Variational Approach to the Modeling and Analysis of Magnetoelastic Materials

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The night is darkest just before the dawn.
And I promise you, the dawn is coming.

Harvey Dent

Contents

Notations

$\mathbb{R}_0^+$	nonnegative real numbers
$\mathbb{R}^+$	positive real numbers
e_i	i-th standard basis vector in $\mathbb{R}^d$ for $1 \leq i \leq d$
δ_{ij}	Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$
S^{d-1}	unit sphere in $\mathbb{R}^d$
$\mathbb{R}^{d\times d}$	$d \times d$-matrices with real entries
$SO(d)$	special orthogonal matrices in $\mathbb{R}^{d\times d}$, i.e., any $\mathcal{R} \in SO(d)$ satisfies $\mathcal{R}^\top \mathcal{R} = I$ and $\det \mathcal{R} = 1$
Ω_0	reference configuration, $\Omega_0 \subset \mathbb{R}^d$
Ω	deformed / current configuration, $\Omega \subset \mathbb{R}^d$
t	time, $t \in \mathbb{R}_0^+$
X	a material point in the Lagrangian coordinate system, $X \in \Omega_0$
x	a spatial point in the Eulerian coordinate system, $x \in \Omega$
v	velocity in the Eulerian coordinate system
$\widetilde{F}$	deformation gradient in the Lagrangian coordinate system
F	deformation gradient in the Eulerian coordinate system
M	magnetization (Eulerian coordinate system)
H	magnetic (stray) field (Eulerian coordinate system)
B	magnetic induction (Eulerian coordinate system)
$\mathbf{n}$	outer normal vector to the boundary of Ω
$A^\top$	the transpose of a matrix $A \in \mathbb{R}^{d\times d}$, i.e., $(A^\top)_{ij} = A_{ji}$
$a_k b_k$, $A_{ik}B_{kj}$	for vectors or matrices the *Einstein summation convention* is used throughout this work: summation sign is omitted and the sum is over all indices which appear twice
$a \cdot b$	defines for $a, b \in \mathbb{R}^d$ the scalarproduct $a \cdot b := \sum_{i=1}^d a_i b_i = a_i b_i$ on the space of vectors
$a \otimes b$	dyadic product defines for $a, b \in \mathbb{R}^d$ the matrix $(a \otimes b)_{i,j} := a_i b_j$
$a \times b$	cross product defines for $a, b \in \mathbb{R}^3$ a vector $a \times b$ perpendicular to a and b such that $a, b, a \times b$ define a right-handed coordinate system; if the cross $\times$ is in the very beginning of a line within an equation or calculation, it simply indicates a multiplication

$A : B$	defines for $A, B \in \mathbb{R}^{d\times d}$ the scalarproduct $A : B := \mathrm{tr}(A^\top B) = \sum_{i,j=1}^d A_{ij}B_{ij} = A_{ij}B_{ij}$ on the space of matrices
$A \odot B$	defines for $A, B \in \mathbb{R}^{d\times d}$ the $d \times d$-matrix $(A \odot B)_{i,j} := (A^\top B)_{i,j} = \sum_{k=1}^d A_{ki}B_{kj} = A_{ki}B_{kj}$
$\nabla A \,\vdots\, \nabla B$	defines for $A, B \in \mathbb{R}^{d\times d}$ the product $\nabla A \,\vdots\, \nabla B := \sum_{i,j,k=1}^d \nabla_k A_{ij} \nabla_k B_{ij} = \nabla_k A_{ij} \nabla_k B_{ij}$
$\mathrm{skew}(A)$	skew symmetric part $\mathrm{skew}(A) = \frac{1}{2}(A - A^\top)$ of $A \in \mathbb{R}^{d\times d}$
$W'(F)$	the first derivative with respect to F of $W : \mathbb{R}^{d\times d} \to \mathbb{R}$, i.e., $W'(F) = \left(\frac{\partial W(F)}{\partial F_{ij}}\right)_{i,j=1}^d \in \mathbb{R}^{d\times d}$
$W''(F)$	the second derivative with respect to F of $W : \mathbb{R}^{d\times d} \to \mathbb{R}$, i.e., $W''(F) = \left(\frac{\partial^2 W(F)}{\partial F_{ij}\partial F_{kl}}\right)_{i,j,k,l=1}^d \in \mathbb{R}^{d\times d\times d\times d}$
X^*	dual space of any space X
${}_{X^*}\langle \cdot, \cdot \rangle_X$	duality pairing
$\mathrm{Tr}\, f$	trace of a function f in a Sobolev space
$W^{1,p}(\Omega; \mathbb{R}^{\tilde{d}})$	Sobolev space of order 1, namely $\{f \in L^p(\Omega; \mathbb{R}^{\tilde{d}}) : \partial_{x_i} f \in L^p(\Omega; \mathbb{R}^{\tilde{d}})$ for $1 \le i \le d\}$
$W^{1,p}(\mathbb{R}^d; \mathbb{R}^{\tilde{d}})$	Sobolev space of order 1 for functions $f : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$
$\mathbf{H}^1(\Omega; \mathbb{R}^{\tilde{d}})$	$W^{1,2}(\Omega; \mathbb{R}^{\tilde{d}})$
$\mathbf{H}^1(\mathbb{R}^d; \mathbb{R}^{\tilde{d}})$	$W^{1,2}(\mathbb{R}^d; \mathbb{R}^{\tilde{d}})$
$\mathbf{H}^1_0(\Omega; \mathbb{R}^{\tilde{d}})$	$\{f \in \mathbf{H}^1(\Omega; \mathbb{R}^{\tilde{d}}) \,:\, \mathrm{Tr}\, f = 0 \text{ on } \partial\Omega\}$
$\mathbf{H}^{-1}(\Omega; \mathbb{R}^{\tilde{d}})$	dual of $\mathbf{H}^1_0(\Omega; \mathbb{R}^{\tilde{d}})$
$\mathbf{H}^1_{\mathbf{n}}(\Omega; \mathbb{R}^{\tilde{d}})$	$\{f \in \mathbf{H}^1(\Omega; \mathbb{R}^{\tilde{d}}) \,:\, \mathrm{Tr}\, f \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$
$\mathbf{H}^2(\Omega; \mathbb{R}^{\tilde{d}})$	Sobolev space of order 2, namely $\{v \in L^2(\Omega; \mathbb{R}^{\tilde{d}}) : \partial_{x_i} v, \partial_{x_j}\partial_{x_i} v \in L^2(\Omega; \mathbb{R}^{\tilde{d}})$ for $1 \le i, j \le \tilde{d}\}$
$\mathbf{H}^2(\mathbb{R}^d; \mathbb{R}^{\tilde{d}})$	Sobolev space of order 2 for functions $f : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$
$\mathbf{H}^2_{\mathbf{n}}(\Omega; \mathbb{R}^{\tilde{d}})$	$\left\{f \in \mathbf{H}^2(\Omega; \mathbb{R}^{\tilde{d}}) \,:\, \frac{\partial f}{\partial \mathbf{n}} := \mathrm{Tr}(\nabla f) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}$
$\mathcal{V}$	$C_0^\infty(\Omega; \mathbb{R}^d) \cap \{v : \nabla \cdot v = 0\}$
$\mathbf{H}$	closure of $\mathcal{V}$ in $L^2(\Omega; \mathbb{R}^d)$
$\mathbf{V}$	closure of $\mathcal{V}$ in $\mathbf{H}^1(\Omega; \mathbb{R}^d)$

1 Introduction

1.1 Aim of this work

Magnetic materials are of great importance in technological applications. In our work at hand, we consider magnetoelastic materials, which are strongly susceptible to the phenomenon of converting applied stresses into changes of the magnetic field and vice versa. These materials are so-called *smart materials*. In general, *smart materials* are constructed materials which have the special property that they react to applied external stimuli in remarkable ways. Generally, such stimuli are, for instance, stresses, temperature, external electric or magnetic fields. In the case of magnetoelastic materials, as mentioned earlier, elastic effects and magnetic properties are taken into account and they are coupled.
Due to this coupling effect, magnetoelastic materials have been of interest for a variety of applications. For instance, they can be found in sensors to measure torque or force (see, e.g., [BS02, BS04, GRRC11]). Here, magnetoelastic materials convert stresses into changes of the magnetic field, whereat the magnitude of the changes depends on the strength of the stresses. Finally, the changes of the magnetic field can be measured and related to the applied force or torque.
Further, magnetoelastic materials are also used in magnetic actuators (see, e.g., [SNR10]) and generators for ultrasonic sound (see, e.g., [BV92]). The magnetoelastic effect is used in these applications the other way round compared to the sensors: changes of applied magnetic fields induce changes of the magnetization of the material which then, due to the coupling, convert to changes in the deformation of the material body. The resulting motion of the material is then used for the specific applications.
Our main motivation is to understand magnetic fluids with immersed particles of a certain *intermediate* size. Common models consider systems with particles of homogeneous magnetization, which is an acceptable presumption for very small particles (5–10 nm; *ferrofluids*; see, e.g., [BAB$^+$99], [AR09, Section 7]), or relatively large particles (1–10 μm; *magnetorheological fluids*; see, e.g., [Wer14]). Under applied magnetic fields, ferrofluids stay in the fluid phase, while the viscosity of magnetorheolocial fluids increases in such a way that they become viscoelastic solids. On the other hand, it is interesting to mathematically describe the behavior of fluids with intermediate-sized particles, which show micromagnetic domains. Such *micromagnetic fluids* might bear a considerable technological potential.
The aim of this work is to start the approach of such magnetic fluids by

- *deriving a general mathematical model* for a *micromagnetic and elastic* particle and by

- *proving existence of weak solutions* to systems of partial differential equations which are deduced from the general model under special assumptions.

Discussing the existence of weak solutions to the resulting highly coupled systems is not only challenging and interesting on its own but also crucial, e.g., for numerical analysis to be meaningful.
The derivation of our general model and the model itself has the following features, of which the combination cannot be found within the existing mathematical literature on magnetoelastic materials (see Section 1.2):

- modeling is focused on the *interplay between Lagrangian and Eulerian coordinate systems* to combine elasticity and magnetism, which are described on different coordinate systems;
- model is *phrased* entirely *in the Eulerian coordinate system*, which makes it easier to extend the model further in future work;
- framework of *micromagnetism* is used to allow for a magnetic domain structure in the materials;
- *variational approach* is employed, where *dissipation* mechanisms and *time evolution* are included;
- *coupling of phenomena* on the *macroscopic scale* (deformation) and the *microscopic scale* (magnetization) happen through coupling on the *energetic level* as well as *transport* relations.

We elaborate on these special features of our modeling approach and the model in the following and highlight their importance.
The transformation of the variables and the energies between Lagrangian and Eulerian coordinate systems is a crucial ingredient in our modeling. Elastic effects are commonly defined in the Lagrangian coordinate system, i.e., on the reference configuration; however, the magnetization and other magnetic variables are usually defined in the Eulerian coordinate system, i.e., on the deformed or current configuration. In order to combine elastic and magnetic effects described on different coordinate systems, we need to make sure that the descriptions fit together on the same coordinate system. We choose to phrase the model entirely in the Eulerian coordinate system (see also [LW01]). This is useful, since it is then unnecessary to assure invertibility of deformations and it allows to incorporate fluid-structure interaction for magnetic fluids later. This is what makes the interplay between Lagrangian and Eulerian coordinate systems so important. For the importance of invertibility questions in (magneto-)elasticity theory, we refer to [RL05, KSZ15].
We consider the micromagnetic framework (see Section 2.3 below) to describe the magnetic properties of the materials. This allows for a heterogeneous magnetization across the material and makes the model applicable to richer settings. Our derivation of the model is based on a variational approach which includes dissipation mechanisms. This approach goes back to the works of John William Strutt (Lord Rayleigh) [Str73] and Lars Onsager [Ons31a, Ons31b] and has been

applied in the derivation of models for complex fluids in, e.g., [LLZ05, SL09, WXL13]. For an introduction to this approach we refer also to [For13] and Section 2.2 below.
Further advantages of this approach are that energy terms can be established relatively easily and that forces within the system are not counted twice, among others. Moreover, using this approach, we can naturally combine effects on different scales (so-called *multi-scale modeling*) – in our case, elasticity on a macroscopic scale and magnetism on a microscopic scale – in a time evolution model within the framework of complex fluids. It is important to note that time evolution is at the core of our approach. This is vital to understand the dynamical behavior of materials.
As mentioned before, a meaningful feature is the cross-scale coupling of the magnetization on the microscopic scale and the deformation on the macroscopic scale. Here, we consider coupling on the energetic level in the anisotropy energy, which connects the easy axes of the magnetization to the actual elastic deformation. Moreover, the transport plays an important role as it couples the macroscopic motion to the microscopic variables. For the details, we refer to Sections 2.3 and 2.4.

1.2 Historical overview and embedding

Next, we give an overview of the history and the development of the theory of magnetoelasticity as well as the theory of micromagnetism. Moreover, we highlight the features of our model compared to what has been done before.
The discovery of magnetoelasticity dates back at least to the 19th century (see, e.g., [Bro66]). It was observed that if a ferromagnetic rod is subject to a magnetizing field, the rod changes not only its magnetization but also its length. Further, the opposite way can also be observed: if the rod experiences tension, its length as well as its magnetization changes. From such experiments it was concluded that there exists an interaction between elastic and magnetic effects. The general term for this class of phenomena is *magnetoelastic interaction* or simply *magnetoelasticity*. More precisely, *magnetostriction* for the shape change during magnetization and *magnetoelastic effect* for the change of magnetization resulting from a mechanical stress.
The importance of magnetoelasticity has been acknowledged starting with the modern theory of ferromagnetism. However, until the 1960's, most of the magnetoelastic derivations were based on works from the early 1930's. Unfortunately, the theory from that time suffered from several flaws (see [Bro66]). Then, in Brown's monograph [Bro66], the first rigorous phenomenological theory of magnetoelasticity was built, using both Lagrangian and Eulerian coordinate systems in the description. Practically concurrently with Brown, who gives an overview of force and energy based methods from the physical point of view in [Bro66], Tiersten presented an essentially equivalent theory for magnetoelastic solids in two papers [Tie64, Tie65]. Both these works consider magnetically saturated media

undergoing large deformations, phrased in the Eulerian coordinate system. The first one is from the viewpoint of differential equations, where a ferromagnetic body is modeled as a superposition of two continua, the *lattice continuum* and the *electronic spin continuum*, which interact by forces and stresses. The second work by Tiersten is in the form of a variational principle, which yields the same equations, but does not employ a rational mathematical derivation.
Compared to our ansatz, it lacks dissipation mechanisms and does not include the theory of *micromagnetics*, which was developed by Brown in [Bro63]. Despite the fact that the systematic development of the framework is due to Brown, some of the main ideas had already been published by Landau and Lifschitz in 1935 [LL35]. Brown's theory of micromagnetics, however, did not experience general acceptance until around the year 1990 [JK90]. Pertinent works are consequently relatively scarce before that time, see, e.g., [MW79, Slo79, Vis85]. However, the situation changed quickly and the field developed rapidly (see [HS98, KP06]), let it be physical modeling, mathematical analysis, model validation, reduction or numerics.
In [DP95], the authors consider evolution equations for liquid crystals and for magnetostrictive solids and show how to study such apparently different and diverse materials within a unified dynamical theory of structured continua. In this work, the approach is based on modeling with forces, i.e., without an advantageous variational approach. Further, the model does not include micromagnetism, but the authors employ a similar micromagnetic balance equation as the *Landau-Lifschitz-Gilbert (LLG) equation*.
In [DP96], the same authors revisit the models of Tiersten and Brown from the 1960's and use again a force-based approach to the modeling over an energetic variational approach. A mathematical analysis including existence of weak solutions of the obtained model (the so-called *soft ferromagnets at rest*) is then performed in [BPV01]. This model is a special form of the LLG equation, which is decoupled from elasticity in that work.
Further prominent works in the field of magnetoelasticity are [DD98] and [DJ02]. In the former article, a model on nonlinear magnetoelasticity is analyzed as a variational problem using convex integration. The latter article uses micromagnetics to derive a variational approach for the macroscopic behavior and equilibrium configurations of materials with high anisotropy. Both articles, however, focus on static problems in magnetoelasticity. In addition, the work [JK93] is on a theory on materials with large magnetostriction, it takes anisotropy from lattice considerations into account and it predicts observed domain structures precisely. Compared to viewing magnetoelastic materials as a continuum, the article [LJL06] presents a static problem of magnetic particles within an elastic matrix. The particles are described by the theory discussed in [DJ02]. One could look at such magnetostrictive composites basically as a magnetic fluid with dehydrated fluid material.
In 1935, Landau and Lifschitz [LL35] derived an equation describing the dynamics of the magnetization. In 1955, it was further improved by Gilbert [Gil55, Gil04] (notice that the first article is only an abstract, the details were published almost 50 years later in the second article) into what is nowadays known as the *Landau-*

Lifschitz-Gilbert (LLG) equation. Detailed reviews of the theory of micromagnetics can be found, for instance, in [HS98, KP06]. Prominent analytical works in the field of micromagnetics are, for example, [Vis85, JK90, AS92, CP01, CF01]. Moreover, micromagnetics and the LLG equation for thin films are studied in, e.g., [GJ97, DKMO00, DKMO02, DKMO06, Kur06a, Kur06b, Mel07, Mel10, CIM14], where Γ-convergence is an important mathematical tool. All the preceding articles treat the magnetic phenomena only, i.e., without magnetoelasticity and thus lack the coupling to elastic effects.
The evolution of the coupling of micromagnetics with elasticity for magnetoelastic materials has already been tackled and analyzed from the viewpoint of existence of solutions, see, e.g., [CISVC09, CEF11]. The former uses an approximation of the LLG equation, which does not give rise to a gradient flow, the latter features coupling in the LLG equation through the effective magnetic field and a stress tensor, which is not derived by means of variational principles. For a recent numerically-oriented work see, e.g., [BPPR14]. We note that, compared to our approach, the models in all these works lack certain coupling of the physical quantities through the transport and material derivatives. Moreover, for recent works from the engineering and experimental point of view see, e.g., [MKR11, ESM15, MVT15, MVT16].

1.3 Overview of this work

The main part of this thesis is organized in two chapters with an additional appendix.
Chapter 2 is dedicated to the modeling of magnetoelastic materials. In the first part of this chapter we fix the notation and recall several notions and concepts from continuum mechanics in Section 2.1. In Section 2.2, we outline the energetic variational approach. Then, we use Section 2.3 to give a brief overview of the theory of micromagnetics. After highlighting the evolution of the variables describing the material via transport in Section 2.4, we continue with discussing the energy dissipation law for our model in Section 2.5.
In Section 2.6, we state the main result of this chapter, viz our mathematical model for magnetoelastic materials. This is a system of partial differential equations which govern the evolution of the entire material. The equations are highly coupled due to interactions between the macroscopic scale and the microscopic scale. The partial differential equations include the equation of motion with stress and pressure terms as well as a dynamical equation for the magnetization. Moreover, the law of conservation of mass and the evolution of the deformation gradient are part of the system of equations. A derivation of this is then provided in Section 2.7.
We start the mathematical analysis of the obtained system of partial differential equations by considering a model for a simplified setting. In this setting, we consider special assumptions on the energy and dissipation terms as well as on the kinematics of the magnetization and the deformation. In Section 2.8 we

highlight all the assumptions in this setting and derive the corresponding model for this case.
In Chapter 3, we then present existence results of weak solutions to the considered model for the simplified setting. We state the existence results right in the beginning of Chapter 3: the first one is Theorem 9. This states the existence of weak solutions to the model for the simplified setting considering gradient flow dynamics on the magnetization. The second one is Theorem 11 which is the existence of weak solutions to the model for the simplified setting considering LLG dynamics for small initial data. For the LLG equation, we also present Lemma 10 in the beginning of Chapter 3, which shows the property of the LLG equation to conserve the length of the magnetization and three equivalent forms of the LLG equation.
Section 3.1 then deals with the proof of Theorem 9. The proof is based on a Galerkin method discretizing the velocity in the equation of motion and a fixed point argument and borrows ideas from [LL95].
In Section 3.2, we present the proof of Theorem 11. The proof is based on the proof presented in Section 3.1 but features special ideas from [CF01] which are necessary due to the more complicated form of the LLG equation. There, the three equivalent forms of the LLG equation from Lemma 10 play a crucial role to obtain uniform a priori energy estimates. Section 3.2 presents the main steps of the proof and highlights the differences compared to the proof presented in Section 3.1.
Moreover, the appendix includes further results on special transport as well as supplementary proofs. We also discuss a version of the model for the simplified setting in two dimensions there, which can be used as a starting point for the analysis of special solutions and numerical simulations to gain more insight into the coupling within the model and to compare the model with experiments in future work.
Finally, in Section 4, we conclude this thesis with an overview of open problems. These include analysis of further generalizations of the models discussed in this work as well as an extension towards fluid-structure interactions. Moreover, numerical analysis and experiments are of big importance to continue the path towards better understanding of the behavior and properties of magnetic materials.

2 Modeling of magnetoelastic materials

In this part of the thesis, we present a model for magnetoelastic materials and derive it from an energy ansatz within a continuum mechanical setting. In order to describe the setting properly, we use the following preliminary sections to explain and fix the notation used in continuum mechanics. Furthermore, we give an overview of the energetic variational approach used and recall facts from the theory of micromagnetics.

2.1 Continuum mechanical setting

In the forthcoming analysis, we work in a continuum mechanical setting (see also [For13, Chapter 2]).

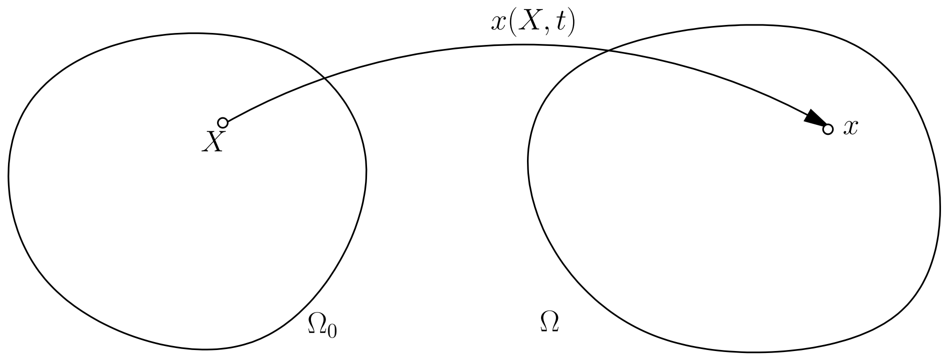

Figure 2.1: Deformation mapping between reference configuration Ω_0 and deformed configuration Ω.

Let $t \in \mathbb{R}_0^+$ be the time variable and t^* a given end time. Let $\Omega_0, \Omega \subset \mathbb{R}^d$, $d = 2, 3$, be the reference (undeformed) and the deformed configuration of the material, respectively. If not otherwise stated, we assume that Ω is a bounded domain with a smooth boundary which has positive and finite Hausdorff measure, $0 < \mathcal{H}^{d-1}(\partial\Omega) < \infty$.
Elasticity is commonly phrased in the Lagrangian coordinate system $X \in \Omega_0$, whereas magnetic quantities are usually defined in the Eulerian coordinate system $x \in \Omega$. In our ansatz, we phrase the mathematical model for magnetoelastic materials entirely in the Eulerian coordinate system. To rewrite elasticity in Eulerian coordinates, we make use of the deformation or flow map (see also Section 2.7 and, e.g., [For13, Section 3.3])

$$x(X, t) : \Omega_0 \times [0, t^*] \to \Omega \tag{2.1}$$

defines the position of particle $X \in \Omega_0$ at time t in the current configuration. We assume that $X \mapsto x(X,t)$ is a bijective mapping at every time $t \in [0,t^*]$. With the flow map, we define the velocity in the Eulerian coordinate system $v : \Omega \times [0,t^*] \to \mathbb{R}^d$ by

$$v(x(X,t),t) = \frac{\partial}{\partial t} x(X,t). \tag{2.2}$$

Moreover, we assume that the deformation gradient $\widetilde{F} : \Omega_0 \times [0,t^*] \to \mathbb{R}^{d\times d}$

$$\widetilde{F}(X,t) := \nabla_X x(X,t) \tag{2.3}$$

has positive determinant $J := \det \widetilde{F} > 0$, i.e., is orientation preserving. We refer to the coordinates $X \in \Omega_0$ as Lagrangian coordinates and to $x \in \Omega$ as Eulerian coordinates. Note that we also use the notion deformation gradient for the push forward $F : \Omega \times [0,t^*] \to \mathbb{R}^{d\times d}$ which is defined by

$$F(x(X,t),t) = \widetilde{F}(X,t). \tag{2.4}$$

$\widetilde{F}(X,t)$ is a quantity in the Lagrangian coordinate system, whereas $F(x,t)$ is a quantity in the Eulerian coordinate system.
The general model that we derive in Section 2.7 includes compressible materials. In Section 2.8, we restrict our analysis to incompressible materials. That is, we assume

$$\det \widetilde{F} \equiv 1 \tag{2.5}$$

in the Lagrangian coordinate system. This then implies

$$\nabla \cdot v = 0 \tag{2.6}$$

in the Eulerian coordinate system. For a proof, see, e.g., [For13, Section 2.3].
For the description of the magnetic properties of the material, we introduce the magnetization

$$M : \Omega \times [0,t^*] \to \mathbb{R}^3.$$

It is extended by zero to the whole space-time $\mathbb{R}^d \times [0,t^*]$.
The magnetization then induces a magnetic field $H : \mathbb{R}^3 \times [0,t^*] \to \mathbb{R}^3$, the so-called stray field, through which the different regions of the material interact with each other over long-ranges. It is given as a solution to Poisson's equation arising from Maxwell's equations for magnetostatics. Details are given in Section 2.3. Notice that the magnetization is only defined on the magnetic body but takes always values in $\mathbb{R}^3$, no matter whether $d = 2$ or $d = 3$. The induced magnetic field is defined in the entire $\mathbb{R}^3$ with values in $\mathbb{R}^3$, where it does not matter which value d takes.
In our modeling, we assume to have two scales. On the one hand, the velocity v and the deformation gradient F define the large scale or macroscopic scale. On the other hand, the magnetic properties and the magnetization M define the "fine" scale or microscopic scale. The communication between these two scales happens through different coupling on the energetic level (see Section 2.3.1) and transport relations (see Section 2.4). Further, we assume separation of scales in the sense that everything that happens between microscopic scale and macroscopic scale is determined by these two scales, and that we may neglect dynamics on one scale when treating the other separately.

2.2 Overview of the energetic variational approach

As already mentioned in the introduction, we apply an energetic variational approach to obtain the balance of momentum equation for the model describing magnetoelastic materials. This allows us to derive an evolutionary system of partial differential equations from a rather easy energy ansatz.
This particular energetic variational approach is used by Chun Liu and coauthors, an overview can be found in, e.g., [Liu11], [HKL10]. It goes back to the works of John William Strutt (Lord Rayleigh) [Str73] and Lars Onsager [Ons31a, Ons31b]. The basic concepts of this approach are briefly outlined below: energy dissipation law, least action principle, maximum dissipation principle, and Newton's force balance law. For a more detailed review we refer to [For13].
The starting point of the energy treatment is the energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{total} = -\Delta_E, \tag{2.7}$$

where E^{total} represents the total energy and Δ_E denotes the dissipation. The first is defined as the sum of kinetic and free internal energy; the latter is modeled, for instance, as a quadratic functional of certain rates, such as velocity. If it holds that $\Delta_E \neq 0$, the system is dissipative. If $\Delta_E = 0$, the system is called conservative or Hamiltonian.
Hamiltonian systems are treated with the *least action principle*. First, we recall the definition of the action functional, see, e.g., [LL76, Chapter I, Section 2].
Let $L = \mathcal{K} - \mathcal{F}$ be the Lagrangian function of a conservative system, where $\mathcal{K}$ is the kinetic energy and $\mathcal{F}$ is the free energy, depending on the state variable $q(t)$ and its derivates. Then, the action functional for this system is defined by

$$\mathcal{A}(q) := \int_0^{t^*} L(t, q(t), q_t(t))\ \mathrm{d}t. \tag{2.8}$$

The least action principle then states that the equation of motion for the Hamiltonian system follows by taking the variation of the action functional with respect to q.
In our modeling approach, the kinetic energy and the free energy can be written in the form of integrals over the domain Ω_0. Moreover, the main state variable is the flow map $x(X,t)$. We express all the quantities, such as the deformation gradient and the magnetization, by means of $x(X,t)$. To calculate the equation of motion, we calculate the variation of the action functional with respect to x. To this end, we consider a variation $x(X,t) + \varepsilon\widetilde{\chi}(X,t)$ of the minimizing trajectory $x(X,t)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$, and $\widetilde{\chi}(X,t)$ an arbitrary test function that is smooth and compactly supported on $\Omega_0 \times [0,t^*]$. The calculation of $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\mathcal{A}(x+\varepsilon\widetilde{\chi}) = 0$ then leads to the Euler-Lagrange equation or equation of motion for the system. When we work under incompressibility constraints in Section 2.8, we need to consider different variations, namely volume preserving diffeomorphisms, see (2.110).
We also denote the equation of motion for the Hamiltonian system by the notion

$\text{force}_{conservative}$. In this case, we formally write

$$\delta E^{total} = \text{force}_{conservative} \cdot \delta x,$$

where δ denotes a virtual change of the respective quantities.
Dissipative systems also satisfy the *maximum dissipation principle.* This leads to the dissipative force of the described system. We do this by a variation of the (scaled) dissipation functional $\frac{1}{2}\Delta_E$ with respect to the rate. Here, we formally write

$$\delta\left(\frac{1}{2}\Delta_E\right) = \text{force}_{dissipative} \cdot \delta x_t.$$

The final step is to combine these forces with *Netwon's force balance law.* The law states that conservative and dissipative forces are equal ("actio" is equal to "reactio")

$$\text{force}_{conservative} = \text{force}_{dissipative}.$$

This final force balance equation is the *equation of motion* for the entire system, also regarded as *balance of momentum.*
In our setting of magnetoelastic materials, the free energies are given as integrals over the domain in Eulerian coordinates of some spatial energy densities. Hence, the action functional involves not only an integral over time, but also an integral over the domain. To calculate the variation of the action, we pull back the integral, i.e., write everything into the Lagrangian coordinate system. At this point, transport equations come in: they tell us how the quantities evolve and provide us with information on how to write these quantities in terms of the Lagrangian coordinate system. Transport equations are discussed in Section 2.4.
Since the starting point of the energetic variational approach is a total energy together with a dissipation, we set up all considered energies and dissipation terms in the following. In Section 2.3, we describe the micromagnetic framework for the magnetic variables and the corresponding energy terms. In Section 2.5, we combine this together with the elasticity part and the dissipative part to obtain the total energy dissipation law.

2.3 Overview of the theory of micromagnetics

In this section, we give a brief introduction to micromagnetics. We refer to [HS98, KP06] for a more detailed review of micromagnetics.
In our modeling ansatz, we assume a quasi-static setting and work with Maxwell's equations for magnetostatics. Moreover, we neglect electric effects and currents by assuming isolating materials. Maxwell's equations for magnetostatics (see, e.g., [Bob00, Kov00]) for the magnetic induction $B : \mathbb{R}^3 \to \mathbb{R}^3$ and the magnetic field $H : \mathbb{R}^3 \to \mathbb{R}^3$ read

$$\nabla \cdot B = 0, \tag{2.9}$$

$$\nabla \times H = 0, \tag{2.10}$$

with the constitutive relation

$$B = \mu_0(M + H), \tag{2.11}$$

where the constant $\mu_0 > 0$ is the magnetic permeability. Due to Maxwell's equation (2.10), we can write

$$H = -\nabla\varphi := -\nabla_x \varphi(M)(x), \tag{2.12}$$

where the scalar potential

$$\varphi(M)(x) = (\nabla N * M)(x) = \int_{\mathbb{R}^3} (\nabla N)(x - y) \cdot M(y) \, \mathrm{d}y \tag{2.13}$$

is the solution to Poisson's equation

$$\Delta\varphi = \nabla \cdot M \qquad \text{in } \mathbb{R}^3 \tag{2.14}$$

arising from (2.9), (2.11) and (2.12) (see, e.g., [HS98, Section 3.2.5]), understood in the sense of distributions (see, e.g., [Gar07, Section 1], and the weak form (2.71) below). The solution is subject to the transition condition $[\![\nabla\varphi \cdot \mathbf{n}]\!] = -M \cdot \mathbf{n}$ on $\partial\Omega$, where $[\![a]\!] = a^+ - a^-$ denotes the difference of outer trace a^+ and inner trace a^- of the quantity a. Further, $N(r) := -\frac{1}{4\pi|r|}$ and $(\nabla N)(r) = \frac{1}{4\pi}\frac{r}{|r|^3}$, $r \neq 0$.
Note that in the case where $\Omega \subset \mathbb{R}^2$, we use a stray field energy term which was derived in [GJ97] for thin films. The advantage of this is that we do not need the stray field explicitly. For details, see Section 2.3.1 below.

2.3.1 Micromagnetic energy

The first ingredient in the theory of micromagnetics is the *micromagnetic energy* $w_{\mu\text{mag}}$, defined on a suitable function space. It reads (see, e.g., [HS98, Section 3.2])

$$w_{\mu\text{mag}}(M) = A \int_\Omega |\nabla M|^2 \, \mathrm{d}x + \int_\Omega \widetilde{\psi}(M) \, \mathrm{d}x + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |H(M)|^2 \, \mathrm{d}x - \mu_0 \int_\Omega M \cdot H_{\text{ext}} \, \mathrm{d}x. \tag{2.15}$$

The first term is the exchange energy term with the exchange constant $A > 0$. This energy reflects the tendency of the magnetic field to align in one direction. The second term is the anisotropy energy. It accounts for the dependence of the energy on the direction of the magnetization relative to the easy axes of the material. The nonnegative anisotropy energy density $\widetilde{\psi} : \mathbb{R}^3 \to \mathbb{R}_0^+,\ M \mapsto \widetilde{\psi}(M)$ is usually defined as a polynomial function reflecting the crystal symmetry of the material [DKMO06]. A simple ansatz for $\widetilde{\psi} := \widetilde{\psi}_{\text{uni}}$ models uniaxial anisotropy, which means that the magnetization prefers one certain direction within the material. In literature on micromagnetics, e.g., [DKMO06], this ansatz reads $\widetilde{\psi}_{\text{uni}}(M) = 1 - (M \cdot e)^2$ for a certain unit vector e. This particular anisotropy energy density penalizes the deviation of the magnetization from the easy axis, i.e., it becomes small when the alignment of M is parallel to e.
The third term is the stray field energy. The magnetic (stray) field $H = H(M)$ is induced by the magnetized body and is a solution to Maxwell's equations of magnetostatics (2.9)–(2.10) and (2.11). The stray field energy can be rewritten in the following way [HS98, Section 3.2.5]:

$$\frac{\mu_0}{2} \int_{\mathbb{R}^3} |H(M)|^2 \, \mathrm{d}x = -\frac{\mu_0}{2} \int_\Omega M \cdot H(M) \, \mathrm{d}x. \tag{2.16}$$

Finally, the last term in (2.15) represents the Zeeman energy due to the externally applied magnetic field $H_{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$.
We extend the usual definition of the micromagnetic energy in our model to the magnetoelastic setting in the following way. Firstly, by the transformation from the Eulerian coordinate system to the Lagrangian coordinate system, we introduce a coupling of the magnetic quantities and the deformation within the micromagnetic energy. Moveover, we couple deformation and magnetization in the anisotropy energy. The usual anisotropy energy in (2.15) does not depend on the deformation gradient F (see, e.g., [HS98, Section 3.2.7]). However, when we consider elastic materials, the crystalline structure of the material changes according to the deformation. So, in order to take this in account, we introduce the anisotropy energy density

$$\psi : \mathbb{R}^{d\times d} \times \mathbb{R}^3 \to \mathbb{R}_0^+, \quad (F, M) \mapsto \psi(F, M),$$

which now depends on the deformation gradient F in the sense that the easy axes e_i of the materials at rest are changed by F. For instance, one could set Fe_i to be the easy axes in the deformed configuration (Cauchy-Born relation, see, e.g., [TM11, Section 11.2.2]).
In the following, however, we stick to the general ψ without focusing on specific materials with specific crystalline structure and dependence on F. Let us assume that ψ is a smooth function on $\mathbb{R}^{d\times d} \times \mathbb{R}^3$.
Moreover, in the case where $\Omega \subset \mathbb{R}^2$, we use a stray field energy for thin films which was derived in [GJ97], defined by

$$\frac{\mu_0}{2} \int_\Omega M_3^2 \, \mathrm{d}x = \frac{\mu_0}{2} \int_\Omega (M \cdot e_3)^2 \, \mathrm{d}x,$$

where M_3 denotes the third component of $M \in \mathbb{R}^3$, and e_3 denotes the third standard basis vector in $\mathbb{R}^3$.
In summary, in our approach, the micromagnetic energy $W_{\mu\text{mag}}$, defined on suitable function spaces, does depend on the deformation gradient as well and reads

$$\begin{aligned} W_{\mu\text{mag}}(F, M) = A \int_\Omega |\nabla M|^2 \, \mathrm{d}x + \int_\Omega \psi(F, M) \, \mathrm{d}x \\ + \int_\Omega E^{(d)}_{\text{stray}}(M) \, \mathrm{d}x - \mu_0 \int_\Omega M \cdot H_{\text{ext}} \, \mathrm{d}x, \end{aligned} \tag{2.17}$$

where

$$E^{(d)}_{\text{stray}}(M) := \begin{cases} \frac{\mu_0}{2}(M \cdot e_3)^2 & \text{for } d = 2, \\ -\frac{\mu_0}{2} M \cdot H(M) & \text{for } d = 3 \end{cases} \tag{2.18}$$

is the stray field energy density.
A characteristic property of micromagnetic materials is the formation of so-called *domains* of magnetization. Typically, under no applied field, the equilibrium configuration of magnetoelastic materials contains these domains on which magnetization is approximately constant (see [HS98]). These *domain patterns* result from the competition of crystal structure (reflected in the anisotropy energy) with long-range magnetic interactions (reflected by the stray field energy). The first is responsible for existence of preferred crystallographic directions (the so-called *easy axes of magnetization*), the latter, however, disadvantages configurations with uniform magnetization throughout the whole body.

2.3.2 Landau-Lifshitz-Gilbert equation

The second ingredient in the theory of micromagnetics is the *Landau-Lifshitz-Gilbert (LLG) equation*, see, e.g., [HS98, Section 3.2.7]. It models the dissipative dynamical behavior of the magnetization M, and reads

$$M_t = -M \times H_{\text{eff}} + \alpha_{damp} M \times M_t, \tag{2.19}$$

where $H_{\text{eff}} := -\frac{\delta W_{\mu\text{mag}}}{\delta M}$ represents the effective magnetic field (see, e.g., [KP06, Section 3.2], [GW07]). The notation $\frac{\delta(\cdot)}{\delta M}$ denotes the variational derivative with respect to the magnetization M. The effective magnetic field H_{eff} is calculated in Section 2.7.2 below. Further, $\alpha_{damp} \geq 0$ is a damping constant. The LLG equation is usually solved together with the boundary condition, see [GCGE03],

$$\frac{\partial M}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega. \tag{2.20}$$

Moreover, in micromagnetics, the length of the magnetization is assumed to be fixed (Heisenberg constraint), i.e., $|M| = M_s$, where $M_s > 0$ is the saturation magnetization (see, e.g., [DKMO06]). For simplicity, we assume $M_s = 1$, so, we have the length constraint

$$|M| = 1 \quad \text{a.e. in } \Omega. \tag{2.21}$$

This constraint enters the variational principle with the help of a Lagrange multiplier in Section 2.7.1.
The LLG equation is a given equation which we do not obtain from a microscopic energy ansatz. For a complete energetic picture of the entire system, however, we derive an energy dissipation law governing the microscopic scale, i.e., the energy dissipation law related to the LLG equation. We assume separation of scales in the following sense: when considering the microscopic scale (magnetization M) in the following calculation and taking the time derivative of the micromagnetic energy, the macroscopic time scale is fixed, so the dependence of F may be neglected. To obtain a governing energy dissipation law, we start by taking the cross product of the LLG equation (2.19) with M. We obtain using the Graßmann identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ for $a, b, c \in \mathbb{R}^3$ and $M \cdot M = 1$:

$$\begin{aligned} M \times M_t &= -M \times (M \times H_{\text{eff}}) + \alpha_{damp} M \times (M \times M_t) \\ &= -(M \cdot H_{\text{eff}})M + H_{\text{eff}} + \alpha_{damp} M \times (M \times M_t), \end{aligned}$$

which leads to

$$H_{\text{eff}} = M \times M_t + (M \cdot H_{\text{eff}})M - \alpha_{damp} M \times (M \times M_t). \tag{2.22}$$

Now, we multiply the LLG equation (2.19) scalarly by $-H_{\text{eff}} = \frac{\delta W_{\mu\text{mag}}}{\delta M}$ and integrate over Ω. Using (2.22), this yields

$$\begin{aligned} &\int_\Omega \frac{\delta W_{\mu\text{mag}}}{\delta M} \cdot M_t \, \mathrm{d}x = -\int_\Omega \alpha_{damp}(M \times M_t) \cdot H_{\text{eff}} \, \mathrm{d}x \\ &= -\int_\Omega \alpha_{damp}(M \times M_t) \cdot (M \times M_t + (M \cdot H_{\text{eff}})M - \alpha_{damp} M \times (M \times M_t)) \, \mathrm{d}x \\ &= -\int_\Omega \alpha_{damp} |M \times M_t|^2 \, \mathrm{d}x. \end{aligned}$$

In view of (2.17) and the definition of a variational derivative, we have that $\frac{\mathrm{d}}{\mathrm{d}t} W_{\mu\text{mag}} = \int_\Omega \frac{\delta W_{\mu\text{mag}}}{\delta M} \cdot M_t \, \mathrm{d}x$. So, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} W_{\mu\text{mag}} = -\int_\Omega \alpha_{damp} |M \times M_t|^2 \, \mathrm{d}x. \tag{2.23}$$

We regard this as the energy dissipation law for the microscopic variable M.

2.4 Kinematics and transport

In this section, we present the *transport equations* for all quantities describing the system. Transport is the evolution of a quantity from the reference configuration to the deformed configuration at time t.
The positive and bounded function $\rho : \Omega \times (0, t^*) \to \mathbb{R}^+$ denotes the mass density of the material. The transport of the mass density is the law of conservation of mass

$$\rho_t + \nabla \cdot (v\rho) = 0 \tag{2.24}$$

in its general strong form. However, in the case of incompressibility $\nabla \cdot v = 0$, we obtain

$$\rho_t + (v \cdot \nabla)\, \rho = 0. \tag{2.25}$$

The conservation of mass can either be expressed as a partial differential equation for the density in the Eulerian coordinate system as in (2.24)–(2.25) or as a pull back formula in the Lagrangian coordinate system. In the compressible case, one can deduce from (2.24) that

$$\rho(x(X,t),t) = \frac{1}{\det \widetilde{F}(X,t)} \rho_0(X), \tag{2.26}$$

where $\rho_0 : \Omega_0 \to \mathbb{R}^+$ denotes the density of the material in the reference configuration. We refer to Appendix A.1 for a proof.
Similarly, in the incompressible case, we see directly from (2.25)

$$\rho(x(X,t),t) = \rho_0(X). \tag{2.27}$$

Next, we discuss the transport of the deformation gradient. This transport is described by an equation which follows from the push forward $\widetilde{F}(X,t) = F(x(X,t),t)$, see (2.4). It reads (a derivation can be found in Appendix A.1)

$$F_t + (v \cdot \nabla)F = \nabla v F. \tag{2.28}$$

Finally, we discuss the transport of the magnetization. We assume a weak transport coupling of the magnetic and the elastic variables.
Another transport coupling is discussed in Appendix A.3. The rotational coupling which we find appropriate for magnetic fluids allows for particle rotations within the carrier fluid. In Appendix A.3, we highlight the corresponding assumption on the transport of the magnetization and the problems and difficulties which arise in the energetic variatonal approach using these particular assumptions.
The weak transport coupling means that we think of the magnetic variable following the elastic deformation by means of a movement of the dipole's center of mass and a volume change in compressible materials. Then, the crystalline structure of the material, changed by the macroscopic deformation, causes the magnetization to relax and adapt to the new easy axes. In this case of weak

transport coupling, we assume simple transport of the form (see also [DD98]; this relates to conservation of mass, see (2.26))

$$M(x(X,t),t) = \frac{1}{\det \widetilde{F}(X,t)} M_0(X) \tag{2.29}$$

in the Lagrangian coordinate system. In the Eulerian coordinate system, we obtain (see Appendix A.1 for a proof)

$$\mathcal{M}_t := M_t + \nabla \cdot (M \otimes v) = M_t + (v \cdot \nabla) M + (\nabla \cdot v) M = 0 \tag{2.30}$$

in the Eulerian coordinate system.

In a next step, we couple the transport of M with the LLG equation. This coupling is meaningful in the following sense: the LLG equation represents the dynamics in the case of no motion by a surrounding elastic body. The transport, in addition, brings in exactly this macroscopic material movement. So, the simple time derivative in the LLG equation (2.19) is replaced by the transport equation (2.30). We obtain

$$\mathcal{M}_t = -M \times H_{\text{eff}} + \alpha_{damp} M \times \mathcal{M}_t \tag{2.31}$$

as a microscopic force balance equation.

2.5 Energy dissipation law

As discussed in Section 2.2, the energy dissipation law reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{total} = -\Delta_E,$$

where E^{total} is the sum of kinetic and free internal energy. In our model, we write

$$E^{total} = \int_\Omega \frac{1}{2}\rho|v|^2 + \frac{1}{\det F}W(F)\ \mathrm{d}x + W_{\mu\mathrm{mag}}(F,M),$$

where $W : \mathbb{R}^{d\times d} \to \mathbb{R}_0^+$ is an elastic energy density. The elastic energy density is usually defined in the Lagrangian coordinate system and we have the integral transformation (see also [For13, Section 3.3])

$$\int_{\Omega_0} W(\widetilde{F})\ \mathrm{d}X = \int_\Omega \frac{1}{\det F}W(F)\ \mathrm{d}x. \tag{2.32}$$

Further, $W_{\mu\mathrm{mag}}(F,M)$ is the micromagnetic energy introduced in (2.17). Finally, the total energy reads

$$\begin{aligned} E^{total} = \int_\Omega \frac{1}{2}\rho|v|^2 + \frac{1}{\det F}W(F) + A|\nabla M|^2 + \psi(F,M) \\ + E^{(d)}_{\mathrm{stray}}(M) - \mu_0 M\cdot H_{\mathrm{ext}}\ \mathrm{d}x, \end{aligned} \tag{2.33}$$

where $E^{(d)}_{\mathrm{stray}}(M)$ is given by (2.18). Moreover, we introduce a viscosity term

$$\Delta_E(v) = \int_\Omega \nu|\nabla v|^2\ \mathrm{d}x \tag{2.34}$$

as the dissipation on the macroscopic scale, where $\nu > 0$ is a viscosity constant. For the microscopic scale, represented by the magnetization M, notice that the LLG equation is dissipative, as discussed in Section 2.3.2. There, we found a microscopic energy dissipation law (2.23). Following the idea of coupling the transport with the LLG equation in Section 2.4, we couple the dissipation in (2.23) with the transport by replacing the simple time derivative with the transport $\mathcal{M}_t$ in (2.30). This yields

$$\Delta_E(M_t) = \int_\Omega \alpha_{damp}|M\times\mathcal{M}_t|^2\ \mathrm{d}x \tag{2.35}$$

as a dissipation on the microscopic scale.
In summary, we obtain for the total energy dissipation law covering the microscopic scale as well as the macroscopic scale

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega \frac{1}{2}\rho|v|^2 + \frac{1}{\det F}W(F) \\ &\qquad + A|\nabla M|^2 + \psi(F,M) + E^{(d)}_{\mathrm{stray}}(M) - \mu_0 M\cdot H_{\mathrm{ext}}\ \mathrm{d}x\bigg) \\ &= -\int_\Omega \nu|\nabla v|^2 + \alpha_{damp}|M\times(M_t + \nabla\cdot(M\otimes v))|^2\ \mathrm{d}x. \end{aligned} \tag{2.36}$$

Note that we introduce a regularization for F later in Section 2.8.2 for mathematical reasons.

2.6 Summary of equations

Our system of partial differential equations for magnetoelastic materials consists of the following equations: firstly, we have the equation of motion (2.37) including the stress tensor (2.39) and the induced pressure term (2.38). These are derived in Section 2.7.1 below by a calculation of the first variation of the corresponding action functional with respect to the domain. The equation of motion also includes the magnetic forces due to the magnetic stray field $H(M)$ (for $d = 3$; this force disappears for $d = 2$) and the externally applied magnetic field H_{ext}. The form of the force terms is different ($\nabla^\top HM$) from what we can find in the literature on magnetic forces ($(M \cdot \nabla)H$), see, e.g., [Bro66, Sch05, SS09]. However, due to the form of the stray field as a gradient of a scalar potential (2.12), we note that the form of the force can be changed from $\nabla^\top HM$ to $(M \cdot \nabla)H$ and vice versa, which we highlight in the proof of Theorem 1 on page 26 (see also [DO14, Section 2.2.5]).
Secondly, we have the microscopic force balance (2.42), i.e., the coupled equation between the transport of M and the Landau-Lifshitz-Gilbert (LLG) equation. This coupled equation comes from (2.31). The effective magnetic field H_{eff} (2.43) which enters the microscopic force balance within the LLG equation is derived in Section 2.7.2.
Furthermore, we need the conservation of mass from (2.24) and the transport for the deformation gradient from (2.28), reflected in (2.40) and (2.41), respectively. The boundary conditions (2.45) and (2.46) together with the initial conditions (2.47)–(2.50) then complete the system.

$$\varrho(v_t + (v \cdot \nabla)v) + \nabla p_{\text{ind}} - \nabla \cdot \tau = \nu \Delta v + (d-2)\mu_0 \nabla^\top H(M)M + \mu_0 \nabla^\top H_{\text{ext}} M \tag{2.37}$$

$$p_{\text{ind}} = -2A\Delta M \cdot M - A\,|\nabla M|^2 + \psi_M(F, M) \cdot M + (3-d)\frac{\mu_0}{2}(M \cdot e_3)^2 - \psi(F, M) + 3\Psi\,|M(x,t)|^2 - \Psi, \tag{2.38}$$

$$\tau = \frac{1}{\det F} W'(F)F^\top - 2A\nabla M \odot \nabla M + \psi_F(F, M)\,F^\top, \tag{2.39}$$

$$\rho_t + \nabla \cdot (v\rho) = 0, \tag{2.40}$$

$$F_t + (v \cdot \nabla)F - \nabla v F = 0, \tag{2.41}$$

$$M_t + \nabla \cdot (M \otimes v) = -M \times H_{\text{eff}} + \alpha_{damp} M \times (M_t + \nabla \cdot (M \otimes v)), \tag{2.42}$$

$$H_{\text{eff}} = 2A\Delta M - \psi_M(F, M) + (d-2)\mu_0 H(M) - (3-d)\mu_0 (M \cdot e_3)e_3 + \mu_0 H_{\text{ext}}, \tag{2.43}$$

$$|M| = 1 \quad \text{a.e.} \tag{2.44}$$

on $\Omega \times (0, t^*)$, where Ψ is a Lagrange multiplier for the length constraint (2.44), and $H(M)$ is defined as in equations (2.13)–(2.14) (for $d = 3$). Further, e_3 denotes the third standard basis vector in $\mathbb{R}^3$. We impose the boundary conditions

$$v = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{2.45}$$

$$\frac{\partial M}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega \times (0, t^*) \tag{2.46}$$

and the initial conditions

$$v(x,0) = v_0(x) \qquad \text{in } \Omega, \tag{2.47}$$

$$\rho(x,0) = \rho_0(x) \qquad \text{in } \Omega, \tag{2.48}$$

$$F(x,0) = F_0(x) = I \qquad \text{in } \Omega, \tag{2.49}$$

$$M(x,0) = M_0(x) \quad \text{in } \Omega, \quad |M_0(x)| = 1 \quad \text{a.e. in } \Omega. \tag{2.50}$$

2.7 Derivation of the model

In the following part, we derive the equation of motion (2.37) together with the stress tensor (2.39) and the induced pressure (2.38) in Section 2.7.1. The effective magnetic field H_{eff} (2.43) for the microscopic force balance is derived in Section 2.7.2.
We prove both Theorem 1 in Section 2.7.1 and Theorem 5 in Section 2.7.2 for $\Omega \subset \mathbb{R}^3$ first. Then we comment on the particular changes which occur in the case where $\Omega \subset \mathbb{R}^2$: the only change is in the handling of the stray field energy density $E^{(d)}_{\text{stray}}$, all other calculations are the same.
In the derivation of the model, we assume that all the quantities are as smooth as necessary to justify the calculations.

