k=1}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} \right)^+ \right\|,
\end{aligned}$$

where the last norm does not contain $\mathbf{M}^0$ anymore.

Another property of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ appears in the special case that no cross-ownership is present in the system, i.e. if

$$\mathbf{M}^m = \mathbf{M}^{m-1} = \ldots = \mathbf{M}^1 = \mathbf{M}^0 = \mathbf{0}_{n\times n}. \qquad (6.41)$$

Corollary 6.9. *Under* (6.41) *it holds that:*

(i) $\mathbf{R}^* = \mathbf{R}_{\text{small}} = \Phi(\mathbf{R}_{\text{great}})$ *and*

(ii) $\|\mathbf{R}_{\text{great}} - \mathbf{R}_{\text{small}}\| = \|\mathbf{a} - \sum_{k=0}^{m} \mathbf{r}^k_{\text{great}}\|$.

Proof. (i) In Corollary 6.7 (ii) it was shown that for the a priori error of $\mathbf{R}_{\text{small}}$ we have

$$\|\Phi(\mathbf{R}_{\text{small}}) - \mathbf{R}_{\text{small}}\| = \left\| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k_{\text{small}} \right\| \overset{(6.41)}{=} \|\mathbf{0}_n\| = 0. \qquad (6.42)$$

Therefore, $\mathbf{R}_{\text{small}}$ fulfills the fixed point property and must consequently be equal to the fixed point $\mathbf{R}^*$, proving the first equality. For the second equality, check that under (6.41) the last n components of $\mathbf{R}_{\text{great}}$ become

$$\mathbf{r}^0_{\text{great}} = \left(\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k \right)^+ \qquad (6.43)$$

and further

$$\Phi(\mathbf{R}_{\text{great}}) = \Phi \begin{pmatrix} \mathbf{r}^m_{\text{great}} \\ \vdots \\ \mathbf{r}^1_{\text{great}} \\ \mathbf{r}^0_{\text{great}} \end{pmatrix} = \Phi \begin{pmatrix} \min\{\mathbf{d}^m, \mathbf{a}\} \\ \vdots \\ \min\{\mathbf{d}^1, (\mathbf{a} - \sum_{k=2}^{m} \mathbf{d}^k)^+\} \\ (\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k)^+ \end{pmatrix} = \mathbf{R}_{\text{small}}. \qquad (6.44)$$

(ii) From part (i) and Corollary 6.7 (iii), it follows that under (6.41)

$$\begin{aligned}
\|\mathbf{R}_{\text{great}} - \mathbf{R}_{\text{small}}\| &= \|\mathbf{R}_{\text{great}} - \Phi(\mathbf{R}_{\text{great}})\| \\
&= \left\| \mathbf{a} + \sum_{k=0}^{m} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} \right\| \qquad (6.45) \\
&= \left\| \mathbf{a} - \sum_{k=0}^{m} \mathbf{r}^k_{\text{great}} \right\|.
\end{aligned}$$

$\square$

In other words, Corollary 6.9 says that if no cross-ownership is present in the financial system, we can obtain the fixed point by simply calculating $\mathbf{R}_{\text{small}}$ or – more elaborate – by applying Φ on $\mathbf{R}_{\text{great}}$. Check that, in line with the statement of Proposition 6.2, we have that under (6.41) $\mathbf{R}_{\text{great}} = \mathbf{R}_{\text{small}} = \mathbf{R}^*$ if and only if $a_i \geq \sum_{k=1}^{m} d_i^k$ for all $i \in \mathcal{N}$. The findings of the previous Corollary can help to better understand why for systems with low degrees of cross-ownership, $\mathbf{R}_{\text{small}}$ is mostly the starting vector that needs less iteration steps to reach $\mathbf{R}^*$.

As we have determined the a priori error in Equation (6.2) with Lemma 6.6 and Corollary 6.7, we want to do this for the error in Equation (6.3) in the same way. To this end, we need the following lemma.

Lemma 6.10. *Let* $\mathbf{R} = (\mathbf{r}^m, \ldots, \mathbf{r}^0) \in \mathbb{R}^{n(m+1)}$ *with* $\mathbf{R}_{\text{small}} \leq \mathbf{R} \leq \mathbf{R}_{\text{great}}$. *Then:*

$$\|\Phi(\mathbf{R}) - \mathbf{R}_{\text{great}}\| = \left\| \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k_{\text{great}} \right\| \tag{6.46}$$

$$\|\Phi(\mathbf{R}) - \mathbf{R}_{\text{small}}\| = \left\| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k \right\|. \tag{6.47}$$

Proof. To prove the validity of Equation (6.46), check that

$$\|\Phi(\mathbf{R}) - \mathbf{R}_{\text{great}}\| = \left\| \begin{pmatrix} \min\{\mathbf{d}^m, & \mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k\} & - \mathbf{r}^m_{\text{great}} \\ \min\{\mathbf{d}^{m-1}, & (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \mathbf{d}^m)^+\} & - \mathbf{r}^{m-1}_{\text{great}} \\ \vdots & & \\ \min\{\mathbf{d}^1, & (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=2}^{m} \mathbf{d}^k)^+\} & - \mathbf{r}^1_{\text{great}} \\ & (\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \cdot \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k)^+ & - \mathbf{r}^0_{\text{great}} \end{pmatrix} \right\|. \tag{6.48}$$

The argumentation follows the one in the proof of Lemma 6.6, i.e. we check the validity componentwise for each firm i. As above, we distinguish between three different scenarios. During the proof, we again use that $d_i^0 := 0$.

(i) Let $\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k\right)_i \leq d_i^m$ which leads to

$$\left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k \right)_i - d_i^m \right| + \sum_{k=0}^{m-1} |-(\mathbf{r}^k_{\text{great}})_i| = \left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k_{\text{great}} \right) \right|_i. \tag{6.49}$$

(ii) Let for $m > j \geq 0$,

$$\left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+1}^{m} \mathbf{d}^k \right)_i \leq d_i^j \text{ and } \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+2}^{m} \mathbf{d}^k \right)_i > d_i^{j+1}. \tag{6.50}$$

Then:

$$\left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=j+1}^{m} \mathbf{d}^k \right)_i - d_i^j \right| + \sum_{k=0}^{j-1} |-(\mathbf{r}^k_{\text{great}})_i| = \left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=0}^{m} \mathbf{r}^k_{\text{great}} \right) \right|_i. \tag{6.51}$$

(iii) The case of $(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=1}^{m}\mathbf{d}^k)_i>0$ is straightforward.

For Equation (6.47), expanding the norm leads to

$$\|\Phi(\mathbf{R})-\mathbf{R}_{\text{small}}\| = \left\|\begin{pmatrix} \min\{\mathbf{d}^m, \quad \mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\cdot\mathbf{r}^k\} & -\min\{\mathbf{d}^m,\mathbf{a}\} \\ \min\{\mathbf{d}^{m-1},(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\cdot\mathbf{r}^k-\mathbf{d}^m)^+\} & -\min\{\mathbf{d}^{m-1},(\mathbf{a}-\mathbf{d}^m)^+\} \\ \vdots & \\ \min\{\mathbf{d}^1, \quad (\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\cdot\mathbf{r}^k-\sum_{k=2}^{m}\mathbf{d}^k)^+\} & -\min\left\{\mathbf{d}^1,\left(\mathbf{a}-\sum_{k=2}^{m}\mathbf{d}^k\right)^+\right\} \\ (\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\cdot\mathbf{r}^k-\sum_{k=1}^{m}\mathbf{d}^k)^+ & -\left(\mathbf{a}-\sum_{k=1}^{m}\mathbf{d}^k\right)^+ \end{pmatrix}\right\|. \tag{6.52}$$

In this situation we have to take four different cases into account.

(i) Let $(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k)_i\le d_i^m$ from which follows that $a_i\le d_i$. (6.52) becomes for the i-th firm:

$$\left|(\mathbf{a}+\sum_{k=0} 1^m\mathbf{M}^k\mathbf{r}^k)_i-a_i\right| = \left|\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k\right|_i. \tag{6.53}$$

(ii) Let for $m>j\ge 0$,

$$\left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=j+1}^{m}\mathbf{d}^k\right)_i\le d_i^j \text{ and } \left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=j+2}^{m}\mathbf{d}^k\right)_i > d_i^{j+1}. \tag{6.54}$$

as well as for $p\ge j$,

$$\left(\mathbf{a}-\sum_{k=p+1}^{m}\mathbf{d}^k\right)_i\le d_i^p \text{ and } \left(\mathbf{a}-\sum_{k=p+2}^{m}\mathbf{d}^k\right)_i > d_i^{p+1}. \tag{6.55}$$

It follows that $\left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=j+1}^{m}\mathbf{d}^k\right)_i>0$ and $\left(\mathbf{a}-\sum_{k=p+1}^{m}\mathbf{d}^k\right)_i>0$ and therefore

$$\left|d_i^p-\left(\mathbf{a}-\sum_{k=p+1}^{m}\mathbf{d}^k\right)_i\right|+\sum_{k=j+1}^{p-1}|d_i^k|+\left|\left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=j+1}^{m}\mathbf{d}^k\right)_i\right| = \left|\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k\right|_i. \tag{6.56}$$

Note that in case of $p=m$, (6.55) means that $a_i\le d_i$ and, hence, that the right part in (6.55) can be ignored.

(iii) Let $\left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=1}^{m}\mathbf{d}^k\right)_i>0$ and for $m\ge j\ge 0$,

$$\left(\mathbf{a}-\sum_{k=j+1}^{m}\mathbf{d}^k\right)_i\le d_i^j \text{ and } \left(\mathbf{a}-\sum_{k=j+2}^{m}\mathbf{d}^k\right)_i > d_i^{j+1}, \tag{6.57}$$

from which follows that

$$\left|d_i^j-\left(\mathbf{a}-\sum_{k=j+1}^{m}\mathbf{d}^k\right)_i\right|+\sum_{k=1}^{j-1}|d_i^k|+\left|\left(\mathbf{a}+\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k-\sum_{k=1}^{m}\mathbf{d}^k\right)_i\right| = \left|\sum_{k=0}^{m}\mathbf{M}^k\mathbf{r}^k\right|_i. \tag{6.58}$$

The comments at the end of part (ii) also hold here if $m=j$.

(iv) If $\left(\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k\right)_i^+ > 0$, we get

$$\left| \left(\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k - \sum_{k=1}^{m} \mathbf{d}^k \right)_i - \left(\mathbf{a} - \sum_{k=1}^{m} \mathbf{d}^k \right)_i \right| = \left| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^k \right|_i , \tag{6.59}$$

which completes the proof.

□

Because of $\Phi(\mathbf{R}^*) = \mathbf{R}^*$, the results of Lemma 6.10 can directly be applied to the fixed point $\mathbf{R}^*$.

Corollary 6.11.

(i) $\|\mathbf{R}^* - \mathbf{R}_{\text{great}}\| = \left\|\mathbf{a} + \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} - \sum_{k=0}^{m} \mathbf{r}_{\text{great}}^k\right\|$.

(ii) $\|\mathbf{R}^* - \mathbf{R}_{\text{small}}\| \leq \left\|\sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}_{\text{great}}^k\right\|$.

Proof. Since $\mathbf{R}^* = \Phi(\mathbf{R}^*)$ is the fixed point of Φ, it follows that $\mathbf{R}^* \in [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$. Thus, we can use the equations given in Lemma 6.10. Part (i) then becomes obvious, part (ii) follows because of Proposition 6.2 and

$$\|\mathbf{R}^* - \mathbf{R}_{\text{small}}\| = \left\| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}^{*,k} \right\| \leq \left\| \sum_{k=0}^{m} \mathbf{M}^k \mathbf{r}_{\text{great}}^k \right\| . \tag{6.60}$$

□

6.1.2 Minimizing the Iteration Error

With the properties of $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ in mind, we can now summarize our findings and take a closer analytical look at the iteration error given in Lemma 6.1. The main goal of our consideration is to minimize the distance of the k-th iteration $\mathbf{R}^k$ to the fixed point $\mathbf{R}^*$. For this purpose, we can use the two inequalities

$$\|\mathbf{R}^* - \mathbf{R}^k\| \leq \begin{cases} \frac{(I^{\max})^k}{1 - I^{\max}} \|\mathbf{R}^1 - \mathbf{R}^0\| \\ (I^{\max})^k \|\mathbf{R}^* - \mathbf{R}^0\|. \end{cases} \tag{6.61}$$

For $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$, together with the Corollaries 6.11 (i) and 6.7 (iii), this leads to

$$\|\mathbf{R}^* - \Phi^k(\mathbf{R}_{\text{great}})\| \leq \begin{cases} \dfrac{(I^{\max})^k}{1 - I^{\max}} \left\| \mathbf{a} + \sum_{l=0}^{m} (\mathbf{M}^l - \mathbf{I}_n) \mathbf{r}_{\text{great}}^l \right\| & (6.62) \\ (I^{\max})^k \left\| \mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=0}^{m} \mathbf{r}_{\text{great}}^l \right\|. & (6.63) \end{cases}$$

Equation (6.63) still contains the unknown fixed point and a plausible upper bound for this norm could not be found. Consequently, we can only use the error in (6.62) for further analysis.

For $\mathbf{R}^0 = \mathbf{R}_{\text{small}}$, we use the Corollaries 6.11 (ii) and 6.7 (ii) and get

$$\|\mathbf{R}^* - \Phi^k(\mathbf{R}_{\text{small}})\| \leq \begin{cases} \dfrac{(I^{\max})^k}{1 - I^{\max}} \left\| \displaystyle\sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{small}} \right\| & (6.64) \\ (I^{\max})^k \left\| \displaystyle\sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} \right\|. & (6.65) \end{cases}$$

In this case, we can use both inequalities in (6.64) and (6.65), since only the information that is a priori available in the financial system $\mathcal{F}$ is needed. To decide in which situation which estimator is more precise, we can compare (6.64) and (6.65) with one another:

$$\begin{aligned} & \frac{(I^{\max})^k}{1 - I^{\max}} \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{small}} \right\| = (I^{\max})^k \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} \right\| \\ \iff \quad & \frac{1}{1 - I^{\max}} \left\| \sum_{l=0}^{m} \mathbf{r}^l_{\text{small}} \right\| = \left\| \sum_{l=0}^{m} \mathbf{r}^l_{\text{great}} \right\| \\ \iff \quad & \frac{1}{1 - I^{\max}} \|\mathbf{a}\| = \|\mathbf{R}_{\text{great}}\|, \end{aligned}$$

where $\| \sum_{l=0}^{m} \mathbf{r}^l_{\text{small}} \| = \|\mathbf{a}\|$ because of Corollary 6.7 (i). Solving this for $I^{\max}$ leads to

$$I^{\max} = 1 - \frac{\|\mathbf{a}\|}{\|\mathbf{R}_{\text{great}}\|} \in [0, 1], \tag{6.66}$$

because of $\|\mathbf{a}\| \leq \|\mathbf{R}_{\text{great}}\|$. Thus, the comparison in (6.66) is no contradiction to Assumption 5.6. Note that for $\|\mathbf{R}_{\text{great}}\| = 0$, the ratio in (6.66) is not defined. But $\|\mathbf{R}_{\text{great}}\|$ can only be zero if $\mathbf{d}^k = \mathbf{0}_n$ for all $k = 1, \ldots, m$, hence we can assume that the ratio exists. The conclusion from the result in (6.66) is that, before comparing $\|\mathbf{R}^* - \mathbf{R}^k\|$ for the two starting vectors, we have to determine $1 - (\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|)$. If $I^{\max}$ is smaller than this difference, the expression in (6.64) is the more exact upper bound for $\|\mathbf{R}^* - \Phi^k(\mathbf{R}_{\text{small}})\|$. If $I^{\max} > 1 - (\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|)$, we take (6.65) as the upper bound.

Example 6.12. We want to demonstrate the principle of the case differentiation outlined in (6.66) for the financial systems defined in (6.14). Set $k = 1$, i.e. we investigate the error after the first iteration $\mathbf{R}^1 = \Phi(\mathbf{R}_{\text{small}})$. For the first system, the iteration error is $\|\mathbf{R}^* - \mathbf{R}^1\| = 0.2647$ and we have that $I^{\max} = 0.75 > 0.3373 = 1 - (\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|)$ from which follows that

$$\frac{I^{\max}}{1 - I^{\max}} \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{small}} \right\| = 4.9125 > 1.4367 = I^{\max} \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} \right\|, \tag{6.67}$$

so the estimate in (6.65) is the more exact one. If we modify the system in (6.14) as follows:

$$\mathbf{d} = \begin{pmatrix} 4 \\ 3.5 \\ 2 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 0.05 & 0.025 \\ 0.01 & 0 & 0.05 \\ 0.01 & 0.025 & 0 \end{pmatrix}, \tag{6.68}$$

and leave $\mathbf{a}$ and $\mathbf{M}^0$ unchanged, we obtain

$$\mathbf{R}^* = (1.1076, 1.1130, 2, 0, 0, 0.0389)^t. \tag{6.69}$$

This leads to $\|\mathbf{R}^* - \mathbf{R}^1\| = 0.0444$ and the two upper bounds are given by

$$\frac{I^{\max}}{1 - I^{\max}} \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{small}}^l \right\| = 0.0444 < 0.0844 = I^{\max} \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{great}}^l \right\|. \tag{6.70}$$

Because of $I^{\max} = 0.1 < 0.5263 = 1 - (\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|)$, the estimate in (6.64) is the more precise one in this case. We observe here that for highly indebted systems the more precise bound tends to be another one than for solvent systems.

Comparing the two bounds of (6.64) and (6.65), it is obvious that

$$\left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{small}}^l \right\| \leq \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{great}}^l \right\|, \tag{6.71}$$

since $\mathbf{R}_{\text{small}} \leq \mathbf{R}_{\text{great}}$. The answer which estimate is smaller must hence only lie in the parameter $I^{\max}$ that can be considered as a scaling factor in this situation. A relatively high value of $I^{\max}$ has the effect that the ratio $\frac{1}{1-I^{\max}}$ becomes much larger than 1. In such cases, the estimate in (6.64) becomes very large as well which makes it a useless bound as seen in Equation (6.67). Moreover, if we take a closer look at $\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|$, we can interpret this ratio as a way to measure the degree of indebtedness of the system. If it is large, the system must more likely be able to satisfy its obligees than in case of a small ratio. This means that $1 - \|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|$ is larger for highly indebted systems and thus is also more likely to be larger than $I^{\max}$ in which case (6.64) is chosen as the better bound. For highly solvent systems, the argumentation leads on an analogously way to the fact that $1 - \|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|$ tends to be smaller than $I^{\max}$ which results in the choice of (6.65).

The insights from above are outlined in the next Proposition.

Proposition 6.13 (Decision Rule)**.** *For the minimization of the upper bound of the k-th iteration error $\|\mathbf{R}^* - \Phi^k(\mathbf{R}^0)\|$, one has to calculate*

$$\Delta_{\text{start}} := \begin{cases} \left\| \mathbf{a} + \sum_{l=0}^{m} (\mathbf{M}^l - \mathbf{I}_n) \mathbf{r}_{\text{great}}^l \right\| - \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{small}}^l \right\| & \text{if } I^{\max} \leq 1 - \frac{\|\mathbf{a}\|}{\|\mathbf{R}_{\text{great}}\|} \quad (6.72) \\ \frac{1}{1 - I^{\max}} \left\| \mathbf{a} + \sum_{l=0}^{m} (\mathbf{M}^l - \mathbf{I}_n) \mathbf{r}_{\text{great}}^l \right\| - \left\| \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}_{\text{great}}^l \right\| & \text{if } I^{\max} > 1 - \frac{\|\mathbf{a}\|}{\|\mathbf{R}_{\text{great}}\|} \quad (6.73) \end{cases}$$

before starting the iteration. If $\Delta_{\text{start}} > 0$ the choice of $\mathbf{R}_{\text{small}}$ as the starting vector is the better one and if $\Delta_{\text{start}} < 0$ the staring vector $\mathbf{R}_{\text{great}}$ should be preferred. In case of $\Delta_{\text{start}} = 0$ there is no preference in one of the two starting vectors and we demand to start with $\mathbf{R}_{\text{great}}$ in such situations. Hence, we define the starting vector

$$\mathbf{R}_{\text{opt}} := \begin{cases} \mathbf{R}_{\text{great}} & \text{if } \Delta_{\text{start}} \leq 0, \\ \mathbf{R}_{\text{small}} & \text{if } \Delta_{\text{start}} > 0 \end{cases} \tag{6.74}$$

and call this vector the optimal starting vector *for the Picard Algorithm.*

The rule behind the choice between $\mathbf{R}_{\text{great}}$ and $\mathbf{R}_{\text{small}}$ in (6.74) based on the definition of Δ_{start} for the optimal starting vector is denoted as *decision rule* in the following.

