Classification and Reduction of Equivariant Star Products on Symplectic Manifolds

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Contents

Chapter 1

Introduction

1.1 Quantum Mechanics

Quantum mechanics was originally conceived as a set of empirical rules to describe experimental observations, which could not be explained by contemporary classical physics at the beginning of the 20th century. The original such case, the one that triggered the subsequent quantum revolution, was certainly Max Planck's explanation of the black body radiation: one knew from experiments that a black body at a fixed temperature emitted a certain spectrum of electromagnetic energy. And one should, in principle, be able to derive said spectrum from the theory of electromagnetism and thermodynamics. However, all attempts at understanding the problem using classical physics failed and it took the radical idea that the energy emitted from the black body at a certain frequency ν had to be an integer multiple of [102]

$$E = h\nu$$

where h is the, today well-known, Planck constant. But despite providing the correct spectrum, the above postulate lacked any theoretical or physical explanation and Max Planck himself described it as "an act of desperation". It took a few years until Albert Einstein built upon Planck's postulate by proposing that not only the energy of the emitted electromagnetic radiation, but also the electromagnetic radiation itself is actually quantized. In todays terms, one would say that light consists of photons and that $E = h\nu$ is precisely the energy of one photon, a statement that Einstein used to beautifully explain the photoelectric effect [41]. These and related ideas were however not only used to explain experimental results that classical physics could not, but also to solve some fundamental paradoxes such as the stability of atoms: before quantum mechanics, atoms were thought of as consisting of a positively charged core, which is circled by electrons. In this model however, the orbiting electrons should continuously lose energy, and with that distance to the core, due to synchrotron radiation. In conclusion, according to this model, atoms should not be stable, which is drastically contradicted by reality. A solution to this paradox was proposed in the form of the Bohr-Rutherford model, whose key feature is again, that the energy levels of electrons orbiting the core cannot assume arbitrary values, but only a certain set of discrete ones [14]. This new model also provided a theoretical explanation of the Rydberg formula, which describes the spectral lines of the hydrogen atom.

These and many other results then culminated, largely from 1925 to 1928, in a proper formulation of quantum mechanics. One of the most popular achievements of this era was the development of wave mechanics by Erwin Schrödinger with the Schrödinger equation [110]

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\Delta + V(x,t)\right)\psi(x,t) \tag{1.1.1}$$

for a (non-relativistic) quantum mechanical particle of mass m in a potential V. For a general quantum system, the differential operator $(-\hbar^2/2m)\Delta + V(x,t)$ has to be replaced by another differential operator $\widehat{H}$, called the Hamiltonian ($\widehat{H}$ then completely describes the system through the Schrödinger equation). The interpretation of this equation requires a plethora of novel and, unfortunately, unintuitive concepts. The most glaring one is certainly, that instead of the particle having (time-dependent) position and momentum, it is described by a complex-valued wave function $\psi(x,t)$. In other words, the particle is no longer sharply localized at a certain position but instead it is potentially spread out over a whole area of space. Similarly, the momentum of the particle is replaced by the derivative of the wave function and hence not localized either. The physical interpretation of the wave function $\psi(x,t)$ is then, that $|\psi(x,t)|^2$ describes the probability density of the particle being found at position x at time t [1]. Or, formulated differently, repeatedly preparing identical systems and measuring the position of the particle will result in the distribution of positions $|\psi(x,t)|^2$. This interpretation reveals one of the core principles of quantum mechanics, namely that quantum mechanics is an inherently statistical theory. In general, the result of a single measurement cannot be predicted. This, of course, starkly contrasts classical physics, where, given sufficiently precise knowledge of initial conditions and computation power, all possible measurement values can be predicted. On the other hand, quantum mechanics is not just a statistical theory, like thermodynamics, either. The well-known Heisenberg uncertainty principle illustrates this point: assuming a one-particle system, one can, over repeated measurements, determine the expectation values of the components of the particles position q^k and momentum p_i. But even more, as with any random variable, position and momentum have associated higher statistical moments, such as variance, denoted by Δq^k and Δp_i respectively. The statement of the uncertainty principle is then, that these variances satisfy the inequality [64]

$$(\Delta q^k)(\Delta p_i) \geq \frac{\hbar}{2}\delta^k_i,$$

which is a novel, purely quantum mechanical effect, that has no analogon in classical statistical theories. The physical interpretation of above inequality can be understood as follows: by trying to narrow down the position of the particle, i.e. by restricting it to a small box, the uncertainty of the particles momentum increases. For the limit case, where the position is pinned to a point, the momentum becomes completely indeterminate. One particular insightful explanation for the Heisenberg uncertainty principle can be found in matrix mechanics (developed 1925 by Born, Heisenberg and Jordan), which was historically the first consistent formulation of quantum mechanics and which was later shown to be equivalent to Schrödinger's wave mechanics. At the core of matrix mechanics lies the equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{O}(t) = \frac{\partial\mathcal{O}}{\partial t} + \frac{1}{\mathrm{i}\hbar}\left[\mathcal{O}(t), \widehat{H}\right] \tag{1.1.2}$$

governing the time-evolution of observables $\mathcal{O}$. The crucial point here is, that the observables are no longer, as in classical mechanics, modelled as functions on a phase space. Instead,

[1] For $|\psi(x,t)|^2$ to be interpretable as a probability density one has to require that ψ is normalized, that is $\int|\psi(x,t)|^2\mathrm{d}x = 1$.

at each point in time t, $\mathcal{O}(t)$ is a self-adjoint operator on a pre-Hilbert[2] space (which depends on the quantum system at hand). $[\ ,\ ]$ is then the commutator with respect to composition of operators and $\widehat{H}$ is the Hamiltonian of the system. In this picture, the Heisenberg uncertainty principle follows from the single fact, that the operators $\widehat{q}^k$ and $\widehat{p}_i$ corresponding to the position and momentum components do not commute [71]. More precisely [19],

$$\left[\widehat{q}^k, \widehat{p}_i\right] = \widehat{q}^k \widehat{p}_i - \widehat{p}_i \widehat{q}^k = \mathrm{i}\hbar \delta_i^k \tag{1.1.3}$$

holds, which in itself is a profound result revealing the noncommutativity of the algebra of observables in quantum mechanics. One physical consequence of noncommutativity is, that the order of measurements taken is now important: first measuring the position of a particle and then its momentum will, in general, result in vastly different results than first measuring its momentum and then its position. Of course, this does not only apply to position and momentum observables, but to all observables, whose associated operators do not commute. Coming back to the Heisenberg uncertainty principle, it should then not be surprising that not only the variances of position and momentum measurements are constrained, but that in fact the variances of all pairs of observables, whose operators do not commute, are constrained in a similar fashion. Other interesting examples of noncommuting observables include the angular momentum components $\widehat{J}_k$ of a particle, for which $[\widehat{J}_i, \widehat{J}_j] = \mathrm{i}\hbar\epsilon_{ijk}\widehat{J}_k$ holds, and, especially important for particle physics, the energy and lifespan of excited states. All these observations however beg one question. If quantum mechanics is so drastically different to classical physics, why do we not observe quantum effects in everyday life? From our experience one can, up to a certain precision, determine position and momentum of a particle, which according to quantum mechanics, should not be possible. The answer to this question lies, of course, in the inconspicuous looking phrase "up to a certain precision". Assuming that $\hbar$ is small compared to all other relevant quantities of the same unit, the right-hand term in the uncertainty inequality $(\Delta q^k)(\Delta p_i) \geq \mathrm{i}\hbar\delta_i^k$ can safely be approximated to zero. But since variances are always non-negative, the inequality is trivially satisfied and hence the quantum effect vanishes. A similar behaviour can be observed for other quantum effects as well. In fact, this observation has been and continues to be one of the guiding principles of quantum mechanics: classical mechanics should be recovered from quantum mechanics by taking the limit case $\hbar \approx 0$, the so-called classical limit of quantum mechanics. This establishes quantum mechanics as a generalization of classical mechanics in the same way in which special relativity is a generalization of the Newtonian equations of motion, which are recovered in the limit $1/c \approx 0$, where c is the speed of light. Again, this approximation is valid in the case where all velocities are small compared to c.

1.2 Quantization

As described in the previous section, the quantum world is governed by wave mechanics or matrix mechanics. However, there is an important problem left for any practical application. Both descriptions require the specification of a Hamiltonian $\widehat{H}$. While for simple systems, such as the single-particle system, it might be feasible to guess a Hamiltonian and then verify it through experimentation, more complex systems cannot be treated that way. Furthermore, knowledge of only the Hamiltonian will only allow (in principle) to obtain

[2]Pre-Hilbert spaces are used to avoid analytical subtleties regarding unbounded operators. In order to compute spectra of observables, completions are necessary.

the wave function describing the system, but not to predict any measurements besides its total energy. For this task, the operators associated to all other observables are needed as well. Consequently it is highly desirable to have at least a set of guidelines, or better a complete mathematical framework, called quantization, on how to obtain the Hamiltonian and the quantum description of all observables of any given quantum system. A first hint as to the nature of such a framework is provided by the Schrödinger equation for the free particle (that is the potential V in (1.1.1) is assumed to be zero). With the ansatz $\psi(x,t) = \psi_0 \exp\{\mathrm{i}S(x,t)/\hbar\}$, for a function S, the Schrödinger equation produces the partial differential equation [108]

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 = \mathrm{i}\frac{\hbar}{2m}\Delta S$$

for S. Taking the classical limit of this equation, that is setting the right-hand side to zero, the remaining equation can be recognized as the Hamilton-Jacobi equation for a classical particle described by the Hamiltonian function $H(q,p) = p^2/2m$. Comparing this Hamiltonian function with the Hamiltonian operator $\widehat{H} = -\hbar^2\Delta/2m$ suggests, that the latter could be obtained by replacing the function p^2 by the differential operator $-\hbar^2\Delta$. Or, as it is more commonly known, by replacing the momentum components according to the rule

$$p_i \qquad \rightsquigarrow \qquad \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^i}. \tag{1.2.1}$$

This observation is the first in an extensive list of formal similarities between the Hamiltonian description of classical mechanics and quantum mechanics. In the former, a classical, mechanical system is conceived as a symplectic manifold M together with a Hamiltonian function $H \in \mathscr{C}^\infty(M)$. A symplectic manifold is a smooth manifold M together with a differential two-form ω which is closed with respect to the de Rham differential and nondegenerate. Through ω one can define the Hamiltonian vector X_f field of a function $f \in \mathscr{C}^\infty(M)$ by $\mathrm{d}f = \mathrm{i}_{X_f}\omega$ and in turn equip $\mathscr{C}^\infty(M)$ with a Lie bracket via

$$\{f,g\} := \omega(X_f, X_g) \qquad \text{for all } f,g \in \mathscr{C}^\infty(M),$$

the so-called Poisson bracket, turning $\mathscr{C}^\infty(M)$ into a Poisson algebra. Both the symplectic manifold M and the Hamiltonian function H can, at least for a large class of systems, be obtained directly from analyzing the generalized coordinates and the Lagrange function associated to the system. An important subclass of the former are the n-point-particle systems with holonomic constraints and regular Legendre transform [108]. In such a system, any point of M is physically interpreted as a possible state of the system. Observables are then nothing but functions on M: if the system is in the state $x \in M$, measuring any observable $f \in \mathscr{C}^\infty(M)$ will produce the outcome $f(x)$. Of course, the positions and momenta of all n particles are observables as well and they can be recognized precisely as the Darboux coordinates $\{q^i\}$ and $\{p_i\}$ in each chart of M. By the Darboux theorem [30], they satisfy

$$\left\{q^i, q^j\right\} = \{p_i, p_j\} = 0 \qquad \text{and} \qquad \left\{q^i, p_j\right\} = \delta^i_j. \tag{1.2.2}$$

Another special observable is the Hamiltonian function H, whose physical interpretation is that of the total energy of the system. H also governs the time-evolution of any observable f via

$$\frac{\mathrm{d}}{\mathrm{d}\,t}f(t) = \frac{\partial f}{\partial t} + \{f(t), H\}. \tag{1.2.3}$$

Note that if the mechanical system described by H is autonomous, that is if H does not explicitly depend on t, then the total time derivative of H vanishes: replacing f with H

itself, the above equation implies that H is time-independent by the antisymmetry of the Poisson bracket. Physically, this means that the total energy of the system is conserved. Comparing the previous two equations with (1.1.3) and (1.1.2), there is a striking similarity:

$$\begin{array}{rcl} \{q^i, p_j\} = \delta^i_j & \rightsquigarrow & [\widehat{q}^i, \widehat{p}_j] = \mathrm{i}\hbar\delta^i_j \\ \frac{\mathrm{d}}{\mathrm{d}t}f(t) = \frac{\partial f}{\partial t} + \{f(t), H\} & \rightsquigarrow & \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = \frac{\partial \mathcal{O}}{\partial t} + \frac{1}{\mathrm{i}\hbar}\left[\mathcal{O}(t), \widehat{H}\right] \end{array}$$

Namely, both correlations are compatible with the assumption that $\widehat{\ }$ is actually a map, assigning quantum observables to classical ones such that the rule

$$\left[\widehat{f}, \widehat{g}\right] = \mathrm{i}\hbar\widehat{\{f, g\}}$$

is respected. This is Dirac's original idea for canonical quantization [34, 35]. Furthermore, if the earlier replacement rule (1.2.1) is supplemented with the expression for $\widehat{q}^i$ acting on any wave function Ψ through

$$q^i \qquad \rightsquigarrow \qquad \big(\widehat{q}^i\Psi\big)(x, t) := x^i\Psi(x, t),$$

then the commutator of $\widehat{q}^i$ and $\widehat{p}_j$ yields precisely (1.1.3) and is thus consistent with canonical quantization. From all these motivational considerations, a set of sensible requirements for any map deserving of the name quantization, can be extracted: first, the domain and codomain of any such map must be the smooth functions $\mathscr{C}^\infty(M)$ on a symplectic manifold and the operators on a pre-Hilbert space $\mathsf{End}(\mathcal{H})$, respectively. Secondly, any quantization map should be injective, meaning that every classical observable has a quantum analogon. The key point to remember here is that quantum mechanics should encompass classical mechanics in the limit $\hbar \longrightarrow 0$, hence all classical observables must be obtainable as a classical limit of its quantum analogon. And thirdly, it certainly would be desirable for a quantization map to respect the algebraic structures on both sides. Since $\mathscr{C}^\infty(M)$ together with the Poisson bracket $\{\ ,\ \}$, as well as $\mathsf{End}(\mathcal{H})$ together with the commutator $[\ ,\ ]$ are unital Poisson algebras, and since $\widehat{\ }$ already respects the bracket structure, it seems natural to require any quantization map to be a homomorphism of unital Poisson algebras. More precisely, any quantization map Q should be linear and respect units, Poisson brackets and multiplications. However, it is immediately clear, that the fourth condition can, in general, not be satisfied. Indeed, since $\mathscr{C}^\infty(M)$, as an associative algebra, is commutative,

$$Q(f) \circ Q(g) = Q(fg) = Q(gf) = Q(g) \circ Q(f)$$

conflicts with the noncommutativity of $\mathsf{End}(\mathcal{H})$ for all pairs of noncommuting operators $f, g \in \operatorname{im} Q$. The most obvious example for two such operators is, of course, the pair of position and momentum observables. So instead of requiring Q to be a homomorphism of unital Poisson algebras, the fourth requirement must be dropped and hence Q is reduced to a Lie algebra homomorphism that respects units:

$$\begin{array}{rcl} Q(1) & = & 1 \\ Q(\lambda f + g) & = & \lambda Q(f) + Q(g) \\ Q(\{f, g\}) & = & \frac{1}{\mathrm{i}\hbar}[Q(f), Q(g)] \end{array} \tag{1.2.4}$$

for all $f, g \in \mathscr{C}^\infty(M)$ and $\lambda \in \mathbb{R}$.

Remark From a purely mathematical standpoint, instead of the fourth, the third requirement could be dropped, leading to Q being a homomorphism of unital associative algebras. However, specifically for position and momentum observables, the Heisenberg uncertainty relation is experimentally verified, hence, physically, at least position and momentum operators must not commute, leaving the above contradiction still unresolved.

From a slightly different mathematical viewpoint then, a quantization map Q should be a faithful Lie algebra representation of $\mathscr{C}^\infty(M)$ on $\mathcal{H}$. But if, in addition, $\mathcal{H}$ is required to be a Hilbert space and Q is required to be irreducible, then the famous Groenewold-van Hove theorem [53, 118] states that such a Q does not exist:

Theorem *No faithful, irreducible representation of* $\operatorname{span}\{1, q^1, \ldots, q^n, p_1, \ldots, p_n\}$ *can be extended to a faithful, irreducible representation of* $\operatorname{Pol}(\mathbb{R}^{2n})$.

For discussions of the Groenewold-van Hove theorem in the context of unbounded operators, consult e.g. [1, 52] and references therein. This shows that, already for only polynomials on the simplest symplectic manifold, a quantization map as required cannot exist. Furthermore, a very similar, purely algebraic result from [120] states:

Proposition *There is no unital, associative algebra $\mathcal{A}$ together with a Lie algebra isomorphism*

$$Q\colon \operatorname{Pol}(\mathbb{R}^{2n}) \longrightarrow \left(\mathcal{A}, \frac{1}{\mathrm{i}\hbar}[\;,\;]\right)$$

where $[\;,\;]$ is the commutator on $\mathcal{A}$.

From the previous two results it is clear that the original proposal has to be weakened even further. However, this time there is no obvious choice of how to do so. One popular ansatz has been found in geometric quantization, first proposed by Kirillov [73], Kostant [80] and Souriau [113]. The path geometric quantization takes can roughly be summarized as requiring $\mathrm{i}\hbar Q(\{f, g\}) = [Q(f), Q(g)]$ to only hold on some suitable subset of all smooth functions. A different approach, the one that will be the main theme of this thesis, namely deformation quantization, originally due to Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer [7, 8], weakens the same requirement, but in a different direction:

$$Q(\{f, g\}) = \frac{1}{\mathrm{i}\hbar}[Q(f), Q(g)] + \mathcal{O}(\hbar). \tag{1.2.5}$$

So instead of requiring Q to respect the Lie brackets exactly, it is only required that the Poisson bracket is mapped to the commutator up to terms of higher orders in $\hbar$. But rather than elaborate on deformation quantization, which will be the subject of the following sections, we would be amiss if, after detailing some of the formal similarities between classical and quantum mechanics, we did not also touch upon some of the profound differences of both physical theories. In fact, understanding these differences is essential in constructing a consistent theory of quantization. This has already been seen in one instance: even though the algebra of classical observables is commutative, the algebra of quantum observables is not. On the one hand, noncommutativity provided insights into physical effects, such as the Heisenberg uncertainty principle, but on the other hand, it poses obstructions to possible quantization maps. Another such difference can be found in the nature of observables. Given any proposition about a classical system, one can ask for all possible states of the system, for which said proposition is true. As an example, consider a point particle moving on a line and the proposition $P =$ "the particles position is contained in the interval $[0,1]$". Then the collection of states for which P is true, clearly is $q^{-1}([0,1])$, the preimage of the given interval under the position observable. So propositions about the system can be encoded by subsets of its symplectic manifold. Furthermore, set-theoretic operations such as intersections, joins and complements directly correspond to conjunction, disjunction and negation of propositions. Consequently, propositions about classical systems have the structure of a boolean lattice. Propositions about quantum systems on the other hand, are represented by projections on the underlying Hilbert space. These projections admit the same types of logical operations, however, the lattice of projections is, in general, not distributive and hence only an orthocomplemented lattice, not a boolean one. A recent attempt at reformulating quantum mechanics (see [22, 39, 68]) replaces the nondistributive lattice of projections by a certain Heyting algebra. One of the physical ramifications thereof is then the existence of propositions, which are neither true nor false.