2.7.1 Equation of motion: variation with respect to the domain

The result of the derivation is given in

Theorem 1. *For a compressible viscoelastic and micromagnetic material subject to an external magnetic field the equation of motion is given by*

$$\begin{aligned}\varrho(v_t + (v \cdot \nabla)v) + \nabla p_{\text{ind}} - \nabla \cdot \tau = \nu \Delta v + (d-2)\mu_0 \nabla^\top H(M)M \\ + \mu_0 \nabla^\top H_{\text{ext}} M \quad \textit{in } \Omega \times (0, t^*), \quad (2.51)\end{aligned}$$

where the induced pressure is given by

$$\begin{aligned}p_{\text{ind}} = -2A\Delta M \cdot M - A\left|\nabla M\right|^2 + \psi_M(F, M) \cdot M + (3-d)\frac{\mu_0}{2}(M \cdot e_3)^2 \\ - \psi(F, M) + 3\Psi \left|M(x,t)\right|^2 - \Psi \qquad (2.52)\end{aligned}$$

and Ψ denotes the Lagrange multiplier for the constraint $|M| = 1$, and the total stress tensor is given by

$$\tau = \frac{1}{\det F} W'(F)F^\top - 2A\nabla M \odot \nabla M + \psi_F(F, M)F^\top. \qquad (2.53)$$

Proof. Firstly, we consider the conservative part of the energy dissipation law. We start by discussing the case where $\Omega \subset \mathbb{R}^3$. The total energy in (2.33) for $d = 3$ inserted into the general action (2.8) yields the action functional

$$\begin{aligned}\hat{\mathcal{A}}(v, F, M) = \int_0^{t^*} \int_\Omega \frac{1}{2}\rho|v|^2 - \frac{1}{\det F}W(F) - A|\nabla M|^2 - \psi(F, M) \\ + \frac{\mu_0}{2} M \cdot H(M) + \mu_0 M \cdot H_{\text{ext}} \,\mathrm{d}x\,\mathrm{d}t. \qquad (2.54)\end{aligned}$$

We consider the length constraint $|M| = 1$ as a side condition which we take care of in the action functional with a Lagrange multiplier $\Psi \in L^2(0, t^*; L^2(\Omega; \mathbb{R}))$. To this end, we introduce the extended action functional

$$\hat{\mathcal{A}}_{ext}(v,F,M) = \int_0^{t^*} \int_\Omega \frac{1}{2}\rho|v|^2 - \frac{1}{\det F}W(F) - A|\nabla M|^2 - \psi(F,M) + \frac{\mu_0}{2}M\cdot H(M) + \mu_0 M\cdot H_{\text{ext}}\,\mathrm{d}x + \int_\Omega \Psi(|M|^2-1)\,\mathrm{d}x\,\mathrm{d}t \tag{2.55}$$

which now takes care of the length constraint on the energetic level. To calculate the variation with respect to the domain $\Omega = x(\Omega_0, t)$, we use variations of the form

$$x^\varepsilon(X,t) := x(X,t) + \varepsilon\widetilde{\chi}(X,t) \tag{2.56}$$

with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\widetilde{\chi}(X,t) = \chi(x(X,t),t)$ smooth and compactly supported on $\Omega_0 \times [0,t^*]$ and

$$\widetilde{F}^\varepsilon(X,t) := \nabla_X x^\varepsilon(X,t). \tag{2.57}$$

These variations vary the domain Ω in the sense that $\Omega_\varepsilon := x^\varepsilon(\Omega_0,t)$.
Since the variations (2.56) are functions on the Lagrangian coordinate system, the extended action functional needs to be transformed into the Lagrangian coordinate system. In this transformation, the integrals change according to the formula $\int_\Omega \cdots\ \mathrm{d}x = \int_{\Omega_0} \cdots \det\widetilde{F}\ \mathrm{d}X$.
All the variables are expressed in terms of $x(X,t)$ with the help of the transport relations. Note that in general $\det\widetilde{F} \neq 1$ and that we have $\rho(x(X,t),t) = \frac{\rho_0(X)}{\det\widetilde{F}}$ from the conservation of mass (2.26) and $M(x(X,t),t) = \frac{1}{\det\widetilde{F}}M_0(X)$ by the transport (2.29). We use these formulas for the transformation between Lagrangian and Eulerian coordinate systems.
Moreover, the gradient changes according to the formula

$$\nabla_X(\cdot)\widetilde{F}^{-1} = \nabla_x(\cdot) = \nabla(\cdot), \quad \text{i.e.,} \quad \nabla_{X_j}(\cdot)\widetilde{F}^{-1}_{jk} = \nabla_{x_k}(\cdot) \tag{2.58}$$

which is a direct consequence of the chain rule (for a proof, see Appendix A.1). Together with the definition of the elastic energy (2.32) and the push-forward formula for the deformation gradient $\widetilde{F}(X,t) = F(x(X,t),t)$, we obtain

$$\hat{\mathcal{A}}_{ext}(v,F,M) = \int_0^{t^*}\int_{\Omega_0} \frac{1}{2}\varrho_0|x_t(X,t)|^2 - W(\widetilde{F})\,\mathrm{d}X\,\mathrm{d}t \tag{2.59}$$

$$+ \int_0^{t^*}\int_{\Omega_0} -A\left|\nabla_X\left(\frac{1}{\det\widetilde{F}}M_0(X)\right)\widetilde{F}^{-1}\right|^2 \det\widetilde{F}\,\mathrm{d}X\,\mathrm{d}t \tag{2.60}$$

$$+ \int_0^{t^*}\int_{\Omega_0} -\psi\left(\widetilde{F}, \frac{1}{\det\widetilde{F}}M_0(X)\right)\det\widetilde{F}\,\mathrm{d}X\,\mathrm{d}t \tag{2.61}$$

$$+ \int_0^{t^*}\int_{\Omega_0} \frac{\mu_0}{2}M_0(X)\cdot H(M)(x(X,t),t)\,\mathrm{d}X\,\mathrm{d}t \tag{2.62}$$

$$+ \int_0^{t^*}\int_{\Omega_0}\left(\mu_0\frac{1}{\det\widetilde{F}}M_0(X)\cdot H_{\text{ext}}(x(X,t),t) + \Psi\left(\left|\frac{1}{\det\widetilde{F}}M_0(X)\right|^2 - 1\right)\right)\det\widetilde{F}\,\mathrm{d}X\,\mathrm{d}t \tag{2.63}$$

$$=: \mathcal{A}_{ext}(x),$$

which reflects that the integrands in (2.59)–(2.63) depend on x, x_t and $\nabla_X x$ only. In the following, we split the variation of the extended action functional into multiple parts for better readability. Since the stray field part (2.62) is more involved, we discuss this part at last.
In the following calculations, we need the formulas

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \det \widetilde{F}^{\varepsilon} = \det \widetilde{F}\,\mathrm{tr}(\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1}), \tag{2.64}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \widetilde{F}^{\varepsilon} = \nabla_X \widetilde{\chi}(X,t), \tag{2.65}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (\widetilde{F}^{\varepsilon})^{-1} = -\widetilde{F}^{-1}\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1}. \tag{2.66}$$

For a proof of (2.64) and (2.66), we refer to Appendix A.1.

Variation of the kinetic and purely elastic parts (2.59). With (2.56) plugged in, we calculate the derivative with respect to ε, using (2.65),

$$\begin{aligned}\mathcal{T}_1 :=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_0^{t^*}\int_{\Omega_0} \frac{1}{2}\varrho_0 |x_t^{\varepsilon}(X,t)|^2 - W(\widetilde{F}^{\varepsilon})\ \mathrm{d}X\ \mathrm{d}t \\ =& \int_0^{t^*}\int_{\Omega_0} \varrho_0 x_t(X,t)\cdot \widetilde{\chi}_t(X,t) \\ &\qquad - \frac{1}{\det \widetilde{F}} W'(\widetilde{F}) : (\nabla_X \widetilde{\chi}(X,t)\underbrace{\widetilde{F}^{-1}\widetilde{F}}_{=I}) \det \widetilde{F}\ \mathrm{d}X\ \mathrm{d}t.\end{aligned}$$

Next, we transform back to the Eulerian coordinate system, using the transformation formula for the elastic energy (2.32) and the gradient formula (2.58), and integrate the very first summand by parts with respect to the time t. We obtain

$$\begin{aligned}\mathcal{T}_1 =& \int_0^{t^*}\int_{\Omega} -\varrho(x,t)\underbrace{(v_t + (v\cdot\nabla)v)}_{=\frac{\mathrm{d}}{\mathrm{d}t}v(x,t)}\cdot\chi(x,t) \\ &\qquad - \left(\frac{1}{\det F} W'(F)F^{\top}\right) : \nabla\chi(x,t)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

In order to apply the fundamental lemma of the calculus of variations (see, e.g., [JLJ08, Lemma 1.1.1]), we integrate by parts with respect to the spatial variable x. We get

$$\begin{aligned}\mathcal{T}_1 =& \int_0^{t^*}\int_{\Omega} -\varrho(x,t)(v_t + (v\cdot\nabla)v)\cdot\chi(x,t) \\ &\qquad + \left(\nabla\cdot\left(\frac{1}{\det F} W'(F)F^{\top}\right)\right)\cdot\chi(x,t)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned} \tag{2.67}$$

Variation of the exchange part (2.60). With (2.56) plugged in, we calculate the derivative with respect to ε, using the product and chain rules, (2.64) and (2.66),

$$
\begin{aligned}
\mathcal{T}_2 &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_0^{t^*} \int_{\Omega_0} -A \left| \nabla_X \left(\frac{1}{\det \widetilde{F}^\varepsilon} M_0(X) \right) (\widetilde{F}^\varepsilon)^{-1} \right|^2 \det \widetilde{F}^\varepsilon \, \mathrm{d}X \, \mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} -2A \left(\nabla_X \frac{1}{\det \widetilde{F}} M_0(X) \widetilde{F}^{-1} \right) : \Bigg(- \nabla_X \frac{1}{(\det \widetilde{F})^2} \det \widetilde{F} \\
&\qquad\qquad \times \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}) M_0(X) \widetilde{F}^{-1} \Bigg) \det \widetilde{F} \\
&\qquad - 2A \left(\nabla_X \frac{1}{\det \widetilde{F}} M_0(X) \widetilde{F}^{-1} \right) : \left(-\nabla_X \frac{1}{\det \widetilde{F}} M_0(X) \widetilde{F}^{-1} \nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1} \right) \\
&\qquad\qquad \times \det \widetilde{F} \\
&\qquad - A \left| \nabla_X \frac{1}{\det \widetilde{F}} M_0(X) \widetilde{F}^{-1} \right|^2 \det \widetilde{F} \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}) \, \mathrm{d}X \, \mathrm{d}t.
\end{aligned}
$$

In the next step, we transform back to the Eulerian coordinate system, using again (2.58). We obtain

$$
\begin{aligned}
\mathcal{T}_2 = \int_0^{t^*} \int_\Omega & -2A \nabla M(x,t) : (-\nabla(\nabla \cdot \chi(x,t) M(x,t))) \\
& - 2A \nabla M(x,t) : (-\nabla M(x,t) \nabla \chi(x,t)) \\
& - A \left| \nabla M(x,t) \right|^2 (\nabla \cdot \chi(x,t)) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Then, we integrate by parts with respect to x (twice in the first summand!) in order to isolate $\chi(x,t)$. We get

$$
\begin{aligned}
\mathcal{T}_2 = \int_0^{t^*} \int_\Omega & 2A(\nabla(\Delta M(x,t) \cdot M(x,t))) \cdot \chi(x,t) \\
& - 2A(\nabla \cdot (\nabla M(x,t) \odot \nabla M(x,t))) \cdot \chi(x,t) \\
& + A \nabla \left| \nabla M(x,t) \right|^2 \cdot \chi(x,t) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{2.68}
$$

Variation of the anisotropy part (2.61). With (2.56) plugged in, we calculate the derivative with respect to ε, using the product and chain rules and (2.64)–(2.66),

$$
\begin{aligned}
\mathcal{T}_3 &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_0^{t^*} \int_{\Omega_0} -\psi \left(\widetilde{F}^\varepsilon, \frac{1}{\det \widetilde{F}^\varepsilon} M_0(X) \right) \det \widetilde{F}^\varepsilon \, \mathrm{d}X \, \mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} -\psi_F \left(\widetilde{F}, \frac{1}{\det \widetilde{F}} M_0(X) \right) : \nabla_X \widetilde{\chi}(X,t) \det \widetilde{F} \\
&\qquad - \psi_M \left(\widetilde{F}, \frac{1}{\det \widetilde{F}} M_0(X) \right) \cdot \left(- \frac{1}{\det \widetilde{F}} \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}) M_0(X) \right) \det \widetilde{F} \\
&\qquad - \psi \left(\widetilde{F}, \frac{1}{\det \widetilde{F}} M_0(X) \right) \det \widetilde{F} \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}) \, \mathrm{d}X \, \mathrm{d}t.
\end{aligned}
$$

In the next step, we transform back to the Eulerian coordinate system, using (2.58), to obtain

$$
\begin{aligned}
\mathcal{T}_3 = \int_0^{t^*} \int_\Omega & - \Big(\psi_F \left(F, M(x,t)\right) F^\top\Big) : \nabla\chi(x,t) \\
& - \psi_M \left(F, M(x,t)\right) \cdot \Big(- \nabla \cdot \chi(x,t) M(x,t)\Big) \\
& - \psi \left(F, M\right) \left(\nabla \cdot \chi(x,t)\right) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

Lastly, we integrate by parts with respect to x to find

$$
\begin{aligned}
\mathcal{T}_3 = \int_0^{t^*} \int_\Omega & \nabla \cdot \Big(\psi_F \left(F, M(x,t)\right) F^\top\Big) \cdot \chi(x,t) \\
& - \nabla\big(\psi_M \left(F, M(x,t)\right) \cdot M(x,t)\big) \cdot \chi(x,t) \\
& + \nabla\psi \left(F, M\right) \cdot \chi(x,t) \,\mathrm{d}x \,\mathrm{d}t. \qquad (2.69)
\end{aligned}
$$

Variation of the external field and Lagrange multiplier parts (2.63). With (2.56) plugged in, we calculate the derivative with respect to ε, using the product and chain rules, (2.64) and (2.65),

$$
\begin{aligned}
\mathcal{T}_5 :=& \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_0^{t^*} \int_{\Omega_0} \mu_0 M_0(X) \cdot H_{\mathrm{ext}}(x^\varepsilon(X,t),t) \\
& \qquad + \Psi\Big(\Big|\frac{1}{\det \widetilde{F}^\varepsilon} M_0(X)\Big|^2 - 1\Big) \det \widetilde{F}^\varepsilon \,\mathrm{d}X \,\mathrm{d}t \\
=& \int_0^{t^*} \int_{\Omega_0} \mu_0 M_0(X) \cdot (\widetilde{\chi}(X,t) \cdot \nabla) H_{\mathrm{ext}}(x(X,t),t) \\
& + 2\Psi \frac{1}{\det \widetilde{F}} M_0(X) \cdot \Big(\frac{1}{(\det \widetilde{F})^2} \det \widetilde{F} \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}) M_0(X)\Big) \det \widetilde{F} \\
& + \Psi\Big(\Big|\frac{1}{\det \widetilde{F}} M_0(X)\Big|^2 - 1\Big) \det \widetilde{F} \operatorname{tr}(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1})\Big) \,\mathrm{d}X \,\mathrm{d}t.
\end{aligned}
$$

In the next step, we transform back to the Eulerian coordinate system, using (2.58), to obtain

$$
\begin{aligned}
\mathcal{T}_5 = \int_0^{t^*} \int_\Omega & \mu_0 (\nabla^\top H_{\mathrm{ext}}(x,t) M(x,t)) \cdot \chi(x,t) \\
& + 2\Psi\Big(M(x,t) \cdot \Big((\nabla \cdot \chi(x,t)) M(x,t)\Big)\Big) \\
& + \Psi\Big(|M(x,t)|^2 - 1\Big) (\nabla \cdot \chi(x,t))\Big) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

Then, we integrate by parts with respect to x. We find

$$\begin{aligned}\mathcal{T}_5 = \int_0^{t^*} \int_\Omega & \mu_0(\nabla^\top H_{\text{ext}}(x,t)M(x,t)) \cdot \chi(x,t) \\ & - 2\nabla \left(\Psi \, |M(x,t)|^2 \right) \cdot \chi(x,t) \\ & - \nabla \left(\Psi \, |M(x,t)|^2 - \Psi \right) \cdot \chi(x,t) \Big) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{2.70}$$

Variation of the stray field part (2.62)**.** At last, we discuss the stray field $H(M)$ and the stray field energy in the following. Since we work in a quasi-static setting with Maxwell's equations of magnetostatics, we drop the explicit time dependence in the notation. The stray field is defined in the Eulerian coordinate system in (2.12)–(2.13). Due to this fact, we start our investigation of the stray field in the Eulerian coordinate system.

The weak form of Poisson's equation (2.14) defining the stray field reads

$$\int_{\mathbb{R}^3} \nabla\varphi(M)(x,t)\cdot\nabla\psi(x) \, \mathrm{d}x = \int_\Omega M(x,t)\cdot\nabla\psi(x) \, \mathrm{d}x \qquad \forall \psi \in \mathbf{H}^1(\mathbb{R}^3;\mathbb{R}). \tag{2.71}$$

Firstly, we are interested in the dependence of the potential $\varphi(M)(x,t)$, when exposed to a variation of the domain in the inner integral, meaning the integral of the convolution

$$\varphi(M)(x,t) = \int_\Omega (\nabla N)(x-y) \cdot M(y,t) \, \mathrm{d}y.$$

When imposing this variation of the domain, which is a variation x^ε of the deformation x, the dependence of the potential $\varphi(M)(x,t)$ on ε is explicit through the domain, i.e., the domain depends on ε:

$$\varphi(M)(x,t,\varepsilon) := \int_{\Omega_\varepsilon} (\nabla N)(x-y) \cdot M(y,t) \, \mathrm{d}y \tag{2.72}$$

for $x \in \mathbb{R}^3$ and $\Omega_\varepsilon = x^\varepsilon(\Omega_0, t)$. We investigate the variation of (2.72) not explicitly, but through its defining equation (2.71). From equation (2.71), we obtain in the Lagrangian coordinate system, using the transport of M and (2.58),

$$\int_{\mathbb{R}^3} \nabla\varphi(M)(x,t,\varepsilon) \cdot \nabla\psi(x) \, \mathrm{d}x = \int_{\Omega_0} M_0(X) \cdot \nabla_X \psi(x^\varepsilon)(\widetilde{F}^\varepsilon)^{-1} \, \mathrm{d}X \tag{2.73}$$

for any $\psi \in \mathbf{H}^1(\mathbb{R}^3;\mathbb{R})$.

In order not to be confused with indices, we use the notation $\frac{\partial(\cdot)}{\partial\varepsilon}$ in the following computations to indicate the partial derivate instead of $(\cdot)_\varepsilon$.

We calculate the variation of (2.71) with respect to the domain by taking the derivative of (2.73) with respect to ε at $\varepsilon = 0$. Note that $\varphi \in \mathbf{H}^2(\Omega;\mathbb{R}^3)$ (this follows immediately from (2.14) and our assumption on smoothness of all quantities; in this case, we need that $M \in \mathbf{H}^1(\Omega;\mathbb{R}^3)$) at least and assume additionally

$\psi \in \mathbf{H}^2(\mathbb{R}^3;\mathbb{R})$ to find out that, using product and chain rules and the formula (2.58),

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla \left(\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \varphi(M)(x,t,\varepsilon) \right) \cdot \nabla\psi(x) \,\mathrm{d}x \\
&= \int_{\Omega_0} M_0(X)_k \nabla_{X_j} \big(\nabla_{x_l}\psi(x(X,t)) \widetilde{\chi}_l(X,t) \big) \widetilde{F}^{-1}_{jk} \\
&\qquad\qquad - M_0(X)_k \nabla_{X_j}\psi(x(X,t)) \widetilde{F}^{-1}_{jl} \nabla_{X_\sigma}\widetilde{\chi}_l(X,t) \widetilde{F}^{-1}_{\sigma k} \,\mathrm{d}X \\
&= \int_{\Omega} M(x,t)_k \nabla_{x_k} \big(\nabla_{x_l}\psi(x)\chi_l(x,t) \big) - M(x,t)_k \nabla_{x_l}\psi(x) \nabla_{x_k}\chi_l(x,t) \,\mathrm{d}x \\
&= \int_{\Omega} M(x,t)_k \nabla_{x_k}\nabla_{x_l}\psi(x)\chi_l(x,t) \,\mathrm{d}x. \qquad (2.74)
\end{aligned}$$

Setting $\psi(x) = \frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\varphi(M)(x,t,\varepsilon)$ in (2.71) and $\psi(x) = \varphi(M)(x,t)$ in (2.74) we immediately obtain

$$\begin{aligned}
&\int_{\Omega} M(x,t) \cdot \nabla \left(\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \varphi(M)(x,t,\varepsilon) \right) \,\mathrm{d}x \\
&= \int_{\Omega} M(x,t)_k \nabla_{x_k}\nabla_{x_l}\varphi(M)(x,t)\chi_l(x,t) \,\mathrm{d}x. \qquad (2.75)
\end{aligned}$$

Finally, we can take the variation of the stray field part with respect to the domain. With

$$H(M)(x,t,\varepsilon) := -\nabla\varphi(x,t,\varepsilon)$$

and (2.56) plugged in, we calculate the derivative with respect to ε, using the product and chain rules and (2.66),

$$\begin{aligned}
\mathcal{T}_4^{(3)} &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_0^{t^*} \int_{\Omega_0} -\frac{\mu_0}{2} M_0(X) \cdot \nabla_X \varphi(M)(x^\varepsilon(X,t),t,\varepsilon)(\widetilde{F}^\varepsilon)^{-1} \,\mathrm{d}X \,\mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} -\frac{\mu_0}{2} M_0(X)_k \Bigg(\nabla_{X_j} \left(\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \varphi(M)(x(X,t),t,\varepsilon) \right. \\
&\qquad\qquad \left. + \nabla_{x_l}\varphi(M)(x(X,t),t)\widetilde{\chi}_l(X,t) \right) \widetilde{F}^{-1}_{jk} \\
&\qquad\qquad - \nabla_{X_j}\varphi(M)(x(X,t),t)(\widetilde{F}^{-1}_{jl} \nabla_{X_\sigma}\widetilde{\chi}_l(X,t)\widetilde{F}^{-1}_{\sigma k}) \Bigg) \,\mathrm{d}X \,\mathrm{d}t.
\end{aligned}$$

In the next step, we transform back to the Eulerian coordinate system, using (2.58), and apply (2.75) to obtain

$$\mathcal{T}_4^{(3)} = \int_0^{t^*} \int_{\Omega} -\mu_0 M_k(x,t) \underbrace{\nabla_{x_k}\nabla_{x_l}\varphi(M)(x,t)}_{=\nabla_{x_l}\nabla_{x_k}\varphi(M)(x,t) = -\nabla_{x_l}H_k(M)(x,t)} \chi_l(x,t) \,\mathrm{d}x \,\mathrm{d}t.$$

Then, we write index notation back into matrix and vector products to find

$$\mathcal{T}_4^{(3)} = \int_0^{t^*} \int_{\Omega} \mu_0 \left(\nabla^\top H(M)(x,t) M(x,t) \right) \cdot \chi(x,t) \,\mathrm{d}x \,\mathrm{d}t. \qquad (2.76)$$

Finally, we put (2.67), (2.68), (2.69), (2.70), and (2.76) together to obtain the expression $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\mathcal{A}_{ext}(x^\varepsilon)$. We find out that

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} &\mathcal{A}_{ext}(x^\varepsilon) = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4^{(3)} + \mathcal{T}_5 \\
= \int_0^{t^*}\int_\Omega & -\varrho(x,t)(v_t + (v\cdot\nabla)v)\cdot\chi(x,t) + \left(\nabla\cdot\left(\frac{1}{\det F}W'(F)F^\top\right)\right)\cdot\chi(x,t) \\
&+ 2A(\nabla(\Delta M(x,t)\cdot M(x,t)))\cdot\chi(x,t) \\
&- 2A(\nabla\cdot(\nabla M(x,t)\odot\nabla M(x,t)))\cdot\chi(x,t) \\
&+ A\nabla\left|\nabla M(x,t)\right|^2\cdot\chi(x,t) \\
&+ \nabla\cdot\left(\psi_F\left(F, M(x,t)\right)F^\top\right)\cdot\chi(x,t) \\
&- \nabla\big(\psi_M\left(F, M(x,t)\right)\cdot M(x,t)\big)\cdot\chi(x,t) \\
&+ \nabla\psi\left(F,M\right)\cdot\chi(x,t) \\
&+ \mu_0\left(\nabla^\top H(M)(x,t)M(x,t)\right)\cdot\chi(x,t) \\
&+ \mu_0(\nabla^\top H_{\text{ext}}(x,t)M(x,t))\cdot\chi(x,t) \\
&- 2\nabla\left(\Psi\left|M(x,t)\right|^2\right)\cdot\chi(x,t) \\
&- \nabla\left(\Psi\left|M(x,t)\right|^2 - \Psi\right)\cdot\chi(x,t)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

From here, we deduce that, due to the properties of χ and with the fundamental lemma of the calculus of variations,

$$\varrho(v_t + (v\cdot\nabla)v) + \nabla p_{\text{ind}} - \nabla\cdot\tau - \mu_0\nabla^\top H(M)M - \mu_0\nabla^\top H_{\text{ext}}M = 0, \quad (2.77)$$

where

$$\begin{aligned}
p_{\text{ind}} = -2A\Delta M\cdot M - A\left|\nabla M\right|^2 + \psi_M\left(F,M\right)\cdot M& \\
- \psi\left(F,M\right) + 3\Psi\left|M(x,t)\right|^2 - \Psi&, \qquad (2.78)
\end{aligned}$$

is the induced pressure and

$$\tau = \frac{1}{\det F}W'(F)F^\top - 2A\nabla M\odot\nabla M + \psi_F\left(F,M\right)F^\top \quad (2.79)$$

is the total stress tensor as in (2.53).
In the case where $\Omega\subset\mathbb{R}^2$, we use the corresponding stray field energy density from (2.18). Since all other terms stay the same, we just calculate the variation with respect to the domain of the stray field part of the action.
Due to the simple transport of M, we transform

$$\begin{aligned}
\mathcal{T}_4^{(2)} :=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_0^{t^*}\int_{\Omega_\varepsilon} -\frac{\mu_0}{2}(M(x,t)\cdot e_3)^2\ \mathrm{d}x\ \mathrm{d}t \\
=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_0^{t^*}\int_{\Omega_0} -\frac{\mu_0}{2}\left(\frac{1}{\det\widetilde{F}^\varepsilon}M_0(X)\cdot e_3\right)^2\det\widetilde{F}^\varepsilon\ \mathrm{d}X\ \mathrm{d}t \\
=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_0^{t^*}\int_{\Omega_0} -\frac{\mu_0}{2}\left(M_0(X)\cdot e_3\right)^2\frac{1}{\det\widetilde{F}^\varepsilon}\ \mathrm{d}X\ \mathrm{d}t.
\end{aligned}$$

Next, we take the derivative with respect to ε, using the formula (2.64) together with the chain rule. We obtain

$$\begin{aligned}\mathcal{T}_4^{(2)} &= \int_0^{t^*} \int_{\Omega_0} \frac{\mu_0}{2}(M_0(X)\cdot e_3)^2 \left(\frac{1}{\det \widetilde{F}^\varepsilon}\right)^2 \det \widetilde{F}\, \mathrm{tr}(\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1})\ \mathrm{d}X\ \mathrm{d}t \\ &= \int_0^{t^*} \int_{\Omega_0} \frac{\mu_0}{2} \left(\frac{1}{\det \widetilde{F}^\varepsilon} M_0(X)\cdot e_3\right)^2 \det \widetilde{F}\, \mathrm{tr}(\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1})\ \mathrm{d}X\ \mathrm{d}t.\end{aligned}$$

Finally, we transform back to the Eulerian coordinate system, using the transport of M, the transformation formula of the gradient (2.58) and an integration by parts with respect to x in the last step. This yields

$$\begin{aligned}\mathcal{T}_4^{(2)} &= \int_0^{t^*} \int_{\Omega} \frac{\mu_0}{2}(M(x,t)\cdot e_3)^2\, \mathrm{tr}(\nabla \chi(x,t))\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^{t^*} \int_{\Omega} \frac{\mu_0}{2}(M(x,t)\cdot e_3)^2 \nabla\cdot \chi(x,t)\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^{t^*} \int_{\Omega} -\frac{\mu_0}{2}\nabla (M(x,t)\cdot e_3)^2 \chi(x,t)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

In view of the calculations from above, this results in a contribution $+\frac{\mu_0}{2}(M\cdot e_3)^2$ to the induced pressure. In summary, with (2.78), this yields the pressure (2.52).

Now, we take care of the viscosity term (2.34) which is the dissipation related to the macroscopic scale in (2.36). As highlighted in Section 2.2, to derive the dissipative part of the equation of motion, we use a variation of the form $v + \varepsilon\tilde{v}$ with $\tilde{v}$ being compactly supported and smooth (see also [For13, Section 3.5]) and calculate

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \frac{1}{2}\Delta_E(v+\varepsilon\tilde{v}) = \int_\Omega \big(-\nabla\cdot(\nu\nabla v)\big)\cdot \tilde{v}\ \mathrm{d}x = \int_\Omega \big(-\nu\Delta v\big)\cdot\tilde{v}\ \mathrm{d}x.$$

At this point, we can again use the fundamental lemma of the calculus of variations to obtain

$$-\nu\Delta v = 0. \tag{2.80}$$

Finally, by the force balance law, we put the conservative part (2.77) and the dissipative part (2.80) together and, by noting that there is no force term due to the stray field in the case $d = 2$, we obtain equation (2.51). This concludes the proof. □

2.7.2 Calculation of the effective magnetic field H_{eff}

This section is dedicated to the calculation of the effective magnetic field H_{eff}. It is derived as the negative variational derivative of the micromagnetic energy $W_{\mu\mathrm{mag}}$ with respect to the magnetization M (see [KP06, Section 3.2] and Section 2.3.2, where the effective magnetic field appears in the LLG equation (2.19)). During the calculation, we need to take care of the nonlocal term that represents

the stray field $H(M)$ (only for $d = 3$) induced by the magnetization of the body. To this end, we apply the following result. The proof of the lemma is based on a regularization of the Newton potential and the single layer potential and is provided in Appendix A.4.
Notice that M also depends on time. For the evaluation of Maxwell's equations for magnetostatics on the microscopic scale, however, the dependence on time can be neglected and we drop the time dependence in the notation as well.

Lemma 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary of positive and finite Hausdorff measure $0 < \mathcal{H}^2(\partial\Omega) < \infty$, let $M \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ and $\widehat{M} \in W_0^{1,\infty}(\Omega;\mathbb{R}^3)$. Then, it holds*

$$\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle \widehat{M}, H(M)\rangle_{L^2(\Omega;\mathbb{R}^3)}, \tag{2.81}$$

where $\langle f, g\rangle_{L^2(\Omega;\mathbb{R}^3)} := \int_\Omega f(x) \cdot g(x)\, \mathrm{d}x$.

Remark 3. *Notice that we take $\widehat{M}$ with zero trace in Lemma 2. This is because we use compactly supported test functions in the proof of Theorem 5.*

For the next result, we need the definition of a variational derivative, which we present adapted for our special case.

Definition 4. *Let $W_{\mu\mathrm{mag}}$ be the micromagnetic energy functional as in (2.17). Further, let $\widehat{M}$ be smooth and compactly supported in space within Ω, and let $M^\varepsilon = M + \varepsilon\widehat{M}$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The variational derivative of $W_{\mu\mathrm{mag}}$ with respect to M, denoted by $\frac{\delta W_{\mu\mathrm{mag}}}{\delta M}$, is defined through*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} W_{\mu\mathrm{mag}}(M^\varepsilon) = \int_\Omega \frac{\delta W_{\mu\mathrm{mag}}}{\delta M} \cdot \widehat{M}\, \mathrm{d}x. \tag{2.82}$$

Now, with the above lemma, we are able to derive the effective magnetic field H_{eff}. The result is stated in

Theorem 5. *For a compressible viscoelastic and micromagnetic material subject to an external magnetic field the effective magnetic field H_{eff} is given by*

$$\begin{aligned} H_{\mathrm{eff}} = -\frac{\delta W_{\mu\mathrm{mag}}}{\delta M} &= 2A\Delta M - \psi_M(F, M) \\ &\quad + (d-2)\mu_0 H(M) - (3-d)\mu_0 (M \cdot e_3)e_3 + \mu_0 H_{\mathrm{ext}}, \end{aligned} \tag{2.83}$$

where $W_{\mu\mathrm{mag}}$ as in (2.17).

Proof. We start by discussing the case where $\Omega \subset \mathbb{R}^3$. We calculate the variational derivative of $W_{\mu\mathrm{mag}}$ with respect to M according to Definition 4. Then we obtain the effective magnetic field as $H_{\text{eff}} = -\frac{\delta W_{\mu\text{mag}}}{\delta M}$. We obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} W_{\mu\text{mag}}(M^\varepsilon)\\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega A|\nabla M^\varepsilon|^2 + \psi\left(F, M^\varepsilon\right) - \frac{\mu_0}{2} M^\varepsilon \cdot H(M^\varepsilon) - \mu_0 M^\varepsilon \cdot H_{\text{ext}} \,\mathrm{d}x\\
&= \int_\Omega 2A\nabla M \nabla \widehat{M} + \psi_M\left(F, M\right) \cdot \widehat{M} - \frac{\mu_0}{2}\left(\widehat{M} \cdot H(M) + M \cdot \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} H(M^\varepsilon)\right)\\
&\qquad - \mu_0 H_{\text{ext}} \cdot \widehat{M} \,\mathrm{d}x.
\end{aligned}
$$

We integrate by parts with respect to x the first term and apply (2.13) in the fourth term to find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} W_{\mu\text{mag}}(M^\varepsilon)\\
&= \int_\Omega -2A(\nabla \cdot \nabla M) \cdot \widehat{M} + \psi_M\left(F, M\right) \cdot \widehat{M}\\
&\qquad - \frac{\mu_0}{2}\left(\widehat{M} \cdot H(M) - M \cdot \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\left(\nabla \int_\Omega (\nabla N)(x-y) \cdot M^\varepsilon(y) \,\mathrm{d}y\right)\right)\\
&\qquad - \mu_0 H_{\text{ext}} \cdot \widehat{M} \,\mathrm{d}x\\
&= \int_\Omega -2A(\nabla \cdot \nabla M) \cdot \widehat{M} + \psi_M\left(F, M\right) \cdot \widehat{M}\\
&\qquad - \frac{\mu_0}{2}\left(\widehat{M} \cdot H(M) - M \cdot \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\left(\nabla \int_\Omega (\nabla N)(x-y) \cdot M(y) \,\mathrm{d}y\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.\left.+ \varepsilon \nabla \int_\Omega (\nabla N)(x-y) \cdot \widehat{M}(y) \,\mathrm{d}y\right)\right)\\
&\qquad - \mu_0 H_{\text{ext}} \cdot \widehat{M} \,\mathrm{d}x.
\end{aligned}
$$

At this point, we can immediately take the derivative with respect to ε due to the linearity in ε. In the second step we apply Lemma 2 to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} W_{\mu\text{mag}}(M^\varepsilon)\\
&= \int_\Omega -2A(\Delta M) \cdot \widehat{M} + \psi_M\left(F, M\right) \cdot \widehat{M}\\
&\qquad - \frac{\mu_0}{2}\left(\widehat{M} \cdot H(M) - M \cdot \underbrace{\left(\nabla \int_\Omega (\nabla N)(x-y) \cdot \widehat{M}(y) \,\mathrm{d}y\right)}_{=-H(\widehat{M})}\right)\\
&\qquad - \mu_0 H_{\text{ext}} \cdot \widehat{M} \,\mathrm{d}x\\
&= \int_\Omega -2A(\Delta M) \cdot \widehat{M} + \psi_M\left(F, M\right) \cdot \widehat{M} - \mu_0 \widehat{M} \cdot H(M) - \mu_0 H_{\text{ext}} \cdot \widehat{M} \,\mathrm{d}x.
\end{aligned}
$$

The last expression is equal to

$$
-\int_\Omega \left(2A\Delta M - \psi_M\left(F, M\right) + \mu_0 H(M) + \mu_0 H_{\text{ext}}\right) \cdot \widehat{M} \,\mathrm{d}x,
$$

from where we immediately deduce that, using (2.82),

$$H_{\text{eff}} = 2A\Delta M - \psi_M(F, M) + \mu_0 H(M) + \mu_0 H_{\text{ext}}. \tag{2.84}$$

In the case where $\Omega \subset \mathbb{R}^2$, only the stray field energy density (2.18) changes. As before, in the calculation of the equation of motion, all other terms stay the same, so we only redo the calculation for the stray field energy term We obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega \frac{\mu_0}{2}(M^\varepsilon \cdot e_3)^2 \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega \frac{\mu_0}{2}((M + \varepsilon\widehat{M}) \cdot e_3)^2 \,\mathrm{d}x \\ = \int_\Omega \mu_0(M \cdot e_3)(\widehat{M} \cdot e_3) \,\mathrm{d}x &= \int_\Omega (\mu_0(M \cdot e_3)e_3) \cdot \widehat{M} \,\mathrm{d}x.\end{aligned}$$

In view of the calculations from above, this results in a contribution $-\mu_0(M \cdot e_3)e_3$ from the stray field to the effective magnetic field H_{eff}. In summary, with (2.84), we obtain the effective magnetic field (2.83). This concludes the proof. □

2.8 The model for a simplified setting

The model describing magnetoelastic materials with micromagnetic domain structure we summarized in Section 2.6 is rather complex to analyze. This is due to the fact that the resulting system of PDEs has the Navier-Stokes system as a subsystem and has many highly coupled equations. So, we seek to simplify our model in order to start the mathematical analysis of the resulting PDE system. To this end, we propose a simplified version of our model in the following. Prominent assumptions are a simpler form of the micromagnetic energy and the inclusion of incompressibility conditions. All assumptions for this model are discussed in Section 2.8.1. The corresponding energy dissipation law is set up in Section 2.8.2. Then, the equations are summarized in Section 2.8.3 and Section 2.8.4 contains derivations necessary to establish these equations. For existence of weak solutions we refer to Chapter 3, and also to [BFGC$^+$16] for an overview.

2.8.1 Simplifying model assumptions

Incompressibility. We impose incompressibility conditions. These conditions were already introduced in Section 2.1 and read (see equations (2.5)–(2.6))

$$J = \det \widetilde{F} = \det F \equiv 1 \qquad \text{and} \qquad \nabla \cdot v = 0 \tag{2.85}$$

in the Lagrangian coordinate system and in the Eulerian coordinate system, respectively. This assumption does not affect the transport equation for the deformation gradient F which reads (see (2.28))

$$F_t + (v \cdot \nabla)F - \nabla v F = 0. \tag{2.86}$$

However, conservation of mass translates to

$$\rho_t + (v \cdot \nabla)\, \rho = 0 \tag{2.87}$$

which we recall from equation (2.25). Since this implies that the mass density is constant along the trajectory, i.e., $\rho(x(X,t),t) = \rho_0(X)$, we set, without ambiguity, $\rho_0(X) \equiv 1$. This reduces the number of variables as well as the number of equations in the system. Moreover, the transport of the magnetization changes to

$$\mathcal{M}_t := M_t + (v \cdot \nabla)\, M = 0 \tag{2.88}$$

which implies that $M(x(X,t),t) = M_0(X)$. The right-hand side of (2.88) is the material derivative. Notice that this is similar to the conservation of mass. We cannot drop this equation just like conservation of mass, because we still consider certain dynamical behavior of the magnetization, see below.

Relaxed length constraint on M. We introduce a penalization term in the micromagnetic energy which punishes the deviation of $|M|$ from 1. We use the term

$$\frac{1}{4\mu^2}\int_\Omega (|M|^2-1)^2 \,\mathrm{d}x, \tag{2.89}$$

where μ is a constant to control the strength of the penalization. Such a penalization is used in, e.g., [Kur04, Section 1.2] and [CISVC09]. The effect of the relaxed length constraint is that there is no need for the Lagrange multiplier any more.

Special energy terms. We assume that there is no external field present, i.e., $H_{\text{ext}} \equiv 0$, and further, we neglect the stray field energy corresponding to long-range interactions.

Remark 6. *A consequence of this simplification is, that we do not model micromagnetic domain patterns. As mentioned in Section 2.3, the formation of domains, where the magnetization is approximately constant (see [HS98]), is considerably influenced by the magnetic stray field: domains result from the competition of crystal structure (anisotropy energy) with long-range magnetic interactions from the stray field.*
Moreover, when studying magnetic fluids as some possible extension to this work, the simplification of dropping the long-range interaction does not seem to be reasonable in the sense that particles immersed in a carrier fluid interact over longer ranges via the stray field.
Finally, the assumption on neglecting the stray field energy allows us to derive the model without the need to distinguish between $d=2$ and $d=3$ in Section 2.8.4.

As for the anisotropy energy, we assume that

$$\psi(F, M) = 0, \tag{2.90}$$

so we neglect the anisotropy and the coupling herein. Notice that there is still a coupling of elasticity and magnetic properties since both effects are described on different coordinate systems. The change of coordinate systems then introduces a coupling within the micromagnetic energy (see also Section 2.3.1). The resulting micromagnetic energy then reads

$$W_{\mu\text{mag}}^{\text{simpl.}} = \int_\Omega A|\nabla M|^2 + \frac{1}{4\mu^2}(|M|^2-1)^2 \,\mathrm{d}x. \tag{2.91}$$

Simplified dynamics of M. We replace the LLG equation by dynamics of gradient flow type (see, e.g., [LS03, LSFY05]). To this end, we set

$$\mathcal{M}_t = -\frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M}, \tag{2.92}$$

where we couple the left-hand side with the transport (2.88). Hence, we obtain

$$M_t + (v \cdot \nabla) M = -\frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M} \tag{2.93}$$

as the microscopic force balance equation in the simplified case. Notice that the right-hand side of this equation corresponds to the effective magnetic field.

2.8.2 Energy dissipation law

As before, we work in this simplified setting with the energy dissipation law (2.7). The total energy E^{total} involves the kinetic energy, the elastic energy as in (2.32) and the micromagnetic energy as in (2.91). We then have

$$E^{total} = \int_\Omega \frac{1}{2}|v|^2 + W(F) + A|\nabla M|^2 + \frac{1}{4\mu^2}(|M|^2 - 1)^2 \, \mathrm{d}x. \tag{2.94}$$

Next, we introduce a regularizing term for F, namely

$$\Delta_E(F) = \int_\Omega \kappa |\nabla F|^2 \, \mathrm{d}x,$$

where $\kappa > 0$ is a regularizing constant and $|\nabla F|^2 = \nabla F \,\vdots\, \nabla F$.
The motivation of introducing this regularizing term for F is of purely mathematical nature: we then obtain more regularity from the regularized F-equation later in the existence proofs in Chapter 3. Without this regularization the proof of existence is even more involved and cannot be done without further assumptions on F (see [LLZ05]).
The dissipative term on M (microscopic scale) is given by the gradient flow type dynamics: assuming again the separation of scales and thus that the micromagnetic energy does not depend on F when considering the microscopic scale, we formally obtain the governing energy dissipation law by multiplying equation (2.92) scalarly with $\frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M}$ and integrating over Ω. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} W_{\mu\text{mag}} = -\int_\Omega \left| \frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M} \right|^2 \, \mathrm{d}x$$

which we regard as energy dissipation law for the microscopic scale in the simplified setting. In summary, we obtain for the total dissipation

$$\Delta_E = \int_\Omega \nu |\nabla v|^2 + \kappa |\nabla F|^2 + \left| \frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M} \right|^2 \, \mathrm{d}x. \tag{2.95}$$

2.8.3 Summary of equations

Our system of partial differential equations for magnetoelastic materials in the simplified setting consists of the following equations: firstly, we have the equation

of motion (2.96) including the stress tensor (2.97), both derived in Section 2.8.4 below.
Secondly, there is the microscopic force balance (2.100) (see (2.93)), i.e., the coupled equation of the transport equation of M and the gradient flow dynamics. In the existence analysis of weak solutions in Chapter 3, we also consider the model including the LLG equation (2.100') with initial condition (2.106') instead of (2.100) with initial condition (2.106).
Furthermore, we have the transport equation of the deformation gradient (2.86) in (2.99). For mathematical reasons, we couple this equation to a regularization term and replace (2.99) by (2.99') for the existence proofs in Chapter 3.
Moreover, we have the incompressibility conditions from (2.85). Due to the additional regularization of F introduced in Section 2.8.2, the solution to the regularized evolution equation for F (2.99') is not the deformation gradient which satisfies the pure transport equation (2.99) without the regularization.
Notice that we dropped the incompressibility condition $\det F = 1$ in the following: in the case of considering the system including equation (2.99), the incompressibility condition $\nabla \cdot v = 0$ implies that $\det F = const.$, and thus $\det F = 1$ directly follows with appropriate initial conditions on F. On the other hand, if we consider (2.99'), as in Chapter 3, the incompressibility condition $\det F = 1$ cannot be satisfied any longer, since the solution of (2.99') is just an approximation of the actual deformation gradient.
The boundary conditions (2.101)–(2.103) and the initial conditions (2.104)–(2.106) finally complete the system of equations.

$$v_t + (v \cdot \nabla)v + \nabla p - \nabla \cdot \tau = \nu \Delta v \qquad \text{in } \Omega \times (0, t^*), \tag{2.96}$$

$$\tau = W'(F)F^\top - 2A\big(\nabla M \odot \nabla M\big) \qquad \text{in } \Omega \times (0, t^*), \tag{2.97}$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega \times (0, t^*), \tag{2.98}$$

$$F_t + (v \cdot \nabla)F - \nabla v F = 0 \qquad \text{in } \Omega \times (0, t^*), \tag{2.99}$$

$$F_t + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F \qquad \text{in } \Omega \times (0, t^*), \tag{2.99'}$$

$$M_t + (v \cdot \nabla)M = 2A\Delta M - \frac{1}{\mu^2}(|M|^2 - 1)M \qquad \text{in } \Omega \times (0, t^*), \tag{2.100}$$

$$M_t + (v \cdot \nabla)M = -M \times \Delta M - M \times (M \times \Delta M) \qquad \text{in } \Omega \times (0, t^*), \tag{2.100'}$$

with boundary conditions

$$v = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{2.101}$$

$$F = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{2.102}$$

$$\frac{\partial M}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega \times (0, t^*) \tag{2.103}$$

and initial conditions

$$
\begin{aligned}
v(x,0) &= v_0(x), \quad \nabla \cdot v_0(x) = 0 && \text{in } \Omega, && (2.104)\\
F(x,0) &= F_0(x) = I && \text{in } \Omega, && (2.105)\\
M(x,0) &= M_0(x) && \text{in } \Omega, && (2.106)\\
M(x,0) &= M_0(x) \quad \text{in } \Omega, \quad |M_0(x)| = 1 && \text{a.e. in } \Omega. && (2.106')
\end{aligned}
$$

Remark 7. *In Section 2.8.4.1, we derive the equation of motion* (2.96) *based on the pure transport* (2.99') *of the deformation gradient F, and on the incompressibility conditions* (2.85)*, i.e., including the condition* $\det F = 1$ *which is more physical. Later, in Chapter 3, we replace equation* (2.99) *by* (2.99') *due to the aforementioned mathematical reason.*

2.8.4 Derivation of the model

In this section, we derive the equation of motion for the system first. Then, the effective magnetic field and the dissipative term of the deformation gradient are considered. We note that several steps in the calculations for the following results are similar to those in Section 2.7 and are therefore shortened.
In the derivation of the model, we assume that all the quantities are as smooth as necessary to justify the calculations.

2.8.4.1 Equation of motion: variation with respect to the domain

Theorem 8. *For an incompressible viscoelastic and micromagnetic material in the simplified setting described in Sections 2.8.1–2.8.2 the equation of motion is given by*

$$v_t + (v \cdot \nabla)v + \nabla p - \nabla \cdot \tau = \nu \Delta v \qquad \text{in } \Omega \times (0, t^*), \tag{2.107}$$

where the total stress tensor is given by the formula

$$\tau = W'(F)F^\top - 2A\big(\nabla M \odot \nabla M\big). \tag{2.108}$$

Proof. We start by considering the conservative part of the energy dissipation law. In view of (2.8), we obtain from the total energy in (2.94) the action functional

$$\mathcal{A}(v, F, M) = \int_0^{t^*} \int_\Omega \frac{1}{2}|v|^2 - W(F) - A|\nabla M|^2 - \frac{1}{4\mu^2}(|M|^2 - 1)^2 \,\mathrm{d}x \,\mathrm{d}t. \tag{2.109}$$

We use volume preserving diffeomorphisms $x^\varepsilon(X,t)$ of class C^2 with deformation gradient $\widetilde{F}^\varepsilon(X,t) := \nabla_X x^\varepsilon(X,t)$ for the variation with respect to the domain such that

$$x^0 = x \quad \text{and} \quad \frac{\mathrm{d}x^\varepsilon}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} := \widetilde{\chi} \quad \text{and} \quad \forall \varepsilon : \; \det \widetilde{F}^\varepsilon \equiv 1 \tag{2.110}$$

and $\widetilde{\chi}$ being any compactly supported (with respect to space and time) test function of class C^∞. Here, $\widetilde{F} = \nabla_X x = \nabla_X x^\varepsilon\big|_{\varepsilon=0}$.
The nonlinear constraint leads to a divergence-free condition for the push forward $\widetilde{\chi}(X,t) = \chi(x(X,t),t)$ (see also [For13, Section 3.5]); using (2.58) and (2.64) we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \det \widetilde{F}^\varepsilon = \det \widetilde{F}\,\mathrm{tr}(\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1}) = \mathrm{tr}\,(\nabla_x \chi(x(X,t),t)) = \nabla\cdot\chi.$$

As for the variation with respect to the domain in the general case in Section 2.7.1, the variations $x^\varepsilon(X,t)$ are again functions defined on the Lagrangian coordinate system, so, the action functional (2.109) needs to be transformed into the Lagrangian coordinate system. We obtain using the push-forward formula for the deformation gradient $\widetilde{F}(X,t) = F(x(X,t),t)$ and the transport $M(x(X,t),t) = M_0(X)$

$$\begin{aligned}\mathcal{A}(x) = \int_0^{t^*}\int_{\Omega_0} &\frac{1}{2}|x_t(X,t)|^2 - W(\widetilde{F}(X,t)) \\ &- A|\nabla_X M_0(X)\widetilde{F}^{-1}(X,t)|^2 - \frac{1}{4\mu^2}(|M_0(X)|^2-1)^2 \,\mathrm{d}X\,\mathrm{d}t. \quad (2.111)\end{aligned}$$

Notice that due to the simple transport of M the last summand in the action functional does not depend on the variation.
Now, we are ready to take the variation of the action functional with respect to the domain. To this end, we plug in the volume preserving diffeomorphisms as described above. We obtain

$$\begin{aligned}\mathcal{A}(x^\varepsilon) = \int_0^{t^*}\int_{\Omega_0} &\frac{1}{2}|x_t^\varepsilon(X,t)|^2 - W(\widetilde{F}^\varepsilon(X,t)) \\ &- A|\nabla_X M_0(X)(\widetilde{F}^\varepsilon)^{-1}(X,t)|^2 - \frac{1}{4\mu^2}(|M_0(X)|^2-1)^2 \,\mathrm{d}X\,\mathrm{d}t. \quad (2.112)\end{aligned}$$

We continue the calculation:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{A}(x^\varepsilon) \\ &= \int_0^{t^*}\int_{\Omega_0} x_t(X,t)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} x_t^\varepsilon(X,t)\right) - W'(\widetilde{F}) : \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\widetilde{F}^\varepsilon\right) \\ &\qquad - 2A\left(\nabla_X M_0(X)\widetilde{F}^{-1}\right) : \left(-\nabla_X M_0(X)\widetilde{F}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\widetilde{F}^\varepsilon\right)\widetilde{F}^{-1}\right)\,\mathrm{d}X\,\mathrm{d}t.\end{aligned}$$

We assume that the variation x^ε is at least C^2. Hence $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\widetilde{F}^\varepsilon = \nabla_X\widetilde{\chi}(X,t)$ and thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{A}(x^\varepsilon) \\ &= \int_0^{t^*}\int_{\Omega_0} x_t(X,t)\cdot\widetilde{\chi}_t(X,t) - W'(\widetilde{F}) : \left(\nabla_X\widetilde{\chi}(X,t)\widetilde{F}^{-1}\widetilde{F}\right) \\ &\qquad - 2A\left(\nabla_X M_0(X)\widetilde{F}^{-1}\right) : \left(-\nabla_X M_0(X)\widetilde{F}^{-1}\nabla_X\widetilde{\chi}(X,t)\widetilde{F}^{-1}\right)\,\mathrm{d}X\,\mathrm{d}t.\end{aligned}$$

We integrate by parts with respect to time in the first summand to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) \\
&= \int_0^{t^*} \int_{\Omega_0} -\frac{\mathrm{d}}{\mathrm{d}t} x_t(X,t) \cdot \widetilde{\chi}(X,t) - W'(\widetilde{F}) : \left(\nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1} \widetilde{F}\right) \\
&\qquad + 2A \left(\nabla_X M_0(X) \widetilde{F}^{-1}\right) : \left(\nabla_X M_0(X) \widetilde{F}^{-1} \nabla_X \widetilde{\chi}(X,t) \widetilde{F}^{-1}\right) \, \mathrm{d}X \, \mathrm{d}t.
\end{aligned}$$

Now, we transform the spatial integral back into the Eulerian coordinate system. Here, we use (2.58) and the push forward formula $\widetilde{\chi}(X,t) = \chi(x(X,t),t)$. We get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) \\
&= \int_0^{t^*} \int_{\Omega} -(v_t(x,t) + (v(x,t) \cdot \nabla) v(x,t)) \cdot \chi(x,t) - (W'(F) F^\top) : \nabla \chi(x,t) \\
&\qquad + 2A \nabla M(x,t) : (\nabla M(x,t) \nabla \chi(x,t)) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Next, we perform an integration by parts with respect to x to isolate χ. The details of these calculations are already carried out above in section 2.7.1. This yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) = \int_0^{t^*} \int_{\Omega} \Bigg(&- (v_t + (v \cdot \nabla) v) + \nabla \cdot \left(W'(F) F^\top\right) \\
&- 2A \nabla \cdot (\nabla M \odot \nabla M) \Bigg) \cdot \chi \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

We successfully isolated χ. Then, we set $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) = 0$ and obtain using the Helmholtz decomposition (see, e.g., [DL00, Chapter IX, Section 1, Propostion 1]) for some $p_1 \in W^{1,2}(\Omega, \mathbb{R})$

$$0 = v_t + (v \cdot \nabla) v + \nabla p_1 - \nabla \cdot \left(W'(F) F^\top\right) + 2A \nabla \cdot \big(\nabla M \odot \nabla M\big).$$

At this point, we rewrite the result and get

$$v_t + (v \cdot \nabla) v + \nabla p_1 - \nabla \cdot \tau = 0 \qquad \text{in } \Omega \times (0, t^*), \tag{2.113}$$

where

$$\tau = W'(F) F^\top - 2A \big(\nabla M \odot \nabla M\big) \tag{2.114}$$

is the total stress as in (2.108).

We proceed with the dissipative part (2.95). Again, we only have a viscosity term as a dissipation related to the velocity. The calculation is almost the same as the one to obtain (2.80) in the general setting. However, since we work under incompressibility conditions here, we use a variation $v+\varepsilon\tilde{v}$ with $\tilde{v}$ being compactly supported, smooth and satisfying $\nabla \cdot \tilde{v} = 0$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \frac{1}{2}\Delta_E(v+\varepsilon\tilde{v}) = \int_\Omega \big(-\nu\Delta v\big)\cdot\tilde{v}\;\mathrm{d}x.$$

At this point, we can again use the Helmholtz decomposition and obtain

$$-\nu\Delta v = \nabla p_2 \tag{2.115}$$

with $p_2 \in W^{1,2}(\Omega,\mathbb{R})$. Finally, we define the total pressure $p := p_1 - p_2$ (see (2.113), (2.115)) and by the force balance law, we put the conservative part (2.113) and the dissipative part (2.115) together to obtain equation (2.107). This concludes the proof. □

2.8.4.2 Effective magnetic field H_{eff} and regularization of F

In the following, we derive additional terms for the incompressible viscoelastic and micromagnetic material in the simplified setting described in Sections 2.8.1–2.8.2. The first term is the effective magnetic field which is obtained as a variational derivative with respect to M of the simplified micromagnetic energy from (2.91), namely

$$W_{\mu\text{mag}}^{\text{simpl.}} = \int_\Omega A|\nabla M|^2 + \frac{1}{4\mu^2}(|M|^2-1)^2\;\mathrm{d}x.$$

The calculations are done analogously to those in Section 2.7.2, so we only give the result here:

$$H_{\text{eff}} = -\frac{\delta W_{\mu\text{mag}}^{\text{simpl.}}}{\delta M} = 2A\Delta M - \frac{1}{\mu^2}(|M|^2-1)M. \tag{2.116}$$

We plug this into the microscopic force balance equation (2.93) to obtain (2.100). Finally, we treat the regularization on F in (2.95). This is done in an analogous way to the treatment of the viscosity part (2.115). We obtain $\kappa\Delta F$ as the regularizing part which we couple with the pure transport equation (2.99) to obtain the force balance equation (2.99').
This establishes the entire system summarized in Section 2.8.3.