Example 6.14. The financial system defined in (6.14) is solvent, i.e. all firms are able to fully service their debt payments. In such a situation, $\mathbf{R}_{\text{great}}$ should obviously be chosen as $\mathbf{R}_{\text{opt}}$ by the decision rule. Because of

$$I^{\max} = 0.75 > 0.3373 = 1 - (\|\mathbf{a}\|/\|\mathbf{R}_{\text{great}}\|), \tag{6.75}$$

we calculate

$$\Delta_{\text{start}} = \frac{1}{1 - I^{\max}} \left\| \mathbf{a} + \sum_{k=0}^{1} (\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} \right\| - \left\| \sum_{k=0}^{1} \mathbf{M}^k \mathbf{r}^k_{\text{great}} \right\| = -1.4356 < 0 \tag{6.76}$$

and therefore choose $\mathbf{R}_{\text{great}}$ as the optimal starting vector. With a tolerance level of $\varepsilon = 10^{-6}$ there are 7 iterations needed for $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ and 8 when $\mathbf{R}_{\text{small}}$ is the starting point.

For the modified system in (6.69), we find that $\Delta_{\text{start}} = 3.2 > 0$ and start the iteration with $\mathbf{R}_{\text{opt}} = \mathbf{R}_{\text{small}}$. It turns out that this was "correct" since with 7 iteration steps, there is one step less necessary to find $\mathbf{R}^*$ than when starting with $\mathbf{R}_{\text{great}}$.

The decision rule also applies for the financial system in Example 6.5. For the system in part (i), $\mathbf{R}_{\text{great}}$ is chosen as optimal and for the parts (ii) and (iii), the starting vector must be $\mathbf{R}_{\text{opt}} = \mathbf{R}_{\text{small}}$ according to the decision rule. As demonstrated therein, these choices result in all cases in a minimization of the number of iteration steps.

For the later simulations, we modify the Picard Iteration in Algorithm 1 and include the decision rule such that $\mathbf{R}_{\text{opt}}$ is always chosen as the starting point.

Algorithm 18 (Optimized Picard Algorithm)**.** *Set* $\varepsilon \geq 0$.

1. *For* $k = 0$, *determine* $\mathbf{R}_{\text{opt}} = \mathbf{R}^0 \in \{\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}\}$ *according to* (6.74).
2. *For* $k \geq 1$, *determine* $\mathbf{R}^k = \Phi(\mathbf{R}^{k-1})$.
3. *If* $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, *stop the algorithm. Else, set* $k = k + 1$ *and proceed with Step 2.*

In Section 7.3 we investigate whether the Optimized Picard Algorithm can really systematically minimize the computational effort compared to the two cases in which $\mathbf{R}_{\text{small}}$ or $\mathbf{R}_{\text{great}}$ are always "blindly" used as the initial iterate for the algorithm.

6.2 Elsinger and Hybrid Algorithm

Utilizing the Elsinger or the Hybrid Algorithms presented in the Sections 4.1.2 and 4.1.3 for the case of $m = 1$ and in the Sections 5.1.2 and 5.1.3 for $m > 1$, we are in the same conflict of choosing an appropriate starting vector for the iteration procedure. For both algorithms, the starting vectors are identical and given by

$$\begin{pmatrix} \mathbf{r}^m_{\text{great}} \\ \vdots \\ \mathbf{r}^1_{\text{great}} \\ \mathbf{r}^0(\mathbf{d}^m, \ldots, \mathbf{d}^1) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{r}^m_{\text{small}} \\ \vdots \\ \mathbf{r}^1_{\text{small}} \\ \mathbf{r}^0(\mathbf{r}^m_{\text{small}}, \ldots, \mathbf{r}^1_{\text{small}}) \end{pmatrix}, \tag{6.77}$$

where the first vector represents the upper and the second vector the lower starting point. To solve this trade-off situation, we want to apply the same ideas we used for the Picard Algorithm, i.e. estimate the a priori error $\|\mathbf{R}^1 - \mathbf{R}^0\|$ and the initial error $\|\mathbf{R}^* - \mathbf{R}^0\|$ and decide based on these estimates which starting vector to choose.

However, we will see in the remainder of this section, that applying the conclusions that we derived for the Picard Algorithm are not one-to-one applicable on the Elsinger and the Hybrid Algorithm. In particular, the distance between two consecutive iterates must not necessarily decrease, which means that Lemma 6.1 cannot serve as the basis for our considerations in this section. Nevertheless, we can still use the principles of optimizing the needed iterations by adapting the methods developed in Section 6.1.2 for the Elsinger and the Hybrid Algorithm. For the sake of simplicity, we present the detected results only for financial systems with one seniority level ($m = 1$). In this context, we omit the superscript for the nominal value of the liabilities and write $\mathbf{d} = \mathbf{d}^1$ instead during this section. Following this notation, the debt components of $\mathbf{R}_{\text{small}}$ become to $\mathbf{r}_{\text{small}} = \mathbf{r}^1_{\text{small}}$ as well as $\mathbf{r}_{\text{great}} = \mathbf{r}^1_{\text{great}}$ for $\mathbf{R}_{\text{great}}$. Moreover, the k-th debt iterate is defined as $\mathbf{r}^k$ and we write $\mathbf{r}^0(\mathbf{r}^k)$ for the corresponding equity iterate.

When calculating the norm of an iterate $\mathbf{R}^k$, the contribution of firm i to the norm is in this section defined as

$$\|\mathbf{R}^k\|_i := \sum_{l=0}^{m} |r_i^{k,l}|. \tag{6.78}$$

Moreover, the contribution of firm i to the norm of the difference between two iterates $\mathbf{R}^1$ and $\mathbf{R}^2$ is given by

$$\|\mathbf{R}^1 - \mathbf{R}^2\|_i := \sum_{l=0}^{m} |r_i^{1,l} - r_i^{2,l}|. \tag{6.79}$$

6.2.1 Behavior of the Distance between consecutive Iterates

Denote by

$$\mathbf{R}^k = \begin{pmatrix} \mathbf{r}^k \\ \mathbf{r}^0(\mathbf{r}^k) \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{d})^+ \end{pmatrix} \tag{6.80}$$

the k-th iterate of the Elsinger Algorithm (Algorithm 3) and by

$$\mathbf{R}^k = \begin{pmatrix} \mathbf{r}^k \\ \mathbf{r}^0(\mathbf{r}^k) \end{pmatrix} = \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{d})^+ \end{pmatrix} \tag{6.81}$$

the k-th iterate of the Hybrid Algorithm (Algorithm 6). The vector $\mathbf{r}^{k-1}$ is in both cases the debt iterate of the corresponding preceding iteration step and $\mathbf{r}^0(\mathbf{r}^{k-1})$ is the associated equity vector.

Recall that for the derivation of the upper bounds for the iteration and the a priori error of the Picard Algorithm, a crucial assumption in Lemma 6.1 was that

$$\|\mathbf{R}^{k+1} - \mathbf{R}^k\| \leq I^{\max}\|\mathbf{R}^k - \mathbf{R}^{k-1}\| \tag{6.82}$$

for all $k \geq 0$. At least for the Increasing Elsinger Algorithm, no such statement is possible as can be seen in the next Example.

Example 6.15. We consider the financial system with $n = 5$ firms and $m = 1$ seniority level. Further,

$$\mathbf{a} = \begin{pmatrix} 3.15 \\ 2.95 \\ 2.88 \\ 2.85 \\ 3.06 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 11.99 \\ 6.60 \\ 4.17 \\ 6.15 \\ 12.94 \end{pmatrix}, \ \mathbf{M}^1 = \begin{pmatrix} 0 & 1/6 & 1/8 & 0 & 0 \\ 1/8 & 0 & 1/8 & 1/4 & 1/6 \\ 1/8 & 0 & 0 & 0 & 1/6 \\ 1/8 & 1/6 & 1/8 & 0 & 1/6 \\ 1/8 & 1/6 & 1/8 & 1/4 & 0 \end{pmatrix} \tag{6.83}$$

and $\mathbf{M}^0 = \frac{1}{2}\mathbf{M}^1$. The fixed point is

$$\mathbf{R}^* = (4.8167, 6.6, 4.1, 6.15, 6.8807, 0, 0.2007, 0.4589, 0.1155, 0)^t \tag{6.84}$$

and the first five iterates are given as

$$\mathbf{R}^0 = \begin{pmatrix} 3.15 \\ 2.95 \\ 2.88 \\ 2.85 \\ 3.06 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}^1 = \begin{pmatrix} 4.0017 \\ 4.9263 \\ 3.7837 \\ 4.6054 \\ 5.0179 \\ 0 \\ 0 \\ 0.0465 \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}^2 = \begin{pmatrix} 4.4469 \\ 5.9138 \\ 4.17 \\ 5.4834 \\ 6.0085 \\ 0 \\ 0 \\ 0.2673 \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}^3 = \begin{pmatrix} 4.6736 \\ 6.4161 \\ 4.17 \\ 5.9309 \\ 6.5103 \\ 0 \\ 0.0469 \\ 0.3792 \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}^4 = \begin{pmatrix} 4.7682 \\ 6.6 \\ 4.17 \\ 6.1375 \\ 6.7451 \\ 0 \\ 0.1630 \\ 0.4302 \\ 0.0819 \\ 0 \end{pmatrix}. \tag{6.85}$$

Because of $I^{\max} = 0.5$, it holds that

$$\begin{aligned} \|\mathbf{R}^2 - \mathbf{R}^1\| &= 3.9084 > 3.7458 = I^{\max}\|\mathbf{R}^1 - \mathbf{R}^0\| \\ \|\mathbf{R}^4 - \mathbf{R}^3\| &= 0.9689 > 0.9186 = I^{\max}\|\mathbf{R}^3 - \mathbf{R}^2\|. \end{aligned} \tag{6.86}$$

Albeit simulation results suggest that (6.82) is fulfilled for Decreasing Elsinger and both versions for the Hybrid Algorithm, no strict formal proof could be found to substantiate this conjecture, neither for the increasing nor for the decreasing version of the procedure. Also no counterexamples could be found to falsify the validity of (6.82) for these procedures.

6.2.2 Estimates for the Iteration Errors

The fact that the property of a decreasing distances of iterates as in (6.82) cannot be ensured or is even violated in some situations, actually prevents the usage of the upper bounds for the Elsinger and the Hybrid Algorithm given in Lemma 6.1, since in the proof, that assumption was crucial. Nevertheless, we still attempt to investigate whether there are upper bounds for the k-th iteration and the a priori error for the mentioned procedures in this subsection to gain a better insight into the problem of choosing an optimal starting vector. As in the section above, we only search upper bounds for $\|\mathbf{R}^1 - \mathbf{R}^0\|$ and $\|\mathbf{R}^* - \mathbf{R}^0\|$ when $m = 1$ to keep the results as simple as possible.

For both procedures, the Elsinger and the Hybrid Algorithm, we did not found any formulas to express the norm $\|\mathbf{R}^1 - \mathbf{R}^0\|$ in a more simple form. However, for the Decreasing Elsinger Algorithm, there exists an upper bound for $\|\mathbf{R}^1 - \mathbf{R}^0\|$.

Proposition 6.16. *For the Decreasing Elsinger Algorithm it holds that*

$$\|\mathbf{R}^1 - \mathbf{R}^0\| \leq \|\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d}\|. \tag{6.87}$$

Proof. We check the norm component-wise for an arbitrary firm $i \in \mathcal{N}$. The norm for the decreasing version of the algorithm constitutes as

$$\|\mathbf{R}^1 - \mathbf{R}^0\| = \left\| \begin{matrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\} - \mathbf{d} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^1) - \mathbf{d})^+ - (\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d})^+ \end{matrix} \right\|. \tag{6.88}$$

For the calculation of this norm we distinguish between three cases. Keep in mind for the following, that $\mathbf{r}^1 \leq \mathbf{d}$ and $\mathbf{r}^0(\mathbf{r}^1) \leq \mathbf{r}^0(\mathbf{d})$.

- Let $i \in \mathcal{D}(\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$. It holds that

$$\|\mathbf{R}^1 - \mathbf{R}^0\|_i = |\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d}|_i. \tag{6.89}$$

- Let $i \notin \mathcal{D}(\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$ and $i \in \mathcal{D}(\mathbf{r}^1, \mathbf{r}^0(\mathbf{r}^1))$. Hence,

$$\|\mathbf{R}^1 - \mathbf{R}^0\|_i = |\mathbf{d} - \mathbf{d}|_i + |\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d}|_i. \tag{6.90}$$

- At last, let $i \notin \mathcal{D}(\mathbf{r}^1, \mathbf{r}^0(\mathbf{r}^1))$ which means that

$$\left(\mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^1)\right)_i \geq (\mathbf{d} - \mathbf{a})_i \tag{6.91}$$

  and thus

$$\begin{aligned} \|\mathbf{R}^1 - \mathbf{R}^0\|_i &= |\mathbf{M}^1\mathbf{r}^1 + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^1) - \mathbf{M}^1\mathbf{d} - \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i \\ &\leq |\mathbf{d} - \mathbf{a} - \mathbf{M}^1\mathbf{d} - \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i \\ &= |\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d}|_i. \end{aligned} \tag{6.92}$$

Assembling the three cases results in Equation (6.87). □

The generalization of Proposition 6.16 for $m > 1$ is straightforward; similar to the proof in case of $m = 1$, more case differentiations have to be considered. It seems natural to apply the upper bound of the Increasing Picard Algorithm from Corollary 6.7 in an analogous way to the Increasing Elsinger Algorithm. This would result in

$$\|\mathbf{R}^1 - \mathbf{R}^0\| \leq \|\mathbf{M}^1\mathbf{r}_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})\|. \tag{6.93}$$

However, this bound does not hold in general, as can easily be checked for the financial system of Example 6.15. Doing so, we get

$$\|\mathbf{R}^1 - \mathbf{R}^0\| = 7.4915 > 7.445 = \|\mathbf{M}^1\mathbf{r}_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})\| \tag{6.94}$$

and we see that this relationship is violated. An alternative plausible upper boundary for the a priori error of the algorithm could not be found.

Moreover, there exists a connection between the a priori error of the Decreasing Elsinger and the Decreasing Hybrid Algorithm. Let the subscript E denote the iterate of the Decreasing Elsinger Algorithm and the subscript H the iterate of the corresponding Hybrid Algorithm. Due to Proposition 4.19, we know that $\mathbf{r}_{\mathrm{H}}^1 \leq \mathbf{r}_{\mathrm{E}}^1$ and therefore $\mathbf{r}^0(\mathbf{r}_{\mathrm{H}}^1) \leq \mathbf{r}^0(\mathbf{r}_{\mathrm{E}}^1)$. Since both algorithms have the same starting vector $\mathbf{R}^0$, it must hold that

$$\|\mathbf{R}_{\mathrm{H}}^1 - \mathbf{R}^0\| \geq \|\mathbf{R}_{\mathrm{E}}^1 - \mathbf{R}^0\|. \tag{6.95}$$

In other words, the distance between the first iterate and the starting vector is for the Decreasing Hybrid Algorithm always larger than its counterpart from the Elsinger Algorithm. Clearly, the connection between both algorithms and the a priori error is same if we consider the increasing versions of the procedures.

Upper bounds for the a priori errors of Increasing and Decreasing Hybrid Algorithm could not be found. An obvious guess for an upper bound of the Decreasing Hybrid Algorithm would be to apply the insights of Corollary 6.7, where we found that

$$\|\Phi(\mathbf{R}_{\mathrm{great}}) - \mathbf{R}_{\mathrm{great}}\| = \|\mathbf{a} + \mathbf{M}^1\mathbf{r}_{\mathrm{great}} + \mathbf{M}^0\mathbf{r}_{\mathrm{great}}^0 - \mathbf{d}\|. \tag{6.96}$$

However, this bound does not hold in general, as Example 6.17 demonstrates. In the same way, the intuitive upper bound for the increasing procedure would be $\|\mathbf{M}^1\mathbf{r}_{\mathrm{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\mathrm{small}})\|$ which, like in the case for the Elsinger Algorithm, also does not hold in general as the next example shows.

Example 6.17. Consider again the financial system from Example 6.15. Following Equation (6.93), we obtain for the Increasing Hybrid Algorithm that

$$\|\mathbf{R}^1 - \mathbf{R}^0\| = 14.3356 > 7.445 = \|\mathbf{M}^1\mathbf{r}_{\mathrm{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\mathrm{small}})\| \tag{6.97}$$

and it becomes clear that an upper bound for this algorithm cannot be the counterpart that was found for the Picard Algorithm.

For the Decreasing Hybrid Algorithm, we can also show that the upper bound of the a priori error is not always the one given in (6.96). To this end, replace in the debt vector $\mathbf{d}$ from the considered system the second component with the value 8 (before, it was 6.6) such that

$$\mathbf{d} = (11.99, 8, 4.17, 6.15, 12.94)^t. \tag{6.98}$$

The solution $\mathbf{R}^*$ slightly changes now (among others, the second firm defaults now which was not the case before), which is of less interest. Important for us is that since

$$\|\mathbf{R}^1 - \mathbf{R}^0\| = 17.8362 > 17.1210 = \|\mathbf{a} + \mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{d}\| \tag{6.99}$$

the potential upper bound of the a priori error does not hold.

After having investigated the a priori error for both algorithms, we want to do so for the initial error $\|\mathbf{R}^* - \mathbf{R}^0\|$ as well. In case of the increasing versions of both algorithms, we can find an upper bound for the error, similar to the results of the Picard Algorithm in Corollary 6.11.

Proposition 6.18. *Let $\mathbf{R}^k$ denote the k-th iterate of either the Increasing Elsinger or the Increasing Hybrid Algorithm. Then it holds that*

$$\|\mathbf{R}^k - \mathbf{R}^0\| \leq \|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\| \tag{6.100}$$

and, thus, $\|\mathbf{R}^* - \mathbf{R}^0\| \leq \|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\|$.

Proof. We show the assertion first for the Elsinger Algorithm which means that the corresponding iteration error is given by

$$\begin{aligned}\|\mathbf{R}^k - \mathbf{R}^0\| &= \left\| \begin{pmatrix} \mathbf{r}^k - \mathbf{r}_{\text{small}} \\ \mathbf{r}^0(\mathbf{r}^k) - \mathbf{r}^0(\mathbf{r}_{\text{small}}) \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})\} - \min\{\mathbf{d}, \mathbf{a}\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{d})^+ - (\mathbf{a} + \mathbf{M}^1\mathbf{r}_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}}) - \mathbf{d})^+ \end{pmatrix} \right\| \end{aligned} \tag{6.101}$$

Similar to the proofs above, we have to do a case differentiation for the i-th component of the norm of $\|\mathbf{R}^k - \mathbf{R}^0\|$.

- Let $i \in \mathcal{D}(\mathbf{r}^k, \mathbf{r}^0(\mathbf{r}^k))$ from which follows in particular that $a_i < d_i$. Hence,

$$\begin{aligned}\|\mathbf{R}^k - \mathbf{R}^0\| &= |\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1}) - \mathbf{a}|_i \\ &= |\mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})|_i \\ &\leq |\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i. \end{aligned} \tag{6.102}$$

- Let next $i \notin \mathcal{D}(\mathbf{r}^k, \mathbf{r}^0(\mathbf{r}^k))$ and $i \in \mathcal{D}(\mathbf{r}^{k-1}, \mathbf{r}^0(\mathbf{r}^{k-1}))$. Consequently it holds that

$$(\mathbf{a} - \mathbf{d})_i < -(\mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1}))_i \tag{6.103}$$

  and thus

$$\begin{aligned}\|\mathbf{R}^k - \mathbf{R}^0\| &= |\mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})|_i + |\mathbf{a} + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{d}|_i \\ &\leq |\mathbf{M}^1\mathbf{r}^{k-1} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1}) + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{M}^1\mathbf{r}^{k-1} - \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^{k-1})|_i \\ &= |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k)|_i \\ &\leq |\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i \end{aligned} \tag{6.104}$$

- Let now $i \notin \mathcal{D}(\mathbf{r}^{k-1}, \mathbf{r}^0(\mathbf{r}^{k-1}))$ and $i \in \mathcal{D}(\mathbf{r}_{\text{small}}, \mathbf{r}^0(\mathbf{r}_{\text{small}}))$:

$$\begin{aligned}\|\mathbf{R}^k - \mathbf{R}^0\| &= |\mathbf{d} - \mathbf{a}|_i + |\mathbf{a} + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{d}|_i \\ &= |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k)|_i \\ &\leq |\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i. \end{aligned} \tag{6.105}$$

- Let $i \notin \mathcal{D}(\mathbf{r}_{\text{small}}, \mathbf{r}^0(\mathbf{r}_{\text{small}}))$ and $i \in \mathcal{D}(\mathbf{0}_n, \mathbf{0}_n)$ from which follows that

$$\left(\mathbf{M}^1\mathbf{r}_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})\right)_i > (\mathbf{d} - \mathbf{a})_i. \tag{6.106}$$

  Consequently,

$$\begin{aligned}\|\mathbf{R}^k - \mathbf{R}^0\| &= |\mathbf{d} - \mathbf{a}|_i + |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{M}^1\mathbf{r}_{\text{small}} - \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})|_i \\ &\leq |\mathbf{M}^1\mathbf{r}_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}}) + \mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{M}^1\mathbf{r}_{\text{small}} - \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})|_i \\ &= |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k)|_i \\ &\leq |\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i. \end{aligned} \tag{6.107}$$

- Finally, suppose that $i \notin \mathcal{D}(\mathbf{0}_n, \mathbf{0}_n)$, i.e. $a_i \geq d_i$, where we have that

$$\begin{aligned} \|\mathbf{R}^k - \mathbf{R}^0\| &= |\mathbf{d} - \mathbf{d}|_i + |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k) - \mathbf{M}^1\mathbf{r}_{\text{small}} - \mathbf{M}^0\mathbf{r}^0(\mathbf{r}_{\text{small}})|_i \\ &\leq |\mathbf{M}^1\mathbf{r}^k + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^k)|_i \\ &\leq |\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})|_i \end{aligned} \tag{6.108}$$

  which completes the proof for the Elsinger Algorithm.