1.3 Deformation Quantization

Deformation quantization is an approach to quantization, that falls into the broader category of deformation theory. The core idea of deformation theory has been summarized very succinctly by Moshé Flato, one of the founding fathers of deformation quantization in [47] as follows:

> What is a deformation? Mathematically one starts with an algebraic structure which is e.g. a Lie algebra or an associative algebra and asks the question: does there exist a 1 (or n) parameter family of similar structures that for an initial value (say zero) of the parameter we get the structure we started with? If such a field of structures exist, we call it a deformation of the original structure. In particular, if for any value of the parameter, the structure is isomorphic with the one we started with we call the deformation trivial.

One of the most prominent examples of deformation theory is certainly the interpretation of special relativity as a deformation of Newtonian physics: one of the cornerstones of Newtonian physics is the invariance of the Newtonian laws under spatial rotations, spatial and temporal translations and uniform motion of spacetime. That is, the theory is invariant under a certain type of coordinate transformations of absolute time and space $\mathbb{R}^1 \times \mathbb{R}^3$. All of these transformations are encoded in the action of the Galilei group $\mathrm{Gal}(1,3)$ on $\mathbb{R}^1 \times \mathbb{R}^3$. Special relativity on the other hand, has, instead of the Galilei group, the Poincaré group as its symmetry group. A connection between the two can be made thus: viewing the speed of light c as a parameter of the Poincaré group, the Galilei group can be recovered from the Poincaré group as the limit $1/c \longrightarrow 0$. Conversely, the Poincaré group can be viewed as a deformation of the Galilei group for the deformation parameter $1/c$ [47, 106, 112]. Obviously, this deformation is nontrivial, as the Galilei and Poincaré groups are not isomorphic. Another remarkable aspect of this particular deformation becomes apparent by considering the subsets of $\mathbb{R}$ of those values for the deformation parameter $1/c$, for which the deformed group is isomorphic to the Galilei group S_{G} and for which it is isomorphic to the Poincaré group S_{P}. It turns out, that, topologically, S_{G} is closed, in fact it is only a point, while S_{P} is open in $\mathbb{R}$. But this means that, starting at the Galilei group, an arbitrarily small displacement of the deformation parameter can be found such that a nonisomorphic group is obtained. On the other hand, for sufficiently small displacements of the deformation parameter the deformed group stays in the isomorphism class of the Poincaré group. One says that the Poincaré group is stable under small deformations, while the Galilei group is not. So the transition from Newtonian physics to special relativity can be seen as a transition to a theory more robust under small changes of its parameters. At first glance, this may seem like purely mathematical sophistication, however, there is a physical argument to be made as well: consider an experiment which is to decide whether reality adopted the Galilei or the Poincaré group as its symmetry group. The deciding factor here would be the precise value of $1/c$. But physical measurements are only ever able to determine values up to certain errors. So even if 0 is determined as the value of $1/c$, no matter how precise the measurement, the error range will always allow values of the deformation parameter in S_{P}. In contrast, if the measured value of $1/c$ is in S_{P}, one can, in principle, achieve a sufficiently high precision to be confident that reality is Poincaré invariant.

Coming back to quantum mechanics, the idea proposed in [7, 8] is to view the algebra of quantum observables as a deformation of the algebra of classical observables with $\hbar$ playing the role of a deformation parameter. In fact, only the pointwise, commutative multiplication μ_0 of $\mathscr{C}^\infty(M)$ will be deformed into a noncommutative multiplication $\mu_\hbar$. Of course, $\mu_\hbar$ is not any arbitrary multiplication. Instead, it must have μ_0 as its classical limit. This is guaranteed by the proposal that $\mu_\hbar$ is actually a power series in $\hbar$ starting with μ_0, that is

$$\mu_\hbar(f,g) = \mu_0(f,g) + \sum_{k=1}^{\infty} \hbar^k \mu_k(f,g) \qquad \text{for all } f,g \in \mathscr{C}^\infty(M) \tag{1.3.1}$$

for any maps μ_k such that $\mu_\hbar$ is an associative multiplication on $\mathscr{C}^\infty(M)$. One of the inherent advantages of this approach becomes clear, if we consider the context of general quantization maps from the previous section. Since the multiplications μ_0 and $\mu_\hbar$ operate on the same underlying set, the quantization map Q can be taken as just the identity (as a set morphism). And from there, the requirements for quantization maps (1.2.4) can directly be translated into requirements for the deformed multiplication $\mu_\hbar$: $\mu_\hbar$ must be a bilinear map and

$$\mu_\hbar(1,f) = f = \mu_\hbar(f,1)$$

must be satisfied for all $f \in \mathscr{C}^\infty(M)$. But, as already mentioned in the previous section, instead of requiring the third part of (1.2.4) to hold precisely, the condition is weakened as displayed in (1.2.5). This weakened condition can be translated into a requirement for $\mu_\hbar$ by calculating

$$\frac{1}{\mathrm{i}\hbar}[f,g]_{\mu_\hbar} = \frac{1}{\mathrm{i}}(\mu_1(f,g) - \mu_1(g,f) + \mathcal{O}(\hbar)) \stackrel{!}{=} \{f,g\} + \mathcal{O}(\hbar),$$

where $[\ ,\ ]_{\mu_\hbar}$ is the commutator with respect to $\mu_\hbar$, and concluding

$$\mathrm{i}\hbar\{f,g\} \stackrel{!}{=} \mu_1(f,g) - \mu_1(g,f)$$

for all $f,g \in \mathscr{C}^\infty(M)$. That is, the antisymmetric part of the first order of $\mu_\hbar$ is completely determined by the Poisson bracket. This realization already has some far-reaching consequences: since the Poisson bracket on a symplectic manifold can never vanish identically (assuming $\dim M > 0$), the multiplication $\mu_\hbar$ is always noncommutative. And with that, the deformation of the classical observable algebra is always nontrivial, as commutative algebras cannot be isomorphic to noncommutative ones. There is, however, an obvious problem with the ansatz so far. Namely, the power series (1.3.1) may not converge for some observables. In fact, according to [120], even for the simple case where the symplectic manifold is a cotangent bundle, that is $M = \mathrm{T}^*Q$, there exist always pairs of observables for which the series (1.3.1) has a radius of convergence of 0. Mathematically, the problem of convergence can be circumvented by replacing the algebra of observables $\mathscr{C}^\infty(M)$ with its formal power series $\mathscr{C}^\infty(M)[\![\hbar]\!]$ in the, now formal, parameter $\hbar$. Both, the original and the deformed product, can then be extended $\hbar$-linearly to $\mathscr{C}^\infty(M)[\![\hbar]\!]$ where they are well defined. The deformed product $\mu_\hbar$ on $\mathscr{C}^\infty(M)[\![\hbar]\!]$ is then called a formal star product and written as $\mu_\hbar(f,g) = f \star g$. Using this approach, the problem of quantization has been split into two parts: first, finding a formal product on the formal power series of observables deforming the classical product and second, analyzing its convergence behaviour, i.e. finding suitable subalgebras of $\mathscr{C}^\infty(M)[\![\hbar]\!]$ on which the formal product converges.

Remark The first part is accordingly called formal deformation quantization, while the second is a subfield of strict deformation quantization. Generally speaking, strict deformation quantization deals with converging deformations of μ_0, not necessarily using formal deformations, see e.g. [13, 17, 82, 105]. This thesis will exclusively treat aspects of formal deformation quantization.

Finally, let us note that we will only use a certain subclass of star products, namely those, for which all maps μ_k in (1.3.1) are bidifferential operators. Differential operators have the advantage of being local operators and with that, the multiplication $\mu_\hbar$ can be restricted from $\mathscr{C}^\infty(M)[\![\hbar]\!]$ to the observable algebra $\mathscr{C}^\infty(U)[\![\hbar]\!]$ on any open subset $U \subseteq M$, implementing a form of the locality principle in physics: objects should only be influenced by its immediate surroundings. Conversely, if we only required the maps μ_k to be local, then they would, by the Peetre theorem [100], at least locally be bidifferential operators. Under this restriction, the notion of differential star products can be finalized:

Definition *Let (M, ω) be a symplectic manifold with associated Poisson bracket $\{\ ,\ \}$. A differential star product on M is an associative, ν-linear product $\star$ on $\mathscr{C}^\infty(M)[\![\nu]\!]$ such that for all $f, g \in \mathscr{C}^\infty(M)$ the product $f \star g$ can be expanded as*

$$f \star g = fg + \sum_{k=1}^{\infty} \nu^k C_k(f,g)$$

with bidifferential operators C_k for all $k \in \mathbb{N}$ and the following equations hold:

$$1 \star f = f = f \star 1 \qquad \textit{and} \qquad f \star g - g \star f = \nu\{f,g\} + \mathcal{O}(\nu).$$

Note that $\mu_\hbar(f,g) \coloneqq f \star g$ is a particular deformation of μ_0. Also note, that the formal parameter has been renamed from $\hbar$ to ν to emphasize its formal nature and to absorb a factor i in order to ease notation. Furthermore, since all star products occurring in this thesis will be differential, we shall drop the explicit qualifier. With the proper definition of star products at hand, the first question to arise should be the following: do star products exist? That is, given any symplectic manifold, can one find a star product on it? One of the earliest known examples of star products, the Weyl-Moyal star product

$$\begin{aligned} f \star_{\mathrm{WM}} g \coloneqq & \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\nu}{2}\right)^r \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} \\ & \sum_{i_1,\dots,i_r} \frac{\partial^r f}{\partial q^{i_1} \dots \partial q^{i_s} \partial p_{i_{s+1}} \dots \partial p_{i_r}} \frac{\partial^r g}{\partial p_{i_1} \dots \partial p_{i_s} \partial q^{i_{s+1}} \dots \partial q^{i_r}} \end{aligned}$$

already gives a positive answer on $\mathbb{R}^{2n}$ and with that on any Darboux chart of any symplectic manifold M. One of the first global results on the matter can be found in [91] wherein it is shown that Weyl-Moyal star products on Darboux charts can be glued together if the third de Rham cohomology of M vanishes. Subsequent publications then established the existence of star products on trivial cotangent bundles [23], on all cotangent bundles [33] and finally on all symplectic manifolds [32]. Additionally, a constructive proof of existence has been given by Fedosov, see [42–45], in the so-called Fedosov construction, which will be detailed in Section 3.1 and used extensively later on.

Remark It is imperative to note, that, whereas the definition of star products can immediately be generalized to Poisson manifolds, the proofs mentioned above for their existence cannot. One of the earliest results on the topic is the Gutt star product [58] on cotangent bundles of Lie groups. The existence (and simultaneously classification) of star products for any Poisson manifold has later been established through Kontsevich's celebrated formality theorem [76, 77, 79], see also [27, 37]. An alternative proof can also be found in [78, 116]

After their existence is guaranteed, the next problem is, naturally, the uniqueness of star products. That is, given a symplectic manifold, how many inequivalent deformations of the algebra of observables are there? Here the notion of inequivalent deformations must be specified first. According to general deformation theory, two star products $\star_1$ and $\star_2$ are equivalent, if the algebras $(\mathscr{C}^\infty(M)[\![\nu]\!], \star_1)$ and $(\mathscr{C}^\infty(M)[\![\nu]\!], \star_2)$ are isomorphic. In the context of differential star products, however, one requires additionally, that the isomorphism can be expanded as a formal series of differential operators starting with the identity:

Definition *Let M be a symplectic manifold. Two star products $\star_1$ and $\star_2$ on M are said to be equivalent, if there exists a formal series*

$$T = \mathsf{id} + \sum_{k=1}^{\infty} \nu^k T_k$$

of differential operators S_k such that

$$T(f \star_1 g) = T(f) \star_2 T(g)$$

holds for all $f, g \in \mathscr{C}^\infty(M)[\![\nu]\!]$.

The additional requirement of T_k being differential operators is set in place in order to remain inside the context of differential star products: if T_k were allowed to be any map, differential star products could be equivalent to nondifferential ones. The task is then to describe the set of equivalence classes $\mathrm{Def}(M, \omega)$ of star products on M. One of the first results already states that this set contains only a single element, if $\mathrm{H}^2_{\mathrm{dR}}(M) = 0$, see [56, 84]. In other words, if the underlying symplectic manifold is topologically trivial enough, then star products are unique up to equivalences. One of the surprising aspects of this result is then, that the uniqueness depends neither on the symplectic structure ω nor on the differential structure of M, but instead only on the topology of M, which will be a common thread throughout all classification results. In fact, the above result has later been recognized to be the special case of the complete classification, due to various authors [12, 31, 92, 93, 122], see also Section 3.2

Theorem *On any symplectic manifold (M, ω) there exists a bijection*

$$c\colon \mathrm{Def}(M, \omega) \longrightarrow \frac{[\omega]}{\nu} + \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!].$$

So not only is the uniqueness of star products controlled by the topology of M, but all equivalence classes are completely characterized by formal series in the second de Rham cohomology of M. For any star product $\star$, the de Rham class $c(\star)$ is then called the characteristic class of $\star$.

1.4 Symmetries of Star Products

As another case, consider that classical mechanical systems often come equipped with gauge symmetries. Such symmetries are encoded in the context of symplectic geometry by symplectic actions of Lie groups: given a Lie group G and a symplectic manifold (M, ω) together with an action of G on M, the action is said to be symplectic, if the symplectic form ω is invariant with respect to it, that is

$$g^*\omega = \omega \qquad \text{for all } g \in G.$$

Here $g^*\omega$ denotes the induced action on differential forms by pullbacks. Note then, that the classical observable algebra also acquires an action of G induced by pullbacks and that the pointwise product μ_0 is equivariant with respect to that action as

$$g^*\mu_0(f, h)(x) = (f \cdot h)(g \triangleright x) = f(g \triangleright x) \cdot h(g \triangleright x) = \mu_0(g^*f, g^*h)(x)$$

holds for all $f, h \in \mathscr{C}^\infty(M)$, $g \in G$ and $x \in M$, where we denoted by $g \triangleright x$ the action of g on x. Considering then, that G also acts, by ν-linear extension, on the algebra of quantum observables $\mathscr{C}^\infty(M)[\![\nu]\!]$, it seems sensible to search for star products $\star$, which are also equivariant with respect to that action, in detail, for which

$$g^*(f \star h) = (g^*f) \star (g^*h)$$

holds. In other words, the gauge symmetry of the classical system can be transferred to its quantum mechanical analogon. These star products are called invariant star products.

Remark To be precise, invariant star products should, and later on will be, called G-invariant star products, since there is also a Lie algebraic notion of invariance,

$$\mathscr{L}_\xi(f \star h) = (\mathscr{L}_\xi f) \star h + f \star (\mathscr{L}_\xi h) \qquad \text{for all } \xi \in \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G. This condition is precisely the derived version of the group theoretic requirement. Of course, if G is connected, both notions of invariance are equivalent.

Obviously, the very same questions for existence and uniqueness of invariant star products arise, just as in the case for general star products. And, just as in the previous case, both can be answered together by a suitable classification theorem. To do so however, a suitable notion of equivalences has to be established first. The common motivation is again, that, by applying this subtype of equivalences to an invariant star product, it should not be possible to obtain a non-invariant one. An easy way to guarantee this behaviour is to require equivalences which commute with the group action. Such equivalences are then called invariant equivalences and the set of equivalence classes of invariant star products up to invariant equivalences is denoted by $\mathrm{Def}^G(M, \omega)$. The corresponding classification theorem, due to Bertelson, Bieliavsky and Gutt [11], see also Section 3.3, then reads:

Theorem *On any symplectic manifold* (M, ω) *equipped with a symplectic* G*-action, there exists a bijection*

$$c^G \colon \operatorname{Def}^G(M, \omega) \longrightarrow \frac{[\omega]^G}{\nu} + \mathrm{H}^{2,G}_{\mathrm{dR}}(M)[\![\nu]\!].$$

So the cohomology theory parametrizing equivalence classes, which was formerly the de Rham cohomology, has been replaced by the invariant de Rham cohomology.

Next, consider the case where a classical system has not only gauge symmetries, but where those symmetries come with an associated momentum map: given a symplectic action of a Lie algebra $\mathfrak{g}$, which may or may not be the derived action of a Lie group action, a linear map $J \colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)$ is called a momentum map of said action, if

$$\mathscr{L}_\xi f = -\{J(\xi), f\} \qquad \text{and} \qquad J([\xi, \eta]) = \{J(\xi), J(\eta)\}$$

hold for all $\xi, \eta \in \mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)$. The interpretation of the first equation is, that the vector field, by which $\xi \in \mathfrak{g}$ acts, is not only symplectic, but in fact Hamiltonian, and that $J(\xi)$ is a Hamiltonian for this vector field. The second equation then states that J is a homomorphism of Lie algebras. This notion of momentum maps can, in a very straightforward manner, be adapted to star products, see [90, 123]: a quantum momentum map is a linear map $\mathbf{J} \colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[\![\nu]\!]$ such that

$$\mathscr{L}_\xi f = -\frac{1}{\nu}[\mathbf{J}(\xi), f]_\star \qquad \text{and} \qquad \mathbf{J}([\xi, \eta]) = \frac{1}{\nu}[\mathbf{J}(\xi), \mathbf{J}(\eta)]_\star$$

hold for all $\xi, \eta \in \mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)[\![\nu]\!]$. The motivation for this definition of quantum momentum maps is immediately clear: all Poisson brackets in the definition of momentum maps have been replaced by commutators. Pairs of an invariant star product and a quantum momentum map of that star product are then called equivariant star products. Expanding the defining equation in the lowest order of ν, it is obvious that $\mathbf{J}\big|_{\nu=0}$ is a classical momentum map. Consequently, $\mathbf{J}$ is called a deformation of its lowest order. Notably, it is possible to have multiple equivariant star products with the same underlying invariant star product or even multiple quantum momentum maps deforming the same classical one. In fact, given an equivariant star product $(\star, \mathbf{J})$, an easy calculation shows that $T\mathbf{J}$ is, for any invariant self-equivalence T of $\star$, again a quantum momentum map of $\star$. And since T starts with id, both $T\mathbf{J}$ and $\mathbf{J}$ deform the same momentum map. A similar result also holds for invariant equivalences between two different invariant star products: quantum momentum maps are mapped to quantum momentum maps under invariant equivalences. This observation motivates the definition of equivariant equivalences. Two equivariant star products, $(\star_1, \mathbf{J}_1)$ and $(\star_2, \mathbf{J}_2)$, are called equivariantly equivalent, if there exists an invariant equivalence T between $\star_1$ and $\star_2$ such that $T\mathbf{J}_1 = \mathbf{J}_2$ holds.