3 Existence of weak solutions

This chapter is dedicated to the existence results of weak solutions to the models derived in Chapter 2. The notation is the common notation used for Navier-Stokes equations in [Tem77] and related models in, e.g., [LL95, SL09]. For the notation of the function spaces used throughout this chapter we refer to page viii in the beginning of this work.
We start our analysis of the model in the simplified setting for magnetoelastic materials proposed in Section 2.8.3. In the following, $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$. For convenience, we set $A = \frac{1}{2}$ to find

$$v_t + (v \cdot \nabla)v + \nabla p - \nabla \cdot \tau = \nu \Delta v \qquad \text{in } \Omega \times (0, t^*), \tag{3.1}$$

$$\tau = W'(F)F^\top - \nabla M \odot \nabla M \qquad \text{in } \Omega \times (0, t^*), \tag{3.2}$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega \times (0, t^*), \tag{3.3}$$

$$F_t + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F \qquad \text{in } \Omega \times (0, t^*), \tag{3.4}$$

$$M_t + (v \cdot \nabla)M = \Delta M - \frac{1}{\mu^2}(|M|^2 - 1)M \qquad \text{in } \Omega \times (0, t^*) \tag{3.5}$$

with boundary conditions

$$v = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{3.6}$$

$$F = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{3.7}$$

$$\frac{\partial M}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{3.8}$$

and initial conditions

$$v(x, 0) = v_0(x), \quad \nabla \cdot v_0(x) = 0 \qquad \text{in } \Omega, \tag{3.9}$$

$$F(x, 0) = F_0(x) = I \qquad \text{in } \Omega, \tag{3.10}$$

$$M(x, 0) = M_0(x) \qquad \text{in } \Omega. \tag{3.11}$$

Moreover, we assume that the elastic energy density $W : \mathbb{R}^{d\times d} \to \mathbb{R}_0^+$ satisfies $W(\mathcal{R}\Xi) = W(\Xi)$ for all $\mathcal{R} \in SO(d)$ (and thus $W'(\mathcal{R}\Xi) = \mathcal{R}W'(\Xi)$; see also [LW01]), and the following conditions for some constants $C_1, C_2, C_3, a > 0$, any $\Xi \in \mathbb{R}^{d\times d}$, and any $\Xi_1, \Xi_2 \in \mathbf{H}^1(\Omega; \mathbb{R}^{d\times d})$

$$W \in C^2(\mathbb{R}^{d\times d}; \mathbb{R}), \tag{3.12}$$

$$C_1|\Xi|^2 \le W(\Xi) \le C_1(|\Xi|^2 + 1), \tag{3.13}$$

$$|W'(\Xi)| \le C_2(1 + |\Xi|), \tag{3.14}$$

$$W'(0) = 0, \tag{3.15}$$

$$|W''(\Xi)| \le C_3, \tag{3.16}$$

$$(W''(\Xi_1)\nabla\Xi_2) \,\vdots\, \nabla\Xi_2 \ge a|\nabla\Xi_2|^2 \quad \text{a.e. in } \Omega, \tag{3.17}$$

where $W''(\Xi)\nabla\Xi = \frac{\partial^2 W(\Xi)}{\partial \Xi_{ij}\partial \Xi_{kl}} \nabla_\sigma \Xi_{kl}$, using index notation. Notice that (3.17) is convexity of W (see, e.g., [GH96, Chapter 4, Section 1.3]) which we assume for simplicity. In the proofs of Corollaries 20 and 29, we only need a weaker variant of this condition, namely

$$\int_\Omega (W''(\Xi_1)\nabla\Xi_1) \,\vdots\, \nabla\Xi_1 \,\mathrm{d}x \geq \int_\Omega a|\nabla\Xi_1|^2 \,\mathrm{d}x. \tag{3.17'}$$

In Section 3.1, we prove the existence of weak solutions (see Definition 14 in Section 3.1.1) to this system, summarized in the following theorem:

Theorem 9. *Let $d = 2, 3$. For any $T > 0$, any $v_0 \in \mathbf{H}$, $F_0 \in L^2(\Omega; \mathbb{R}^{d\times d})$, $M_0 \in \mathbf{H}^1(\Omega; \mathbb{R}^3)$ and W satisfying (3.12)–(3.17), the system (3.1)–(3.11) has a weak solution (v, F, M) in $\Omega \times (0, T)$.*

Our approach to the proof of existence is based on the work in [LL95, SL09]. Next to a Galerkin approximation method which is also used for time-dependent Navier-Stokes equations in [Tem77, Chapter III] we use a fixed point argument to establish the existence of weak solutions.
In Section 3.2 the model for the simplified setting gets altered in the sense that the gradient flow dynamics for the magnetization in (3.5) is replaced by the Landau-Lifshitz-Gilbert (LLG) equation and the length constraint $|M| = 1$. We obtain

$$M_t + (v\cdot\nabla)M = -M \times \Delta M - M \times (M \times \Delta M) \tag{3.18}$$

for the microscopic force balance equation. Since we do not relax the length constraint $|M| = 1$ here, the effective magnetic field reduces to $H_{\text{eff}} = \Delta M$ (compare the simplified micromagnetic energy (2.91) and the resulting effective magnetic field (2.116), both including a term accounting for the length constraint).
At this point, we need to comment on the different forms of (2.31) and (3.18). It is a special porperty of the LLG equation that, under certain assumptions on the form of the effective field H_{eff}, which is the case here, the following lemma holds true (this idea is also used in [BPV01, CF01]):

Lemma 10. *If M solves*

$$\begin{cases} M_t + (v\cdot\nabla)M = -M \times \Delta M - M \times (M \times \Delta M) & \text{in } \Omega \times (0, t^*), \\ \frac{\partial M}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0, t^*), \\ M(x, 0) = M_0(x), \quad |M_0| = 1 \text{ a.e.} & \text{in } \Omega, \end{cases} \tag{3.19}$$

where v is divergence-free and vanishes on $\partial\Omega$, then the length of M is conserved, i.e., $|M| = 1$ a.e. in $\Omega \times (0, t^*)$. Moreover, the following equations are equivalent:

$$\begin{aligned} M_t + (v\cdot\nabla)M &= -M \times \Delta M - M \times (M \times \Delta M), && (3.20) \\ M_t + (v\cdot\nabla)M &= -M \times \Delta M + |\nabla M|^2 M + \Delta M, && (3.21) \\ M_t + (v\cdot\nabla)M &= -2M \times \Delta M + M \times (M_t + (v\cdot\nabla)M). && (3.22) \end{aligned}$$

Proof. Firstly, by multiplying $(3.19)_1$ by M we obtain (up to a constant factor of $\frac{1}{2}$)

$$\left(|M|^2\right)_t + (v \cdot \nabla)|M|^2 = 0.$$

Then, we prove that solutions to

$$\begin{cases} \theta_t + (v \cdot \nabla)\theta = 0 & \text{in } \Omega \times (0, t^*), \\ \theta(x, 0) = \theta_0 & \text{a.e. in } \Omega \end{cases}$$

are unique. To this end, let $\theta_1 \neq \theta_2$ be two solutions. Subtracting the respective ODEs, multiplying by $\theta_1 - \theta_2$ and integrating over Ω yields

$$\begin{aligned} & \int_\Omega (\theta_1 - \theta_2)_t \cdot (\theta_1 - \theta_2) \, \mathrm{d}x + \int_\Omega (v \cdot \nabla)(\theta_1 - \theta_2) \cdot (\theta_1 - \theta_2) \, \mathrm{d}x = 0 \\ \iff \quad & \frac{1}{2} \int_\Omega \left(|\theta_1 - \theta_2|^2\right)_t \, \mathrm{d}x + \underbrace{\frac{1}{2} \int_\Omega (v \cdot \nabla)|\theta_1 - \theta_2|^2 \, \mathrm{d}x}_{\overset{\nabla \cdot v = 0}{\equiv} 0} = 0 \\ \iff \quad & \|\theta_1 - \theta_2\|^2_{L^2(\Omega)}(t) = \|\theta_1 - \theta_2\|^2_{L^2(\Omega)}(0) = 0, \end{aligned}$$

which concludes the proof of uniqueness. Since $\theta(x, t) = |M(x, t)|^2$ and the constant solution $\theta(x, t) \equiv 1$ solve this equation, they must be the same a.e. Thus, $|M|^2 = 1$ a.e., which is equivalent to $|M| = 1$ a.e.
Knowing this, we apply the Laplace on both sides of $|M| = 1$ to find out that $M \cdot \Delta M = -|\nabla M|^2$. Thus, we get with the application of the Graßmann identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ for $a, b, c \in \mathbb{R}^3$

$$\begin{aligned} & (3.20) \\ \iff & M_t + (v \cdot \nabla)M = -M \times \Delta M - (M \cdot \Delta M)M + \Delta M \\ \iff & M_t + (v \cdot \nabla)M = -M \times \Delta M + |\nabla M|^2 M + \Delta M \\ \iff & (3.21). \end{aligned}$$

The next equivalence is a bit more involved: Since $M \times M = 0$, we have

$$\begin{aligned} & (3.20) \\ \iff & M_t + (v \cdot \nabla)M = -M \times \Delta M - M \times (\underbrace{-|\nabla M|^2 M + M \times \Delta M}_{\overset{(3.21)}{=} -(M_t + (v \cdot \nabla)M) + \Delta M}) \\ \iff & M_t + (v \cdot \nabla)M = -M \times \Delta M + M \times (M_t + (v \cdot \nabla)M) - M \times \Delta M \\ \iff & (3.22). \end{aligned}$$

This concludes the proof of the lemma. □

The proof of existence of weak solutions to the system including the LLG equation then involves methods from the existence theory used for the LLG equation alone, i.e., not coupled to elastic behavior in materials. Here, we apply ideas from [CF01] involving also the results from Lemma 10 in order to adapt the proof from Section 3.1 to work for the system with LLG dynamics. In the setting where we analyze the system including the LLG equation, we consider the case where $d = 2$ only. This is due to the estimates (3.187)–(3.190) used in the proof of Lemma 26 to ensure estimate (3.167) needed to extend the approximate solution of the magnetization while keeping its $\mathbf{H}^2$-regularity, and due to the Sobolev estimate (3.227), valid only for $d = 2$. It is applied in the proof of uniform energy estimates in Corollary 29. The setting is comparable to the situation in [LLW10], where the authors proof existence and regularity of global weak solutions for liquid crystals: the domain is also two-dimensional and the liquid crystals are vectors on S^2, the unit sphere in $\mathbb{R}^3$. The governing dynamics in the liquid crystal case naturally differ from the LLG equation considered in our magnetic case. The existence result for weak solutions (see Definition 24 in Section 3.2.1) is summarized in

Theorem 11. *Let $d = 2$. For any $T > 0$, any $v_0 \in \mathbf{H}$, $F_0 \in L^2(\Omega;\mathbb{R}^{2\times 2})$, $M_0 \in \mathbf{H}^2(\Omega;S^2)$ satisfying*

$$\|v_0\|^2_{L^2(\Omega)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega)} < \frac{1}{C(\Omega)} \tag{3.23}$$

for some constant $C(\Omega)$ and W satisfying (3.12)–(3.17)*, the system* (3.1)–(3.4)*,* (3.18)*,* (3.6)–(3.11) *has a weak solution (v, F, M) in $\Omega \times (0, T)$.*

The corresponding proof is presented in Section 3.2.

Remark 12. *The smallness condition* (3.23) *is there to ensure $\mathbf{H}^2$-regularity of the magnetization. Notice that there is also a smallness condition on the initial data in [LLW10] to ensure regularity.*

Remark 13. *In the following sections, we focus on the existence of weak solutions. The reconstruction of the pressure p is not in scope of this work.*

3.1 System for simplified setting including magnetic gradient flow

In this section, we present the proof of Theorem 9. In the entire section, $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$.

3.1.1 Definition of a weak solution

At first, we need to define the notion of a weak solution to the system (3.1)–(3.11). We multiply equations (3.1) and (3.4)–(3.5) by test functions $\zeta \in W^{1,\infty}(0, t^*; \mathbb{R})$ with $\zeta(t^*) = 0$ and $\xi \in \mathbf{V}$, $\Xi \in \mathbf{H}_0^1(\Omega; \mathbb{R}^{d\times d})$, $\varphi \in \mathbf{H}^1(\Omega; \mathbb{R}^3)$, respectively, integrate over time and space and obtain via integrations by parts

$$\int_0^{t^*} \int_\Omega -v \cdot (\zeta'\xi) + (v \cdot \nabla)v \cdot (\zeta\xi) + \Big(W'(F)F^\top - \nabla M \odot \nabla M\Big) : (\zeta\nabla\xi) \, \mathrm{d}x \, \mathrm{d}t \\ - \int_\Omega v(0) \cdot (\zeta(0)\xi) \, \mathrm{d}x = - \int_0^{t^*} \int_\Omega \nu\nabla v : (\zeta\nabla\xi) \, \mathrm{d}x \, \mathrm{d}t, \tag{3.24}$$

$$\int_0^{t^*} \int_\Omega -F : (\zeta'\Xi) + (v \cdot \nabla)F : (\zeta\Xi) - (\nabla vF) : (\zeta\Xi) \, \mathrm{d}x \, \mathrm{d}t \\ - \int_\Omega F(0) : (\zeta(0)\Xi) \, \mathrm{d}x = - \int_0^{t^*} \int_\Omega \kappa\nabla F \,\vdots\, (\zeta\nabla\Xi) \, \mathrm{d}x \, \mathrm{d}t, \tag{3.25}$$

$$\int_0^{t^*} \int_\Omega -M \cdot (\zeta'\varphi) + (v \cdot \nabla)M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega M(0) \cdot (\zeta(0)\varphi) \, \mathrm{d}x \\ = \int_0^{t^*} \int_\Omega -\nabla M : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M|^2 - 1)M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t. \tag{3.26}$$

Now, we are able to give a definition of the weak solution:

Definition 14. *The triple (v, F, M) is called a weak solution to the system* (3.1)–(3.11) *in $\Omega \times [0, t^*]$ provided that*

$$v \in L^\infty(0, t^*; \mathbf{H}) \cap L^2(0, t^*; \mathbf{V}),$$
$$F \in L^\infty(0, t^*; L^2(\Omega; \mathbb{R}^{d\times d})) \cap L^2(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^{d\times d})),$$
$$M \in L^\infty(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^3)) \cap L^2(0, t^*; \mathbf{H}^2(\Omega; \mathbb{R}^3)),$$

and if it satisfies (3.24)–(3.26) *together with the boundary conditions* (3.6)–(3.8) *in the sense of traces (see, e.g., [Eva02, Section 5.5]) and the initial conditions* (3.9)–(3.11) *in the sense*

$$v(\cdot, t) \xrightarrow{w-L^2(\Omega)} v_0(\cdot), \quad F(\cdot, t) \xrightarrow{w-L^2(\Omega)} F_0(\cdot), \quad M(\cdot, t) \xrightarrow{w-\mathbf{H}^1(\Omega)} M_0(\cdot) \quad \text{as } t \to 0^+.$$

3.1.2 Galerkin approximation: definition of the approximate problem

In this section, we discretize the PDE for the velocity by means of the Galerkin method following [LL95]. To this end, we construct solutions to approximate problems by means of a projection onto finite dimensional subspaces of $\mathbf{H}$.
Let $\{\xi_i\}_{i=1}^{\infty} \subset C^\infty(\overline{\Omega}; \mathbb{R}^d)$ be an orthonormal basis of $\mathbf{H}$ and an orthogonal basis of $\mathbf{V}$ satisfying

$$\Delta \xi_i + \nabla p_i = -\lambda_i \xi_i \tag{3.27}$$

in Ω and vanishing on the boundary. Here, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$ with $\lambda_m \xrightarrow{m\to\infty} \infty$. The functions ξ_i, $i \in \mathbb{N}$, are eigenfunctions of the Stokes operator (existence of these functions can be shown by means of the Hilbert-Schmidt theorem, see, e.g., [RR04, Theorem 8.94], with a method similar to the one used in [Eva02, Section 6.5.1]). The reason why we consider this particular basis is that the ODE (3.44) below has a linear first term which is due to (3.27).
Now, let

$$\mathbf{H}_m := \operatorname{span}\{\xi_1, \xi_1, \ldots, \xi_m\} \tag{3.28}$$

and

$$P_m : \mathbf{H} \to \mathbf{H}_m \tag{3.29}$$

be the orthonormal projection. We consider an approximate problem which is obtained from the original problem, now considered for functions $v_m \in \mathbf{H}_m$:

$$\begin{aligned}
(v_m)_t = P_m\Big(&\nu\Delta v_m - (v_m\cdot\nabla)v_m \\
&+ \nabla\cdot\big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big)\Big) && \text{in } \Omega\times(0,t^*), && (3.30)\\
v_m \in \mathbf{H}_m &\Longrightarrow \nabla\cdot v_m = 0, && && (3.31)\\
(F_m)_t + (v_m\cdot\nabla)F_m - \nabla v_m F_m &= \kappa\Delta F_m && \text{in } \Omega\times(0,t^*), && (3.32)\\
(M_m)_t + (v_m\cdot\nabla)M_m &= \Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m && \text{in } \Omega\times(0,t^*), && (3.33)\\
v_m &= 0 && \text{on } \partial\Omega\times(0,t^*), && (3.34)\\
F_m &= 0 && \text{on } \partial\Omega\times(0,t^*), && (3.35)\\
\frac{\partial M_m}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega\times(0,t^*), && (3.36)\\
v_m(x,0) &= P_m(v_0(x)) && \text{in } \Omega, && (3.37)\\
F_m(x,0) &= F_0(x) = I && \text{in } \Omega, && (3.38)\\
M_m(x,0) &= M_0(x) && \text{in } \Omega. && (3.39)
\end{aligned}$$

This system is meant to hold in a weak sense, i.e., boundary and initial conditions (3.34)–(3.39) hold and the following integral equations are satisfied

$$\int_\Omega (v_m)_t \cdot \xi + (v_m \cdot \nabla) v_m \cdot \xi + \big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big) : \nabla \xi \,\mathrm{d}x$$
$$= - \int_\Omega \nu \nabla v_m : \nabla \xi \,\mathrm{d}x, \tag{3.40}$$

$${}_{\mathbf{H}^{-1}}\Big\langle (F_m)_t, \Xi \Big\rangle_{\mathbf{H}_0^1} + \int_\Omega (v_m \cdot \nabla) F_m : \Xi - (\nabla v_m F_m) : \Xi \,\mathrm{d}x$$
$$= - \int_\Omega \kappa \nabla F_m \,\vdots\, \nabla \Xi \,\mathrm{d}x, \tag{3.41}$$

$$\int_\Omega (M_m)_t \cdot \varphi + (v_m \cdot \nabla) M_m \cdot \varphi \,\mathrm{d}x$$
$$= \int_\Omega \Delta M_m \cdot \varphi - \frac{1}{\mu^2}(|M_m|^2 - 1) M_m \cdot \varphi \,\mathrm{d}x, \tag{3.42}$$

for a.e. t, where $\xi \in \mathbf{V} \cap \mathbf{H}_m = \mathbf{H}_m$ (the equality holds due to the smoothness of $\{\xi_i\}_{i=1}^\infty$), $\Xi \in \mathbf{H}_0^1(\Omega; \mathbb{R}^{d\times d})$, $\varphi \in L^2(\Omega; \mathbb{R}^3)$.

3.1.3 Galerkin approximation: existence of weak solutions to the approximate problem

First, we define the notion of a weak solution to the approximate problem.

Definition 15. *We call* (v_m, F_m, M_m) *a weak solution to the system* (3.30)–(3.39) *provided that*

$$\begin{aligned} v_m &\in L^\infty(0, t^*; \mathbf{H}) \cap L^2(0, t^*; \mathbf{V}), \\ F_m &\in L^\infty(0, t^*; L^2(\Omega; \mathbb{R}^{d\times d})) \cap L^2(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^{d\times d})) \\ M_m &\in L^\infty(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^3)) \cap L^2(0, t^*; \mathbf{H}^2(\Omega; \mathbb{R}^3)) \end{aligned}$$

and that the system (3.30)–(3.39) *is satisfied in the weak sense* (3.40)–(3.42)*.*

The following theorem states that the approximate problem has indeed a weak solution.

Theorem 16. *For any* $0 < T < \infty$ *and any* $m > 0$, $v_0 \in \mathbf{H}$, $F_0 \in L^2(\Omega; \mathbb{R}^{d\times d})$, $M_0 \in \mathbf{H}^1(\Omega; \mathbb{R}^3)$ *and* W *satisfying* (3.12)–(3.17)*, the system* (3.30)–(3.39) *has a weak solution* (v_m, F_m, M_m) *in* $\Omega \times (0, T)$*.*

In the following, we prepare the proof of Theorem 16. To this end, we first relate the approximate equation of motion (3.40) to an ODE system. Since we look for a solution v_m satisfying $v_m(\cdot, t) \in \mathbf{H}_m$ for all $t \in (0, T)$, we write

$$v_m(x, t) = \sum_{i=1}^m g_m^i(t) \xi_i(x). \tag{3.43}$$

We plug this discretization into (3.40). For its left-hand side we obtain, setting $\xi = \xi_i$ which is orthonormal in $L^2(\Omega;\mathbb{R}^d)$ to every ξ_j, $j \neq i$,

$$
\begin{aligned}
&\int_\Omega \left(\sum_{j=1}^m g_m^j(t)\xi_j(x)\right)_t \cdot \xi_i(x)\,\mathrm{d}x \\
&\quad + \int_\Omega \left(\left(\sum_{j=1}^m g_m^j(t)\xi_j(x)\right)\cdot\nabla\right)\left(\sum_{k=1}^m g_m^k(t)\xi_k(x)\right)\cdot \xi_i(x)\,\mathrm{d}x \\
&\quad + \int_\Omega \big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big) : \nabla\xi_i(x)\,\mathrm{d}x \\
=&\sum_{j=1}^m \frac{\mathrm{d}}{\mathrm{d}t} g_m^j(t) \underbrace{\int_\Omega \xi_j(x)\cdot\xi_i(x)\,\mathrm{d}x}_{=\delta_{ij}} \\
&\quad + \sum_{j,k=1}^m g_m^j(t) g_m^k(t) \int_\Omega (\xi_j(x)\cdot\nabla)(\xi_k(x))\cdot\xi_i(x)\,\mathrm{d}x \\
&\quad + \int_\Omega \big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big) : \nabla\xi_i(x)\,\mathrm{d}x.
\end{aligned}
$$

For the right-hand side of (3.40) we obtain, setting $\xi = \xi_i$, and using integration by parts and (3.27),

$$
\begin{aligned}
-\int_\Omega \nu\nabla v_m : \nabla\xi_i(x)\,\mathrm{d}x &= \int_\Omega \nu\Delta v_m \cdot \xi_i(x)\,\mathrm{d}x \\
&= \int_\Omega \nu\Delta\left(\sum_{j=1}^m g_m^j(t)\xi_j(x)\right)\cdot\xi_i(x)\,\mathrm{d}x \\
&= \sum_{j=1}^m g_m^j(t)\int_\Omega \nu \underbrace{\Delta\xi_j(x)}_{=-\nabla p_j - \lambda_j\xi_j} \cdot\xi_i(x)\,\mathrm{d}x \\
&= -\sum_{j=1}^m g_m^j(t)\Bigg(\underbrace{\int_\Omega \nu\nabla p_j\cdot\xi_i(x)\,\mathrm{d}x}_{\overset{\nabla\cdot\xi_j=0}{=}0} + \int_\Omega \nu\lambda_j\xi_j\cdot\xi_i(x)\,\mathrm{d}x\Bigg) \\
&= -\sum_{j=1}^m \nu\lambda_j g_m^j(t)\underbrace{\int_\Omega \xi_j(x)\cdot\xi_i(x)\,\mathrm{d}x}_{=\delta_{ij}} = -\nu\lambda_i g_m^i(t).
\end{aligned}
$$

We put both parts together and obtain from there the ODE system

$$
\frac{\mathrm{d}}{\mathrm{d}t} g_m^i(t) = -\nu\lambda_i g_m^i(t) + \sum_{j,k=1}^m g_m^j(t) g_m^k(t) A_{jk}^i + D_m^i(t), \qquad i = 1,\dots,m, \quad (3.44)
$$

where

$$A^i_{jk} \quad = \quad -\int_\Omega (\xi_j(x)\cdot\nabla)\xi_k(x)\cdot\xi_i(x)\;\mathrm{d}x, \tag{3.45}$$

$$D^i_m(t) \quad = \quad -\int_\Omega \big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big) : \nabla\xi_i\;\mathrm{d}x. \tag{3.46}$$

The first summand on the right-hand side of (3.44) is due to the fact that the ξ_i are eigenvectors of the Stokes operator satisfying (3.27). The term ∇p_i vanishes with an integration by parts, since ξ_i is divergence-free. Moreover, from (3.37) we obtain the initial condition

$$g^i_m(0) = \int_\Omega v_0(x)\cdot\xi_i(x)\;\mathrm{d}x \tag{3.47}$$

for $i = 1,\ldots,m$.

3.1.3.1 Weak solutions to the sub-problem

We are not yet able to construct a solution v_m. But, for a fixed velocity v the following lemma provides us with unique weak solutions to the PDEs for the deformation gradient F and the magnetization M. These solutions are then used to solve for the velocity in the balance of momentum equation in a next step.

Lemma 17. *For $v \in L^\infty(0,t^*;W^{2,\infty}(\Omega))$ satisfying $v = 0$ on $\partial\Omega\times(0,t^*)$ and $v(x,0) = v_0(x)$ and $\nabla\cdot v = 0$, there exists a time $0 < \tilde{t} \le t^*$ such that the system*

$$\begin{aligned}
&F_t + (v\cdot\nabla)F - \nabla vF = \kappa\Delta F && \text{in } \Omega\times(0,\tilde{t}),\\
&M_t + (v\cdot\nabla)M = \Delta M - \frac{1}{\mu^2}(|M|^2-1)M && \text{in } \Omega\times(0,\tilde{t}),\\
&F = 0 && \text{on } \partial\Omega\times(0,\tilde{t}),\\
&\frac{\partial M}{\partial\mathbf{n}} = 0 && \text{on } \partial\Omega\times(0,\tilde{t}),\\
&F(x,0) = F_0(x) = I && \text{in } \Omega,\\
&M(x,0) = M_0(x) && \text{in } \Omega
\end{aligned}$$

has a unique weak solution such that

$$\begin{aligned}
&\|F\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^{d\times d}))} + \|F\|_{L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^{d\times d}))} + \|F_t\|_{L^2(0,\tilde{t};\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d}))} \le C(v),\\
&\|M\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} + \|M\|_{L^4(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))} + \|M\|_{L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3))} \le C,\\
&\|M\|_{L^\infty(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3))} + \|M\|_{L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))}\\
&\quad + \|M\|_{\mathbf{H}^1(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} + \|M\|_{L^\infty(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))} \le C(v),
\end{aligned}$$

where $C(v) = C(\|v\|_{L^\infty(0,\tilde{t};W^{2,\infty}(\Omega;\mathbb{R}^d))})$ and C is independent of v.

Proof. Notice that the partial differential equations are decoupled. Consequently, we can prove existence separately.

Existence of a weak solution to the F-equation. This is again done by a Galerkin approximation. To this end, let $\{\Xi_i\}_{i=1}^{\infty} \subset C^{\infty}(\overline{\Omega}; \mathbb{R}^{d\times d})$ be an orthonormal basis of $L^2(\Omega; \mathbb{R}^{d\times d})$ and an orthogonal basis of $H_0^1(\Omega; \mathbb{R}^{d\times d})$ satisfying

$$\Delta \Xi_i = -\mu_i \Xi_i \tag{3.48}$$

in Ω and vanishing on the boundary. Here, $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$ with $\mu_n \xrightarrow{n\to\infty} \infty$ (existence of these functions can be shown by means of the Hilbert-Schmidt theorem, see, e.g., [RR04, Theorem 8.94], with a method similar to the one used in [Eva02, Section 6.5.1]).
Let

$$L_n^2 := \operatorname{span}\{\Xi_1, \Xi_2, \ldots, \Xi_n\} \tag{3.49}$$

and

$$\overline{P}_n : L^2(\Omega; \mathbb{R}^{d\times d}) \to L_n^2 \tag{3.50}$$

be the orthonormal projection. We consider the original problem for functions in L_n^2 and show existence of a weak solution to

$$F_t + \overline{P}_n\big[(v\cdot\nabla)F - \nabla v F\big] = \kappa \Delta F \qquad \text{in } \Omega \times (0, t^*), \tag{3.51}$$

$$F = 0 \qquad \text{on } \partial\Omega \times (0, t^*), \tag{3.52}$$

$$F(x, 0) = \overline{P}_n(F_0(x)) = I \qquad \text{in } \Omega. \tag{3.53}$$

For a fixed $n \in \mathbb{N}$, we look for a function $F_n : [0, t^*] \to L_n^2$ of the form

$$F_n(x, t) = \sum_{i=1}^{n} d_n^i(t) \Xi_i(x). \tag{3.54}$$

The solution must satisfy (3.41), so, we plug the discretization for F_n into this equation to obtain for $\Xi = \Xi_i$ the ODE system (the derivation is similar to (3.44))

$$\frac{\mathrm{d}}{\mathrm{d}t} d_n^i(t) = -\kappa \mu_i d_n^i(t) + \sum_{j=1}^{n} d_n^j(t) \tilde{A}_j^i(t), \qquad i = 1, \ldots, n, \tag{3.55}$$

where

$$\tilde{A}_j^i(t) = -\int_{\Omega} (v(x,t)\cdot\nabla)\Xi_j(x) : \Xi_i(x) - (\nabla v(x,t)\Xi_j(x)) : \Xi_i(x) \; \mathrm{d}x. \tag{3.56}$$

The initial condition becomes

$$d_n^i(0) = \int_{\Omega} F_0(x) : \Xi_i(x) \; \mathrm{d}x \tag{3.57}$$

for $i = 1, \ldots, n$. We apply Carathéodory's existence theorem (see Theorem 30 in Appendix A.2) to obtain a solution $d_n^i(t)$ of (3.55).

Since the first summand on the right-hand side of (3.55) does not depend on t (looking at t and d_n^i as distinct variables) and the second summand is measurable in t, the entire right-hand side is measurable in t for any d_n^i.
Furthermore, the terms on the right-hand side of (3.55) are linear in d_n^i, so the right-hand side is continuous in d_n^i for a.e. t.
In addition, for $t \in [0, t^*]$ and $\|d_n - d_n(0)\| \leq \tilde{b}$, where $d_n = (d_n^1, \ldots, d_n^n)$, we can bound the right-hand side of (3.55) by the L^1-function

$$(2\tilde{b} + \|d_n(0)\|) \left(-\kappa\mu_i + \sum_{j=1}^{n} \tilde{A}_j^i \right).$$

Finally, Carathéodory's theorem yields the existence of a value $\tilde{t}$ with $0 < \tilde{t} \leq t^*$ such that the ODE system (3.55) has a unique (since the right-hand side of the ODE is locally Lipschitz, see Theorem 31 in Appendix A.2) and absolutely continuous solution $\{d_n^i(t)\}_{i=1}^n$ on $[0, \tilde{t}]$ satisfying (3.57).
We prepare the passage to the limit as $n \to \infty$ with a priori estimates. To this end, we first multiply

$$(F_n)_t + (v \cdot \nabla)F_n - \nabla v F_n = \kappa \Delta F_n \tag{3.58}$$

by F_n (which is the solution obtained with $\{d_n^i(t)\}_{i=1}^n$ from (3.54)) and integrate over both Ω and $[0, t]$ for $t \leq \tilde{t}$ to find

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |F_n|^2 \, \mathrm{d}x \\
&= - \int_0^t \underbrace{\int_\Omega (v \cdot \nabla) \frac{|F_n|^2}{2} \, \mathrm{d}x}_{\overset{\nabla \cdot v = 0}{=} 0} \, \mathrm{d}s + \int_0^t \int_\Omega \nabla v : (F_n F_n^\top) \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad - \int_0^t \int_\Omega \kappa |\nabla F_n|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |\overline{P}_n(F_0)|^2 \, \mathrm{d}x.
\end{aligned}$$

We rearrange and, since $|\overline{P}_n(F_0)| \leq |F_0|$, we obtain

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |F_n|^2 \, \mathrm{d}x + \int_0^t \int_\Omega \kappa |\nabla F_n|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&= \int_0^t \int_\Omega \nabla v : (F_n F_n^\top) \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |\overline{P}_n(F_0)|^2 \, \mathrm{d}x \\
&\leq \int_0^t \int_\Omega |\nabla v : (F_n F_n^\top)| \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |F_0|^2 \, \mathrm{d}x \\
&\leq \underbrace{\|\nabla v\|_{L^\infty(0,T;L^\infty(\Omega))}}_{\leq C(v)} \int_0^t \int_\Omega \underbrace{|F_n F_n^\top|}_{=|F_n|^2} \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |F_0|^2 \, \mathrm{d}x.
\end{aligned} \tag{3.59}$$

Applying Gronwall's inequality yields

$$\begin{aligned}
\int_\Omega |F_n|^2(t) \, \mathrm{d}x &\leq \left(\frac{1}{2} \int_\Omega |F_0|^2 \, \mathrm{d}x \right) \mathrm{e}^{\|\nabla v\|_{L^\infty(0,T;L^\infty(\Omega))} t} \\
&\leq \left(\frac{1}{2} \int_\Omega |F_0|^2 \, \mathrm{d}x \right) \mathrm{e}^{C(v) t}
\end{aligned} \tag{3.60}$$

and then, by taking the supremum over all $t \in [0,T]$ and since $\tilde{t} \leq t^*$ is bounded, we get

$$\sup_{0 \leq t \leq \tilde{t}} \|F_n\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(t) \leq C(v)\|F_0\|^2_{L^2(\Omega;\mathbb{R}^{d\times 3})}. \tag{3.61}$$

This gives us the bound

$$\|F_n\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^{d\times d}))} \leq C(v). \tag{3.62}$$

Moreover, from (3.59) and (3.62) we see that

$$\|F_n\|_{L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^{d\times d}))} \leq C(v). \tag{3.63}$$

Next, we estimate the time derivative $(F_n)_t$ in $L^2(0,\tilde{t};\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d}))$, using $\|\overline{P}_n(\Xi)\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})} \leq \|\Xi\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})} \leq 1$,

$$\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1 \\ \|\Xi\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_0^{\tilde{t}} {}_{\mathbf{H}^{-1}}\Big\langle (F_n)_t, \Xi \Big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1 \\ \|\Xi\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_0^{\tilde{t}} {}_{\mathbf{H}^{-1}}\Big\langle (F_n)_t, \overline{P}_n(\Xi) \Big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1 \\ \|\Xi\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_0^{\tilde{t}} \int_\Omega -(v\cdot\nabla)F_n : (\zeta\overline{P}_n(\Xi)) + (\nabla v F_n) : (\zeta\overline{P}_n(\Xi)) \\
&\qquad\qquad - \kappa\nabla F_n \,\vdots\, (\zeta\nabla\overline{P}_n(\Xi)) \,\mathrm{d}x \,\mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1 \\ \|\Xi\|_{\mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_0^{\tilde{t}} \|v\|_{L^\infty(\Omega;\mathbb{R}^d)}\|\nabla F_n\|_{L^2(\Omega;\mathbb{R}^{d\times d\times d})}|\zeta|\|\overline{P}_n(\Xi)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
&\qquad\qquad + \|\nabla v\|_{L^\infty(\Omega;\mathbb{R}^d)}\|F_n\|_{L^2(\Omega;\mathbb{R}^{d\times d})}|\zeta|\|\overline{P}_n(\Xi)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
&\qquad\qquad + \kappa\|\nabla F_n\|_{L^2(\Omega;\mathbb{R}^{d\times d})}|\zeta|\|\nabla\overline{P}_n(\Xi)\|_{L^2(\Omega;\mathbb{R}^{d\times d\times d})} \,\mathrm{d}t \\
&\underset{\text{Young}}{\overset{\text{Hölder}}{\leq}} \sup_{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1} \Bigg(\|v\|_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^d))} \int_0^{\tilde{t}} \frac{1}{2}\|\nabla F_n\|^2_{L^2(\Omega;\mathbb{R}^{d\times d\times d})} + \frac{1}{2}|\zeta|^2 \,\mathrm{d}t \\
&\qquad\qquad + \|\nabla v\|_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^{d\times d}))} \int_0^{\tilde{t}} \frac{1}{2}\|F_n\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \frac{1}{2}|\zeta|^2 \,\mathrm{d}t \\
&\qquad\qquad + \int_0^{\tilde{t}} \frac{\kappa}{2}\|\nabla F_n\|^2_{L^2(\Omega;\mathbb{R}^{d\times d\times d})} + \frac{\kappa}{2}|\zeta|^2 \,\mathrm{d}t \Bigg) \\
&\overset{(3.63)}{\leq} C(v).
\end{aligned}$$

In summary, we get from the above estimate

$$\|(F_n)_t\|_{L^2(0,\tilde{t};\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d}))} \leq C(v). \tag{3.64}$$

From the preceding estimates, we see that there is a subsequence (never relabeled!) satisfying the convergence results below (see Theorem 33 in Appendix A.2).

$$F_n \rightharpoonup F \qquad \text{in } L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{d\times d})), \tag{3.65}$$
$$(F_n)_t \rightharpoonup (F)_t \qquad \text{in } L^2(0,\tilde{t};\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d})), \tag{3.66}$$
$$\nabla F_n \rightharpoonup \nabla F \qquad \text{in } L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{d\times d\times d})). \tag{3.67}$$

Since the weak solution to the approximate problem is defined using test functions from the projected spaces L^2_n, we need to pass to the limit with these particular test functions (only in space), too. However, for any test function $\Xi \in \mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})$ we use the sequence of approximate test functions defined by $\Xi_n := \overline{P}_n(\Xi) \in L^2_n$ which converges strongly to Ξ in $\mathbf{H}^1(\Omega;\mathbb{R}^{d\times d})$. In the following, we will use this particular sequence of test functions. Moreover, let $\zeta \in W^{1,\infty}(0,\tilde{t})$. Then, it is clear that the equation

$$\int_0^{\tilde{t}} {}_{\mathbf{H}^{-1}}\Big\langle (F_n)_t, \Xi_n \Big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t + \int_0^{\tilde{t}} \int_\Omega (v\cdot\nabla)F_n : (\zeta\Xi_n) - (\nabla v F_n) : (\zeta\Xi_n) \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^{\tilde{t}} \int_\Omega \nabla F_n \,\vdots\, (\zeta\nabla\Xi_n) \,\mathrm{d}x\,\mathrm{d}t$$

converges to the equation

$$\int_0^{\tilde{t}} {}_{\mathbf{H}^{-1}}\Big\langle F_t, \Xi \Big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t + \int_0^{\tilde{t}} \int_\Omega (v\cdot\nabla)F : (\zeta\Xi) - (\nabla v F) : (\zeta\Xi) \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^{\tilde{t}} \int_\Omega \nabla F \,\vdots\, (\zeta\nabla\Xi) \,\mathrm{d}x\,\mathrm{d}t,$$

where $\zeta \in L^2(0,\tilde{t})$ and $\Xi \in \mathbf{H}^1_0(\Omega;\mathbb{R}^{d\times d})$, as $n \to \infty$. All the integral terms are linear, so the weak convergences from above together with the strong convergence of the test functions yield the convergence of the entire equation. Thus, we obtain a weak solution to the system (3.51)–(3.53).

Notice that the estimates (3.62), (3.63) and (3.64) for the approximate solution obtained above still hold in the limit, since norms are weakly lower semicontinuous.

Furthermore, the solution is unique. This can be seen directly from (3.61) and the linearity of the problem which yields that a solution for initial data being equal to zero is itself identically zero.

Existence of a weak solution to the M-equation. For the Galerkin approximation, let $\{\eta_i\}_{i=1}^\infty \subset C^\infty(\overline{\Omega};\mathbb{R}^3)$ be an orthonormal basis of $L^2(\Omega;\mathbb{R}^3)$ and an orthogonal basis of $\mathbf{H}^2_{\mathbf{n}}(\Omega;\mathbb{R}^3)$ (for details on this space and the basis, including existence, we refer to Appendix A.6) satisfying

$$\Delta^2\eta_i + \eta_i = \tilde{\mu}_i\eta_i \tag{3.68}$$

in Ω and $\frac{\partial \eta_i}{\partial \mathbf{n}} = 0$ and $\frac{\partial \Delta \eta_i}{\partial \mathbf{n}} = 0$ in a weak sense on the boundary. Here, it holds that $0 < \tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots \leq \tilde{\mu}_n \leq \cdots$ with $\tilde{\mu}_n \xrightarrow{n\to\infty} \infty$.
Let

$$\widetilde{L}^2_n := \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_n\} \tag{3.69}$$

and

$$\widetilde{P}_n : L^2(\Omega; \mathbb{R}^3) \to \widetilde{L}^2_n \tag{3.70}$$

be the orthonormal projection. We consider the original problem for functions in $\widetilde{L}^2_n$ and finally show existence of a unique weak solution to

$$M_t = \widetilde{P}_n\Big[-(v\cdot\nabla)M + \Delta M - \frac{1}{\mu^2}(|M|^2-1)M\Big] \qquad \text{in } \Omega\times(0,t^*), \tag{3.71}$$

$$\frac{\partial M}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega\times(0,t^*), \tag{3.72}$$

$$M(x,0) = \widetilde{P}_n(M_0(x)) \qquad \text{in } \Omega. \tag{3.73}$$

For a fixed $n \in \mathbb{N}$, we look for a function $M_n : [0,t^*] \to \widetilde{L}^2_n$ of the form

$$M_n(x,t) = \sum_{i=1}^{n} h^i_n(t)\eta_i(x). \tag{3.74}$$

The solution must satisfy (3.42), so we plug the discretization M_n into this equation to obtain for $\varphi = \eta_i$ the ODE system (the derivation is similar to (3.44))

$$\frac{\mathrm{d}}{\mathrm{d}t}h^i_n(t) = \frac{1}{\mu^2}h^i_n(t) + \sum_{j=1}^{n} h^j_n(t)\hat{A}^i_j(t) + \sum_{j,k,l=1}^{n} h^j_n(t)h^k_n(t)h^l_n(t)\hat{B}^i_{jkl}, \qquad i = 1,\ldots,n, \tag{3.75}$$

where

$$\hat{A}^i_j(t) = -\int_\Omega \big((v(x,t)\cdot\nabla)\eta_j(x) - \Delta\eta_j(x)\big)\cdot\eta_i(x)\,\mathrm{d}x, \tag{3.76}$$

$$\hat{B}^i_{jkl} = -\int_\Omega (\eta_k(x)\cdot\eta_j(x))(\eta_l(x)\cdot\eta_i(x))\,\mathrm{d}x. \tag{3.77}$$

The initial condition becomes

$$h^i_n(0) = \int_\Omega M_0(x)\cdot\eta_i(x)\,\mathrm{d}x, \qquad i = 1,\ldots,n. \tag{3.78}$$

We apply Carathéodory's existence theorem again to obtain a solution $h^i_n(t)$ of (3.75).
Since the first and the third summand on the right-hand side of (3.75) are not depending on t (looking at t and h^i_n as distinct variables) and the dependence on t of the second summand is just within a Lipschitz function, the right-hand side is measurable in t for any h^i_n.
Furthermore, the terms on the right-hand side of (3.75) are linear and cubic in

h_n^i, so the right-hand side is continuous in h_n^i for any t.
In addition, for $t \in [0, t^*]$ and $\|h_n - h_n(0)\| \leq \hat{b}$, where $h_n = (h_n^1, \dots, h_n^n)$, we can bound the right-hand side of (3.75) by the L^1-function

$$(2\hat{b} + \|h_n(0)\|) \left(\frac{1}{\mu^2} + \sum_{j=1}^{n} \hat{A}_j^i \right) + (2\hat{b} + \|h_n(0)\|)^3 \sum_{j,k,l=1}^{n} \hat{B}_{jkl}^i(t).$$

Finally, Carathéodory's theorem yields the existence of a value $\tilde{t}$ with $0 < \tilde{t} \leq t^*$ such that the ODE system (3.75) has a unique (since the right-hand side of the ODE is locally Lipschitz) and absolutely continuous solution $\{h_n^i(t)\}_{i=1}^n$ on $[0, \tilde{t}]$ satisfying (3.78).
Now, we prepare the passage to the limit as $n \to \infty$ with uniform estimates. To this end, we first multiply

$$(M_n)_t + (v \cdot \nabla) M_n = \Delta M_n - \frac{1}{\mu^2}(|M_n|^2 - 1) M_n \tag{3.79}$$

by M_n (which is the solution obtained with $\{h_n^i(t)\}_{i=1}^n$ from (3.74)) and integrate over both Ω and $[0, t]$ for $t \leq \tilde{t}$ to find

$$\begin{aligned}
\frac{1}{2} \int_\Omega |M_n|^2 \, \mathrm{d}x \quad &= \quad - \int_0^t \underbrace{\int_\Omega (v \cdot \nabla) \frac{|M_n|^2}{2} \, \mathrm{d}x}_{\overset{\nabla \cdot v = 0}{=} 0} \, \mathrm{d}s - \int_0^t \int_\Omega |\nabla M_n|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \frac{1}{\mu^2}(|M_n|^2 - 1)|M_n|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |\widetilde{P}_n(M_0)|^2 \, \mathrm{d}x \\
&\overset{\text{Young}}{\leq} \quad - \int_0^t \int_\Omega |\nabla M_n|^2 \, \mathrm{d}x \, \mathrm{d}s - \frac{1}{\mu^2} \int_0^t \int_\Omega |M_n|^4 \, \mathrm{d}x \, \mathrm{d}s \\
&\quad + \frac{1}{\mu^2} \int_0^t \int_\Omega \frac{1}{2} + \frac{1}{2}|M_n|^4 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_\Omega |\widetilde{P}_n(M_0)|^2 \, \mathrm{d}x.
\end{aligned}$$

We rearrange to obtain

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |M_n|^2 \, \mathrm{d}x + \int_0^t \int_\Omega |\nabla M_n|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad + \frac{1}{2\mu^2} \int_0^t \int_\Omega |M_n|^4 \, \mathrm{d}x \, \mathrm{d}s \quad \leq \quad \frac{t \, |\Omega|}{2\mu^2} + \frac{1}{2} \int_\Omega |\widetilde{P}_n(M_0)|^2 \, \mathrm{d}x.
\end{aligned}$$

By taking the supremum over all $t \in [0, \tilde{t}]$ we get

$$\begin{aligned}
&\sup_{0 \leq t \leq \tilde{t}} \|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2(t) + 2\|\nabla M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times d}))}^2 + \frac{1}{\mu^2}\|M_n\|_{L^4(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))}^4 \\
\leq \quad & \frac{\tilde{t} \, |\Omega|}{\mu^2} + \|\widetilde{P}_n(M_0)\|_{L^2(\Omega;\mathbb{R}^3)}^2.
\end{aligned} \tag{3.80}$$

Since $\|\widetilde{P}_n(M_0)\|_{L^2(\Omega;\mathbb{R}^3)} \leq \|M_0\|_{L^2(\Omega;\mathbb{R}^3)}$, this gives us the bound

$$\|M_n\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} + \|\nabla M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} + \|M_n\|_{L^4(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))}^4 \leq C, \tag{3.81}$$

where the constant is depending on Ω, μ and the final time $\tilde{t}$. Next, we multiply (3.79) by $-\Delta M_n$, integrate over both Ω and $[0,t]$ for $t \leq \tilde{t}$ and use Young's inequality to obtain the estimate

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla M_n|^2 \,\mathrm{d}x \\
&= \int_0^t \int_\Omega (v\cdot\nabla)M_n \cdot \Delta M_n \,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |\Delta M_n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \frac{1}{\mu^2}\int_0^t\int_\Omega (|M_n|^2-1)M_n\cdot\Delta M_n \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \,\mathrm{d}x \\
&= \int_0^t \int_\Omega (v\cdot\nabla)M_n \cdot \Delta M_n \,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |\Delta M_n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega \left[|M_n|^2\nabla M_n + \nabla|M_n|^2\otimes M_n\right] : \nabla M_n \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega M_n\cdot\Delta M_n \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \,\mathrm{d}x \\
&= \int_0^t \int_\Omega (v\cdot\nabla)M_n \cdot \Delta M_n \,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |\Delta M_n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega |M_n|^2|\nabla M_n|^2 + 2\underbrace{(M_n)_k\nabla_j(M_n)_k(M_n)_i\nabla_j(M_n)_i}_{=\left|\nabla\frac{|M_n|^2}{2}\right|^2} \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega M_n\cdot\Delta M_n \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \,\mathrm{d}x.
\end{aligned}
$$

Further, by Young's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla M_n|^2 \,\mathrm{d}x \\
&\overset{\text{Young}}{\leq} \int_0^t\int_\Omega |(v\cdot\nabla)M_n|^2 + \frac{1}{4}|\Delta M_n|^2 - |\Delta M_n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega |M_n|^2|\nabla M_n|^2 + 2\left|\nabla\frac{|M_n|^2}{2}\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^t\int_\Omega \frac{1}{\mu^4}|M_n|^2 + \frac{1}{4}|\Delta M_n|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \,\mathrm{d}x \\
&\leq \int_0^t \|(v\cdot\nabla)M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s - \frac{1}{2}\int_0^t \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\qquad - \frac{1}{\mu^2}\int_0^t\int_\Omega |M_n|^2|\nabla M_n|^2 + 2\left|\nabla\frac{|M_n|^2}{2}\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \frac{1}{\mu^4}\int_0^t \|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s + \frac{1}{2}\|\nabla \widetilde{P}_n(M_0)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}.
\end{aligned}
$$

Moreover, since $v \in L^\infty(0,t^*;W^{2,\infty}(\Omega;\mathbb{R}^d))$, we can estimate

$$\begin{aligned}
&\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) + \int_0^t \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\qquad\qquad + \frac{2}{\mu^2}\int_0^t \int_\Omega |M_n|^2|\nabla M_n|^2 + 2\left|\nabla \frac{|M_n|^2}{2}\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
\leq\ & C(v)\int_0^t \|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})} \,\mathrm{d}s \\
&\qquad + \frac{2}{\mu^4}\int_0^t \|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s + \|\nabla \widetilde{P}_n(M_0)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}, \qquad (3.82)
\end{aligned}$$

where the constant $C(v)$ depends only on v. Now, we can apply Gronwall's inequality to get

$$\begin{aligned}
&\sup_{0\leq t\leq \tilde{t}} \|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) \\
\leq\ & \Bigg(\frac{2}{\mu^4}\underbrace{\|M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))}}_{\leq \|M_n\|^2_{L^2(0,t^*;L^2(\Omega;\mathbb{R}^3))}} + \|\nabla \widetilde{P}_n(M_0)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}\Bigg)\mathrm{e}^{C(v)\tilde{t}}. \qquad (3.83)
\end{aligned}$$

Since $\tilde{t} \leq t^*$ is bounded, the right-hand side of (3.83) is bounded independently of $\tilde{t}$. This, together with (3.81) and $\|\nabla \widetilde{P}_n(M_0)\|_{L^2(\Omega;\mathbb{R}^{3\times d})} \leq \|\nabla M_0\|_{L^2(\Omega;\mathbb{R}^{3\times d})}$, tells us that

$$\|M_n\|_{L^\infty(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3))} \leq C(v). \qquad (3.84)$$

Furthermore, if we integrate $\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}$ over time and use (3.82), (3.84), we obtain

$$\|\Delta M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \leq C(v). \qquad (3.85)$$

From $\frac{\partial M_n}{\partial \mathbf{n}} = 0$, we obtain, using integration by parts

$$\begin{aligned}
\|\Delta M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} &= \int_\Omega \partial_i\partial_i(M_n)_k\partial_j\partial_j(M_n)_k \,\mathrm{d}x \\
&= \int_\Omega \partial_i\partial_j(M_n)_k\partial_i\partial_j(M_n)_k \,\mathrm{d}x \\
&= \|\nabla^2 M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2\times 2}))},
\end{aligned}$$

which implies, together with (3.84) and (3.85), that

$$\|M_n\|_{L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} \leq C(v). \qquad (3.86)$$

Finally, we test (3.79) with $(M_n)_t$. To this end, we need to verify that $(M_n)_t$ is actually admissible as a test function, i.e., in $L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))$, using the fact that $\|\widetilde{P}_n(\varphi)\|_{L^2(\Omega;\mathbb{R}^3)} \leq \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} \leq 1$,

$$
\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^{\tilde{t}} \int_\Omega (M_n)_t \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^{\tilde{t}} \int_\Omega (M_n)_t \cdot (\zeta\widetilde{P}_n(\varphi)) \,\mathrm{d}x \,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^{\tilde{t}} \int_\Omega -(v\cdot\nabla)M_n \cdot (\zeta\widetilde{P}_n(\varphi)) + \Delta M \cdot (\zeta\widetilde{P}_n(\varphi)) \\
&\qquad\qquad - \frac{1}{\mu^2}(|M|^2-1)M\cdot(\zeta\widetilde{P}_n(\varphi)) \,\mathrm{d}x \,\mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^{\tilde{t}} \|(v\cdot\nabla)M_n\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\widetilde{P}_n(\varphi)\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\widetilde{P}_n(\varphi)\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \frac{1}{\mu^2}\|(|M|^2-1)M\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\widetilde{P}_n(\varphi)\|_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}t.
\end{aligned}
$$

Another application of Hölder's inequality yields

$$
\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^{\tilde{t}} \int_\Omega (M_n)_t \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \sup_{\|\zeta\|_{L^2(0,\tilde{t})}\leq 1} \Big(\|v_m\|_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^d))}\|\nabla M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times d}))}\|\zeta\|_{L^2(0,\tilde{t})} \\
&\qquad + \|\Delta M\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^2(0,\tilde{t})} \\
&\qquad + \frac{1}{\mu^2}\|(|M|^2-1)M\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^2(0,\tilde{t})} \Big) \\
&\leq \|v\|_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^d))}\|\nabla M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times d}))} + \|\Delta M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \\
&\qquad + \frac{1}{\mu^2}\underbrace{\||M_n|^3\|_{L^2(0,\tilde{t};L^2(\Omega))}}_{=\|M_n\|^3_{L^6(0,\tilde{t};L^6(\Omega;\mathbb{R}^3))}} + \frac{1}{\mu^2}\|M_n\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \\
&\leq C(v) + \frac{1}{\mu^2}\|M_n\|^3_{L^6(0,\tilde{t};L^6(\Omega;\mathbb{R}^3))} \leq C(v),
\end{aligned}
$$

where we used the continuous Sobolev embedding $\mathbf{H}^1 \subset L^6$ (valid for $d = 2, 3$) and (3.84) in the last step. In summary, we get from the above estimate

$$
\|(M_n)_t\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \leq C(v). \tag{3.87}
$$

Now, we can multiply (3.79) with $(M_n)_t$, integrate over both Ω and $[0,t]$ for $t \le \tilde{t}$ and use again Young's inequality to get

$$
\begin{aligned}
&\int_0^t \int_\Omega |(M_n)_t|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla M_n|^2 \, \mathrm{d}x \\
&= \; -\int_0^t \int_\Omega (v\cdot\nabla) M_n \cdot (M_n)_t \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad - \int_0^t \int_\Omega \underbrace{\left(\frac{1}{\mu^2}(|M_n|^2-1)M_n\right)\cdot (M_n)_t}_{=\left(\frac{(|M_n|^2-1)^2}{4\mu^2}\right)_t} \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \, \mathrm{d}x \\
&\le \; \frac{1}{2}\int_0^{\tilde{t}} \|(v\cdot\nabla)M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}s + \frac{1}{2}\int_0^t \|(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}s \\
&\qquad - \frac{1}{4\mu^2}\int_\Omega |M_n|^4 - 2|M_n|^2 + 1 \, \mathrm{d}x \\
&\qquad + \int_\Omega \frac{(|\widetilde{P}_n(M_0)|^2-1)^2}{4\mu^2} \, \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla \widetilde{P}_n(M_0)|^2 \, \mathrm{d}x.
\end{aligned}
$$

Then, due to $v \in L^\infty(0,t^*;W^{2,\infty}(\Omega;\mathbb{R}^d))$, the bound (3.84) and the assumption on the initial data $M_0 \in \mathbf{H}^1(\Omega;\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
&\int_0^t \|(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}s + \|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) + \frac{1}{2\mu^2}\|M_n\|^4_{L^4(\Omega;\mathbb{R}^3)}(t) \\
&\le \; \|(v\cdot\nabla)M_n\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + \int_\Omega \frac{(|\widetilde{P}_n(M_0)|^2-1)^2}{2\mu^2} \, \mathrm{d}x \\
&\qquad + \int_\Omega |\nabla M_0|^2 \, \mathrm{d}x + \frac{1}{\mu^2}\|M_n\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + C(\Omega) \\
&\le \; C(v).
\end{aligned}
$$