For the Hybrid Algorithm, the argumentation is analogous, i.e. we consider again five different cases. □

Similar to the propositions above in this section, a generalization of Proposition 6.18 for systems with $m > 1$ is also possible. For the decreasing versions of Elsinger and Hybrid Algorithm, and similar as in the case of the Picard Algorithm, we found no upper bounds for $\|\mathbf{R}^* - \mathbf{R}^0\|$.

Recapitulating the results of this section, we first found that in most cases it is either difficult or impossible to give an upper bound for the iteration error. Secondly, some bounds that were derived for the Picard Algorithm, cannot be established in the same manner for the Elsinger and the Hybrid procedure. This essentially complicates a substantiated derivation of the k-th iteration error and therefore the search for an optimal starting point for both algorithms – not to mention the fact that the basis of all considerations for the Picard iterates was the validity of (6.82) which does not hold in general for the Elsinger and the Hybrid Algorithm. For these reasons we decided to apply the results of Proposition 6.13 to calculate the optimal starting point in a slightly adapted version for the Elsinger and the Hybrid Algorithm as well.

Definition 6.19. For the Elsinger and the Hybrid Algorithm, we define the starting vector

$$\mathbf{R}_{\text{opt}} := \begin{cases} (\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})) & \text{if } \Delta_{\text{start}} \leq 0, \\ (\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})) & \text{if } \Delta_{\text{start}} > 0, \end{cases} \tag{6.109}$$

where Δ_{start} is given by

$$\Delta_{\text{start}} := \begin{cases} \left\|\mathbf{a} + \sum_{k=1}^{m}(\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} + (\mathbf{M}^0 - \mathbf{I}_n)\mathbf{r}^0(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})\right\| \\ \quad - \left\|\sum_{k=1}^{m}\mathbf{M}^k\mathbf{r}^k_{\text{small}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})\right\| & \text{if } I^{\max} \leq x \quad (6.110) \\ \frac{1}{1 - I^{\max}}\left\|\mathbf{a} + \sum_{k=1}^{m}(\mathbf{M}^k - \mathbf{I}_n)\mathbf{r}^k_{\text{great}} + (\mathbf{M}^0 - \mathbf{I}_n)\mathbf{r}^0(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})\right\| \\ \quad - \left\|\sum_{k=1}^{m}\mathbf{M}^k\mathbf{r}^k_{\text{great}} + \mathbf{M}^0\mathbf{r}^0(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})\right\| & \text{if } I^{\max} > x \quad (6.111) \end{cases}$$

and

$$x = 1 - \frac{\|\mathbf{a}\|}{\sum_{k=1}^{m}\|\mathbf{r}^k_{\text{great}}\| + \|\mathbf{r}^0(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}})\|} \tag{6.112}$$

We are aware of the fact that this relies on no strict mathematical derivation, however the simulation results in Section 7.3 suggest that the decision rule in (6.109) leads to appropriate results concerning the "correct" choice of one of the two possible starting vectors.

Example 6.20. We want to check the goodness of the decision rule in (6.109) in a first example with the financial system given in (6.14). We find that

$$I^{\max} = 0.75 > 0.3223 = 1 - \frac{\|\mathbf{a}\|}{\|\mathbf{d}^1\| + \|\mathbf{r}^0(\mathbf{d}^1)\|} \tag{6.113}$$

and chose Equation (6.111) for the decision rule which leads to

$$\frac{1}{1 - I^{\max}} \|\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d}) - \mathbf{r}^0(\mathbf{d})\| - \|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0(\mathbf{d})\| = -1.9022 < 0. \tag{6.114}$$

We therefore chose $(\mathbf{d}^1, \mathbf{r}^0(\mathbf{d}^1))$ as the starting vector and observe that only 1 iteration step is necessary instead of 3 steps when the procedure is started with $(\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$. For the Hybrid Algorithm, the numbers are identical.

If we consider the modified system from (6.68), we use Equation (6.110) for the decision rule, since

$$I^{\max} = 0.1 < 0.5181 = 1 - \frac{\|\mathbf{a}\|}{\|\mathbf{d}\| + \|\mathbf{r}^0(\mathbf{d})\|}. \tag{6.115}$$

We get the value $\Delta_{\text{start}} = 3.06$, so $(\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ should be the starting vector according to the decision rule. This choice minimizes with 6 steps the number of iterations because one step more is needed for the decreasing version. The Hybrid Algorithm needs 4 steps for both directions to reach $\mathbf{R}^*$.

For the sake of completeness, we give in the following the optimized versions of the considered algorithms similar to the Optimized Picard Algorithm in the section above.

Algorithm 19 (Optimized Elsinger Algorithm)**.** *Set* $\varepsilon > 0$.

1. *For* $k = 0$, *choose*

$$\left(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1}\right) \in \left\{(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}}), (\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})\right\} \tag{6.116}$$

   according to (6.109) *and determine* $\mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1})$ *using Algorithm 2 or its modification for* $m > 1$ *given in* (5.14)*. Denote the iterate* $\mathbf{R}^0$ *by* $\mathbf{R}^0 = (\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,0})$.

2. *For* $k \geq 1$, *set for* $1 \leq j \leq m$,

$$\mathbf{r}^{k,j} = \min\left\{\mathbf{d}^j, \left(\mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{k-1,l} - \sum_{l=j+1}^{m} \mathbf{d}^l\right)^+\right\}, \tag{6.117}$$

   calculate $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,m}, \dots, \mathbf{r}^{k,1})$ *and set* $\mathbf{R}^k = (\mathbf{r}^{k,m}, \dots, \mathbf{r}^{k,0})$.

3. *If* $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, *stop the algorithm. Else, set* $k = k + 1$ *and proceed with Step 2.*

Algorithm 20 (Optimized Hybrid Algorithm ($m > 1$))**.** *Set* $\varepsilon > 0$.

1. *For* $k = 0$, *choose*

$$\left(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1}\right) \in \left\{(\mathbf{r}^m_{\text{great}}, \dots, \mathbf{r}^1_{\text{great}}), (\mathbf{r}^m_{\text{small}}, \dots, \mathbf{r}^1_{\text{small}})\right\} \tag{6.118}$$

   according to (6.109) *and determine* $\mathbf{r}^{0,0} = \mathbf{r}^0(\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,1})$ *using Algorithm 2 or its modification for* $m > 1$ *given in* (5.14)*. Denote the iterate* $\mathbf{R}^0$ *by* $\mathbf{R}^0 = (\mathbf{r}^{0,m}, \dots, \mathbf{r}^{0,0})$.

2. *For $k \geq 1$, determine for every $1 \leq j \leq m$ the fixed point $\mathbf{r}^{k,j}$ of the mapping in* (5.15)*, where for the mapping Φ^j the fixed vectors $\mathbf{r}^{k-1,m}, \ldots, \mathbf{r}^{k-1,j+1}, \mathbf{r}^{k-1,j-1}, \ldots, \mathbf{r}^{k-1,0}$ are taken from the preceding iterate. Calculate $\mathbf{r}^{k,0} = \mathbf{r}^0(\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,1})$ and set $\mathbf{R}^k = (\mathbf{r}^{k,m}, \ldots, \mathbf{r}^{k,0})$.*
3. *If $\|\mathbf{R}^{k-1} - \mathbf{R}^k\| < \varepsilon$, stop the algorithm. Else, set $k = k + 1$ and proceed with Step 2.*

7 Simulation Studies

The current chapter is about verifying the theoretical results of the preceding chapters by means of several simulation studies. In Chapter 6 we attempted to find a criteria to minimize the number of iteration steps for the non-finite algorithms by introducing a decision rule that chooses an optimal starting vector for the iteration procedure. This results in the Optimized Picard, Elsinger and Hybrid Algorithm. The question whether the optimized algorithms actually lead to a minimization of the number of needed iteration steps, compared to when always the maximum or the minimum possible starting vector is chosen, is addressed in Section 7.3. The advantage of the Trial-and-Error Algorithms (cf. Section 4.2 and 5.2) compared to the non-finite procedures is the fact that $\mathbf{R}^*$ is reached in a finite number of steps. A drawback of these algorithms is that the choice of the lag value l is a typical tradeoff conflict between the speed of convergence and the minimization of calculation effort. Section 7.4 of this chapter tries to find a solution of this conflict by giving an optimal lag value that minimizes the calculation time on the one hand and still guarantees that the lag value is not chosen too small to avoid situations in which the algorithm stops when having found a potential default set that is actually not equal to $\mathcal{D}^*$. In the last Section, we compare all developed algorithms in this work to investigate the algorithm efficiency trying to identify a potential most efficient valuation algorithm or at least trying to find most efficient algorithms for particular types of financial systems. A crucial influence in all simulations is the form of the underlying financial system, in particular the structure of the ownership matrices and the liability vectors. In Section 7.1, these terms are discussed in more detail, where we introduce some general expressions that are based on the ones given in Acemoglu et al. (2015) and Elliott et al. (2014). Section 7.2 contains an overview of the employed simulation parameters and a definition of the output parameters of the study. Note that all numbers in this chapter are rounded to only three decimal places due to space limitations.

7.1 Simulation Framework

During the entire simulation study, we only consider financial systems without a seniority structure. Because of $m = 1$, we denote the liability vector simply by $\mathbf{d} \in (\mathbb{R}_0^+)^n$. Sometimes we will talk about the ownership matrices of the k-th seniority level for $k = 0, 1$, which for $k = 0$ means that the equity ownership matrix is addressed. Beside its size n, the form of a financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$ is characterized by the structure of the exogenous asset vector $\mathbf{a}$, the liability vector $\mathbf{d}$ and by of the ownership matrices that are contained in $\mathbf{M} = (\mathbf{M}^1, \mathbf{M}^0) \in (\mathbb{R}_0^+)^{n \times 2n}$.

7.1.1 Asset and Debt Structure

The exogenous asset values are considered as fixed throughout all simulations and each firm has the same amount of exogenous assets with value 1, i.e. $\mathbf{a} = (a_1, \ldots, a_n)^t = \mathbf{1}_n$. We assume that the nominal liabilities in the vector $\mathbf{d}$ consist of a part $d > 0$ which is identical for all firms, to which some random variation is added, viz.

$$\mathbf{d} = \left((d, \dots, d)^t + (\varepsilon_1, \dots, \varepsilon_n)^t\right)^+ . \tag{7.1}$$

The ε_i are iid random variables with $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma > 0$, for all $i \in \mathcal{N}$. To ensure a non-negative liability vector, the positive part is taken in (7.1).

7.1.2 Ownership Structure

In order to define the structure of $\mathbf{M}^1$ and $\mathbf{M}^0$, we have to introduce some general terminology connected with ownership matrices. The following definitions are mainly based on the two works of Acemoglu et al. (2015) and Elliott et al. (2014). The authors of the former article introduce matrices without random influence, whereas the latter paper includes the aspect of randomness for the generation of a network structure.

Definition 7.1. For a financial system $\mathcal{F} = (\mathbf{a}, \mathbf{M}, \mathbf{d})$ and $k = 0, 1$, the *integration level of the k-th seniority* is defined as

$$\rho^k = \max_j \sum_{i=1}^n M_{ij}^k = \|\mathbf{M}^k\|. \tag{7.2}$$

For $k = 0$, we use the expression *equity integration level* and *debt integration level* for $k = 1$.

The integration level measures the degree or the extent of intersystem cross-holdings within each seniority level. It clearly holds that $\rho^k \in [0, 1]$ and, if Assumption 5.6 holds (cf. Chapter 6), even $\rho^k \in [0, 1)$. Note that the integration level can, under circumstances be a very "inhomogeneous" measure for the extend of cross-holdings. This is because of the fact that the integration level can be very close to its maximum value 1 even for very sparse ownership matrices where the ownership fractions are very small as long as at least one firm raised a substantial part of its debt or equity from other members of the system. However, in all of our simulation studies, we will ensure that by definition of the ownership matrices in the following, the column sums of the entries in a matrix $\mathbf{M}^k$ will be equal (if all entries are non-zero) and therefore also equal to ρ^k. Consequently, we try to keep the form of the ownership matrices as homogeneous as possible.

The second parameter to characterize the ownership structure is the actual number of present cross-holdings in a seniority level and is called diversification level.

Definition 7.2. As in Definition 7.1, we consider the financial system $\mathcal{F}$. The *diversification level of the k-th seniority* of $\mathcal{F}$ is defined as

$$\phi^k = \max_j \left| \left\{ i \in \mathcal{N} : M_{ij}^k > 0 \right\} \right| . \tag{7.3}$$

For $k = 0$ and $k = 1$, ϕ^0 and ϕ^1 are referred to as *equity diversification level* and *debt diversification level*, respectively.

For the debt ownership matrix, clearly $\phi^1 \leq n - 1$ and $\phi^0 \leq n$ for equity ownership matrix since $M_{ii}^0 > 0$ is not explicitly excluded in our model (cf. Definition 2.2). Practically, we will not allow the diagonal entries of $\mathbf{M}^0$ to be different from zero in the simulations, so the diversification level will also not be larger than $n - 1$. Similar to the definition of the integration level, the diversification level does not provide an explicit insight of the detailed connection structure of a matrix. In its most extreme from, the entries of $\mathbf{M}^k$ can all be zero in $n - 1$ columns and all be non-zero (except for the diagonal entry) in one single column resulting in a diversification level of $n-1$. For the purpose of getting homogeneous matrices, we keep the number of cross-holdings for non-random matrices identical for each firm. If random-ownership matrices are considered, this cannot be guaranteed, as we will see in the following.

After having established a framework for ownership matrices, we can now become more detailed and describe how such a matrix can actually be defined. Doing so, we distinguish between two different approaches that are presented in the next subsections. An elementary way to define the entries of an ownership matrix is to exclude all random influence and to determine the entries only based on the integration level which results in a non-random ownership matrix. In a second method, the entries of the matrix are specified randomly.

Non-random Ownership Structure

Ownership matrices of this type are matrices in which the entries are fixed and completely specified by their given input parameters. The subsequent considerations are based on the following definition of two particular non-random ownership matrices.

Definition 7.3. An ownership matrix $\mathbf{M}$ is called

(i) a *ring ownership matrix* if for $i = 1, \ldots, n-1$, the entries $M_{i+1,i}$ and M_{1n} are equal and greater than zero, and all other entries of $\mathbf{M}$ are zero.

(ii) a *complete ownership matrix* if all entries, except of the diagonal entries, are larger than zero and of the same size.

In other words, the debt ownership matrix $\mathbf{M}^1$ is a ring matrix, if firm $i+1$ is the only creditor of firm i within the system and firm 1 is the only creditor of firm n. Suppose that the equity matrix $\mathbf{M}^0$ is a ring matrix, this implies that the i-th firm has only one single shareholder within the system, namely the $(i+1)$-th firm. Analogous to debt matrices, the only shareholder of firm n is firm 1.

The diversification level of ring ownership matrices is always 1 whereas the diversification level of complete ownership matrices is $n-1$. For both types of matrices, the integration level ρ^k is equal to an arbitrary column sum of $\mathbf{M}^k$. Ring and complete ownership matrices can be considered as extreme forms of interconnectedness of financial systems. To construct intermediate type of networks, we will make use of the next definition.

Definition 7.4. Let $\overline{\mathbf{M}}$ be a ring ownership matrix and $\widetilde{\mathbf{M}}$ be a complete ownership matrix. A λ-*convex combination* of $\overline{\mathbf{M}}$ and $\widetilde{\mathbf{M}}$ is defined as the matrix $\mathbf{M}$ with entries

$$M_{ij} = \lambda \overline{M}_{ij} + (1-\lambda)\widetilde{M}_{ij}, \tag{7.4}$$

where $\lambda \in [0, 1]$.

The more λ decreases, the more assimilate the ownership fractions in the corresponding ownership matrix. For values of λ near the maximum 1, member $i+1$ of the system will be the main owner of i's equity or debt.

Example 7.5. For the size of the system we set $n = 4$, the level of integration is $\rho = 0.6$ for both the ring and the complete ownership matrix and $\lambda = 0.5$. The ring ownership matrix $\overline{\mathbf{M}}$, the complete ownership matrix $\widetilde{\mathbf{M}}$ and the λ-convex combination matrix $\mathbf{M}$ of them are

$$\overline{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 & .6 \\ .6 & 0 & 0 & 0 \\ 0 & .6 & 0 & 0 \\ 0 & 0 & .6 & 0 \end{pmatrix}, \quad \widetilde{\mathbf{M}} = \begin{pmatrix} 0 & .2 & .2 & .2 \\ .2 & 0 & .2 & .2 \\ .2 & .2 & 0 & .2 \\ .2 & .2 & .2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} 0 & .1 & .1 & .4 \\ .4 & 0 & .1 & .1 \\ .1 & .4 & 0 & .1 \\ .1 & .1 & .4 & 0 \end{pmatrix}. \tag{7.5}$$

In our simulation studies, non-random ownership matrices will always be λ-convex combinations of ring and complete matrices as in (7.4). Therefore, the integration level of the ring and of the complete matrix as well as the value of λ are sufficient for the generation of the corresponding ownership matrix. We summarize the findings of this subsection in the next Corollary.

Corollary 7.6. *Let $\lambda \in [0,1]$, $r \in [0,1)$ and let $\overline{\mathbf{M}}$ be a ring ownership matrix with non-zero elements r and let $\widetilde{\mathbf{M}}$ be a complete ownership matrix with integration level r, i.e. with non-diagonal elements $\frac{r}{n-1}$. The tuple (λ, r) then completely specifies the form of the non-random ownership matrix $\mathbf{M} = \lambda\overline{\mathbf{M}} + (1-\lambda)\widetilde{\mathbf{M}}$. It holds for $\mathbf{M}$ and its corresponding integration level ρ that $\rho = r$ and that*

$$\phi = \begin{cases} 1, & \lambda = 1 \\ n-1, & 0 \leq \lambda < 1, \end{cases} \tag{7.6}$$

where ϕ is the diversification level of $\mathbf{M}$.

Clearly, in a λ-convex combination, the underlying ring and complete ownership matrices can have differing integration levels. For our simulations we assume that their integration levels are equal, for convenience. The value r is sometimes referred to as the *integration parameter* of $\mathbf{M}$ or, more sloppy, as the *integration* of $\mathbf{M}$. The convention $r < 1$ ensures that Assumption 5.6 is valid which guarantees a unique solution of the system. In the remainder, we will use the expressions non-random and fixed ownership matrix equivalently and mean that all entries of the matrix are fully determined given the tuple (λ, r).

Random Ownership Structure

While in Section 7.1.2, the entries of the ownership matrices can be arbitrary but non-random, we will now also allow that the entries can be determined randomly. A random ownership matrix is based on a random network matrix. Let $k \in \{0,1\}$ and $p^k \in [0,1]$. The entries of the random network matrix $\mathbf{G}^k$ for the corresponding seniority level are defined as

$$G_{ij}^k = \begin{cases} 1, & \text{with probability } p^k, \\ 0, & \text{with probability } 1-p^k, \end{cases} \tag{7.7}$$

for $i \neq j$ and $G_{ii}^k = 0$ for all $i \in \mathcal{N}$. Hence, the entries of $\mathbf{G}^k$ are Bernoulli-distributed random variables with $\mathbf{E}[G_{ij}^k] = p^k$ for $i \neq j$. As the name says, the matrix $\mathbf{G}^k$ defines connections between two firms in the system. If $G_{ij}^k = 1$, then firm i owns a fraction of firm j's debt if $k = 1$, or some of the shares of firm j, if $k = 0$. The actual amount of debt or shares is defined at a later stage. The probability p^k for such a connection is sometimes called *Erdös-Rényi probability* (cf. Nier et al., 2007). Note that $\mathbf{G}^k$ is not necessarily symmetric so that it is possible that firm i owns a proportion of j's debt payments but firm j receives no debt payments from i. The random network matrix is the basis for the definition of a random ownership matrix.