Denoting the corresponding set of equivalence classes deforming a given momentum map J by $\operatorname{Def}_{\mathfrak{g}}(M, \omega, J)$, it has been recently shown by Waldmann and the author, that a classification result similar to the former two holds, see [104] and Section 3.4:

Main Theorem I *On any symplectic manifold* (M, ω) *equipped with a symplectic* $\mathfrak{g}$*-action and momentum map* J*, there exists a bijection*

$$c_{\mathfrak{g}} \colon \operatorname{Def}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \frac{[\omega - J]_{\mathfrak{g}}}{\nu} + \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!].$$

Again, the cohomology theory parametrizing equivalence classes had to be modified in order to accommodate for the more specialized setting. The cohomology appearing in the above theorem is the cohomology of the Cartan complex $\Omega_{\mathfrak{g}}(M)$ of equivariant differential forms, which is discussed in detail in Section 2.3.2, see also [55]. An important property of the Cartan complex is, that there is a canonical map $\mathrm{H}^2_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M)$ to the $\mathfrak{g}$-invariant cohomology, which allows for a particularly beautiful result on the existence of quantum momentum maps, due to [60,62,90]: given a symplectic Lie algebra action with an associated momentum map J and an invariant star product $\star$, then $\star$ admits a quantum momentum map deforming J if and only if the invariant characteristic class $c^{\mathfrak{g}}(\star)$ lies in the image of the above map.

As a final point, let us revisit the classical side once more. One consequence of a classical system having gauge symmetries is, that the systems description is using more degrees of freedom then strictly necessary. One of the prime examples here is the Kepler problem of two massive bodies interacting through Newtonian gravity. Since Newton's laws are invariant with respect to Galilei transformations, it is always possible to find an inertial frame of reference, where the barycenter lies at the origin of said reference frame. By doing so, the spatial degrees of freedom of the system have been reduced from six (the positions of both bodies) to three (the relative position of the two bodies) without losing any information about the system. The question, in general, is then, whether it is always possible to find a minimal set of coordinates. The space of such minimal sets of coordinates would be called the reduced phase space. Given any system modelled by a symplectic manifold M with gauge symmetries modelled as the action of a group G, the naive approach would be to consider the quotient M/G. For general smooth Lie group actions however, this quotient is, in general, no longer a manifold. And even if it is, it is in general only Poisson, not symplectic. But, in the special case where the action on M admits a momentum map J, there is a well-known symmetry reduction procedure, called Marsden-Weinstein reduction [85, 86], which gives the reduced symplectic manifold as $M_{\mathrm{red}} = C/G$, where C is the momentum-level-set $C := \{x \in M \mid \forall \xi \in \mathfrak{g} \colon J(\xi)(x) = 0\}$. Going back to deformation quantization then, there are a priori two choices: finding a star product on M_{red}, or finding (preferably an equivariant) one on M. Physically though, since the transition from M to M_{red} only removed unnecessary degrees of freedom, this choice should not matter. Or, at least, there should be a consistent way of choosing a star product on M and one on M_{red}. In other words, given a star product on M, there should be a preferred star product on M_{red}. And in fact, a reduction map $_{\mathrm{red}} \colon \mathrm{Star}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \mathrm{Star}(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ from the set of equivariant star products on M to all star products on M_{red} has been developed in [16, 61]. Such maps are commonly referred to as quantum reduction. An easy calculation, see Lemma 4.3.1, then shows, that the reduction map on star products induces a map on equivalence classes of star products $_{\mathrm{red}} \colon \mathrm{Def}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \mathrm{Def}(M_{\mathrm{red}}, \omega_{\mathrm{red}})$. The obvious question here is, given an equivariant star product $(\star, \mathbf{J})$ on M, is there any relation between its characteristic class $c_{\mathfrak{g}}(\star, \mathbf{J})$ and the characteristic class of the corresponding reduced star product $c(\star_{\mathrm{red}})$? A partial, positive answer to this question has already been given by Bordemann in [15]: denoting the inclusion map of C into M by ι and the projection map from C onto M_{red} by π, the characteristic classes of $\star$ and $\star_{\mathrm{red}}$ are related by

$$\iota^* c(\star) = \pi^* c(\star_{\mathrm{red}}).$$

However, the above result does not take any quantum momentum maps into account. And moreover, there are examples, where the second de Rham cohomology $\mathrm{H}^2_{\mathrm{dR}}(C)$ is trivial, but neither $\mathrm{H}^2_{\mathrm{dR}}(M)$ nor $\mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}})$ are. Thus the above equation can only be a necessary requirement for $\star_{\mathrm{red}}$ to be reduced from $\star$.

The novel result by the author, see [103] and Section 4.3, amends this result, by providing means to explicitly obtain the characteristic class $c(\star_{\mathrm{red}})$ from the equivariant class $c_{\mathfrak{g}}(\star, \mathbf{J})$:

Main Theorem II *Given a symplectic manifold M and an equivariant star product $(\star, \mathbf{J})$ on M, there is a surjective map*

$$K \colon \mathrm{H}^2_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}})$$

such that

$$K(c_{\mathfrak{g}}(\star, \mathbf{J})) = c(\star_{\mathrm{red}})$$

holds, if K is extended ν-linearly.

The map K in the above theorem is a Cartan-model analogue of the Kirwan map for equivariant cohomology [74]. The full construction of K is detailed in Section 2.3.3. The main idea however is quite simple: the map $\pi^* \colon \mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(C)$ induced on de Rham cohomology by the pullback with the projection π is, in general, not bijective. However, π^* can also be viewed as a map from $\mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}})$ to $\mathrm{H}^2_{\mathfrak{g}}(C)$, which turns out to be bijective. K is then nothing but the concatenation

$$\mathrm{H}^2_{\mathfrak{g}}(M) \xrightarrow{\iota^*} \mathrm{H}^2_{\mathfrak{g}}(C) \xrightarrow{(\pi^*)^{-1}} \mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}}).$$

One of the major features of K is its surjectivity, which has an obvious yet captivating interpretation: every star product on M_{red} can, up to equivalence, be obtained as the reduction of an equivariant star product on M. However, K is, in general, not injective, as ι^* is not.

Chapter 2

Homological Algebra

2.1 Homology and Cohomology of Chain Complexes

Homological algebra (or at least parts of it) will be a central tool being employed throughout this thesis. Hence we shall provide a brief introduction to some of the necessary subtopics. Excellent introductions to homological algebra include [107, 121] (we will primarily follow the latter), while some very useful applications to geometric settings can be found in [20]. Generally, homological algebra concerns itself with the study of Abelian categories, which for our purposes will always be categories of modules over rings[1]. Two of the central tools to achieve this, are chain complexes and (their associated) homology/cohomology theories, on both of which we will focus here. Let us begin by giving a proper definition for chain complexes of modules over a ring:

Definition 2.1.1 (Chain complex) *Let R be a ring. A homological chain complex (over R) $(C_\bullet, \mathrm{d}_\bullet)$ is a sequence of R-modules $\dots C_{-1}, C_0, C_1, \dots$ together with homomorphisms $\mathrm{d}_k \colon C_k \longrightarrow C_{k-1}$ such that $\mathrm{d}_{k-1} \circ \mathrm{d}_k = 0$ for all $k \in \mathbb{Z}$. A cohomological chain complex $(C^\bullet, \mathrm{d}^\bullet)$ is a sequence of R-modules $\dots, C^{-1}, C^0, C^1, \dots$ together with homomorphisms $\mathrm{d}^k \colon C^k \longrightarrow C^{k+1}$ such that $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$ for all $k \in \mathbb{Z}$. The maps $\mathrm{d}_\bullet$ and $\mathrm{d}^\bullet$ are called differentials.*

Of course, the differentiation between homological and cohomological chain complexes is merely a matter of choice and we can easily switch from one to the other by relabeling. Henceforth we shall use the term chain complex to mean either. It should be mentioned, however, that chain complexes usually come in pairs of a homological and a cohomological chain complex: given a homological chain complex $(C_\bullet, \mathrm{d}_\bullet)$ over a ring R, one can construct a cohomological chain complex through

$$C^k \coloneqq \mathsf{Hom}_R(C_k, R) \qquad \text{and} \qquad \mathrm{d}^k(f) \coloneqq f \circ \mathrm{d}^k \tag{2.1.1}$$

for all $f \in C^k$. One prominent such pair is the singular chain complex of a topological space together with it's singular cochain complex. Also, we shall frequently refer to a chain complex $(C_\bullet, \mathrm{d}_\bullet)$ just by C, if the grading and differential are clear from the context. As so often in abstract algebra, the introduction of a new mathematical object demands a notion of suitable morphisms between those objects. Here morphisms between chain complexes are called chain maps or chain morphisms and defined thus:

[1] By the Freyd-Mitchell embedding theorem, all small Abelian categories can be fully-faithfully embedded into a module category [49].

Definition 2.1.2 (Chain map) *Let $(C_\bullet, \mathrm{d}^C_\bullet)$ and $(D_\bullet, \mathrm{d}^D_\bullet)$ be chain complexes over a ring R. A chain map $f\colon C \longrightarrow D$ is a sequence of R-module morphisms $f_k\colon C_k \longrightarrow D_k$ such that $\mathrm{d}^D_k \circ f_k = f_{k-1} \circ \mathrm{d}^C_k$ for all $k \in \mathbb{Z}$.*

Definition 2.1.2 can be pictorially summarized by requiring that all squares in the following diagram commute:

$$\begin{tikzcd}
\dots & C_{-1} \arrow[l, "\mathrm{d}^C_{-1}"'] & C_0 \arrow[l, "\mathrm{d}^C_0"'] & C_1 \arrow[l, "\mathrm{d}^C_1"'] & \dots \arrow[l, "\mathrm{d}^C_2"'] \\
\dots & D_{-1} \arrow[l, "\mathrm{d}^D_{-1}"] \arrow[u, "f_{-1}"'] & D_0 \arrow[l, "\mathrm{d}^D_0"] \arrow[u, "f_0"'] & D_1 \arrow[l, "\mathrm{d}^D_1"] \arrow[u, "f_1"'] & \dots \arrow[l, "\mathrm{d}^D_2"]
\end{tikzcd}$$

Remark 2.1.3 We will often encounter chain complexes that are bounded on one side, that is chain complexes of the form

$$\dots \longleftarrow 0 \longleftarrow 0 \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \dots$$

We shall notate such chain complexes by C_n for $n \in \mathbb{N}$ and understand to extend it implicitly by zeros.

Often, notation can be simplified by not viewing chain complexes and chain maps as sequences of modules and homomorphisms respectively, but instead considering the direct sums

$$C \coloneqq \bigoplus_{k\in\mathbb{Z}} C_k \qquad \text{and} \qquad \mathrm{d}\big|_{C_k} \coloneqq \mathrm{d}_k \tag{2.1.2}$$

where d has been implicitly defined (by the universal property of the direct sum) as the map, that restricts on each submodule C_k of the direct sum C precisely to the differential d_k. The equations $\mathrm{d}_k \circ \mathrm{d}_{k+1}$ can then collectively be written as $\mathrm{d} \circ \mathrm{d} = 0$ or, even shorter, as $\mathrm{d}^2 = 0$. As hinted at earlier, every homological/cohomological chain complex has an associated homology/cohomology. It is constructed after noticing that the requirement $\mathrm{d}^2 = 0$ immediately implies

$$\mathrm{im}(\mathrm{d}) \subseteq \ker(\mathrm{d}).$$

This allows us to define the subquotients

$$\mathrm{H}_k(C_\bullet, \mathrm{d}_\bullet) \coloneqq \frac{\ker(\mathrm{d}_k)}{\mathrm{im}(\mathrm{d}_{k+1})} \qquad \text{and} \qquad \mathrm{H}(C_\bullet, \mathrm{d}_\bullet) \coloneqq \bigoplus_{k\in\mathbb{Z}} H_k(C_\bullet, \mathrm{d}_\bullet)$$

These quotients are called the homology (resp. cohomology) of the chain complex C. Since for our purposes later on, mostly cohomological chain complexes will be important, we shall only use the term cohomology here and implicitly understand to also mean homology under a certain relabeling of indices.

Remark 2.1.4 Given a pair of a homological chain complex and a cohomological chain complex over a ring R of the form (2.1.1), one can relate the homology of $C_\bullet$ with the cohomology of $C^\bullet$ by the universal coefficient theorem for cohomology, which states that, under certain conditions, there is a split exact sequence [121]

$$0 \longrightarrow \mathrm{Ext}^1_R(\mathrm{H}_{k-1}(C_\bullet), R) \longrightarrow \mathrm{H}^k(C^\bullet) \longrightarrow \mathsf{Hom}(\mathrm{H}_k(C_\bullet), R) \longrightarrow 0,$$

where Ext^n_R are the Ext-functors, that is the right-derived functors of $\mathsf{Hom}_R(A, \bullet)$.

One of the crucial aspects to note, is that chain complexes together with chain maps form an Abelian category, and that cohomology is actually a functor from the category of chain complexes to the category of modules. To see this, we only have to demonstrate that each chain map f descends to a map on the cohomology quotient. This, however, follows directly from the definitions: let $f\colon C \longrightarrow D$ be a chain map and let $c' \in [c] \in \mathrm{H}(C)$ for any $c \in \ker(\mathrm{d}^C)$. Then, by

$$(\mathrm{d}^D \circ f)(c) = (f \circ \mathrm{d}^C)(c) = 0,$$

we see that $f(c) \in \ker(\mathrm{d}^D)$ and furthermore, from $c' - c = \mathrm{d}^C b$ for some $b \in C$,

$$f(c') = f(c) + f\big(\mathrm{d}^C b\big) = f(c) + \mathrm{d}^D f(b)$$

and consequently $f(c') \in [f(c)]$ holds. Thus the map

$$\mathrm{H}(f)\colon \mathrm{H}(C) \longrightarrow \mathrm{H}(D)\colon [c] \longmapsto [f(c)] \tag{2.1.3}$$

is well defined. $\mathrm{H}(f)$ is often denoted by just f. Of course, one has still to check, whether H preserves identities and is compatible with concatenations, but this follows easily from (2.1.3). The definition of $\mathrm{H}(f)$ allows to define a particular distinguished subset of chain maps, the so-called quasi-isomorphisms:

Definition 2.1.5 (Quasi-isomorphism) *Let $f\colon C \longrightarrow D$ be a chain map. If the induced map*

$$\mathrm{H}(f)\colon \mathrm{H}(C) \longrightarrow \mathrm{H}(D)$$

on cohomology is an isomorphism, then f is called a quasi-isomorphism.

Furthermore, H being a functor already enables one of the central applications of homology/cohomology theory. Continuing the example of singular homology from earlier, assume that we are given two topological spaces X and Y. If those spaces were homeomorphic, that is there exists an isomorphism $X \longrightarrow Y$ in the category of topological spaces, then there also exists a bijective chain map between the singular chain complexes $C_{\mathrm{Sing}}(X)$ and $C_{\mathrm{Sing}}(Y)$ (the construction of singular chain complexes from topological spaces is a functor). Finally, using the functoriality of H, one can conclude that there exists an isomorphism of modules $\mathrm{H}_{\mathrm{Sing}}(X) = \mathrm{H}(C_{\mathrm{Sing}}(X)) \longrightarrow \mathrm{H}(C_{\mathrm{Sing}}(Y)) = \mathrm{H}_{\mathrm{Sing}}(Y)$. Consequently, whenever X and Y are homeomorphic, their corresponding singular homologies are isomorphic. And vice versa, whenever $\mathrm{H}_{\mathrm{Sing}}(X)$ and $\mathrm{H}_{\mathrm{Sing}}(Y)$ are distinct, X and Y cannot be homeomorphic [63]. Thus singular cohomology can be used as a tool to distinguish topological spaces. Unfortunately, the reverse is not true. Topological spaces with isomorphic singular homologies need not be homeomorphic. However, there are situations, where a homology module completely classifies certain objects, that is two objects are isomorphic if and only if their corresponding homology modules coincide. We will encounter three such situation later on in the classification of various types of star products on symplectic manifolds.

Another frequent appearance of homology/cohomology modules is as obstructions to certain problems. For a particularly accessible example, consider a group G acting on two sets M and N and assume that we are given an surjective, equivariant map $f\colon M \longrightarrow N$ (equivariant means that f commutes with the group action). Then we can consider the sets of invariants under the group action, denoted by M^G and N^G respectively. It is clear, that f descends to a map $f^G\colon M^G \longrightarrow N^G$. One could then ask the question, whether f^G is still surjective. As it turns out, f^G is not surjective in general, however, it is surjective whenever the first group cohomology $\mathrm{H}^1(G)$ vanishes, that is $\mathrm{H}^1(G) = 0$ [121]. In such a situation, one says that the first group cohomology is an obstruction to f^G being surjective.

Finally, let us turn towards a particular type of chain complexes, that will become important in the following parts. As seen in (2.1.2), chain complexes can be viewed as modules themselves. Thus it is entirely possible to have chain complexes, where each component is itself a chain complex, which are called double complexes. Consequently, such a chain complex comes equipped with two differentials and one usually requires them to anticommute. The complete definition in basic terms is then as follows [121]:

Definition 2.1.6 (Double complex) *Let R be a ring. A double complex $\left(C_{\bullet,\bullet}, \mathrm{d}^h_{\bullet,\bullet}, \mathrm{d}^v_{\bullet,\bullet}\right)$ (over R) is a family $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ of R-modules together with homomorphisms*

$$\mathrm{d}^h_{p,q}\colon C_{p,q} \longrightarrow C_{p-1,q} \qquad \textit{and} \qquad \mathrm{d}^v_{p,q}\colon C_{p,q} \longrightarrow C_{p,q-1}$$

such that $\mathrm{d}^h \circ \mathrm{d}^h = \mathrm{d}^v \circ \mathrm{d}^v = \mathrm{d}^h \circ \mathrm{d}^v + \mathrm{d}^v \circ \mathrm{d}^h = 0$. We will refer to p as the horizontal and to q as the vertical degree of the double complex. Accordingly, d^h is called the horizontal and d^v is called the vertical differential.

Remark 2.1.7 Of course, as in the case of chain complexes, one can easily relabel the modules of a double complex to account for differentials that increase horizontal/vertical degrees.

Remark 2.1.8 Instead of requiring the horizontal and vertical differentials to anticommute, the vertical differential can be equipped with an additional sign

$$\tilde{\mathrm{d}}^v_{p,q} \coloneqq (-1)^p\ \mathrm{d}^v_{p,q}.$$

This immediately entails the identity $\mathrm{d}^h_{p,q-1} \circ \tilde{\mathrm{d}}^v_{p,q} = \tilde{\mathrm{d}}^v_{p-1,q} \circ \mathrm{d}^h_{p,q}$. Furthermore, using this sign convention, one can identify the category of double complexes with the category of chain complexes over chain complexes [121].