We take the supremum over all $t \in [0,\tilde{t}]$ to find out that

$$
\begin{aligned}
\|(M_n)_t\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} &+ \sup_{0\le t\le \tilde{t}} \|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) \\
&+ \sup_{0\le t\le \tilde{t}} \frac{1}{2\mu^2}\|M_n\|^4_{L^4(\Omega;\mathbb{R}^3)}(t) \;\le\; C(v).
\end{aligned}
$$

So, we see that

$$\|M_n\|_{\mathbf{H}^1(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \le C(v) \tag{3.88}$$

and, furthermore,

$$\|M_n\|_{L^\infty(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))} \le C(v). \tag{3.89}$$

Finally, we pass to the limit as $n \to \infty$ to obtain a weak solution to the system (3.71)–(3.73). We need the convergence results

$$
\begin{aligned}
M_n &\to M && \text{in } L^4(0,\tilde{t};L^4(\Omega;\mathbb{R}^3)), && (3.90)\\
(M_n)_t &\rightharpoonup M_t && \text{in } L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3)), && (3.91)\\
\nabla M_n &\rightharpoonup \nabla M && \text{in } L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times d})). && (3.92)
\end{aligned}
$$

The weak convergence results follow directly from the estimates obtained above for a subsequence (not relabeled; see Theorem 33 in Appendix A.2). For the strong convergence (3.90), we have to argue a bit more: From the embeddings $\mathbf{H}^1(\Omega;\mathbb{R}^3) \overset{c}{\subset} L^4(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3)$ (the first embedding is compact since $d < 4$, the second one is continuous), the fact that $M_n \in L^4(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3))$, and (3.88), we conclude by the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2) the compact embedding

$$\left\{M \in L^4(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3)) : \; M_t \in L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))\right\} \overset{c}{\subset} L^4(0,\tilde{t};L^4(\Omega;\mathbb{R}^3)).$$

This yields the strong convergence (3.90) (up to subsequence) of $\{M_n\}_n$.
Again, as the weak solution to the approximate problem is defined using test functions from the projected spaces $\widetilde{L}^2_n$, we also need to pass to the limit with these particular test functions (only in space). However, for any test function $\varphi \in \mathbf{H}^1(\Omega;\mathbb{R}^3)$ we use the sequence of approximate test functions defined by $\varphi_n := \widetilde{P}_n(\varphi) \in \widetilde{L}^2_n$ which converges strongly to φ in $\mathbf{H}^1(\Omega;\mathbb{R}^3)$. In the following, we use this particular sequence of test functions. Moreover, let $\zeta \in L^\infty(0,\tilde{t})$.
So, the equation

$$\begin{aligned}
&\int_0^{\tilde{t}}\int_\Omega (M_n)_t \cdot (\zeta\varphi_n) + (v\cdot\nabla)M_n \cdot (\zeta\varphi_n)\;\mathrm{d}x\;\mathrm{d}t \\
=&\int_0^{\tilde{t}}\int_\Omega -\nabla M_n : (\zeta\nabla\varphi_n) - \frac{1}{\mu^2}(|M_n|^2-1)M_n\cdot(\zeta\varphi_n)\;\mathrm{d}x\;\mathrm{d}t
\end{aligned}$$

converges to the equation

$$\begin{aligned}
&\int_0^{\tilde{t}}\int_\Omega M_t \cdot (\zeta\varphi) + (v\cdot\nabla)M \cdot (\zeta\varphi)\;\mathrm{d}x\;\mathrm{d}t \\
=&\int_0^{\tilde{t}}\int_\Omega -\nabla M : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M|^2-1)M\cdot(\zeta\varphi)\;\mathrm{d}x\;\mathrm{d}t
\end{aligned}$$

as $n\to\infty$. All the integral terms on the left-hand side and the first term on the right-hand side are linear, so the weak convergences from above together with the strong convergence of the test functions yield the convergence of these terms. For the last term, we need the strong convergence of $\{M_n\}_n$. We proceed by calculating and add zeroes in the first step in order to factor out neighboring summands in the second step:

$$\begin{aligned}
&\left|\int_0^{\tilde{t}}\int_\Omega (|M_n|^2-1)M_n\cdot(\zeta\varphi_n) - (|M|^2-1)M\cdot(\zeta\varphi)\;\mathrm{d}x\;\mathrm{d}t\right| \\
=&\;\Bigg|\int_0^{\tilde{t}}\int_\Omega (|M_n|^2-1)M_n\cdot(\zeta\varphi_n) - (|M_n|^2-1)M\cdot(\zeta\varphi_n) \\
&\quad + (|M_n|^2-1)M\cdot(\zeta\varphi_n) - (|M|^2-1)M\cdot(\zeta\varphi_n) \\
&\quad + (|M|^2-1)M\cdot(\zeta\varphi_n) - (|M|^2-1)M\cdot(\zeta\varphi)\;\mathrm{d}x\;\mathrm{d}t\Bigg| \\
=&\;\Bigg|\int_0^{\tilde{t}}\int_\Omega (|M_n|^2-1)(M_n-M)\cdot(\zeta\varphi_n) \\
&\quad + (|M_n|^2-|M|^2)M\cdot(\zeta\varphi_n) + (|M|^2-1)M\cdot(\zeta(\varphi_n-\varphi))\;\mathrm{d}x\;\mathrm{d}t\Bigg|
\end{aligned}$$

$$
\begin{aligned}
= \quad & \Big| \int_0^{\tilde t} \int_\Omega |M_n|^2 (M_n - M)\cdot(\zeta\varphi_n) - (M_n - M)\cdot(\zeta\varphi_n) \\
& \qquad + (|M_n|^2 - |M|^2) M \cdot (\zeta\varphi_n) \\
& \qquad + |M|^2 M \cdot (\zeta(\varphi_n - \varphi)) - M\cdot(\zeta(\varphi_n-\varphi)) \,\mathrm{d}x\,\mathrm{d}t \Big| \\
\leq \quad & \int_0^{\tilde t}\int_\Omega |M_n|^2 |M_n - M|\,|\zeta\varphi_n| + |M_n - M|\,|\zeta\varphi_n| \\
& \qquad + \big||M_n|^2 - |M|^2\big|\,|M|\,|\zeta\varphi_n| \\
& \qquad + |M|^2|M|\,\big|\zeta(\varphi_n-\varphi)\big| + |M|\,\big|\zeta(\varphi_n-\varphi)\big| \,\mathrm{d}x\,\mathrm{d}t \\
\overset{\text{Hölder}}{\leq} \quad & \underbrace{\||M_n|^2\|_{L^4(0,\tilde t;L^2(\Omega))}}_{=\|M_n\|^2_{L^8(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}\leq C(v)} \|M_n - M\|_{L^4(0,\tilde t;L^4(\Omega;\mathbb{R}^3))} \underbrace{\|\zeta\varphi_n\|_{L^2(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}}_{\leq C} \\
& + \|M_n - M\|_{L^2(0,\tilde t;L^2(\Omega;\mathbb{R}^3))} \underbrace{\|\zeta\varphi_n\|_{L^2(0,\tilde t;L^2(\Omega;\mathbb{R}^3))}}_{\leq C} \\
& + \big\||M_n|^2 - |M|^2\big\|_{L^2(0,\tilde t;L^2(\Omega))} \underbrace{\|M\|_{L^\infty(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}}_{\leq C(v)} \underbrace{\|\zeta\varphi_n\|_{L^2(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}}_{\leq C} \\
& + \underbrace{\||M|^2\|_{L^4(0,\tilde t;L^2(\Omega))}}_{=\|M\|^2_{L^8(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}\leq C(v)} \|M\|_{L^4(0,\tilde t;L^4(\Omega;\mathbb{R}^3))} \|\zeta(\varphi_n-\varphi)\|_{L^2(0,\tilde t;L^4(\Omega;\mathbb{R}^3))} \\
& + \|M\|_{L^2(0,\tilde t;L^2(\Omega;\mathbb{R}^3))} \|\zeta(\varphi_n - \varphi)\|_{L^2(0,\tilde t;L^2(\Omega;\mathbb{R}^3))} \\
& \xrightarrow{n\to\infty} 0,
\end{aligned}
$$

where we used Hölder's inequality in the expression $\big\||M_n|^2 - |M|^2\big\|_{L^2(0,\tilde t;L^2(\Omega))}$ $\leq \Big(\|M_n\|_{L^4(0,\tilde t;L^4(\Omega;\mathbb{R}^3))} + \|M\|_{L^4(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}\Big)\|M_n - M\|_{L^4(0,\tilde t;L^4(\Omega;\mathbb{R}^3))}$.
Thus, we obtain a weak solution to the system (3.71)–(3.73).
Notice that all the estimates for the approximate solution obtained above still hold in the limit due to the weak lower semicontinuity of norms.
Furthermore, the solution is unique. Let us assume that we have two solutions $M_1 \neq M_2$. The difference $M_1 - M_2$ then solves

$$
\begin{aligned}
& (M_1 - M_2)_t + (v\cdot\nabla)(M_1 - M_2) \\
& \quad = \quad \Delta(M_1 - M_2) + \frac{1}{\mu^2}(M_1 - M_2) - \frac{1}{\mu^2}(|M_1|^2 M_1 - |M_2|^2 M_2).
\end{aligned}
$$

This equation we multiply by $(M_1 - M_2)$ and integrate over Ω to find

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla(M_1 - M_2)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})} \\
& \quad \underbrace{+\frac{1}{\mu^2}\int_\Omega (|M_1|^2 M_1 - |M_2|^2 M_2)\cdot(M_1 - M_2)\,\mathrm{d}x}_{=:I} \quad = \quad \frac{1}{\mu^2}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)}.
\end{aligned}
$$

Notice that integration by parts does not yield any boundary terms here, since the gradients of M_1 and M_2 both vanish on the boundary. Now, we take care of the integral term I. Firstly, we have

$$\begin{aligned}
|M_1|^2M_1 - |M_2|^2M_2 &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(|M_2 + (M_1 - M_2)s|^2(M_2 - (M_1 - M_2)s)\right)\,\mathrm{d}s \\
&= \int_0^1 |M_2 + (M_1 - M_2)s|^2(M_1 - M_2) \\
&\qquad\qquad + 2|M_2 + (M_1 - M_2)s|^2(M_1 - M_2)\,\mathrm{d}s \\
&= \int_0^1 3|M_2 + (M_1 - M_2)s|^2(M_1 - M_2)\,\mathrm{d}s.
\end{aligned}$$

Then, we obtain

$$I = \frac{3}{\mu^2}\int_\Omega \int_0^1 |M_2 + (M_1 - M_2)s|^2\,\mathrm{d}s|M_1 - M_2|^2\,\mathrm{d}x \geq 0.$$

This allows us to estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq \frac{1}{\mu^2}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)},$$

where we apply Gronwall's inequality to find

$$\sup_{0\leq t\leq \tilde{t}} \|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} = 0.$$

Thus, the solution is unique. This concludes the proof of Lemma 17. □

3.1.3.2 Weak solutions to the approximate problem for a short time using a fixed point argument

The next result yields a weak solution to the approximate problem which exists only for a certain (short) time t_0^*. The main ingredient of the proof is an application of Schauder's fixed point theorem.

Lemma 18. *For any $m > 0$ and W satisfying* (3.12)–(3.17)*, there exists a time t_0^* depending on v_0, M_0, Ω, and m such that the system* (3.30)–(3.39) *has a weak solution (v_m, F_m, M_m) in $\Omega \times (0, t_0^*)$.*

Proof. In this proof, $m > 0$ is fixed, which allows us to use the simpler notation $v = v_m$ and $\tilde{v} = \tilde{v}_m$, respectively.
We choose $t_1^* > 0$ and any Galerkin approximation of the velocity v by $v(x,t) = \sum_{i=1}^m g_m^i(t)\xi_i(x)$ with $g_m^i(0) = \int_\Omega v_0(x)\cdot\xi_i(x)\,\mathrm{d}x$ and $\left(\sum_{i=1}^m |g_m^i(t)|^2\right)^{\frac{1}{2}} \leq N$ for any $t \in [0, t_1^*]$, where N is a suitably large constant which we choose later. Since $v \in L^\infty(0, t_1^*; W^{2,\infty}(\Omega))$, by Lemma 17 we obtain a unique weak solution (F, M) to

$$F_t + (v\cdot\nabla)F - \nabla v F = \kappa\Delta F,$$

$$M_t + (v\cdot\nabla)M = \Delta M - \frac{1}{\mu^2}(|M|^2 - 1)M$$

on $[0, t_1^*]$ satisfying

$$\|F\|_{L^\infty(0,t_1^*;L^2(\Omega;\mathbb{R}^{d\times d}))} \leq C(v), \tag{3.93}$$

$$\|M\|_{L^\infty(0,t_1^*;\mathbf{H}^1(\Omega;\mathbb{R}^3))} \leq C(v). \tag{3.94}$$

This unique solution (F, M) is used in the following to solve the equation of motion for v which is rewritten in the ODE system (3.44). To this end, we apply Carathéodory's existence theorem again.
From (3.14), (3.93) and (3.94) we directly obtain for $D_m^i(t)$ from (3.46)

$$D_m^i(t) \in L^\infty(0, t_1^*). \tag{3.95}$$

Since the first two summands on the right-hand side of (3.44) are independent of t (looking at t and g_m^i as distinct variables) and the third summand is in $L^\infty(0, t_1^*)$, the right-hand side is measurable in t for any g_m^i.
Furthermore, the $D_m^i(t)$ are independent of g_m^i and the first and second summand of the right-hand side of (3.44) are linear and quadratic in g_m^i, respectively, so the right-hand side is continuous in g_m^i for any t.
In addition, for $t \in [0, t_1^*]$ and $\|g_m - g_m(0)\| \leq b$, where $g_m = (g_m^1, \ldots, g_m^m)$, we can bound the right-hand side of (3.44) by the L^1-function

$$-\nu\lambda_i(2b + \|g_m(0)\|) + (2b + \|g_m(0)\|)^2 \sum_{j,k=1}^{m} A_{jk}^i + D_m^i(t).$$

Finally, Carathéodory's theorem (see Theorem 30 in Appendix A.2) yields the existence of a value t_2^* with $0 < t_2^* \leq t_1^*$ so that the ODE system (3.44) has an absolutely continuous and unique (since the right-hand side of the ODE is locally Lipschitz, see Theorem 31 in Appendix A.2) solution $\{\tilde{g}_m^i(t)\}_{i=1}^m$ on $[0, t_2^*]$ satisfying (3.47).
We define the velocity through these time-dependent coefficients by the sum $\tilde{v}(x,t) = \sum_{i=1}^m \tilde{g}_m^i(t)\xi_i(x)$. We can get the following estimate for $\tilde{v}(x,t)$

$$\|\tilde{v}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}(t) \leq \|v_0\|_{L^2(\Omega;\mathbb{R}^d)} + C_1(m) + C_2(m)t\exp(C_3(m)\|v\|_{L^\infty(0,t_2^*;L^2(\Omega;\mathbb{R}^d))}t).$$

Indeed, with

$$\begin{aligned} \|\nabla\tilde{v}\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}(t) &\leq C(m) \max_{i=1,\ldots,m} |\tilde{g}_m^i(t)| \\ &\leq C(m) \left(\sum_{i=1}^m \left|\tilde{g}_m^i(t)\right|^2 \right)^{\frac{1}{2}} \\ &= C(m)\|\tilde{v}\|_{L^2(\Omega;\mathbb{R}^d)}(t), \end{aligned} \tag{3.96}$$

we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde v\|^2_{L^2(\Omega;\mathbb{R}^d)}(t)\\
&= -2\underbrace{\int_\Omega (\tilde v\cdot\nabla)\tilde v\cdot\tilde v\;\mathrm{d}x}_{=0} -2\nu\|\nabla\tilde v\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\\
&\qquad\qquad -2\int_\Omega (W'(F)F^\top-\nabla M\odot\nabla M):\nabla\tilde v\;\mathrm{d}x\\
&\le -2\int_\Omega (W'(F)F^\top-\nabla M\odot\nabla M):\nabla\tilde v\;\mathrm{d}x\\
&\le 2C(m)\|\tilde v\|_{L^2(\Omega;\mathbb{R}^d)}(t)\left(C+C\|F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(t)+\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t)\right).
\end{aligned}
$$

Without loss of generality, let $\|\tilde v\|_{L^2(\Omega)}>0$ on $[0,t_2^*]$. Otherwise, if $v(T_0)=0$ for some $T_0\in[0,t_2^*]$, then $v(t)=0$ for any $t\ge T_0$ due to uniqueness which follows immediately from the local Lipschitz property of the right-hand side of (3.44). Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde v\|_{L^2(\Omega)}(t) &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde v\|^2_{L^2(\Omega;\mathbb{R}^d)}(t)}{2\|\tilde v\|_{L^2(\Omega;\mathbb{R}^d)}(t)}\\
&\le C(m)\left(C+C\|F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(t)+\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t)\right),
\end{aligned}
$$

from which we can deduce, using the obtained estimates of Lemma 17 and (3.60) from the lemma's proof,

$$
\begin{aligned}
&\|\tilde v\|_{L^2(\Omega;\mathbb{R}^d)}(t)\\
&\le \underbrace{\|P(v_0)\|_{L^2(\Omega;\mathbb{R}^d)}}_{\le\|v_0\|_{L^2(\Omega;\mathbb{R}^d)}}+C(m)\int_0^t C+C\|F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(s)+\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(s)\;\mathrm{d}s\\
&\le \|v_0\|_{L^2(\Omega;\mathbb{R}^d)}+C_1(m)+C_2(m)\int_0^t \exp(\underbrace{\|\nabla v\|_{L^\infty(0,t_2^*;L^\infty(\Omega;\mathbb{R}^{d\times d}))}}_{\overset{(3.96)}{\le} C_3(m)\|v\|_{L^\infty(0,t_2^*;L^2(\Omega;\mathbb{R}^d))}}\,s)\;\mathrm{d}s\\
&\le \|v_0\|_{L^2(\Omega;\mathbb{R}^d)}+C_1(m)+C_2(m)t\exp(C_3(m)\|v\|_{L^\infty(0,t_2^*;L^2(\Omega;\mathbb{R}^d))}t).
\end{aligned}
$$

Now, let $N=\|v_0\|_{L^2(\Omega;\mathbb{R}^d)}+C_1(m)+1$ and let $0<t_0^*\le t_2^*$ be such that

$$
C_2(m)t_0^*\exp(C_3(m)Nt_0^*)\le 1.
$$

Then, it holds that if $\|v\|_{L^2(\Omega;\mathbb{R}^d)}(t)\le N$ on $(0,t_0^*)$ then also $\|\tilde v\|_{L^2(\Omega;\mathbb{R}^d)}(t)\le N$ on $(0,t_0^*)$.
Next, we define a map $\mathcal{L}:V_m(t_0^*)\to V_m(t_0^*),\ v\mapsto\tilde v$ on the set

$$
\begin{aligned}
V_m(t_0^*)=\Bigg\{v(x,t)=\sum_{i=1}^m g_m^i(t)\xi_i(x):\ \left(\sum_{i=1}^m\left|g_m^i(t)\right|^2\right)^{\frac12}\le N\text{ for }0\le t\le t_0^*,\\
g_m^i\text{ continuous},\ g_m^i(0)=\int_\Omega v_0(x)\cdot\xi_i(x)\;\mathrm{d}x\Bigg\}.
\end{aligned}
$$

Notice that, due to the construction above, $\mathcal{L}$ maps $V_m(t_0^*)$ into itself.
The set $V(t_0^*)$ is a closed, convex subset of $C([0,t_0^*];\mathbf{H}_m) \subset C([0,t_0^*];L^2(\Omega;\mathbb{R}^d))$. Let us show that $\mathcal{L}(V_m(t_0^*))$ is precompact there. Since the dimension of $\mathbf{H}_m$ is finite, boundedness is the same as precompactness, and the ξ_i, $i = 1,\ldots,m$, are bounded in $\mathbf{H}_m$. Next, due to the choice of N, all the $g_m^i(t)$ are uniformly bounded, and from (3.44) and (3.95) we get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}g_m^i(t)\right| &= \left|-\nu\lambda_i g_m^i(t) + \sum_{j,k=1}^m g_m^j(t)g_m^k(t)A_{jk}^i + D_m^i(t)\right| \\ &\leq CN + C(m)N^2 + C \leq C(N,m), \end{aligned} \tag{3.97}$$

from where we obtain equicontinuity of all the $g_m^i(t)$. Now, the Arzelà-Ascoli theorem gives us the precompactness of all the $g_m^i(t)$ in $C([0,t_0^*])$. So, in summary, we have that $\mathcal{L}(V_m(t_0^*))$ is a precompact set in $C([0,t_0^*];\mathbf{H}_m)$, i.e., also in $C([0,t_0^*];L^2(\Omega;\mathbb{R}^d))$.
We also show that $\mathcal{L}$ is a continuous map on $V_m(t_0^*)$ in the topology of the space $C([0,t_0^*];L^2(\Omega;\mathbb{R}^d))$. To this end, let $\{v_l\}_l \subset V_m(t_0^*)$ converge to some $v \in V_m(t_0^*)$ in the sense

$$\begin{aligned} V_m(t_0^*) \ni v_l &\xrightarrow{l\to\infty} v \in V_m(t_0^*) \\ \Longleftrightarrow (g_m^i)_l &\xrightarrow{l\to\infty} g_m^i \text{ in } C([0,t_0^*]),\ i = 1,\ldots,m. \end{aligned} \tag{3.98}$$

Remark 19. *Notice that the constants $C(v)$ obtained in Lemma 17 are uniform over $V_m(t_0^*)$ (only depending on the particular m and N) since the time-dependent coefficients $g_m^i(t)$ are uniformly bounded. Thus, these constants do not depend on the index l of the sequence $\{v_l\}_l \subset V_m(t_0^*)$ chosen to prove continuity of $\mathcal{L}$.*

Now, we show that the solutions F_l and M_l guaranteed by Lemma 17 for v_l converge strongly to those for v in $L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^{d\times d}))$ and $L^\infty(0,t_0^*;\mathbf{H}^1(\Omega;\mathbb{R}^3))$, respectively.

Convergence of $\{F_l\}_l$. We obtain from the partial differential equation for F

$$(F_l-F)_t+(v_l\cdot\nabla)(F_l-F)+((v_l-v)\cdot\nabla)F-\nabla v_l(F_l-F)-(\nabla v_l-\nabla v)F = \kappa\Delta(F_l-F).$$

By multiplying this equation by $(F_l - F)$, integrating over both Ω and $[0,t]$ for $t \leq t_0^*$, we get the estimate

$$\begin{aligned} &\frac{1}{2}\int_\Omega |F_l - F|^2(t)\ \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |F_l - F|^2(0)\ \mathrm{d}x - \int_0^t\int_\Omega (v_l\cdot\nabla)(F_l-F) : (F_l - F)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad - \int_0^t\int_\Omega ((v_l - v)\cdot\nabla)F : (F_l - F)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_0^t\int_\Omega \nabla v_l(F_l - F) : (F_l - F)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_0^t\int_\Omega (\nabla v_l - \nabla v)F : (F_l - F)\ \mathrm{d}x\ \mathrm{d}s - \int_0^t\int_\Omega \kappa|\nabla(F_l - F)|^2\ \mathrm{d}x\ \mathrm{d}s. \end{aligned}$$

An application of Young's inequality yields

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |F_l - F|^2(t)\ \mathrm{d}x \\
&\leq \frac{1}{2}\|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(0) + \int_0^t \int_\Omega \kappa|\nabla(F_l - F)|^2 + \frac{1}{4\kappa}|(v_l)_k(F_l - F)_{ij}|^2 \\
&\quad + \frac{1}{2}|((v_l - v)\cdot\nabla)F|^2 + \frac{1}{2}|F_l - F|^2 + \frac{1}{2}|\nabla v_l(F_l - F)|^2 + \frac{1}{2}|F_l - F|^2 \\
&\quad + \frac{1}{2}|(\nabla v_l - \nabla v)F|^2 + \frac{1}{2}|F_l - F|^2 - \kappa|\nabla(F_l - F)|^2\ \mathrm{d}x\ \mathrm{d}s \\
&= \frac{1}{2}\|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(0) + \frac{1}{4\kappa}\int_0^t \|(v_l)_k(F_l - F)_{ij}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d\times d})}\ \mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|((v_l - v)\cdot\nabla)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s + \frac{1}{2}\int_0^t \|\nabla v_l(F_l - F)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|(\nabla v_l - \nabla v)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s + \frac{3}{2}\int_0^t \|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s.
\end{aligned}
$$

Moreover, since v_l is smooth in space, we can estimate

$$
\begin{aligned}
&\|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(t) \\
&\quad \leq \|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(0) \\
&\qquad + \underbrace{\int_0^t \|((v_l - v)\cdot\nabla)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|(\nabla v_l - \nabla v)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s}_{\text{non-decreasing}} \\
&\qquad + \int_0^t C\|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(s)\ \mathrm{d}s, \qquad (3.99)
\end{aligned}
$$

where the constant C depends on Ω, N and κ. Note that, since we have the same initial data when solving for F_l and F, i.e., $F_l(0) = F(0)$, the first term on the right-hand side of (3.99) is zero. Now, we can apply Gronwall's inequality to get

$$
\begin{aligned}
\sup_{0\leq t\leq t_0^*} \|F_l - F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}(t) \leq & \int_0^{t_0^*} \|((v_l - v)\cdot\nabla)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
& + \|(\nabla v_l - \nabla v)F\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}s\ \mathrm{e}^{Ct_0^*}. \quad (3.100)
\end{aligned}
$$

Due to (3.98) we can pass to the limit as $l \to \infty$ to see that

$$
F_l \xrightarrow{l\to\infty} F \text{ in } L^\infty(0, t_0^*; L^2(\Omega;\mathbb{R}^{d\times d})). \qquad (3.101)
$$

Convergence of $\{M_l\}_l$. We check the strong convergence of $\{M_l\}_l$ in the space $L^\infty(0, t_0^*; \mathbf{H}^1(\Omega;\mathbb{R}^3))$. To this end, we first obtain from the partial differential equation for M

$$
\begin{aligned}
&(M_l - M)_t + (v_l\cdot\nabla)(M_l - M) + ((v_l - v)\cdot\nabla)M \\
&\quad = \Delta(M_l - M) - \frac{1}{\mu^2}\left((|M_l|^2 - 1)M_l - (|M|^2 - 1)M\right) \qquad (3.102)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&(M_l - M)_t + (v_l \cdot \nabla)(M_l - M) + ((v_l - v) \cdot \nabla)M \\
= \quad &\Delta(M_l - M) - \frac{1}{\mu^2}(|M|^2 - 1)(M_l - M) - \frac{1}{\mu^2}(|M_l|^2 - |M|^2)M. \quad (3.103)
\end{aligned}
$$

By multiplying equation (3.102) with $(M_l - M)$, integrating over both Ω and $[0, t]$ for $t \le t_0^*$ and using Young's inequality and the inequality

$$
\begin{aligned}
&\int_\Omega \left(|y|^{p-2}y - |z|^{p-2}z\right) \cdot (y - z) \, \mathrm{d}x \\
\ge \quad &\left(\|y\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)} - \|z\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}\right) \left(\|y\|_{L^p(\Omega;\mathbb{R}^n)} - \|z\|_{L^p(\Omega;\mathbb{R}^n)}\right)
\end{aligned}
$$

for $y, z \in L^p(\Omega; \mathbb{R}^n)$ (see [Rou13, (2.141), p.76] at $(*)$, we obtain the estimate

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |M_l - M|^2(t) \, \mathrm{d}x \\
= \quad &\frac{1}{2}\int_\Omega |M_l - M|^2(0) \, \mathrm{d}x \\
&- \int_0^t \int_\Omega ((v_l - v) \cdot \nabla)M \cdot (M_l - M) \, \mathrm{d}x \, \mathrm{d}s \\
&+ \int_0^t \int_\Omega \Delta(M_l - M) \cdot (M_l - M) \, \mathrm{d}x \, \mathrm{d}s \\
&- \int_0^t \frac{1}{\mu^2} \underbrace{\int_\Omega \left(|M_l|^2 M_l - |M|^2 M\right) \cdot (M_l - M) \, \mathrm{d}x}_{\overset{(*)}{\ge} \left(\|M_l\|^3_{L^4(\Omega;\mathbb{R}^3)} - \|M\|^3_{L^4(\Omega;\mathbb{R}^3)}\right)\left(\|M_l\|_{L^4(\Omega;\mathbb{R}^3)} - \|M\|_{L^4(\Omega;\mathbb{R}^3)}\right) \ge 0} \, \mathrm{d}s \\
&+ \int_0^t \int_\Omega \frac{1}{\mu^2}|M_l - M|^2 \, \mathrm{d}x \, \mathrm{d}s \\
\overset{\text{Young}}{\le} \quad &\frac{1}{2}\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}(0) \\
&+ \int_0^t \int_\Omega \frac{\mu^2}{2}|((v_l - v) \cdot \nabla)M|^2 + \frac{1}{2\mu^2}|M_l - M|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&- \int_0^t \int_\Omega |\nabla(M_l - M)|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \int_0^t \int_\Omega \frac{1}{\mu^2}|M_l - M|^2 \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}
$$

Rearranging yields, also since $M_l(0) = M(0)$,

$$
\begin{aligned}
&\int_\Omega |M_l - M|^2(t) \, \mathrm{d}x \\
\le \quad &\underbrace{\int_0^t \mu^2 \|((v_l - v) \cdot \nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}s}_{\text{non-decreasing}} + \int_0^t \frac{3}{\mu^2}\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}s. \quad (3.104)
\end{aligned}
$$

We apply Gronwall's inequality to obtain

$$\begin{aligned} &\sup_{0\le t\le t_0^*} \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) \\ &\le \left(\int_0^{t_0^*} \mu^2 \|((v_l - v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s)\ \mathrm{d}s\right) \mathrm{e}^{\frac{3}{\mu^2}t_0^*}. \end{aligned} \tag{3.105}$$

Due to (3.98) we can pass to the limit as $l \to \infty$ to see that

$$M_l \xrightarrow{l\to\infty} M \text{ in } L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^3)). \tag{3.106}$$

We are left to prove the convergence of ∇M_l in $L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^{3\times d}))$. We need an estimate established with the Gagliardo-Nirenberg interpolation inequality (see, e.g., [Nir59, Bre11]) for $d = 2,3$

$$\begin{aligned} &\|M_l - M\|^2_{L^6(\Omega;\mathbb{R}^3)} \\ &\overset{\text{Gagliardo-}}{\underset{\text{Nirenberg}}{\le}} \Big(C_1\|\nabla(M_l - M)\|_{L^2(\Omega;\mathbb{R}^{3\times d})} + C_2\|M_l - M\|_{L^2(\Omega;\mathbb{R}^3)}\Big)^2 \\ &\le C(\Omega)\Big(\|\nabla(M_l - M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})} + \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ &\qquad + \|\nabla(M_l - M)\|_{L^2(\Omega;\mathbb{R}^{3\times d})}\|M_l - M\|_{L^2(\Omega;\mathbb{R}^3)}\Big) \\ &\overset{\text{Young}}{\le} C(\Omega)\Big(\|\nabla(M_l - M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})} + \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big). \end{aligned} \tag{3.107}$$

Now, by multiplying equation (3.103) with $-\Delta(M_l - M)$, integrating over both Ω and $[0,t]$ for $t \le t_0^*$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\ \mathrm{d}x \\ &= \int_0^t\int_\Omega (v_l\cdot\nabla)(M_l - M)\cdot\Delta(M_l - M)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_0^t\int_\Omega ((v_l - v)\cdot\nabla)M\cdot\Delta(M_l - M)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad - \int_0^t\int_\Omega |\Delta(M_l - M)|^2\ \mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_0^t\int_\Omega \frac{1}{\mu^2}(|M|^2 - 1)(M_l - M)\cdot\Delta(M_l - M)\ \mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_0^t\int_\Omega \frac{1}{\mu^2}(|M_l|^2 - |M|^2)M\cdot\Delta(M_l - M)\ \mathrm{d}x\ \mathrm{d}s. \end{aligned}$$

By applying Young's inequality and the bounds obtained in Lemma 17 (see also Remark 19), we find

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\ \mathrm{d}x \\
&\overset{\text{Young}}{\leq} \int_0^t \int_\Omega |(v_l\cdot\nabla)(M_l-M)|^2 + \frac{1}{4}|\Delta(M_l-M)|^2 \\
&\qquad + |((v_l - v)\cdot\nabla)M|^2 + \frac{1}{4}|\Delta(M_l-M)|^2 \\
&\qquad - |\Delta(M_l-M)|^2 \\
&\qquad + \frac{1}{\mu^4}(|M|^2-1)^2|M_l-M|^2 + \frac{1}{4}|\Delta(M_l-M)|^2 \\
&\qquad + \frac{1}{\mu^4}\underbrace{(|M_l|^2-|M|^2)^2}_{=(|M_l|+|M|)^2\underbrace{(|M_l|-|M|)^2}_{\leq |M_l-M|^2}}|M|^2 + \frac{1}{4}|\Delta(M_l-M)|^2\ \mathrm{d}x\ \mathrm{d}s \\
&\leq \int_0^t \|(v_l\cdot\nabla)(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s + \int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \frac{1}{\mu^4}(|M|^4 - 2|M|^2+1)|M_l-M|^2 \\
&\qquad + \frac{1}{\mu^4}(|M_l|+|M|)^2|M|^2|M_l-M|^2\ \mathrm{d}x\ \mathrm{d}s.
\end{aligned}
$$

An Application of Hölder's inequality yields

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\ \mathrm{d}x \\
&\leq \int_0^t \|(v_l\cdot\nabla)(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s + \int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad + \frac{1}{\mu^4}\int_0^t \|M\|^4_{L^6(\Omega;\mathbb{R}^3)}\|M_l-M\|^2_{L^6(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad - \frac{2}{\mu^4}\int_0^t\int_\Omega |M|^2|M_l-M|^2\ \mathrm{d}x\ \mathrm{d}s + \int_0^t \|M_l-M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad + \frac{1}{\mu^4}\int_0^t \overbrace{\||M_l||M|+|M|^2\|^2_{L^3(\Omega)}}^{\leq \underbrace{\||M_l||M|\|^2_{L^3(\Omega)}}_{\leq \|M_l\|^2_{L^6(\Omega;\mathbb{R}^3)}\|M\|^2_{L^6(\Omega;\mathbb{R}^3)}} + \underbrace{\||M|^2\|^2_{L^3(\Omega)}}_{=\|M\|^4_{L^6(\Omega;\mathbb{R}^3)}} + \underbrace{2\||M_l||M|\|_{L^3(\Omega)}}_{\leq 2\|M_l\|_{L^6(\Omega;\mathbb{R}^3)}\|M\|_{L^6(\Omega;\mathbb{R}^3)}}\underbrace{\||M|^2\|_{L^3(\Omega)}}_{=\|M\|^2_{L^6(\Omega;\mathbb{R}^3)}}} \|M_l-M\|^2_{L^6(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\leq \int_0^t \|(v_l\cdot\nabla)(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s + \int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad + \int_0^t \|M_l-M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\quad + \int_0^t C\|M_l-M\|^2_{L^6(\Omega;\mathbb{R}^3)}\ \mathrm{d}s.
\end{aligned}
$$

Next, using the estimate (3.107), we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\,\mathrm{d}x \\
&\quad\le \int_0^t \|(v_l\cdot\nabla)(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s + \int_0^t \|((v_l - v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s \\
&\qquad + \int_0^t \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s \\
&\qquad + \int_0^t C(\Omega)\Big(\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})} + \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big)\,\mathrm{d}s.
\end{aligned}$$

Moreover, since $\{v_l\}_l$ is uniformly bounded in $L^\infty(0,t_0^*;L^\infty(\Omega;\mathbb{R}^d))$, we have

$$\begin{aligned}
&\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) \\
&\quad\le \underbrace{\int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s) + C\|M_l-M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s)\,\mathrm{d}s}_{\text{non-decreasing}} \\
&\qquad\qquad + \int_0^t C\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(s)\,\mathrm{d}s,
\end{aligned} \tag{3.108}$$

where the constant C only depends on Ω, N, m and μ. We apply Gronwall's inequality to get

$$\begin{aligned}
&\sup_{0\le t\le t_0^*}\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(t) \\
&\le \int_0^{t_0^*}\|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s) + C\|M_l-M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s)\,\mathrm{d}s\;\mathrm{e}^{Ct_0^*}.
\end{aligned} \tag{3.109}$$

Due to (3.98) and (3.106) we can pass to the limit as $l\to\infty$ to see that, in summary,

$$M_l \xrightarrow{l\to\infty} M \text{ in } L^\infty(0,t_0^*;\mathbf{H}^1(\Omega;\mathbb{R}^3)). \tag{3.110}$$

Convergence of the solutions $\mathcal{L}(v_l)$. We have the continuity of the mapping $W' : L^p(0,t_0^*;L^2(\Omega;\mathbb{R}^{d\times d}))\to L^p(0,t_0^*;L^2(\Omega;\mathbb{R}^{d\times d}))$ for any $1\le p<\infty$ by (3.14) and (3.101) (see, e.g., [Rou13, Theorem 1.43] for Nemytskiĭ mappings in Bochner spaces), which tells us $W'(F_l)\xrightarrow{l\to\infty} W'(F)$ in $L^p(0,t_0^*;L^2(\Omega;\mathbb{R}^{d\times d}))$. So, together with (3.101) and (3.110) we obtain the strong convergence of $\{(D_m^i(t))_l\}_l$ in $L^p(0,t_0^*)$ for any $1\le p<\infty$ to the appropriate $D_m^i(t)$ which comes from the v (see (3.46)). We use this to prove the convergence of the solutions to the ODE (3.44), namely $\mathcal{L}(v_l)$, in the following. We first write the ODE system in vector form using the notation

$$\begin{aligned}
(\tilde g_m)_l &= ((\tilde g_m^1)_l,\dots,(\tilde g_m^m)_l),\\
\tilde g_m &= (\tilde g_m^1,\dots,\tilde g_m^m),\\
(D_m)_l &= ((D_m^1)_l,\dots,(D_m^m)_l),\\
D_m &= (D_m^1,\dots,D_m^m),\\
A^i &= (A^i_{jk})_{j,k=1}^m\in\mathbb{R}^{m\times m},\quad i=1,\dots,m,
\end{aligned}$$

and then subtract the corresponding ODE system (3.44) to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((\tilde g_m(t))_l - \tilde g_m(t)\right)\\
&\quad= \; -\nu\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)\left((\tilde g_m(t))_l - \tilde g_m(t)\right)\\
&\qquad\quad +\big((A^1(\tilde g_m)_l)\cdot(\tilde g_m)_l,\dots,(A^m(\tilde g_m)_l)\cdot(\tilde g_m)_l\big)\\
&\qquad\qquad\quad -\big((A^1\tilde g_m)\cdot\tilde g_m,\dots,(A^m\tilde g_m)\cdot\tilde g_m\big)\\
&\qquad\quad +(D_m(t))_l - D_m(t)\\
&\quad= \; -\nu\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)\left((\tilde g_m(t))_l - \tilde g_m(t)\right)\\
&\qquad\quad +\big((A^1(\tilde g_m)_l)\cdot(\tilde g_m)_l,\dots,(A^m(\tilde g_m)_l)\cdot(\tilde g_m)_l\big)\\
&\qquad\qquad\quad -\big((A^1(\tilde g_m)_l)\cdot\tilde g_m,\dots,(A^m(\tilde g_m)_l)\cdot\tilde g_m\big)\\
&\qquad\qquad\quad +\big((A^1(\tilde g_m)_l)\cdot\tilde g_m,\dots,(A^m(\tilde g_m)_l)\cdot\tilde g_m\big)\\
&\qquad\qquad\quad -\big((A^1\tilde g_m)\cdot\tilde g_m,\dots,(A^m\tilde g_m)\cdot\tilde g_m\big)\\
&\qquad\quad +(D_m(t))_l - D_m(t)\\
&\quad= \; -\nu\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)\left((\tilde g_m(t))_l - \tilde g_m(t)\right)\\
&\qquad\quad +\big((A^1(\tilde g_m)_l)\cdot\big((\tilde g_m)_l-\tilde g_m\big),\dots,(A^m(\tilde g_m)_l)\cdot\big((\tilde g_m)_l-\tilde g_m\big)\big)\\
&\qquad\qquad\quad +\big((A^1((\tilde g_m)_l-\tilde g_m))\cdot\tilde g_m,\dots,(A^m((\tilde g_m)_l-\tilde g_m))\cdot\tilde g_m\big)\\
&\qquad\quad +(D_m(t))_l - D_m(t).
\end{aligned}
$$

This expression we integrate in time, take the absolute value (the norm in $\mathbb{R}^m$ and the associated matrix norm) and estimate

$$
\begin{aligned}
&|(\tilde g_m(t))_l - \tilde g_m(t)|\\
&\quad\le \; \underbrace{|(\tilde g_m(0))_l - \tilde g_m(0)|}_{=0} + \nu|\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)|\int_0^t |(\tilde g_m(s))_l - \tilde g_m(s)|\;\mathrm{d}s\\
&\qquad + \int_0^t \Big|\big((A^1(\tilde g_m)_l)\cdot\big((\tilde g_m)_l-\tilde g_m\big),\dots,(A^m(\tilde g_m)_l)\cdot\big((\tilde g_m)_l-\tilde g_m\big)\big)\\
&\qquad\qquad\qquad +\big((A^1((\tilde g_m)_l-\tilde g_m))\cdot\tilde g_m,\dots,(A^m((\tilde g_m)_l-\tilde g_m))\cdot\tilde g_m\big)\Big|\;\mathrm{d}s\\
&\qquad + \int_0^t |(D_m(s))_l - D_m(s)|\;\mathrm{d}s\\
&\quad\le \; C(m)\int_0^t |(\tilde g_m(s))_l - \tilde g_m(s)|\;\mathrm{d}s\\
&\qquad + \int_0^t \max_{i=1,\dots,m}\big\{\big|(A^i(\tilde g_m)_l)\cdot\big((\tilde g_m)_l-\tilde g_m\big)\big|\big\}\\
&\qquad\qquad\qquad + \max_{i=1,\dots,m}\big\{\big|(A^i((\tilde g_m)_l-\tilde g_m))\cdot\tilde g_m\big|\big\}\;\mathrm{d}s\\
&\qquad + \int_0^t |(D_m(s))_l - D_m(s)|\;\mathrm{d}s.
\end{aligned}
$$

An application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&|(\tilde{g}_m(t))_l - \tilde{g}_m(t)| \\
&\leq \; C(m) \int_0^t |(\tilde{g}_m(s))_l - \tilde{g}_m(s)| \; \mathrm{d}s \\
&\quad + \int_0^t \max_{i=1,\dots,m} \Big\{ \underbrace{|A^i|}_{\leq C(m)} \underbrace{|(\tilde{g}_m)_l|}_{\leq N} |(\tilde{g}_m)_l - \tilde{g}_m| \Big\} \\
&\qquad\qquad + \max_{i=1,\dots,m} \Big\{ \underbrace{|A^i|}_{\leq C(m)} |(\tilde{g}_m)_l - \tilde{g}_m| \underbrace{|\tilde{g}_m|}_{\leq N} \Big\} \; \mathrm{d}s \\
&\quad + \int_0^t |(D_m(s))_l - D_m(s)| \; \mathrm{d}s \\
&\leq \; C(m) \int_0^t |(\tilde{g}_m(s))_l - \tilde{g}_m(s)| \; \mathrm{d}s + C(N,m) \int_0^t |(\tilde{g}_m)_l - \tilde{g}_m| \; \mathrm{d}s \\
&\quad + \int_0^t |(D_m(s))_l - D_m(s)| \; \mathrm{d}s \\
&\leq \; C(N,m) \int_0^t |(\tilde{g}_m(s))_l - \tilde{g}_m(s)| \; \mathrm{d}s + \underbrace{\int_0^t |(D_m(s))_l - D_m(s)| \; \mathrm{d}s}_{\text{non-decreasing}} .
\end{aligned}
$$

We apply Gronwall's inequality to obtain

$$
\big|(\tilde{g}_m(t))_l - \tilde{g}_m(t)\big| \leq \left(\int_0^t \big|(D^i_m(s))_l - D^i_m(s)\big| \; \mathrm{d}s \right) \mathrm{e}^{C(N,m)t}.
$$

Due to the convergence of $\big\{ \big(D^i_m(s)\big)_l \big\}_l$ the right-hand side of the inequality tends to zero as $l \to \infty$, so $(\tilde{g}_m(t))_l \xrightarrow{l\to\infty} \tilde{g}_m(t)$ uniformly. In view of (3.98), this is equivalent to $\mathcal{L}(v_l) \xrightarrow{l\to\infty} \mathcal{L}(v)$. Hence, $\mathcal{L}$ is continuous on $V_m(t_0^*)$.
Thus, by Schauder's fixed point theorem, $\mathcal{L}$ has a fixed point, denoted by $\overline{v}_m$, which is together with the corresponding $\overline{F}_m$ and $\overline{M}_m$ a local weak solution to the system (3.30)–(3.39). This completes the proof of Lemma 18, i.e., of the local existence of weak approximate solutions. □

3.1.3.3 Energy estimates for short time weak solutions to the approximate problem

We continue the analysis of the weak approximate solutions with energy estimates. These energy estimates are necessary to extend the solution beyond time t_0^* while keeping its regularity. We obtain

Corollary 20. *Let (v_m, F_m, M_m) be the weak solution to the approximate problem (3.30)–(3.39) in $\Omega \times (0, t_0^*)$ obtained in Lemma 18. Then, we have*

$$
\begin{aligned}
&\sup_{0\le t\le t_0^*}\left(\int_\Omega |v_m|^2 + C|F_m|^2 + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2 \,\mathrm{d}x\right)\\
&\qquad + 2\int_0^{t_0^*}\int_\Omega \nu|\nabla v_m|^2 + a\kappa|\nabla F_m|^2 + \left|\Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m\right|^2 \,\mathrm{d}x\,\mathrm{d}s\\
&\le \sup_{0\le t\le t_0^*}\left(\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2 \,\mathrm{d}x\right)\\
&\qquad + 2\int_0^{t_0^*}\int_\Omega \nu|\nabla v_m|^2 + a\kappa|\nabla F_m|^2 + \left|\Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m\right|^2 \,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_\Omega |v_0|^2 + 2W(F_0) + |\nabla M_0|^2 + \frac{1}{2\mu^2}(|M_0|^2-1)^2 \,\mathrm{d}x
\end{aligned}
\tag{3.111}
$$

and, in particular,

$$v_m \in L^\infty(0, t_0^*; \mathbf{H}) \cap L^2(0, t_0^*; \mathbf{V}), \tag{3.112}$$
$$F_m \in L^\infty(0, t_0^*; L^2(\Omega; \mathbb{R}^{d\times d})) \cap L^2(0, t_0^*; \mathbf{H}^1(\Omega; \mathbb{R}^{d\times d})) \tag{3.113}$$
$$M_m \in L^\infty(0, t_0^*; \mathbf{H}^1(\Omega; \mathbb{R}^3)) \cap L^2(0, t_0^*; \mathbf{H}^2(\Omega; \mathbb{R}^3)) \tag{3.114}$$

uniformly with respect to $m > 0$.

Proof. Notice that the following calculations are reasonable due to the regularity obtained in Lemma 17.
We multiply equation (3.30) by v_m, equation (3.32) by $W'(F_m)$, equation (3.33) by $-\Delta M_m + \frac{1}{\mu^2}(|M_m|^2 - 1)M_m$ and integrate all the equations over both Ω and $(0, t)$ for $t \le t_0^*$. Notice that $W'(F_m)$ is an admissible test function: $W'(F_m)$ is in $\mathbf{H}_0^1(\Omega; \mathbb{R}^{d\times d})$. Indeed, due to (3.14), it holds $W'(F_m) \in L^2(\Omega; \mathbb{R}^{d\times d})$ if $F_m \in L^2(\Omega; \mathbb{R}^{d\times d})$, which is guaranteed by Lemma 17. Moreover, since $W''(\cdot)$ is bounded by (3.16), we have that $\nabla W'(F_m) = W''(F_m)\nabla F_m$ is in $L^2(\Omega; \mathbb{R}^{d\times d\times d})$ if $\nabla F_m \in L^2(\Omega; \mathbb{R}^{d\times d\times d})$, which is again guaranteed by Lemma 17 where a bound on F_m in $L^2(0, t_0^*; \mathbf{H}^1(\Omega; \mathbb{R}^{d\times d}))$ is obtained. Finally, due to the continuity of the trace operator and (3.15), we know that $W'(F_m) = 0$ on $\partial\Omega$. For the tests, we

find (after using integration by parts)

$$
\begin{aligned}
\int_\Omega \frac{1}{2}|v_m|^2 \,\mathrm{d}x = \int_0^t \int_\Omega \Big(&- \nu |\nabla v_m|^2 - (v_m\cdot\nabla)v_m\cdot v_m \\
&+ \Big(\nabla\cdot\big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big)\Big)\cdot v_m\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_\Omega \frac{1}{2}|P_m(v_0)|^2\,\mathrm{d}x,
\end{aligned} \tag{3.115}
$$

$$
\begin{aligned}
\int_\Omega W(F_m)\,\mathrm{d}x + \int_0^t\int_\Omega (v_m\cdot\nabla)F_m : W'(F_m)\,\mathrm{d}x\,\mathrm{d}s& \\
- \int_0^t\int_\Omega (\nabla v_m F_m) : W'(F_m)\,\mathrm{d}x\,\mathrm{d}s& \\
= - \int_0^t\int_\Omega \kappa\nabla F_m \,\vdots\, \nabla W'(F_m)\,\mathrm{d}x\,\mathrm{d}s &+ \int_\Omega W(F_0)\,\mathrm{d}x,
\end{aligned} \tag{3.116}
$$

$$
\begin{aligned}
\int_\Omega \frac{1}{2}|\nabla M_m|^2 + \frac{1}{4\mu^2}(|M_m|^2-1)^2\,\mathrm{d}x &- \int_0^t\int_\Omega (v_m\cdot\nabla)M_m\cdot\Delta M_m\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \underbrace{\int_\Omega (v_m\cdot\nabla)M_m\cdot\left(\frac{1}{\mu^2}(|M_m|^2-1)M_m\right)\mathrm{d}x}_{=\int_\Omega (v_m\cdot\nabla)\left(\frac{1}{4\mu^2}(|M_m|^2-1)^2\right)\,\mathrm{d}x}\,\mathrm{d}s \\
&= -\int_0^t\int_\Omega \left|\Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m\right|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_\Omega \frac{1}{2}|\nabla M_0|^2 + \frac{1}{4\mu^2}(|M_0|^2-1)^2\,\mathrm{d}x.
\end{aligned} \tag{3.117}
$$

Notice that, due to $\nabla\cdot v_m = 0$ and $v_m = 0$ on $\partial\Omega$, we have

$$
\int_\Omega (v_m\cdot\nabla)v_m\cdot v_m\,\mathrm{d}x = 0
$$

In addition, we have

$$
\begin{aligned}
\int_0^t\int_\Omega \big(\nabla\cdot W'(F_m)F_m^\top\big)\cdot v_m\,\mathrm{d}x\,\mathrm{d}s &= -\int_0^t\int_\Omega \big(\nabla v_m W'(F_m)\big) : F_m\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t\int_\Omega \big(\nabla v_m F_m\big) : W'(F_m)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla\cdot\big(\nabla M_m\odot\nabla M_m\big) &= \nabla_j\big(\nabla_i (M_m)_k \nabla_j (M_m)_k\big) \\
&= \nabla\frac{|\nabla M_m|^2}{2} + \nabla^\top M_m \Delta M_m
\end{aligned}
$$

and

$$
(\nabla^\top M_m \Delta M_m)\cdot v_m = (v_m\cdot\nabla)M_m\cdot\Delta M_m.
$$

Next, we sum equations (3.115)–(3.117) and with the above calculations obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2 \,\mathrm{d}x \\
&\quad + \int_0^t \int_\Omega \left(\nabla \frac{|\nabla M_m|^2}{2} + \nabla W(F_m) + \nabla \left(\frac{1}{4\mu^2}(|M_m|^2-1)^2 \right) \right) \cdot v_m \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \int_\Omega -(\nabla \cdot W'(F_m)F_m^\top) \cdot v_m + \left(\nabla^\top M_m \Delta M_m \right) \cdot v_m \\
&\qquad\qquad - (\nabla v_m F_m) : W'(F_m) - (v_m \cdot \nabla) M_m \cdot \Delta M_m \,\mathrm{d}x\,\mathrm{d}s \\
&= - \int_0^t \int_\Omega \nu |\nabla v_m|^2 + \kappa \nabla F_m \,\vdots\, \nabla W'(F_m) + \left| \Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m \right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_\Omega |P_m(v_0)|^2 + 2W(F_0) + |\nabla M_0|^2 + \frac{1}{2\mu^2}(|M_0|^2-1)^2 \,\mathrm{d}x.
\end{aligned}
$$

Since $\nabla \cdot v_m = 0$ and $v_m = 0$ on $\partial\Omega$, the terms on the second line vanish. Notice that $\nabla W'(F_m) = W''(F_m)\nabla F_m$ in the sense $\nabla_\sigma W'(F_m)_{ij} = W''(F_m)_{ijkl}\nabla_\sigma (F_m)_{kl}$. We obtain by using (3.17), simplifying and rearranging

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2 \,\mathrm{d}x \\
&\quad + \int_0^t \int_\Omega \nu |\nabla v_m|^2 + a\kappa |\nabla F_m|^2 + \left| \Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m \right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
\leq\ &\frac{1}{2}\int_\Omega |P_m(v_0)|^2 + 2W(F_0) + |\nabla M_0|^2 + \frac{1}{2\mu^2}(|M_0|^2-1)^2 \,\mathrm{d}x.
\end{aligned}
$$

We calculate the supremum over all $t \in [0, t_0^*]$ on both sides of this equality and, since $\|P_m(v_0)\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|v_0\|_{L^2(\Omega;\mathbb{R}^d)}$, the second inequality in estimate (3.111) follows. Applying (3.13), the first inequality follows immediately. The improved regularities in (3.112) and (3.113) and their uniformity in m are a direct consequence of the preceding estimate. The regularity result (3.114) follows from the preceding estimate and an application of Young's inequality together with the boundedness of Ω:

$$
\|M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} = \int_\Omega |M_m|^2 - 1 + 1 \,\mathrm{d}x \leq \int_\Omega \frac{1}{2\mu^2}(|M_m|^2-1)^2 \,\mathrm{d}x + \left(\frac{\mu^2}{2} + 1 \right) |\Omega|.
$$

This concludes the proof. □

3.1.3.4 Weak solutions to the approximate problem by time extension

What remains to prove for Theorem 16 is the extension of the time interval, where solutions exist. We achieve this task using Corollary 20, thus ultimately justifying Theorem 16.

Proof. Let $0 < T < \infty$ be fixed. We first define

$$
\widetilde{C} := \int_\Omega |v_0|^2 + 2W(F_0) + |\nabla M_0|^2 + \frac{1}{2\mu^2}(|M_0|^2-1)^2 \,\mathrm{d}x
$$

to be the right-hand side of (3.111). If (v_m, F_m, M_m) is a solution to the system (3.30)–(3.39) in $\Omega \times (0, \tilde{t})$ for some $0 < \tilde{t} < t_0^*$, then

$$\|v_m\|^2_{L^2(\Omega;\mathbb{R}^d)}(\tilde{t}) + 2\int_\Omega W(F_m)(\tilde{t})\ \mathrm{d}x \\ + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times d})}(\tilde{t}) + \int_\Omega \frac{1}{4\mu^2}(|M_m(\tilde{t})|^2 - 1)^2\ \mathrm{d}x \le \widetilde{C}$$

due to (3.111).