Definition 7.7. Let $\mathbf{G}^k$ be a random network matrix as in (7.7) and $r^k \in [0,1)$ for $k = 0,1$. The corresponding *random ownership matrix* $\mathbf{M}^k$ for seniority level k is then given by

$$M_{ij}^k = \begin{cases} r^k \frac{G_{ij}^k}{G_j^k} & \text{if } G_j^k > 0, \\ 0, & \text{if } G_j^k = 0, \end{cases} \tag{7.8}$$

where $G_j^k = \sum_{i=1}^n G_{ij}^k$ is the sum of non-zero entries in the j-th column of $\mathbf{G}^k$.

Note that the definition in (7.8) ensures that the column sum of $\mathbf{M}^k$ will always be smaller or equal to r^k:

$$\sum_{i=1}^n M_{ij}^k \leq r^k \sum_{i=1}^n \frac{G_{ij}^k}{G_j^k} = \frac{r^k}{G_j^k} \sum_{i=1}^n G_{ij}^k = r^k < 1. \tag{7.9}$$

The equality in the equation above holds for $G_j \neq 0$. For this reason, r^k is the upper bound for the integration level of $\mathbf{M}^k$ which is why we will use the term *integration parameter* for r^k similar to the case of non-random ownership matrices.

Corollary 7.8. *Let $r \in [0,1)$ and $p \in [0,1]$ be the Erdös-Rényi probability for a random network matrix $\mathbf{G}$. Beside the size n, the tuple (r,p) is the only information necessary to generate a random ownership matrix as defined in* (7.8)*. It holds for $\mathbf{M}$ that*

$$\rho = \begin{cases} 0, & \textit{if } G_1 = \ldots = G_n = 0, \\ r, & \textit{else}. \end{cases} \tag{7.10}$$

Of course, the diversification level ϕ of a random ownership matrix as the maximum number of cross-holdings in every column is random itself. The expected number of cross-holdings in the j-th column of $\mathbf{M}$ is the expected number of non-zero entries in the corresponding column of $\mathbf{G}$ for which we know that:

$$\mathbf{E}\left[\sum_{j=1}^n G_{ij}\right] = \sum_{j=1}^n \mathbf{E}\left[G_{ij}\right] = (n-1)p. \tag{7.11}$$

This means that the higher the Erdös-Rényi probability p is chosen, the higher the diversification level of $\mathbf{M}$ tends to be. Clearly, if $p = 1$, we obtain a complete ownership matrix with $\phi = n-1$. On the other hand, if $p = 0$, no cross-holdings will be present and therefore $\phi = 0$. That is why we call the parameter p the *diversification parameter* or, simply, the *diversification* of $\mathbf{M}$ in the following.

In contrast to the expected value of an entry of $\mathbf{G}^k$, the expectation of the entries of $\mathbf{M}^k$ are a little bit more complicated to calculate.

Proposition 7.9. *Let $B_{n,p}(l)$ be the probability for $l \in \{0,1,\ldots,n\}$ successes of a binomial distributed variable with parameters n and $p \in [0,1]$, i.e.*

$$B_{n,p}(l) = \binom{n}{l} p^l (1-p)^{n-l}. \tag{7.12}$$

For given integration and diversification parameters r^k and p^k, the expectation of the entry M_{ij}^k based on the random network matrix $\mathbf{G}^k$ is given by

$$\mathbf{E}\left[M_{ij}^k\right] = \frac{r^k}{n-1}\left(1 - B_{n-1,p^k}(0)\right) = \frac{r^k}{n-1}\left(1 - \left(1 - p^k\right)^{n-1}\right) \tag{7.13}$$

for $i \neq j$ and by $\mathbf{E}[M^k_{ii}] = 0$ for all $i \in \{1, \dots, n\}$. The variance of M^k_{ij} is

$$\mathbf{Var}\left[M^k_{ij}\right] = \frac{(r^k)^2}{n-1} \sum_{l=1}^{n-1} \frac{1}{l} B_{n-1,p^k}(l) - \mathbf{E}\left[M^k_{ij}\right]^2 \tag{7.14}$$

for $i \neq j$ and $\mathbf{Var}[M^k_{ii}] = 0$ for all $i \in \{1, \dots, n\}$.

Proof. For the sake of simplicity, we omit the seniority level and write $\mathbf{M}$, r and p instead. Because of (7.8), the set of possible values for the non-diagonal elements M_{ij} is

$$\mathcal{A} := \left\{0, r, \frac{r}{2}, \frac{r}{3}, \dots, \frac{r}{n-1}\right\}. \tag{7.15}$$

Let $x \in \mathcal{A}\backslash\{0\}$. Then $x = \frac{r}{l}$ with $l \in \{1, \dots, n-1\}$ and

$$\begin{aligned} x \cdot \mathbb{P}[M_{ij} = x] &= \frac{r}{l} \cdot \mathbb{P}[G_j = l, G_{ij} = 1] \\ &= \frac{r}{l} \cdot \mathbb{P}\left[\sum_{s=1, s\neq j}^{n-1} G_{sj} = l-1\right] \mathbb{P}\left[G_{ij} = 1\right] \\ &= \frac{r}{l} \cdot B_{n-2,p}(l-1) \cdot p \\ &= \frac{r}{n-1} \cdot B_{n-1,p}(l). \end{aligned} \tag{7.16}$$

Using this result, the expectation of M_{ij} is for $i \neq j$ given by

$$\begin{aligned} \mathbf{E}\left[M_{ij}\right] &= \sum_{x \in \mathcal{A}} x \mathbb{P}[M_{ij} = x] = \frac{r}{n-1} \sum_{l=1}^{n-1} B_{n-1,p}(l) \\ &= \frac{r}{n-1}\left(1 - B_{n-1,p}(0)\right) = \frac{r}{n-1}\left(1 - (1-p)^{n-1}\right). \end{aligned} \tag{7.17}$$

For the variance, first note that for $x \in \mathcal{A}$, $\mathbb{P}[M^2_{ij} = x^2] = \mathbb{P}[M_{ij} = x]$ and using a similar argumentation as in (7.16), it holds that

$$x^2 \cdot \mathbb{P}\left[M^2_{ij} = x^2\right] = \frac{r^2}{l^2} \cdot \mathbb{P}[M_{ij} = x] = \frac{r^2}{(n-1)l} \cdot B_{n-1,p}(l), \tag{7.18}$$

for $x \in \mathcal{A}\backslash\{0\}$. Let $\mu := \mathbf{E}[M_{ij}] = \frac{r}{n-1}(1 - (1-p)^{n-1})$. Then,

$$\begin{aligned} \mathbf{Var}[M_{ij}] &= \mathbf{E}\left[M^2_{ij}\right] - \mu^2 \\ &= \sum_{x \in \mathcal{A}} x^2 \cdot \mathbb{P}\left[M^2_{ij} = x^2\right] - \mu^2 \\ &= \frac{r^2}{n-1} \sum_{l=1}^{n-1} \frac{1}{l} \cdot B_{n-1,p}(l) - \mu^2. \end{aligned} \tag{7.19}$$

□

The factor $1 - B_{n-1,p^k}(0)$ in (7.13) can be interpreted as a correction factor. If p^k is near 1, we have that $B_{n-1,p^k}(0) \approx 0$ and hence $\mathbf{E}[M^k_{ij}] \approx \frac{r^k}{n-1}$. But if p^k is small, the probability of the event $G_j = 0$ for some $j \in \{1, \dots, n\}$ is relatively high. In such cases, the entries of the corresponding column in $\mathbf{M}^k$ will be zero which is why the ratio $\frac{r^k}{n-1}$ has to be corrected by $1 - B_{n-1,p^k}(0) < 1$.

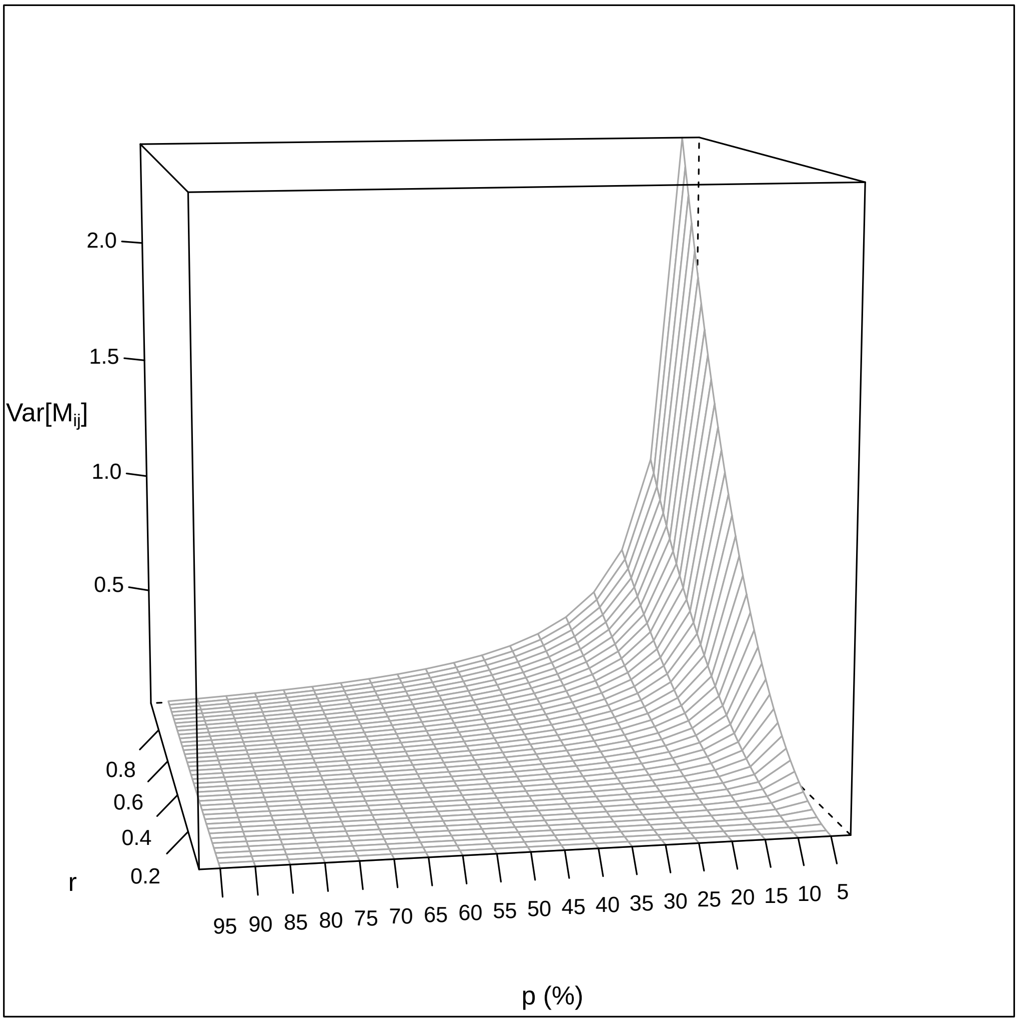
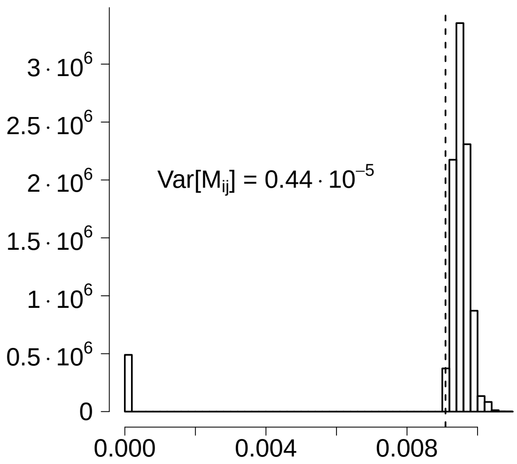
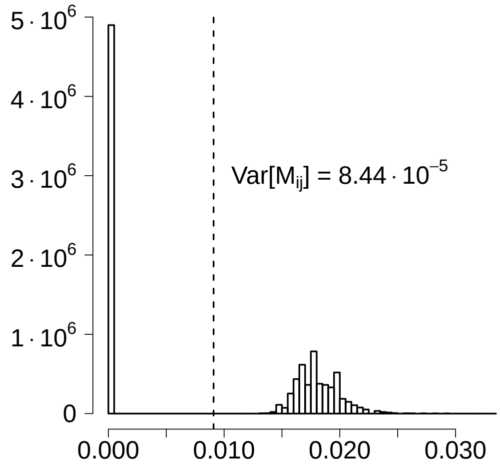


Figure 7.1: Plot of the variance of M_{ij} given by formula (7.14) for different integration parameters r and diversification parameters p for a system entailing $n = 100$ firms. The labels for $\mathbf{Var}[M_{ij}]$ on the vertical axis are multiplied with 10^3 for better readability, i.e. the label 2 on the axis actually stands for $2 \cdot 10^{-3}$. Note that the diversification parameter p is shown in %, i.e. the value 95 means $p = 0.95$.

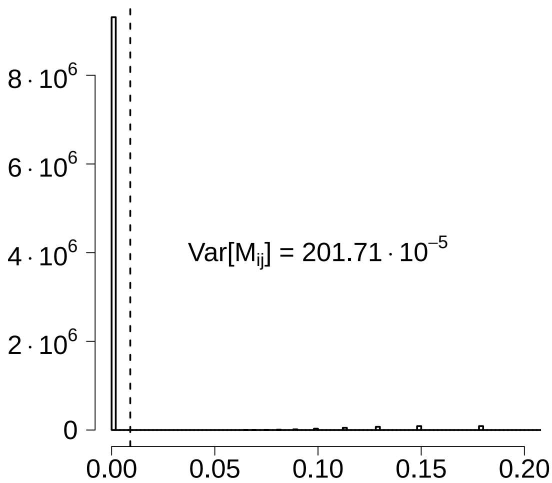
Example 7.10. To get a better insight, we want to visualize the behavior of the variance of M_{ij} as well as the distribution for some given values of r and p. In Figure 7.1 the variance is shown according to Equation (7.14) for some combinations of integration and diversification parameters and a system size of $n = 100$. We observe that the variance is relatively small in case of a high diversification and a low integration parameter. For an increasing integration parameter, the variance does only very slightly increase if the diversification parameter stays high. If the integration parameter becomes high and the diversification parameter becomes low at the same time, a disproportional high increase of the variance is the consequence. To get a numeric impression, for $r = 0.05$ and $p = 0.99$, we have that $\mathbf{Var}[M_{ij}] = 2.60 \cdot 10^{-9}$. If $r = 0.99$ and $p = 0.05$, the variance is given by $\mathbf{Var}[M_{ij}] = 2.44 \cdot 10^{-3}$, which is an approximate increase by a factor of 10^6.



(a) $r = 0.9$ and $p = 0.95$



(b) $r = 0.9$ and $p = 0.5$



(c) $r = 0.9$ and $p = 0.05$

Figure 7.2: Histograms of the entries M_{ij} of 1000 randomly generated ownership matrices of size $n = 100$, i.e. 9 900 000 values in total. The integration parameter was $r = 0.9$ every time, the diversification parameter was given by $p = 0.95$ in part (a), $p = 0.5$ in part (b) and $p = 0.05$ in part (c). Note that in case of $p = 0.05$, the limit of the x-axis is truncated to a maximum value of 0.2 for a better visibility. The fraction of observations M_{ij} with higher values than 0.2 is given by 1.36%. The bar height denotes the frequency, i.e. the number of observations that are contained in every interval. The vertical dotted line denotes the expectation $\mathbf{E}[M_{ij}]$ for the corresponding parameter combination. Note that in all three considered cases, the expectation of M_{ij} is approximately equal, viz. $\mathbf{E}[M_{ij}] = 0.0090$ if $p = 0.05$ and $\mathbf{E}[M_{ij}] = 0.0091$ for $p = 0.5$ and $p = 0.95$.

In Figure 7.2, we generated for $r = 0.9$ and three different diversification parameters 1 000 ownership matrices and created a histogram for every scenario. Since $n = 100$ is chosen, every matrix contains $100^2 - 100 = 9\,900$ entries – the entries on the diagonal are excluded – and every histogram contains $9\,900 \cdot 1\,000 = 9\,900\,000$ values. Even for a constant integration parameter, the variance of M_{ij} is for $p = 0.05$ about 450 times higher than for $p = 0.95$. It becomes clear that the probability of the event $M_{ij} = 0$ becomes very high for small diversification levels. When $p = 0.05$, the value was zero for about 95% of the observations. A consequence is that for relatively small diversification parameters, the ownership matrices become very sparse.

7.2 Simulation Parameters

7.2.1 Input Parameters

With the background of Section 7.1 and the therein defined parameters of a financial system in mind, we now want to list the range of the specific parameters that are considered in the study. In Table 7.1, the values are listed for a compact overview.

For the system size n we choose five different values, viz. $n \in \{5, 10, 25, 50, 100\}$. A system with only 5 or 10 firms can be considered as relatively small whereas networks with $n = 25$ or $n = 50$ are regarded as medium-sized. Small systems are investigated for example in Gouriéroux et al. (2012) ($n = 5$), Rogers and Veraart (2013) ($n = 6$), Elsinger et al. (2006a) and Cifuentes et al. (2005) ($n = 10$ both). An example of a medium-sized system is Nier et al. (2007) with 20 firms. Further, we add networks with 100 firms into our study to also include an example of a larger system. Existing studies for such sizes are for instance Elliott et al. (2014) and Müller (2006) that entail networks with 100, and 200 firms, respectively. Note that there are also studies that investigate systems of much higher size like Gai and Kapadia (2010) with $n = 1000$. Works with large systems are often empirical studies of real-life networks like the ones of Elsinger et al. (2006b) and Cont et al. (2010) where $n = 881$ and $n = 2400$, respectively.

As mentioned above, the exogenous asset values are assumed to have the fixed value 1, following the approach in Karl and Fischer (2014). The range for the nominal debt values is given by $d \in \{0.5, 1, 1.5, 2\}$. Nier et al. (2007) consider a ratio $\mathbf{d}$ to $\mathbf{a}$ of about 1.15 in their article ("benchmark experiment") and Cifuentes et al. (2005) a ratio of about 1.3 which justifies the interval of our values of d. On the liability vector, we add independently normally distributed "shocks" ε_i on each value separately, as described in (7.1). The mean of ε_i is zero and for the standard deviation σ, we use the set $\sigma \in \{0.5, 1\}$. Note that for σ there exist, by our best knowledge, no benchmark values since in most studies, the values in $\mathbf{d}$ are assumed to be fixed and the random variation is included in the vector $\mathbf{a}$.

When the debt ownership matrix is generated, integration parameters within the range of $r^1 \in \{0, 0.05, 0.5, 0.95\}$ are used. Possible integration parameters for equity ownership matrices are taken from $r^0 \in \{0, 0.025, 0.25, 0.475\}$, where each value is half the associated debt integration. The justification for this approach is that equity cross-ownership is probably commonly less pronounced than debt cross-ownership. Further, we want to avoid possible cross-ownership entries lager that 0.5 since this would mean that a firm is owned by majority by another firm in the system. Our choice of the interval of possible integration levels is based on the work of Elliott et al. (2014) who use integration parameters from 0.1 to 0.9. Note that in Gouriéroux et al. (2012) there is an example of a financial system with $m = 1$ consisting out of the $n = 5$ largest French banks. In this system, the integration levels of equity and debt are given by

$\rho^0 = 0.036$ and $\rho^1 = 0.0502$. Even though this is only a small system, it indicates that the integration levels are probably much smaller in practice.

For non-random ownership matrices, the remaining parameter to define is λ. Beside ring ($\lambda = 1$) and complete ($\lambda = 0$) ownership matrices, we use the intermediate value of $\lambda = 0.5$ in our study. In random ownership matrices, we further need to specify the diversification parameter p. We choose with $p \in \{0.05, 0.5, 0.95\}$ three different parameter values. Note that for $p = 0$ we obtain ownership matrices with only zero entries and that for $p = 1$ we will get a complete ownership matrix. To retain the random character of the ownership matrices, we therefore choose values strictly larger than zero and strictly smaller than one, unlike other studies such as Nier et al. (2007) who allow p to become 0 and 1.

For every possible combination of the input parameters shown in Table 7.1, we generated $N = 1000$ realizations of the same financial system and one such realization is denoted as a *simulated system* in the following. The number of $N = 1000$ is the same than in Elliott et al. (2014). Additional simulations show that the results are fairly stable for $N = 1000$, which is why we view the number of 1000 repetitions as reliable. Note that in other papers, smaller numbers of repetitions are also considered such as $N = 100$ as in Nier et al. (2007).

Table 7.1: Overview of the investigated parameters of the simulation study and the ranges of the utilized values.