One of the most common types of double complexes are the first-quadrant double complexes, which are distinguished by $C_{p,q} = 0$ for all p, q with $p < 0$ or $q < 0$, similar to bounded chain complexes. We will frequently depict such a double complex as

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
C_{0,1} \arrow[d] & C_{1,1} \arrow[l] \arrow[d] & \dots \arrow[l] \\
C_{0,0} & C_{1,0} \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

and implicitly understand to extend it by zeros again. Starting with any double complex C, there is a common construction to extract a chain complex from C, called the total complex:

$$(C_{\mathrm{Tot}})_n \coloneqq \bigoplus_{p+q=n} C_{p,q} \qquad \mathrm{d}_{\mathrm{Tot}} \coloneqq \mathrm{d}^h + \mathrm{d}^v$$

And indeed, $\mathrm{d}^2_{\mathrm{Tot}} = 0$ since

$$\mathrm{d}^2_{\mathrm{Tot}} = (\mathrm{d}^v)^2 + (\mathrm{d}^h)^2 + (\mathrm{d}^v\mathrm{d}^h + \mathrm{d}^h\mathrm{d}^v) = 0$$

by Definition 2.1.6. Accordingly, the cohomology of the total complex of a double complex is called the total cohomology and is denoted by

$$\mathrm{H}_{\mathrm{Tot}}(C) \coloneqq \mathrm{H}(C_{\mathrm{Tot}}, \mathrm{d}_{\mathrm{Tot}}).$$

One of the most important examples of the total cohomology of a double complex for this thesis will be the cohomology of the Cartan (double) complex, see Definition 2.3.11. Another prominent example from differential geometry would be the Čech-de Rham complex $C(\mathfrak{U}, \Omega)$ of a good cover $\mathfrak{U}$ of a manifold M. The leftmost column of the Čech-de Rham complex is the complex of differential forms $\Omega(M)$, while the bottom row is the Čech-complex of the cover. The remaining parts interpolate between both. It's total cohomology is then used to construct an isomorphism between the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(M)$ and the Čech cohomology $\check{\mathrm{H}}(\mathfrak{U})$ of M via [20]

$$\mathrm{H}_{\mathrm{dR}}(M) \cong \mathrm{H}_{\mathrm{Tot}}(C(\mathfrak{U}, \Omega)) \cong \check{\mathrm{H}}(\mathfrak{U}).$$

2.2 Hochschild Cohomology

Since the beginnings of deformation quantization, Hochschild cohomology has been a central tool and consequently we will be using it throughout the later chapters, especially to obtain the most fundamental classification result for star products. As such, we shall give a brief introduction and collect some necessary results on the topic. The general first part will essentially follow the treatise in [121], while the specialized version of the Hochschild-Kostant-Rosenberg theorem for manifolds is taken from [120]. Throughout this section we will assume that k is a commutative ring, R is a unital k-algebra and M is an R-bimodule. Using the shorthand notation $R^{\otimes n} = R \otimes \ldots \otimes R$ for the n-fold tensor product of R over k, we define the Hochschild complex $\mathrm{HC}^{\bullet}(R, M)$ as

$$\mathrm{HC}^n(R, M) \coloneqq \mathsf{Hom}\big(R^{\otimes n}, M\big) \qquad \mathrm{HC}^0(R, M) \coloneqq M \qquad \mathrm{HC}^{-n}(R, M) \coloneqq 0$$

for all $n \in \mathbb{N}$. On $\mathrm{HC}^n(R, M)$ we define the following coface maps $\partial^i \colon \mathrm{HC}^n(R, M) \longrightarrow \mathrm{HC}^{n+1}(R, M)$ through

$$(\partial^i f)(r_0 \otimes \ldots \otimes r_n) \coloneqq \begin{cases} r_0 \cdot f(r_1 \otimes r_2 \otimes \ldots \otimes r_n) & \text{for } i = 0 \\ f(r_0 \otimes \ldots \otimes r_{i-1} r_i \otimes \ldots \otimes r_n) & \text{for } 0 < i < n \\ f(r_0 \otimes \ldots \otimes r_{n-1}) \cdot r_n & \text{for } i = n \end{cases}$$

for all $f \in \mathrm{HC}^n(R, M)$ and $r_0, \ldots, r_n \in R$. The name coface map comes from the fact that the ∂_i together with the so called codegeneracy maps $(\sigma^i f)(r_1 \otimes \ldots \otimes r_{n-1}) \coloneqq f(r_1 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n)$ forms a cosimplicial k-module $M \otimes R^{\otimes \bullet}$ (see [121]). We will however only be interested in the associated cochain complex, given by $\mathrm{HC}(R, M)$ together with the differential

$$\partial \colon \mathrm{HC}_n(R, M) \longrightarrow \mathrm{HC}_{n+1}(R, M) \colon\ \partial \coloneqq \sum_{i=0}^{n} (-1)^i \, \partial^i$$

(the fact that $\mathrm{HC}^{\bullet}(R, M)$ together with the coface and codegeneracy maps is a cosimplicial k-module guarantees that ∂ is a differential. One could also check this directly by an easy combinatorial calculation). Accordingly, Hochschild cohomology $\mathrm{HH}^{\bullet}(R, M)$ is defined as

$$\mathrm{HH}^n(R, M) \coloneqq \frac{\ker\big(\partial \colon \mathrm{HC}^n(R, M) \longrightarrow \mathrm{HC}^{n+1}(R, M)\big)}{\mathrm{im}\big(\partial \colon \mathrm{HC}^{n-1}(R, M) \longrightarrow \mathrm{HC}^n(R, M)\big)}.$$

As usual, it is instructive to explicitly calculate $\mathrm{HC}^n(R, M)$ for some small n. For $n = 0$ we have $\mathrm{HC}^0(R, M) = M$ and the coface maps are given by

$$\partial^0(m) = r \longmapsto r \cdot m$$
$$\partial^1(m) = r \longmapsto m \cdot r$$

from which we can immediately conclude that $\mathrm{HH}^0(R, M)$ is precisely the center $\mathscr{Z}(M)$ of M:

$$\mathrm{HH}^0(R, M) = \ker\big(\partial^0 - \partial^1\big) = \{m \in M \mid rm = mr \quad \text{for all } r \in R\} = \mathscr{Z}(M).$$

For $n = 1$ on the other hand, we have the three coface maps

$$(\partial^0 f)(r_0 \otimes r_1) = r_0 \cdot f(r_1)$$
$$(\partial^1 f)(r_0 \otimes r_1) = f(r_0 \cdot r_1)$$
$$(\partial^2 f)(r_0 \otimes r_1) = f(r_0) \cdot r_1$$

and consequently

$$\ker\big(\partial^0 - \partial^1 + \partial^2\big) = \{f \in \mathsf{Hom}(R, M) \mid f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1\} = \mathsf{Der}_k(R, M)$$

where $\mathsf{Der}_k(R, M)$ denotes the M-valued derivations of R. Furthermore, from the discussion of $\mathrm{HH}^0(R, M)$ we see immediately that

$$\begin{aligned} \mathrm{im}\big(\partial^0 - \partial^1\big) &= \{f \in \mathsf{Hom}(R, M) \mid \exists m \in M \colon f(r) = mr - rm \quad \text{for all } r \in R\} \\ &= \mathsf{InnDer}_k(R, M), \end{aligned}$$

where $\mathsf{InnDer}_k(R, M)$ denotes the inner M-valued derivations of R. We see then that

$$\mathrm{HH}^1(R, M) = \frac{\mathsf{Der}_k(R, M)}{\mathsf{InnDer}_k(R, M)}.$$

One can show that in general $\mathrm{HH}^\bullet(R, M)$ can be calculated as a relative Ext [121]:

Lemma 2.2.1 *Hochschild cohomology is isomorphic to relative* Ext *for the ring map $k \longrightarrow R^{\mathrm{e}} = R \otimes R^{\mathrm{op}}$:*

$$\mathrm{HH}^\bullet(R, M) \cong \mathrm{Ext}^\bullet_{R^{\mathrm{e}}/k}(R, M).$$

This description of Hochschild cohomology however, is in many cases still too complicated. Fortunately, we will not need Hochschild cohomology in full generality, but only a slightly different variant of it in the following very specific setting: let us assume for the remainder of this chapter that k is a field, $\mathcal{A}$ a unital, commutative k-algebra and that $R = M = \mathcal{A}$. We begin by defining multidifferential operators on $\mathcal{A}$ of order $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ for any $n \in \mathbb{N}$ with values in $\mathcal{A}$ recursively by

$$\mathrm{DiffOp}^K(\mathcal{A}^{\otimes n}, \mathcal{A}) := 0$$

if $k_i < 0$ for at least one $1 \leq i \leq n$ and

$$\begin{aligned} &\mathrm{DiffOp}^K(\mathcal{A}^{\otimes n}, \mathcal{A}) \\ &\qquad := \Big\{D \in \mathsf{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A}) \;\Big|\; \forall a \in \mathcal{A}\, \forall i \colon L_a \circ D - D \circ L_a^{(i)} \in \mathrm{DiffOp}^{K - \mathrm{e}_i}(\mathcal{A}^{\otimes n}, \mathcal{A})\Big\}, \end{aligned}$$

where we used $K - \mathrm{e}_i := (k_1, \dots, k_i - 1, \dots, k_n)$, denoted by L_a the left-multiplication with a in $\mathcal{A}$ and by $L_a^{(i)}$ the left multiplication on the i-th tensor factor in $\mathcal{A}^{\otimes n}$. Finally, denote by

$$\mathrm{DiffOp}^\bullet(\mathcal{A}^{\otimes n}, \mathcal{A}) := \bigcup_{K \in \mathbb{Z}^n} \mathrm{DiffOp}^K(\mathcal{A}^{\otimes n}, \mathcal{A})$$

the union of differential operators of all orders. We can now define the complex of smooth Hochschild cochains $\mathrm{HC}^\bullet_{\mathrm{diff}}(\mathcal{A}, \mathcal{A})$ (where we will shorten notation to $\mathrm{HC}^\bullet_{\mathrm{diff}}(\mathcal{A})$) by

$$\mathrm{HC}^n_{\mathrm{diff}}(\mathcal{A}) := \mathrm{DiffOp}^\bullet(\mathcal{A}^{\otimes n}, \mathcal{A}).$$

Clearly, $\mathrm{HC}^\bullet_{\mathrm{diff}}(\mathcal{A})$ is a subcomplex of $\mathrm{HC}^\bullet(\mathcal{A}, \mathcal{A})$. We denote by $\mathrm{HH}^\bullet_{\mathrm{diff}}(\mathcal{A})$ the corresponding smooth Hochschild cohomology. Specializing the above setting even further to the case $\mathcal{A} = \mathscr{C}^\infty(M)$ for any manifold M, one can obtain an explicit expression of the smooth Hochschild cohomology by a variation on the famous Hochschild-Kostant-Rosenberg theorem [66, 120], which was first mentioned in [119], where we use the shorthand notation $\mathfrak{X}^\bullet(M) := \Gamma^\infty(\Lambda^\bullet TM)$ for multivectorfields on M.

Theorem 2.2.2 (Hochschild-Kostant-Rosenberg) *The Hochschild-Kostant-Rosenberg map*

$$\mathcal{U} \colon \mathfrak{X}^\bullet(M) \longrightarrow \mathrm{HC}^\bullet_{\mathrm{diff}}(\mathscr{C}^\infty(M))$$

given by

$$(\mathcal{U}X)(f_1, \dots, f_n) := X(\mathrm{d}f_1, \dots, \mathrm{d}f_n)$$

induces an isomorphism on cohomology (where $\mathfrak{X}^\bullet(M)$ is equipped with the 0-differential)

$$\mathcal{U} \colon \mathfrak{X}^\bullet(M) \cong \mathrm{HH}^\bullet_{\mathrm{diff}}(\mathscr{C}^\infty(M))$$

Remark 2.2.3 There are different variants of the Hochschild-Kostant-Rosenberg theorem that apply to different settings. For example, if $\mathcal{A}$ is a commutative algebra over a field k subject to some technical conditions, then there exists an isomorphism [51, 121]

$$\Lambda^\bullet \operatorname{\mathsf{Der}}(\mathcal{A}, \mathcal{A}) \cong \mathrm{HH}^\bullet(\mathcal{A}, \mathcal{A})$$

between the exterior algebra of $\mathcal{A}$-valued derivations on $\mathcal{A}$ and the Hochschild cohomology of $\mathcal{A}$. The similarity to Theorem 2.2.2 becomes apparent, once one recognizes vector fields on M as $\mathscr{C}^\infty(M)$-valued derivations of $\mathscr{C}^\infty(M)$ and uses the isomorphism $\Gamma^\infty(\Lambda^\bullet TM) \cong \Lambda^\bullet \operatorname{\mathsf{Der}}(\mathscr{C}^\infty(M), \mathscr{C}^\infty(M))$. Other variants of the HKR-theorem can be found e.g. in [2], [28], [87] or [101].

Remark 2.2.4 $\mathfrak{X}^\bullet(M)$ [97, 109] and $\mathrm{HC}^\bullet_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ [50] both come equipped with a differential graded Lie algebra structure (the latter even with a Gerstenhaber structure). However, the HKR-map $\mathcal{U}$ from Theorem 2.2.2 is not a morphism of differential graded Lie algebras. The celebrated formality theorem by Kontsevich [76] (see also [4, 25–27, 36–38, 77–79, 115]) extends the HKR-map to an L_∞-quasi-isomorphism that has $\mathcal{U}$ as its first Taylor coefficient.

We will conclude this section with an application of the HKR-Theorem [57, 98, 119]:

Corollary 2.2.5 *Any smooth Hochschild cocycle $C \in \mathrm{HC}^k_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ can be written as*

$$C(f_1, \dots, f_k) = \partial c(f_1, \dots, f_k) + X(\mathrm{d}f_1, \dots, \mathrm{d}f_k)$$

with $c \in \mathrm{HC}^{k-1}_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ and $X \in \mathfrak{X}^\bullet(M)$.

Proof: Consider $[C] \in \mathrm{HH}^k_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ and denote its pre-image in $\mathfrak{X}^k(M)$ under the HKR-map for cohomology by X_C. Then $\mathcal{U}X_C$ and C are clearly cohomologous and hence there exists a ∂-exact ∂c in $\mathrm{HC}^k_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ such that

$$\mathcal{U}X_C - C = \partial c. \qquad \square$$

2.3 Equivariant Cohomology

As mentioned briefly in Section 2.1, cohomology modules can play an important role in classifying topological spaces and manifolds. During this thesis however, we will rarely be interested in bare manifolds or topological spaces. Rather, our focus lies with manifolds that come equipped with a smooth Lie group action. Similarly, for the motivational Section 2.3.1, the main object of interest are topological spaces on which a topological group acts continuously. In these situations, it is often the case that singular or de Rham cohomology do not capture as many features as one would like. Hence the need arises to find cohomology theories more adapted to group actions. So one may, for example, use invariant or equivariant cohomology, the latter of which will be detailed throughout the following parts. First, however, for the convenience of the reader as well as to fix notations, we shall give a brief introduction into the necessary concepts surrounding Lie group actions on manifolds. For clarification, whenever we speak of actions, we will always implicitly mean left actions (if not stated otherwise). Given any group action $\triangleright\colon G \times S \longrightarrow S$ for a group G and a set S, we will frequently write $g \triangleright s$ or just gs instead of $\triangleright(g, s)$. We shall also omit any mention of the action $\triangleright$ itself, if it is clear from the context. We shall also agree, that actions of Lie groups on manifolds are always smooth and that actions of topological groups on topological spaces are always continuous. Also, since each $g \in G$ can be viewed as an automorphism of M through $m \longmapsto g \triangleright m$, one always has an induced right action on tensor fields t on M by the pullback with said automorphism, which we denote by g^*t. Of course, there exists a plethora of types of actions, two of which will be particularly useful to us later on. The first ones are free actions:

Definition 2.3.1 (Free action) *Let $\triangleright\colon G \times S$ be an action of a group G on a set S. $\triangleright$ is called free if $g \triangleright s = h \triangleright s$ implies $g = h$ for all $g, h \in G$ and $s \in S$.*

One of the most commonly used property of free actions is then the following: let G be a group, S_1 and S_2 sets and $\triangleright_i\colon G \times S_i \longrightarrow S_i$ actions. We can combine both actions into one by setting

$$\triangleright\colon G \times (S_1 \times S_2) \longrightarrow S_1 \times S_2\colon (g, s_1, s_2) \longmapsto (gs_1, gs_2).$$

If either $\triangleright_1$ or $\triangleright_2$ is a free action, then so is $\triangleright$. In Section 2.3.1, the definition of equivariant cohomology of topological spaces equipped with continuous actions of topological groups relies heavily on this fact. The second important type is that of a proper action, which only applies to continuous group actions:

Definition 2.3.2 (Proper action) *Let $\triangleright\colon G \times M \longrightarrow M$ be a continuous action. $\triangleright$ is called proper, if the map*

$$G \times M \longrightarrow M \times M\colon (g, m) \longmapsto (gm, m)$$

is a proper map, that is if the pre-images of compact subsets are compact.

In the later parts, we will often use smooth actions of Lie groups G on manifolds M that are free and proper, for the sole reason, that the quotient space

$$M/G := \{G \triangleright m \mid m \in M\}$$

is then again a manifold. Staying with Lie groups for the remainder of this section, there are a few particularly important concepts special to smooth actions. So let G be a Lie group acting on a manifold M and let $\mathfrak{g}$ be the Lie algebra of G together with its exponential map $\exp\colon \mathfrak{g} \longrightarrow G$. For any Lie algebra element $\xi \in \mathfrak{g}$ and point $m \in M$ we can then define a curve in M by

$$\exp(-t \cdot \xi) \triangleright m \qquad \text{for } t \in \mathbb{R}. \tag{2.3.1}$$

Furthermore, we can derive γ with respect to t in $t = 0$ to obtain a tangent vector at m. Repeating this process for all m, one obtains a smooth vector field

$$X_\xi(m) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(-t \cdot \xi) \triangleright m) \tag{2.3.2}$$

on M, called the fundamental vector field of ξ. It is also commonly denoted by ξ_M. One can then show, that the map

$$X_\bullet\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)\colon \xi \longmapsto X_\xi \tag{2.3.3}$$

is a Lie algebra anti-homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields $\mathfrak{X}(M)$ on M.

Remark 2.3.3 Note, that the sign in the definition of fundamental vector fields is entirely up to convention. If we used a plus in (2.3.1) and (2.3.2), then the map $X_\bullet$ would turn out to be a Lie algebra homomorphism. Both conventions are commonly used.

The definition of fundamental vector fields, and especially the homomorphism $X_\bullet$, then directly motivates the concept of Lie algebra actions:

Definition 2.3.4 (Lie algebra action) *Let $\mathfrak{g}$ be a Lie algebra and M be a manifold. A Lie algebra action of $\mathfrak{g}$ on M is a Lie algebra anti-homomorphism*

$$\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M).$$

We will frequently abbreviate the Lie derivative $\mathscr{L}_{\rho(\xi)}$ to just $\mathscr{L}_\xi$, if the action ρ is clear from the context. Via fundamental vector fields, we can always obtain a Lie algebra action from a given Lie group action. The converse, however, is only true under special circumstances, as detailed in a famous theorem by Palais [99] (see also [88, Thm. 6.5]):

Theorem 2.3.5 (Palais) *Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, M be a manifold and*

$$\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$$

be a Lie algebra action. Then ρ integrates to a Lie group action $\triangleright\colon G \times M \longrightarrow M$ if and only if all $X \in \operatorname{im}(\rho)$ have a complete flow.

Instead of actions on pure manifolds however, we will mainly be interested in actions on symplectic manifolds. And in this case, one requires the actions to be compatible with the additional symplectic structure. A Lie group action is said to be symplectic, if pullbacks with group elements preserve the symplectic two-form and Lie algebra actions are said to be symplectic if the flow of all elements preserves the symplectic structure. We will refer to symplectic manifolds equipped with symplectic Lie group or Lie algebra actions as G- and $\mathfrak{g}$-spaces respectively:

Definition 2.3.6 (Symplectic G-space) *Let (M, ω) be a symplectic manifold, G a Lie group and $\triangleright$ a Lie group action of G on M. $\triangleright$ is called symplectic if*

$$g^*\omega = \omega$$

holds for all $g \in G$. The tuple $(M, \omega, G, \triangleright)$ is then called a symplectic G-space.