Following the proof of Lemma 18, we conclude that there exists a constant δ which depends only on m and $\widetilde{C}$ (due to the L^∞-bounds obtained from the energy estimate (3.111) this δ does not depend on the time $\tilde{t}$) such that the system (3.30)–(3.39) has a solution $(\tilde{v}_m, \tilde{F}_m, \tilde{M}_m)$ on $\Omega \times [\tilde{t}, \tilde{t} + \delta]$ satisfying $(\tilde{v}_m, \tilde{F}_m, \tilde{M}_m)(\tilde{t}) = (v_m, F_m, M_m)(\tilde{t})$. We can continue this extension and finally obtain a solution (v_m, F_m, M_m) on $\Omega \times (0, T)$.

Notice that, due to the regularity of the solutions, the new initial data have always the same regularity as before.

Moreover, we have the energy estimate

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\int_\Omega |v_m|^2 + C|F_m|^2 + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2\ \mathrm{d}x\right) \\
&\qquad + 2\int_0^T\int_\Omega \nu|\nabla v_m|^2 + a\kappa|\nabla F_m|^2 + \left|\Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m\right|^2\ \mathrm{d}x\ \mathrm{d}s \\
&\le \sup_{0\le t\le T}\left(\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 + \frac{1}{2\mu^2}(|M_m|^2-1)^2\ \mathrm{d}x\right) \\
&\qquad + 2\int_0^T\int_\Omega \nu|\nabla v_m|^2 + a\kappa|\nabla F_m|^2 + \left|\Delta M_m - \frac{1}{\mu^2}(|M_m|^2-1)M_m\right|^2\ \mathrm{d}x\ \mathrm{d}s \\
&\le \int_\Omega |v_0|^2 + 2W(F_0) + |\nabla M_0|^2 + \frac{1}{2\mu^2}(|M_0|^2-1)^2\ \mathrm{d}x,
\end{aligned} \tag{3.118}$$

implying

$$v_m \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \tag{3.119}$$

$$F_m \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d})) \cap L^2(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^{d\times d})), \tag{3.120}$$

$$M_m \in L^\infty(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^3)) \cap L^2(0,T;\mathbf{H}^2(\Omega;\mathbb{R}^3)) \tag{3.121}$$

for any $m > 0$. This concludes the proof of Theorem 16. □

3.1.4 Existence of weak solutions to the original problem

In this part, we prove that the approximate solutions have a limit and that this limit is actually a solution to the original system (3.1)–(3.11). That means, in the following, we finish the proof of Theorem 9.

Proof of Theorem 9. We start by preparing passing to the limit as $m \to \infty$ to obtain a weak solution to the original system (3.1)–(3.11). To this end, the following convergence results are necessary. Their proof is given in Section 3.1.4.1 below.

$$v_m \to v \qquad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^d)), \tag{3.122}$$
$$\nabla v_m \rightharpoonup \nabla v \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{3.123}$$
$$M_m \to M \qquad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^3)), \tag{3.124}$$
$$\nabla M_m \to \nabla M \qquad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times d})), \tag{3.125}$$
$$F_m \to F \qquad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^{d\times d})), \tag{3.126}$$
$$\nabla F_m \rightharpoonup \nabla F \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d})). \tag{3.127}$$

3.1.4.1 Convergence results for the approximate weak solutions

The weak convergences (3.123) and (3.127) (up to subsequences, not relabeled) follow directly from the energy estimates (3.119) and (3.120), respectively (see Theorem 33 in Appendix A.2).

To obtain the strong convergences, we estimate the time derivatives of the respective quantities and rely on the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2).

Firstly, we show that $(v_m)_t \in L^{\frac{4}{3}}(0,T;\mathbf{V}^*)$. Using (3.24) and the fact that $\|P_m(\xi)\|_{\mathbf{V}} \le \|\xi\|_{\mathbf{V}} \le 1$, we obtain

$$\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\xi\|_{\mathbf{V}}\le 1}} \int_0^T \int_\Omega (v_m)_t \cdot (\zeta\xi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\xi\|_{\mathbf{V}}\le 1}} \int_0^T \int_\Omega (v_m)_t \cdot (\zeta P_m(\xi))\,\mathrm{d}x\,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\xi\|_{\mathbf{V}}\le 1}} \int_0^T \int_\Omega -(v_m\cdot\nabla)v_m \cdot (\zeta P_m(\xi)) \\
&\qquad\qquad - \left(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\right) : (\zeta \nabla P_m(\xi)) \\
&\qquad\qquad - \nu \nabla v_m : (\zeta\nabla P_m(\xi))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

An application of Hölder's inequality yields

$$\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\xi\|_{\mathbf{V}}\le 1}} \int_0^T \int_\Omega (v_m)_t \cdot (\zeta\xi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\xi\|_{\mathbf{V}}\le 1}} \Bigg(\int_0^T \|v_m\|_{L^3(\Omega)} \|\nabla v_m\|_{L^2(\Omega;\mathbb{R}^{d\times d})} |\zeta| \underbrace{\|P_m(\xi)\|_{L^6(\Omega;\mathbb{R}^d)}}_{\overset{\text{Sobolev}}{\underset{\text{embedding}}{\le}} C\|P_m(\xi)\|_{\mathbf{H}^1(\Omega;\mathbb{R}^d)}} \,\mathrm{d}t
\end{aligned}$$

$$+\int_0^T \Big(\underbrace{\|W'(F_m)F_m\|_{L^2(\Omega;\mathbb{R}^{d\times d})}}_{\overset{(3.14)}{\leq} C+\|F_m\|^2_{L^4(\Omega;\mathbb{R}^{d\times d})}}$$

$$+ \|\nabla M_m\|^2_{L^4(\Omega;\mathbb{R}^{3\times d})}\Big)|\zeta|\|\nabla P_m(\xi)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}t$$

$$+ \int_0^T \nu\|\nabla v_m\|_{L^2(\Omega;\mathbb{R}^{d\times d})}|\zeta|\|\nabla P_m(\xi)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\ \mathrm{d}t\Big)$$

$$\overset{\text{Hölder}}{\leq} C\|v_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^d))}\|\nabla v_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}$$
$$+CT + \|F_m\|^2_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))} + \|\nabla M_m\|^2_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times d}))}$$
$$+C\nu\|\nabla v_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}.$$

From the regularities (3.119)–(3.121) and interpolation inequalities (see Proposition 34 in Appendix A.2) we get the boundedness of the norms $\|v_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^d))}$, $\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times d}))}$ and $\|F_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))}$. Moreover, we have that $\|\nabla v_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}$ is bounded since $[0,T]$ is a bounded interval. In summary, we obtain

$$(v_m)_t \in L^{\frac{4}{3}}(0,T;\mathbf{V}^*) \tag{3.128}$$

uniformly in m.

From the embeddings $\mathbf{V} \overset{c}{\subset} \overline{\mathcal{V}}^{L^4(\Omega;\mathbb{R}^d)} \subset \mathbf{V}^*$, where the first embedding is compact and the second one is continuous, and the fact that $v_m \in L^2(0,T;\mathbf{H}^1_0(\Omega;\mathbb{R}^d))$ by (3.119), we conclude by the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2) that the embedding

$$\Big\{v \in L^2(0,T;\mathbf{V}) :\ v_t \in L^{\frac{4}{3}}(0,T;\mathbf{V}^*)\Big\} \overset{c}{\subset} L^2\left(0,T;\overline{\mathcal{V}}^{L^4(\Omega;\mathbb{R}^d)}\right)$$

is compact. This yields the strong convergence (3.122) (up to subsequence) of $\{v_m\}_m$.

For the convergence result (3.125) of the magnetization we apply the same technique as above. We estimate $(M_m)_t$ and obtain from there an estimate on $(\nabla M_m)_t$:

$$\sup_{\substack{\|\zeta\|_{L^4(0,T)}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^T \int_\Omega (M_m)_t\cdot(\zeta\varphi)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\leq 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\leq 1}} \int_0^T \int_\Omega -(v_m\cdot\nabla)M_m\cdot(\zeta\varphi) + \Delta M_m\cdot(\zeta\varphi)$$
$$-\frac{1}{\mu^2}(|M_m|^2-1)M_m\cdot(\zeta\varphi)\ \mathrm{d}x\ \mathrm{d}t.$$

An application of Hölder's inequality yields

$$
\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \int_\Omega (M_m)_t\cdot(\zeta\varphi)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \|v_m\|_{L^4(\Omega;\mathbb{R}^d)}\|\nabla M_m\|_{L^4(\Omega;\mathbb{R}^{3\times d})}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\\
&\qquad + \|\Delta M_m\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\\
&\qquad + \frac{1}{\mu^2}\|(|M_m|^2-1)M_m\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}t.
\end{aligned}
$$

By another application of Hölder's inequality, we get

$$
\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \int_\Omega (M_m)_t\cdot(\zeta\varphi)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \sup_{\|\zeta\|_{L^4(0,T)}\le 1}\Big(\|v_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^d))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times d}))}\|\zeta\|_{L^4(0,T)}\\
&\qquad + \|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^4(0,T)}\\
&\qquad + \frac{1}{\mu^2}\|(|M_m|^2-1)M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^4(0,T)}\Big)\\
&\le \|v_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^d))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times d}))} + \|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}\\
&\qquad + \frac{1}{\mu^2}\underbrace{\||M_m|^3\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega))}}_{=\|M_m\|^3_{L^4(0,T;L^6(\Omega;\mathbb{R}^3))}} + \frac{1}{\mu^2}\|M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}.
\end{aligned}
$$

From the regularities (3.119)–(3.121), and interpolation inequalities (see Proposition 34 in Appendix A.2) and the boundedness of the interval $(0,T)$, we get that the right-hand side is bounded. Thus,

$$(M_m)_t \in L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3)) \tag{3.129}$$

uniformly in m. This then implies that

$$(\nabla M_m)_t \in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{3\times d})) \tag{3.130}$$

uniformly in m.

Remark 21. *In fact, in general it holds that* $f\in L^p(0,T;L^q(\Omega;\mathbb{R}^n))$ *implies* $\nabla f\in L^p(0,T;(W_0^{1,q'}(\Omega;\mathbb{R}^{n\times d}))^*) = L^p(0,T;W^{-1,q}(\Omega;\mathbb{R}^{n\times d}))$*, for* $p,q\in(1,\infty)$

with $\frac{1}{q} + \frac{1}{q'} = 1$*, which is a direct consequence of the following calculation:*

$$\begin{aligned}\int_0^T \|\nabla f\|^p_{W^{-1,q}(\Omega;\mathbb{R}^{n\times d})} \,\mathrm{d}t &= \int_0^T \sup_{\|\varphi\|_{W^{1,q'}}\le 1} \left| {}_{W^{-1,q}}\langle \nabla f, \varphi\rangle_{W_0^{1,q'}}\right|^p \,\mathrm{d}t \\ &= \int_0^T \sup_{\|\varphi\|_{W^{1,q'}}\le 1} \left|\int_\Omega f\nabla\cdot\varphi \,\mathrm{d}x\right|^p \,\mathrm{d}t \\ &\le \int_0^T \sup_{\|\varphi\|_{W^{1,q'}}\le 1} \|f\|^p_{L^q(\Omega;\mathbb{R}^n)}\|\nabla\cdot\varphi\|^p_{L^{q'}(\Omega;\mathbb{R}^n)} \,\mathrm{d}t \\ &\le \int_0^T \|f\|^p_{L^q(\Omega;\mathbb{R}^n)} \,\mathrm{d}t.\end{aligned}$$

Following the above arguments for the convergence of $\{v_m\}_m$, we obtain the strong convergence results (3.124) and (3.125) (up to subsequences, respectively). For the convergence result (3.126) of the deformation gradient, we apply the same technique once again. To this end, using (3.41), we estimate $(F_m)_t$ in $L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d}))$:

$$\begin{aligned}&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\Xi\|_{\mathbf{H}_0^1(\Omega;\mathbb{R}^{d\times d})}\le 1}} \int_0^T {}_{\mathbf{H}^{-1}}\Big\langle (F_m)_t, \Xi\Big\rangle_{\mathbf{H}_0^1} \zeta \,\mathrm{d}t \\ &= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\Xi\|_{\mathbf{H}_0^1(\Omega;\mathbb{R}^{d\times d})}\le 1}} \int_0^T\int_\Omega -(v_m\cdot\nabla)F_m : (\zeta\Xi) + (\nabla v_m F_m) : (\zeta\Xi) \\ &\qquad - \kappa\nabla F_m \,\vdots\, (\zeta\nabla\Xi) \,\mathrm{d}x\,\mathrm{d}t \\ &\overset{\text{Hölder}}{\le} \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\Xi\|_{\mathbf{H}_0^1(\Omega;\mathbb{R}^{d\times d})}\le 1}} \int_0^T \|v_m\|_{L^3(\Omega)}\|\nabla F_m\|_{L^2(\Omega;\mathbb{R}^{d\times d\times d})}|\zeta|\|\Xi\|_{L^6(\Omega;\mathbb{R}^{d\times d})} \\ &\qquad + \|\nabla v_m\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|F_m\|_{L^3(\Omega;\mathbb{R}^{d\times d})}|\zeta|\|\Xi\|_{L^6(\Omega;\mathbb{R}^{d\times d})} \\ &\qquad + \kappa\|\nabla F_m\|_{L^2(\Omega;\mathbb{R}^{d\times d\times d})}|\zeta|\|\nabla\Xi\|_{L^2(\Omega;\mathbb{R}^{d\times d\times d})} \,\mathrm{d}t \\ &\overset{\text{Hölder}}{\le} \sup_{\|\zeta\|_{L^2(0,T)}\le 1} \Big(\|v_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^{d\times d}))}\|\nabla F_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))}\|\zeta\|_{L^4(0,T)} \\ &\qquad + \|\nabla v_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}\|F_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^{d\times d}))}\|\zeta\|_{L^4(0,T)} \\ &\qquad + \kappa\|\nabla F_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))}\|\zeta\|_{L^4(0,T)}\Big) \\ &\le \|v_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^d))}\|\nabla F_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))} \\ &\qquad + \|\nabla v_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}\|F_m\|_{L^4(0,T;L^3(\Omega;\mathbb{R}^{d\times d}))} \\ &\qquad + \kappa\|\nabla F_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))}.\end{aligned}$$

Again, from the regularities (3.119)–(3.121), and interpolation inequalities (see Proposition 34 in Appendix A.2) and the boundedness of $(0,T)$, we get that the right-hand side is bounded. Thus,

$$(F_m)_t \in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d})) \tag{3.131}$$

uniformly in m. Like above, we obtain the strong convergence (3.126) (up to subsequences).

3.1.4.2 Convergence to the weak formulations of the original problem

Up to now, we made sure that the solutions to the approximate problems converge to some limit. In the following, we need to show that the limit also satisfies the weak formulation of the system (3.1), (3.4), (3.5) in $\Omega \times (0,T)$.
To this end, we insert the solutions of the approximate problem and approximate test functions into the weak formulation (3.24)–(3.26) and pass to the limit as $m \to \infty$. The boundary conditions (3.6)–(3.8) hold for the limit, since the approximate solutions are constructed satisfying these conditions and they are in a closed subspace of the respective spaces for the solutions. The attainment of the initial data (3.9)–(3.11) is then shown in the final step of the entire proof. Notice that since the weak solution v_m to the approximate problem is defined using test functions from the projected spaces $\mathbf{H}_m$ in (3.24), we also need to pass to the limit with these test functions (only in space). However, for any test function $\xi \in \mathbf{V}$ we immediately find a sequence of approximate test functions $\xi_m := P_m(\xi) \in \mathbf{H}_m$ which converges strongly to ξ. In the following, we will use this particular sequence of test functions. Moreover, $\zeta \in W^{1,\infty}(0,T)$ is a test function satisfying $\zeta(T) = 0$.

Convergence of the v-equation (3.24). We need to show that with the convergence results (3.122)–(3.127) the equation

$$\begin{aligned}\int_0^T \int_\Omega & -v_m \cdot (\zeta' \xi_m) + (v_m \cdot \nabla) v_m \cdot (\zeta \xi_m) \\ & + \Big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\Big) : (\zeta \nabla \xi_m) \,\mathrm{d}x \,\mathrm{d}t \\ & - \int_\Omega v_m(0) \cdot (\zeta(0)\xi_m) \,\mathrm{d}x = -\int_0^T \int_\Omega \nu \nabla v_m : (\zeta \nabla \xi_m) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.132}$$

converges to the equation

$$\begin{aligned}\int_0^T \int_\Omega & -v \cdot (\zeta' \xi) + (v \cdot \nabla) v \cdot (\zeta \xi) \\ & + \Big(W'(F)F^\top - \nabla M \odot \nabla M\Big) : (\zeta \nabla \xi) \,\mathrm{d}x \,\mathrm{d}t \\ & - \int_\Omega v_0 \cdot (\zeta(0)\xi) \,\mathrm{d}x = -\int_0^T \int_\Omega \nu \nabla v : (\zeta \nabla \xi) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.133}$$

as $m \to \infty$. We examine each term individually; for the first term, we obtain

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega v_m \cdot (\zeta' \xi_m) - v \cdot (\zeta' \xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
&= \quad \left|\int_0^T \int_\Omega v_m \cdot (\zeta' \xi_m) - v \cdot (\zeta' \xi_m) + v \cdot (\zeta' \xi_m) - v \cdot (\zeta' \xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
&\leq \quad \int_0^T \int_\Omega \big|(v_m - v) \cdot (\zeta' \xi_m)\big| + |v \cdot (\zeta'(\xi_m - \xi))| \, \mathrm{d}x \, \mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \quad \|v_m - v\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \underbrace{\|\zeta' \xi_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}}_{\leq C} \\
&\qquad + \underbrace{\|v\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}}_{\leq C} \|\zeta'(\xi_m - \xi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \xrightarrow{m \to \infty} 0.
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega (v_m \cdot \nabla) v_m \cdot (\zeta \xi_m) - (v \cdot \nabla) v \cdot (\zeta \xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
&= \quad \left|\int_0^T \int_\Omega (v_m \cdot \nabla) v_m \cdot (\zeta \xi_m) - (v \cdot \nabla) v_m \cdot (\zeta \xi)\right. \\
&\qquad \left. + (v \cdot \nabla) v_m \cdot (\zeta \xi) - (v \cdot \nabla) v \cdot (\zeta \xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
&\leq \quad \int_0^T \int_\Omega \Big|\big(v_m \otimes (\zeta \xi_m) - v \otimes (\zeta \xi)\big) : \nabla^\top v_m\Big| \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \left|\int_0^T \int_\Omega (\nabla v_m - \nabla v) : \underbrace{(v \otimes (\zeta \xi))}_{\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d \times d}))} \, \mathrm{d}x \, \mathrm{d}t\right| \\
&\overset{\text{Hölder}}{\leq} \quad \|v_m \otimes (\zeta \xi_m) - v \otimes (\zeta \xi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d \times d}))} \underbrace{\|\nabla v_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d \times d}))}}_{\leq C} \\
&\qquad + \left|\int_0^T \int_\Omega (\nabla v_m - \nabla v) : (v \otimes (\zeta \xi)) \, \mathrm{d}x \, \mathrm{d}t\right| \xrightarrow{m \to \infty} 0.
\end{aligned}
$$

Considering the third term on the left-hand side for the stress tensor, we look at all the summands separately. Since we have the continuity of the mapping $W' : L^2(0,T;L^4(\Omega;\mathbb{R}^{d \times d})) \to L^2(0,T;L^4(\Omega;\mathbb{R}^{d \times d}))$ by (3.14) and (3.120) (see, e.g., [Rou13, Theorem 1.43] for Nemytskiĭ mappings in Bochner spaces), we get for the F part

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega \Big(W'(F_m) F_m^\top\Big) : (\zeta \nabla \xi_m) - \Big(W'(F) F^\top\Big) : (\zeta \nabla \xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
&= \quad \left|\int_0^T \int_\Omega \Big(W'(F_m) F_m^\top\Big) : (\zeta \nabla \xi_m) - \Big(W'(F) F^\top\Big) : (\zeta \nabla \xi_m)\right. \\
&\qquad \left. + \Big(W'(F) F^\top\Big) : (\zeta \nabla \xi_m) - \Big(W'(F) F^\top\Big) : (\zeta \nabla \xi) \, \mathrm{d}x \, \mathrm{d}t\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^T \int_\Omega \Big|(W'(F_m)F_m^\top - W'(F)F^\top) : (\zeta\nabla\xi_m)\Big| \\
&\qquad\qquad + \Big|W'(F)F^\top : (\zeta(\nabla\xi_m - \nabla\xi))\Big| \;\mathrm{d}x\;\mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \underbrace{\|W'(F_m)\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \|F_m - F\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))} \\
&\qquad\qquad \times \underbrace{\|\zeta\nabla\xi_m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \\
&\qquad + \|W'(F_m) - W'(F)\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))} \\
&\qquad\qquad \times \underbrace{\|F\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))}\|\zeta\nabla\xi_m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \\
&\qquad + \underbrace{\|W'(F)F^\top\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \|\zeta(\nabla\xi_m - \nabla\xi)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}
$$

We obtain for the M part

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega (\nabla M_m \odot \nabla M_m) : (\zeta\nabla\xi_m) - (\nabla M \odot \nabla M) : (\zeta\nabla\xi) \;\mathrm{d}x\;\mathrm{d}t\right| \\
&= \Bigg| \int_0^T \int_\Omega (\nabla M_m \odot \nabla M_m) : (\zeta\nabla\xi_m) - (\nabla M \odot \nabla M) : (\zeta\nabla\xi_m) \\
&\qquad + (\nabla M \odot \nabla M) : (\zeta\nabla\xi_m) - (\nabla M \odot \nabla M) : (\zeta\nabla\xi) \;\mathrm{d}x\;\mathrm{d}t\Bigg| \\
&\leq \int_0^T \int_\Omega |(\nabla M_m \odot \nabla M_m - \nabla M \odot \nabla M) : (\zeta\nabla\xi_m)| \\
&\qquad + |(\nabla M \odot \nabla M) : (\zeta(\nabla\xi_m - \nabla\xi))| \;\mathrm{d}x\;\mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \|\nabla M_m \odot \nabla M_m - \nabla M \odot \nabla M\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \underbrace{\|\zeta\nabla\xi_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \\
&\qquad + \underbrace{\|\nabla M \odot \nabla M\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \|\zeta(\nabla\xi_m - \nabla\xi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}
$$

The last terms on the left-hand side of (3.132) and (3.133) yield

$$
\begin{aligned}
&\left|\int_\Omega v_m(0)\cdot(\zeta(0)\xi_m) - v_0\cdot(\zeta(0)\xi) \;\mathrm{d}x\right| \\
&= \left|\int_\Omega v_m(0)\cdot(\zeta(0)\xi_m) - v_m(0)\cdot(\zeta(0)\xi) + v_m(0)\cdot(\zeta(0)\xi) - v_0\cdot(\zeta(0)\xi) \;\mathrm{d}x\right| \\
&\leq \int_\Omega |(v_m(0)\zeta(0))\cdot(\xi_m - \xi)| + |(v_m(0) - v_0)\cdot(\zeta(0)\xi)| \;\mathrm{d}x
\end{aligned}
$$

$$\overset{\text{Hölder}}{\leq} \underbrace{\|v_0\zeta(0)\|_{L^2(\Omega)}}_{\leq C} \|\xi_m - \xi\|_{L^2(\Omega;\mathbb{R}^d)} + \|v_m(0) - v_0\|_{L^2(\Omega;\mathbb{R}^d)} \underbrace{\|\zeta(0)\xi\|_{L^2(\Omega;\mathbb{R}^d)}}_{\leq C}$$
$$\xrightarrow{m\to\infty} 0.$$

Finally, we see that the right-hand side of (3.132) converges, too (we omit the constant ν):

$$\left|\int_0^T \int_\Omega \nabla v_m : (\zeta\nabla\xi_m) - \nabla v : (\zeta\nabla\xi) \,\mathrm{d}x \,\mathrm{d}t\right|$$
$$= \left|\int_0^T \int_\Omega \nabla v_m : (\zeta\nabla\xi_m) - \nabla v : (\zeta\nabla\xi_m) + \nabla v : (\zeta\nabla\xi_m) - \nabla v : (\zeta\nabla\xi) \,\mathrm{d}x \,\mathrm{d}t\right|$$
$$\leq \int_0^T \int_\Omega |(\nabla v_m - \nabla v) : \underbrace{(\zeta\nabla\xi_m)}_{\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} | + |\nabla v : (\zeta(\nabla\xi_m - \nabla\xi))| \,\mathrm{d}x \,\mathrm{d}t$$
$$\overset{\text{Hölder}}{\leq} \int_0^T \int_\Omega |(\nabla v_m - \nabla v) : (\zeta\nabla\xi_m)| \,\mathrm{d}x \,\mathrm{d}t + \underbrace{\|\nabla v\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \|\zeta(\nabla\xi_m - \nabla\xi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}$$
$$\xrightarrow{m\to\infty} 0.$$

Thus, the v-equation (3.24) converges.

Convergence of the F-equation (3.25). We need to prove that with the convergence results (3.122)–(3.127) the equation

$$\int_0^T \int_\Omega -F_m : (\zeta'\Xi) + (v_m \cdot \nabla)F_m : (\zeta\Xi) \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega F_m(0) : (\zeta(0)\Xi) \,\mathrm{d}x$$
$$= -\int_0^T \int_\Omega \kappa\nabla F_m \,\vdots\, (\zeta\nabla\Xi) \,\mathrm{d}x \,\mathrm{d}t \tag{3.134}$$

converges to the equation

$$\int_0^T \int_\Omega -F : (\zeta'\Xi) + (v \cdot \nabla)F : (\zeta\Xi) \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega F_0 : (\zeta(0)\Xi) \,\mathrm{d}x$$
$$= -\int_0^T \int_\Omega \kappa\nabla F \,\vdots\, (\zeta\nabla\Xi) \,\mathrm{d}x \,\mathrm{d}t \tag{3.135}$$

as $m \to \infty$. Notice that we integrated by parts with respect to time, so the dual pairing becomes an integral. Moreover, the test functions $\zeta\Xi$ are taken from the same spaces for both the approximate problem and the original problem. Thus, the third term on the left-hand side of the equation converges since $F_m(0) \to F_0$ strongly in $L^2(\Omega)$ by construction.

The first term on the left-hand side and the right-hand side of the equation are linear, so, the convergence is directly provided by the weak convergence of the sequences.
Estimates for the second term yield

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega (v_m \cdot \nabla)F_m : (\zeta\Xi) - (v \cdot \nabla)F : (\zeta\Xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
= \quad & \left|\int_0^T \int_\Omega (v_m \cdot \nabla)F_m : (\zeta\Xi) - (v \cdot \nabla)F_m : (\zeta\Xi)\right. \\
& \qquad \left. + (v \cdot \nabla)F_m : (\zeta\Xi) - (v \cdot \nabla)F : (\zeta\Xi) \, \mathrm{d}x \, \mathrm{d}t\right| \\
\leq \quad & \int_0^T \int_\Omega |((v_m - v) \cdot \nabla)F_m : (\zeta\Xi)| + |(v \cdot \nabla)(F_m - F) : (\zeta\Xi)| \, \mathrm{d}x \, \mathrm{d}t \\
\overset{\text{Hölder}}{\leq} \quad & \|v_m - v\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^d))} \underbrace{\|\nabla F_m\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))} \|\zeta\Xi\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^{d\times d}))}}_{\leq C} \\
& \qquad + \|\nabla F_m - \nabla F\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))} \underbrace{\|v(\zeta\Xi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d\times d}))}}_{\leq C} \\
& \xrightarrow{m\to\infty} 0.
\end{aligned}
$$

Thus, the F-equation (3.25) converges. Next, we prove

Convergence of the M-equation (3.26). We need to show that with the convergence results (3.122)–(3.127) the equation

$$
\begin{aligned}
&\int_0^T \int_\Omega -M_m \cdot (\zeta'\varphi) + (v \cdot \nabla)M_m \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega M_m(0) \cdot (\zeta(0)\varphi) \, \mathrm{d}x \\
=& \int_0^T \int_\Omega -\nabla M_m : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M_m|^2 - 1)M_m \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
\tag{3.136}
$$

converges to the equation

$$
\begin{aligned}
&\int_0^T \int_\Omega -M \cdot (\zeta'\varphi) + (v \cdot \nabla)M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega M_0 \cdot (\zeta(0)\varphi) \, \mathrm{d}x \\
=& \int_0^T \int_\Omega -\nabla M : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M|^2 - 1)M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
\tag{3.137}
$$

as $m \to \infty$. Notice that we integrated by parts with respect to time, so, the dual form becomes an integral. Moreover, the test functions $\zeta\varphi$ are taken from the same spaces for both the approximate problem and the original problem. Thus, the third term on the left-hand side of the equation converges since $M_m(0) \to M_0$ strongly in $L^2(\Omega)$ by construction.

For the first term we obtain

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega M_m \cdot (\zeta'\varphi) - M \cdot (\zeta'\varphi) \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \quad \int_0^T \int_\Omega |(M_m - M) \cdot (\zeta'\varphi)| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \overset{\text{Hölder}}{\leq} \quad \|M_m - M\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \underbrace{\|\zeta'\varphi\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}}_{\leq C} \xrightarrow{m\to\infty} 0.
\end{aligned}
$$

Estimates for the second term yield

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega (v_m \cdot \nabla) M_m \cdot (\zeta\varphi) - (v \cdot \nabla) M \cdot (\zeta\varphi) \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \quad \Bigg|\int_0^T \int_\Omega (v_m \cdot \nabla) M_m \cdot (\zeta\varphi) - (v \cdot \nabla) M_m \cdot (\zeta\varphi) \\
&\qquad\qquad + (v \cdot \nabla) M_m \cdot (\zeta\varphi) - (v \cdot \nabla) M \cdot (\zeta\varphi) \,\mathrm{d}x\,\mathrm{d}t\Bigg| \\
&\quad \leq \quad \int_0^T \int_\Omega |((v_m - v) \cdot \nabla) M_m \cdot (\zeta\varphi)| + |(v \cdot \nabla)(M_m - M) \cdot (\zeta\varphi)| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \overset{\text{Hölder}}{\leq} \quad \|v_m - v\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^d))} \underbrace{\|\nabla M_m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times d}))} \|\zeta\varphi\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))}}_{\leq C} \\
&\qquad\qquad + \|\nabla M_m - \nabla M\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times d}))} \underbrace{\|(\zeta\varphi) \otimes v\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times d}))}}_{\leq C} \\
&\quad \xrightarrow{m\to\infty} 0.
\end{aligned}
$$

Next, we see that the first term on the right-hand side of the equation converges, too, since it is linear and thus the weak convergence directly provides the result. Finally, we obtain for the second term on the right-hand side of the equation (again, we omit the constant for brevity)

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega (|M_m|^2 - 1) M_m \cdot (\zeta\varphi) - (|M|^2 - 1) M \cdot (\zeta\varphi) \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \quad \Bigg|\int_0^T \int_\Omega (|M_m|^2 - 1) M_m \cdot (\zeta\varphi) - (|M|^2 - 1) M_m \cdot (\zeta\varphi) \\
&\qquad\qquad + (|M|^2 - 1) M_m \cdot (\zeta\varphi) - (|M|^2 - 1) M \cdot (\zeta\varphi) \,\mathrm{d}x\,\mathrm{d}t\Bigg| \\
&\quad = \quad \Bigg|\int_0^T \int_\Omega (|M_m|^2 - |M|^2) M_m \cdot (\zeta\varphi) \\
&\qquad\qquad + (|M|^2 - 1)(M_m - M) \cdot (\zeta\varphi) \,\mathrm{d}x\,\mathrm{d}t\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \int_0^T \int_\Omega |(|M_m| + |M|) \underbrace{(|M_m| - |M|)}_{\leq |M_m - M|} M_m \cdot (\zeta\varphi)| \\
& \qquad + ||M|^2 (M_m - M) \cdot (\zeta\varphi)| + |(M_m - M) \cdot (\zeta\varphi)| \, \mathrm{d}x \, \mathrm{d}t \\
\leq \quad & \int_0^T \int_\Omega \big||M_m||M_m - M| M_m \cdot (\zeta\varphi)\big| + \big||M||M_m - M| M_m \cdot (\zeta\varphi)\big| \\
& \qquad + \big||M|^2 (M_m - M) \cdot (\zeta\varphi)\big| + \big|(M_m - M) \cdot (\zeta\varphi)\big| \, \mathrm{d}x \, \mathrm{d}t \\
\overset{\text{Hölder}}{\leq} \quad & \|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|M_m - M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \\
& \qquad \times \|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \\
& + \|M\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|M_m - M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \\
& \qquad \|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \\
& + \underbrace{\||M|^2\|_{L^\infty(0,T;L^2(\Omega))}}_{= \|M\|^2_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))}} \|M_m - M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \\
& + \|M_m - M\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \xrightarrow{m \to \infty} 0.
\end{aligned}
$$

Notice that $\|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \leq C$ and $\|M\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \leq C$ due to (3.121) and the lower semicontinuity of norms. Thus, the M-equation converges.

3.1.4.3 Attainment of initial data for the weak solution to the original problem

Finally, we are left to prove that the initial data is actually attained by the solution. We already obtained

$$
\begin{aligned}
(v_m)_t &\in L^{\frac{4}{3}}(0,T;\mathbf{V}^*), \\
(M_m)_t &\in L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3)), \\
(\nabla M_m)_t &\in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{3\times d})), \\
(F_m)_t &\in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{d\times d}))
\end{aligned}
$$

from the estimates in (3.128), (3.129), (3.130), and (3.131), respectively. Now, we treat the quantities v_m, M_m, ∇M_m and F_m together (in the sense that we omit the target spaces) to establish the attainment result for the initial data. The regularity results (3.119)–(3.121) provide us with the fact that

$$v_m, M_m, \nabla M_m, F_m \in L^4(0,T;\mathbf{H}^{-1}(\Omega)).$$

With the regularities on the time derivatives obtained above and the help of Gelfand's triple (see Lemma 36 in Appendix A.2) we get that

$$v_m, M_m, \nabla M_m, F_m \in C^0(0,T;\mathbf{H}^{-1}(\Omega)).$$

Now, since it also follows from (3.119)–(3.121) that

$$v_m, M_m, \nabla M_m, F_m \in L^\infty(0,T;L^2(\Omega)),$$

and since $L^2(\Omega)$ is a reflexive Banach space densely and compactly embedded into $\mathbf{H}^{-1}(\Omega)$, we obtain that

$$v_m, M_m, \nabla M_m, F_m \in C^0(0,T; L^2_w(\Omega)),$$

where the index w indicates weak topology in L^2 (for the general result and a proof we refer to [Tem77, Chapter III, Lemma 1.4]). Therefore, it makes sense, to talk about the attainment of initial data. Now, we prove that the initial values of the solutions coincide with the initial data in the L^2-sense.

Attainment for v. We start with the velocity and show that $v(0) = v_0$. We firstly integrate the first term in (3.132) by parts with respect to time to obtain

$$\begin{aligned}
&\int_0^T {}_{\mathbf{H}^{-1}}\big\langle (v_m)_t, \xi_m \big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (v_m \cdot \nabla) v_m \cdot (\zeta \xi_m) + \Big(W'(F_m) F_m^\top - \nabla M_m \odot \nabla M_m \Big) : (\zeta \nabla \xi_m) \,\mathrm{d}x \,\mathrm{d}t \\
&= - \int_0^T \int_\Omega \nu \nabla v_m : (\zeta \nabla \xi_m) \,\mathrm{d}x \,\mathrm{d}t. \qquad (3.138)
\end{aligned}$$

From the obtained regularity for $(v_m)_t$, it is a direct consequence that the first term in (3.138) converges to $\int_0^T {}_{\mathbf{H}^{-1}}\langle v_t, \xi \rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t$, so, with the previous convergence results it is clear that (3.138) converges to

$$\begin{aligned}
&\int_0^T {}_{\mathbf{H}^{-1}}\big\langle v_t, \xi \big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (v \cdot \nabla) v \cdot (\zeta \xi) + \Big(W'(F) F^\top - \nabla M \odot \nabla M \Big) : (\zeta \nabla \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&= - \int_0^T \int_\Omega \nu \nabla v : (\zeta \nabla \xi) \,\mathrm{d}x \,\mathrm{d}t. \qquad (3.139)
\end{aligned}$$

Integrating by parts with respect to t in (3.133) (a new boundary term is showing up) and comparing the equation with (3.139), we see that (we choose $\zeta(0) = 1$)

$$\int_\Omega (v(0) - v_0) \cdot \xi \,\mathrm{d}x = 0$$

for any $\xi \in \mathbf{H}^1_0(\Omega; \mathbb{R}^d)$. This then proves the attainment for v.

Attainment for F. For the deformation gradient F, we show that $F(0) = F_0$. We integrate the first term in (3.134) by parts with respect to time to obtain

$$\begin{aligned}
\int_0^T {}_{\mathbf{H}^{-1}}\Big\langle (F_m)_t, \Xi \Big\rangle_{\mathbf{H}^1_0} \zeta \,\mathrm{d}t + \int_0^T \int_\Omega (v_m \cdot \nabla) F_m : (\zeta \Xi) \,\mathrm{d}x \,\mathrm{d}t \\
= - \int_0^T \int_\Omega \kappa \nabla F_m \,\vdots\, (\zeta \nabla \Xi) \,\mathrm{d}x \,\mathrm{d}t. \quad (3.140)
\end{aligned}$$

From the obtained regularity for $(F_m)_t$ it is a direct consequence that the first term in (3.140) converges to $\int_0^T {}_{\mathbf{H}^{-1}}\langle F_t, \xi\rangle_{\mathbf{H}_0^1}\zeta \ \mathrm{d}t$, so, (3.140) converges to

$$\int_0^T {}_{\mathbf{H}^{-1}}\Big\langle F_t, \Xi\Big\rangle_{\mathbf{H}_0^1} \zeta \ \mathrm{d}t + \int_0^T \int_\Omega (v\cdot\nabla)F : (\zeta\Xi) \ \mathrm{d}x \ \mathrm{d}t$$
$$= -\int_0^T \int_\Omega \kappa\nabla F \ \vdots\ (\zeta\nabla\Xi) \ \mathrm{d}x \ \mathrm{d}t. \qquad (3.141)$$

Integration by parts with respect to t in (3.135) (again, an additional boundary term is showing up) and comparison of the resulting equation with (3.141) leads to (we choose again $\zeta(0) = 1$)

$$\int_\Omega (F(0) - F_0) : \Xi \ \mathrm{d}x = 0$$

for any $\Xi \in \mathbf{H}_0^1(\Omega; \mathbb{R}^{d\times d})$. This proves the attainment for F.

Attainment for M. Lastly, for M, we show that $M(0) = M_0$. We integrate the first term in (3.136) by parts with respect to time to get

$$\int_0^T \int_\Omega (M_m)_t \cdot (\zeta\varphi) + (v\cdot\nabla)M_m \cdot (\zeta\varphi) \ \mathrm{d}x \ \mathrm{d}t$$
$$= \int_0^T \int_\Omega -\nabla M_m : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M_m|^2 - 1)M_m \cdot (\zeta\varphi) \ \mathrm{d}x \ \mathrm{d}t. \qquad (3.142)$$

From the obtained regularity for $(M_m)_t$, it is a direct consequence that the first term in (3.142) converges to $\int_0^T \int_\Omega M_t \cdot (\zeta\varphi) \ \mathrm{d}x \ \mathrm{d}t$, so, it is clear that (3.142) converges to

$$\int_0^T \int_\Omega M_t \cdot (\zeta\varphi) + (v\cdot\nabla)M \cdot (\zeta\varphi) \ \mathrm{d}x \ \mathrm{d}t$$
$$= \int_0^T \int_\Omega -\nabla M : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M|^2 - 1)M \cdot (\zeta\varphi) \ \mathrm{d}x \ \mathrm{d}t. \qquad (3.143)$$

Integrating by parts with respect to t in (3.137) (here, an additional boundary term is showing up, too) and comparing the equation with (3.143), we see that (we choose also $\zeta(0) = 1$)

$$\int_\Omega (M(0) - M_0)\cdot\varphi \ \mathrm{d}x = 0$$

for any $\varphi \in \mathbf{H}^1(\Omega; \mathbb{R}^3)$. This then proves the attainment for M in L^2.
Since $\nabla M(t)$ converges as $t \to 0$ and we have $M(t) \xrightarrow{t\to 0} M_0$ in L^2, we immediately get that $\nabla M(t) \xrightarrow{t\to 0} \nabla M_0$. This concludes the proof of Theorem 9. □

3.2 System for simplified setting including LLG

This section is dedicated to the proof of Theorem 11 on page 44. The structure of the proof is essentially the same as the proof of Theorem 9 from Section 3.1. However, due to the more complicated form of the Landau-Lifshitz-Gilbert equation (3.18) compared to the gradient flow (3.5), which reflects in stronger nonlinearities in the LLG equation, we have to use different techniques. The first specialty is that we need more regularity of the magnetization to obtain a weak solution to the LLG equation for a fixed velocity, see Lemma 26, which follows ideas from [CF01]. Further, the energy estimates are more involved and also invoke methods used in [CF01], where the small data assumption (3.23) is important. In the following, let $\Omega \subset \mathbb{R}^2$, which we assume in order to apply certain Sobolev estimates in Lemma 26 and Corollary 29 to ensure $\mathbf{H}^2$-regularity of the magnetization.

3.2.1 Definition of a weak solution

We start with the definition of the notion of a weak solution to the simplified system with LLG.

Definition 22. *The triple* (v, F, M) *is called a weak solution to the system* (3.1)–(3.4)*,* (3.18)*,* (3.6)–(3.11) *in* $\Omega \times [0, t^*]$ *provided that*

$$\begin{aligned} v &\in L^\infty(0, t^*; \mathbf{H}) \cap L^2(0, t^*; \mathbf{V}), \\ F &\in L^\infty(0, t^*; L^2(\Omega; \mathbb{R}^{2\times 2})) \cap L^2(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^{2\times 2})), \\ M &\in L^\infty(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^3)) \cap L^2(0, t^*; \mathbf{H}^2(\Omega; \mathbb{R}^3)), \end{aligned}$$

and if it satisfies

$$\begin{aligned} \int_0^{t^*} \int_\Omega -v \cdot (\zeta' \xi) + (v \cdot \nabla) v \cdot (\zeta \xi) + \Big(W'(F) F^\top - \nabla M \odot \nabla M \Big) : (\zeta \nabla \xi) \, \mathrm{d}x \, \mathrm{d}t \\ - \int_\Omega v(0) \cdot (\zeta(0) \xi) \, \mathrm{d}x = - \int_0^{t^*} \int_\Omega \nu \nabla v : (\zeta \nabla \xi) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{3.144}$$

$$\begin{aligned} \int_0^{t^*} \int_\Omega -F : (\zeta' \Xi) + (v \cdot \nabla) F : (\zeta \Xi) - (\nabla v F) : (\zeta \Xi) \, \mathrm{d}x \, \mathrm{d}t \\ - \int_\Omega F(0) : (\zeta(0) \Xi) \, \mathrm{d}x = - \int_0^{t^*} \int_\Omega \kappa \nabla F \,\vdots\, (\zeta \nabla \Xi) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{3.145}$$

$$\begin{aligned} &\int_0^{t^*} \int_\Omega -M \cdot (\zeta' \varphi) + (v \cdot \nabla) M \cdot (\zeta \varphi) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega M(0) \cdot (\zeta(0) \varphi) \, \mathrm{d}x \\ &= \int_0^{t^*} \int_\Omega -(M \times \Delta M) \cdot (\zeta \varphi) + |\nabla M|^2 M \cdot (\zeta \varphi) - \nabla M : (\zeta \nabla \varphi) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{3.146}$$

where $\zeta : [0, t^*] \to \mathbb{R}$ *is any* $W^{1,\infty}$*-function with* $\zeta(t^*) = 0$ *and* $\xi \in \mathbf{V}$*,* $\Xi \in \mathbf{H}^1_0(\Omega; \mathbb{R}^{2\times 2})$*,* $\varphi \in \mathbf{H}^1(\Omega; \mathbb{R}^3)$*, together with the boundary conditions* (3.6)–(3.8) *in the sense of traces and with the initial conditions* (3.9)–(3.11) *in the sense*

$$v(\cdot, t) \xrightarrow{w-L^2(\Omega)} v_0(\cdot), \quad F(\cdot, t) \xrightarrow{w-L^2(\Omega)} F_0(\cdot), \quad M(\cdot, t) \xrightarrow{w-\mathbf{H}^1(\Omega)} M_0(\cdot) \ \textit{as } t \to 0^+.$$

Remark 23. *Notice that the weak form* (3.146) *of the LLG equation is motivated by the equivalent versions of the LLG equation from Lemma 10.*

3.2.2 Galerkin approximation: definition of the approximate problem

The construction of solutions to an approximate problem starts – just like in the model for the simplified setting without the LLG equation in Section 3.1.2 – by projecting the velocity v onto finite dimensional subspaces $\mathbf{H}_m$ of $\mathbf{H}$ following [LL95].
We refer to Section 3.1.2 for the details on the Stokes operator (see also (3.27)). The approximate problem, where the equation for the magnetization is the coupled LLG equation and the initial condition is supposed to satisfy the length constraint, reads

$$
\begin{aligned}
(v_m)_t = P_m\Big(&\nu\Delta v_m - (v_m\cdot\nabla)v_m \\
&+ \nabla\cdot\big(W'(F_m)F_m^\top - \nabla M_m\odot\nabla M_m\big)\Big) && \text{in } \Omega\times(0,t^*), \quad (3.147)\\
v_m \in \mathbf{H}_m &\Longrightarrow \nabla\cdot v_m = 0, && (3.148)\\
(F_m)_t + (v_m\cdot\nabla)F_m &- \nabla v_m F_m = \kappa\Delta F_m && \text{in } \Omega\times(0,t^*), \quad (3.149)\\
(M_m)_t + (v_m\cdot\nabla)M_m &= -M_m\times\Delta M_m \\
&\quad + |\nabla M_m|^2 M_m + \Delta M_m && \text{in } \Omega\times(0,t^*), \quad (3.150)\\
v_m &= 0 && \text{on } \partial\Omega\times(0,t^*), \quad (3.151)\\
F_m &= 0 && \text{on } \partial\Omega\times(0,t^*), \quad (3.152)\\
\frac{\partial M_m}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega\times(0,t^*), \quad (3.153)\\
v_m(x,0) &= P_m(v_0(x)) && \text{in } \Omega, \quad (3.154)\\
F_m(x,0) &= I && \text{in } \Omega, \quad (3.155)\\
M_m(x,0) &= M_0(x) \qquad \text{in } \Omega, \qquad |M_0| = 1 && \text{a.e. in } \Omega. \quad (3.156)
\end{aligned}
$$

Again, this approximating system is meant to hold in a weak sense, i.e., boundary and initial conditions (3.151)–(3.156) hold and the following integral equations are satisfied

$$
\begin{aligned}
&\int_\Omega (v_m)_t\cdot\xi + (v_m\cdot\nabla)v_m\cdot\xi + \big(W'(F_m)F_m^\top - \nabla M_m\odot\nabla M_m\big) : \nabla\xi \, \mathrm{d}x \\
&= -\int_\Omega \nu\nabla v_m : \nabla\xi \, \mathrm{d}x, \qquad (3.157)
\end{aligned}
$$

$$\underset{\mathbf{H}^{-1}}{}\Big\langle (F_m)_t, \Xi \Big\rangle_{\mathbf{H}_0^1} + \int_\Omega (v_m \cdot \nabla) F_m : \Xi - (\nabla v_m F_m) : \Xi \, \mathrm{d}x$$

$$= - \int_\Omega \kappa \nabla F_m \,\vdots\, \nabla \Xi \, \mathrm{d}x, \tag{3.158}$$

$$\int_\Omega (M_m)_t \cdot \varphi + (v_m \cdot \nabla) M_m \cdot \varphi \, \mathrm{d}x$$

$$= \int_\Omega -(M_m \times \Delta M_m) \cdot \varphi + |\nabla M_m|^2 M_m \cdot \varphi + \Delta M_m \cdot \varphi \, \mathrm{d}x, \tag{3.159}$$

for a.e. t, where $\xi \in \mathbf{V} \cap \mathbf{H}_m = \mathbf{H}_m$, $\Xi \in \mathbf{H}_0^1(\Omega; \mathbb{R}^{2\times 2})$, $\varphi \in L^2(\Omega; \mathbb{R}^3)$.

3.2.3 Galerkin approximation: existence of weak solutions to the approximate problem

We start by defining the notion of a weak solution to the approximate problem.

Definition 24. *We call* (v_m, F_m, M_m) *a weak solution to the system* (3.147)–(3.156) *provided that*

$$\begin{aligned} v_m &\in L^\infty(0, t^*; \mathbf{H}) \cap L^2(0, t^*; \mathbf{V}), \\ F_m &\in L^\infty(0, t^*; L^2(\Omega; \mathbb{R}^{2\times 2})) \cap L^2(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^{2\times 2})) \\ M_m &\in L^\infty(0, t^*; \mathbf{H}^1(\Omega; \mathbb{R}^3)) \cap L^2(0, t^*; \mathbf{H}^2(\Omega; \mathbb{R}^3)) \end{aligned}$$

and that the system (3.147)–(3.156) *is satisfied in the weak sense* (3.157)–(3.159).

The following result states the existence of a weak solution to the approximate problem.

Theorem 25. *For any* $0 < T < \infty$ *and any* $m > 0$, $v_0 \in \mathbf{H}$, $F_0 \in L^2(\Omega; \mathbb{R}^{2\times 2})$, $M_0 \in \mathbf{H}^2(\Omega; S^2)$ *satisfying*

$$\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} < \frac{1}{C(\Omega)} \tag{3.160}$$

for some constant $C(\Omega)$ *and* W *satisfying* (3.12)–(3.17), *the system* (3.147)–(3.156) *has a weak solution* (v_m, F_m, M_m) *in* $\Omega \times (0, T)$.

We prepare the proof of Theorem 25. The approach is – as in Section 3.1.3 – to convert the PDE for the velocity v, i.e., the balance of momentum equation (3.147), to an ODE system. From the very same discretization of the velocity (3.43), we obtain for $\xi = \xi_i$ also the same ODE system which was derived in Section 3.1.3

$$\frac{\mathrm{d}}{\mathrm{d}t} g_m^i(t) = -\nu \lambda_i g_m^i(t) + \sum_{j,k=1}^m g_m^j(t) g_m^k(t) A_{jk}^i + D_m^i(t), \qquad i = 1, \ldots, m, \tag{3.161}$$

where

$$A_{jk}^i = -\int_\Omega (\xi_j(x)\cdot\nabla)\xi_k(x)\cdot\xi_i(x)\,\mathrm{d}x, \tag{3.162}$$

$$D_m^i(t) = -\int_\Omega \big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big) : \nabla\xi_i\,\mathrm{d}x, \tag{3.163}$$

and the initial condition

$$g_m^i(0) = \int_\Omega v_0(x)\cdot\xi_i(x)\,\mathrm{d}x \tag{3.164}$$

for $i = 1, \ldots, m$.

3.2.3.1 Weak solutions to the sub-problem

The following lemma mimics Lemma 17 in the model without the LLG equation. This yields unique weak solutions to the PDEs for the deformation gradient F and the magnetization M for fixed velocity v. However, the crucial difference is that we need to obtain more regularity to converge the LLG equation. As for the solution to the equation for F, there is no difference in the proof compared to Lemma 17.

Lemma 26. *For $v \in L^\infty(0,t^*;W^{2,\infty}(\Omega;\mathbb{R}^2))$ satisfying $v = 0$ on $\partial\Omega\times(0,t^*)$ and $v(x,0) = v_0(x)$ and $\nabla\cdot v = 0$, there exists a time $0 < \tilde{t} < t^*$ such that the system*

$$\begin{aligned}
&F_t + (v\cdot\nabla)F - \nabla v F = \kappa\Delta F && \text{in } \Omega\times(0,\tilde{t}),\\
&M_t + (v\cdot\nabla)M = -M\times\Delta M + |\nabla M|^2 M + \Delta M && \text{in } \Omega\times(0,\tilde{t}),\\
&F = 0 && \text{on } \partial\Omega\times(0,\tilde{t}),\\
&\frac{\partial M}{\partial \mathbf{n}} = 0 && \text{on } \partial\Omega\times(0,\tilde{t}),\\
&F(x,0) = F_0(x) = I && \text{in } \Omega,\\
&M(x,0) = M_0(x) && \text{in } \Omega
\end{aligned}$$

has a unique weak solution such that

$$\begin{aligned}
&\|F\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^{2\times 2}))} + \|F\|_{L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^{2\times 2}))}\\
&\qquad + \|F_t\|_{L^2(0,\tilde{t};\mathbf{H}^{-1}(\Omega;\mathbb{R}^{2\times 2}))} \le C(v), && (3.165)\\
&\|M\|_{L^\infty(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} + \|M\|_{L^2(0,\tilde{t};\mathbf{H}^3(\Omega;\mathbb{R}^3))} + \|M_t\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))}\\
&\qquad + \|\nabla M_t\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2}))} \le C(v,M_0), && (3.166)
\end{aligned}$$

where the constants are given by

$$\begin{aligned}
C(v) &= C(\|v\|_{L^\infty(0,\tilde{t};W^{2,\infty}(\Omega;\mathbb{R}^2))}),\\
C(v,M_0) &= C(\|v\|_{L^\infty(0,\tilde{t};W^{2,\infty}(\Omega:\mathbb{R}^2))}, \|M_0\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)}).
\end{aligned}$$

Moreover, it holds that

$$
\begin{aligned}
&\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) \\
&\leq \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + C(v)\int_0^t \Big(\|\nabla M\|^8_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + \Big(1+\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big)\,\|\Delta M\|^4_{L^2(\Omega;\mathbb{R}^3)}\Big)\mathrm{d}s \qquad (3.167)
\end{aligned}
$$

for any $0 \leq t \leq \tilde{t}$.

Proof. The proof of existence for the equation for F works as before. Hence, for this, we refer to the first part of the proof of Lemma 17.