General Setting	
Number of simulations	$N = 1000$
System size	$n \in \{5, 10, 25, 50, 100\}$
Exogenous assets	$\mathbf{a} = \mathbf{1}_n = (1, \ldots, 1)^t$
Tolerance level	$\varepsilon = 10^{-3}$
Liability structure	
Debt values	$d \in \{0.5, 1, 1.5, 2\}$
Standard deviation of ε_i	$\sigma \in \{0.5, 1\}$
Ownership structure	
Debt integration	$r^1 \in \{0, 0.05, 0.5, 0.95\}$
Equity integration	$r^0 \in \{0, 0.025, 0.25, 0.475\}$
Diversification	$p^0, p^1 \in \{0.05, 0.5, 0.95\}$
Mixing parameter	$\lambda^0, \lambda^1 \in \{0, 0.5, 1\}$

Notes: The diversification parameters p^0 and p^1 are used only for financial systems with a random ownership structure, t he mixing parameters λ^0 and λ^1 only for systems with a fixed ownership structure. The setting $\varepsilon = 10^{-3}$ was used in every algorithm or subalgorithm in which the specification of a tolerance level is needed.

As shown in Table 7.1, the following parameters are needed for the simulation of a financial system: n, d, σ, r^1, r^0 and either λ^0, λ^1 in case of non-random ownership matrices or p^0, p^1 for random matrices. Every possible combination of parameters is considered in the simulation study. Since there are five possible values for n, four for d, two for σ, four for each value of r^1

and r^0 and three values for each diversification level, this leads to

$$2 \cdot \Big(5 \cdot 4 \cdot 2 \cdot 4 \cdot 4 \cdot 3 \cdot 3\Big) = 11520 \tag{7.20}$$

different scenarios to define the shape of $\mathcal{F}$ in total. The factor 2 stems from the fact that we consider random and non-random ownership matrices. Note that we exclude the parameter combinations in which both $r^0 = r^1 = 0$ since then, no cross-holdings are present. In total, this concerns $720 = 2 \cdot (5 \cdot 4 \cdot 2 \cdot 3 \cdot 3)$ scenarios which is why only 10800 scenarios of different parameter combinations are considered.

To get a better impression about the question in how many of the scenarios the firms are more likely to be in default or more likely to be solvent, we calculated for fixed ownership matrices and a value of $\sigma = 0$ the corresponding fixed points for every possible parameter combination. Since in this situation all firms have identical recovery values, we checked whether one of the n firms is in default or solvent. The result is that for $n = 5, 10, 50$ the firms are in default in 25% of all scenarios and for $n = 25, 100$, about 29% of the firms are not solvent. This means that we can assume that for the majority of the considered scenarios, most firms are solvent which seems realistic for practical purposes.

7.2.2 Output Parameters

The main goal of the simulation study in Section 7.5 is to compare the different algorithms with one another to find out whether there is a most efficient algorithm. We therefore want to find the algorithm that minimizes the calculation effort to find the solution $\mathbf{R}^*$. The calculation costs of a single iteration step, i.e. the total costs that are needed to get from iterate $\mathbf{R}^k$ to $\mathbf{R}^{k+1}$, are quantified with the Landau symbol (Big $\mathcal{O}$ notation), where $\mathcal{O}(n)$ means that the time $T(n)$ to compute a problem of size n grows at the rate n. For the Picard Iteration, the most expensive calculations are multiplications of a matrix with a vector. According to Dahlquist and Björck (2008), this results in costs of $\mathcal{O}(n^2)$. The two other iteration techniques given by the Elsinger and the Hybrid Algorithm are more expensive, since linear equation systems have to be solved with costs up to $\mathcal{O}(n^3)$ (cf. Dahlquist and Björck, 2008).

Beside the costs of an iteration step, the convergence speed is the second crucial parameter to assess the algorithm efficiency. Even though Elsinger and Hybrid Algorithm are computationally more intensive to conduct, we have seen in the Propositions 4.11 and 4.19 that they will not require more iteration steps to reach $\mathbf{R}^*$ sufficiently close than the Picard Algorithm. Therefore, a tradeoff-situation is given between computational costs and convergence speed of the Picard, the Elsinger and the Hybrid Algorithm. The convergence speed is usually described by the convergence order.

Definition 7.11 (Dahlquist and Björck (2008))**.** A series of iterates $\mathbf{R}^k$ with $\lim_{k\to\infty} \mathbf{R}^k = \mathbf{R}^*$ is said to have *convergence order equal to* $q \geq 1$ if for some constant $0 < c < \infty$,

$$\lim_{k\to\infty} \frac{\|\mathbf{R}^{k+1} - \mathbf{R}^*\|}{\|\mathbf{R}^k - \mathbf{R}^*\|^q} = c. \tag{7.21}$$

For $q = 1$ and $c < 1$, we have a *linear convergence order.*

In order to show that the three non-finite algorithms have at least a linear convergence, we need to prove the following Lemma first.

Lemma 7.12. *For the Picard Algorithm (Algorithm 1), for the Elsinger Algorithm (Algorithms 3 and 11) and for the Hybrid Algorithm (Algorithms 6 and 12) it holds for the iterates $\mathbf{R}^k$ that*

$$\|\mathbf{R}^{k+1} - \mathbf{R}^*\| \leq I^{\max}\|\mathbf{R}^k - \mathbf{R}^*\|, \tag{7.22}$$

where $I^{\max} = \max\{\|\mathbf{M}^m\|, \dots, \|\mathbf{M}^0\|\}$ *and* $\mathbf{R}^*$ *is the solution of the corresponding financial system.*

Proof. Let us first consider the case of only one seniority class ($m = 1$) and the Picard Algorithm, i.e. we have the norm

$$\|\mathbf{R}^{k+1} - \mathbf{R}^*\| = \left\| \begin{pmatrix} \min\{\mathbf{d}^1, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0}\} - \min\{\mathbf{d}^1, \mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0}\} \\ (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0} - \mathbf{d}^1)^+ - (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}^1)^+ \end{pmatrix} \right\|. \tag{7.23}$$

We check validity of (7.23) for each firm $i \in \mathcal{N}$ separately and start with the decreasing version of the algorithm. To this end, we need to consider the following cases.

(i) Let $i \in \mathcal{D}^*$ and $i \in \mathcal{D}(\mathbf{R}^k)$:

$$\begin{aligned} \|\mathbf{R}^{k+1} - \mathbf{R}^*\|_i &= |\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0} - \mathbf{a} - \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{M}^0\mathbf{r}^{*,0}|_i \\ &= \Big| \underbrace{\Big(\mathbf{M}^1(\mathbf{r}^{k,1} - \mathbf{r}^{*,1}) + \mathbf{M}^0(\mathbf{r}^{k,0} - \mathbf{r}^{*,0})\Big)_i}_{\geq 0} \Big|. \end{aligned} \tag{7.24}$$

(ii) Let $i \in \mathcal{D}^*$ and $i \notin \mathcal{D}(\mathbf{R}^k)$:

$$\begin{aligned} \|\mathbf{R}^{k+1} - \mathbf{R}^*\|_i &= |\mathbf{d}^1 - \mathbf{a} - \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{M}^0\mathbf{r}^{*,0}|_i + |\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0} - \mathbf{d}^1|_i \\ &= \Big| \underbrace{\Big(\mathbf{M}^1(\mathbf{r}^{k,1} - \mathbf{r}^{*,1}) + \mathbf{M}^0(\mathbf{r}^{k,0} - \mathbf{r}^{*,0})\Big)_i}_{\geq 0} \Big|. \end{aligned} \tag{7.25}$$

(iii) Let $i \notin \mathcal{D}^*$:

$$\begin{aligned} \|\mathbf{R}^{k+1} - \mathbf{R}^*\|_i &= |\mathbf{d}^1 - \mathbf{d}^1|_i + |\mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0} - \mathbf{M}^1\mathbf{r}^{*,1} - \mathbf{M}^0\mathbf{r}^{*,0}|_i \\ &= \Big| \underbrace{\Big(\mathbf{M}^1(\mathbf{r}^{k,1} - \mathbf{r}^{*,1}) + \mathbf{M}^0(\mathbf{r}^{k,0} - \mathbf{r}^{*,0})\Big)_i}_{\geq 0} \Big|. \end{aligned} \tag{7.26}$$

It has therefore been established that

$$\begin{aligned} \|\mathbf{R}^{k+1} - \mathbf{R}^*\| &= \|\mathbf{M}^1(\mathbf{r}^{k,1} - \mathbf{r}^{*,1}) + \mathbf{M}^0(\mathbf{r}^{k,0} - \mathbf{r}^{*,0})\| \\ &\leq \|\mathbf{M}^1\| \cdot \|\mathbf{r}^{k,1} - \mathbf{r}^{*,1}\| + \|\mathbf{M}^0\| \cdot \|\mathbf{r}^{k,0} - \mathbf{r}^{*,0}\| \\ &\leq I^{\max}\|\mathbf{R}^k - \mathbf{R}^*\|. \end{aligned} \tag{7.27}$$

For the increasing version, we also have to take three cases into account.

(i) Let $i \in \mathcal{D}(\mathbf{R}^k)$ and $i \in \mathcal{D}^*$. This is analogous to case (i) of the decreasing version but with the difference that the i-th component of the sum in the norm is smaller or equal to zero.

(ii) Let $i \in \mathcal{D}(\mathbf{R}^k)$ and $i \notin \mathcal{D}^*$:

$$\|\mathbf{R}^{k+1} - \mathbf{R}^*\|_i = |\mathbf{a} + \mathbf{M}^1\mathbf{r}^{k,1} + \mathbf{M}^0\mathbf{r}^{k,0} - \mathbf{d}^1|_i + | - (\mathbf{a} + \mathbf{M}^1\mathbf{r}^{*,1} + \mathbf{M}^0\mathbf{r}^{*,0} - \mathbf{d}^1)|_i$$
$$= \Bigg|\underbrace{\Big(\mathbf{M}^1(\mathbf{r}^{k,1} - \mathbf{r}^{*,1}) + \mathbf{M}^0(\mathbf{r}^{k,0} - \mathbf{r}^{*,0})\Big)_i}_{\leq 0}\Bigg|. \tag{7.28}$$

(iii) Let $i \notin \mathcal{D}(\mathbf{R}^k)$. This is equivalent to case (iii) of the decreasing version with the same change of the sign as in (i).

For the other algorithms (Elsinger, Hybrid) the argumentation is very similar. If systems with $m > 1$ are considered, the argumentation also stays the same with the difference that more case differentiations have to be taken into account. □

Corollary 7.13. *Under Assumption 5.6, Picard, Elsinger and Hybrid Algorithm have at least a linear convergence order.*

Proof. Together with Lemma 7.12 it holds for the three algorithms that

$$\lim_{k\to\infty} \frac{\|\mathbf{R}^{k+1} - \mathbf{R}^*\|}{\|\mathbf{R}^k - \mathbf{R}^*\|} \leq I^{\max} < 1 \tag{7.29}$$

from which the linear convergence follows immediately. □

However, we could not show a higher convergence order which prevents the application of an efficiency index to compare the different iteration techniques, see Dahlquist and Björck (2008) for more details. For Trial-and-Error Algorithms the comparison of the procedures on an analytical way becomes even more complicated because of the additional aspect that a pseudo solution with costs $\mathcal{O}(n^3)$ is calculated after some undefined number of iteration steps.

Because of these reasons, the comparison of the different methods on an analytical basis seems impossible. This is why we measured the time that was needed to execute an algorithm and considered this value as the primary output of our simulation. Though this measure strongly depends on the processor speed and memory capacity of the computer, it allows an objective comparison of the different algorithms. The simulations were conducted on a computer with 2.4 GHz and 32 GB RAM, the software used was R (R Core Team, 2014). The runtime in the subsequent sections is defined as the total time measured in seconds that was needed to determine the $N = 1000$ solutions of each simulated system for a given combination of input parameters. Beside the runtime, we also documented the iteration and calculation steps needed to reach the solution of the current financial system. Recall that in Section 4.2 we introduced the expressions *iteration step* and *calculation step*, where a calculation step was – basically – the solution of a linear equation system and an iteration step included all type of computations to get from iterate $\mathbf{R}^k$ to the next iterate $\mathbf{R}^{k+1}$, $k \geq 0$.

In Section 7.3, another simulation target is to investigate whether the decision rules in (6.74) and (6.109) lead to acceptable results. Apart from the actual runtime, we therefore also assessed the goodness of the decision rule by comparing the number of iterations that are needed to approach the fixed point $\mathbf{R}^*$ sufficiently close. For the Picard Algorithm, these numbers are defined as

$$\begin{aligned} k_{\text{great}} &:= \min\{k \in \mathbb{N} : \|\Phi^k(\mathbf{R}_{\text{great}}) - \Phi^{k-1}(\mathbf{R}_{\text{great}})\| < \varepsilon\}, \\ k_{\text{small}} &:= \min\{k \in \mathbb{N} : \|\Phi^k(\mathbf{R}_{\text{small}}) - \Phi^{k-1}(\mathbf{R}_{\text{small}})\| < \varepsilon\}, \\ k_{\text{opt}} &:= \min\{k \in \mathbb{N} : \|\Phi^k(\mathbf{R}_{\text{opt}}) - \Phi^{k-1}(\mathbf{R}_{\text{opt}})\| < \varepsilon\}, \end{aligned} \tag{7.30}$$

with $\mathbf{R}_{\text{opt}}$ defined as in (6.74). The choice of $\mathbf{R}_{\text{opt}}$ was "right" if $k_{\text{opt}} = \min\{k_{\text{great}}, k_{\text{small}}\}$ which leads to the *error rate* of the Picard Algorithm defined as

$$\epsilon_{\text{P}} := \frac{\left|\{l \in \{1, \ldots, N\} : k^l_{\text{opt}} \neq \min\{k^l_{\text{great}}, k^l_{\text{small}}\}\}\right|}{N} \in [0, 1], \tag{7.31}$$

where the index $l = 1, \ldots, N$ denotes the corresponding simulated system. In a similar way, the error rates ϵ_{E} and ϵ_{H} are determined for the Elsinger and the Hybrid algorithm, where $\mathbf{R}_{\text{great}}$ is replaced by $(\mathbf{r}^1_{\text{great}}, \mathbf{r}^0(\mathbf{r}^1_{\text{great}}))$, $\mathbf{R}_{\text{small}}$ by $(\mathbf{r}^1_{\text{small}}, \mathbf{r}^0(\mathbf{r}^1_{\text{small}}))$ and $\mathbf{R}_{\text{opt}}$ is calculated using Equation (6.109).

In Section 7.4, we address the issue of finding an optimal lag value. Suppose that for a given combination of input parameters, we have generated N simulated systems. We ignore for a moment the direction of the Trial-and-Error Algorithm and assume that we have determined for the Trial-and-Error Picard (TP), the Trial-and-Error Elsinger (TE) and the Trial-and-Error Hybrid Algorithm (TH) for a lag value $2 \leq l \leq 5$ the first potential default set $\bar{\mathcal{D}}^j_{\text{TP}}(l)$, $\bar{\mathcal{D}}^j_{\text{TE}}(l)$ and $\bar{\mathcal{D}}^j_{\text{TH}}(l)$ where $j = 1, \ldots, N$. In case of the Trial-and-Error Picard Algorithm we define

$$\epsilon^j_{\text{TP}}(l) = \begin{cases} 1, & \text{if } \bar{\mathcal{D}}^j_{\text{TP}}(l) \neq \mathcal{D}^*, \\ 0, & \text{else,} \end{cases} \tag{7.32}$$

and analogously $\epsilon^j_{\text{TE}}(l)$ and $\epsilon^j_{\text{TH}}(l)$ for the TE and TH Algorithm, respectively. The *error rate* for the TP Algorithm for the lag value l is then given by

$$\epsilon_{\text{TP}}(l) = \frac{1}{N} \sum_{j=1}^{N} \epsilon^j_{\text{TP}}(l) \in [0, 1]. \tag{7.33}$$

In the same way the error rates $\epsilon_{\text{TE}}(l)$ and $\epsilon_{\text{TH}}(l)$ are defined.

Before we report the simulation results, we introduce abbreviations for the algorithms. The short name of an algorithm consists out of three different components. In the first component, the direction is determined, where we distinguish between the two directions "Decreasing" (D) and "Increasing" (I). The second component describes the algorithm class for which "Trial-and-Error" (T), "Sandwich" (S) or no letter at all is possible for the non-finite algorithms of Section 4.1. Finally, we have to specify the iteration technique, where "Picard" (P), "Elsinger" (E) and "Hybrid" (H) are possible. Putting these letters together leads to a particular algorithm of Chapter 4. For instance, DP stands for the Decreasing Picard Algorithm or ITH for the Increasing Trial-and-Error Hybrid Algorithm. If we use the letter O instead of one of the two directions D or I, this means that the optimized version of the Algorithm is used. OP therefore stands for the Optimized Picard Algorithm (Algorithm 18) and OTE means that the Optimized Trial-and-Error version of the Elsinger Algorithm is applied, where the starting vector was calculated via the decision rule in (6.109). Note that for the Sandwich Algorithms the first letter that denotes the direction is suppressed since no particular direction has to be determined for this type of procedure. The abbreviation SP therefore stands for the Sandwich Picard Algorithm.

7.3 Optimizing Non-finite Algorithms

Before presenting the results of the simulation study in detail, we point out that when the error rate as in (7.31) is the output parameter of interest, situations in which $\mathbf{a} \geq \mathbf{d}$ must not be taken into account.

Lemma 7.14. *For financial systems $\mathcal{F}$ with $\mathbf{a} \geq \mathbf{d}$, it holds that $\mathbf{R}^* = \mathbf{R}_{\text{great}}$. Further, both decision rules in* (6.74) *and* (6.109) *have as a consequence that the initial starting vector is equal to $\mathbf{R}^*$.*

Proof. Check that if $\mathbf{a} \geq \mathbf{d}$, it holds that

$$\mathbf{r}^0_{\text{great}} = (\mathbf{I}_n - \mathbf{M}^0)^{-1}(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d}) = \mathbf{r}^0(\mathbf{d}^1) \tag{7.34}$$

and, obviously, $\mathbf{R}_{\text{great}} = \mathbf{R}^*$. Moreover, $\mathbf{R}_{\text{small}} = (\mathbf{d}, (\mathbf{a} - \mathbf{d}))$ and

$$\mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{r}^0_{\text{great}} = -(\mathbf{I}_n - \mathbf{M}^0)\mathbf{r}^0_{\text{great}} = -\left(\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d}\right). \tag{7.35}$$

It follows that Δ_{start} defined as in (6.74) becomes

$$\begin{aligned}\Delta_{\text{start}} &= \begin{cases} \|\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{r}^0_{\text{great}}\| - \|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0(\mathbf{a} - \mathbf{d})\| \\ \frac{I^{\max}}{1-I^{\max}}\|\mathbf{a} + \mathbf{M}^1\mathbf{d} - \mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}} - \mathbf{r}^0_{\text{great}}\| - \|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}}\| \end{cases} \\ &= \begin{cases} -\|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0(\mathbf{a} - \mathbf{d})\| \\ -\|\mathbf{M}^1\mathbf{d} + \mathbf{M}^0\mathbf{r}^0_{\text{great}}\| \end{cases} \\ &\leq 0. \end{aligned} \tag{7.36}$$

Therefore, $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ for the Picard Algorithm. In case of the Elsinger or the Hybrid Algorithm, $\mathbf{R}^0$ is chosen according to (6.109), where we obtain with a similar argumentation that $\mathbf{R}^0 = (\mathbf{d}, \mathbf{r}^0(\mathbf{d})) = \mathbf{R}^*$. □

Firms with $a_i \geq d_i$ are referred to as *super-hedged* in Fischer (2015). A consequence from Lemma 7.14 is that if the financial system consists only out of super-hedged firms, no empirical analysis about the decision rule is necessary. However, since the entries of $\mathbf{d}$ in the study are not chosen fixed but with a random variation, we also include scenarios in which $d \leq 1$ into our simulation.

7.3.1 Results

In case of financial systems with a non-random ownership structure the mean error rate over all considered 5400 scenarios is 0.094 for the Picard Algorithm, 0.050 for the Elsinger and 0.017 for the Hybrid Algorithm. Table 7.2 lists on the left hand side these overall error rates and the error rates grouped by every input parameter for systems with non-random ownership matrices. Beside these data, we also found that there are 390 parameter combinations for the Picard Algorithm with an error rate higher than 0.5 and corresponding numbers of 167 and 20 for the Elsinger and the Hybrid Algorithm (numbers not shown in Table 7.2). A first insight is that the size of a financial system has almost no influence on the error rate since the rates are for every algorithm almost constant for the different values of n. Only a slight tendency that the

Table 7.2: Mean error rates defined in (7.31) grouped by the simulation parameters for systems with fixed and random ownership structure. In the last three rows of the table with the label “total”, the mean error rates over all parameters are shown.