Definition 2.3.7 (Symplectic $\mathfrak{g}$-space) *Let (M, ω) be a symplectic manifold, $\mathfrak{g}$ a Lie algebra and $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ a Lie algebra action of $\mathfrak{g}$ on M. ρ is called symplectic if*

$$\mathscr{L}_{\rho(\xi)}\omega = 0$$

holds for all $\xi \in \mathfrak{g}$. The tuple $(M, \omega, \mathfrak{g}, \rho)$ is then called a symplectic $\mathfrak{g}$-space.

As per our conventions, the explicit notation of the action in symplectic G- and $\mathfrak{g}$-spaces is usually dropped. Also, recall that any vector field $X \in \mathfrak{X}(M)$ on M is called symplectic, if $\mathscr{L}_X\omega = 0$ holds. Hence the definition of symplectic Lie algebra actions could be reformulated to require ρ to be a map $\mathfrak{g} \longrightarrow \mathfrak{X}_{\mathrm{sympl}}(M)$ into the symplectic vector fields instead of all vector fields.

2.3.1 Equivariant Cohomology in Topology

Equivariant cohomology was originally devised as a tool to handle group actions on topological spaces for cases where the orbit space is not sufficiently "nice". To elaborate, let X be a topological space and G a topological group acting on X. Also, for this part only, let G be compact. If in addition the action is free, the orbit space X/G usually turns out to be well-behaved. For example, if X is Hausdorff, so is X/G. If X is a manifold and G a Lie group acting smoothly on X, then X/G is again a manifold. With this background, equivariant cohomology is a result from the search for a cohomology theory HE_G that agrees with singular cohomology $\mathrm{H}_{\mathrm{Sing}}$ on quotients of free actions, but is better behaved in the non-free case where singular cohomology is often pathological. So given any coefficient ring R,

$$\mathrm{HE}_G(X, R) = \mathrm{H}_{\mathrm{Sing}}(X/G, R)$$

is required to hold whenever the action of G on X is free.

Remark 2.3.8 The notation HE_G for equivariant cohomology is non-standard. We chose it solely to avoid confusion with the cohomology of the Cartan complex.

The actual construction of HE_G centers around two facts. For one, any two topological spaces that are homotopy equivalent have isomorphic singular cohomology. And second, the cartesian product of X with any topological space E with free G action is equipped with the diagonal action of G which is free. So the main idea is to search for a contractible topological space E with a free G action (contractability guarantees that $X \times E$ is homotopy equivalent to X). Assuming the existence of such an E, equivariant cohomology can be defined as

$$\mathrm{HE}_G(X, R) := \mathrm{H}_{\mathrm{Sing}}\left(\frac{X \times E}{G}, R\right).$$

To show that HE_G is well-defined, we employ a standard classification theorem, which can be found in many textbooks on algebraic topology, i.e. [117].

Theorem 2.3.9 *Let G be a topological group. Then there exists a topological space BG (the classifying space of G) and a natural isomorphism*

$$\iota_B\colon [B, BG] \longrightarrow \mathrm{GBun}(B, G),$$

where $[B, BG]$ denotes the set of homotopy classes of continuous maps $B \longrightarrow BG$ and $\mathrm{GBun}(B, G)$ *the set of isomorphism classes of numerable G-principal bundles over B.*

We will not concern ourselves with the technical details of that theorem, however, one particular consequence is the next theorem [55]:

Theorem 2.3.10 *If E_1 and E_2 are contractible spaces on which G acts freely, they are equivalent as G-spaces. In other words there exist G-equivariant maps*

$$\phi\colon E_1 \longrightarrow E_2, \qquad \psi\colon E_2 \longrightarrow E_1$$

with G-equivariant homotopies

$$\psi \circ \phi \simeq \mathsf{id}_{E_1}, \qquad \phi \circ \psi \simeq \mathsf{id}_{E_2}.$$

This immediately assures that HE_G is indeed well-defined. As for the existence of a suitable space E, one has to turn to the proof of Theorem 2.3.9, where a universal bundle $EG \longrightarrow BG$, called the classifying bundle of G, is constructed. The total space EG is then equipped with a free G-action and is contractible. In hindsight, one can view $EG \longrightarrow BG$ as $\iota_{BG}(\mathsf{id}_{BG})$. Finally, let us note that if G acts freely on X then the map

$$\frac{X \times E}{G} \longrightarrow X/G$$

induced by the projection onto the first factor is a homotopy equivalence and as such we have

$$\mathrm{HE}_G(X, R) = \mathrm{H}_{\mathrm{Sing}}\left(\frac{X \times E}{G}, R\right) \cong \mathrm{H}_{\mathrm{Sing}}(X/G, R).$$

2.3.2 The Cartan Model of Equivariant Cohomology

As seen in the previous part, the very definition of equivariant cohomology requires finding a contractible space on which G acts freely (i.e. the classifying bundle of G), which is notoriously difficult. As such, one is highly motivated to find alternative constructions for equivariant cohomology. One of these is the Cartan model. The Cartan model is defined via differential forms and hence only applicable to the case of Lie group or Lie algebra actions on smooth manifolds. We will start by defining the Lie algebraic chain complex of equivariant differential forms, also called the Cartan complex [55]:

Definition 2.3.11 (Cartan complex) *Let M be a smooth manifold, $\mathfrak{g}$ a Lie algebra and $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ a Lie algebra action of $\mathfrak{g}$ on M. The complex of $\mathfrak{g}$-equivariant differential forms, or Cartan complex, $\Omega_{\mathfrak{g}}(M)$ is defined as*

$$\Omega^k_{\mathfrak{g}}(M) := \left(\bigoplus_{2i+j=k} \left[\mathrm{S}^i(\mathfrak{g}^*) \otimes \Omega^j(M)\right]^{\mathfrak{g}}, \quad \mathrm{d}_{\mathfrak{g}} := \mathrm{d} + \mathrm{i}_\bullet \right).$$

Invariants are taken with respect to the representation

$$\xi \triangleright (p \otimes \alpha) := \big(-\operatorname{ad}^*_\xi p\big) \otimes \alpha + p \otimes (-\mathscr{L}_\xi \alpha) \qquad \textit{for all } p \in \mathrm{S}(\mathfrak{g}^*),\ \alpha \in \Omega(M).$$

We will refer to $k = 2i + j$ as the total, i the symmetric and j the exterior degree of any $\alpha \in [\mathrm{S}^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^{\mathfrak{g}}$.

Here we denoted by $\Omega(M)$ the de Rham complex of differential forms on M with de Rham differential d, by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, by $S^i(\mathfrak{g}^*)$ the i-th component of the symmetric tensor algebra over $\mathfrak{g}^*$, by $\mathrm{i}_\bullet$ the insertion of vector fields into the first component of the differential form part, by $-\operatorname{ad}^*$ the coadjoint representation of $\mathfrak{g}$

$$(-\operatorname{ad}^*_\xi \alpha)(\eta) \coloneqq \alpha([\xi,\eta]) \qquad \text{for all} \quad \xi,\eta\in\mathfrak{g} \quad \text{and} \quad \alpha\in\mathfrak{g}^*$$

extended to $\mathrm{S}(\mathfrak{g}^*)$ as the tensor product representation and by $[\ldots]^{\mathfrak{g}}$ the set of $\mathfrak{g}$-invariant elements with respect to $\triangleright$. Obviously, $\triangleright$ is also nothing but the tensor product of the coadjoint representation $-\operatorname{ad}^*$ on $\mathrm{S}(\mathfrak{g}^*)$ and $-\mathscr{L}_\bullet$ on $\Omega(M)$. A particularly useful point of view on equivariant differential forms will be that of equivariant polynomials on $\mathfrak{g}$ with values in $\Omega(M)$: any $\alpha \in [\mathrm{S}^i(\mathfrak{g}^*)\otimes\Omega^j(M)]^{\mathfrak{g}}$ can be seen as a degree i polynomial map $\alpha\colon \mathfrak{g}\longrightarrow \Omega^j(M)$ such that

$$(\operatorname{ad}^*_\xi \alpha)(\eta) = -\mathscr{L}_\xi[\alpha(\eta)]$$

holds for all $\xi,\eta\in\mathfrak{g}$. To complete Definition 2.3.11, we have to demonstrate that $\mathrm{d}_{\mathfrak{g}}$ maps again into the equivariant differential forms and that $\mathrm{d}_{\mathfrak{g}}$ squares to zero. For the first part, let $\alpha\in\Omega_{\mathfrak{g}}(M)$ and $\xi,\eta\in\mathfrak{g}$. Since $\mathscr{L}_\xi$ commutes with d, we only have to show that $\mathrm{i}_\bullet$ is equivariant:

$$\begin{aligned}
-\mathscr{L}_\xi[(\mathrm{i}_\bullet\,\alpha)(\eta)] &= -\mathscr{L}_\xi[\mathrm{i}_\eta\,\alpha(\eta)] = \mathrm{i}_{[\xi,\eta]}\,\alpha(\eta) + \mathrm{i}_\eta\,\mathscr{L}_\xi\alpha(\eta)\\
&= \mathrm{i}_{[\xi,\eta]}\,\alpha(\eta) - \mathrm{i}_\eta(\operatorname{ad}^*_\xi\alpha)(\eta) = \big[\operatorname{ad}^*_\xi(\mathrm{i}_\bullet\,\alpha)\big](\eta).
\end{aligned}$$

To conclude, let us show that $\mathrm{d}_{\mathfrak{g}}$ is a differential. Using $\mathrm{d}^2 = \mathrm{i}_\xi^2 = 0$ for all $\xi\in\mathfrak{g}$, we can calculate

$$[\mathrm{d}_{\mathfrak{g}}\alpha](\xi) = \mathrm{d}^2\alpha(\xi) + (\mathrm{d}\,\mathrm{i}_\xi + \mathrm{i}_\xi\,\mathrm{d})\alpha(\xi) + \mathrm{i}_\xi^2\,\alpha(\xi) = \mathscr{L}_\xi\alpha(\xi) = (-\operatorname{ad}^*_\xi\alpha)(\xi) = 0. \tag{2.3.4}$$

To explore some of the properties of $\Omega_{\mathfrak{g}}(M)$, let us briefly note that $\Omega_{\mathfrak{g}}$ can be expressed as the following concatenation of functors

$$\Omega_{\mathfrak{g}} = \big(\bullet^G\big)\circ(\mathrm{S}(\mathfrak{g}^*)\otimes\bullet)\circ\Omega,$$

which immediately implies that $\Omega_{\mathfrak{g}}$ is a functor. We will abuse notation a little bit here and write

$$f^* = \Omega_{\mathfrak{g}}(f)\colon \Omega_{\mathfrak{g}}(M)\longrightarrow\Omega_{\mathfrak{g}}(N)\colon p\otimes\alpha\longmapsto p\otimes f^*\alpha$$

for any equivariant map $f\colon N\longrightarrow M$. Furthermore, it is possible to view $\Omega_{\mathfrak{g}}$ as a double complex with vertical differential $\mathrm{i}_\bullet$ and horizontal differential d, which will be important later on. To be explicit, define

$$\Omega^{i,j}_{\mathfrak{g}}(M) \coloneqq \big(\big[\mathrm{S}^i(\mathfrak{g}^*)\otimes\Omega^{i+j}(M)\big]^{\mathfrak{g}},\quad \mathrm{d}^v\coloneqq\mathrm{i}_\bullet,\ \mathrm{d}^h\coloneqq\mathrm{d}\big).$$

We use the standard convention that $\Omega^j(M)=0$ for all $j<0$. And indeed, the horizontal and vertical differentials anticommute, which can be shown easily by using Cartan's formula together with the last part of (2.3.4)

$$\big[(\mathrm{d}^v\mathrm{d}^h + \mathrm{d}^h\mathrm{d}^v)\alpha\big](\xi) = [\mathscr{L}_\bullet\alpha](\xi) = \mathscr{L}_\xi\alpha(\xi) = 0.$$

Remark 2.3.12 In the above definition, one might wonder about the fact that $\Omega_{\mathfrak{g}}^{i,j}(M)$ has $\Omega^{i+j}(M)$ instead of $\Omega^j(M)$ as the second tensor factor. This is done to match the double complex conventions that the differentials map as

$$\mathrm{d}_h\colon \Omega_{\mathfrak{g}}^{i,j}(M) \longrightarrow \Omega_{\mathfrak{g}}^{i,j+1}(M) \qquad\qquad \mathrm{d}_v\colon \Omega_{\mathfrak{g}}^{i,j}(M) \longrightarrow \Omega_{\mathfrak{g}}^{i+1,j}(M)$$

If we had chosen the other variant, the vertical differential would map from $\Omega_{\mathfrak{g}}^{i,j}$ to $\Omega_{\mathfrak{g}}^{i+1,j-1}$. In the end, this convention is purely cosmetical in nature and will not impact any results.

Finally, let us turn towards equivariant cohomology. By this point it should come as no surprise that equivariant cohomology in the Cartan model is defined as follows [55]:

Definition 2.3.13 (Equivariant cohomology) *Let M be a smooth manifold and $\mathfrak{g}$ a Lie algebra acting on M. The equivariant cohomology of M with respect to $\mathfrak{g}$ is defined as*

$$\mathrm{H}_{\mathfrak{g}}(M) \coloneqq \frac{\ker \mathrm{d}_{\mathfrak{g}}}{\operatorname{im} \mathrm{d}_{\mathfrak{g}}}.$$

In other words, using the standard cohomology functor H on chain complexes [121], we have $\mathrm{H}_{\mathfrak{g}} = \mathrm{H} \circ \Omega_{\mathfrak{g}}$. Viewing the complex of equivariant differential forms as a double complex, we can equivalently state that $\mathrm{H}_{\mathfrak{g}} = \mathrm{H}_{\mathrm{tot}} \circ \Omega_{\mathfrak{g}}^{\bullet,\bullet}$ where $\mathrm{H}_{\mathrm{tot}}$ is the total cohomology functor on double complexes. Especially the last two formulations immediately clarify the functoriality of $\mathrm{H}_{\mathfrak{g}}$. In addition, since d and $\mathrm{i}_\bullet$ are natural, we know that for any equivariant map of manifolds $f\colon N \longrightarrow M$ the pullback $\Omega_{\mathfrak{g}}(f)\colon \Omega_{\mathfrak{g}}(M) \longrightarrow \Omega_{\mathfrak{g}}(N)$ on equivariant differential forms descends to cohomology as

$$f^* = \mathrm{H}_{\mathfrak{g}}(f)\colon \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathfrak{g}}(N)\colon [\alpha]_{\mathfrak{g}} \longmapsto [f^*\alpha]_{\mathfrak{g}}.$$

In complete analogy to the Lie algebraic case, there is a Lie group version of the Cartan complex. Its definition, at its core, is nothing but Definition 2.3.11 with all formulas involving Lie algebra elements integrated:

Definition 2.3.14 (Cartan complex) *Let M be a smooth manifold and G be a Lie group with Lie algebra $\mathfrak{g}$ acting on M. The complex of G-equivariant differential forms $\Omega_G(M)$ is defined as*

$$\Omega_G^k(M) \coloneqq \left(\bigoplus_{2i+j=k} \big[\mathrm{S}^i(\mathfrak{g}^*) \otimes \Omega^j(M)\big]^G, \quad \mathrm{d}_G \coloneqq \mathrm{d} + \mathrm{i}_\bullet \right).$$

Invariants are taken with respect to the action

$$g \triangleright (p \otimes \alpha) \coloneqq (\mathrm{Ad}_g^* p) \otimes ((g^{-1})^*\alpha).$$

Similarly to the Lie algebraic Cartan complex, it has to be established that d_G maps into the G-equivariant differential forms,

$$\begin{aligned}(\mathrm{d}_G\alpha)(\mathrm{Ad}_g^*\xi) &= \mathrm{d}\alpha(\mathrm{Ad}_g^*\xi) + \mathrm{i}_{\mathrm{Ad}_g^*\xi}\,\alpha(\mathrm{Ad}_g^*\xi) = \mathrm{d}(g^{-1})^*\alpha(\xi) + \big((g^{-1})^*\,\mathrm{i}_\xi\, g^*\big)\big((g^{-1})^*\alpha(\xi)\big)\\ &= (g^{-1})^*(\mathrm{d}_G\alpha)(\xi),\end{aligned}$$

and that d_G is a differential:

$$\begin{aligned}\big[\mathrm{d}_G^2\alpha\big](\xi) &= \mathscr{L}_\xi\alpha(\xi) = -\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\exp(-t\xi)^*\alpha(\xi) = -\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha\big(\mathrm{Ad}_{\exp(-t\xi)}\,\xi\big)\\ &= -\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha(\exp(-t\,\mathrm{ad}_\xi)\xi) = -\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha(\xi) = 0.\end{aligned}$$

Note that every action of a Lie group G has an associated derived action of its Lie algebra $\mathfrak{g}$ by fundamental vector fields. If G is additionally connected, then any element $\alpha \in \mathrm{S}(\mathfrak{g}^*) \otimes \Omega(M)$ is G-invariant, if and only if it is $\mathfrak{g}$-invariant, hence in this particular case, the Lie algebraic Cartan complex $\Omega_{\mathfrak{g}}(M)$ coincides with the group theoretic Cartan complex $\Omega_G(M)$. However, all results in Section 3.4 will be valid for all Lie algebra actions, even those that do not arise as fundamental vector fields of a global action.

With all necessary definitions in place, it is now that we are able to make the originally sought after connection to equivariant cohomology in the form of the following theorem, originally due to [24], see also [55, 69, 96]:

Theorem 2.3.15 (Cartan) *Let G be a compact Lie group acting on a compact manifold M. Then equivariant cohomology and the cohomology of the Cartan complex are isomorphic.*

$$\mathrm{HE}_G(M) \cong \mathrm{H}_G(M).$$

It is vital to remark that for noncompact groups the above theorem generally fails.

Remark 2.3.16 Throughout this thesis we will use HE_G exclusively for motivational purposes. This will allow us to circumvent the cumbersome expression "cohomology of the complex of equivariant differential forms" for $\mathrm{H}_{\mathfrak{g}}$ and H_G, which we will instead just call equivariant cohomology. One must be aware at all times that this terminology is not at all justified, since we will be working with noncompact groups in general and hence Theorem 2.3.15 may not apply. Even more, in the case of nonintegrable Lie algebra actions, see Theorem 2.3.5, HE_G simply cannot be constructed. Occasionally we will refer to HE_G as the topological or Borel model for equivariant cohomology.

2.3.3 Equivariant Cohomology of Principal Bundles

One main point of interest later on will be the equivariant cohomology of G-principal bundles for any connected, finite-dimensional Lie group G. Standard definitions of principal bundles can be found in most books on differential geometry (e.g. [75]). For our purposes however, another characterization of principal bundles [40] will prove to be tremendously useful, hence we will elevate it to a definition

Definition 2.3.17 (Principal bundle) *Let G be a Lie group. A G-principal bundle is a smooth manifold P equipped with a free and proper action of G.*

The connection to the standard definition is essentially that the canonical projection $P \xrightarrow{\pi} P/G$ is a fibre bundle (which also implies that π is a surjective submersion). From the topological model of equivariant cohomology we can already hypothesize as to what the Cartan model of equivariant cohomology on principal bundles will turn out. Remember that if G acts freely on a topological space X, one could construct an isomorphism $\mathrm{HE}_G(X) \cong \mathrm{H}_{\mathrm{Sing}}(X/G)$. On smooth manifolds M we can additionally employ the de Rham isomorphism $\mathrm{H}_{\mathrm{Sing}}(M) \cong \mathrm{H}_{\mathrm{dR}}(M)$ [20, 89]. Combining both for the case of a principal bundle $P \longrightarrow P/G$ then yields

$$\mathrm{HE}_G(P) \cong \mathrm{H}_{\mathrm{Sing}}(P/G) \cong \mathrm{H}_{\mathrm{dR}}(P/G).$$

If the Lie group G and P/G were compact, then it immediately followed that there is an isomorphism

$$\mathrm{H}_G(P) \cong \mathrm{H}_{\mathrm{dR}}(P/G) \tag{2.3.5}$$

due to Theorem 2.3.15. However, since we do not restrict to compact Lie groups, the situation is more complicated. We will devote the rest of this section to show that (2.3.5) holds nevertheless.