Existence of a weak solution to the M-equation. For the Galerkin approximation, let $\{\eta_i\}_{i=1}^\infty \subset C^\infty(\overline{\Omega};\mathbb{R}^3)$ be an orthonormal basis of $L^2(\Omega;\mathbb{R}^3)$ and an orthogonal basis of $\mathbf{H}^2_{\mathbf{n}}(\Omega;\mathbb{R}^3)$ (for details we refer to Appendix A.6) satisfying

$$\Delta^2\eta_i + \eta_i = \tilde{\mu}_i\eta_i \qquad (3.168)$$

in Ω and $\frac{\partial\eta_i}{\partial\mathbf{n}} = 0$ and $\frac{\partial\Delta\eta_i}{\partial\mathbf{n}} = 0$ in the weak sense on the boundary. Here, $0 < \tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots \leq \tilde{\mu}_n \leq \cdots$ with $\tilde{\mu}_n \xrightarrow{n\to\infty} \infty$.
Let

$$\widetilde{L}^2_n := \operatorname{span}\{\eta_1,\eta_2,\ldots,\eta_n\} \qquad (3.169)$$

and

$$\widetilde{P}_n : L^2(\Omega;\mathbb{R}^3) \to \widetilde{L}^2_n \qquad (3.170)$$

be the orthonormal projection. We consider the original problem for functions in $\widetilde{L}^2_n$ and show existence of a weak solution to

$$M_t = \widetilde{P}_n\big[-(v\cdot\nabla)M - M\times\Delta M + |\nabla M|^2 M + \Delta M\big] \quad \text{in } \Omega\times(0,t^{**}), \qquad (3.171)$$

$$\frac{\partial M}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega\times(0,t^{**}), \qquad (3.172)$$

$$M_n(x,0) = \widetilde{P}_n(M_0(x)) \quad \text{in } \Omega. \qquad (3.173)$$

For a fixed $n\in\mathbb{N}$, we look for a function $M_n : [0,t^{**}] \to \widetilde{L}^2_n$ of the form

$$M_n(x,t) = \sum_{i=1}^n h^i_n(t)\eta_i(x). \qquad (3.174)$$

The solution must satisfy (3.159), so we plug this discretization into (3.159) to obtain for $\varphi = \eta_i$ the ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t}h^i_n(t) = \sum_{j=1}^n h^j_n(t)\hat{A}^i_j(t) + \sum_{j,k=1}^n h^j_n(t)h^k_n(t)\hat{B}^i_{jk} + \sum_{j,k,l=1}^n h^j_n(t)h^k_n(t)h^l_n(t)\hat{C}^i_{jkl}, \quad i=1,\ldots,n, \qquad (3.175)$$

where

$$\hat{A}_j^i(t) = -\int_\Omega \big((v(x,t)\cdot\nabla)\eta_j(x) - \Delta\eta_j(x)\big)\cdot\eta_i(x)\;\mathrm{d}x, \tag{3.176}$$

$$\hat{B}_{jk}^i = -\int_\Omega (\eta_j(x)\times\Delta\eta_k(x))\cdot\eta_i(x)\;\mathrm{d}x, \tag{3.177}$$

$$\hat{C}_{jkl}^i = \int_\Omega (\nabla\eta_j(x):\nabla\eta_k(x))(\eta_l(x)\cdot\eta_i(x))\;\mathrm{d}x. \tag{3.178}$$

The initial condition becomes

$$h_n^i(0) = \int_\Omega M_0(x)\cdot\eta_i(x)\;\mathrm{d}x, \qquad i = 1,\dots,n. \tag{3.179}$$

We also apply here Carathéodory's existence theorem (see Theorem 30 in Appendix A.2) to obtain a solution $h_n^i(t)$ of (3.175).
Since the second and the last summand on the right-hand side of (3.175) are not depending on t (looking at t and h_n^i as distinct variables) and the dependence on t of the first summand is just within a Lipschitz function, the right-hand side is measurable in t for any h_n^i.
Furthermore, the terms on the right-hand side of (3.175) are linear, quadratic and cubic in h_n^i, respectively, so the right-hand side is continuous in h_n^i for any t.
In addition, for $t\in[0,t^*]$ and $\|h_n - h_n(0)\|\le\hat{b}$, where $h_n = (h_n^1,\dots,h_n^n)$, we can bound the right-hand side of (3.175) by the L^1-function

$$(2\hat{b}+\|h_n(0)\|)n\hat{A} + \tilde{\mu}_n(2\hat{b}+\|h_n(0)\|)^2\sum_{j,k=1}^n \hat{B}_{jk}^i + (2\hat{b}+\|h_n(0)\|)^3\sum_{j,k,l=1}^n \hat{C}_{jkl}^i,$$

where we can choose the constant $\hat{A}$ in such a way that it is independent of v, which then makes the above function independent of v.
Finally, Carathéodory's theorem yields the existence of a value t^{**} (independent of v) with $0<t^{**}\le t^*$ such that the ODE system (3.175) has a unique (since the right-hand side of the ODE is locally Lipschitz) and absolutely continuous solution $\{h_n^i(t)\}_{i=1}^n$ on $[0,t^{**}]$ satisfying (3.179).
Now, we prepare the passage to the limit as $n\to\infty$ with uniform estimates. To this end, we first multiply (3.171) by M_n and integrate over Ω to find out that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})} = 2\int_\Omega |\nabla M_n|^2|M_n|^2\;\mathrm{d}x\\ \overset{\text{Hölder}}{\le}\;& 2\|M_n\|^2_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}.\end{aligned} \tag{3.180}$$

Next, we multiply (3.171) by $\Delta^2 M_n$ (notice that $\widetilde{P}_n(\Delta^2 M_n) = \Delta^2 M_n$ by (3.168)) and integrate over Ω to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&= \int_\Omega -(v\cdot\nabla)M_n\cdot\Delta^2 M_n - (M_n\times\Delta M_n)\cdot\Delta^2 M_n + |\nabla M_n|^2 M_n\cdot\Delta^2 M_n \,\mathrm{d}x \\
&\le \underbrace{\int_\Omega |(\nabla v\nabla^\top M_n):\nabla\Delta M_n| + |(v\cdot\nabla)\nabla M_n : \nabla\Delta M_n| \,\mathrm{d}x}_{=:I_1} \\
&\quad + \underbrace{\int_\Omega |(\nabla M_n\times\Delta M_n):\nabla\Delta M_n| \,\mathrm{d}x}_{=:I_2} \\
&\quad + \underbrace{\int_\Omega |(2M_n\otimes(\nabla M_n\nabla^2 M_n)):\nabla\Delta M_n| + |\nabla M_n|^2|\nabla M_n:\nabla\Delta M_n| \,\mathrm{d}x}_{=:I_3}.
\end{aligned} \tag{3.181}$$

We need to estimate the integrals I_1, I_2, and I_3 separately. To do so, we utilize some estimates also used in [CF01]: there is a constant $C > 0$ such that for all $M \in \mathbf{H}^2_{\mathbf{n}}(\Omega;\mathbb{R}^3)$

$$\|M\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{3.182}$$

$$\|\nabla M\|_{\mathbf{H}^1(\Omega;\mathbb{R}^{3\times 2})} \le C\left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{3.183}$$

$$\|M\|_{L^\infty(\Omega;\mathbb{R}^3)} \le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{3.184}$$

$$\|\nabla M\|_{L^6(\Omega;\mathbb{R}^{3\times 2})} \le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{3.185}$$

and that for all $M \in \mathbf{H}^2_{\mathbf{n}}(\Omega;\mathbb{R}^3)\cap\mathbf{H}^3(\Omega;\mathbb{R}^3)$

$$\begin{aligned}
&\|\nabla^2 M\|_{L^3(\Omega;\mathbb{R}^{3\times 2\times 2})} \\
&\le C\Bigg(\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}} \\
&\qquad + \left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{4}}\|\nabla\Delta M\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Bigg).
\end{aligned} \tag{3.186}$$

Moreover, since $\Omega \subset \mathbb{R}^2$, we also have

$$\|\nabla M\|_{L^4(\Omega;\mathbb{R}^{3\times 2})} \leq C\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{1}{2}} \times \left(\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2 + \|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{1}{4}}, \tag{3.187}$$

$$\|\nabla M\|_{L^6(\Omega;\mathbb{R}^{3\times 2})} \leq C\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{1}{3}} \times \left(\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2 + \|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{1}{3}}, \tag{3.188}$$

$$\|\nabla M\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \leq C\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{1}{2}} \times \left(\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2 + \|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\nabla \Delta M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2\right)^{\frac{1}{4}}, \tag{3.189}$$

$$\|\Delta M\|_{L^4(\Omega;\mathbb{R}^3)} \leq C\|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}^{\frac{1}{2}} \times \left(\|\Delta M\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\nabla \Delta M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2\right)^{\frac{1}{4}}. \tag{3.190}$$

We start to estimate the term I_1 and get, since $v \in L^\infty(0, t^*; W^{2,\infty}(\Omega;\mathbb{R}^2))$,

$$\begin{aligned} I_1 &\overset{\text{Hölder}}{\leq} \|\nabla v\|_{L^3(\Omega;\mathbb{R}^{2\times 2})}\|\nabla M_n\|_{L^6(\Omega;\mathbb{R}^{3\times 2})}\|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\quad + \|v\|_{L^6(\Omega;\mathbb{R}^2)}\|\nabla^2 M_n\|_{L^3(\Omega;\mathbb{R}^{3\times 2\times 2})}\|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\underset{(3.186)}{\overset{(3.185)}{\leq}} C(v)\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{1}{2}} \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\quad + C(v)\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{1}{4}} \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{3}{2}}. \end{aligned} \tag{3.191}$$

For the integral term I_2, we obtain

$$\begin{aligned} I_2 &\overset{\text{Hölder}}{\leq} \|\nabla M_n\|_{L^6(\Omega;\mathbb{R}^{3\times 2})}\|\Delta M_n\|_{L^3(\Omega;\mathbb{R}^3)}\|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\underset{(3.186)}{\overset{(3.185)}{\leq}} C\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right) \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\quad + C\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{3}{4}} \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{3}{2}}. \end{aligned} \tag{3.192}$$

We estimate the integral term I_3 and find out that

$$\begin{aligned} I_3 &\overset{\text{Hölder}}{\leq} 2\|M_n\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_n\|_{L^6(\Omega;\mathbb{R}^{3\times 2})} \\ &\qquad \times \|\nabla^2 M_n\|_{L^3(\Omega;\mathbb{R}^{3\times 2\times 2})}\|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\quad + \|\nabla M_n\|_{L^6(\Omega;\mathbb{R}^{3\times 2})}^3\|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\underset{\substack{(3.185)\\(3.186)}}{\overset{(3.184)}{\leq}} C\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right)^{\frac{3}{2}} \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\ &\quad + C\left(\|M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \|\Delta M_n\|_{L^2(\Omega;\mathbb{R}^3)}^2\right) \|\nabla \Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^{\frac{3}{2}}. \end{aligned} \tag{3.193}$$

Summing (3.191)–(3.193), we obtain from (3.181) and an iterative application of Young's inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}$$
$$\leq C(v)\left(1+\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{3}{2}}\right)\|\nabla\Delta M_n\|^{\frac{3}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})}. \quad (3.194)$$

Next, we sum (3.180) and (3.194) and apply (3.182), (3.184) and Young's inequality to find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad\qquad + 2\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + 2\|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
\overset{(3.182)}{\underset{(3.184)}{\leq}}\ & C(v)\left(1+\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{3}{2}}\right)\|\nabla\Delta M_n\|^{\frac{3}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&+C\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^2 \\
\overset{\text{Young}}{\leq}\ & C(v)\frac{27C(v)^3}{256}\left(1+\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{3}{2}}\right)^4 \\
&+\|\nabla\Delta M_n\|^{\frac{3}{2}\cdot\frac{4}{3}}_{L^2(\Omega;\mathbb{R}^{3\times 2})} + C\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{2\cdot 3} + C(\Omega) \\
\overset{\text{Young}}{\leq}\ & C(v)\left(1+\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^6\right) + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad\qquad + 2\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
\leq\ & C(v)\left(1+\left(\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^6\right). \qquad (3.195)
\end{aligned}$$

In the next step, we make use of the following classical comparison lemma (see, e.g., [CF01, Lemma 2.4]) which we state without a proof:

Lemma 27. *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be C^1 and nondecreasing in its second variable. Assume further that $y : I \subset \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the inequality $y(t) \leq y_0 + \int_0^t f(s, y(s))\ \mathrm{d}s$ for all $t > 0$. Let $z : I \to \mathbb{R}$ be the solution of $z'(t) = f(t, z(t))$, $z(0) = y_0$. Then, it holds $y(t) \leq z(t)$ for all $t > 0$.*

From (3.195) and Lemma 27 we deduce the existence of a time $0 < T^* \leq t^{**}$ and a constant $C(v, M_0) = C(\|M_0\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)}, \|v\|_{L^\infty(0,t^*;W^{2,\infty}(\Omega;\mathbb{R}^2))})$ which is independent of n, such that for any $\tilde{t} < T^*$

$$\sup_{0\leq t\leq\tilde{t}}\|M_n\|^2_{\mathbf{H}^2(\Omega;\mathbb{R}^3)}(t)$$
$$+\int_0^{\tilde{t}} 2\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(t) + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(t)\ \mathrm{d}t \leq C(v, M_0),$$

which tells us that

$$\|M_n\|_{L^\infty(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} + \|M_n\|_{L^2(0,\tilde{t};\mathbf{H}^3(\Omega;\mathbb{R}^3))} \leq C(v, M_0). \tag{3.196}$$

Moreover, we need to multiply (3.171) by $(M_n)_t$ and $-\Delta(M_n)_t$ and integrate over Ω. To do so, we have to verify that the time derivatives of the temporal coefficients of M_n are in $L^2(0,\tilde{t})$. This we obtain directly from the LLG equation (3.171): the temporal part of the entire right-hand side is at least in $L^2(0,\tilde{t})$, since the temporal coefficients of M_n are continuous. So, the time derivatives of the temporal coefficients of M_n are also in $L^2(0,\tilde{t})$.
Now, we are able to continue the estimates and multiply (3.171) by $(M_n)_t$ and integrate over Ω to obtain, using Young's inequality,

$$\begin{aligned}
&\|(M_n)_t\|_{L^2(\Omega;\mathbb{R}^3)} \\
&= \int_\Omega -(v\cdot\nabla)M_n\cdot(M_n)_t - (M_n\times\Delta M_n)\cdot(M_n)_t \\
&\qquad + |\nabla M_n|^2 M_n\cdot(M_n)_t + \Delta M_n\cdot(M_n)_t \,\mathrm{d}x \\
&\overset{\text{Young}}{\leq} 2\int_\Omega |(v\cdot\nabla)M_n|^2 + |M_n\times\Delta M_n|^2 + |\nabla M_n|^4|M_n|^2 + |\Delta M_n|^2 \,\mathrm{d}x \\
&\qquad + \frac{4}{8}\int_\Omega |(M_n)_t|^2 \,\mathrm{d}x.
\end{aligned}$$

From there, we get

$$\begin{aligned}
&\|(M_n)_t\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\overset{\text{Hölder}}{\leq} 4\Big(\|v\|_{L^\infty(0,t^*;L^\infty(\Omega;\mathbb{R}^2))}\|M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|M_n\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\nabla M_n\|^4_{L^4(\Omega;\mathbb{R}^{3\times 2})}\|M_n\|^2_{L^\infty(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big),
\end{aligned}$$

where we take the supremum over all $t \in [0,\tilde{t}]$ to find, using (3.196),

$$\begin{aligned}
\sup_{0\leq t\leq\tilde{t}} \|(M_n)_t\|_{L^2(\Omega;\mathbb{R}^3)} &\leq 4\Big(\|v\|_{L^\infty(0,t^*;L^\infty(\Omega;\mathbb{R}^2))}\|M_n\|^2_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \\
&\qquad + \|M_n\|_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^3))}\|\Delta M_n\|^2_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \\
&\qquad + \|\nabla M_n\|^4_{L^\infty(0,\tilde{t};L^4(\Omega;\mathbb{R}^{3\times 2}))}\|M_n\|^2_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^3))} \\
&\qquad + \|\Delta M_n\|^2_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))}\Big) \\
&\leq C(v, M_0).
\end{aligned}$$

This gives us the bound

$$\|(M_n)_t\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^3))} \leq C(v, M_0). \tag{3.197}$$

Next, we multiply (3.171) by $-\Delta(M_n)_t$ and integrate over both Ω and $[0,t]$ for $t \leq \tilde{t}$ to find out that, since $\|\widetilde{P}_n(M_0)\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \leq \|M_0\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)}$, and using integration by parts with respect to x in the second step,

$$\begin{aligned}
&\int_0^t \|\nabla(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\ \mathrm{d}s + \frac{1}{2}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&= \frac{1}{2}\|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \int_\Omega (v\cdot\nabla)M_n \cdot \Delta(M_n)_t + (M_n \times \Delta M_n)\cdot\Delta(M_n)_t \\
&\qquad\qquad - |\nabla M_n|^2 M_n \cdot \Delta(M_n)_t\ \mathrm{d}x\ \mathrm{d}s \\
&= \frac{1}{2}\|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\quad + \int_0^t \int_\Omega -(\nabla v \nabla^\top M_n) : \nabla(M_n)_t - (v\cdot\nabla)\nabla M_n : \nabla(M_n)_t \\
&\qquad - (\nabla M_n \times \Delta M_n) : \nabla(M_n)_t - (M_n \times \nabla\Delta M_n) : \nabla(M_n)_t \\
&\qquad + (2M_n \otimes (\nabla M_n \nabla^2 M_n)) : \nabla(M_n)_t \\
&\qquad + |\nabla M_n|^2 \nabla M_n : \nabla(M_n)_t\ \mathrm{d}x\ \mathrm{d}s \\
&\overset{\text{Young}}{\le} \frac{1}{2}\|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{6}{12}\int_0^t \|\nabla(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad + 3\int_0^t \int_\Omega |\nabla v \nabla^\top M_n|^2 + |(v\cdot\nabla)\nabla M_n|^2 \\
&\qquad + |(\nabla M_n \times \Delta M_n)|^2 + |M_n \times \nabla\Delta M_n|^2 \\
&\qquad + 4|M_n \otimes (\nabla M_n \nabla^2 M_n)|^2 + |\nabla M_n|^6\ \mathrm{d}x\ \mathrm{d}s.
\end{aligned}$$

An application of Hölder's inequality yields

$$\begin{aligned}
&\int_0^t \|\nabla(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\ \mathrm{d}s + \frac{1}{2}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\le \frac{1}{2}\|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\int_0^t \|\nabla(M_n)_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad + 3\Big(\|\nabla v\|_{L^\infty(0,t^*;L^\infty(\Omega;\mathbb{R}^{2\times 2}))}\|\nabla M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad + \|v\|_{L^\infty(0,t^*;L^\infty(\Omega;\mathbb{R}^2))}\|\nabla^2 M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2\times 2}))} \\
&\qquad + \|\nabla M_n\|_{L^\infty(0,\tilde{t};L^4(\Omega;\mathbb{R}^{3\times 2}))}\|\Delta M_n\|_{L^2(0,\tilde{t};L^4(\Omega;\mathbb{R}^3))} \\
&\qquad + \|M_n\|^2_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^3))}\|\nabla\Delta M_n\|^2_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad + 4\|M_n\|^2_{L^\infty(0,\tilde{t};L^\infty(\Omega;\mathbb{R}^3))}\|\nabla M_n\|_{L^\infty(0,\tilde{t};L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad\qquad \times \|\nabla^2 M_n\|_{L^2(0,\tilde{t};L^4(\Omega;\mathbb{R}^{3\times 2\times 2}))} \\
&\qquad + \|\nabla M_n\|^6_{L^6(0,\tilde{t};L^6(\Omega;\mathbb{R}^{3\times 2}))}\Big).
\end{aligned}$$

Taking the supremum over all $t \in [0,\tilde{t}]$ and using (3.196), we get the bound

$$\|\nabla(M_n)_t\|_{L^2(0,\tilde{t};L^2(\Omega;\mathbb{R}^{3\times 2}))} \le C(v, M_0). \tag{3.198}$$

Next, we estimate the integral terms in (3.181), using (3.187)–(3.190). This is necessary to extend the solution in time in Section 3.2.3.4. We start with the term I_1 and obtain, since $v \in L^\infty(0,t^*;W^{2,\infty}(\Omega;\mathbb{R}^2))$,

$$\begin{aligned}
I_1 &\overset{\text{Hölder}}{\leq} \|\nabla v\|_{L^4(\Omega;\mathbb{R}^{2\times 2})}\|\nabla M_n\|_{L^4(\Omega;\mathbb{R}^{3\times 2})}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + \|v\|_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla^2 M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2\times 2})}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\underset{(3.187)}{\overset{(3.183)}{\leq}} C(v)\|\nabla M_n\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{4}} \\
&\qquad\qquad \times \|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + C(v)\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned} \tag{3.199}$$

For the term I_2, we find out that

$$\begin{aligned}
I_2 &\overset{\text{Hölder}}{\leq} \|\nabla M_n\|_{L^4(\Omega;\mathbb{R}^{3\times 2})}\|\Delta M_n\|_{L^4(\Omega;\mathbb{R}^3)}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\underset{(3.190)}{\overset{(3.187)}{\leq}} C\|\nabla M_n\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})}\|\Delta M_n\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad \times \left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{4}} \\
&\qquad \times \left(\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)^{\frac{1}{4}} \\
&\qquad \times \|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned} \tag{3.200}$$

We estimate the term I_3 and get

$$\begin{aligned}
I_3 &\overset{\text{Hölder}}{\leq} 2\|M_n\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_n\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad \times \|\nabla^2 M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2\times 2})}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad + \|\nabla M_n\|^3_{L^6(\Omega;\mathbb{R}^{3\times 2})}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\underset{(3.188),\,(3.189)}{\overset{(3.183)}{\leq}} C\|M_n\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_n\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad \times \left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)^{\frac{1}{4}} \\
&\qquad \times \left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}\|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad + C\|\nabla M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad \times \left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad \times \|\nabla\Delta M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned} \tag{3.201}$$

Summing (3.199)–(3.201), we obtain from (3.181), an iterative application of

Young's inequality, and an integration over $[0,t]$ for $0 \le t \le \tilde{t}$ that

$$
\begin{aligned}
&\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla \Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\,\mathrm{d}s \\
&\le \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\quad + \int_0^t C(v)\|\nabla M_n\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}} \\
&\quad + C(v)\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\quad + C\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\quad + C\|M_n\|^4_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^2 \\
&\quad + C\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\quad + C\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^2 \mathrm{d}s. \qquad (3.202)
\end{aligned}
$$

We continue this estimate after passing to the limit as $n \to \infty$ to finally prove (3.167). Now, we pass to the limit as $n \to \infty$ to obtain a weak solution to the system (3.171)–(3.173). We need the convergence results

$$
\begin{aligned}
M_n &\to M && \text{in } L^p(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3)), \quad 1 < p < \infty, && (3.203)\\
M_n &\rightharpoonup M && \text{in } L^2(0,\tilde{t};\mathbf{H}^3(\Omega;\mathbb{R}^3)), && (3.204)\\
M_n &\overset{*}{\rightharpoonup} M && \text{in } L^\infty(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3)), && (3.205)\\
(M_n)_t &\rightharpoonup M_t && \text{in } L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3)). && (3.206)
\end{aligned}
$$

The weak (and weak-$*$) convergence results follow directly from the estimates obtained above for a subsequence (not relabeled; see Theorems 32 and 33 in Appendix A.2). The strong convergence (3.203) we obtain from an application of the Aubin-Lions lemma and Hölder's inequality: from the embeddings $\mathbf{H}^3(\Omega;\mathbb{R}^3) \overset{c}{\subset} \mathbf{H}^2(\Omega;\mathbb{R}^3) \subset \mathbf{H}^1(\Omega;\mathbb{R}^3)$ (the first embedding is compact since $d < 4$, the second one is continuous), the fact that $M_n \in L^2(0,t^*;\mathbf{H}^3(\Omega;\mathbb{R}^3))$, and (3.197), (3.198), we conclude by the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2) the compact embedding

$$
\left\{M \in L^2(0,\tilde{t};\mathbf{H}^3(\Omega;\mathbb{R}^3)) : \ M_t \in L^2(0,\tilde{t};\mathbf{H}^1(\Omega;\mathbb{R}^3))\right\} \overset{c}{\subset} L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3)).
$$

This yields the strong convergence of $\{M_n\}_n$ in $L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))$ (up to a subsequence). The final step is to combine the result with Hölder's inequality and

(3.205):

$$
\begin{aligned}
\|M_n - M\|^p_{L^p(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} &= \int_0^{\tilde{t}} \|M_n - M\|^p_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}t \\
&= \int_0^{\tilde{t}} \|M_n - M\|^2_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \|M_n - M\|^{p-2}_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}t \\
&\leq \int_0^{\tilde{t}} \|M_n - M\|^2_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} \left(\|M_n\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)} + \|M\|_{\mathbf{H}^2(\Omega;\mathbb{R}^3)}\right)^{p-2} \,\mathrm{d}t \\
&\leq \underbrace{\left(\|M_n\|_{L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} + \|M\|_{L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))}\right)^{p-2}}_{\leq C(v,M_0)} \|M_n - M\|^2_{L^2(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

This proves the strong convergence (3.203).
Again, as the weak solution to the approximate problem is defined using test functions from the projected spaces $\widetilde{L}^2_n$, we also need to pass to the limit with these particular test functions (only in space). However, for any test function $\varphi \in \mathbf{H}^1(\Omega;\mathbb{R}^3)$ we use the sequence of approximate test functions defined by $\varphi_n := \widetilde{P}_n(\varphi) \in \widetilde{L}^2_n$ which converges strongly to $\varphi \in \mathbf{H}^1(\Omega;\mathbb{R}^3)$. In the following, we use this particular sequence of test functions. Moreover, let $\zeta \in L^\infty(0,\tilde{t})$.
So, the equation

$$
\begin{aligned}
&\int_0^{\tilde{t}} \int_\Omega (M_n)_t \cdot (\zeta\varphi_n) + (v\cdot\nabla)M_n \cdot (\zeta\varphi_n) \,\mathrm{d}x \,\mathrm{d}t \\
=& \int_0^{\tilde{t}} \int_\Omega -(M_n \times \Delta M_n)\cdot(\zeta\varphi_n) + |\nabla M_n|^2 M_n \cdot (\zeta\varphi_n) - \nabla M_n : (\zeta\nabla\varphi_n) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

converges to the equation

$$
\begin{aligned}
&\int_0^{\tilde{t}} \int_\Omega M_t \cdot (\zeta\varphi) + (v\cdot\nabla)M \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t \\
=& \int_0^{\tilde{t}} \int_\Omega -(M \times \Delta M)\cdot(\zeta\varphi) + |\nabla M|^2 M \cdot (\zeta\varphi) - \nabla M : (\zeta\nabla\varphi) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

as $n \to \infty$. All the integral terms on the left-hand side and the last term on the right-hand side are linear, so the weak convergences from above together with the strong convergence of the test functions are sufficient to obtain the convergence of these terms. The two remaining terms, viz the first and the second on the right-hand side, are converged with the help of the strong convergence (3.203): it follows directly from Hölder's inequality and the fact that we have strong convergence for each factor in $L^p(0,\tilde{t};\mathbf{H}^2(\Omega;\mathbb{R}^3))$ for a suitable $1 < p < \infty$, keeping the Sobolev embedding $\mathbf{H}^2(\Omega) \subset L^\infty(\Omega)$ in mind. Thus, we obtain a weak solution to the system (3.171)–(3.173).
Notice that all the estimates for the approximate solution obtained above still hold in the limit due to the weak lower semicontinuity of norms.

Furthermore, the solution is unique. Let us assume that we have two solutions $M_1 \neq M_2$. The difference $M_1 - M_2$ then solves

$$\begin{aligned}
&(M_1 - M_2)_t + (v \cdot \nabla)(M_1 - M_2) \\
&\quad = \; -(M_1 - M_2) \times \Delta M_1 + M_2 \times (\Delta(M_1 - M_2)) \\
&\qquad + \left(|\nabla M_1|^2 - |\nabla M_2|^2\right) M_1 + |\nabla M_2|^2 (M_1 - M_2) + \Delta(M_1 - M_2).
\end{aligned}$$

We multiply this equation by $(M_1 - M_2)$, integrate over Ω and use the identity $(a \times b) \cdot c = (b \times c) \cdot a$ to find out that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla(M_1 - M_2)\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\quad = \int_\Omega \big(M_2 \times (\Delta(M_1 - M_2))\big) \cdot (M_1 - M_2) \\
&\qquad + \left(|\nabla M_1|^2 - |\nabla M_2|^2\right) M_1 \cdot (M_1 - M_2) + |\nabla M_2|^2 |M_1 - M_2|^2 \,\mathrm{d}x \\
&\quad = \int_\Omega \underbrace{\big((\Delta(M_1 - M_2)) \times (M_1 - M_2)\big)}_{=\nabla\cdot\left((\nabla(M_1-M_2))\times(M_1-M_2)\right)} \cdot M_2 \\
&\qquad + \left((\nabla M_1 - \nabla M_2) \cdot (\nabla M_1 + \nabla M_2)\right) M_1 \cdot (M_1 - M_2) \\
&\qquad + |\nabla M_2|^2 |M_1 - M_2|^2 \,\mathrm{d}x \\
&\quad = \int_\Omega -\big((\nabla(M_1 - M_2)) \times (M_1 - M_2)\big) : \nabla M_2 \\
&\qquad + \left((\nabla M_1 - \nabla M_2) : (\nabla M_1 + \nabla M_2)\right) M_1 \cdot (M_1 - M_2) \\
&\qquad + |\nabla M_2|^2 |M_1 - M_2|^2 \,\mathrm{d}x \\
&\quad \overset{\text{Young}}{\leq} \int_\Omega \frac{1}{2}|\nabla(M_1 - M_2)|^2 + \frac{1}{2}|M_1 - M_2|^2 |\nabla M_2|^2 \\
&\qquad + \frac{1}{2}|\nabla(M_1 - M_2)|^2 + \frac{1}{2}|\nabla M_1 + \nabla M_2|^2 |M_1|^2 |M_1 - M_2|^2 \\
&\qquad + |\nabla M_2|^2 |M_1 - M_2|^2 \,\mathrm{d}x \\
&\quad \overset{\text{Hölder}}{\leq} \|\nabla(M_1 - M_2)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \frac{3}{2}\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} \|\nabla M_2\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + \frac{1}{2}\|\nabla M_1 + \nabla M_2\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \|M_1\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)}.
\end{aligned}$$

We then integrate over $[0, t]$ for $t \leq \tilde{t}$ and obtain employing $M_1(0) = M_2(0)$ and the regularity (3.196)

$$\begin{aligned}
&\|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) \\
&\leq \int_0^t \underbrace{\left(3\|\nabla M_2\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} + \|\nabla M_1 + \nabla M_2\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \|M_1\|^2_{L^\infty(\Omega;\mathbb{R}^3)}\right)}_{\in L^1(0,\tilde{t})} \\
&\qquad\qquad \times \|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}t,
\end{aligned}$$

where we apply Gronwall's inequality to find

$$\sup_{0 \leq t \leq \tilde{t}} \|M_1 - M_2\|^2_{L^2(\Omega;\mathbb{R}^3)} = 0.$$

Thus, the solution is unique.
Finally, we converge the inequality in (3.202). Since norms are lower semicontinuous, since we have the strong convergence (3.203), and since it holds that $|M| \equiv 1$ in the limit, we obtain from (3.202)

$$
\begin{aligned}
&\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) + \int_0^t \|\nabla \Delta M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \,\mathrm{d}s \\
&\le \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\quad + \int_0^t C(v)\|\nabla M\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}} \\
&\qquad + C(v)\left(\|\nabla M_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad + C\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)} \left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad + C\|M\|^4_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^2 \\
&\qquad + C\left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad + C\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^2 \,\mathrm{d}s. \qquad (3.207)
\end{aligned}
$$

Applying Young's inequality, we find out that

$$
\begin{aligned}
&\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) + \int_0^t \|\nabla \Delta M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \,\mathrm{d}s \\
&\le \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + C(v)\int_0^t 1 + \left(\|\nabla M\|^8_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right. \\
&\qquad\qquad \left. + \left(1 + \|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)\|\Delta M\|^4_{L^2(\Omega;\mathbb{R}^3)}\right) \mathrm{d}s, \qquad (3.208)
\end{aligned}
$$

which then implies (3.167).This concludes the proof of Lemma 26. ☐

3.2.3.2 Weak solutions to the approximate problem for a short time using a fixed point argument

From the next result, we obtain a weak solution to the approximate problem. This is the counterpart to Lemma 18 for the system without LLG. The solution also exists only for a certain short time t_0^* and its existence is also proven using Schauder's fixed point theorem.

Lemma 28. *For any $m > 0$ and W satisfying* (3.12)–(3.17)*, there exists a time t_0^* depending on v_0, M_0, Ω, and m such that the system* (3.147)–(3.156) *has a weak solution (v_m, F_m, M_m) in $\Omega \times (0, t_0^*)$.*

Proof. The reasoning in this proof is the same as in the proof for Lemma 18. The first obvious difference is that we look at the LLG equation

$$M_t + (v \cdot \nabla)M = -M \times \Delta M + |\nabla M|^2 M + \Delta M \qquad (3.209)$$

on $[0, t_1^*]$ satisfying

$$\|M\|_{L^\infty(0,t_1^*;\mathbf{H}^2(\Omega;\mathbb{R}^{3\times 2}))} \leq C(v, M_0). \tag{3.210}$$

This ensures the applicability of Carathéodory's existence theorem to the ODE for v.
After choosing the time t_0^* accordingly, we need to prove the continuity of the solution operator $\mathcal{L}$. This is done in exactly the same way, with the only difference that the convergence of M_l to M has to be shown using the LLG equation and the higher regularity obtained for the magnetization in Lemma 26.

Convergence of $\{M_l\}_l$. We check the strong convergence of $\{M_l\}_l$ in the space $L^\infty(0, t_0^*; \mathbf{H}^1(\Omega; \mathbb{R}^3))$. To this end, we first obtain from the LLG

$$\begin{aligned}
&(M_l - M)_t + (v_l \cdot \nabla)(M_l - M) + ((v_l - v) \cdot \nabla)M \\
&= \; -(M_l - M) \times \Delta M_l + M \times (\Delta(M_l - M)) \\
&\quad + \left(|\nabla M_l|^2 - |\nabla M|^2\right) M_l + |\nabla M|^2 (M_l - M) + \Delta(M_l - M).
\end{aligned} \tag{3.211}$$

By multiplying equation (3.211) with $(M_l - M)$, integrating over both Ω and $[0, t]$ for $t \leq t_0^*$ and using Young's inequality and the identity $(a \times b) \cdot c = (b \times c) \cdot a$ we find out that, since $M_l(0) = M(0)$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |M_l - M|^2(t) \,\mathrm{d}x + \int_0^t \int_\Omega |\nabla(M_l - M)|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_0^t \int_\Omega -((v_l - v) \cdot \nabla)M \cdot (M_l - M) + (M \times (\Delta(M_l - M))) \cdot (M_l - M) \\
&\qquad + \left(|\nabla M_l|^2 - |\nabla M|^2\right) M_l \cdot (M_l - M) \\
&\qquad + |\nabla M|^2 (M_l - M) \cdot (M_l - M) \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_0^t \int_\Omega -((v_l - v) \cdot \nabla)M \cdot (M_l - M) - (\nabla(M_l - M) \times (M_l - M)) : \nabla M \\
&\qquad + ((\nabla(M_l - M)) : (\nabla M_l + \nabla M))\, M_l \cdot (M_l - M) \\
&\qquad + |\nabla M|^2 |M_l - M|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\overset{\text{Young}}{\leq} \int_0^t \int_\Omega \frac{1}{2}|((v_l - v) \cdot \nabla)M|^2 + \frac{1}{2}|M_l - M|^2 \\
&\qquad + \frac{1}{2}|\nabla(M_l - M)|^2 + \frac{3}{2}|M_l - M|^2 |\nabla M|^2 \\
&\qquad + \frac{1}{2}|\nabla(M_l - M)|^2 + \frac{1}{2}|\nabla M_l + \nabla M|^2 |M_l|^2 |M_l - M|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\overset{\text{Hölder}}{\leq} \int_0^t \frac{1}{2}\|((v_l - v) \cdot \nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\nabla(M_l - M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \frac{3}{2}\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \|\nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + \frac{1}{2}\|\nabla M_l + \nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \|M_l\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s.
\end{aligned}$$

We deduce with the regularity (3.196) that

$$\int_\Omega |M_l - M|^2(t)\ \mathrm{d}x \leq \underbrace{\int_0^t \|((v_l - v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s}_{\text{non-decreasing}}$$

$$+ \int_0^t \underbrace{\left(1 + 3\|\nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} + \|\nabla M_l + \nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\|M_l\|^2_{L^\infty(\Omega;\mathbb{R}^3)}\right)}_{\in L^1(0,t_0^*)} \times \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s, \quad (3.212)$$

where we apply Gronwall's inequality to find

$$\sup_{0\leq t\leq t_0^*} \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq \left(\int_0^t \|((v_l - v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s\right) \mathrm{e}^{C(v,M_0)t_0^*}. \quad (3.213)$$

Due to the convergence of the velocities v_l to v (3.98) we can pass to the limit as $l \to \infty$ to obtain

$$M_l \xrightarrow{l\to\infty} M \text{ in } L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^3)). \quad (3.214)$$

We are left to prove the convergence of ∇M_l in $L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^{3\times 2}))$. To this end, we multiply equation (3.211) with $-\Delta(M_l - M)$, integrating over both Ω and $[0,t]$ for $t \leq t_0^*$ and using Young's inequality and the bounds obtained in Lemma 26, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\ \mathrm{d}x \\
&= \int_0^t \int_\Omega (v_l\cdot\nabla)(M_l - M)\cdot\Delta(M_l - M) \\
&\quad + ((v_l - v)\cdot\nabla)M\cdot\Delta(M_l - M) \\
&\quad + ((M_l - M)\times\Delta M_l)\cdot\Delta(M_l - M) \\
&\quad + \left(|\nabla M_l|^2 - |\nabla M|^2\right) M_l\cdot\Delta(M_l - M) \\
&\quad + |\nabla M|^2(M_l - M)\cdot\Delta(M_l - M) - |\Delta(M_l - M)|^2\ \mathrm{d}x\ \mathrm{d}s \\
&\overset{\text{Young}}{\leq} \int_0^t \int_\Omega \frac{5}{4}|(v_l\cdot\nabla)(M_l - M)|^2 + \frac{1}{5}|\Delta(M_l - M)|^2 \\
&\quad + \frac{5}{4}|((v_l - v)\cdot\nabla)M|^2 + \frac{1}{5}|\Delta(M_l - M)|^2 \\
&\quad + \frac{5}{4}|(M_l - M)\times\Delta M_l|^2 + \frac{1}{5}|\Delta(M_l - M)|^2 \\
&\quad + \frac{5}{4}|\left((\nabla(M_l - M)) : (\nabla M_l + \nabla M)\right) M_l|^2 + \frac{1}{5}|\Delta(M_l - M)|^2 \\
&\quad + \frac{5}{4}|\nabla M|^4|M_l - M|^2 + \frac{1}{5}|\Delta(M_l - M)|^2 - |\Delta(M_l - M)|^2\ \mathrm{d}x\ \mathrm{d}s.
\end{aligned}$$

By applying Hölder's inequality, we find

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(M_l - M)|^2(t)\,\mathrm{d}x \\
&\quad\le \frac{5}{4}\int_0^t \|(v_l\cdot\nabla)(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \underbrace{\|M_l - M\|^2_{L^6(\Omega;\mathbb{R}^3)}}_{\le C(\Omega)\left(\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}+\|M_l-M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)} \|\Delta M_l\|^2_{L^3(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\|\nabla M_l + \nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\|M_l\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\nabla M\|^4_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s.
\end{aligned}
$$

Since $\{v_l\}_l$ is uniformly bounded in $L^\infty(0,t_0^*;L^\infty(\Omega;\mathbb{R}^2))$, we obtain

$$
\begin{aligned}
&\|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(t) \le \underbrace{\frac{5}{2}\int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s}_{\text{non-decreasing}} \\
&+ \underbrace{\frac{5}{2}\int_0^t \left(C + C(\Omega)\|\Delta M_l\|^2_{L^3(\Omega;\mathbb{R}^3)} + \|\nabla M\|^4_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\right)\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s}_{\text{non-decreasing}} \\
&+ \int_0^t \Big(\underbrace{C + C(\Omega)\|\Delta M_l\|^2_{L^3(\Omega;\mathbb{R}^3)} + \|\nabla M_l + \nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\|M_l\|^2_{L^\infty(\Omega;\mathbb{R}^3)}}_{=:g(t)\in L^1(0,t_0^*)}\Big) \\
&\qquad\qquad \times \|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(s)\,\mathrm{d}s, \quad (3.215)
\end{aligned}
$$

where we apply Gronwall's inequality to find out that

$$
\begin{aligned}
\sup_{0\le t\le t_0^*} \|\nabla(M_l-M)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(t) &\le \mathrm{e}^{\int_0^{t_0^*} g(t)\,\mathrm{d}t}\Bigg(\frac{5}{2}\int_0^t \|((v_l-v)\cdot\nabla)M\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&+ \left(C + C(\Omega)\|\Delta M_l\|^2_{L^3(\Omega;\mathbb{R}^3)} + \|\nabla M\|^4_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\right)\|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\Bigg).
\end{aligned}
\quad (3.216)
$$

Due to (3.98) and (3.214) we can pass to the limit as $l\to\infty$ to see that, in summary,

$$
M_l \xrightarrow{l\to\infty} M \text{ in } L^\infty(0,t_0^*;\mathbf{H}^1(\Omega;\mathbb{R}^3)). \quad (3.217)
$$

This ensures the continuity of the operator $\mathcal{L}$ and the applicability of Schauder's fixed point theorem.

Following the further reasoning from the proof of Lemma 18, we complete the proof of Lemma 28, i.e., of the local existence of weak approximate solutions to the LLG system. □

3.2.3.3 Energy estimates for short time weak solutions to the approximate problem

We continue the analysis of the weak approximate solutions with the establishment of energy estimates. These energy estimates are necessary to extend the solution beyond time t_0^* while keeping certain regularity. For the system including the LLG equation, we need two energy laws ensuring all necessary regularity. The smallness condition (3.220) is crucial at this point to obtain $\mathbf{H}^2$-regularity for the magnetization. We obtain

Corollary 29. *Let (v_m, F_m, M_m) be the weak solution to the approximate problem (3.147)–(3.156) in $\Omega \times (0, t_0^*)$ obtained in Lemma 28. Then, we have*

$$\begin{aligned}
&\sup_{0\le t\le t_0^*} \Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + C\|F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \Big) \\
&\quad + \int_0^{t_0^*} \|(M_m)_t + (v_m\cdot\nabla)M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\qquad + 2\int_0^{t_0^*} \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} \,\mathrm{d}s \\
&\le \sup_{0\le t\le t_0^*} \Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_m)\|_{L^1(\Omega)} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \Big) \\
&\quad + \int_0^{t_0^*} \|(M_m)_t + (v_m\cdot\nabla)M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\qquad + 2\int_0^{t_0^*} \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} \,\mathrm{d}s \\
&\le \|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^3)}.
\end{aligned} \tag{3.218}$$

and, moreover,

$$\begin{aligned}
&\sup_{0\le t\le t_0^*} \Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_m)\|_{L^1(\Omega)} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \Big) \\
&\quad + 2\int_0^{t_0^*} \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} \\
&\qquad + \Bigg(1 - C_1(\Omega)\Bigg(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} \\
&\qquad\qquad + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \Bigg)\Bigg) \|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\le C_2(\Omega) \Big(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \Big)^2 t_0^* \\
&\qquad + \Big(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \Big)
\end{aligned} \tag{3.219}$$

as long as the initial data satisfies the condition

$$\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} < \frac{1}{C_1(\Omega)} \tag{3.220}$$

for some constant $C_1(\Omega)$. Then, in particular,

$$v_m \in L^\infty(0,t_0^*;\mathbf{H}) \cap L^2(0,t_0^*;\mathbf{V}), \tag{3.221}$$
$$F_m \in L^\infty(0,t_0^*;L^2(\Omega;\mathbb{R}^{2\times 2})) \cap L^2(0,t_0^*;\mathbf{H}^1(\Omega;\mathbb{R}^{2\times 2})) \tag{3.222}$$
$$M_m \in L^\infty(0,t_0^*;\mathbf{H}^1(\Omega;\mathbb{R}^3)) \cap L^2(0,t_0^*;\mathbf{H}^2(\Omega;\mathbb{R}^3)) \tag{3.223}$$

for any $m > 0$.

Proof. The calculation of the energy estimate with an LLG type equation is based on the ideas used in [CF01].
Notice that the following calculations are reasonable due to the regularity obtained in Lemma 26.
We multiply equation (3.147) by v_m, equation (3.149) by $W'(F_m)$ (to see that this test function is admissible, we refer to the proof of Corollary 20), equation (3.150) by $-\Delta M_m$ and integrate all the equations over both Ω and $(0,t)$ for $t \leq t_0^*$ to find (after using integration by parts)

$$\begin{aligned}&\int_\Omega \frac{1}{2}|v_m|^2 \,\mathrm{d}x \\ &= \int_0^t \int_\Omega \Big(-\nu|\nabla v_m|^2 + \Big(\nabla\cdot\big(W'(F_m)F_m^\top - \nabla M_m \odot \nabla M_m\big)\Big)\cdot v_m\Big) \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_\Omega \frac{1}{2}|P_m(v_0)|^2 \,\mathrm{d}x,\end{aligned} \tag{3.224}$$

$$\begin{aligned}&\int_\Omega W(F_m)\,\mathrm{d}x - \int_0^t\int_\Omega (\nabla v_m F_m) : W'(F_m)\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_0^t\int_\Omega \kappa\nabla F_m \,\vdots\, \nabla W'(F_m)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega W(F_0)\,\mathrm{d}x,\end{aligned} \tag{3.225}$$

$$\begin{aligned}&\int_\Omega \frac{1}{2}|\nabla M_m|^2\,\mathrm{d}x - \int_0^t\int_\Omega (v_m\cdot\nabla)M_m\cdot\Delta M_m\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_0^t\int_\Omega |\nabla M_m|^2 \underbrace{M_m\cdot\Delta M_m}_{=-|\nabla M_m|^2}\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |\Delta M_m|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_\Omega\frac{1}{2}|\nabla M_0|^2\,\mathrm{d}x.\end{aligned} \tag{3.226}$$

Next, we sum equations (3.224)–(3.226). Since v_m is divergence-free and vanishes on the boundary and due to the identities

$$\int_0^t\int_\Omega \big(\nabla\cdot W'(F_m)F_m^\top\big)\cdot v_m\,\mathrm{d}x\,\mathrm{d}s = -\int_0^t\int_\Omega \big(\nabla v_m F_m\big) : W'(F_m)\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\nabla\cdot\big(\nabla M_m \otimes \nabla M_m\big) = \nabla\frac{|\nabla M_m|^2}{2} + \nabla^\top M_m \Delta M_m,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 \,\mathrm{d}x \\
&\qquad\qquad + \int_0^t \int_\Omega \nu |\nabla v_m|^2 + \kappa \nabla F_m \,\vdots\, \nabla W'(F_m) + |\Delta M_m|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_0^t \int_\Omega |\nabla M_m|^4 \,\mathrm{d}x \,\mathrm{d}s + \frac{1}{2}\int_\Omega |P_m(v_0)|^2 + 2W(F_0) + |\nabla M_0|^2 \,\mathrm{d}x.
\end{aligned}$$

Since $\Omega \subset \mathbb{R}^2$, we have the following Sobolev estimate (see, e.g., [CF01, Section 5, Equation (5.6)])

$$\begin{aligned}
&\|\nabla M\|_{L^4(\Omega;\mathbb{R}^{3\times 2})} \\
&\leq C(\Omega)\|\nabla M\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})} \left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{4}}. \qquad (3.227)
\end{aligned}$$

Thus, we obtain, using also the identity $\nabla W'(F_m) = W''(F_m)\nabla F_m$ (which reads, using index notation, $\nabla_\sigma W'(F_m)_{ij} = W''(F_m)_{ijkl}\nabla_\sigma (F_m)_{kl}$) and (3.17),

$$\begin{aligned}
&\frac{1}{2}\left(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_m)\,\mathrm{d}x + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right) \\
&\qquad + \int_0^t \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} + \|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\leq \int_0^t C(\Omega)\,\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \mathrm{d}s \\
&\qquad + \frac{1}{2}\left(\|P_m(v_0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\,\mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Rearranging yields the first LLG energy estimate

$$\begin{aligned}
&\frac{1}{2}\left(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_m)\,\mathrm{d}x + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right) \\
&\qquad + \int_0^t \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} \\
&\qquad\qquad + \left(1 - C(\Omega)\,\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)\|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)} \,\mathrm{d}s \\
&\leq \int_0^t C(\Omega)\,\|\nabla M_m\|^4_{L^2(\Omega;\mathbb{R}^{3\times 2})} \,\mathrm{d}s \\
&\qquad + \frac{1}{2}\left(\|P_m(v_0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\,\mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right). \qquad (3.228)
\end{aligned}$$

To continue, we create another energy estimate by using the equivalent forms of the LLG equation from Lemma 10, which can be used for M_m since the initial datum M_0 has also length 1.
We multiply (3.20) (with v_m and M_m plugged in) by $((M_m)_t + (v_m \cdot \nabla)M_m)$ and (3.22) by $-\Delta M_m$, integrate the equations over both Ω and $(0,t)$ for $t \leq t_0^*$ to

find

$$\begin{aligned}\int_0^t \int_\Omega |(M_m)_t + (v_m \cdot \nabla)M_m|^2 \ \mathrm{d}x \ \mathrm{d}s \\ = \ -\int_0^t \int_\Omega (M_m \times \Delta M_m) \cdot ((M_m)_t + (v_m \cdot \nabla)M_m) \ \mathrm{d}x \ \mathrm{d}s \\ -\int_0^t \int_\Omega (M_m \times (M_m \times \Delta M_m)) \cdot \Big((M_m)_t + (v_m \cdot \nabla)M_m\Big) \ \mathrm{d}x \ \mathrm{d}s,\end{aligned} \tag{3.229}$$

and

$$\begin{aligned}\int_\Omega \frac{1}{2}|\nabla M_m|^2 \ \mathrm{d}x - \int_0^t \int_\Omega (v_m \cdot \nabla)M_m \cdot \Delta M_m \ \mathrm{d}x \ \mathrm{d}s \\ = \ -\int_0^t \int_\Omega (M_m \times ((M_m)_t + (v_m \cdot \nabla)M_m)) \cdot \Delta M_m \ \mathrm{d}x \ \mathrm{d}s \\ + \int_\Omega \frac{1}{2}|\nabla M_0|^2 \ \mathrm{d}x.\end{aligned} \tag{3.230}$$

The last term on the right-hand side of (3.229) can be rewritten using the Graßmann identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ for $a, b, c \in \mathbb{R}^3$

$$\begin{aligned}-\int_0^t \int_\Omega (M_m \times (M_m \times \Delta M_m)) \cdot \Big((M_m)_t + (v_m \cdot \nabla)M_m\Big) \ \mathrm{d}x \ \mathrm{d}s \\ = \ -\int_0^t \int_\Omega (M_m \cdot \Delta M_m) \underbrace{M_m \cdot ((M_m)_t + (v_m \cdot \nabla)M_m)}_{=\left(\frac{|M_m|^2}{2}\right)_t + (v_m \cdot \nabla)\frac{|M_m|^2}{2} = 0} \ \mathrm{d}x \ \mathrm{d}s \\ -\int_0^t \int_\Omega -\Delta M_m \cdot ((M_m)_t + (v_m \cdot \nabla)M_m) \ \mathrm{d}x \ \mathrm{d}s \\ = \ -\int_\Omega \frac{1}{2}|\nabla M_m|^2 \ \mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla M_0|^2 \ \mathrm{d}x \\ + \int_0^t \int_\Omega (v_m \cdot \nabla)M_m \cdot \Delta M_m \ \mathrm{d}x \ \mathrm{d}s.\end{aligned}$$

Now, summing up (3.229) and (3.230) with 2·(3.224) and 2·(3.225) and using the identity $(a \times b) \cdot c = -(a \times c) \cdot b$ we obtain

$$\begin{aligned}\int_\Omega |v_m|^2 + 2W(F_m) + |\nabla M_m|^2 \ \mathrm{d}x + \int_0^t \int_\Omega |(M_m)_t + (v_m \cdot \nabla)M_m|^2 \ \mathrm{d}x \ \mathrm{d}s \\ + 2\int_0^t \int_\Omega \nu|\nabla v_m|^2 + a\kappa|\nabla F_m|^2 \ \mathrm{d}x \ \mathrm{d}s \\ \leq \ \int_\Omega |P_m(v_0)|^2 + 2W(F_0) + |\nabla M_0|^2 \ \mathrm{d}x,\end{aligned}$$

or, equivalently

$$\begin{aligned}
&\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_m)\ \mathrm{d}x + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\qquad + \int_0^t \|(M_m)_t + (v_m\cdot\nabla)M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
&\qquad + 2\int_0^t \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})}\ \mathrm{d}s \\
\leq\ & \|P_m(v_0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\ \mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}. \qquad (3.231)
\end{aligned}$$

This second LLG energy estimate proves the second inequality in estimate (3.218) (as $\|P_m(v_0)\|_{L^2(\Omega;\mathbb{R}^2)} \leq \|v_0\|_{L^2(\Omega;\mathbb{R}^2)}$); the first inequality follows from an application of (3.13). Moreover, (3.231) helps the first LLG energy estimate (3.228) to become

$$\begin{aligned}
&\frac{1}{2}\left(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_m)\ \mathrm{d}x + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right) \\
&\qquad + \int_0^t \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})} \\
&\qquad + \Bigg(1 - C_1(\Omega)\Bigg(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\ \mathrm{d}x \\
&\qquad\qquad + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Bigg)\Bigg)\|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\ \mathrm{d}s \\
\leq\ & C_2(\Omega)\left(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\ \mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)^2 t \\
&\qquad + \frac{1}{2}\left(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\ \mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)
\end{aligned}$$

for small initial data satisfying

$$\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\int_\Omega W(F_0)\ \mathrm{d}x + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} < \frac{1}{C_1(\Omega)}.$$

We take the supremum over all $t \in [0, t_0^*]$ on both sides of this equality and the estimate (3.219) together with the condition (3.220) follows. The improved regularities in (3.221) and (3.222) and their uniformity in m are a direct consequence of the preceding estimates.
The regularity result (3.223) follows from the preceding estimates on the gradient of the magnetization and from the uniformly conserved length $|M_m| = 1$, which yields $M_m \in L^\infty(0, t_0^*; L^2(\Omega;\mathbb{R}^3))$. □

3.2.3.4 Weak solutions to the approximate problem by time extension

In order to prove the existence of weak solutions to the approximate problem, it remains to show the extension of the time interval, where solutions exist. We achieve this task using Corollary 29, together with an estimate derived from (3.167), thus ultimately proving Theorem 25.

Before we head to the proof of the theorem, owing to estimate (3.167), we strengthen the estimate $M_m \in L^2(0, t_0^*; \mathbf{H}^2(\Omega; \mathbb{R}^3))$ from (3.223) in the sense, that, albeit not uniformly in the Galerkin variable m, it is L^∞ in time, which is then sufficient to extend the approximate solution in time.

Indeed, notice that since $\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(t)$ is bounded uniformly by the initial data on $(0, t_0^*)$ through (3.218), we may rewrite (3.167) as

$$\begin{aligned}\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}(t) &\leq \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + C(v, \|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^3)}) \\ &\qquad\qquad \times \int_0^t \left(1 + \|\Delta M\|^4_{L^2(\Omega;\mathbb{R}^3)}(s)\right) \mathrm{d}s,\end{aligned}$$

whence we obtain by Gronwall's inequality, since $\int_0^{t_0^*} \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}(s)\mathrm{d}s$ is bounded by (3.219), that for all $t \in [0, t_0^*)$

$$\begin{aligned}&\|\Delta M(t)\|_{L^2(\Omega;\mathbb{R}^3)} \\ &\leq \ C(v, \|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^3)}) \\ &\qquad\qquad \times (\|\Delta M_0\|_{L^2(\Omega;\mathbb{R}^3)} + T), \quad (3.232)\end{aligned}$$

where $0 < T < \infty$ is the end time given in Theorem 25. Now, we continue with the proof of this theorem.

Proof of Theorem 25. Let $0 < T < \infty$ be fixed. We first define

$$\widetilde{C} := \|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}$$

to be the right-hand side of (3.218). If (v_m, F_m, M_m) is a solution to the system (3.147)–(3.156) in $\Omega \times (0, \tilde{t})$ for some $0 < \tilde{t} < t_0^*$, then

$$\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)}(\tilde{t}) + 2\|W(F_m)\|_{L^1(\Omega)}(\tilde{t}) + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}(\tilde{t}) \leq \widetilde{C}$$

due to (3.218).