	Fixed ownership matrices					Random ownership matrices				
	n					n				
	5	10	25	50	100	5	10	25	50	100
P	0.080	0.098	0.095	0.097	0.098	0.093	0.104	0.116	0.114	0.094
E	0.040	0.045	0.052	0.056	0.055	0.057	0.060	0.075	0.074	0.061
H	0.017	0.015	0.013	0.016	0.023	0.022	0.023	0.021	0.019	0.022

	r^0				r^0			
	0	0.025	0.25	0.475	0	0.025	0.25	0.475
P	0.104	0.092	0.105	0.076	0.117	0.100	0.114	0.088
E	0.104	0.074	0.031	0.003	0.119	0.080	0.049	0.028
H	0.002	0.032	0.024	0.005	0.003	0.031	0.028	0.020

	r^1				r^1			
	0	0.05	0.5	0.95	0	0.05	0.5	0.95
P	0.051	0.014	0.024	0.275	0.045	0.016	0.041	0.299
E	0.000	0.012	0.026	0.148	0.003	0.015	0.042	0.186
H	0.000	0.005	0.020	0.039	0.003	0.005	0.025	0.049

	λ^0			p^0		
	0	0.5	1	0.05	0.5	0.95
P	0.087	0.089	0.104	0.111	0.102	0.099
E	0.050	0.049	0.049	0.075	0.062	0.060
H	0.016	0.015	0.019	0.024	0.021	0.020

	λ^1			p^1		
	0	0.5	1	0.05	0.5	0.95
P	0.079	0.092	0.110	0.133	0.100	0.080
E	0.044	0.050	0.055	0.082	0.065	0.050
H	0.012	0.015	0.024	0.034	0.018	0.013

	d				d			
	0.5	1	1.5	2	0.5	1	1.5	2
P	0.096	0.101	0.090	0.087	0.104	0.110	0.101	0.101
E	0.041	0.045	0.054	0.058	0.053	0.057	0.071	0.081
H	0.010	0.011	0.019	0.027	0.017	0.015	0.024	0.030

	σ		σ	
	0.5	1	0.5	1
P	0.017	0.170	0.050	0.158
E	0.006	0.093	0.032	0.099
H	0.004	0.030	0.014	0.029

	total	total
P	0.094	0.104
E	0.050	0.065
H	0.017	0.022

error rates increase for increasing n can be observed. The same statement holds for the equity integration r^0 as well, with the difference that the error rates here tend to decrease for higher integration values. One of the strongest influence parameters is the debt integration which can be concluded from a disproportional high increase of the error rate when r^1 reaches its highest value of 0.95 compared to lower values. Note that this change is much smaller for the Hybrid Algorithm. The parameters λ^0 and λ^1 seem to be minor influence parameters that hardly affect the error rate. We indeed detect increasing error rates for increasing values of both λ^0 and λ^1, but the growth is relatively small. Though the error rates for the differing values of the debt level d reveal no noticeable results, we see that the standard deviation of the random values that are added on d has a strong effect on the rates. Mean error rates for $\sigma = 1$ tremendously increase compared to $\sigma = 0.5$. In Table B.1 in the Appendix, the error rates are listed in more detail grouped by the system size, the debt values and the values for σ. In particular, we list the mean rates for all parameters combinations in which $r^1 = 0.95$ is given to stress the fact that for the combination of a high debt integration and a value of $\sigma = 1$, there seems to be some kind of interaction effect on the error rate.

If the ownership matrices are randomly generated we get very similar impressions. At a global level, the Hybrid Algorithm has with 0.022 the lowest mean error rate over all scenarios, followed by the Elsinger (0.065) and the Picard Algorithm (0.104), as can be seen on the right side of Table 7.2. The number of all possible 5400 scenarios with mean error rates over 0.5 was a bit higher compared to systems with a fixed ownership structure. For the Picard Algorithm, the number was 414, for the Elsinger Algorithm 216 and 26 for the Hybrid Algorithm. Investigating the relevant columns of Table 7.2, we detect that the assertions made on the influence of the simulation parameters for fixed ownership matrices also hold for the most parts if random matrices are considered. The values for n, p^0, p^1, r^0 and d have no severe influence on the error rates, the debt integration level and the values of σ, however, do have a more significant influence. As was the case for non-random ownership matrices, the rates heavily increase for $r^1 = 0.95$ and $\sigma = 1$. We expected some interesting results for high integration values and low values of the corresponding diversification due to an disproportionately increased variance of the entries in the ownership matrices in these situations (see Example 7.10) but there were no noticeable results, as Table B.2 in the Appendix demonstrates.

Concerning the second output parameter, the runtime of the procedures, we find somewhat contradictory results. In case of non-random matrices, the optimized versions of the algorithms are mostly slower than their increasing and decreasing counterparts. For instance, there are only 9 parameter combinations where the calculation time of the optimized version of the Picard Algorithm was less than for the increasing and the decreasing version simultaneously. For the Elsinger Algorithm, there was one such combination and even no combination for the Hybrid Algorithm. The same statements hold for the simulations with random ownership matrices. Here, there are 6 combinations in which the Optimized Picard Algorithm is faster than the other two versions, two for the Elsinger and no ones for the Hybrid Algorithm. See also Table 7.3 where the mean runtimes are grouped by the system size that obviously has a strong influence on the runtime. For the other simulation parameters we found no remarkable effects on the mean runtime which is why we omitted further tables.

7.3.2 Discussion

The error rates as the prior output of this part of the simulation study show, that the optimized versions for the algorithms clearly work with respect to the minimization of iteration steps. For

Table 7.3: Mean Runtime grouped by the system size n for systems with fixed and random ownership structure.

	n (Fixed ownership matrices)					n (Random ownership matrices)				
	5	10	25	50	100	5	10	25	50	100
DP	1.541	1.864	2.377	3.413	8.550	1.404	1.721	2.363	3.507	8.641
IP	1.986	2.242	2.692	3.618	8.085	1.594	1.912	2.588	3.691	8.214
OP	2.474	2.859	3.482	4.837	11.637	2.294	2.681	3.479	4.977	11.669
DE	1.597	1.913	2.701	4.617	14.237	1.568	1.908	2.830	4.820	14.514
IE	1.833	2.185	3.027	5.098	15.519	1.634	2.037	3.048	5.240	15.722
OE	3.166	3.576	4.754	7.824	23.838	3.071	3.562	4.899	8.024	23.278
DH	1.837	2.151	3.056	5.801	19.452	1.816	2.105	3.070	5.477	17.207
IH	2.133	2.508	3.304	5.091	13.718	1.872	2.262	3.253	5.007	13.081
OH	4.171	4.690	6.142	10.561	32.667	4.006	4.506	6.152	9.993	29.743

all three algorithms we find only small error rates in most considered scenarios. In particular the error rates for the Elsinger and the Hybrid Algorithms are even smaller than for the Picard Algorithm. This is a strong indication that the approach of choosing an optimal starting point in Section 6.2 is justified regarding the minimization of iteration steps, even though we identified some mathematical problems in association with the decision rule for the two algorithms. In line with these findings is the fact that the number of needed iterations to reach $\mathbf{R}^*$ is also minimized as shown in Table B.9 in the Appendix.

A common property for both types of scenarios, with random and non-random ownership matrices, is that a higher standard deviation of σ leads to higher error rates. We also identify that an increase of the debt integration up to $r^1 = 0.95$ strongly increases the error rates at least for the Picard and the Elsinger Algorithm. For most parameter combinations with error rates lager than 0.5, i.e. in which a wrong selection appears in the majority of times, the minimum starting vector is chosen by the decision rule instead of the maximum starting vector for which the iteration number would have been minimized. However, we also notice that the consequences of such a wrong decision are not too severe. For systems with non-random ownership matrices, the maximum number of additional iteration steps that have to be performed unnecessarily is 4. In case of random ownership matrices, the highest discrepancy is with 3 unnecessary steps even smaller.

Taking the runtime of the procedures as an orientation, we have to state that the usage of the optimized versions should not be recommended since the runtime is for almost all scenarios higher compared to the decreasing and the increasing versions. We show in our simulation that almost no additional iteration steps have to be performed when using the optimized versions. Therefore, the only reason for the higher runtime must be that checking the decision rule is computational more costly and does not compensate the savings of the runtime.

In order to decrease the computational effort of the decision rule, a possible improvement could be to omit the case differentiation when determining Δ_{start} in (6.72) – (6.73) and in (6.110) – (6.111). This is because additional results suggest that in most cases, the differentiation is unnecessary since both equations would lead to a value of Δ_{start} with the same sign and hence to the same decision for the starting point. Another modification concerns the estimate of the iteration error for $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$ in (6.63) which was for the k-iteration given by

$$(I^{\max})^k \left\| \mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^{*,l} - \sum_{l=0}^{m} \mathbf{r}^l_{\text{great}} \right\|. \tag{7.37}$$

Since (7.37) contains the fixed point $\mathbf{R}^*$, the norm cannot be calculated a priori which is why we only used (6.62) as an upper bound of the k-th iteration error in case of $\mathbf{R}^0 = \mathbf{R}_{\text{great}}$. However, an upper bound for (7.37) is given by

$$(I^{\max})^k \left\| \max\left\{ \left| \mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{small}} - \sum_{l=0}^{m} \mathbf{r}^l_{\text{great}} \right|, \left| \mathbf{a} + \sum_{l=0}^{m} \mathbf{M}^l \mathbf{r}^l_{\text{great}} - \sum_{l=0}^{m} \mathbf{r}^l_{\text{great}} \right| \right\} \right\|, \tag{7.38}$$

i.e. we set one time $\mathbf{R}^* = \mathbf{R}_{\text{small}}$ and one time $\mathbf{R}^* = \mathbf{R}_{\text{great}}$ in (7.37) and take in every component the maximum of the absolute values of the two results. Additional simulations suggest that in some situations, the estimate in (7.38) is indeed smaller than the one in (6.62) that we used in the simulation study. This might further increase the accuracy of the decision rule.

Because of the fact that the runtime is almost always higher for the optimized versions, we do not use the Optimized Picard, Elsinger and Hybrid Algorithm for our investigation of the algorithm efficiency in Section 7.5 and use only the increasing and the decreasing versions of the Picard, Elsinger and Hybrid Algorithm.

7.4 Optimizing Trial-and-Error Algorithms

Recall that when utilizing the Trial-and-Error Algorithms of Section 4.2.1 to calculate $\mathbf{R}^*$, we talk of the Optimized Trial-and-Error Algorithm if $\mathbf{R}_{\text{opt}}$ is used as the initial iterate $\mathbf{R}^0$. In case of the Optimized Trial-and-Error Picard Algorithm, $\mathbf{R}_{\text{opt}}$ is given in (6.74) and for the Elsinger and Hybrid version of the Optimized Trial-and-Error Algorithm, the initial iterate is determined via (6.109). The error rate considered in this section is defined in (7.33), and the optimization relates to a minimization of this error rate with respect to the lag value.

7.4.1 Results

The overall mean error rates for both non-random and random ownership matrices are given in Table 7.4. As expected, the fraction of events where the first potential default set is not the actual default set, decreases for higher lag values. Comparing the error rates for the different versions of the algorithms in Table 7.4 with each other, we obtain the common picture that the decreasing version has the smallest rates compared to its increasing and optimized counterparts. Another result is that for both types of ownership matrices we notice that there is no simulation parameter with a strong effect on the error rate. For all possible levels of every simulation parameters the rates stay approximately constant which is why we omit tables that list the error rates grouped by the single parameters.

Taking the runtime as the output into account, we first see that the overall means are smallest for the smallest lag values (cf. Table 7.5). Second, we observe that algorithms that start with $\mathbf{R}_{\text{opt}}$ have a lower performance concerning the runtime. In almost all scenarios of combinations, the optimized version of the algorithms needs more time than the increasing or the decreasing version. For non-random ownership matrices and a lag value of $l = 2$ there are 8 out of 5400

Table 7.4: Mean error rates defined in (7.33) over all simulation parameters grouped by the lag value for systems with fixed and random ownership structure.

	Fixed ownership matrices				Random ownership matrices			
	$l=2$	$l=3$	$l=4$	$l=5$	$l=2$	$l=3$	$l=4$	$l=5$
DTP	0.058	0.018	0.006	0.002	0.058	0.017	0.006	0.002
ITP	0.092	0.029	0.010	0.004	0.080	0.024	0.008	0.003
OTP	0.063	0.020	0.007	0.003	0.060	0.018	0.006	0.002
DTE	0.019	0.003	0	0	0.019	0.003	0	0
ITE	0.031	0.006	0.002	0	0.028	0.005	0.001	0
OTE	0.020	0.003	0.001	0	0.020	0.003	0.001	0
DTH	0.002	0	0	0	0.002	0	0	0
ITH	0.002	0	0	0	0.002	0	0	0
OTH	0.002	0	0	0	0.002	0	0	0

scenarios in which the optimized version of the Trial-and-Error Picard Algorithm was fastest, 7 for the Elsinger and no combination of the Trial-and-Error Hybrid Algorithm. If the matrices were randomly generated, the associated numbers are 14 (Picard), 6 (Elsinger) and 0 (Hybrid). Moreover, we observe that there is – beside the size n of the system – no other influence parameter that strongly affects the runtime. The mean runtimes grouped by n for every value of l are listed in Table B.3 in the Appendix.

Table 7.5: Mean runtime over all simulation parameters grouped by the lag value for systems with fixed and random ownership structure.

	Fixed ownership matrices				Random ownership matrices			
	$l=2$	$l=3$	$l=4$	$l=5$	$l=2$	$l=3$	$l=4$	$l=5$
DTP	4.620	4.996	5.460	5.913	4.323	4.680	5.101	5.543
ITP	4.681	5.011	5.453	5.913	4.358	4.684	5.099	5.531
OTP	6.290	6.681	7.148	7.649	5.919	6.294	6.725	7.183
DTE	6.859	7.846	8.831	9.819	6.563	7.493	8.458	9.412
ITE	7.289	8.266	9.257	10.286	6.962	7.888	8.873	9.838
OTE	9.781	10.752	11.738	12.745	9.320	10.271	11.251	12.221
DTH	4.908	6.517	8.158	9.762	4.590	6.117	7.644	9.178
ITH	4.805	5.953	7.056	8.122	4.581	5.692	6.781	7.814
OTH	10.133	11.504	12.833	14.120	9.842	11.128	12.415	13.676

7.4.2 Discussion

A consequence of the relatively small error rates is that all Trial-and-Error Algorithms converge very fast in the sense that the first potential default set is in the majority of the considered scenarios also the actual default set $\mathcal{D}^*$. Even for the smallest lag value of $l = 2$ the overall mean error rates are, according to Table 7.4, for no procedure lager than 10%. Another conclusion concerns the optimized versions of the algorithms: using $\mathbf{R}_{\text{opt}}$ does not affect the error rate in the sense of a minimization. Compared to the decreasing versions of the algorithm, the rates are even slightly larger. Similar to the insights of Section 7.3, we also can observe here that the saved iteration steps due to the choice of an optimal starting point does not compensate the calculation effort that is needed when determining $\mathbf{R}_{\text{opt}}$. Together with the fact that all three optimized versions of the algorithms require more computation time than their increasing and decreasing counterparts, the usage of the Optimized Trial-and-Error Picard, Elsinger and Hybrid Algorithm is not recommended.

For these reasons, we exclude the optimized versions in the part of the study where the algorithm efficiency is investigated (Section 7.5) and use for this purpose only the Increasing and Decreasing Trial-and-Error Algorithms. Additionally, we choose a setting of $l = 2$ for the lag value for every algorithm since on the one hand the error rates are acceptable and on the other hand, even more important, the overall mean runtime as the primary output is minimized for the smallest possible lag value.

7.5 Analysis of Algorithm Efficiency

In line with the insights of the Sections 7.3 and 7.4, we exclude the optimized versions of each procedure since no essential improvement of the runtime is achieved when $\mathbf{R}^0 = \mathbf{R}_{\text{opt}}$. For all Trial-and-Error Algorithms, a lag value of $l = 2$ is used.

7.5.1 Results

In Table 7.6 we see an overview of the runtimes grouped by the system size which is the strongest influence parameter on the runtime. As expected, the mean runtime needed to find $\mathbf{R}^*$ increases with increasing system sizes. The minimum runtime for all algorithms with non-random ownership matrices is 0.35 seconds, in a scenario with $n = 5$ where the Decreasing Trial-and-Error Hybrid Algorithm was applied and the maximum runtime is 52.834 seconds for $n = 100$ and the Decreasing Hybrid Algorithm. In systems with random ownership matrices, the minimum is 0.349 seconds ($n = 5$ and Decreasing Trial-and-Error Hybrid Algorithm) and the maximum runtime is given by 61.8 seconds for $n = 100$ and the Increasing Elsinger Algorithm. Beyond the runtimes for single parameter combinations, we observe in Table 7.6 that the mean runtime over all considered scenarios becomes minimal for the Sandwich Picard Algorithms for systems with $n = 5, 10, 25$ compared to all other algorithms. For large systems, i.e. for $n = 50, 100$ the fastest method is the Picard Algorithm, where for $n = 50$, the decreasing and for $n = 100$, the increasing version of the procedure yields the best performance. These statements hold for systems with non-random as well as for systems with random ownership matrices.

In Section B.3 in the Appendix, the results when investigating the influence of the other simulation parameters on the runtime are shown. The integration parameter seems to have an influence on the algorithm efficiency. We observe increasing runtimes for increasing equity integration levels and an increase in the runtime for debt integration levels up to a level of $r^1 = 0.5$ as shown in the Tables B.5 and B.6. The same findings can be reported for the debt values, i.e. error rates tend to be higher for higher values of d (cf. Table B.4). In Table B.4 it also becomes visible that the runtime increases if $\sigma = 1$ compared to situations in which $\sigma = 0.5$ The differences of the runtimes for different diversification parameters are negligible small (see Tables B.7 and B.8) which is why we conclude that the diversification does not affect the algorithm efficiency.

Table 7.6: Mean runtime for each considered algorithm grouped by the system size n for systems with fixed and random ownership structure.

	n (Fixed ownership matrices)					n (Random ownership matrices)				
	5	10	25	50	100	5	10	25	50	100
DP	1.541	1.864	2.377	3.413	8.550	1.404	1.721	2.363	3.507	8.641
IP	1.986	2.242	2.692	3.618	8.085	1.594	1.912	2.588	3.691	8.214
DE	1.597	1.913	2.701	4.617	14.237	1.568	1.908	2.830	4.820	14.514
IE	1.833	2.185	3.027	5.098	15.519	1.634	2.037	3.048	5.240	15.772
DH	1.837	2.151	3.056	5.801	19.452	1.816	2.105	3.070	5.477	17.207
IH	2.133	2.508	3.304	5.091	13.718	1.872	2.262	3.253	5.007	13.081
DTP	1.775	1.940	2.478	4.112	12.794	1.715	1.914	2.469	3.985	11.532
ITP	1.884	2.099	2.672	4.290	12.460	1.811	2.020	2.621	4.125	11.213
DTE	2.450	2.789	3.582	6.239	19.236	2.361	2.688	3.529	5.934	18.306
ITE	2.757	3.036	3.844	6.626	20.180	2.619	2.879	3.734	6.257	19.318
DTH	1.267	1.603	2.329	4.453	14.886	1.335	1.646	2.387	4.304	13.278
ITH	1.872	2.218	2.906	4.618	12.409	1.739	2.063	2.877	4.536	11.691
SP	1.205	1.389	1.939	3.474	10.373	1.144	1.317	1.912	3.381	10.034
SE	1.290	1.513	2.185	4.113	13.183	1.218	1.425	2.143	4.036	13.365
SH	1.489	1.792	2.607	4.773	14.933	1.370	1.680	2.580	4.713	14.692

7.5.2 Discussion

The simulation results suggest that the Picard Iteration is for the three classes of algorithms (non-finite, Trial-and-Error, Sandwich) in almost all situations the most efficient method to obtain the next iterate. For the class of Trial-and-Error Algorithms, the Hybrid versions of the algorithms are sometimes faster than their counterparts from the Picard version, as can be seen in Table 7.6. But for the other two classes, i.e. the non-finite and the Sandwich Algorithms the mean runtime over all parameters always becomes minimal for one version of the Picard Algorithm. The Elsinger and the Hybrid Algorithms resulted in a series that will not need more iterates than the Picard method. This faster convergence, however, was accompanied with the fact that the single iteration steps have higher computational costs. With the results of the simulation in mind we can now say that in the majority of considered situations, the higher convergence speed does not compensate the additional computational costs. Another main results is that the size of the financial system determines which algorithm class should be chosen to find the solution. For small sample sizes, the Sandwich Algorithms have the smallest runtimes and the larger n becomes, the better becomes the performance of the non-finite algorithms – at least the Picard Algorithm.