We will be needing two additional definitions for the final result. One of them is the notion of basic differential forms on a surjective submersion of manifolds:

Definition 2.3.18 (Basic differential form) *Let $\pi\colon M \longrightarrow N$ be a surjective submersion between smooth manifolds. A differential form $\mu \in \Omega(M)$ is called basic if*

$$\mathrm{i}_Y\, \mu = 0 \qquad \textit{and} \qquad \mathscr{L}_Y \mu = 0$$

for all $Y \in \ker(\mathrm{d}\pi)$. We will denote the complex of basic differential forms on M by $\Omega_{\mathrm{bas}}(M)$.

If we specialize the above definition for the case that M is a G-principal bundle P and N the corresponding base manifold P/G then one can note that $\ker(\mathrm{d}\pi) = X(\mathfrak{g})$ and thus $\mu \in \Omega_{\mathrm{bas}}(P)$ is equivalent to

$$\mathrm{i}_\xi\, \mu = 0 \qquad \text{and} \qquad \mathscr{L}_\xi \mu = 0$$

for all $\xi \in \mathfrak{g}$ (here we used the map X which maps lie algebra elements to their fundamental vector fields). One easy lemma on basic differential forms is the following:

Lemma 2.3.19 *Let $\pi\colon M \longrightarrow N$ be a surjective submersion such that $\pi^{-1}(y)$ is a connected submanifold of M for all $y \in N$. Then $\mu \in \Omega(M)$ is basic if and only if there exists a unique $\nu \in \Omega(N)$ with*

$$\mu = \pi^*\nu$$

Proof: We will directly define ν in two independent ways. First, given any point $y \in N$ and tangent vectors $Z_1, \ldots, Z_n \in T_yN$ define

$$\nu_y(Z_1, \ldots, Z_n) \coloneqq \mu_p(X_1, \ldots, X_n)$$

for any tangent vectors $X_1, \ldots, X_n \in T_pM$ for which $\mathrm{d}\pi_p X_k = Z_k$ holds for any choice of $p \in \pi^{-1}(y)$. Such tangent vectors exist, since π is a surjective submersion. To show that ν is well-defined, consider open neighbourhood $U \subseteq N$ of y such that there exists a local section $\sigma\colon U \longrightarrow M$ (that is $\pi \circ \sigma = \mathsf{id}_U$), which exists due to the constant rank theorem, see e.g. [75, Lemma 2.2]. Evaluating $\sigma^*\mu$ at y for the tangent vectors Z_k from above yields

$$(\sigma^*\mu)_y(Z_1, \ldots, Z_n) = \mu_{\sigma(y)}(\mathrm{d}\sigma_y Z_1, \ldots, \mathrm{d}\sigma_y Z_n)$$

and hence, by the identity $\mathrm{d}\pi_{\sigma(y)}\mathrm{d}\sigma_y Z_k = \mathrm{d}(\pi \circ \sigma)_y Z_k = Z_k$, we conclude that $\sigma^*\mu$ agrees with ν, as defined above, at y. This clearly establishes the independence of ν from the choices of $p \in \pi^{-1}(y)$ and $X_1, \ldots, X_n \in T_pM$. On the other hand, for any additional local section $\tilde{\sigma}\colon V \longrightarrow M$, $\sigma^*\mu$ and $\tilde{\sigma}^*\mu$ agree on the overlap $U \cap V$ since both yield ν at every point in $U \cap V$. Hence for any open cover $\mathfrak{U}$ with local sections $\{\sigma_U\colon U \longrightarrow M\}_{U\in\mathfrak{U}}$ the local differential forms $\{\sigma_U^*\mu\}_{U\in\mathfrak{U}}$ can be glued together to a global differential form on N and at each point, and thus globally, this differential form equals ν. Finally, we can calculate pointwise for any $q \in M$ and $Y_1, \ldots, Y_n \in T_qM$

$$(\pi^*\nu)_q(Y_1, \ldots, Y_n) = \nu_{\pi(q)}(\mathrm{d}\pi_q Y_1, \ldots, \mathrm{d}\pi_q Y_n) = \mu_q(Y_1, \ldots, Y_n).$$

The uniqueness of ν is clear from the injectivity of π^*. Conversely, given a differential form $\mu = \pi^*\nu$ for any $\nu \in \Omega(N)$, and any $Y \in \ker \mathrm{d}\pi$, μ is obviously basic. First, calculate

$$(\pi^*\nu)_p(Y, X_2, \ldots, X_n) = \nu_{\pi(p)}(\mathrm{d}\pi_p Y, \mathrm{d}\pi_p X_2, \ldots, \mathrm{d}\pi_p X_n) = 0 \tag{2.3.6}$$

for any $p \in M$ and $X_2, \ldots, X_n \in T_pM$, to show that $\mathrm{i}_Y(\pi^*\nu) = 0$. To establish $\mathscr{L}_Y(\pi^*\nu) = 0$, use Cartan's formula $\mathscr{L}_Y = \mathrm{d}\,\mathrm{i}_Y + \mathrm{i}_Y\,\mathrm{d}$ to see that $\mathrm{d}\,\mathrm{i}_Y(\pi^*\nu) = 0$ by (2.3.6) and that $\mathrm{i}_Y\,\mathrm{d}(\pi^*\nu) = \mathrm{i}_Y(\pi^*\mathrm{d}\nu) = 0$ due to a similar calculation to (2.3.6). □

Since π^* can be viewed as a chain map $\pi^*\colon \Omega(N) \longrightarrow \Omega(M)$, we immediately have the following corollary:

Corollary 2.3.20 *Let $\pi\colon M \longrightarrow N$ be a surjective submersion such that $\pi^{-1}(y)$ is a connected submanifold of M for all $y \in N$. Then π^* induces an isomorphism of chain complexes*

$$\pi^*\colon \Omega(N) \xrightarrow{\sim} \Omega_{\mathrm{bas}}(M).$$

Note that for a connected Lie group G and a G-principal bundle P we obtain the chain isomorphism $\pi^*\colon \Omega(P/G) \xrightarrow{\sim} \Omega_{\mathrm{bas}}(P)$ (Corollary 2.3.20 applies since $\pi^{-1}(y) \cong G$ for all $y \in P/G$). The final ingredients will be principal connections [75]:

Definition 2.3.21 (Principal connection) *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and P a G-principal bundle. A principal connection on P is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ such that*

$$\mathrm{Ad}_g\,\omega = g^*\omega \qquad \text{and} \qquad \omega(X_\xi) = \xi$$

for all $g \in G$ and $\xi \in \mathfrak{g}$.

Given any G-principal bundle $\pi\colon P \longrightarrow P/G$ for a connected Lie group G, we can choose a principal connection $\omega \in \Omega(P) \otimes \mathfrak{g}$ on P (for the existence of principal connections consult e.g. [5, 75]) and define the following map for $k \geq 1$

$$h_\omega\colon \mathrm{S}^k(\mathfrak{g}^*) \otimes \Omega(P) \longrightarrow \mathrm{S}^{k-1}(\mathfrak{g}^*) \otimes \Omega(P)\colon \left(\prod_{i=1}^k p_i\right) \otimes \alpha \longrightarrow \sum_{j=1}^k \prod_{i\neq j} p_i \otimes (p_j(\omega) \wedge \alpha)$$

where $p_j(\omega)$ denotes the application of $p_j \in \mathfrak{g}^*$ to the second tensor factor of $\omega \in \Omega(P) \otimes \mathfrak{g}$. The significance of h_ω will become clear, once we consider the following augmentation of the chain complex of equivariant differential forms $\widetilde{\Omega_G}(P)$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega_G^{0,0}(P) \arrow[r, "\mathrm{d}"] \arrow[u, "\mathrm{i}_\bullet"] & \Omega_G^{0,1}(P) \arrow[r, "\mathrm{d}"] \arrow[u, "\mathrm{i}_\bullet"] & \Omega_G^{0,2}(P) \arrow[r, "\mathrm{d}"] \arrow[u, "\mathrm{i}_\bullet"] & \cdots \\
 & \Omega_{\mathrm{bas}}^0(P) \arrow[r, "\mathrm{d}"] \arrow[u, hook] & \Omega_{\mathrm{bas}}^1(P) \arrow[r, "\mathrm{d}"] \arrow[u, hook] & \Omega_{\mathrm{bas}}^2(P) \arrow[r, "\mathrm{d}"] \arrow[u, hook] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the bottom row maps are the canonical inclusion of $\Omega^i_{\mathrm{bas}}(P) = \ker \mathrm{i}_\bullet$ into $\Omega^{0,i}_G(P) = \Omega^i(P)^G$. With the help of h_ω we can now show that the columns of this complex are exact [103]:

Lemma 2.3.22 *Let G be a connected Lie group and P a G-principal bundle. For all $n \geq 0$ h_ω is a homotopy of the rows $C^k := \Omega^{k,n}_G(P)$ (with differential $\mathrm{i}_\bullet$).*

PROOF: We have to establish two facts. First, that h_ω is an equivariant map and second that it is indeed a homotopy. We will only prove the lemma for $k = 1$ since all other cases essentially reduce to it. Let us begin with the equivariance. For all $p \otimes \alpha \in C^1$ we have

$$\begin{aligned} h_\omega(g \triangleright p \otimes \alpha) &= h_\omega\big(\mathrm{Ad}^*_g p \otimes (g^{-1})^*\alpha\big) = \big(\mathrm{Ad}^*_g p\big)(\omega) \wedge (g^{-1})^*\alpha = p(\mathrm{Ad}_{g^{-1}} \omega) \wedge (g^{-1})^*\alpha \\ &= p\big((g^{-1})^*\omega\big) \wedge (g^{-1})^*\alpha = (g^{-1})^*(p(\omega) \wedge \alpha) = g \triangleright h_\omega(p \otimes \alpha). \end{aligned}$$

As for the homotopy property, we notice that

$$\mathrm{i}_\xi\, p(\omega) = p(\mathrm{i}_\xi\, \omega) = p(\xi)$$

to finally show

$$\mathrm{i}_\bullet\, h_\omega(p \otimes \alpha) = \mathrm{i}_\bullet\, p(\omega) \wedge \alpha = (\mathrm{i}_\bullet\, p(\omega)) \wedge \alpha - p(\omega) \wedge \mathrm{i}_\bullet\, \alpha = p \otimes \alpha - h_\omega(\mathrm{i}_\bullet(p \otimes \alpha)). \qquad \square$$

Using Lemma 2.3.22 it is clear that also the augmented complex $\widetilde{\Omega_G}(P)$ has exact rows since the bottom row was precisely defined to be the kernel of $\mathrm{i}_\bullet$. We can now employ a standard argument for double complexes with exact columns (see e.g. [20]) to prove that the total cohomology of $\Omega_G(P)$, which is nothing but the equivariant cohomology in the Cartan model, is isomorphic to the cohomology of the bottom row complex of the augmented complex $\widetilde{\Omega_G}(P)$, explicitly given by $\Omega_{\mathrm{bas}}(P)^G$ with differential d, see [24, 55, 69, 96]

Theorem 2.3.23 *Let G be a finite-dimensional, connected Lie group with Lie algebra $\mathfrak{g}$ and P a G-principal bundle. Then*

$$\mathrm{H}_{\mathfrak{g}}(P) \cong \mathrm{H}_G(P) \cong \mathrm{H}(\Omega_{\mathrm{bas}}(P), \mathrm{d}).$$

PROOF: First note that, since G is connected, $\mathrm{H}_{\mathfrak{g}}(P) \cong \mathrm{H}_G(P)$ holds automatically. The main idea is then to find for any $[\alpha]_G \in \mathrm{H}_G(P)$ a representative $\alpha' \in [\alpha]_G$ such that $\alpha' \in \Omega_{\mathrm{bas}}(P)$, by successively reducing the symmetric degree of α using the exactness of the columns of $\Omega^{\bullet,\bullet}_G(P)$. To begin, we assume that α has total degree k and remember from Definition 2.3.14 that we can write

$$\alpha = \sum_{2i+j=k} \alpha_{i,j} \qquad \text{with} \qquad \alpha_{i,j} \in \big[\mathrm{S}^i(\mathfrak{g}^*) \otimes \Omega^j(P)\big]^G.$$

In order for α to define a class $[\alpha]_G \in \mathrm{H}_G(P)$ we obviously need α to be d_G-closed. In detail, d_G-closedness is equivalent to the following chain of equations:

$$\mathrm{d}\alpha_{i,j} = \mathrm{i}_\bullet\, \alpha_{i-1,j+2}$$

(with the express convention $\alpha_{i,j} = 0$ for $i < 0$ or $j < 0$). We can visualize those equations for the example $\alpha = \alpha_{2,0} + \alpha_{1,2} + \alpha_{0,4}$ as follows

$$
\begin{tikzcd}
0 & & & \\
\alpha_{2,0} \arrow[u, "\mathrm{i}_\bullet"] \arrow[r, "\mathrm{d}"] & {} & & \\
& \alpha_{1,2} \arrow[u, "\mathrm{i}_\bullet"] \arrow[r, "\mathrm{d}"] & {} & \\
& & \alpha_{0,4} \arrow[u, "\mathrm{i}_\bullet"] \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}
$$

Since the total degree of α is finite, there must be a constituent $\alpha_{n,m}$ with maximal symmetric degree $n \in \mathbb{N}_0$ such that $\alpha_{n,m} \neq 0$. By d_G-closedness of α we can immediately conclude that $\mathrm{i}_\bullet\, \alpha_{n,m} = 0$. But then, by exactness of the columns (see Lemma 2.3.22) there must be a $\beta_{n-1,m+1} \in [\mathrm{S}^{n-1}(\mathfrak{g}^*) \otimes \Omega^{m+1}(P)]^G$ such that $\mathrm{i}_\bullet\, \beta_{n-1,m+1} = \alpha_{n,m}$. Now consider $\alpha^1 := \alpha - \mathrm{d}_G \beta_{n-1,m+1}$. Clearly we have $[\alpha]_G = [\alpha^1]_G \in \mathrm{H}^k_G(P)$. However, by construction, the symmetric degree n part of α^1 vanishes. Repeating this process (possibly less than) $n-1$ times yields $\alpha^n \in \Omega^k(P)$ with $[\alpha]_G = [\alpha^n]_G \in \mathrm{H}^k_G(P)$. The d_G-closedness condition for α^n now reads $\mathrm{d}\alpha^n = 0$ and $\mathrm{i}_\bullet\, \alpha^n = 0$. The second equation tells us that $\alpha^n \in \ker \mathrm{i}_\bullet \subseteq \Omega^k(P)^G$. Using the fact that the invariance of α^n with respect to the action of G implies that $\mathscr{L}_\xi \alpha^n = 0$ for all $\xi \in \mathfrak{g}$, we can immediately conclude that $\alpha^n \in \Omega^k_{\mathrm{bas}}(P)$. Conversely, every d-closed $\alpha \in \Omega^k_{\mathrm{bas}}(P)$ defines a class $[\alpha]_G \in \mathrm{H}^k_G(P)$, concluding this proof. $\square$

Remark 2.3.24 Since in Lemma 2.3.22 we not only showed that the rows of $\Omega_G^{\bullet,\bullet}$ are exact, but even found a homotopy h_ω, we can explicitly construct a bijection $\mathrm{H}_G(P) \longrightarrow \mathrm{H}(\Omega_{\mathrm{bas}}(P), \mathrm{d})$. Thereto consider the map

$$\phi\colon Z^k_G(P) \longrightarrow Z^k_G(P)\colon \alpha \longmapsto \alpha - \mathrm{d}_{\mathfrak{g}} h_\omega(\alpha).$$

on the d_G-closed equivariant differential forms $Z_G(P)$. Obviously $\phi = \mathsf{id}_{\mathrm{H}^k_G(P)}$ on $\mathrm{H}^k_G(P)$. However, ϕ does implement the reduction of symmetric degree in the proof of Theorem 2.3.23 on representatives (the only difference is that ϕ possibly also alters parts of nonmaximal symmetric degree). Finally, define the map

$$\Phi := \prod_{i=1}^{\infty} \phi\colon Z^k_G(P) \longrightarrow Z^k_{\mathrm{bas}}(P)$$

into the d-closed, basic differential forms $Z_{\mathrm{bas}}(P)$. There are no convergence problems since on $Z^k_G(P)$ the infinite concatenation $\prod_{i=1}^{\infty} \phi$ stabilizes after at most k applications of ϕ. And again, $\Phi = \mathsf{id}_{\mathrm{H}^k_G(P)}$ if viewed as $\Phi\colon \mathrm{H}^k_G(P) \longrightarrow \mathrm{H}^k_G(P)$.

2.4 The Homological Perturbation Lemma

As a last tool of the trade, before coming to deformation quantization, we shall provide a brief exposition on the homological perturbation lemma. Roughly, the thrust of the homological perturbation lemma is, that, given two homotopy equivalent chain complexes (of certain kinds), one can "perturb" the differential of one of them by a "small" amount, such that there exists a perturbed differential on the second chain complex and that those two new chain complexes are again homotopy equivalent. Even more, the homological perturbation lemma provides explicit formulas for the perturbed objects. In this exposition, we will follow [29], other applications and references can be found e.g. in [54, 67, 70, 81]. We begin by defining homotopy equivalence data, deformation retracts and special deformation retracts:

Definition 2.4.1 (Homotopy equivalence data) *A homotopy equivalence data (or HE data) consists of two chain complexes (C, d_C) and (D, d_D) over a commutative ring R together with two quasi-isomorphisms*

$$i\colon (C, \mathrm{d}_C) \longrightarrow (D, \mathrm{d}_D) \qquad \text{and} \qquad p\colon (D, \mathrm{d}_D) \longrightarrow (C, \mathrm{d}_C)$$

and a homotopy

$$h\colon D \longrightarrow D \qquad \text{with} \qquad \mathsf{id}_D - ip = \mathrm{d}_D h + h \mathrm{d}_D$$

between id_D and ip.

To shorten notation, we will throughout summarize any given HE data, such as in Definition 2.4.1, as

$$i : (C, \mathrm{d}_C) \leftrightarrows (D, \mathrm{d}_D) : p,\ h.$$

Definition 2.4.2 (Deformation retract) *A deformation retract (or DR) is a HE data, such that additionally*

$$pi = \mathsf{id}_C$$

holds.

Definition 2.4.3 (Special deformation retract) *A special DR is a deformation retract, such that additionally*

$$hi = 0, \qquad ph = 0 \qquad \text{and} \qquad h^2 = 0$$

hold.

Remark 2.4.4 Note, that in Definition 2.4.1 we are using a slightly different sign convention compared to [29]. One can however easily pass from one to the other by replacing h with $-h$. We chose this convention purely to better fit the application of the homological perturbation lemma in Section 4.2.