Following the proof of Lemma 28, we conclude that there exists a constant δ which depends only on m and $\widetilde{C}$ (due to the L^∞-bounds obtained from the energy estimate (3.218) this δ does not depend on the time $\tilde{t}$) such that the system (3.147)–(3.156) has a solution $(\tilde{v}_m, \tilde{F}_m, \tilde{M}_m)$ on $\Omega \times [\tilde{t}, \tilde{t} + \delta]$ satisfying $(\tilde{v}_m, \tilde{F}_m, \tilde{M}_m)(\tilde{t}) = (v_m, F_m, M_m)(\tilde{t})$. Moreover, due to (3.232), we can assure that $M_m(\tilde{t})$ is bounded in the $\mathbf{H}^2$-norm by a constant that only depends on m and the initial data.

Then, we can continue this extension and finally obtain a solution (v_m, F_m, M_m) on $\Omega \times (0, T)$.

Notice that, due to the regularity of the solutions, the new initial data has the same regularity as before. Moreover, if the initial data satisfies the smallness condition (3.220) then so does the solution at any following time. Thus, we have the energy estimates

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + C\|F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big)\\
&\quad + \int_0^T \|(M_m)_t + (v_m\cdot\nabla)M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\qquad\qquad + 2\nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + 2a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})}\,\mathrm{d}s\\
&\le \sup_{0\le t\le T}\Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_m)\|_{L^1(\Omega)} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big)\\
&\quad + \int_0^T \|(M_m)_t + (v_m\cdot\nabla)M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\qquad\qquad + 2\nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + 2a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})}\,\mathrm{d}s\\
&\le \|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned}
\tag{3.233}
$$

and

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\|v_m\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_m)\|_{L^1(\Omega)} + \|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big)\\
&\quad + 2\int_0^T \nu\|\nabla v_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + a\kappa\|\nabla F_m\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2\times 2})}\\
&\qquad + \Bigg(1 - C_1(\Omega)\Bigg(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)}\\
&\qquad\qquad + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Bigg)\Bigg)\|\Delta M_m\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C_2(\Omega)\Big(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big)^2 T\\
&\qquad + \Big(\|v_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + 2\|W(F_0)\|_{L^1(\Omega)} + \|\nabla M_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big).
\end{aligned}
\tag{3.234}
$$

From here we directly deduce that

$$ v_m \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \tag{3.235} $$

$$ F_m \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})) \cap L^2(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^{2\times 2})), \tag{3.236} $$

$$ M_m \in L^\infty(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^3)) \cap L^2(0,T;\mathbf{H}^2(\Omega;\mathbb{R}^3)) \tag{3.237} $$

uniformly for any $m > 0$. This concludes the proof of Theorem 25. □

3.2.4 Existence of weak solutions to the original problem

Finally, we prove that the limit of the Galerkin approximations is a solution to the original system (3.1)–(3.4), (3.18), (3.6)–(3.11). So, in the following we provide the final step of the proof of Theorem 11.

Proof of Theorem 11. We prepare passing to the limit as $m \to \infty$. To establish this, we need the following convergence results

$$v_m \to v \quad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^3)), \tag{3.238}$$
$$\nabla v_m \rightharpoonup \nabla v \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})), \tag{3.239}$$
$$F_m \to F \quad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^{2\times 2})), \tag{3.240}$$
$$\nabla F_m \rightharpoonup \nabla F \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{2\times 2\times 2})), \tag{3.241}$$
$$M_m \to M \quad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^3)), \tag{3.242}$$
$$\nabla M_m \to \nabla M \quad \text{in } L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2})), \tag{3.243}$$
$$\Delta M_m \rightharpoonup \Delta M \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),. \tag{3.244}$$

3.2.4.1 Convergence results for the approximate weak solutions

The convergence results for the velocity (3.238)–(3.239) and the convergence results for the deformation gradient (3.240)–(3.241) are established in exactly the same way as for the system without the LLG equation in Section 3.1.4.1. For this reason, we omit these details here and only take care of the convergence results for the magnetization (3.243)–(3.244). We rely on the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2) to obtain the strong convergence (3.243). To this end, we estimate $(M_m)_t$ in $L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))$ which then leads to an estimate on the time derivative of ∇M_m:

$$\begin{aligned}
&\sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \int_\Omega (M_m)_t \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \\
&= \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \int_\Omega -(v_m\cdot\nabla)M_m\cdot(\zeta\varphi) - (M_m\times\Delta M_m)\cdot(\zeta\varphi) \\
&\qquad\qquad + |\nabla M_m|^2 M_m\cdot(\zeta\varphi) + \Delta M_m\cdot(\zeta\varphi)\, \mathrm{d}x\, \mathrm{d}t \\
&\overset{\text{Hölder}}{\le} \sup_{\substack{\|\zeta\|_{L^4(0,T)}\le 1\\ \|\varphi\|_{L^2(\Omega;\mathbb{R}^3)}\le 1}} \int_0^T \|v_m\|_{L^4(\Omega;\mathbb{R}^2)}\|\nabla M_m\|_{L^4(\Omega;\mathbb{R}^{3\times 2})}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|M_m\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\Delta M_m\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\nabla M_m\|^2_{L^4(\Omega;\mathbb{R}^{3\times 2})}\|M_m\|_{L^\infty(\Omega;\mathbb{R}^3)}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\qquad + \|\Delta M_m\|_{L^2(\Omega;\mathbb{R}^3)}|\zeta|\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \\
&\overset{\text{Hölder}}{\le} \sup_{\|\zeta\|_{L^4(0,T)}\le 1} \Big(\|v_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^2))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))}\|\zeta\|_{L^4(0,T)} \\
&\qquad + \|M_m\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}\|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^4(0,T)} \\
&\qquad + \|M_m\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))}\|\zeta\|_{L^4(0,T)} \\
&\qquad + \|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}\|\zeta\|_{L^4(0,T)} \Big)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \|v_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^2))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
& \qquad + \|M_m\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}\|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))} \\
& \qquad + \|M_m\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}\|\nabla M_m\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
& \qquad + \|\Delta M_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))}.
\end{aligned}
$$

From the regularities (3.235)–(3.237), and interpolation inequalities (see Proposition 34 in Appendix A.2), the boundedness of $(0,T)$, and the length constraint of M we get that the right-hand side is bounded. Thus, we obtain

$$
(M_m)_t \in L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3)) \tag{3.245}
$$

uniformly in m. This then implies that (see Remark 21)

$$
(\nabla M_m)_t \in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{3\times 2})) \tag{3.246}
$$

uniformly in m.
From the embeddings $\mathbf{H}^1(\Omega;\mathbb{R}^3) \overset{c}{\subset} L^4(\Omega;\mathbb{R}^3) \subset \mathbf{H}^{-1}(\Omega;\mathbb{R}^3)$ and $\mathbf{H}^1_{\mathbf{n}}(\Omega;\mathbb{R}^{3\times 2}) \overset{c}{\subset} L^4(\Omega;\mathbb{R}^{3\times 2}) \subset \mathbf{H}^{-1}(\Omega;\mathbb{R}^{3\times 2})$, where the first embedding is compact and the second one is continuous, respectively, and the fact that $M_m \in L^2(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^3))$ and $\nabla M_m \in L^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega;\mathbb{R}^{3\times 2}))$, we conclude by the Aubin-Lions Lemma (see Lemma 35 in Appendix A.2) the compact embeddings

$$
\Big\{M \in L^2(0,T;\mathbf{H}^1(\Omega;\mathbb{R}^3)):\ M_t \in L^{\frac{4}{3}}(0,T;L^2(\Omega;\mathbb{R}^3))\Big\} \overset{c}{\subset} L^2(0,T;L^4(\Omega;\mathbb{R}^3))
$$

and

$$
\begin{aligned}
& \Big\{\nabla M \in L^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega;\mathbb{R}^{3\times 2})):\ (\nabla M)_t \in L^{\frac{4}{3}}(0,T;\mathbf{H}^{-1}(\Omega;\mathbb{R}^{3\times 2}))\Big\} \\
& \overset{c}{\subset} L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2})),
\end{aligned}
$$

respectively. This yields the strong convergence results (3.242) and (3.243) (up to subsequence) of $\{M_m\}_m$ and $\{\nabla M_m\}_m$. The convergence (3.244) is a direct consequence of the regularity (3.237) and the convergence of $\{\nabla M_m\}_m$.

3.2.4.2 Convergence to the weak formulations of the original problem

After having made sure that the solution to the approximate problem converges, we have to prove that the limit also satisfies the weak formulation of the system (3.1), (3.4), (3.18) in $\Omega \times (0,T)$.
To establish this, we insert the solutions of the approximate problem and approximate test functions into the weak formulation (3.144)–(3.146) and pass to the limit as $m \to \infty$. The boundary conditions (3.6)–(3.8) hold for the limit, since the approximate solutions are constructed satisfying these conditions. The attainment of the initial data (3.9)–(3.11) is then shown in a final step of the entire proof.

Notice that, since the weak solution v_m to the approximate problem is defined using test functions from the projected spaces $\mathbf{H}_m$ in (3.144), we also need to pass to the limit with the test functions (only in space). But for any test function $\xi \in \mathbf{V}$ we use the sequence of approximate test functions $\xi_m := P_m(\xi) \in \mathbf{H}_m$ which converges strongly to ξ. In the following, we will use this particular sequence of test functions. Moreover, $\zeta \in W^{1,\infty}(0,T)$ is a test function with $\zeta(T) = 0$.

Convergence of the v-equation (3.144) **and the F-equation** (3.145)**.** Since these equations do not differ substantially from the corresponding ones in the system without LLG, the reasoning here is the same as in Section 3.1.4.2. The difference is that the approximate solution for the magnetization has two indices m and n, but this does not affect the argument. In detail we prove

Convergence of the M-equation (3.146)**.** Here, we need to show that with the convergence results (3.238)–(3.241) the equation

$$\begin{aligned} &\int_0^T \int_\Omega -M_m \cdot (\zeta' \varphi) + (v \cdot \nabla) M_m \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega M_m(0) \cdot (\zeta(0)\varphi) \,\mathrm{d}x \\ = &\int_0^T \int_\Omega -(M_m \times \Delta M_m) \cdot (\zeta\varphi) \\ &\qquad + |\nabla M_m|^2 M_m \cdot (\zeta\varphi) - \nabla M_m : (\zeta \nabla\varphi) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.247}$$

converges to the equation

$$\begin{aligned} &\int_0^T \int_\Omega -M \cdot (\zeta' \varphi) + (v \cdot \nabla) M \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega M_0 \cdot (\zeta(0)\varphi) \,\mathrm{d}x \\ = &\int_0^T \int_\Omega -(M \times \Delta M) \cdot (\zeta\varphi) + |\nabla M|^2 M \cdot (\zeta\varphi) - \nabla M : (\zeta \nabla\varphi) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.248}$$

as $m \to \infty$. Notice that we integrated by parts with respect to time, so the dual form becomes an integral again. The third term on the left-hand side of the equation converges since $M_m(0) \to M_0$ strongly in $L^2(\Omega; \mathbb{R}^3)$ by construction. For the first and the second term on the left-hand side of the equation we obtain the convergence immediately from the strong convergence results (for details on the convergence of these two terms which are the same as in the gradient flow equation, we refer to Section 3.1.4.2).
Next, we see that the last term on the right-hand side of the equation converges, too, since it is linear and thus the weak convergence directly provides this result. For the first term on the right-hand side, we get

$$\begin{aligned} &\left| \int_0^T \int_\Omega (M_m \times \Delta M_m) \cdot (\zeta\varphi) - (M \times \Delta M) \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t \right| \\ &= \left| \int_0^T \int_\Omega (M_m \times \Delta M_m) \cdot (\zeta\varphi) - (M \times \Delta M_m) \cdot (\zeta\varphi) \right. \\ &\qquad\qquad \left. + (M \times \Delta M_m) \cdot (\zeta\varphi) - (M \times \Delta M) \cdot (\zeta\varphi) \,\mathrm{d}x \,\mathrm{d}t \right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T \int_\Omega |((M_m - M) \times \Delta M_m) \cdot (\zeta\varphi)| \\
&\qquad\qquad + (M \times (\Delta M_m - \Delta M)) \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \|M_m - M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \|\Delta M_m)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \\
&\qquad\qquad + \int_0^T \int_\Omega \underbrace{((\zeta\varphi) \times M)}_{\in L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \cdot (\Delta M_m - \Delta M) \, \mathrm{d}x \, \mathrm{d}t \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}$$

Finally, we obtain for the second term on the right-hand side of the equation (again, we omit the constant for brevity)

$$\begin{aligned}
&\left| \int_0^T \int_\Omega |\nabla M_m|^2 M_m \cdot (\zeta\varphi) - |\nabla M|^2 M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \right| \\
&= \Big| \int_0^T \int_\Omega |\nabla M_m|^2 M_m \cdot (\zeta\varphi) - |\nabla M|^2 M_m \cdot (\zeta\varphi) \\
&\qquad\qquad + |\nabla M|^2 M_m \cdot (\zeta\varphi) - |\nabla M|^2 M \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&= \Big| \int_0^T \int_\Omega (|\nabla M_m|^2 - |\nabla M|^2) M_m \cdot (\zeta\varphi) \\
&\qquad\qquad + |\nabla M|^2 (M_m - M) \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\leq \int_0^T \int_\Omega |(|\nabla M_m| + |\nabla M|) \underbrace{(|\nabla M_m| - |\nabla M|)}_{\leq |\nabla M_m - \nabla M|} M_m \cdot (\zeta\varphi)| \\
&\qquad\qquad + |\nabla M|^2 (M_m - M) \cdot (\zeta\varphi) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \int_\Omega \big| |\nabla M_m| |\nabla M_m - \nabla M| M_m \cdot (\zeta\varphi) \big| \\
&\qquad\qquad + \big| |\nabla M| |\nabla M_m - \nabla M| M_m \cdot (\zeta\varphi) \big| \\
&\qquad\qquad + |\nabla M|^2 |(M_m - M) \cdot (\zeta\varphi)| \, \mathrm{d}x \, \mathrm{d}t \\
&\overset{\text{Hölder}}{\leq} \|\nabla M_m\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \|\nabla M_m - \nabla M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad\qquad \times \|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \\
&\quad + \|\nabla M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \|\nabla M_m - \nabla M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad\qquad \times \|M_m\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^\infty(0,T;L^4(\Omega;\mathbb{R}^3))} \\
&\quad + \underbrace{\||\nabla M|^2\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}}_{= \|\nabla M\|^2_{L^4(0,T;L^3(\Omega;\mathbb{R}^{3\times 2}))}} \|M_m - M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^3))} \|\zeta\varphi\|_{L^\infty(0,T;L^{12}(\Omega;\mathbb{R}^3))} \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}$$

Notice that the right-hand side is bounded due to (3.237), interpolation inequalities (see Proposition 34 in Appendix A.2) and the conservation of the length. Moreover, we have $\varphi \in \mathbf{H}^1(\Omega;\mathbb{R}^3) \subset L^{12}(\Omega;\mathbb{R}^3)$ (in fact, it holds that $\varphi \in \mathbf{H}^1 \subset L^p$ for any $p \in [2,+\infty)$) since $\Omega \subset \mathbb{R}^2$. Thus, the M-equation converges.

3.2.4.3 Attainment of initial data for the weak solution to the original problem

Finally, we have left to prove that the initial data is actually attained by the solution. The arguments are again the same as in Section 3.1.4.3 for the system without the LLG equation. Notice that the different form of the LLG equation compared to the gradient flow equation does not affect the reasoning: this is due to the fact that only the time derivative is investigated during the analysis of the attainment of the initial data. This concludes the proof of Theorem 11. □

4 Conclusion

In Chapter 2 of this work, we derived models for magnetoelastic materials. This was done by utilizing variational principles in a continuum mechanical setting. We included elasticity as well as the theory of micromagnetics in our models. Our approach features the interplay between Lagrangian and Eulerian coordinate systems, which is important to combine elasticity, usually described in Lagrangian coordinates, and magnetism, usually described in Eulerian coordinates, into models for magnetoelasticity. Further, we coupled these effects both on the energetic level within the anisotropy energy and through transport relations.
In Chapter 3 we proved existence of weak solutions to models for a specific setting obtained in Chapter 2. One model includes gradient flow dynamics on the magnetic variable. The proof of existence for this model is based on a Galerkin method and a fixed point argument and uses ideas from [LL95]. The second model handles the more involved Landau-Lifshitz-Gilbert (LLG) equation instead of the gradient flow. The proof of existence for the model including the LLG equation additionally borrows special ideas from [CF01] which are needed to analyze the more complicated form of the LLG equation. In the following, we highlight open problems and present further possible research directions that go beyond the results of this work.
At first, the models, derived in Section 2.8 and then mathematically analyzed in Chapter 3, are based on simplifying assumptions. Open problems are to get rid of these assumptions to get closer to the full model presented in Section 2.6. We

1. neglect the stray field energy and the anisotropy energy in the full micromagnetic energy (2.17), and
2. incorporate the regularizing term $\kappa\Delta F$ in the transport equation of the deformation gradient F (2.99').
3. Further, we set $H_{\text{ext}} = 0$ in this work. The problem where $H_{\text{ext}} \neq 0$ is discussed in [BFLS16].

The first assumption results in the fact that the long-range interactions are not considered. However, these are a key feature in magnetic effects, nonetheless, especially in micromagnetics, where the domain patterns result from the interaction of the crystal structure (easy axes of magnetization), reflected in the anisotropy energy, with the long-range magnetic effects. To be also able to describe magnetic domains, stray field energy and anisotropy energy should be considered in an extension to this work. There is also a mathematical theory for the stray field around, see, e.g., [JK90, CF01], which one could try to extend to the setting of magnetoelasticity. Further, the crystal anisotropy, coupling the magnetic variable M to the elastic variable F, can be incorporated, for instance, in the form of a polynomial function as suggested in Section 2.3.1.

The second assumption is very strong as it basically destroys the character of the solution to the non-regularized F-equation (2.99) in the sense that the solution is no longer the actual deformation gradient. To overcome this, one has to set $\kappa = 0$ and see, whether it is possible to obtain solutions to the system without the regularization. As mentioned in Section 2.8.2, the proof of existence is then more involved and can not be done without further assumptions on F (see [LLZ05]), so this is a demanding open problem.

As highlighted in the introduction, the models we derived in this work set the basis for future work on magnetic fluids with immersed intermediate-sized particles: this is the reason why we phrased the model entirely in the Eulerian coordinate system. This way, it is possible to introduce a phase field parameter to model the fluid-structure interaction in the Eulerian coordinate system which is commonly used in fluid dynamics. Another point in the case of magnetic fluids is the rotational transport which allows for particle rotations. Some results on the variational approach using the rotational transport are stated in Appendix A.3.

Finally, a physical verification of our mathematical model is a meaningful topic for future discussions. On the one hand, from the analytical point of view, the analysis of special solutions in two dimensions can give insight on the strength of the coupling of deformation and magnetism within the partial differential equations, for instance. Regarding these special solutions, we already started the discussion with Carlos García-Cervera and Chun Liu within the joint DAAD project with my advisor Anja Schlömerkemper. On the other hand, numerical simulations are left for future research. Numerical results are very important to compare the model with actual experiments. Then, one could again achieve a big leap forward towards better understanding of magnetoelastic materials.

A Appendix

A.1 On special calculations and formulas

This part of the appendix is devoted to details on formulas needed in the modeling part in Chapter 2.

Conservation of mass in the Lagrangian coordinate system. Conservation of mass for compressible materials in the Eulerian coordinate system is given by equation (2.24). We want an explicit formula for the push-forward of the mass density, i.e., $\rho(x(X,t),t)$ in terms of the mass density $\rho_0(X)$ in the reference configuration and the deformation gradient $\widetilde{F}$ (see also [For13, Section 2.4]). To this end, we consider the mass contained within a subdomain $\omega_0 \subset \Omega_0$ given by

$$m_0 = m(0) = \int_{\omega_0} \rho_0(X) \, \mathrm{d}X.$$

Since the mass is conserved, the mass of any deformed configuration $\omega \subset \Omega \subset \mathbb{R}^d$ must be equal to m_0. Thus, we obtain

$$\int_{\omega_0} \rho_0(X) \, \mathrm{d}X = \int_{\omega} \rho(x,t) \, \mathrm{d}x.$$

Next, we transform into the Lagrangian coordinate system on the right-hand side and get

$$\int_{\omega_0} \rho_0(X) \, \mathrm{d}X = \int_{\omega_0} \rho(x(X,t),t) \, \det \widetilde{F} \, \mathrm{d}X.$$

This is equivalent to

$$\int_{\omega_0} \left(\rho_0(X) - \rho(x(X,t),t) \, \det \widetilde{F} \right) \, \mathrm{d}X = 0.$$

Since this is true for all subbodies ω_0 of Ω_0, it must be satisfied pointwise (Lebesgue-Besicovitch differentiation theorem [EG92, Section 1.7.1]), thus

$$\rho(x(X,t),t) = \frac{\rho_0(X)}{\det \widetilde{F}(X,t)}.$$

Transport equation of the deformation gradient F. We derive the transport of the deformation gradient from the push forward $\widetilde{F}(X,t) = F(x(X,t),t)$, see (2.4). We calculate the time derivative on both sides, then an application of the chain rule and (2.2)–(2.3) lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x(X,t),t) = \frac{\partial}{\partial t}F(x(X,t),t) + (v(x(X,t),t)\cdot\nabla)\, F(x(X,t),t)$$

and, assuming enough regularity,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{F}(X,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_X x(X,t)\right) = \nabla_X\left(\frac{\partial}{\partial t}x(X,t)\right) = \nabla_X v(x(X,t),t)\\ &= \nabla v(x(X,t),t)\nabla_X x(X,t) = \nabla v(x(X,t),t)\cdot\widetilde{F}(X,t).\end{aligned}$$

In view of the push forward formula $\widetilde{F}(X,t) = F(x(X,t),t)$, we write everything in the Eulerian coordinate system to find out that

$$F_t + (v\cdot\nabla)F = \nabla vF.$$

Transport equation of the magnetization M. We derive the (simple) transport equation of the magnetization in the Eulerian coordinate system from the proposed transport in the Lagrangian coordinate system (2.29), i.e.

$$M(x(X,t),t) = \frac{1}{\det\widetilde{F}(X,t)}M_0(X). \tag{A.1}$$

Taking the total time derivative of (A.1), we find with the formula

$$\frac{\mathrm{d}(\det\widetilde{F})}{\mathrm{d}\widetilde{F}} = (\det\widetilde{F})\widetilde{F}^{-T} \tag{A.2}$$

(a proof of this basic formula can be found in, e.g., [For13, Appendix A.3])

$$\begin{aligned}&M_t(x(X,t),t) + (v(x(X,t),t)\cdot\nabla)M(x(X,t),t)\\ =\quad& -\frac{1}{(\det\widetilde{F})^2}\det\widetilde{F}\left(\widetilde{F}^{-T} : \frac{\mathrm{d}}{\mathrm{d}t}\nabla_X x(X,t)\right)M_0(X)\\ =\quad& -\frac{1}{(\det\widetilde{F})^2}\det\widetilde{F}\,\mathrm{tr}\left(\widetilde{F}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_X x(X,t)\right)M_0(X)\\ =\quad& -\frac{1}{\det\widetilde{F}}\,\mathrm{tr}\left(\sum_{k=1}^{n}\frac{\partial X^k}{\partial x^i}\frac{\partial x_t^j}{\partial X^k}\right)M_0(X)\\ =\quad& -\frac{1}{\det\widetilde{F}}\big(\nabla\cdot v(x(X,t),t)\big)M_0(X)\\ =\quad& -(\nabla\cdot v(x(X,t),t))M(x(X,t),t).\end{aligned}$$

This is equivalent to

$$M_t + (v\cdot\nabla)M + (\nabla\cdot v)M = 0$$

in the Eulerian coordinate system and can also be rewritten in the form

$$M_t + \nabla\cdot(M\otimes v) = 0.$$

Gradient transformation formula. We derive a formula which allows to transform gradients with respect to x in the Eulerian coordinate system into gradients with respect to X in the Lagrangian coordinate system.
To this end, let $Q(x,t)$ be some quantity in the Eulerian coordinate system, which may be scalar-, vector-, or matrix-valued (even higher order tensors are fine). By inserting the deformation $x(X,t)$ from (2.1), we obtain a quantity in the Lagrangian coordinate system $Q(x(X,t),t)$. Next, we calculate the gradient of this with respect to X. We obtain, using the chain rule and the definition of the deformation gradient (2.3),

$$\nabla_X Q(x(X,t),t) = \nabla_x Q(x(X,t),t)\nabla_X x(X,t) = \nabla_x Q(x(X,t),t)\widetilde{F}.$$

Since it is important to get the dimensions right, we use the index notation to find out that

$$\nabla_{X_j} Q(x(X,t),t) = \nabla_{x_k} Q(x(X,t),t)\nabla_{X_j} x_k(X,t) = \nabla_{x_k} Q(x(X,t),t)\widetilde{F}_{kj}.$$

This form is particularly convenient when Q is a higher order tensor.
Finally, we multiply both equations by the inverse of $\widetilde{F}$ to get

$$\nabla_X Q(x(X,t),t)\widetilde{F}^{-1} = \nabla_x Q(x(X,t),t) = \nabla Q(x(X,t),t)$$

and

$$\nabla_{X_j} Q(x(X,t),t)\widetilde{F}^{-1}_{jk} = \nabla_{x_k} Q(x(X,t),t).$$

A formula for $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \det \widetilde{F}^\varepsilon$. We use the definitions (2.56)–(2.57) and (A.2). We obtain by an application of the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \det \widetilde{F}^\varepsilon &= (\det \widetilde{F})\widetilde{F}^{-\top} : \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \widetilde{F}^\varepsilon\right)\\ &= (\det \widetilde{F})\widetilde{F}^{-\top} : \nabla_X \widetilde{\chi}(X,t)\\ &= (\det \widetilde{F})\widetilde{\mathrm{tr}}(\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1}).\end{aligned}$$

A formula for $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} (\widetilde{F}^\varepsilon)^{-1}$. We start from the identity

$$I = (\widetilde{F}^\varepsilon)^{-1}\widetilde{F}^\varepsilon, \tag{A.3}$$

which holds true for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. We calculate the derivative with respect to ε at $\varepsilon = 0$ on both sides of (A.3), using the product rule, to find

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} I = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \left((\widetilde{F}^\varepsilon)^{-1}\widetilde{F}^\varepsilon\right) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (\widetilde{F}^\varepsilon)^{-1}\widetilde{F} + \widetilde{F}^{-1}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \widetilde{F}^\varepsilon.$$

Finally, we rearrange this, multiply by $\widetilde{F}^{-1}$ from the right and apply the definitions (2.56)–(2.57) to get

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (\widetilde{F}^\varepsilon)^{-1} = -\widetilde{F}^{-1}\nabla_X \widetilde{\chi}(X,t)\widetilde{F}^{-1}.$$

A.2 Supplementary results from the literature

In this part of the appendix, we list some important results needed in this thesis. For proofs, we refer to the cited literature.

Theorem 30. *(Carathéodory's existence theorem) [Fil88, Chapter 1, Theorem 1] For $t_0 \le t \le t_0 + a$, $a > 0$, and $\|x - x_0\| \le b$, $b > 0$ let the function $f(t,x)$ satisfy the Carathéodory conditions:*

- *let $f(t,x)$ be defined and continuous in x for almost all t;*
- *let $f(t,x)$ be measurable in t for each x;*
- *$|f(t,x)| \le m(t)$, the function $m(t)$ being L^1 (locally, if t is unbounded in the domain of definition D of $f(t,x)$).*

Then, on a closed interval $[t_0, t_0 + d]$, where $d > 0$, there exists a solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(t,x), \qquad x(t_0) = x_0. \tag{A.4}$$

In this case, one can take an arbitrary real number d which satisfies the inequalities

$$0 < d \le a, \qquad \phi(t_0 + d) \le b, \quad \textit{where} \quad \phi(t) := \int_{t_0}^{t} m(s)\ \mathrm{d}s.$$

Theorem 31. *[Fil88, Chapter 1, Theorem 2] Let $(t_0, x_0) \in D$ and let there exist an L^1-function $l(t)$ such that for any points $(t,x), (t,y) \in D$ it holds*

$$|f(t,x) - f(t,y)| \le l(t)|x - y|.$$

Then, in the domain D there exists at most one solution of the problem (A.4).

Theorem 32. *(Banach-Alaoglu-Bourbaki) [Bre11, Theorem 3.16] Let X be a Banach space and X^* be its dual space. The closed unit ball*

$$B_{X^*} := \{f \in X^* :\ \|f\|_{X^*} \le 1\}$$

is compact in the weak-$*$ topology, i.e., every sequence in B_{X^*} has a weakly-$*$ converging subsequence.

Theorem 33. *[Bre11, Theorem 3.18] Let X be a reflexive Banach space and let $\{x_n\}_n$ be a bounded sequence in X. Then there exists a subsequence $\{x_{n_k}\}_k$ that converges in the weak topology.*

Proposition 34. *(Interpolation in Bochner spaces) [Rou13, Proposition 1.41] Let $I \subset \mathbb{R}$ be a bounded interval. Let $p_1, p_2, q_1, q_2 \in [1, +\infty]$, $\lambda \in [0,1]$, and $f \in L^{p_1}(I; L^{q_1}(\Omega)) \cap L^{p_2}(I; L^{q_2}(\Omega))$. Then*

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2} \quad \textit{and} \quad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}$$

implies that

$$\|v\|_{L^p(I;L^q(\Omega))} \le \|v\|^{\lambda}_{L^{p_1}(I;L^{q_1}(\Omega))} \|v\|^{1-\lambda}_{L^{p_2}(I;L^{q_2}(\Omega))}.$$

Lemma 35. *(Aubin-Lions) [Rou13, Lemma 7.7] Let* $I \subset \mathbb{R}$ *be a bounded interval. Let* V_1, V_2 *be Banach spaces, and* V_3 *be a metrizable Hausdorff locally convex space,* V_1 *be separable and reflexive,* $V_1 \overset{c}{\subset} V_2$ *(a compact embedding),* $V_2 \subset V_3$ *(a continuous embedding),* $1 < p < +\infty$*,* $1 \leq q \leq +\infty$*. Then*

$$\{f \in L^p(I; V_1) : \ f_t \in L^q(I; V_3)\} \overset{c}{\subset} L^p(I; V_2)$$

is a compact embedding.

Lemma 36. *(Gelfand's triple) [Rou13, Lemma 7.3] Let* $I \subset \mathbb{R}$ *be a bounded interval. Let* H *be a Hilbert space identified with its own dual space* $H \equiv H^*$*. Let the embedding* $V \subset H$ *be continuous and dense (it follows that the embedding* $H \subset V^*$ *is continuous and dense). Let* $p' = \frac{p}{p-1}$ *be the conjugate exponent to* p*. Then*

$$\left\{f \in L^p(I; V) : \ f_t \in L^{p'}(I; V^*)\right\} \subset C(I; H)$$

is a continuous embedding.

A.3 On the rotational transport of magnetization and its coupling to elasticity

In this part of the appendix, we discuss the strong rotational coupling of the magnetic and the elastic variables already mentioned in Section 2.4. We find this transport coupling appropriate for magnetic fluids with immersed particles. In our model for magnetoelastic materials derived in Chapter 2, we have a single particle in mind, without a surrounding fluid, where the particle can move and rotate. Thus, we do not need such kind of transport in our setting. This rotational transport is important for fluids with immersed particles, which could be studied in a possible extension to this work. We understand the rotational transport and the underlying coupling in the following way.
When the particles are moved and deformed within the fluid, the magnetization follows the motion instantaneously. The magnetic dipoles are attached to the particles, so the center of mass of the dipoles follow the particle's motion, and, moreover, the angle of the dipoles are also changed. However, since the magnetization is supposed to be of unit length within the theory of micromagnetics which we involve in our modeling (see Section 2.3), the dipoles should not be stretched. Thus, the transport we find to be suitable takes the form

$$M(x(X,t),t) = \mathcal{R}M_0(X) \tag{A.5}$$

in the Lagrangian coordinate system, where $\mathcal{R} = \mathcal{R}(x(X,t),t)$ is a field of rotations. To satisfy the condition that $\mathcal{R}$ is indeed a field of rotations, we have to assume that

$$\dot{\mathcal{R}} = \mathcal{R}_t + (v \cdot \nabla)\mathcal{R} = \Omega_v \mathcal{R} \tag{A.6}$$

holds for $\mathcal{R}$, where $\Omega_v = \frac{\nabla v - \nabla^\top v}{2}$ denotes the skew-symmetric velocity gradient. Notice that the justification would work for any skew-symmetric matrix. In our case, however, Ω_v is chosen in accordance to molecular transport (see, e.g., [SL09, WXL12] and [For13, Remark 26]).
Then, we take the total time derivative of (A.5), and with (A.6) we find

$$\mathcal{M}_t := \underbrace{M_t + (v \cdot \nabla)M}_{\text{center of mass moving}} \qquad \underbrace{-\Omega_v M}_{\text{accounts for rotation}} \qquad = 0 \tag{A.7}$$

in the Eulerian coordinate system. Equivalently, one can also multiply (A.6) by $M_0(X)$ and use (A.5) to find (A.7).

It is a straightforward calculation to prove that with the assumed PDE (A.6) the field $\mathcal{R}(x(X,t),t)$ is a field of rotations. (A.6) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}^\top \mathcal{R}) = \dot{\mathcal{R}}^\top \mathcal{R} + \mathcal{R}^\top \dot{\mathcal{R}} = -\mathcal{R}^\top \Omega_v \mathcal{R} + \mathcal{R}^\top \Omega_v \mathcal{R} = 0$$

and since it holds true that $\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \det A(t)\,\mathrm{tr}(A^\top(t)A_t(t))$ (see, e.g., [For13, A.1–A.2]), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\det \mathcal{R}) &= \det \mathcal{R}\big(\mathcal{R} : \Omega_v \mathcal{R}\big) = \frac{1}{2}\det \mathcal{R}\big(\mathcal{R} : (\nabla v - \nabla^\top v)\mathcal{R}\big)\\ &= \frac{1}{2}\det \mathcal{R}\big(\mathcal{R} : (\nabla v \mathcal{R}) - \mathcal{R} : (\nabla^\top v \mathcal{R})\big)\\ &= \frac{1}{2}\det \mathcal{R}\big(\mathcal{R} : (\nabla v \mathcal{R}) - (\nabla v \mathcal{R}) : \mathcal{R}\big) = 0.\end{aligned}$$

Since it is natural to set

$$\mathcal{R}(x(X,0),0) = I,$$

as the deformation at time $t = 0$ is simply the identity, we obtain that $\mathcal{R}^\top\mathcal{R} \equiv I$ and $\det \mathcal{R} \equiv 1$ along the trajectory. Hence $\mathcal{R}(x,t) \in SO(3)$ is a rotation for any $(x,t) \in \Omega \times (0,t^*)$.

In the following, we present mainly two different approaches to derive the equation of motion for the system including the rotational transport. For this presentation, we neglect the stray field term in the micromagnetic energy and the purely elastic term. The remainder is sufficient to highlight the problems which arise.

A.3.1 Principal of virtual work

We discuss in this part an application of the *principal of virtual work* (see [DE88, FSL00]). This method is applied in the context of complex fluids in, e.g., [YFLS05, BLQS14]. When applying this method, we calculate the stress and pressure terms by means of the variation δW of the internal free energy

$$W = \int_\Omega \frac{1}{2}A|\nabla M|^2 + \psi(F,M)\ \mathrm{d}x \tag{A.8}$$

without the need to transform the integral with respect to x:

$$\delta W = \int_\Omega A\nabla M : \delta\nabla M + \psi_F(F,M)\delta F + \psi_M(F,M)\delta M\ \mathrm{d}x. \tag{A.9}$$

The goal is to obtain some expression (force term) multiplied by $\delta x = v\delta t$ [YFLS05, BLQS14] within the integral $\int_\Omega \cdots\ \mathrm{d}x$, where δx represents a virtual displacement or the variation of x.

In order to obtain δx in the equation, we need to substitute the expressions $\delta\nabla M$, δF, and δM.

From the transport equation for M (A.7) and the chain rule for F (2.28), we obtain an equation relating the above mentioned expressions with the variation of x by applying $\delta(\cdot) = ((\cdot)_t)\,\delta t + (\delta x \cdot \nabla)(\cdot)$ from [YFLS05, Section 2.3] or $\delta(\cdot) = ((\cdot)_t)\,\delta t$ from [BLQS14, Section 2], where the latter is a formal multiplication by δt.

It is not clear a priori which approach is correct. However, the latter seems to

be most consistent with the definition $\delta x = v\delta t$. The difference strongly reminds us of the difference between temporal and material derivative and different coordinate systems. We use the latter ansatz in the following calculations. Since it has the additional convection term, we can easily track what happens to those terms and compare the outcome for both definitions.
From (A.7), we obtain

$$\delta M_i = -(\delta x \cdot \nabla) M_i + \frac{\nabla_k \delta x_i - \nabla_i \delta x_k}{2} M_k. \tag{A.10}$$

Differentiating (A.7), we get

$$\begin{aligned}\nabla_j (M_t)_i = &-\nabla_j (u \cdot \nabla) M_i - (u \cdot \nabla)\nabla_j M_i + \frac{\nabla_j \nabla_k u_i - \nabla_j \nabla_i u_k}{2} M_k \\ &+ \frac{\nabla_k u_i - \nabla_i u_k}{2} \nabla_j M_k\end{aligned}$$

and hence

$$\begin{aligned}\delta \nabla_j M_i = &-\big(\nabla_j (\delta x \cdot \nabla)\big) M_i - (\delta x \cdot \nabla)\nabla_j M_i + \frac{\nabla_j \nabla_k \delta x_i - \nabla_j \nabla_i \delta x_k}{2} M_k \\ &+ \frac{\nabla_k \delta x_i - \nabla_i \delta x_k}{2} \nabla_j M_k.\end{aligned} \tag{A.11}$$

From the chain rule for the deformation gradient, we obtain

$$\delta F_{ij} = -(\delta x \cdot \nabla) F_{ij} + \nabla_k \delta x_i F_{kj}. \tag{A.12}$$

We plug (A.7), (A.11) and (A.12) into (A.9) to find

$$\begin{aligned}\delta W = &\int_\Omega A \nabla M : \delta \nabla M + \psi_F(F, M)\delta F + \psi_M(F, M)\delta M \ \mathrm{d}x \\ = &\int_\Omega A \nabla_j M_i \delta \nabla M \bigg(- \big(\nabla_j (\delta x \cdot \nabla)\big) M_i - (\delta x \cdot \nabla)\nabla_j M_i \\ &\quad + \frac{\nabla_j \nabla_k \delta x_i - \nabla_j \nabla_i \delta x_k}{2} M_k + \frac{\nabla_k \delta x_i - \nabla_i \delta x_k}{2} \nabla_j M_k \bigg) \\ &- (\psi_F)_{ij} (\delta x \cdot \nabla) F_{ij} + (\psi_F)_{ij} \nabla_k \delta x_i F_{kj} \\ &- (\psi_M)_i (\delta x \cdot \nabla) M_i + (\psi_M)_i \frac{\nabla_k \delta x_i - \nabla_i \delta x_k}{2} M_k \ \mathrm{d}x.\end{aligned}$$

The next step is to isolate δx with the help of integration by parts:

$$\begin{aligned}\delta W = &\int_\Omega A \nabla_j \big(\nabla_k M_i \nabla_j M_i\big) \delta x_k - A(\delta x \cdot \nabla) \frac{|\nabla M|^2}{2} \\ &\quad + \frac{A}{2} (\nabla_k \nabla_j (\nabla_j M_i M_k)) \delta x_i - \frac{A}{2} (\nabla_k \nabla_j (\nabla_j M_k M_i)) \delta x_i \\ &\quad \underbrace{- \frac{A}{2} \nabla_k (\nabla_j M_i \nabla_j M_k) \delta x_i + \frac{A}{2} \nabla_i (\nabla_j M_i \nabla_j M_k) \delta x_k}_{=0} \\ &- (\delta x \cdot \nabla) \psi(F, M) - \nabla_k \big((\psi_F)_{ij} F_{kj}\big) \delta x_i \\ &+ \frac{1}{2} \nabla_k \left(\big((\psi_M)_k M_i\big) - \big((\psi_M)_i M_k\big) \right) \delta x_i \ \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
= \int_\Omega & A\left(\nabla\cdot\left(\nabla M\odot\nabla M\right)\right)\cdot\delta x - A(\delta x\cdot\nabla)\frac{|\nabla M|^2}{2}\\
&+ A\left(\nabla\cdot\operatorname{skew}(\Delta M\otimes M)\right)\cdot\delta x\\
&- (\delta x\cdot\nabla)\psi(F,M) - \Big(\nabla\cdot\left(\psi_F F^\top\right)\Big)\cdot\delta x\\
&+ \nabla\cdot\left(\operatorname{skew}\left(M\otimes\psi_M\right)\right)\cdot\delta x \,\mathrm{d}x.
\end{aligned}
$$

This results in the total stress tensor according to the definition from [YFLS05, BLQS14]

$$
\begin{aligned}
\tau_{\text{total}} = &-A\nabla M\odot\nabla M + A\frac{|\nabla M|^2}{2}I - A\operatorname{skew}(\Delta M\otimes M)\\
&+ \psi(F,M)I + \psi_F F^\top - \operatorname{skew}\left(M\otimes\psi_M\right),
\end{aligned}
$$

where we define the elastic part of the total stress by

$$
\tau_{\text{elastic}} = -A\nabla M\odot\nabla M - A\operatorname{skew}(\Delta M\otimes M) + \psi_F F^\top - \operatorname{skew}\left(M\otimes\psi_M\right),
$$

and the isotropic part by

$$
\tau_{\text{isotropic}} = \left(A\frac{|\nabla M|^2}{2} + \psi(F,M)\right)I.
$$

The latter can be absorbed into the pressure. A difference of an isotropic stress tensor is still comparable according to [DE88, p.71], where it is stated that two stresses are regarded as equal, if the difference is an isotropic stress tensor.
However, notice that if we used the definition of the variation from [YFLS05, Section 2.3], namely $\delta(\cdot) = ((\cdot)_t)\,\delta t + (\delta x\cdot\nabla)(\cdot)$, then the outcome would not only lack the entire isotropic part of the stress tensor – which would still be a comparable result – but also the first summand of the elastic stress would vanish.
Moreover, if we applied this method with the definition of the variation from [YFLS05, Section 2.3] in the case of weak coupling and used the simple transport for the magnetization, the stress tensor would reduce to $\tau_{\text{total}} = \psi_F F^\top$, so there would be no magnetic contribution in the stress tensor at all.
To conclude this investigation, we state that it is not clear, which approach is the correct one for the principle of virtual work. This principle is often used, but there seems to be some ambiguity related to this approach.

A.3.2 Variation with respect to the domain: classical variation

In this part, we look at the variation with respect to the domain by a *classical variation*. Hereby, we mean the way of applying the least action principle as described in Section 2.2.
Since we need to transform the spatial integrals into the Lagrangian coordinate system, we use the transport (A.5) to express the magnetization M in terms of the Lagrangian coordinate system. It follows directly that

$$
\nabla M(x(X,t),t) = \left(\nabla\mathcal{R}(x(X,t),t)\right)M_0(X), \tag{A.13}
$$

where $\nabla\mathcal{R}(x(X,t),t)$ is a third order tensor and ∇ denotes the spatial gradient with respect to x.

We denote as above the reference configuration by Ω_0 and the deformed configuration by Ω and consider the exemplary action functional

$$\begin{aligned}\mathcal{A} &= \int_0^{t^*}\int_\Omega \frac{1}{2}\rho|u|^2 - \frac{1}{2}A|\nabla M|^2 - \psi(F,M)\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^{t^*}\int_{\Omega_0} \frac{1}{2}\rho_0|x_t|^2 - \frac{1}{2}A\left|\nabla_X(\mathcal{R}M_0)F^{-1}\right|^2 - \psi(F,\mathcal{R}M_0)\ \mathrm{d}X\ \mathrm{d}t, \qquad \text{(A.14)}\end{aligned}$$

where the first term is the kinetic energy, and the second and third term are the exchange energy term and the anisotropy term from the micromagnetic energy (2.17), respectively.

We calculate the variation of the action with respect to the flow map using volume preserving diffeomorphisms $x^\varepsilon(X,t)$ (due to incompressibility, see (2.110)) with $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}x^\varepsilon = \widetilde{\chi}$ and $\widetilde{\chi}(X,t) = \chi(x(X,t),t)$ being compactly supported with respect to space and time and smooth. We obtain, using (2.64) and (2.66) to find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{A}(x^\varepsilon)\\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_0^{t^*}\int_{\Omega_0}\frac{1}{2}\rho_0|x_t^\varepsilon|^2 - \frac{1}{2}A\left|\nabla_X(\mathcal{R}(x^\varepsilon,t)M_0)\left(\nabla_X x^\varepsilon\right)^{-1}\right|^2\\ &\qquad\qquad - \psi\left(\nabla_X x^\varepsilon, \mathcal{R}(x^\varepsilon,t)M_0\right)\ \mathrm{d}X\ \mathrm{d}t\\ &= \int_0^{t^*}\int_{\Omega_0}\rho_0 x_t\cdot\widetilde{\chi}_t - A\left(\nabla_X(\mathcal{R}M_0)F^{-1}\right) : \Big(\nabla_X((\nabla\mathcal{R}\widetilde{\chi})M_0)F^{-1}\\ &\qquad\qquad + \nabla_X(\mathcal{R}M_0)\left(-F^{-1}\nabla_X\widetilde{\chi}F^{-1}\right)\Big)\\ &\qquad\qquad - \psi_F : \nabla_X\widetilde{\chi} - \psi_M\cdot(\nabla\mathcal{R}M_0\widetilde{\chi})\ \mathrm{d}X\ \mathrm{d}t.\end{aligned}$$

The next step is to transform the integral and the variables to the Eulerian coordinate system and to pull out χ from all the summands in the last step. We get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{A}(x^\varepsilon)\\ &= \int_0^{t^*}\int_\Omega -\rho\frac{\mathrm{d}}{\mathrm{d}t}u\cdot\chi - A\left(\nabla M\right) : \left(\nabla((\nabla M)\chi) - \nabla M\nabla\chi\right)\\ &\qquad\qquad - \psi_F F^\top : \nabla\chi - \psi_M\cdot(\nabla M\chi)\ \mathrm{d}x\ \mathrm{d}t\\ &= \int_0^{t^*}\int_\Omega -\rho\frac{\mathrm{d}}{\mathrm{d}t}u\cdot\chi - A\left(\nabla M\right) : \left(\nabla\nabla M\chi\right)\\ &\qquad\qquad - \psi_F F^\top : \nabla\chi - \psi_M\cdot(\nabla M\chi)\ \mathrm{d}x\ \mathrm{d}t\\ &= \int_0^{t^*}\int_\Omega \left(-\rho\frac{\mathrm{d}}{\mathrm{d}t}u - A\nabla\frac{|\nabla M|^2}{2} + \nabla\cdot(\psi_F F^\top) - \nabla^\top M\psi_M\right)\cdot\chi\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

We find that the first term is the acceleration term and the third term has divergence form similar to the stress term. The second term is a total gradient

and can thus be absorbed into the induced pressure term.
However, the last term has neither divergence form nor is it a total gradient. This seems to indicate that the handling of the rotational transport is not correct. Moreover, there is no contribution to the stress from the microscopic variable M here. This does not seem to be physical as the microscopic variable should lead to a stress contribution. Hence, (2.29) seems to be the most reasonable choice. Note that this is also used in, e.g., [DD98].

A.3.3 Further investigation of the field of rotations

In a further attempt to tackle the problem with the rotation, we try to investigate the field of rotations in more detail. We calculate the variation of the action with respect to the flow map using volume preserving diffeomorphisms $x^\varepsilon(X,t)$ (due to incompressibility, see (2.110)) with $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} x^\varepsilon = \widetilde{\chi}$ and $\widetilde{\chi}(X,t) = \chi(x(X,t),t)$ being smooth and compactly supported with respect to space and time. We define $\mathcal{R}_\varepsilon(t) = \mathcal{R}_\varepsilon(x^\varepsilon(X,t),t)$ as a solution to

$$\begin{cases} \dot{\mathcal{R}}_\varepsilon = \Omega_{v_\varepsilon}\mathcal{R}_\varepsilon & t > 0 \\ \mathcal{R}_\varepsilon(0) = \mathcal{R}_\varepsilon(x^\varepsilon(X,0),0) = I & t = 0, \end{cases} \tag{A.15}$$

where $\Omega_{v_\varepsilon} = \operatorname{skew}(\nabla v_\varepsilon)$ and $v_\varepsilon = (x^\varepsilon)_t$.
We define $S := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\mathcal{R}_\varepsilon$. Since (if we assume that $\mathcal{R}_\varepsilon$ is at least of class C^2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{R}_\varepsilon = \Omega_{v_\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{R}_\varepsilon + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Omega_{v_\varepsilon}\mathcal{R}_\varepsilon, \tag{A.16}$$

we have

$$\dot{S} = \Omega_v S + \Omega_u \mathcal{R}_0, \tag{A.17}$$

where $u = y_t$ and $\mathcal{R}_0 = \mathcal{R}$ is the non-perturbed rotation matrix.
For S, we consider the ansatz

$$S(t) = \mathcal{R}(t)\mathcal{A}(t) \tag{A.18}$$

for some quadratic and time-dependent matrix $\mathcal{A}(t)$. For this matrix, we try to find a solution. We have

$$\begin{aligned} \dot{S}(t) &= \dot{\mathcal{R}}(t)\mathcal{A}(t) + \mathcal{R}(t)\dot{\mathcal{A}}(t) = \Omega_v\mathcal{R}(t)\mathcal{A}(t) + \mathcal{R}(t)\dot{\mathcal{A}}(t) \\ &= \Omega_v S(t) + \mathcal{R}(t)\dot{\mathcal{A}}(t) \end{aligned} \tag{A.19}$$

and, together with (A.17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t) = \dot{\mathcal{A}}(t) = \mathcal{R}(t)^{-1}\Omega_u\mathcal{R}(t), \tag{A.20}$$

where we can integrate to get

$$\mathcal{A}(t) = \int_0^t \mathcal{R}(s)^{-1}\Omega_{u(s)}\mathcal{R}(s)\ \mathrm{d}s + \mathcal{A}(0). \tag{A.21}$$

We plug this into (A.18) and find

$$S(t) = \mathcal{R}(t) \int_0^t \mathcal{R}(s)^{-1} \Omega_{u(s)} \mathcal{R}(s)\ \mathrm{d}s + \mathcal{R}(t)\mathcal{A}(0). \tag{A.22}$$

Since by (A.18) it holds that $\mathcal{A}(0) = S(0)$ and by (A.15) we have at least formally

$$S(0) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{R}_\varepsilon(0) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} I = 0,$$

the solution for S simplifies to

$$S(t) = \mathcal{R}(t) \int_0^t \mathcal{R}(s)^{-1} \Omega_{u(s)} \mathcal{R}(s)\ \mathrm{d}s. \tag{A.23}$$

This solution is used in the variation of the action. The problem we run into is that the test function $\chi(x,t)$ remains within a time integral, so there is no isolation of the variation possible. We further simplify the energy terms by neglecting the anisotropy. However, the calculations are presented for the compressible case, which does not affect the problem. We define $J^\varepsilon = \det F^\varepsilon$. This results in

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_0^{t^*} \int_{\Omega_0} \left|\nabla_X(\mathcal{R}_\varepsilon M_0)\left(\nabla_X x^\varepsilon\right)^{-1}\right|^2\ \det\left(\nabla_X x^\varepsilon\right)\ \mathrm{d}X\ \mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} \bigg(2\nabla_X(\mathcal{R}M_0)F^{-1} : \nabla_X(SM_0)F^{-1} \\
&\qquad\qquad + 2\nabla_X(\mathcal{R}M_0)F^{-1} : \nabla_X(\mathcal{R}M_0) \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F_\varepsilon^{-1} \bigg) J\ \mathrm{d}X\ \mathrm{d}t \\
&\quad + \int_0^{t^*} \int_{\Omega_0} \left|\nabla_X(\mathcal{R}M_0)F^{-1}\right|^2 \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} J^\varepsilon\ \mathrm{d}X\ \mathrm{d}t.
\end{aligned}$$

We plug in the solution for S and apply differentiation rules for the inverse and the determinant of a matrix. With the formula $\chi(x(X,t),t) = \widetilde{\chi}(X,t)$ and the transport $\mathcal{R}(s)M_0 = M(x(X,s),s)$, this yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}(x^\varepsilon) \\
&= \int_0^{t^*} \int_{\Omega_0} \bigg(2\nabla_X(\mathcal{R}M_0)F^{-1} : \nabla_X \left(\mathcal{R}(t) \int_0^t \mathcal{R}(s)^{-1}\Omega_{u(s)}\mathcal{R}(s)\ \mathrm{d}s M_0 \right) F^{-1} \\
&\qquad\qquad - 2\nabla_X(\mathcal{R}M_0)F^{-1} : \nabla_X(\mathcal{R}M_0)F^{-1}\nabla_X\widetilde{\chi}F^{-1} \bigg) J\ \mathrm{d}X\ \mathrm{d}t \\
&\qquad + \int_0^{t^*} \int_{\Omega_0} \left|\nabla_X(\mathcal{R}M_0)F^{-1}\right|^2 \operatorname{tr}(\nabla_X\widetilde{\chi}F^{-1}) J\ \mathrm{d}X\ \mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} 2\nabla M : \nabla \left(\mathcal{R}(t) \int_0^t \mathcal{R}(s)^{-1}\Omega_{u(s)} M(x,s)\ \mathrm{d}s \right)\ \mathrm{d}X\ \mathrm{d}t \\
&\qquad - \int_0^{t^*} \int_{\Omega_0} 2\nabla M : \nabla M \nabla\chi\ \mathrm{d}X\ \mathrm{d}t + \int_0^{t^*} \int_{\Omega_0} |\nabla M|^2\, \nabla\cdot\chi\ \mathrm{d}X\ \mathrm{d}t.
\end{aligned} \tag{A.24}$$

Since the second and third integral are straightforward to calculate, we continue with the first integral. This we denote by $\mathcal{I}$ and transform it into the Lagrangian coordinate system. Then, we obtain with the notation $\widetilde{\nabla M} = \nabla_X(\mathcal{R}M_0)F^{-1}$

$$
\begin{aligned}
\mathcal{I} &= \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \int_0^t \mathcal{R}(s)^{-1} \Bigg(\nabla_X \widetilde{\chi}_t F^{-1} \\
&\qquad\qquad - \left(\nabla_X \widetilde{\chi}_t F^{-1}\right)^\top \Bigg) \mathcal{R}(s) \,\mathrm{d}s M_0 \Bigg) F^{-1} J \,\mathrm{d}X \,\mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[\int_0^t \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi}_t F^{-1} \mathcal{R}(s) M_0 \,\mathrm{d}s \\
&\qquad\qquad - \int_0^t \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi}_t F^{-1}\right)^\top \mathcal{R}(s) M_0 \,\mathrm{d}s \Bigg] \Bigg) F^{-1} J \,\mathrm{d}X \,\mathrm{d}t.
\end{aligned}
$$

From here, we proceed with integration by parts with respect to time within the inner integrals (since $\widetilde{\chi}$ is assumed to be compactly supported and smooth, also the derivatives and gradients are compactly supported, thus the boundary terms vanish) and the help of the chain rule:

$$
\begin{aligned}
\mathcal{I} &= \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[\int_0^t \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi}_t F^{-1} \mathcal{R}(s) M_0 \,\mathrm{d}s \\
&\qquad\qquad - \int_0^t \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi}_t F^{-1}\right)^\top \mathcal{R}(s) M_0 \,\mathrm{d}s \Bigg] \Bigg) F^{-1} J \,\mathrm{d}X \,\mathrm{d}t \\
&= \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[- \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \mathcal{R}(s) M_0 \\
&\qquad\qquad + \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} \frac{\mathrm{d}}{\mathrm{d}s} F^{-1} \mathcal{R}(s) M_0 \\
&\qquad\qquad + \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{R}(s) M_0 \,\mathrm{d}s \\
&\qquad\qquad + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1}\right)^\top \mathcal{R}(s) M_0 \\
&\qquad\qquad + \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} \frac{\mathrm{d}}{\mathrm{d}s} F^{-1}\right)^\top \mathcal{R}(s) M_0 \\
&\qquad\qquad + \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1}\right)^\top \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{R}(s) M_0 \,\mathrm{d}s \Bigg] \Bigg) F^{-1} J \,\mathrm{d}X \,\mathrm{d}t.
\end{aligned}
$$

Next, we apply the derivative of an inverse of a matrix field (see (2.66) and [For13, Appendix A.5]), the transport of F (2.28) and the ODE (A.15) to obtain

$$
\begin{aligned}
\mathcal{I} = \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[- \int_0^t & -\mathcal{R}(s)^{-1} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{R}(s)\mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \frac{\mathrm{d}}{\mathrm{d}s} F F^{-1} \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \\
+ \int_0^t & -\mathcal{R}(s)^{-1} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{R}(s)\mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \frac{\mathrm{d}}{\mathrm{d}s} F F^{-1} \right)^\top \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \Bigg] \Bigg) F^{-1} J \ \mathrm{d}X \ \mathrm{d}t \\
= \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[- \int_0^t & -\mathcal{R}(s)^{-1} \Omega_{v(s)} \mathcal{R}(s)\mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \nabla_x u F F^{-1} \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \\
+ \int_0^t & -\mathcal{R}(s)^{-1} \Omega_{v(s)} \mathcal{R}(s)\mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \nabla_x u F F^{-1} \right)^\top \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \Bigg] \Bigg) F^{-1} J \ \mathrm{d}X \ \mathrm{d}t.
\end{aligned}
$$

Then, we simplify the terms in each line and finally transform the integral back to the Eulerian coordinate system in the last step. We get

$$
\begin{aligned}
\mathcal{I} = \int_0^{t^*} \int_{\Omega_0} \widetilde{\nabla M} : \nabla_X \Bigg(\mathcal{R}(t) \Bigg[+ \int_0^t & \mathcal{R}(s)^{-1} \Omega_{v(s)} \nabla_X \widetilde{\chi} F^{-1} \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \nabla_x v \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \nabla_X \widetilde{\chi} F^{-1} \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \\
- \int_0^t & +\mathcal{R}(s)^{-1} \Omega_{v(s)} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \mathcal{R}(s) M_0 \\
& + \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \nabla_x v \right)^\top \mathcal{R}(s) M_0 \\
& - \mathcal{R}(s)^{-1} \left(\nabla_X \widetilde{\chi} F^{-1} \right)^\top \Omega_{v(s)} \mathcal{R}(s) M_0 \ \mathrm{d}s \Bigg] \Bigg) F^{-1} J \ \mathrm{d}X \ \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
= \int_0^{t^*} \int_{\Omega_0} \nabla_x M : \nabla\Bigg(\mathcal{R}(t) \Bigg[\int_0^t & \mathcal{R}(s)^{-1} \Omega_{v(s)} \nabla\chi M(x,s) \\
& + \mathcal{R}(s)^{-1} \nabla\chi \nabla_x v M(x,s) \\
& - \mathcal{R}(s)^{-1} \nabla\chi \Omega_{v(s)} M(x,s) \, \mathrm{d}s \\
- \int_0^t & + \mathcal{R}(s)^{-1} \Omega_{v(s)} \left(\nabla\chi\right)^\top M(x,s) \\
& + \mathcal{R}(s)^{-1} \left(\nabla\chi \nabla_x v\right)^\top M(x,s) \\
& - \mathcal{R}(s)^{-1} \left(\nabla\chi\right)^\top \Omega_{v(s)} M(x,s) \, \mathrm{d}s \Bigg] \Bigg) \, \mathrm{d}X \, \mathrm{d}t,
\end{aligned}
$$

where we used a coordinate transformation back into the Eulerian coordinate system in the last step.