Let us also mention the separate comparison of increasing and decreasing version of each algorithm. We observe that the runtime for the Decreasing Elsinger and the Decreasing Trial-and-Error Elsinger is always less than the time of their increasing counterparts. With the small exception for $n = 100$, the same statement holds for the Picard and the Trial-and-Error Picard Algorithm as well. This result can probably be explained by the fact that the situations in which many or all firms of the system are in default are relatively rare, see also the comments at the end of Section 7.2.1. In such cases, the starting vector $\mathbf{R}_{\text{great}}$ or $(\mathbf{d}, \mathbf{r}^0(\mathbf{d}))$ that assumes maximal debt payments tends to be closer to the solution than the starting vector of the increasing version and needs less iteration steps and therefore less runtime to reach the stopping criteria. For the Hybrid and the Trial-and-Error Hybrid Algorithm, however, we see that for $n = 100$ the performance for the increasing version becomes essentially better than the one for the decreasing version. An explanation for this behavior might be that for the increasing version, a Picard-type algorithm (Algorithm 5) is used to obtain the next debt components instead of Algorithm 4 that solves linear equation systems which is utilized in the decreasing version. One of the main results of the simulation is that for large systems, the simple application of the mapping is preferable to procedures in which potentially many linear equation systems have to be solved which is reflected in this situation too.

Table 7.7: Mean runtime over all parameters for Decreasing Picard, Elsinger and Hybrid Algorithm for different tolerance levels ε.

		ε (Fixed ownership matrices)				ε (Random ownership matrices)			
		10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$n = 5$	DP	1.541	1.811	2.081	2.356	1.404	1.601	1.795	1.985
	DE	1.597	1.758	1.952	2.140	1.568	1.707	1.843	1.973
	DH	1.837	1.920	2.060	2.162	1.816	1.928	2.019	2.105
$n = 10$	DP	1.864	2.165	2.500	2.826	1.721	1.971	2.239	2.505
	DE	1.913	2.151	2.392	2.638	1.908	2.078	2.284	2.487
	DH	2.151	2.269	2.434	2.589	2.105	2.226	2.358	2.484
$n = 25$	DP	2.337	2.764	3.183	3.582	2.363	2.717	3.102	3.485
	DE	2.701	3.007	3.320	3.661	2.830	3.085	3.438	3.788
	DH	3.056	3.233	3.452	3.688	3.070	3.291	3.534	3.767
$n = 50$	DP	3.413	3.934	4.492	5.007	3.507	4.017	4.572	5.141
	DE	4.617	5.035	5.586	6.136	4.820	5.226	5.825	6.416
	DH	5.801	6.040	6.394	6.904	5.477	5.930	6.323	6.829
$n = 100$	DP	8.550	9.557	10.675	11.810	8.641	9.629	10.794	11.976
	DE	14.237	14.805	16.397	17.990	14.514	15.885	17.609	19.308
	DH	19.452	19.346	20.544	22.268	17.207	19.066	20.196	22.050
all n	DP	3.549	4.046	4.586	5.116	3.527	3.987	4.500	5.018
	DE	5.013	5.351	5.929	6.513	5.128	5.596	6.200	6.794
	DH	6.460	6.562	6.977	7.522	5.935	6.488	6.886	7.447

We close this section by discussing the influence of the tolerance level ε of the algorithms. Note that for the Sandwich and the Trial-and-Error Algorithms (except for the Increasing Trial-and-Error Hybrid Algorithm) the tolerance level plays no role. For the non-finite algorithms, however, the value of ε is obviously a crucial parameter that strongly affects the runtime. In order to assess this influence more detailed, we performed an additional simulation study that contains the same parameters than all other simulations above and measured the runtime for three additional values of ε. For clarity, we only take the decreasing versions of Picard, Elsinger and Hybrid Algorithm into account in this simulation. The mean runtimes over all considered parameters grouped by the tolerance level and system size are shown in Table 7.7.

The mean runtimes increase for the algorithms for increasing tolerances levels. For $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-6}$ this increase is that high that the Decreasing Picard Algorithm does not have the best performance for large financial systems ($n = 50, 100$) anymore as this is the case if $\varepsilon = 10^{-3}$, see Table 7.6. We can therefore state that a high tolerance level leads to differing results as the ones in the paragraphs above since then, the Sandwich Algorithms should be preferred even for large financial systems which was not the case for a value of $\varepsilon = 10^{-3}$. Moreover, we only considered financial systems with $\mathbf{a} = \mathbf{1}_n$ where this described tendency might only be small. For systems with much larger values of $\mathbf{a}$ and $\mathbf{d}$, we observed that the number of needed iterations for the Picard Algorithm strongly increases. This is why we expect that for more realistic examples or examples with real data, where $\mathbf{a}$ and $\mathbf{d}$ can take values up to one billion or higher, the outperformance of the Sandwich Picard Algorithm will become even more obvious.

8 Summary and Outlook

This thesis represents a contribution to bring some order into the research area of systemic risk and financial interconnectedness for two reasons. Firstly, we aimed at establishing a definition of a financial system that is as general as possible and give associated model assumptions for a unique payment equilibrium. We showed that the standard model defined in Section 2.2 is the most general and flexible system and presented with the Elsinger Property the crucial condition that has to be valid to ensure a unique fixed point. Moreover, if this property is violated, we revealed the additional assumptions that have to hold to ensure the uniqueness of the solution. It is our hope that the standard model can be established in literature to avoid confusion about the compatibility of differing models and also to avoid that authors come to results about existence and uniqueness of certain models being unaware of the fact that some other authors already have came to similar results before. Secondly, this work is an attempt to survey the existing valuation procedures to obtain the solution of a financial system. We described the properties and the connections between the existing valuation algorithms on this field and generalized the calculation techniques by mentioning both an upside and a downside version of the procedures. While the existing algorithms generate iterates who are, under some circumstances, denied to reach the payment equilibrium exactly, we developed a new class of algorithms that overcomes this drawback. In a simulation study we discussed in which situation which algorithm should be preferred to minimize the computational effort. The results suggest that the Picard Iteration seems to be the most efficient method to obtain the next iterate and also that the Sandwich Picard Algorithm tends to be the fastest calculation method for medium and small system sizes. If in particular a high tolerance level for the class of non-finite algorithms is chosen, the statement about the Sandwich Picard Algorithm even holds for large systems as well.

There are several issues that remain open for future directions of research. We mentioned in Remark 3.16 that the stated regularity conditions, no matter which one, are only *sufficient* conditions for the uniqueness of a financial system. The question whether there are *necessary* conditions for the uniqueness of a pricing equilibria still remains open. Such informations could be of interest for a regulatory authority (e.g. a central bank) since if a potential necessary condition on a network is traceable not fulfilled, this means that the prices in a system are not unique, i.e. the different firms eventually calculate with different prices in their balance sheets leading to a potential instability of a financial network. An open problem is also the extension of the standard model to a multi-period model in which prices are not only calculated at maturity but at more than one clearing date. This issue is already mentioned in Eisenberg and Noe (2001) but, as far as we know, has not been treated in a particular article yet. Concerning the different iteration techniques (Picard, Elsinger, Hybrid), we think that it would be desirable to detect higher convergence orders than only the linear convergence showed in Section 7.2.2. It would also be interesting to investigate whether the conclusions of the simulation study stay the same for other types of financial networks. This can be for instance networks with a core-periphery structure in which the system consists of a small amount of larger firms and a relatively large amount of small firms, see Elliott et al. (2014) or Awiszus and Weber (2015) for details. Another network type is a so-called star formation described in Nier et al. (2007) that simulates a banking system with a central bank. Two last questions are clearly whether the simulation results of this work also hold for financial systems with higher values of **a** and **d** and whether the findings can be confirmed for systems with more than one seniority class.

A Auxiliary Results

Lemma A.1. *Let $||\cdot||$ be a not necessarily strictly convex norm on $\mathbb{R}^n$, and let Φ be a map on a nonempty convex and compact set $\mathcal{C} \subset \mathbb{R}^n$ which is non-expansive with respect to the norm-induced metric. The set of fixed points of Φ in $\mathcal{C}$ is then nonempty, closed, and either a singleton, or uncountable.*

Proof. Much-refined versions of this result are known (e.g. Bruck, 1973). For convenience, a short proof is given. Non-expansiveness implies that Φ is (1-Lipschitz) continuous. The set of fixed points is hence closed, and the Brouwer-Schauder Fixed Point Theorem (e.g. Rudin, 1990) provides the existence of at least one fixed point. Assume now that $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ are two distinct fixed points of Φ. For $\mathbf{v} \in \mathcal{C}$ and $\varepsilon > 0$, $\mathcal{B}_\varepsilon(\mathbf{v}) = \{\mathbf{w} \in \mathcal{C} : ||\mathbf{w} - \mathbf{v}|| \leq \varepsilon\}$ is a non-empty, convex and compact subset of $\mathcal{C}$. For $\lambda \in (0, 1)$, the intersection

$$\mathcal{C}_\lambda = \mathcal{B}_{\lambda||\mathbf{y}-\mathbf{x}||}(\mathbf{x}) \cap \mathcal{B}_{(1-\lambda)||\mathbf{y}-\mathbf{x}||}(\mathbf{y}) \tag{A.1}$$

is non-empty (as it contains $(1 - \lambda)\mathbf{x} + \lambda\mathbf{y})$), convex and compact, and it contains neither $\mathbf{x}$, nor $\mathbf{y}$. By the triangle inequality, $\mathcal{C}_{\lambda_1} \cap \mathcal{C}_{\lambda_2} = \emptyset$ for $\lambda_1 \neq \lambda_2$. Non-expansiveness implies that $\Phi(\mathcal{C}_\lambda) \subset \mathcal{C}_\lambda$. By Brouwer-Schauder, there exists a fixed point of Φ in $\mathcal{C}_\lambda$. Hence there exist uncountably many fixed points of Φ in $\mathcal{C}$. □

Lemma A.2. *Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be an ownership matrix that possesses the Elsinger Property. Then $\rho(\mathbf{M}) < 1$, where*

$$\rho(\mathbf{M}) = \max\{|\lambda_i| : \lambda_i \textit{ eigenvalue of } \mathbf{M}\} \tag{A.2}$$

is the spectral radius of $\mathbf{M}$.

Proof. A well known result (cf. Rudin, 1990) is that $\rho(\mathbf{M}) \leq \|\mathbf{M}\| \leq 1$. In case of $\|\mathbf{M}\| < 1$ there is nothing to show, so we assume that $\|\mathbf{M}\| = 1$ which is no contradiction to the Elsinger Property of $\mathbf{M}$. We will show the claim by contradiction. To this end, assume that $\rho(\mathbf{M}) = 1$. For the corresponding eigenvalue $\mathbf{v}$ is must hold that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{M}\mathbf{v} = \rho(\mathbf{M})\mathbf{v} = \mathbf{v}$. We can formulate this equation alternatively as

$$(\mathbf{I}_n - \mathbf{M})\mathbf{v} = \mathbf{0}_n. \tag{A.3}$$

Since $\mathbf{M}$ has the Elsinger Property, it follows by Elsinger (2009, Lemma 1), that $(\mathbf{I}_n - \mathbf{M})$ is invertible. But that means that there exists no vector $\mathbf{v} \neq \mathbf{0}$ such that (A.3) is true. Hence, $\mathbf{v} = \mathbf{0}$ which is a contradiction and from which follows that $\rho(\mathbf{M}) < 1$. □

Lemma A.3. *Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be an ownership matrix that possesses the Elsinger property. Then $(\mathbf{I}_n - \mathbf{M})^{-1}$ exists and can be obtained via the* Neuman expansion*:*

$$(\mathbf{I}_n - \mathbf{M})^{-1} = \sum_{n=0}^{\infty} \mathbf{M}^n, \tag{A.4}$$

where $\mathbf{M}^0 = \mathbf{I}_n$. Consequently, the diagonal entries of $(\mathbf{I}_n - \mathbf{M})^{-1}$ are greater than or equal to 1 and the other entries are all non-negative.

Proof. See Rudin (1990) for a proof. □

Proof of **Theorem 2.7**: A proof of Theorem 2.7 is necessary because related proofs in Suzuki (2002), Gouriéroux et al. (2012) and Fischer (2014) rely on the assumption that $\|\mathbf{M}^k\| < 1$ for all $k = 0, \ldots, m$, which is a stronger condition than the Elsinger Property, while Elsinger (2009) considers an equation system which slightly differs from (2.9) – (2.11). In the equation system of Elsinger, the recovery values are $\mathbf{r}^k$ are subtracted instead of the nominal liabilities $\mathbf{d}^k$ as in (2.9) – (2.11). First, note that (2.9) – (2.11) can only have non-negative solutions. This is shown in Fischer (2014, Lemma 3.5) under stricter matrix conditions, but because of Lemma A.3, it is straightforward to see that the proof works in the same manner under the Elsinger Property. The interval $[\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ is convex and compact, where $\mathbf{R}_{\text{small}}$ is defined in (4.6) (for $m = 1$) or (5.3) (for $m > 1$) and $\mathbf{R}_{\text{great}}$ is defined in (4.5) (for $m = 1$) or (5.1) (for $m > 1$), and $\Phi(\mathbf{R})$ is continuous in $\mathbf{R}$. From Proposition 4.2, it follows that $\Phi([\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]) \subset [\mathbf{R}_{\text{small}}, \mathbf{R}_{\text{great}}]$ for $m = 1$ which can obviously be extended to systems with $m > 1$. Together with the Brouwer-Schauder Fixed Point Theorem, it follows that at least one solution exists. Furthermore, Φ as in (2.12) is a non-expansive mapping. This follows from Fischer (2014, Lemma 4.1), where a strict contraction property is shown under stricter matrix conditions, but again it is straightforward to see how the corresponding proof implies non-expansiveness under the Elsinger Property for all ownership matrices. Since it follows from Proposition 5.10 that for $m \geq 1$ there can be a maximum of $(m+1)^n$ possible solutions of (2.9) – (2.11), the uniqueness follows from Proposition 4.2 and Lemma A.1. □

Theorem A.4 (Tarski Fixed Point Theorem). *Let $\mathcal{X}$ be a complete lattice and $f : \mathcal{X} \to \mathcal{X}$ an increasing function. Then there exists a greatest and a least fixed point of f, i.e. there are $\mathbf{x}^* \in \mathcal{X}$ and $\mathbf{x}_* \in \mathcal{X}$ with $f(\mathbf{x}^*) = \mathbf{x}^*$ and $f(\mathbf{x}_*) = \mathbf{x}_*$ such that for any other fixed point $\mathbf{x} \in \mathcal{X}$ it holds that $\mathbf{x}_* \leq \mathbf{x} \leq \mathbf{x}^*$.*

Proof. For a proof, we refer to the original work of Tarski (1955). □

Theorem A.5 (Banach Contraction Mapping Theorem). *Let $(\mathcal{X}, d)$ be a complete metric space with norm d and $f : \mathcal{X} \to \mathcal{X}$ a strict contraction on $\mathcal{X}$, i.e. there exists a number $0 \leq \lambda < 1$ such that*

$$d\left(f(\mathbf{x}), f(\mathbf{y})\right) \leq \lambda d(\mathbf{x}, \mathbf{y}) \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathcal{X}. \tag{A.5}$$

Then f has a unique fixed point $\mathbf{x}^* \in \mathcal{X}$.

Proof. A proof is given for example in Banach (1922). □

Lemma A.6. *Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be an ownership matrix as in Lemma A.3 and the matrix $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$ be defined as*

$$(\mathbf{\Lambda})_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{N}_0, \\ 0, & \text{else,} \end{cases} \tag{A.6}$$

where $\mathcal{N}_0 \subset \mathcal{N}$. Then it holds that

$$(\mathbf{I}_n - \mathbf{\Lambda}\mathbf{M}\mathbf{\Lambda})^{-1}\mathbf{\Lambda} \leq \mathbf{\Lambda}(\mathbf{I}_n - \mathbf{M})^{-1}\mathbf{\Lambda}. \tag{A.7}$$

Proof. Note that

$$(\mathbf{I}_n - \mathbf{\Lambda})^k = (\mathbf{I}_n - \mathbf{\Lambda}) \quad \text{for } k \in \mathbb{N} \tag{A.8}$$

and that $\mathbf{M}^0 = \mathbf{I}_n$. Using Lemma A.3 we have that

$$\begin{aligned}
(\mathbf{I}_n - \mathbf{\Lambda M \Lambda})^{-1}\mathbf{\Lambda} &= \left(\sum_{n=0}^{\infty}(\mathbf{\Lambda M \Lambda})^n\right)\mathbf{\Lambda} \\
&= (\mathbf{I}_n + \mathbf{\Lambda M \Lambda} + \mathbf{\Lambda M \Lambda M \Lambda} + \mathbf{\Lambda M \Lambda M \Lambda M \Lambda} + \ldots)\mathbf{\Lambda} \\
&= \mathbf{\Lambda} + \mathbf{\Lambda}(\mathbf{M} + \underbrace{\mathbf{M \Lambda M}}_{\leq \mathbf{M}^2} + \underbrace{\mathbf{M \Lambda M \Lambda M}}_{\leq \mathbf{M}^3} + \ldots)\mathbf{\Lambda} \\
&\leq \mathbf{\Lambda} + \mathbf{\Lambda}\left(\sum_{n=1}^{\infty}\mathbf{M}^n\right)\mathbf{\Lambda} \\
&= \mathbf{\Lambda}\left(\sum_{n=0}^{\infty}\mathbf{M}^n\right)\mathbf{\Lambda} \\
&= \mathbf{\Lambda}(\mathbf{I}_n - \mathbf{M})^{-1}\mathbf{\Lambda}.
\end{aligned} \tag{A.9}$$

□

Lemma A.7. *The set of all* $\mathbf{a}$*'s for which the pseudo solution contains at least one firm on borderline, i.e. one* $i \in \mathcal{N}$ *such that* $r_i^1 = d_i$ *and* $r_i^0 = 0$ *has Lebesgue measure zero.*

Proof. First note that it suffices to show the claim for the set $A(\mathcal{I})$ of all $\mathbf{a}$'s for which $r_i^1 = d_i$ and $r_i^0 = 0$ for each $i \in \mathcal{I} \subset \mathcal{N}$, since the number of subsets of $\{1, ..., n\}$ is finite and a finite union of sets of Lebesgue measure zero has Lebesgue measure zero. We first show that $A(\mathcal{I})$ is a Borel set and hence Lebesgue measurable. For this, note that it is shown in Fischer (2014, 2015) that the mapping $\Psi : \mathbf{a} \mapsto \mathbf{R}^*(\mathbf{a})$ that maps any price vector of the exogenous assets onto the corresponding solution of (4.1) and (4.2) is Borel measurable. Let now $H(\mathcal{I})$ denote the $2(n - |\mathcal{I}|)$-dimensional hyperplane in $\mathbb{R}^{2n}$ for which

$$H(\mathcal{I}) = \{(\mathbf{r}^1, \mathbf{r}^0) : r_i^1 = d \text{ and } r_i^0 = 0 \text{ for all } i \in \mathcal{I}\}. \tag{A.10}$$

Clearly, $H(\mathcal{I})$ is a Borel set. One obtains

$$A(\mathcal{I}) = \Psi^{-1}(H(\mathcal{I}) \cap (\mathbb{R}_0^+)^{2n}), \tag{A.11}$$

which must be Borel-measurable, too. Observe now that if $\mathbf{a}_2 \gg \mathbf{a}_1$ ($\mathbf{a}_2$ strictly larger than $\mathbf{a}_1$ in all components), then $\Phi_{\mathbf{a}_2}(\mathbf{R}) \gg \Phi_{\mathbf{a}_1}(\mathbf{R})$ for any non-negative $\mathbf{R}$. Hence, by the Picard Iteration, $\mathbf{R}^*(\mathbf{a}_2) \geq \mathbf{R}^*(\mathbf{a}_1)$. From (4.1) and (4.2) it follows now that if $\mathbf{a}_1, \mathbf{a}_2 \in A(\mathcal{I})$ and $\mathbf{a}_2 \gg \mathbf{a}_1$, then $r_i^{*,0}(\mathbf{a}_2) > r_i^{*,0}(\mathbf{a}_1)$, which is a contradiction. Thus, $\mathbf{a}_2 \gg \mathbf{a}_1$ can hold for no pair $\mathbf{a}_1, \mathbf{a}_2 \in A(\mathcal{I})$. This means that $A(\mathcal{I})$ bears some resemblance to a Pareto set (a Pareto frontier) – indeed, it would be a Pareto set if the statement was true for $\mathbf{a}_2 > \mathbf{a}_1$. It follows that the set $A(\mathcal{I})$ intersects any straight line parallel to the vector $(1, \ldots, 1)$ either once, or not at all. As such, and since the Lebesgue measure is rotation invariant, the problem reduces now to the one which is shown in the next lemma. □

Lemma A.8. *Let* $\mathcal{B}$ *be a Borel set in* $\mathbb{R}^n$ *such that one has* $|\mathcal{B}_\omega| \leq 1$ *for any* $\omega \in \mathbb{R}^{n-1}$*, where* $\mathcal{B}_\omega = \{x \in \mathbb{R} : (x, \omega) \in \mathcal{B}\}$*. Then* $\mathcal{B}$ *has Lebesgue measure zero.*

Proof. Let λ_m denote the Lebesgue measure on $\mathbb{R}^m$. For any Borel set $\mathcal{B}$, it follows from the definition of product measures (e.g. Billingsley (1995)) and the fact that $\lambda_n = \lambda_1 \otimes \lambda_{n-1}$ that

$$\lambda_n(\mathcal{B}) = \int \lambda_1(\mathcal{B}_\omega) d\lambda_{n-1}(\omega). \tag{A.12}$$

Since $\lambda_1(\mathcal{B}_\omega) = 0$, the result follows. $\square$

B Additional Tables

All numbers in Chapter B are rounded to three decimal places.