Given any HE data (we will throughout use the notation from Definition 2.4.1), we will say that any linear map $\delta\colon D \longrightarrow D$ with $\deg \delta = \deg \mathrm{d}_D$ and $(\mathrm{d}_D + \delta)^2 = 0$ is a perturbation of the HE data. We will say that the perturbation δ is small, if $(\mathsf{id}_D + \delta h)$ is invertible. One particular case of perturbations that we will encounter later on is the following: Given any HE data, consider the new HE data

$$(C[\![\nu]\!], \mathrm{d}_C) \rightleftarrows (D[\![\nu]\!], \mathrm{d}_D),\ h \tag{2.4.1}$$

where d_C, d_D, i, p and h are extended ν-linearly. Clearly $C[\![\nu]\!]$ and $D[\![\nu]\!]$ come equipped with a filtration

$$\mathcal{F}^p C[\![\nu]\!] := \nu^p C[\![\nu]\!] \qquad \text{and} \qquad \mathcal{F}^p D[\![\nu]\!] := \nu^p D[\![\nu]\!].$$

Then, if we modify d_D by adding any map δ of the same degree that strictly increases the filtration degree, that is

$$\delta\colon \mathcal{F}^p D^q[\![\nu]\!] \longrightarrow \mathcal{F}^{p+1} D^{q+\deg \mathrm{d}_D}[\![\nu]\!],$$

we see clearly that $\mathsf{id}_D + \delta h$ is invertible as a formal power series by the von Neumann formula (since h is ν-homogeneous, δh strictly increases the filtration degree). Thus we obtained a small perturbation of the initial HE data (after extending as power series). Regardless of the specifics of the small perturbation, the homological perturbation lemma states, that the perturbed HE data is again a HE data [29]:

Lemma 2.4.5 (Homological perturbation lemma) *Given any HE data*

$$i : (C, \mathrm{d}_C) \leftrightarrows (D, \mathrm{d}_D) : p, \ h$$

and any small perturbation δ of d_D, then the following perturbed data

$$I : (C, \mathfrak{d}_C) \leftrightarrows (D, \mathfrak{d}_D) : P, \ H$$

with

$$\begin{array}{rclcrclcrcl} A & := & (\mathsf{id}_D + \delta h)^{-1}\delta & \quad & \mathfrak{d}_D & := & \mathrm{d}_D + \delta & \quad & \mathfrak{d}_C & := & \mathrm{d}_C + pAi \\ I & := & i - hAi & & P & := & p - pAh & & H & := & h - hAh \end{array}$$

is again a HE data.

Proof: We shall here, mostly for convenience, repeat the proof of the homological perturbation lemma from [29], adapted to our sign convention and notation. We will drop most indices, such as the ones of d_D and d_C, to simplify notation. To prepare, we will need three auxiliary results, the first of which is

$$\delta h A = A h \delta = \delta - A, \tag{2.4.2}$$

that holds due to

$$\begin{aligned} (\mathsf{id} + \delta h)A = \delta \quad &\Rightarrow \quad A + \delta h A = \delta \\ &\Rightarrow \quad \delta h A = \delta - A \end{aligned}$$

and

$$\begin{aligned} \delta h \delta = (\mathsf{id} + \delta h)\delta - \delta \quad &\Rightarrow \quad (\mathsf{id} + \delta h)^{-1}\delta h \delta = \delta - (\mathsf{id} + \delta h)^{-1}\delta = \delta - A \\ &\Rightarrow \quad A \delta h = \delta - A. \end{aligned}$$

The second result explicitly gives the inverse of $(\mathsf{id} + \delta h)$ and $(\mathsf{id} + h\delta)$ as

$$(\mathsf{id} + \delta h)^{-1} = \mathsf{id} - Ah \qquad \text{and} \qquad (\mathsf{id} + h\delta)^{-1} = \mathsf{id} - hA, \tag{2.4.3}$$

what can be proven by explicitly calculating all four possible concatenations while using (2.4.2):

$$\begin{array}{rclclcl} (\mathsf{id} + \delta h)(\mathsf{id} - Ah) & = & \mathsf{id} - \delta h A h - A h + \delta h & = & \mathsf{id} + (\delta - A - \delta h A)h & = & \mathsf{id} \\ (\mathsf{id} - Ah)(\mathsf{id} + \delta h) & = & \mathsf{id} + \delta h - Ah - A h \delta h & = & \mathsf{id} + (\delta - A - A h \delta)h & = & \mathsf{id} \\ (\mathsf{id} + h\delta)(\mathsf{id} - hA) & = & \mathsf{id} - hA + h\delta - h\delta h A & = & \mathsf{id} + h(\delta - A - \delta h A) & = & \mathsf{id} \\ (\mathsf{id} - hA)(\mathsf{id} + h\delta) & = & \mathsf{id} + h\delta - hA - hAh\delta & = & \mathsf{id} + h(\delta - A - A h \delta) & = & \mathsf{id} \end{array}$$

For the last relation

$$AipA + A\mathrm{d} + \mathrm{d}A = 0 \tag{2.4.4}$$

we will make use of both (2.4.2) and (2.4.3):

$$\begin{aligned} AipA + A\mathrm{d} + \mathrm{d}A &= A(\mathsf{id} - \mathrm{d}h - h\mathrm{d})A + A\mathrm{d} + \mathrm{d}A \\ &= A^2 + A\mathrm{d}(\mathsf{id} - hA) + (\mathsf{id} - Ah)\mathrm{d}A \\ &= (\mathsf{id} + \delta h)^{-1}\big[(\mathsf{id} + \delta h)A^2(\mathsf{id} + h\delta) + (\mathsf{id} + \delta h)A\mathrm{d} + \mathrm{d}A(\mathsf{id} + h\delta)\big](\mathsf{id} + h\delta)^{-1} \\ &= (\mathsf{id} + \delta h)^{-1}[(A + \delta hA)(A + Ah\delta) + (A + \delta hA)\mathrm{d} + \mathrm{d}(A + Ah\delta)](\mathsf{id} + h\delta)^{-1} \\ &= (\mathsf{id} + \delta h)^{-1}\big[\delta^2 + \delta\mathrm{d} + \mathrm{d}\delta\big](\mathsf{id} + h\delta)^{-1} = 0. \end{aligned}$$

The main part of the proof consists then in five steps, in which it is shown that the perturbed data is again a HE data. In detail, one needs to prove that $\mathfrak{d}_C$ is a differential, that I and P are chain maps, that H is a homotopy between id and IP and that I and P are quasi-isomorphisms. We will go through these parts in order, by calculating

$$\begin{aligned} \mathfrak{d}_C^2 &= (\mathrm{d} + pAi)(\mathrm{d} + pAi) = \mathrm{d}^2 + \mathrm{d}pAi + pAi\mathrm{d} + pAipAi \\ &= \mathrm{d}pAi + pAi\mathrm{d} + p(-\mathrm{d}A - A\mathrm{d})i = 0 \end{aligned}$$

where we used (2.4.4), as well as the fact that i and p are chain maps, hence $\mathrm{d}p = p\mathrm{d}$ and $i\mathrm{d} = \mathrm{d}i$ hold. Secondly, let us show that I is a chain map:

$$\begin{aligned} I\mathfrak{d}_C - \mathfrak{d}_D I &= (i - hAi)(\mathrm{d} + pAi) - (\mathrm{d} + \delta)(i - hAi) \\ &= i\mathrm{d} + ipAi - hAi\mathrm{d} - hAipAi - \mathrm{d}i + \mathrm{d}hAi - \delta i + \delta hAi \\ &\overset{(2.4.2),(2.4.4)}{=} ipAi - hAi\mathrm{d} - h(-A\mathrm{d} - \mathrm{d}A)i + \mathrm{d}hAi - \delta i + (\delta - A)i \\ &= (ip + h\mathrm{d} + \mathrm{d}h - \mathsf{id})Ai = 0. \end{aligned}$$

Third, P is also a chain map:

$$\begin{aligned} \mathfrak{d}_C P - P\mathfrak{d}_D &= (\mathrm{d} + pAi)(p - pAh) - (p - pAh)(\mathrm{d} + \delta) \\ &= \mathrm{d}p - \mathrm{d}pAh + pAip - pAipAh - p\mathrm{d} - p\delta + pAh\mathrm{d} + pAh\delta \\ &\overset{(2.4.2),(2.4.4)}{=} -\mathrm{d}pAh + pAip - p(-\mathrm{d}A - A\mathrm{d})h - p\delta + pAh\mathrm{d} + p(\delta - A) \\ &= pA(ip + \mathrm{d}h + h\mathrm{d} - \mathsf{id}) = 0. \end{aligned}$$

Fourth, H is a homotopy between id and IP:

$$\begin{aligned} &IP + \mathfrak{d}_D H + H\mathfrak{d}_D - \mathsf{id} \\ &\quad = (i - hAi)(p - pAh) + (\mathrm{d} + \delta)(h - hAh) + (h - hAh)(\mathrm{d} + \delta) - \mathsf{id} \\ &\quad = ip - ipAh - hAip + hAipAh + \mathrm{d}h - \mathrm{d}hAh \\ &\qquad + \delta h - \delta hAh + h\mathrm{d} + h\delta - hAh\mathrm{d} - hAh\delta - \mathsf{id} \\ &\quad = -ipAh - hAip + h(-\mathrm{d}A - A\mathrm{d})h - \mathrm{d}hAh + \delta h - (\delta - A)h + h\delta - hAh\mathrm{d} - h(\delta - A) \\ &\quad = -(ip + \mathrm{d}h + h\mathrm{d} - \mathsf{id})Ah - hA(ip + \mathrm{d}h + h\mathrm{d} - \mathsf{id}) = 0. \end{aligned}$$

To show that I is a quasi-isomorphism is slightly more involved. First, it is clear from $\mathsf{id} - IP = \mathfrak{d}_D H + H\mathfrak{d}_D$ that P is a right-inverse of I on cohomology, hence the induced map $I\colon \mathrm{H}(D, \mathfrak{d}_D) \longrightarrow \mathrm{H}(C, \mathfrak{d}_C)$ is surjective. To see that I is also injective on cohomology, take any $x \in C$ that is $\mathfrak{d}_C$-closed and is mapped under I to 0 on cohomology, that is $I(x)$

is $(\mathrm{d}+\delta)$-exact. Let furthermore $y \in D$ be any element with $\mathfrak{d}_D y = I(x)$. Evaluating $\mathfrak{d}_C = \mathrm{d} + pAi$ on x then yields

$$(\mathrm{d} + pAi)(x) = \mathfrak{d}_C(x) = 0. \tag{2.4.5}$$

Similarly, from the definition of y, we have

$$(i - hAi)(x) = I(x) = (\mathrm{d} + \delta)(y). \tag{2.4.6}$$

An immediate consequence of (2.4.6) is

$$\begin{aligned} \delta i(x) - \delta hAi(x) = \delta\mathrm{d}(y) + \delta^2(y) \quad &\Rightarrow \quad \delta i(x) - (\delta - A)i(x) = -\mathrm{d}\delta(y) \\ &\Rightarrow \quad Ai(x) = -\mathrm{d}\delta(y). \end{aligned}$$

By inserting this identity in (2.4.6) , one can easily observe that $x - p\delta(y)$ is d-closed through

$$0 = \mathrm{d}(x) + pAi(x) = \mathrm{d}(x) - p\mathrm{d}\delta(y) = \mathrm{d}(x - p\delta(y)).$$

On the other hand, reinserting the same identity into (2.4.6) produces

$$\begin{aligned} i(x) = -h\mathrm{d}\delta(y) + \mathrm{d}(y) + \delta(y) \quad &\Rightarrow \quad i(x) = -(\mathsf{id} - ip - \mathrm{d}h)\delta(y) + \mathrm{d}(y) + \delta(y) \\ &\Rightarrow \quad i(x - p\delta(y)) = \mathrm{d}h\delta(y) + \mathrm{d}(y). \end{aligned} \tag{2.4.7}$$

This last identity proves that $i(x - p\delta(y))$ is d-exact. Furthermore, since i is a quasi-isomorphism, there exists $z \in C$ such that $x - p\delta(y) = \mathrm{d}(z)$ holds. This in turn allows us to deduce with the help of the first step in (2.4.7)

$$\begin{aligned} x = p\delta(y) + \mathrm{d}(z) \quad &\Rightarrow \quad i(x) = ip\delta(y) + i\mathrm{d}(z) \\ &\Rightarrow \quad -h\mathrm{d}\delta(y) + \mathrm{d}(y) + \delta(y) = ip\delta(y) + i\mathrm{d}(z) \\ &\Rightarrow \quad (ip - \mathsf{id} + \mathrm{d}h)\delta(y) + \mathrm{d}(y) + \delta(y) = ip\delta(y) + i\mathrm{d}(z) \\ &\Rightarrow \quad \mathrm{d}(h\delta(y) + y - i(z)) = 0. \end{aligned}$$

Again, from i being a quasi-isomorphism, we can assume the existence of $\alpha \in C$ and $\beta \in D$ such that $\mathrm{d}(\alpha) = 0$ and $i(\alpha) = i(z) - (\mathsf{id} + h\delta)(y) - \mathrm{d}(\beta)$ holds. Applying pA to this equation then gives

$$\begin{aligned} pAi(\alpha) &= pA(i(z) - (\mathsf{id} + h\delta)(y) - \mathrm{d}(\beta)) \\ &= pAi(z) - p\delta(y) - pA\mathrm{d}(\beta) \\ &= pAi(z) - p\delta(y) - p(-AipA - \mathrm{d}A)(\beta), \end{aligned}$$

from which $p\delta(y) = pAi(z - \alpha) + pAipA(\beta) + p\mathrm{d}A(\beta)$ follows directly. Finally, we can insert this identity into the defining equation of z, to obtain

$$\begin{aligned} x &= p\delta(y) + \mathrm{d}(z) = pAi(z - \alpha) + pAipA(\beta) + p\mathrm{d}A(\beta) + \mathrm{d}(z) \\ &= \mathrm{d}(z + pA(\beta)) + pAi(z - \alpha + pA(\beta)) \\ &= \mathfrak{d}_C(z + pA(\beta) - \alpha), \end{aligned}$$

concluding that x is $\mathfrak{d}_C$-exact and hence that I is injective on cohomology. But since we established earlier that P is a right-inverse of I on cohomology, we know immediately that P is actually the inverse of I on cohomology. □

Remark 2.4.6 According to [29], perturbations of special deformation retracts are again special deformation retracts. For deformation retracts, however, the perturbed data is again a deformation retract, if and only if

$$p\big(Ah^2A - Ah - hA\big)i = 0$$

holds.

Instead of the full homological perturbation lemma, we will only ever encounter a special situation, namely that the complex C is concentrated in degree 0 and $D_n = 0$ for $n < 0$. This situation can be summarized diagrammatically as

$$\begin{tikzcd}
0 & D_0 \arrow[l] \arrow[r, shift left, "\mathrm{d}_D"] \arrow[d, shift left, "p_0"] & D_1 \arrow[l, shift left, "h_0"] \arrow[r, shift left, "\mathrm{d}_D"] & \ldots \arrow[l, shift left, "h_1"] \\
0 & C_0 \arrow[l] \arrow[u, shift left, "i_0"] & 0 \arrow[l]
\end{tikzcd} \tag{2.4.8}$$

In this case, one can readily observe that, for a small perturbation δ, the corresponding deformed HE data according to Lemma 2.4.5 is

$$I_0 = i_0 \qquad P_0 = p_0 - p_0(\mathsf{id}_D + \delta_1 h_0)^{-1}\delta_1 h_0 \qquad H = h - h(\mathsf{id}_D + \delta h)^{-1}\delta h$$

or, simplified slightly,

$$I_0 = i_0 \qquad P_0 = p_0(\mathsf{id}_D + \delta_1 h_0)^{-1} \qquad H = h(\mathsf{id}_D + \delta h)^{-1}, \tag{2.4.9}$$

where we denoted by $i_0 \colon C_0 \longrightarrow D_0$ the degree 0 component of i and analogously for p, I and P. Of course, all other components of P and I are 0. Also note that, even though deformation retracts are, in general, not preserved under deformation, in the case (2.4.8), the additional condition of $h_0 i_0 = 0$ is sufficient to guarantee that the deformed HE data is again a deformation retract, as can be easily verified by

$$P_0 I_0 = \big[p_0 - p_0(\mathsf{id}_D + \delta_1 h_0)^{-1}\delta_1 h_0\big] i_0 = p_0 i_0 - 0 = \mathsf{id}_C.$$

Chapter 3

Classification of Star Products

As motivated in Section 1.3, deformation quantization is an approach to providing a mathematical rigorous framework to quantize classical systems. The central notion of deformation quantization is that of differential star products, which we repeat for convenience here:

Definition 3.0.7 (Differential star product) *Let (M, ω) be a symplectic manifold with associated Poisson bracket $\{\ ,\ \}$. A differential star product on M is an associative, ν-bilinear product $\star$ on $\mathscr{C}^\infty(M)[\![\nu]\!]$ such that for all $f, g \in \mathscr{C}^\infty(M)[\![\nu]\!]$ the product $f \star g$ can be expanded as*

$$f \star g = fg + \sum_{k=1}^{\infty} \nu^k C_k(f, g)$$

with bidifferential operators C_k for all $k \in \mathbb{N}$ and the following equations hold:

$$1 \star f = f \star 1 = f \qquad \textit{and} \qquad f \star g - g \star f = \nu\{f, g\} + \mathcal{O}(\nu^2).$$

Also remember, that we will use the term star product to refer to differential star products throughout. The overarching goal of this chapter is to detail various classification results of different types of star products. The first such result will be the classification of pure star products in Section 3.2 by the de Rham cohomology. Building on top of that, Section 3.3 shows how invariant star products, star products compatible with a Lie group or Lie algebra action on the underlying manifold, are classified by the invariant de Rham cohomology. Finally, in Section 3.4, we present the contribution by Waldmann and the author to the field in the form of a classification result of equivariant star products [104], invariant star products together with quantum momentum maps, by equivariant cohomology (of the Cartan model). Before approaching these classification results however, we shall display one of the main tools used in the proofs shown here, which is the Fedosov construction of star products on symplectic manifolds.

3.1 The Fedosov Construction

One of the central tools of formal deformation quantization certainly is the Fedosov construction [45] of which we will make extensive use. It provides one with the means to construct star products on symplectic manifolds from very simple input data: a symplectic covariant derivative and a formal series of closed two forms on the symplectic manifold. This fact alone already gives a positive answer to the question whether or star products exist on any symplectic manifold. Of course, we have to guarantee the existence of the input data first, which is subject of the Heß-trick (see e.g. [120]):

Proposition 3.1.1 *Let $\widetilde{\nabla}$ be a torsion-free, covariant derivative on a symplectic manifold (M, ω). Then the covariant derivative defined by*

$$\omega(\nabla_X Y, Z) = \omega\Big(\widetilde{\nabla}_X Y, Z\Big) + \frac{1}{3}\Big(\widetilde{\nabla}_X \omega\Big)(Y, Z) + \frac{1}{3}\Big(\widetilde{\nabla}_Y \omega\Big)(X, Z)$$

is a torsion-free, symplectic, covariant derivative.

One significant problem in deformation quantization on Poisson manifolds stems from the fact that the proof of Proposition 3.1.1 relies on ω being a symplectic form. Even more, in general there is no covariant derivative on a Poisson manifold with respect to which the Poisson tensor is constant. Consequently one cannot use the Fedosov construction and has to resort to more advanced means, as shown by Kontsevich [79] and Tamarkin [115]. For our purposes, the single most important result about the Fedosov construction is that it constructs essentially all star products, meaning that for every (invariant, equivariant) star product there is an (invariantly, equivariantly) equivalent one obtained by the Fedosov construction. This will allow us to reduce problems about equivalence classes of star products to the equivalence classes of Fedosov star products (all those obtained by the Fedosov construction).