A next step would be integration by parts with respect to the spatial variable x to isolate the test function χ. However, the problem that χ is still within the third integral $\int_0^t \cdots ds$ still remains.

Another idea could be to go back and continue at equation (A.24). One could use there the fundamental theorem of calculus to pull $(\nabla\widetilde{\chi} - \nabla^\top \widetilde{\chi})$ out of the integral $\int_0^t \cdots ds$. However, this does not seem to be doable.

A.4 Proof of Lemma 2

Since M is supported on Ω, we can use integration by parts to rewrite the scalar potential (2.13). We obtain

$$\varphi(M)(x) = \int_\Omega N(x-y)(\nabla\cdot M)(y)\ \mathrm{d}y + \int_{\partial\Omega} N(x-y)\left(M\cdot\mathbf{n}\right)(y)\ \mathrm{d}\sigma_y, \quad \text{(A.25)}$$

where $\mathrm{d}\sigma_y$ denotes the surface measure. We use the following abbreviations:

$$\mathcal{V}(M)(x) := \int_\Omega N(x-y)(\nabla\cdot M)(y)\ \mathrm{d}y, \quad \text{(A.26)}$$

$$\mathcal{S}(M)(x) := \int_{\partial\Omega} N(x-y)\left(M\cdot\mathbf{n}\right)\ \mathrm{d}\sigma_y, \quad \text{(A.27)}$$

where $\mathcal{V}$ is called Newton potential and $\mathcal{S}$ is called single layer potential (see, e.g., [Sch08]).

Proof of Lemma 2 on page 29. For preciseness, we mark the gradient with the corresponding variable as an index, i.e., we write ∇_x and ∇_y instead of just ∇ in both cases.
Firstly, we introduce regularizations $N_\delta(x-y)$ and $(\nabla N)_\delta(x-y)$ of the kernel $N(x-y)$ and its gradient, respectively, as done in [Sch08]. To this end, let $\eta : [0,\infty] \to \mathbb{R}$ be a smooth function such that $\eta(r) = 0$ if $0 \le r \le \frac{1}{2}$ and $\eta(r) = 1$ if $r \ge 1$. Then, we set

$$N_\delta(x-y) := \eta\left(\frac{|x-y|}{\delta}\right) N(x-y).$$

It is clear that $N_\delta \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Now, we define

$$\mathcal{V}_\delta(M)(x) := \int_\Omega N_\delta(x-y)(\nabla\cdot M)(y)\ \mathrm{d}y.$$

and

$$\mathcal{S}_\delta(M)(x) := \int_{\partial\Omega} N_\delta(x-y)\left(M\cdot\mathbf{n}\right)(y)\ \mathrm{d}\sigma_y.$$

Secondly, we look at the convergence of $\mathcal{V}_\delta(M)(x)$. It follows from [Sch08, Section 2] that $|\mathcal{V}_\delta(M)(x) - \mathcal{V}(M)(x)| \le c\delta^2$ and thus $\mathcal{V}_\delta(M)(x)$ converges uniformly to $\mathcal{V}(M)(x)$.
Moreover, since $\nabla\cdot M, \nabla\cdot\widehat{M} \in L^\infty(\Omega)$, we obtain that $\mathcal{V}_\delta(M)(x)\big(\nabla_x\cdot\widehat{M}\big)(x)$ converges uniformly to $\mathcal{V}(M)(x)\big(\nabla_x\cdot\widehat{M}\big)(x)$ as $\delta \to 0$. The same holds if we

exchange M and $\widehat{M}$. Hence, we obtain

$$\begin{aligned}
&\Bigg|\int_\Omega \int_\Omega N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y)\big(\nabla_x \cdot M\big)(x) \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad\qquad - \int_\Omega \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y)\big(\nabla_x \cdot M\big)(x) \, \mathrm{d}y \, \mathrm{d}x\Bigg| \\
= \; &\Bigg|\int_\Omega \int_\Omega N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \, \mathrm{d}y \, \big(\nabla_x \cdot M\big)(y) \, \mathrm{d}x \\
&\qquad\qquad - \int_\Omega \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \, \mathrm{d}y \, \big(\nabla_x \cdot M\big)(x) \, \mathrm{d}x\Bigg| \\
= \; &\left|\int_\Omega \mathcal{V}_\delta(M)(x) \, \big(\nabla_x \cdot M\big)(x) \, \mathrm{d}x - \int_\Omega \mathcal{V}(M)(x) \, \big(\nabla_x \cdot M\big)(x) \, \mathrm{d}x\right| \\
= \; &\left|\int_\Omega \mathcal{V}_\delta(M)(x) \, \big(\nabla_x \cdot M\big)(x) - \mathcal{V}(M)(x) \, \big(\nabla_x \cdot M\big)(x) \, \mathrm{d}x\right| \\
\leq \; &\left|\int_\Omega [\mathcal{V}_\delta(M)(x) - \mathcal{V}(M)(x)] \, \big\|\nabla_x \cdot M\big\|_{L^\infty(\Omega)} \, \mathrm{d}x\right| \\
\leq \; &|\Omega| \cdot c\delta^2 \cdot \big\|\nabla_x \cdot M\big\|_{L^\infty(\Omega)} \xrightarrow{\delta \to 0} 0, \qquad \text{(A.28)}
\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω. This convergence is necessary in the calculation of the product $\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)}$.
Moreover, since $\mathcal{H}^2(\partial\Omega) < \infty$ and $M \cdot \mathbf{n} \in L^\infty(\partial\Omega)$ by $M \in W^{1,\infty}(\Omega)$ and the trace theorem, see, e.g., [Bre11, Corollary 9.14]), we can prove in an analogous manner that

$$\begin{aligned}
&\lim_{\delta \to 0} \int_{\partial\Omega} \int_\Omega N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y)\big(M \cdot \mathbf{n}\big)(x) \, \mathrm{d}y \, \mathrm{d}\sigma_x \\
&\qquad\qquad = \int_{\partial\Omega} \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y)\big(M \cdot \mathbf{n}\big)(x) \, \mathrm{d}y \, \mathrm{d}\sigma_x.
\end{aligned}$$

Next, we look at the convergence of $\nabla \mathcal{S}_\delta(M)$. Since $\nabla \mathcal{S}_\delta(M) \longrightarrow \nabla \mathcal{S}(M)$ in $L^1(\Omega)$ as $\delta \to 0$ (see [Sch08, Proposition 3.1] for a proof) and $\widehat{M} \in L^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\left|\int_\Omega \nabla \mathcal{S}_\delta(M)(y) \cdot \widehat{M}(y) \, \mathrm{d}y - \int_\Omega \nabla \mathcal{S}(M) \cdot \widehat{M}(y) \, \mathrm{d}y\right| \\
= \; &\left|\int_\Omega \big(\nabla \mathcal{S}_\delta(M)(y) - \nabla \mathcal{S}(M)\big) \cdot \widehat{M}(y) \, \mathrm{d}y\right| \\
\leq \; &\|\nabla \mathcal{S}_\delta(M)(y) - \nabla \mathcal{S}(M)\|_{L^1\Omega} \, \|\widehat{M}\|_{L^\infty(\Omega)} \xrightarrow{\delta \to 0} 0.
\end{aligned}$$

Finally, we calculate $\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)}$, using integration by parts and the convergence results (A.28). We get, since $\widehat{M} \in W_0^{1,\infty}(\Omega;\mathbb{R}^3)$,

$$\begin{aligned}
&\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)} \\
= \; &\int_\Omega M(x) \cdot H(\widehat{M})(x) \, \mathrm{d}x \overset{(2.12)}{=} \int_\Omega -M(x) \cdot \nabla_x \varphi(\widehat{M})(x) \, \mathrm{d}x \\
\overset{(A.25)}{=} \; &\int_\Omega -M(x) \cdot \nabla_x \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \, \mathrm{d}y \, \mathrm{d}x.
\end{aligned}$$

Then, integration by parts with respect to x yields

$$\begin{aligned}
&\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)} \\
&= \int_\Omega \big(\nabla_x \cdot M\big)(x) \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \int_\Omega N(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \,\mathrm{d}y \,\big(M \cdot \mathbf{n}\big)(x) \,\mathrm{d}\sigma_x \\
&= \int_\Omega \int_\Omega \lim_{\delta\to 0} N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(\nabla_x \cdot M\big)(x) \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \int_\Omega \lim_{\delta\to 0} N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(M \cdot \mathbf{n}\big)(x) \,\mathrm{d}y \,\mathrm{d}\sigma_x \\
&\overset{(A.28)}{=} \lim_{\delta\to 0} \int_\Omega \int_\Omega N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(\nabla_x \cdot M\big)(x) \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad + \lim_{\delta\to 0} \int_{\partial\Omega} \int_\Omega N_\delta(x-y) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(M \cdot \mathbf{n}\big)(x) \,\mathrm{d}y \,\mathrm{d}\sigma_x.
\end{aligned}$$

Next, we use Fubini's Theorem (see [Bre11, Theorem 4.5]) to exchange the integrals. This is possible since, by the regularity assumptions on M and $\widehat{M}$, the functions $x \mapsto (\nabla_x \cdot M)(x)$ and $y \mapsto \left(\nabla_y \cdot \widehat{M}\right)(y)$ are in $L^2(\Omega)$. Because both functions do not depend on the other variable y and x, resp., and because Ω is bounded, we obtain that both functions $(x,y) \mapsto (\nabla_x \cdot M)(x)$ and $(x,y) \mapsto \left(\nabla_y \cdot \widehat{M}\right)(y)$ are elements of $L^2(\Omega \times \Omega)$. Then, due to Hölder's inequality and the fact that $N_\delta(x-y) \in L^\infty(\Omega \times \Omega)$, we get that the function $(x,y) \mapsto N_\delta(x-y)\left(\nabla_y \cdot \widehat{M}\right)(y)\big(\nabla_x \cdot M\big)(x)$ is in $L^1(\Omega \times \Omega)$.
We can argue similarly for the double integral involving the boundary $\partial\Omega$. Note that $M \cdot \mathbf{n} \in L^\infty(\partial\Omega)$ by the trace theorem, see, e.g. [Bre11, Corollary 9.14]. Hence, the function $(x,y) \mapsto N_\delta(x-y)\left(\nabla_y \cdot \widehat{M}\right)(y)\big(M \cdot \mathbf{n}\big)(x)$ is in $L^1(\partial\Omega \times \Omega)$. So, we obtain, using $N(-r) = N(r)$,

$$\begin{aligned}
&\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)} \\
&= \lim_{\delta\to 0} \int_\Omega \int_\Omega N_\delta(y-x) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(\nabla_x \cdot M\big)(x) \,\mathrm{d}x \,\mathrm{d}y \\
&\qquad + \lim_{\delta\to 0} \int_\Omega \int_{\partial\Omega} N_\delta(y-x) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(M \cdot \mathbf{n}\big)(x) \,\mathrm{d}\sigma_x \,\mathrm{d}y \\
&= \int_\Omega \int_\Omega \lim_{\delta\to 0} N_\delta(y-x) \left(\nabla_y \cdot \widehat{M}\right)(y) \big(\nabla_x \cdot M\big)(x) \,\mathrm{d}x \,\mathrm{d}y \\
&\qquad + \lim_{\delta\to 0} \int_\Omega \underbrace{\int_{\partial\Omega} N_\delta(y-x) \big(M \cdot \mathbf{n}\big)(x) \,\mathrm{d}\sigma_x}_{=\mathcal{S}_\delta(M)(y)} \left(\nabla_y \cdot \widehat{M}\right)(y) \,\mathrm{d}y \\
&= \int_\Omega \left(\nabla_y \cdot \widehat{M}\right)(y) \int_\Omega N(y-x) \big(\nabla_x \cdot M\big)(x) \,\mathrm{d}x \,\mathrm{d}y \\
&\qquad + \lim_{\delta\to 0} \int_\Omega \mathcal{S}_\delta(M)(y) \left(\nabla_y \cdot \widehat{M}\right)(y) \,\mathrm{d}y.
\end{aligned}$$

Next, integration by parts with respect to y before and after the limiting process yields, since $\widehat{M} \in W_0^{1,\infty}(\Omega;\mathbb{R}^3)$,

$$\begin{aligned}
&\langle M, H(\widehat{M})\rangle_{L^2(\Omega;\mathbb{R}^3)} \\
&\quad= \int_\Omega -\widehat{M}(y)\cdot\nabla_y \int_\Omega N(y-x)\big(\nabla_x\cdot M\big)(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad + \lim_{\delta\to 0}\int_\Omega -\nabla_y \mathcal{S}_\delta(M)(y)\cdot\widehat{M}(y)\,\mathrm{d}y \\
&\quad= \int_\Omega -\widehat{M}(y)\cdot\nabla_y \mathcal{V}(M)(y)\,\mathrm{d}y + \int_\Omega -\nabla_y\mathcal{S}(M)(y)\cdot\widehat{M}(y)\,\mathrm{d}y \\
&\quad= \int_\Omega -\widehat{M}(y)\cdot\nabla_y\varphi(M)(y)\,\mathrm{d}y \\
&\quad\overset{(A.26)}{\underset{(A.27)}{=}} \int_\Omega \widehat{M}(y)\cdot H(M)(y)\,\mathrm{d}y \\
&\quad= \langle \widehat{M}, H(M)\rangle_{L^2(\Omega;\mathbb{R}^3)}.
\end{aligned}$$

This concludes the proof. □

A.5 On the model for the simplified setting in 2D (magnetic gradient flow)

This appendix is dedicated to a special 2D (meaning $d = 2$ and target space of the magnetization is $\mathbb{R}^2$) version of our model for the simplified setting in 3D (meaning target space of the magnetization is $\mathbb{R}^3$) which has gradient flow dynamics on the magnetization and a regularized transport equation for the deformation gradient, and is summarized in Section 2.8.3.
The purpose of this dimension reduction is to get started with the mathematical analysis of special solutions in a possible extension to this work.
In the following, we derive this 2D version of our model and we obtain a set of decoupled scalar equations for the deformation gradient $F \in \mathbb{R}^{2\times 2}$ and the magnetization $M \in \mathbb{R}^2$.
The derivation uses ideas from [LLZ05], where the authors derive a 2D system for viscoelastic materials.
In this 2D case, we make a special choice for the elastic energy density, i.e., we set $W(F) = \frac{\sigma_{\mathrm{el}}}{2}|F^2|$. As a result, we have $W'(F)F^\top = \sigma_{\mathrm{el}} F F^\top$ in the stress tensor (2.97).
For the derivation of the two dimensional system, we assume that all functions are smooth, so all the calculations are justified. We start by setting

$$\overline{M} = (\cos\theta, \sin\theta)^\top, \tag{A.29}$$

where $\theta = \theta(x,t)$ is the angle of the magnetization. It is clear that this vector satisfies the length constraint. Thus, the penalization term $\frac{1}{\mu^2}(|\overline{M}|^2 - 1)\overline{M}$ in the microscopic force balance equation (2.100) drops out.
If we plug in $\overline{M}$ into equation (2.100), we obtain using the chain rule and the identity $\overline{M}^\perp = (-\sin\theta, \cos\theta)^\top$

$$\begin{aligned}
&\Big((\cos\theta, \sin\theta)^\top\Big)_t + (v\cdot\nabla)(\cos\theta,\sin\theta)^\top = 2A\Delta(\cos\theta,\sin\theta)^\top \\
\Longleftrightarrow\ & \theta_t\overline{M}^\perp + \Big(\overline{M}^\perp \otimes \nabla\theta\Big)v = 2A\Big(\Delta\theta\overline{M}^\perp - |\nabla\theta|^2\overline{M}\Big) \\
\Longleftrightarrow\ & \theta_t\overline{M}^\perp + \big((v\cdot\nabla)\theta\big)\overline{M}^\perp = 2A\Big(\Delta\theta\overline{M}^\perp - |\nabla\theta|^2\overline{M}\Big).
\end{aligned}$$

Multiplying this equation with $\overline{M}^\perp$, we obtain

$$\theta_t + (v\cdot\nabla)\theta = 2A\Delta\theta \quad \text{in } \Omega\times(0,t^*),$$

which is the microscopic force balance for the angle of the magnetization vector. The special property is that this equation is just one dimensional. We also seek to derive a corresponding condition for the angle θ from the boundary condition (2.103). To this end, we plug in the form of $\overline{M}$ to find

$$0 = \frac{\partial\overline{M}}{\partial\mathbf{n}} = (\nabla\overline{M})\mathbf{n} = \Big(\overline{M}^\perp\otimes\nabla\theta\Big)\mathbf{n} = (\nabla\theta\cdot\mathbf{n})\overline{M}^\perp = \left(\frac{\partial\theta}{\partial\mathbf{n}}\right)\overline{M}^\perp.$$

Multiplying this expression with $\overline{M}^\perp$, we get

$$\frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

as the boundary condition for the magnetic variable.

Now, we investigate the form of the deformation gradient in two dimensions. At first we prove that

$$(\nabla \cdot F^\top)_t + (v \cdot \nabla)(\nabla \cdot F^\top) = 0. \tag{A.30}$$

Indeed, we obtain from the transport of F (2.28) (without the regularizing term $\kappa\Delta F$) by taking the divergence on both sides of the transposed equation, by using the summation convention and exchanging some indices,

$$\begin{aligned}
&\nabla \cdot (F^\top)_t + \nabla \cdot \Big((v \cdot \nabla)F^\top\Big) = \nabla \cdot (F^\top \nabla^\top v) \\
\iff &\nabla_j (F_{ji})_t + \nabla_j \left(v_l(\nabla_l F_{ji})\right) = \nabla_j (F_{ki} \nabla_k v_j) \\
\iff &\nabla_j (F_{ji})_t + \nabla_j v_l (\nabla_l F_{ji}) + v_l (\nabla_j \nabla_l F_{ji}) = (\nabla_j F_{ki}) \nabla_k v_j + F_{ki} \underbrace{\nabla_j \nabla_k v_j}_{=\nabla_k(\nabla_j v_j)=0} \\
\iff &\nabla_j (F_{ji})_t + \nabla_k v_j (\nabla_j F_{ki}) + v_l (\nabla_j \nabla_l F_{ji}) = (\nabla_j F_{ki}) \nabla_k v_j \\
\iff &\nabla_j (F_{ji})_t + v_l \nabla_l (\nabla_j F_{ji}) = 0 \\
\iff &\nabla \cdot (F^\top)_t + (v \cdot \nabla)(\nabla \cdot F^\top) = 0.
\end{aligned}$$

Since the initial condition $F(x,0) = I$ satisfies $\nabla \cdot F(x,0)^\top = 0$, we have that $\nabla \cdot F^\top \equiv 0$ (this actually holds true in the case where $d = 3$ as well).

Next, let $f_1^\top, f_2^\top \in \mathbb{R}^2$ be the columns of $F = \left(f_1^\top, f_2^\top\right)$. We find

$$\nabla \cdot F^\top = \nabla \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \nabla \cdot f_1 \\ \nabla \cdot f_2 \end{pmatrix} = \begin{pmatrix} \nabla_1 f_{11} + \nabla_2 f_{12} \\ \nabla_1 f_{21} + \nabla_2 f_{22} \end{pmatrix} = 0. \tag{A.31}$$

We look at the (two dimensional) curls of $f_1^\perp = (-f_{12}, f_{11})^\top$ and $f_2^\perp = (-f_{22}, f_{21})^\top$. We obtain

$$\operatorname{curl} f_1^\perp = \nabla_1 f_{11} - \nabla_2(-f_{12}), \tag{A.32}$$

$$\operatorname{curl} f_2^\perp = \nabla_1 f_{21} - \nabla_2(-f_{22}), \tag{A.33}$$

where we see that these expressions vanish due to (A.31). Thus, we can represent $f_1^\perp$ and $f_2^\perp$ as gradients of scalar-valued functions, i.e.,

$$f_1^\perp = \nabla\phi_1 \qquad \text{and} \qquad f_2^\perp = \nabla\phi_2. \tag{A.34}$$

This yields a representation of F in terms of ϕ_1, ϕ_2. To this end, we have to get back to f_1, f_2 by (notice that $(\cdot)^\perp$ rotates a vector by $\frac{\pi}{2}$ and to get back we have to rotate by $-\frac{\pi}{2}$)

$$f_1 = -(\nabla\phi_1)^\perp \qquad \text{and} \qquad f_2 = -(\nabla\phi_2)^\perp, \tag{A.35}$$

so we finally obtain (see [LLZ05, Section 2])

$$F = \begin{pmatrix} -\nabla_2\phi_1 & -\nabla_2\phi_2 \\ \nabla_1\phi_1 & \nabla_1\phi_2 \end{pmatrix}. \tag{A.36}$$

It is clear that $\Phi := (\phi_1, \phi_2)^\top$ is volume preserving since $\det \nabla\Phi = \det F = 1$ and the last equality holds by assumption (this is only true without the regularizing term $\kappa\Delta F$).
A further step is to look how the transport equation for F (2.28) translates to the variables ϕ_1, ϕ_2. We plug (A.36) in the transport equation and obtain for the first column

$$\begin{aligned}
&\begin{pmatrix} -\nabla_2(\phi_1)_t - v_1\nabla_1\nabla_2\phi_1 - v_2\nabla_2^2\phi_1 + \nabla_1 v_1\nabla_2\phi_1 - \nabla_2 v_1\nabla_1\phi_1 \\ \nabla_1(\phi_1)_t + v_1\nabla_1^2\phi_1 + v_2\nabla_2\nabla_1\phi_1 + \nabla_1 v_2\nabla_2\phi_1 - \nabla_2 v_2\nabla_1\phi_1 \end{pmatrix} \\
= &\begin{pmatrix} -\nabla_2\big[(\phi_1)_t + v_1\nabla_1\phi_1 + v_2\nabla_2\phi_1\big] \\ \nabla_1\big[(\phi_1)_t + v_1\nabla_1\phi_1 + v_2\nabla_2\phi_1\big] \end{pmatrix} \\
&+ \begin{pmatrix} \nabla_2 v_1\nabla_1\phi_1 + \underbrace{\nabla_2 v_2\nabla_2\phi_1 + \nabla_1 v_1\nabla_2\phi_1}_{=0,\text{ since } \nabla_1 v_1 + \nabla_2 v_2 = \nabla\cdot v = 0} - \nabla_2 v_1\nabla_1\phi_1 \\ -\nabla_1 v_1\nabla_1\phi_1 \underbrace{-\nabla_1 v_2\nabla_2\phi_1 + \nabla_1 v_2\nabla_2\phi_1}_{=0} - \nabla_2 v_2\nabla_1\phi_1 \end{pmatrix} \\
= &\begin{pmatrix} -\nabla_2\big[(\phi_1)_t + v\cdot\nabla\phi_1\big] + \underbrace{\nabla_2 v_1\nabla_1\phi_1 - \nabla_2 v_1\nabla_1\phi_1}_{=0} \\ \nabla_1\big[(\phi_1)_t + v\cdot\nabla\phi_1\big] \underbrace{-\nabla_1 v_1\nabla_1\phi_1 - \nabla_2 v_2\nabla_1\phi_1}_{=0,\text{ since } \nabla_1 v_1 + \nabla_2 v_2 = \nabla\cdot v = 0} \end{pmatrix} \\
= &\begin{pmatrix} -\nabla_2\big[(\phi_1)_t + v\cdot\nabla\phi_1\big] \\ \nabla_1\big[(\phi_1)_t + v\cdot\nabla\phi_1\big] \end{pmatrix} = \nabla^\perp\big[(\phi_1)_t + v\cdot\nabla\phi_1\big] = 0.
\end{aligned}$$

For the second column, we get analogously

$$\begin{aligned}
&\begin{pmatrix} -\nabla_2(\phi_2)_t - v_1\nabla_1\nabla_2\phi_2 - v_2\nabla_2^2\phi_2 + \nabla_1 v_1\nabla_2\phi_2 - \nabla_2 v_1\nabla_1\phi_2 \\ \nabla_1(\phi_2)_t + v_1\nabla_1^2\phi_2 + v_2\nabla_2\nabla_1\phi_2 + \nabla_1 v_2\nabla_2\phi_2 - \nabla_2 v_2\nabla_1\phi_2 \end{pmatrix} \\
= &\ \nabla^\perp\big[(\phi_2)_t + v\cdot\nabla\phi_2\big] = 0.
\end{aligned}$$

From these calculations, we see that for $i = 1, 2$ the term $(\phi_i)_t + v\cdot\nabla\phi_i$ is equal to a constant that may be time-dependent, i.e.,

$$(\phi_i(x,t))_t + v(x,t)\cdot\nabla\phi_i(x,t) = c_i(t) \quad \forall x \in \Omega, i = 1, 2. \tag{A.37}$$

We are allowed to set $c_i(t) \equiv 0$ by the following argument: due to the special form (A.36) of F, addition of time-dependent constants to the ϕ_i's does not matter, i.e., we generate the same F with $\tilde{\phi}_i = \phi_i + \tilde{c}_i(t)$. We plug $\tilde{\phi}_i$ into (A.37) and obtain

$$(\tilde{\phi}_i(x,t))_t + \tilde{c}_i'(t) + v(x,t)\cdot\nabla\tilde{\phi}_i(x,t) = c_i(t) \quad \forall x \in \Omega, i = 1, 2. \tag{A.38}$$

Here, we can set $\tilde{c}_i'(t) = c_i(t)$ or equivalently (up to a constant) $\tilde{c}_i(t) = \int_0^t c_i(s)\,\mathrm{d}s$. In other words, we are able to find a time-dependent constant $\tilde{c}_i(t)$ that cancels

$c_i(t)$. Thus, we can set $c_i(t) \equiv 0$.
Finally, the transport equations for ϕ_i read

$$(\phi_i)_t + v \cdot \nabla\phi_i = 0 \quad i = 1, 2. \tag{A.39}$$

Next, we look at the regularization term $\kappa\Delta F$. Again, we use the representation (A.36) and obtain for the Laplace term

$$\begin{aligned}
\Delta F &= \begin{pmatrix} -(\nabla_1^2\nabla_2\phi_1 + \nabla_2^3\phi_1) & -(\nabla_1^2\nabla_2\phi_2 + \nabla_2^3\phi_2) \\ \nabla_1^3\phi_1 + \nabla_2^2\nabla_1\phi_1 & \nabla_1^3\phi_1 + \nabla_2^2\nabla_1\phi_1 \end{pmatrix} \\
&= \begin{pmatrix} -\nabla_2(\nabla_1^2\phi_1 + \nabla_2^2\phi_1) & -\nabla_2(\nabla_1^2\phi_2 + \nabla_2^2\phi_2) \\ \nabla_1(\nabla_1^2\phi_1 + \nabla_2^2\phi_1) & \nabla_1(\nabla_1^2\phi_1 + \nabla_2^2\phi_1) \end{pmatrix} \\
&= \begin{pmatrix} -\nabla_2\Delta\phi_1 & -\nabla_2\Delta\phi_2 \\ \nabla_1\Delta\phi_1 & \nabla_1\Delta\phi_2 \end{pmatrix} \\
&= \begin{pmatrix} \nabla^\perp\Delta\phi_1 & \nabla^\perp\Delta\phi_2 \end{pmatrix}.
\end{aligned} \tag{A.40}$$

From this calculation and the fact that addition of time-dependent constants to the ϕ_i does not affect ΔF, too, we see that the same arguments as above can be applied to the regularized transport equation (2.99') for F. So, we get the regularized transport equations for ϕ_i

$$(\phi_i)_t + v \cdot \nabla\phi_i = \kappa\Delta\phi_i \quad i = 1, 2. \tag{A.41}$$

Now, we calculate the stress tensor (or its divergence) in terms of the variables θ, ϕ_1 and ϕ_2. We first plug (A.29) into the first part of the stress tensor (2.97) and obtain

$$\begin{aligned}
&\nabla M \odot \nabla M \\
&= \begin{pmatrix} \sin^2\theta(\nabla_1\theta)^2 + \cos^2\theta(\nabla_1\theta)^2 & \sin^2\theta(\nabla_1\theta\nabla_2\theta) + \cos^2\theta(\nabla_1\theta\nabla_2\theta) \\ \sin^2\theta(\nabla_1\theta\nabla_2\theta) + \cos^2\theta(\nabla_1\theta\nabla_2\theta) & \sin^2\theta(\nabla_2\theta)^2 + \cos^2\theta(\nabla_2\theta)^2 \end{pmatrix} \\
&= \begin{pmatrix} (\nabla_1\theta)^2 & \nabla_1\theta\nabla_2\theta \\ \nabla_1\theta\nabla_2\theta & (\nabla_2\theta)^2 \end{pmatrix} \\
&= \nabla\theta \otimes \nabla\theta.
\end{aligned} \tag{A.42}$$

For the second part of the stress tensor (2.97) we rewrite $\nabla \cdot (FF^\top)$ at first (c.f. [LLZ05]):

$$\begin{aligned}
&\nabla \cdot (FF^\top) \\
&= \nabla \cdot \begin{pmatrix} (\nabla_2\phi_1)^2 + (\nabla_2\phi_2)^2 & -\nabla_2\phi_1\nabla_1\phi_1 - \nabla_2\phi_2\nabla_1\phi_2 \\ -\nabla_2\phi_1\nabla_1\phi_1 - \nabla_2\phi_2\nabla_1\phi_2 & (\nabla_1\phi_1)^2 + (\nabla_1\phi_2)^2 \end{pmatrix} \\
&= \nabla \cdot \left(\begin{pmatrix} (\nabla_2\phi_1)^2 & -\nabla_2\phi_1\nabla_1\phi_1 \\ -\nabla_2\phi_1\nabla_1\phi_1 & (\nabla_1\phi_1)^2 \end{pmatrix} + \begin{pmatrix} (\nabla_2\phi_2)^2 & -\nabla_2\phi_2\nabla_1\phi_2 \\ -\nabla_2\phi_2\nabla_1\phi_2 & (\nabla_1\phi_2)^2 \end{pmatrix} \right) \\
&= -\nabla \cdot (\nabla\phi_1 \otimes \nabla\phi_1 + \nabla\phi_2 \otimes \nabla\phi_2).
\end{aligned} \tag{A.43}$$

So, we can rewrite the equation of motion based on the previous calculation as follows:

$$\begin{aligned}
v_t + (v \cdot \nabla)v + \nabla p + 2A\nabla \cdot (\nabla\theta \otimes \nabla\theta)& \\
+ \sigma_{\mathrm{el}}\nabla \cdot (\nabla\phi_1 \otimes \nabla\phi_1 + \nabla\phi_2 \otimes \nabla\phi_2) &= \nu\Delta v.
\end{aligned} \tag{A.44}$$

At this point, we summarize the model for the simplified setting in two dimensions. The equations read

$$v_t + (v \cdot \nabla)v + \nabla p + 2A\nabla \cdot (\nabla\theta \otimes \nabla\theta) \\ + \sigma_{\mathrm{el}}\nabla \cdot (\nabla\phi_1 \otimes \nabla\phi_1 + \nabla\phi_2 \otimes \nabla\phi_2) = \nu\Delta v, \tag{A.45}$$

$$\nabla \cdot v = 0, \tag{A.46}$$

$$(\phi_i)_t + v \cdot \nabla\phi_i = \kappa\Delta\phi_i, \quad i = 1, 2, \tag{A.47}$$

$$\theta_t + (v \cdot \nabla)\theta = 2A\Delta\theta, \tag{A.48}$$

in $\Omega \times (0, t^*) \subset \mathbb{R}^2 \times \mathbb{R}$ with boundary conditions ($\Phi = (\phi_1, \phi_2)^\top$)

$$v = 0 \quad \text{on } \partial\Omega \times (0, t^*), \tag{A.49}$$

$$\Phi = 0 \quad \text{on } \partial\Omega \times (0, t^*), \tag{A.50}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega \times (0, t^*) \tag{A.51}$$

and initial conditions

$$v(x, 0) = v_0(x), \quad \nabla \cdot v_0(x) = 0 \quad \text{in } \Omega, \tag{A.52}$$

$$\Phi(x, 0) = \Phi_0(x) \quad \text{in } \Omega, \tag{A.53}$$

$$\theta(x, 0) = \theta_0(x) \quad \text{in } \Omega. \tag{A.54}$$

We obtain the following existence result of weak solutions for the above two-dimensional system:

Theorem 37. *For any $T > 0$, $v_0 \in \mathbf{H}$, $\Phi_0(x) \in \mathbf{H}^1(\Omega; \mathbb{R}^2)$ and $\theta_0 \in \mathbf{H}^1(\Omega; \mathbb{R})$ the system* (A.45)–(A.54) *has a weak solution $(v, \phi_1, \phi_2, \theta)$ in $\Omega \times (0, T)$.*

Proof. The proof of Theorem 37 follows from the proof of Theorem 9 presented in Section 3.1. □

We note further, that the 2D model (A.45)–(A.54) is important to study special solutions. We already started the discussion on these special solutions with Carlos García-Cervera and Chun Liu within the joint DAAD project with Anja Schlömerkemper, which we highlighted as an open problem in Chapter 4.

A.6 On an L^2 basis in the Galerkin approximation for the magnetization

In this appendix, we give some more details about the basis used in the proofs of Lemma 17 and Lemma 26.
We have that $\{\eta_i\}_{i=1}^{\infty} \subset C^{\infty}(\overline{\Omega}; \mathbb{R}^3)$ satisfies

$$\begin{cases} \Delta^2 \eta_i + \eta_i = \tilde{\mu}_i \eta_i & \text{in } \Omega, \\ \frac{\partial \eta_i}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \frac{\partial \Delta \eta_i}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

for $0 < \tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots \leq \tilde{\mu}_n \leq \cdots$ with $\tilde{\mu}_n \xrightarrow{n\to\infty} \infty$. This set of functions is an orthonormal basis of $L^2(\Omega; \mathbb{R}^3)$, which can be shown by means of the Hilbert-Schmidt theorem (see, e.g., [RR04, Theorem 8.94]), similar to [Eva02, Section 6.5.1].
Furthermore, $\{\eta_i\}_{i=1}^{\infty}$ is an orthogonal basis of $\mathbf{H}^2_{\mathbf{n}}(\Omega; \mathbb{R}^3)$, which is a closed subspace of $\mathbf{H}^2(\Omega; \mathbb{R}^3)$. We equip it with the scalar product

$$((f, g)) := (f, g)_{L^2(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)} := \int_\Omega f \cdot g \, \mathrm{d}x + \int_\Omega \Delta f \cdot \Delta g \, \mathrm{d}x.$$

The induced norm $|||\cdot|||$ is equivalent to the usual norm $\|\cdot\|_{\mathbf{H}^2}$.

Bibliography

[AR09] D. Andelman and R.E. Rosensweig, *The Phenomenology of Modulated Phases: From Magnetic Solids and Fluids to Organic Films and Polymers*, Polymers, Liquids and Colloids in Electric Fields: Interfacial instabilities, orientation, and phase-transitions (Y. Tsori and U. Steiner, eds.), Series in Soft Condensed Matter, vol. 2, World Scientific, 2009, pp. 1–56.

[AS92] F. Alouges and A. Soyeur, *On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness*, Nonlinear Anal., Theory Methods Appl. **18** (1992), no. 11, 1071–1084.

[BAB+99] P. Berger, N.B. Adelman, K.J. Beckman, D.J. Campbell, A.B. Ellis, and G.C. Lisensky, *Preparation and Properties of an Aqueous Ferrofluid*, Journal of Chemical Education **76** (1999), no. 7, 943.

[BFGC+16] B. Benešová, J. Forster, C.J. García-Cervera, C. Liu, and A. Schlömerkemper, *Analysis of the flow of magnetoelastic materials*, PAMM, accepted, 2016.

[BFLS16] B. Benešová, J. Forster, C. Liu, and A. Schlömerkemper, *Existence of weak solutions to an evolutionary model for magnetoelasticity*, ArXiv e-prints (2016), 1–29, `http://arxiv.org/abs/1608.02992`.

[BLQS14] J. Brannick, C. Liu, T. Qian, and H. Sun, *Diffuse Interface Methods for Multiple Phase Materials: An Energetic Variational Approach*, ArXiv e-prints (2014), 1–15, `https://arxiv.org/abs/1402.5375`.

[Bob00] S. Bobbio, *Electrodynamics of Materials: Forces, Stresses, and Energies in Solids and Fluids*, Academic Press, San Diego, 2000.

[BPPR14] L. Baňas, M. Page, D. Praetorius, and J. Rochat, *A decoupled and unconditionally convergent linear FEM integrator for the Landau-Lifshitz-Gilbert equation with magnetostriction*, IMA J. Numer. Anal. **34** (2014), no. 4, 1361–1385.

[BPV01] M. Bertsch, P. Podio-Guidugli, and V. Valente, *On the dynamics of deformable ferromagnets. I: Global weak solutions for soft ferromagnets at rest*, Ann. Mat. Pura Appl. (4) **179** (2001), 331–360.

[Bre11] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York, NY: Springer, 2011.

[Bro63] W.F. Brown, *Micromagnetics*, Interscience Publishers, New York, 1963.

[Bro66] ______, *Magnetoelastic Interactions*, Springer-Verlag, Berlin, 1966.

[BS02] A. Bieńkowski and R. Szewczyk, *New Possibility of Utilizing Amorphous Ring Cores as Stress Sensor*, physica status solidi (a) **189** (2002), no. 3, 787–790.

[BS04] ______, *The possibility of utilizing the high permeability magnetic materials in construction of magnetoelastic stress and force sensors*, Sensors and Actuators A: Physical **113** (2004), no. 3, 270–276, New materials and Technologies in Sensor Applications, Proceedings of the European Materials Research Society 2003 - Symposium N.

[BV92] V.D. Buchel'nikov and A.N. Vasil'ev, *Electromagnetic generation of ultrasound in ferromagnets*, Soviet Physics Uspekhi **35** (1992), no. 3, 192.

[CEF11] G. Carbou, M.A. Efendiev, and P. Fabrie, *Global weak solutions for the Landau-Lifschitz equation with magnetostriction*, Math. Methods Appl. Sci. **34** (2011), no. 10, 1274–1288.

[CF01] G. Carbou and P. Fabrie, *Regular solutions for Landau-Lifschitz equation in a bounded domain*, Differ. Integral Equ. **14** (2001), no. 2, 213–229.

[CIM14] R. Côte, R. Ignat, and E. Miot, *A thin-film limit in the Landau-Lifshitz-Gilbert equation relevant for the formation of Néel walls*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 241–272.

[CISVC09] M. Chipot, I. I. Shafrir, V. Valente, and G.V. Caffarelli, *On a hyperbolic-parabolic system arising in magnetoelasticity*, J. Math. Anal. Appl. **352** (2009), no. 1, 120–131.

[CP01] C. Carstensen and A. Prohl, *Numerical analysis of relaxed micromagnetics by penalised finite elements*, Numer. Math. **90** (2001), no. 1, 65–99.

[DD98] A. DeSimone and G. Dolzmann, *Existence of Minimizers for a Variational Problem in Two-Dimensional Nonlinear Magnetoelasticity*, Archive for Rational Mechanics and Analysis **144** (1998), no. 2, 107–120.

[DE88] M. Doi and S.F. Edwards, *The Theory of Polymer Dynamics*, International series of monographs on physics, Clarendon Press, 1988.

[DJ02] A. DeSimone and R.D. James, *A constrained theory of magnetoelasticity*, Journal of the Mechanics and Physics of Solids **50** (2002), no. 2, 283–320.

[DKMO00] A. DeSimone, R.V. Kohn, S. Müller, and F. Otto, *Magnetic microstructures – a paradigm of multiscale problems*, ICIAM 99. Proceedings of the 4th international congress on industrial & applied mathematics, Edinburgh, GB, July 5–9, 1999, Oxford: Oxford University Press, 2000, pp. 175–190.

[DKMO02] ———, *A reduced theory for thin-film micromagnetics.*, Commun. Pure Appl. Math. **55** (2002), no. 11, 1408–1460.

[DKMO06] ———, *Recent analytical developments in micromagnetics*, The Science of Hysteresis (G. Bertotti and I. Mayergoyz, eds.), vol. II, Academic Press, 2006, pp. 269–383.

[DL00] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Volume 3: Spectral theory and applications. With the collaboration of Michel Artola and Michel Cessenat. Transl. from the French by John C. Amson. 2nd printing*, Berlin: Springer, 2000.

[DO14] A.L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, Springer US, New York, 2014.

[DP95] A. DeSimone and P. Podio-Guidugli, *Inertial and self interactions in structured continua: Liquid crystals and magnetostrictive solids*, Meccanica **30** (1995), no. 5, 629–640.

[DP96] ———, *On the continuum theory of deformable ferromagnetic solids.*, Arch. Ration. Mech. Anal. **136** (1996), no. 3, 201–233.

[EG92] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, Boca Raton: CRC Press, 1992.

[ESM15] G. Ethiraj, A. Sridhar, and C. Miehe, *Variational modeling and homogenization in dissipative magneto-mechanics*, GAMM-Mitteilungen **38** (2015), no. 1, 75–101.

[Eva02] L.C. Evans, *Partial differential equations*, 2nd ed., Providence, RI: American Mathematical Society, 2002.

[Fil88] A.F. Filippov, *Differential equations with discontinuous right-hand sides. Ed. by F. M. Arscott. Transl. from the Russian*, Dordrecht etc.: Kluwer Academic Publishers, 1988.

[For13] J. Forster, *Mathematical Modeling of Complex Fluids*, Master's thesis, Universität Würzburg, 2013, `http://opus.bibliothek.uni-wuerzburg.de/frontdoor/index/index/docId/1273`.

[FSL00] J.J. Feng, G. Sgalari, and L.G. Leal, *A theory for flowing nematic polymers with orientational distortion*, Journal of Rheology **44** (2000), no. 5, 1085–1101.

[Gar07] C.J. García-Cervera, *Numerical micromagnetics: a review*, Bol. Soc. Esp. Mat. Apl., S$\vec{e}$MA **39** (2007), 103–135.

[GCGE03] C.J. García-Cervera, Z. Gimbutas, and W. E, *Accurate numerical methods for micromagnetics simulations with general geometries*, J. Comput. Phys. **184** (2003), no. 1, 37–52.

[GH96] M. Giaquinta and S. Hildebrandt, *Calculus of variations 1. The Lagrangian formalism*, Berlin: Springer-Verlag, 1996.

[Gil55] T.L. Gilbert, *A Lagrangian formulation of the gyromagnetic equation of the magnetic field*, Phys. Rev. **100** (1955), 1243.

[Gil04] ______, *A phenomenological theory of damping in ferromagnetic materials*, IEEE Transactions on Magnetics **40** (2004), no. 6, 3443–3449.

[GJ97] G. Gioia and R.D. James, *Micromagnetics of very thin films*, Proc. R. Soc. Lond. A **453** (1997), 213–223.

[GRRC11] C.A. Grimes, S.C. Roy, S. Rani, and Q. Cai, *Theory, Instrumentation and Applications of Magnetoelastic Resonance Sensors: A Review*, Sensors **11** (2011), no. 3, 2809–2844.

[GW07] C.J. García-Cervera and X. Wang, *Spin-polarized transport: existence of weak solutions*, Discrete Contin. Dyn. Syst., Ser. B **7** (2007), no. 1, 87–100.

[HKL10] Y. Hyon, D.Y. Kwak, and C. Liu, *Energetic variational approach in complex fluids: maximum dissipation principle*, Discrete Contin. Dyn. Syst. **26** (2010), no. 4, 1291–1304.

[HS98] A. Hubert and R. Schäfer, *Magnetic domains. The analysis of magnetic microstructures*, Springer-Verlag, New York, 1998.

[JK90] R.D. James and D. Kinderlehrer, *Frustration in ferromagnetic materials*, Continuum Mech. Thermodyn. (1990), 215–239.

[JK93] ______, *Theory of magnetostriction with applications to $Tb_xDy_{1-x}Fe_2$*, Philosophical Magazine Part B **68** (1993), no. 2, 237–274.

[JLJ08] J. Jost and X. Li-Jost, *Calculus of variations. Paperback reprint of the hardback edition 1998*, Cambridge: Cambridge University Press, 2008.

[Kov00] A. Kovetz, *Electromagnetic Theory*, Oxford University Press, 2000.

[KP06] M. Kružík and A. Prohl, *Recent developments in the modeling, analysis, and numerics of ferromagnetism*, SIAM Rev. **48** (2006), no. 3, 439–483.

[KSZ15] M. Kružík, U. Stefanelli, and J. Zeman, *Existence results for incompressible magnetoelasticity*, Discrete Contin. Dyn. Syst. **35** (2015), no. 6, 2615–2623.

[Kur04] M.W. Kurzke, *Analysis of boundary vortices in thin magnetic films*, Ph.D. thesis, Fakultät für Mathematik und Informatik der Universität Leipzig, 2004.

[Kur06a] M. Kurzke, *A nonlocal singular perturbation problem with periodic well potential*, ESAIM, Control Optim. Calc. Var. **12** (2006), 52–63.

[Kur06b] ______, *Boundary vortices in thin magnetic films*, Calc. Var. Partial Differ. Equ. **26** (2006), no. 1, 1–28.

[Liu11] C. Liu (ed.), *Energetic Variational Approaches for Complex Fluids*, Faculty of Science, Hokkaido University, August 2011.

[LJL06] L.P. Liu, R.D. James, and P.H. Leo, *Magnetostrictive composites in the dilute limit*, J. Mech. Phys. Solids **54** (2006), no. 5, 951–974.

[LL35] L. Landau and E. Lifshitz, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies*, Phys. Z. Sowjetunion **8** (1935), 153–169.

[LL76] L.D. Landau and E.M. Lifshitz, *Mechanics*, Butterworth Heinemann, Butterworth-Heinemann, 1976.

[LL95] F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Commun. Pure Appl. Math. **48** (1995), no. 5, 501–537.

[LLW10] F.-H. Lin, J. Lin, and C. Wang, *Liquid Crystal Flows in Two Dimensions*, Arch. Ration. Mech. Anal. **197** (2010), 297–336.

[LLZ05] F.-H. Lin, C. Liu, and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Commun. Pure Appl. Math. **58** (2005), no. 11, 1437–1471.

[LS03] C. Liu and J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Physica D: Nonlinear Phenomena **179** (2003), no. 3–4, 211–228.

[LSFY05] C. Liu, J. Shen, J.J. Feng, and P. Yue, *Variational Approach in Two-Phase Flows of Complex Fluids: Transport and Induced Elastic Stress*, Mathematical methods and models in phase transitions (A. Miranville, ed.), New York, NY: Nova Science Publishers, 2005, pp. 259–278.

[LW01] C. Liu and N.J. Walkington, *An Eulerian description of fluids containing viscoelastic particles*, Arch. Ration. Mech. Anal. **159** (2001), no. 3, 229–252.

[Mel07] C. Melcher, *A dual approach to regularity in thin film micromagnetics*, Calc. Var. Partial Differ. Equ. **29** (2007), no. 1, 85–98.

[Mel10] ______, *Thin-film limits for Landau-Lifshitz-Gilbert equations*, SIAM J. Math. Anal. **42** (2010), no. 1, 519–537.

[MKR11] C. Miehe, B. Kiefer, and D. Rosato, *An incremental variational formulation of dissipative magnetostriction at the macroscopic continuum level*, International Journal of Solids and Structures **48** (2011), no. 13, 1846–1866.

[MVT15] C. Miehe, D. Vallicotti, and S. Teichtmeister, *Homogenization and multiscale stability analysis in finite magneto-electro-elasticity*, GAMM-Mitteilungen **38** (2015), no. 2, 313–343.

[MVT16] ______, *Homogenization and multiscale stability analysis in finite magneto-electro-elasticity. Application to soft matter EE, ME and MEE composites*, Computer Methods in Applied Mechanics and Engineering **300** (2016), 294–346.

[MW79] W.L. Miranker and B.E. Willner, *Global analysis of magnetic domains*, Q. Appl. Math. **37** (1979), 219–238.

[Nir59] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. **13** (1959), 115–162.

[Ons31a] L. Onsager, *Reciprocal relations in irreversible processes. I.*, Phys. Rev., II. Ser. **37** (1931), 405–426.

[Ons31b] ______, *Reciprocal relations in irreversible processes. II.*, Phys. Rev., II. Ser. **38** (1931), 2265–2279.

[RL05] P. Rybka and M. Luskin, *Existence of energy minimizers for magnetostrictive materials*, SIAM J. Math. Anal. **36** (2005), no. 6, 2004–2019.

[Rou13] T. Roubíček, *Nonlinear partial differential equations with applications*, 2nd ed., Basel: Birkhäuser, 2013.

[RR04] M. Renardy and R.C. Rogers, *An introduction to partial differential equations*, 2nd ed., New York, NY: Springer, 2004.

[Sch05] A. Schlömerkemper, *Mathematical derivation of the continuum limit of the magnetic force between two parts of a rigid crystalline material*, Arch. Ration. Mech. Anal. **176** (2005), no. 2, 227–269.

[Sch08] A. Schlömerkemper, *About solutions of Poisson's equation with transition condition in non-smooth domains*, Z. Anal. Anwend. **27** (2008), no. 3, 253–281.

[SL09] H. Sun and C. Liu, *On energetic variational approaches in modeling the nematic liquid crystal flows*, Discrete Contin. Dyn. Syst. **23** (2009), no. 1-2, 455–475.

[Slo79] J.C. Slonczewski, *Force, momentum and topology of a moving magnetic domain*, J. Magn. Magn. Mat. **12** (1979), no. 2, 108–122.

[SNR10] R.L. Snyder, V.Q. Nguyen, and R.V. Ramanujan, *The energetics of magnetoelastic actuators is analogous to phase transformations in materials*, Acta Materialia **58** (2010), no. 17, 5620–5630.

[SS09] A. Schlömerkemper and B. Schmidt, *Discrete-to-continuum limit of magnetic forces: Dependence on the distance between bodies*, Arch. Ration. Mech. Anal. **192** (2009), no. 3, 589–611.

[Str73] J.W. Strutt, *Some general theorems relating to vibrations*, Proc. of L. M. S. IV, (1873), 357–368.

[Tem77] R. Temam, *Navier-Stokes Equations: Theory and numerical Analysis*, North-Holland Publishing Company, 1977.

[Tie64] Tiersten, H.F., *Coupled Magnetomechanical Equations for Magnetically Saturated Insulators*, Journal of Mathematical Physics **5** (1964), no. 9, 1298–1318.

[Tie65] H.F. Tiersten, *Variational Principle for Saturated Magnetoelastic Insulators*, Journal of Mathematical Physics **6** (1965), no. 5, 779–787.

[TM11] E.B. Tadmor and R.E. Miller, *Modeling materials. Continuum, atomistic and multiscale techniques*, Cambridge: Cambridge University Press, 2011.

[Vis85] A. Visintin, *On Landau-Lifshitz' equations for ferromagnetism*, Japan J. Appl. Math. **2** (1985), 69–84.

[Wer14] N.M. Wereley (ed.), *Magnetorheology*, RSC Smart Materials, The Royal Society of Chemistry, 2014.

[WXL12] H. Wu, X. Xu, and C. Liu, *Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties*, Calc. Var. Partial Differ. Equ. **45** (2012), no. 3-4, 319–345.

[WXL13] ———, *On the general Ericksen-Leslie system: Parodi's relation, well-posedness and stability*, Arch. Ration. Mech. Anal. **208** (2013), no. 1, 59–107.

[YFLS05] P. Yue, J.J. Feng, C. Liu, and J. Shen, *Diffuse-interface simulations of drop coalescence and retraction in viscoelastic fluids*, J. Non-Newton. Fluid Mech. **129** (2005), no. 3, 163–176.

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