B.1 Error Rates for Non-finite Algorithms

Table B.1: Error rates defined in (7.31) grouped by system size n and values of d and σ for systems with a fixed ownership structure.

		$d = 0.5$		$d = 1$		$d = 1.5$		$d = 2$	
		$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$
	P	0.042	0.122	0.035	0.131	0.017	0.134	0.015	0.147
$n = 5$	E	0.010	0.041	0.010	0.057	0.009	0.081	0.010	0.104
	H	0.010	0.022	0.007	0.020	0.002	0.025	0.017	0.037
	P	0.064	0.167	0.044	0.167	0.011	0.161	0.005	0.163
$n = 10$	E	0.013	0.051	0.007	0.072	0.005	0.097	0.004	0.111
	H	0.009	0.022	0.004	0.017	0.001	0.022	0.006	0.039
	P	0.030	0.164	0.026	0.179	0.005	0.179	0.001	0.176
$n = 25$	E	0.020	0.070	0.006	0.095	0.002	0.108	0.001	0.115
	H	0.007	0.012	0.001	0.014	0.000	0.027	0.002	0.042
	P	0.003	0.188	0.024	0.189	0.003	0.190	0.000	0.183
$n = 50$	E	0.020	0.090	0.004	0.100	0.000	0.116	0.000	0.116
	H	0.004	0.007	0.000	0.017	0.000	0.042	0.001	0.053
	P	0.002	0.180	0.017	0.197	0.002	0.201	0.000	0.184
$n = 100$	E	0.007	0.089	0.001	0.102	0.000	0.122	0.000	0.117
	H	0.002	0.008	0.000	0.032	0.000	0.067	0.003	0.069
	P	0.028	0.164	0.029	0.172	0.007	0.173	0.004	0.170
all n	E	0.014	0.068	0.005	0.085	0.003	0.105	0.003	0.112
	H	0.006	0.014	0.002	0.020	0.001	0.037	0.006	0.048
	P	0.018	0.396	0.024	0.539	0.022	0.605	0.016	0.583
$r^1 = 0.95$	E	0.005	0.142	0.008	0.274	0.012	0.363	0.010	0.367
	H	0.002	0.022	0.001	0.059	0.002	0.106	0.002	0.116

Notes: For systems with random ownership matrices, the results are very similar.

Table B.2: Error rates defined in (7.31) for systems with a random ownership structure grouped by integration and diversification levels of equity and debt.

	p^0/p^1	r^0				r^1			
		0	0.025	0.25	0.475	0	0.05	0.5	0.95
Picard	0.05	0.117	0.098	0.121	0.109	0.045	0.023	0.071	0.369
	0.5	0.117	0.103	0.112	0.081	0.046	0.014	0.031	0.295
	0.95	0.117	0.101	0.107	0.075	0.045	0.012	0.020	0.233
Elsinger	0.05	0.119	0.084	0.061	0.045	0.003	0.014	0.073	0.217
	0.5	0.119	0.078	0.043	0.020	0.003	0.015	0.035	0.191
	0.95	0.119	0.077	0.041	0.017	0.003	0.015	0.018	0.151
Hybrid	0.05	0.003	0.031	0.032	0.026	0.003	0.007	0.036	0.081
	0.5	0.003	0.031	0.027	0.017	0.003	0.004	0.021	0.040
	0.95	0.003	0.030	0.027	0.015	0.003	0.004	0.017	0.026

Notes: The labels of the row refer either to the equity or the debt diversification parameter depending on which part of the table is used. In the left hand part, i.e. the columns that refer to the equity integration parameter r^0, the corresponding diversification parameter is p^0. For the right hand part of the table, the debt diversification parameter p^1 is regarded.

B.2 Runtime for Trial-and-Error Algorithms

Table B.3: Mean runtime for Trial-and-Error Algorithms over all considered simulation parameters grouped by lag value l and system size n for systems with fixed and random ownership structure.

		Fixed ownership matrices				Random ownership matrices			
		$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
$n = 5$	DTP	1.775	1.977	2.185	2.426	1.715	1.915	2.123	2.332
	ITP	1.884	2.037	2.235	2.437	1.811	1.983	2.189	2.377
	OTP	2.923	3.145	3.409	3.648	2.836	3.049	3.294	3.533
	DTE	2.450	2.764	3.088	3.429	2.361	2.681	2.983	3.304
	ITE	2.757	3.077	3.453	3.818	2.619	2.961	3.314	3.675
	OTE	4.010	4.347	4.721	5.091	3.900	4.234	4.586	4.946
	DTH	1.267	1.634	2.022	2.426	1.335	1.759	2.191	2.623
	ITH	1.872	2.302	2.717	3.129	1.739	2.174	2.594	3.001
	OTH	3.849	4.253	4.659	5.051	3.822	4.263	4.682	5.092

Table B.3 (continued):

		fixed ownership matrices				random ownership matrices			
		$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
	DTP	1.940	2.172	2.394	2.647	1.914	2.125	2.348	2.589
	ITP	2.099	2.282	2.502	2.733	2.020	2.207	2.429	2.666
	OTP	3.095	3.320	3.583	3.856	3.022	3.248	3.487	3.742
	DTE	2.789	3.192	3.579	3.972	2.688	3.045	3.419	3.794
$n = 10$	ITE	3.036	3.428	3.825	4.253	2.879	3.249	3.648	4.041
	OTE	4.404	4.773	5.181	5.629	4.192	4.588	4.991	5.400
	DTH	1.603	2.122	2.649	3.173	1.646	2.189	2.734	3.275
	ITH	2.218	2.757	3.250	3.726	2.063	2.583	3.075	3.540
	OTH	4.330	4.882	5.355	5.843	4.264	4.779	5.275	5.767
	DTP	2.478	2.724	3.018	3.315	2.469	2.704	2.983	3.281
	ITP	2.672	2.885	3.155	3.441	2.621	2.834	3.111	3.401
	OTP	3.708	3.955	4.260	4.566	3.681	3.923	4.207	4.503
	DTE	3.582	4.081	4.598	5.147	3.529	4.028	4.536	5.051
$n = 25$	ITE	3.844	4.333	4.878	5.416	3.734	4.239	4.762	5.267
	OTE	5.370	5.876	6.446	6.965	5.270	5.777	6.299	6.830
	DTH	2.329	3.099	3.887	4.671	2.387	3.189	3.986	4.781
	ITH	2.906	3.589	4.233	4.866	2.877	3.559	4.204	4.808
	OTH	5.544	6.226	6.931	7.602	5.606	6.325	7.020	7.706
	DTP	4.112	4.450	4.917	5.301	3.985	4.304	4.697	5.139
	ITP	4.290	4.595	5.010	5.434	4.125	4.418	4.805	5.248
	OTP	5.633	6.018	6.399	6.867	5.483	5.806	6.212	6.662
	DTE	6.239	7.159	8.044	8.999	5.934	6.767	7.648	8.562
$n = 50$	ITE	6.626	7.548	8.418	9.393	6.257	7.100	8.007	8.896
	OTE	8.885	9.782	10.692	11.628	8.414	9.260	10.198	11.079
	DTH	4.453	5.989	7.506	8.972	4.304	5.750	7.172	8.654
	ITH	4.618	5.749	6.756	7.773	4.536	5.623	6.631	7.645
	OTH	9.328	10.621	11.813	13.060	9.137	10.350	11.592	12.756
	DTP	12.794	13.655	14.786	15.879	11.532	12.353	13.353	14.374
	ITP	12.460	13.255	14.362	15.522	11.213	11.978	12.963	13.966
	OTP	16.094	16.967	18.090	19.307	14.571	15.444	16.424	17.475
	DTE	19.236	22.033	24.845	27.548	18.306	20.941	23.704	26.351
$n = 100$	ITE	20.180	22.946	25.711	28.552	19.318	21.893	24.634	27.310
	OTE	26.233	28.982	31.649	34.412	24.823	27.495	30.181	32.849
	DTH	14.886	19.741	24.727	29.569	13.278	17.696	22.137	26.559
	ITH	12.409	15.369	18.323	21.119	11.691	14.519	17.402	20.078
	OTH	27.615	31.535	35.409	39.042	26.382	29.923	33.505	37.062

B.3 Additional Tables for the Runtime

Table B.4: Mean runtimes for each considered algorithms of Section 7.5 over all simulation parameters grouped by debt values d and values of σ for systems with fixed and random ownership structure.

		$d = 0.5$		$d = 1$		$d = 1.5$		$d = 2$	
		$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 1$
FOS	DP	3.032	3.738	3.233	3.896	3.236	3.928	3.360	3.969
	IP	3.789	3.825	3.575	3.835	3.464	3.830	3.594	3.883
	DE	3.618	5.012	4.302	5.551	4.612	5.973	4.943	6.091
	IE	4.202	5.312	4.918	5.951	5.327	6.322	5.687	6.542
	DH	4.961	6.717	5.804	7.225	6.024	7.456	6.038	7.452
	IH	4.497	5.498	4.977	5.842	5.051	6.010	4.961	5.970
	DTP	4.283	4.619	4.483	4.821	4.517	4.886	4.435	4.913
	ITP	4.157	4.627	4.479	4.789	4.636	4.892	4.878	4.990
	DTE	6.965	6.993	6.844	7.123	6.585	7.106	6.283	6.975
	ITE	7.060	7.216	7.216	7.425	7.190	7.493	7.260	7.451
	DTH	3.716	4.944	4.420	5.437	4.785	5.703	4.553	5.704
	ITH	3.783	4.671	4.501	5.048	4.907	5.210	5.027	5.292
	SP	2.958	3.471	3.416	3.763	3.635	3.987	3.962	4.216
	SE	3.191	3.916	3.903	4.509	4.580	5.044	5.052	5.462
	SH	3.301	4.529	4.542	5.375	5.470	5.916	5.609	6.210
ROS	DP	2.969	3.652	3.243	3.781	3.276	3.882	3.447	3.968
	IP	3.623	3.695	3.445	3.699	3.351	3.711	3.485	3.790
	DE	3.694	5.263	4.361	5.753	4.693	6.045	5.039	6.175
	IE	4.176	5.483	4.919	6.079	5.264	6.326	5.552	6.490
	DH	4.585	6.188	5.364	6.547	5.637	6.760	5.727	6.672
	IH	4.282	5.281	4.780	5.569	4.841	5.662	4.785	5.561
	DTP	4.002	4.304	4.199	4.478	4.259	4.579	4.169	4.593
	ITP	3.889	4.272	4.193	4.419	4.345	4.533	4.575	4.637
	DTE	6.499	6.791	6.549	6.910	6.321	6.830	5.943	6.664
	ITE	6.621	7.028	6.897	7.238	6.848	7.189	6.799	7.073
	DTH	3.451	4.649	4.177	5.043	4.496	5.258	4.325	5.320
	ITH	3.589	4.575	4.382	4.818	4.617	4.953	4.674	5.044
	SP	2.823	3.385	3.305	3.625	3.550	3.851	3.858	4.064
	SE	3.198	4.035	3.948	4.560	4.503	5.005	4.902	5.349
	SH	3.308	4.587	4.536	5.245	5.274	5.692	5.442	5.972

Notes: The upper part of the table lists the mean runtimes for systems with a fixed ownership structure (FOS), the lower part shows the runtimes for systems with a random ownership structure (ROS).

Table B.5: Mean runtime for each considered algorithm of Section 7.5 over all simulation parameters grouped by equity integration r^0 for systems with fixed ownership structure (FOS) and random ownership structure (ROS).

	r^0 (FOS)				r^0 (ROS)			
	0	0.025	0.25	0.475	0	0.025	0.25	0.475
DP	2.598	2.845	3.659	4.857	2.702	2.863	3.627	4.710
IP	2.724	2.668	3.829	5.426	2.794	2.653	3.686	4.065
DE	3.538	4.524	5.736	5.884	3.556	4.683	5.843	6.036
IE	3.776	4.765	6.328	6.822	3.722	4.868	6.321	6.781
DH	4.089	5.827	7.506	7.823	3.611	5.300	6.903	7.345
IH	2.951	4.301	6.412	7.139	2.785	4.131	6.097	6.810
DTP	4.028	4.531	4.694	5.077	3.754	4.239	4.232	4.734
ITP	4.131	4.369	4.757	5.330	3.879	4.089	4.429	4.914
DTE	5.211	6.553	7.452	7.808	4.869	6.325	7.167	7.476
ITE	5.468	6.861	7.917	8.453	5.090	6.626	7.600	8.062
DTH	4.556	4.862	5.179	4.946	4.086	4.484	4.899	4.766
ITH	3.851	4.436	5.204	5.491	3.585	4.231	4.991	5.269
SP	3.067	3.174	3.753	4.559	3.015	3.114	3.618	4.348
SE	3.286	4.032	4.950	5.268	3.190	4.094	4.972	5.182
SH	4.288	4.559	5.546	5.876	4.106	4.487	5.459	5.750

Table B.6: Mean runtime for each considered algorithm of Section 7.5 over all simulation parameters grouped by debt integration r^1 for systems with fixed ownership structure (FOS) and random ownership structure (ROS).

	r^1 (FOS)				r^1 (ROS)			
	0	0.05	0.5	0.95	0	0.05	0.5	0.95
DP	2.851	3.165	4.405	3.600	2.808	3.094	4.315	3.713
IP	2.441	2.742	4.680	4.714	2.419	2.692	4.455	4.538
DE	2.983	4.638	7.155	4.769	2.947	4.478	7.180	5.362
IE	3.006	4.241	7.885	6.367	2.924	4.070	7.690	6.807
DH	4.401	6.378	8.299	6.245	3.944	5.683	7.497	6.118
IH	3.441	4.194	7.082	6.209	3.167	3.881	6.624	6.227
DTP	4.357	4.410	5.164	4.482	4.056	4.106	4.791	4.272
ITP	4.187	4.146	5.307	4.960	3.900	3.871	4.867	4.679
DTE	6.186	6.257	7.788	7.038	5.866	5.896	7.453	6.865
ITE	6.413	6.162	8.309	8.051	6.093	5.832	7.848	7.856
DTH	3.884	5.031	6.195	4.265	3.538	4.565	5.692	4.302
ITH	3.684	3.943	5.955	5.357	3.431	3.668	5.615	5.323
SP	2.805	2.983	4.637	4.061	2.733	2.901	4.371	4.019
SE	2.626	3.153	6.206	5.386	2.603	3.085	5.986	5.617
SH	2.632	4.021	7.146	6.055	2.600	3.867	6.822	6.137

Table B.7: Mean runtime for each considered algorithm of Section 7.5 over all simulation parameters grouped by equity diversification λ^0 or p^0 for systems with fixed ownership structure (FOS) and random ownership structure (ROS).

	λ^0 (FOS)			p^0 (ROS)		
	0	0.5	1	0.05	0.5	0.95
DP	3.582	3.565	3.500	3.325	3.611	3.646
IP	3.734	3.715	3.724	3.380	3.696	3.724
DE	5.424	5.443	4.172	4.791	5.290	5.303
IE	5.977	5.989	4.632	5.106	5.734	5.769
DH	6.858	6.820	5.701	5.598	6.096	6.111
IH	5.738	5.722	4.592	4.663	5.297	5.325
DTP	4.690	4.681	4.488	4.193	4.384	4.392
ITP	4.714	4.726	4.603	4.259	4.407	4.408
DTE	7.173	7.215	6.190	6.215	6.734	6.741
ITE	7.626	7.677	6.563	6.564	7.152	7.168
DTH	5.109	5.105	4.509	4.479	4.645	4.646
ITH	5.023	5.029	4.363	4.390	4.674	4.679
SP	3.719	3.710	3.599	3.431	3.626	3.616
SE	4.751	4.815	3.806	4.143	4.586	4.584
SH	5.367	5.407	4.583	4.754	5.120	5.147

Table B.8: Mean runtime for each considered algorithm of Section 7.5 over all simulation parameters grouped by debt diversification λ^1 or p^1 for systems with fixed ownership structure (FOS) and random ownership structure (ROS).

	λ^1 (FOS)			p^1 (ROS)		
	0	0.5	1	0.05	0.5	0.95
DP	3.555	3.568	3.524	3.520	3.544	3.517
IP	3.786	3.750	3.637	3.407	3.682	3.711
DE	5.212	5.127	4.699	5.078	5.163	5.142
IE	5.769	5.659	5.169	5.285	5.665	5.658
DH	6.646	6.672	6.061	5.875	5.878	6.053
IH	5.419	5.393	5.240	5.059	5.127	5.099
DTP	4.624	4.627	4.608	4.350	4.309	4.309
ITP	4.707	4.689	4.646	4.332	4.375	4.366
DTE	6.876	6.866	6.837	6.558	6.575	6.558
ITE	7.348	7.312	7.206	6.837	7.032	7.016
DTH	5.022	5.070	4.631	4.580	4.557	4.633
ITH	4.913	4.889	4.612	4.457	4.638	4.650
SP	3.722	3.682	3.625	3.508	3.586	3.579
SE	4.622	4.521	4.229	4.225	4.538	4.549
SH	5.337	5.287	4.733	4.713	5.108	5.200

B.4 Iteration Numbers of the Algorithms

Table B.9: Median values of the number of iteration steps needed to reach the solution over all simulation parameters grouped by the system size n for systems with fixed and random ownership structure. The number in parentheses are the median values of the associated calculation steps.

	n (Fixed ownership matrices)					n (Random ownership matrices)				
	5	10	25	50	100	5	10	25	50	100
DP	4	5	6	6	7	4	4	5	6	7
IP	5	5	6	7	7	3	4	6	7	7
OP	4	4	5	6	7	3	4	5	6	7
DE	2 (4)	3 (4)	3 (6)	4 (6)	4 (6)	2 (3)	3 (4)	4 (6)	4 (6)	4 (7)
IE	3 (4)	3 (5)	4 (6)	4 (6)	4 (7)	2 (3)	3 (4)	3 (6)	4 (6)	4 (8)
OE	2 (4)	3 (4)	3 (5)	3 (6)	4 (6)	2 (3)	3 (4)	3 (5)	4 (6)	4 (6)
DH	2 (5)	2 (5)	2 (7)	2 (8)	2 (8)	2 (5)	2 (5)	2 (7)	2 (8)	3 (8)
IH	2 (3)	2 (3)	2 (4)	2 (4)	2 (6)	2 (3)	2 (3)	2 (4)	2 (4)	3 (6)
OH	2 (3)	2 (4)	2 (5)	2 (6)	2 (6)	2 (3)	2 (3)	2 (5)	2 (6)	2 (6)
DTP	1 (1)	1 (1)	1 (1)	1 (1)	2 (1)	1 (1)	1 (1)	1 (1)	1 (1)	2 (1)
ITP	1 (1)	1 (1)	1 (1)	2 (1)	2 (1)	1 (1)	1 (1)	1 (1)	2 (1)	2 (1)
OTP	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
DTE	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)
ITE	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)
OTE	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)	1 (3)	1 (3)	1 (4)	1 (5)	1 (5)
DTH	1 (4)	1 (4)	1 (6)	1 (6)	1 (7)	3 (3)	3 (3)	3 (3)	3 (3)	3 (3)
ITH	1 (3)	1 (3)	1 (4)	1 (4)	1 (5)	2 (2)	3 (3)	3 (3)	3 (3)	4 (4)
OTH	1 (4)	1 (4)	1 (4)	1 (5)	1 (5)	3 (3)	3 (3)	3 (3)	3 (3)	3 (3)
SP	0	0	1	1	1	0	0	1	1	1
SE	0 (2)	0 (2)	0 (4)	0 (4)	1 (4)	0 (2)	0 (2)	0 (4)	0 (4)	1 (5)
SH	0 (2)	0 (2)	0 (4)	0 (4)	1 (5)	0 (2)	0 (2)	0 (4)	0 (4)	1 (5)

Notes: For the Trial-and-Error Algorithms, a lag value of $l = 2$ was used for each algorithm.

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