Turning towards the construction itself, its main idea can be summarized roughly as follows. Find a suitable $\mathbb{C}[\![\nu]\!]$-module $\mathcal{W} \otimes \Lambda(M)$ that admits a noncommutative, associative algebra structure, and a subalgebra K isomorphic to $\mathscr{C}^\infty(M)[\![\nu]\!]$ as $\mathbb{C}[\![\nu]\!]$-module. A star product on M can then be obtained by pulling back the associative product on K to $\mathscr{C}^\infty(M)[\![\nu]\!]$. Furthermore, we will be able to construct different star products through different choices of K. Throughout this section we will be using notation from and following the exposition in [94] and [120], while many proofs are based on [37]. The original reference is, of course, [45]. To begin, let us define $\mathcal{W} \otimes \Lambda$

$$\mathcal{W} \otimes \Lambda(M) := \prod_{k=0}^{\infty} \big(\mathbb{C} \otimes \Gamma^\infty\big(\mathrm{S}^k T^*M \otimes \Lambda^\bullet T^*M\big)\big)[\![\nu]\!]$$

as the formal power series with values in the cartesian product of the complexified spaces of sections of the tensor product of the symmetric and the exterior vector bundle over the cotangent bundle T^*M of M.

Remark 3.1.2 At first it is not obvious whether $\mathcal{W} \otimes \Lambda(M)$ can be viewed as the (formal power series of) global sections $\Gamma^\infty(W \otimes \Lambda)$ for any vector bundle, since for the "obvious" choice $W \otimes \Lambda = \prod \mathrm{S}^k T^*M \otimes \Lambda T^*M$ the fibres, and in turn the total space, would be clearly infinite-dimensional and one would have to deal with subtleties regarding the smooth structure on $W \otimes \Lambda$. This can be done, as mentioned in [120], however, the whole Fedosov construction operates solely $\mathcal{W} \otimes \Lambda(M)$ and does not require an underlying vector bundle. With the sheaf

$$\mathscr{C}^\infty[\![\nu]\!] \colon U \longmapsto \mathscr{C}^\infty(U)[\![\nu]\!]$$

defined on all open subsets $U \subseteq M$, it will suffice to say that

$$\mathcal{W} \otimes \Lambda \colon U \longmapsto \prod_{k=0}^{\infty} \big(\mathbb{C} \otimes \Gamma^\infty\big(\mathrm{S}^k T^*U \otimes \Lambda^\bullet T^*U\big)\big)[\![\nu]\!]$$

is a $\mathscr{C}^\infty[\![\nu]\!]$-module sheaf of associative algebras on M. Furthermore, since everything will be compatible with restrictions to open subsets, we will frequently refer to $\mathcal{W} \otimes \Lambda(U)$ for any open $U \subseteq M$ just as $\mathcal{W} \otimes \Lambda$ and similarly to $\mathscr{C}^\infty(U)[\![\nu]\!]$ just by $\mathscr{C}^\infty[\![\nu]\!]$.

$\mathcal{W} \otimes \Lambda$ already comes equipped with an associative product: by the Serre-Swan-Theorem [111, 114] we have for any k that $\Gamma^\infty\big(\mathrm{S}^k T^*M \otimes \Lambda^\bullet T^*M\big) \cong \Gamma^\infty\big(\mathrm{S}^k T^*M\big) \otimes \Gamma^\infty(\Lambda^\bullet T^*M)$. Next, note that $\Gamma^\infty(\Lambda^\bullet T^*M)$ is a finitely generated and (again by Serre-Swan) projective $\mathscr{C}^\infty(M)$-module. As such $\bullet \otimes \Gamma^\infty(\Lambda^\bullet T^*M)$ is a right-adjoint functor [83], hence commutes with limits and we have

$$\prod_{k=0}^\infty \big[\Gamma^\infty\big(\mathrm{S}^k T^*M\big) \otimes \Gamma^\infty(\Lambda^\bullet T^*M)\big] \cong \left[\prod_{k=0}^\infty \Gamma^\infty\big(\mathrm{S}^k T^*M\big)\right] \otimes \Gamma^\infty(\Lambda^\bullet T^*M).$$

Both tensor factors on the right hand side already have an associative product: the symmetrized tensor product $\vee$ and the wedge-product $\wedge$ respectively. So in conclusion, we can equip $\mathcal{W} \otimes \Lambda$ with the tensor product of these two, which we will denote by μ. Furthermore, $\mathcal{W} \otimes \Lambda$ already has various natural gradings. Given that any element of $\mathcal{W} \otimes \Lambda$ can be written as sums over factoring tensors $a = (X \otimes \alpha)\nu^k$ with $X \in \mathrm{S}^l T^*M$, $\alpha \in \Lambda^m T^*M$ and $k \in \mathbb{N}_0$ we define the corresponding degree maps by

$$\deg_{\mathrm{s}} a = la \qquad \deg_{\mathrm{a}} a = ma \qquad \deg_\nu a = ka \qquad \mathrm{Deg} = \deg_{\mathrm{s}} + 2\deg_\nu$$

to which we will refer to as the symmetric, the antisymmetric, the formal and the total degree respectively. We will also occasionally denote by $\mathcal{W}^k \otimes \Lambda^l$ the subspace of elements $a \in \mathcal{W} \otimes \Lambda$ with $\deg_{\mathrm{s}} a = ka$ and $\deg_{\mathrm{a}} a = la$. Accordingly, we will denote the projection maps onto symmetric, antisymmetric, formal and total degree k respectively by pr^s_k, pr^a_k, pr^ν_k and Pr_k respectively. Finally, we will say that an element $a \in \mathcal{W} \otimes \Lambda$ is of, or has, symmetric (antisymmetric, formal, total) order k if $pr^s_l a = 0$ ($\mathrm{pr}^a_l a = 0$, $\mathrm{pr}^\nu_l a = 0$, $\mathrm{Pr}_l a = 0$) for all $l < k$ and write $o_s(a) = k$ ($o_a(a) = k$, $o_\nu(a) = k$, $\mathcal{O}(a) = k$) if k is the minimal number such that a is of symmetric (antisymmetric, formal, total) order k. The product μ is clearly graded commutative with respect to the antisymmetric degree. Moreover μ is independent of the symplectic structure on M. Recalling that any star product $\star$ on M is required to have $[f,g]_\star = \nu\{f,g\} + \mathcal{O}(\nu^2)$ and that we aimed to pull back a product on $\mathcal{W} \otimes \Lambda$ to a star product on $\mathscr{C}^\infty(M)[\![\nu]\!]$, μ clearly seems ill-suited for the task. However, we can use μ to define the following noncommutative product on $\mathcal{W} \otimes \Lambda$:

$$a \circ_{\mathrm{F}} b := \mu \circ \exp\Big\{\frac{\nu}{2}\omega^{ij}\, \mathrm{i_s}(\partial_i) \otimes \mathrm{i_s}(\partial_j)\Big\}(a \otimes b), \tag{3.1.1}$$

where $\omega^{ik}\omega_{jk} = \delta^i_j$ and $\mathrm{i_s}(X)a$ denotes the insertion of a vector field $X \in \mathfrak{X}(M)$ into the first component of the symmetric part of $a \in \mathcal{W} \otimes \Lambda$. Here a word of caution seems appropriate. Strictly speaking, $\circ_{\mathrm{F}}$ is only well defined on $\mathcal{W} \otimes \Lambda(U)$ for coordinate charts $U \subseteq M$. However, since only pairings of ω^{ij} and two insertions of vector fields appear in $\circ_{\mathrm{F}}$, it is invariant under coordinate changes. Hence for any covering of M by coordinate charts $\{U_i\}_{i \in I}$ and sections $a, b \in \mathcal{W} \otimes \Lambda(M)$ we have for all $i, j \in I$

$$\Big(a\big|_{U_i} \circ_{\mathrm{F}} b\big|_{U_i}\Big)\Big|_{U_i \cap U_j} = \Big(a\big|_{U_j} \circ_{\mathrm{F}} b\big|_{U_j}\Big)\Big|_{U_i \cap U_j}$$

thus by $\mathcal{W} \otimes \Lambda$ being a sheaf, there exists a global section $s \in \mathcal{W} \otimes \Lambda(M)$ such that

$$s\big|_{U_i} = a\big|_{U_i} \circ_{\mathrm{F}} b\big|_{U_i}$$

holds for all $i \in I$ and we define $a \circ_{\mathrm{F}} b := s$. This definition is clearly independent of the choice of cover of M. In the following we will frequently define objects or perform calculations locally, which can then be extended to $\mathcal{W} \otimes \Lambda(M)$ by similar arguments. Here let us briefly note that $\circ_{\mathrm{F}}$ is $\deg_{\mathrm{a}}$ graded, but neither $\deg_{\mathrm{s}}$ nor $\deg_\nu$ graded. It is however Deg graded: the two insertions of vector fields in the exponent of $\circ_{\mathrm{F}}$ reduce the symmetric degree by 2 and the factor ν raises the formal degree by 1. Since the latter is counted twice in the total degree, the exponent is a map of total degree 0. Other essential parts of the Fedosov construction will be the following operators on $\mathcal{W} \otimes \Lambda$

$$\delta = \mu\big(1 \otimes \mathrm{d}x^i, \bullet\big) \circ \mathrm{i_s}(\partial_i) \qquad \delta^* = \mu\big(\mathrm{d}x^i \otimes 1, \bullet\big) \circ \mathrm{i_a}(\partial_i) \qquad D_\nabla = \mu\big(1 \otimes \mathrm{d}x^i, \bullet\big) \circ \nabla_{\partial_i} \tag{3.1.2}$$

which we will henceforth abbreviate by dropping μ and writing

$$\delta = \big(1 \otimes \mathrm{d}x^i\big)\, \mathrm{i_s}(\partial_i) \qquad \delta^* = \big(\mathrm{d}x^i \otimes 1\big)\, \mathrm{i_a}(\partial_i) \qquad D_\nabla = \big(1 \otimes \mathrm{d}x^i\big)\nabla_{\partial_i}. \tag{3.1.3}$$

Note that the explicit appearance of 1 is essential, since it distinguishes $1 \otimes \mathrm{d}x^i$, where $\mathrm{d}x^i$ is viewed as an element in $\Lambda^1 T^*M$, from $\mathrm{d}x^i \otimes 1$, where $\mathrm{d}x^i$ is viewed as an element in $\mathrm{S}^1 T^*M$. One can easily recognize δ and δ^* to be differentials of $(\mathcal{W} \otimes \Lambda, \circ_{\mathrm{F}})$, which we will briefly demonstrate for δ. First, δ is clearly a $\deg_{\mathrm{a}}$-graded derivation of μ: let $X \otimes \alpha, Y \otimes \beta \in \mathcal{W} \otimes \Lambda$, then

$$\begin{aligned} \delta\mu(X \otimes \alpha, Y \otimes \beta) &= [(\mathrm{i_s}(\partial_i)X) \vee Y + X \vee \mathrm{i_s}(\partial_i)Y] \otimes \mathrm{d}x^i \wedge \alpha \wedge \beta \\ &= \mu(\delta(X \otimes \alpha), Y \otimes \beta) + (-1)^{\deg_{\mathrm{a}} X \otimes \alpha} \mu(X \otimes \alpha, \delta(Y \otimes \beta)). \end{aligned}$$

Since multiplication with $(1 \otimes \mathrm{d}x^i)$ commutes with symmetric insertions and symmetric insertions commute with each other, we clearly have

$$\Big[(\delta \otimes \mathsf{id}), \exp\Big\{\frac{\nu}{2}\omega^{ij}\, \mathrm{i_s}(\partial_i) \otimes \mathrm{i_s}(\partial_j)\Big\}\Big] = 0 \quad \text{and} \quad \Big[(\mathsf{id} \otimes \delta), \exp\Big\{\frac{\nu}{2}\omega^{ij}\, \mathrm{i_s}(\partial_i) \otimes \mathrm{i_s}(\partial_j)\Big\}\Big] = 0,$$

and thus δ is also a $\deg_{\mathrm{a}}$-graded derivation of $\circ_{\mathrm{F}}$. Finally, let us calculate

$$\delta^2(X \otimes \alpha) = \mathrm{i_s}(\partial_i)\, \mathrm{i_s}(\partial_j) X \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \alpha = -\, \mathrm{i_s}(\partial_j)\, \mathrm{i_s}(\partial_i) X \otimes \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \alpha = -\delta^2(X \otimes \alpha)$$

to see $\delta^2 = 0$. Similarly, δ^* can be shown to be a $\deg_{\mathrm{a}}$-graded derivation of μ and $\circ_{\mathrm{F}}$ with $(\delta^*)^2 = 0$. δ and δ^* themselves are also graded maps with respect to $\deg_{\mathrm{s}}$ and $\deg_{\mathrm{a}}$:

$$\begin{aligned} [\deg_{\mathrm{s}}, \delta] &= -\delta & \qquad [\deg_{\mathrm{a}}, \delta] &= \delta \\ [\deg_{\mathrm{s}}, \delta^*] &= \delta^* & \qquad [\deg_{\mathrm{a}}, \delta^*] &= -\delta^* \end{aligned}$$

that is δ has symmetric degree -1 and antisymmetric degree $+1$ while δ^* has symmetric degree $+1$ and antisymmetric degree -1. Instead of δ^* however, we will use a normalized version defined on homogeneous elements $a \in \mathcal{W}^k \otimes \Lambda^l$ as

$$\delta^{-1} a := \begin{cases} \frac{1}{k+l}\delta^* a & \text{for } k + l \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Together with the projection onto symmetric and antisymmetric degree 0

$$\sigma \colon \mathcal{W} \otimes \Lambda \longrightarrow \mathscr{C}^\infty[\![\nu]\!]$$

we can state a small lemma [37, 120]:

Lemma 3.1.3 *The cohomology* $\mathrm{H}^l(\mathcal{W} \otimes \Lambda^\bullet, \delta)$ *of the complex* $\big(\mathcal{W} \otimes \Lambda^l, \delta\big)$ *is given by*

$$\mathrm{H}^0(\mathcal{W} \otimes \Lambda^\bullet, \delta) = \mathscr{C}^\infty[\![\nu]\!] \qquad \textit{and} \qquad \mathrm{H}^l(\mathcal{W} \otimes \Lambda^\bullet, \delta) = 0 \quad \textit{for } l \geq 1.$$

Moreover we have

$$\delta\delta^{-1} + \delta^{-1}\delta = \mathsf{id}_{\mathcal{W}\otimes\Lambda} - \sigma.$$

Proof: We first show the second statement. Let $a = X \otimes \alpha \in \mathcal{W}^k \otimes \Lambda^l$ and calculate

$$\begin{aligned}(\delta\delta^* + \delta^*\delta)a &= \mathrm{i_s}(\partial_i)\big(\mathrm{d}x^j \vee X\big) \otimes \mathrm{d}x^i \wedge \mathrm{i_a}(\partial_j)\alpha + \mathrm{d}x^j \vee \mathrm{i_s}(\partial_i)X \otimes \mathrm{i_a}(\partial_j)\big(\mathrm{d}x^i \wedge \alpha\big)\\ &= \delta_i^j X \otimes \mathrm{d}x^i \wedge \mathrm{i_a}(\partial_j)\alpha + \delta_j^i \mathrm{d}x^j\, \mathrm{i_s}(\partial_i)X \otimes \alpha\\ &\qquad + \mathrm{d}x^j \wedge \mathrm{i_s}(\partial_i)X \otimes \mathrm{d}x^i \wedge \mathrm{i_a}(\partial_j)\alpha + (-1)^1 \mathrm{d}x^j \vee \mathrm{i_s}(\partial_i) \otimes \mathrm{d}x^i \wedge \mathrm{i_a}(\partial_j)\alpha\\ &= (\deg_{\mathrm{s}} + \deg_{\mathrm{a}})a = (l+k)a\end{aligned}$$

where we used

$$\mathrm{d}x^i \vee \mathrm{i_s}(\partial_i)X = \deg_{\mathrm{s}} X = k \qquad \text{and} \qquad \mathrm{d}x^i \wedge \mathrm{i_a}(\partial_i)\alpha = \deg_{\mathrm{a}} \alpha = l.$$

On the other hand, for $a \in \mathcal{W}^0 \otimes \Lambda^0$ we know $\sigma(a) = \mathsf{id}(a)$ and $\delta a = \delta^* a = 0$, thus $\delta\delta^* + \delta^*\delta = \mathsf{id} - \sigma$ follows trivially. Since both δ and δ^* preserve the sum of symmetric and antisymmetric degree, we obtain the statement for δ^{-1}. This shows in particular that δ^{-1} is a contraction of the complex $\mathcal{W} \otimes \Lambda^\bullet$ everywhere except in degree 0, hence it follows immediately that $\mathrm{H}^l = 0$ for $l \geq 1$. For $l = 0$ we know $\mathrm{H}^0(\mathcal{W} \otimes \Lambda^\bullet, \delta) = \ker(\delta\colon \mathcal{W} \otimes \mathscr{C}^\infty[\![\nu]\!] \longrightarrow \mathcal{W} \otimes \Lambda^1)$. Now let $a \in \mathcal{W} \otimes \mathscr{C}^\infty[\![\nu]\!]$. Then $\delta a = 0$ is equivalent to $\mathrm{i_s}(\partial_i)a = 0$ for all i what in turn is equivalent to $\deg_{\mathrm{s}} a = 0$, concluding the proof. □

Remark 3.1.4 Strictly speaking, we are viewing H^l only as a presheaf since we will be interested in the actual quotients $(\ker\delta)(U)/(\operatorname{im}\delta)(U)$ for all open sets $U \subseteq M$ and not the sheafification of the resulting presheaf. However, one can readily observe that H^l already happens to be a sheaf since $\mathscr{C}^\infty[\![\nu]\!]$ is a sheaf and H^l for $l \geq 1$ is the constant trivial sheaf.

Turning to D_∇ from (3.1.2) and (3.1.3), we see immediately that D_∇ is a $\deg_{\mathrm{a}}$-graded derivation of μ since the covariant derivative ∇_{∂_i} is a derivation of $\otimes$, $\wedge$ and $\vee$. The following multiplication with $(1 \otimes \mathrm{d}x^i)$ is then responsible for the grading, that is

$$D_\nabla \mu(a, b) = \mu(D_\nabla a, b) + (-1)^{\deg_{\mathrm{a}} a} \mu(a, D_\nabla b)$$

for $a, b \in \mathcal{W} \otimes \Lambda$. Furthermore, D_∇ obviously has symmetric, total and formal degree 0. With a little auxiliary calculation for $\alpha \in \Lambda^1$ with local expression $\alpha = \alpha_i \mathrm{d}x^i$ and the Christoffel-symbols Γ_{ij}^k corresponding to ∇

$$(1 \otimes \mathrm{d}x^i)\nabla_{\partial_i}\alpha = \mathrm{d}x^i \wedge \left(\frac{\partial \alpha_i}{\partial x^i} - \alpha_k \Gamma_{ji}^k\right)\mathrm{d}x^j = \mathrm{d}\alpha,$$

(what can be extended to all differential forms since both sides are derivations of $\wedge$ and ΛT^*M is generated by one-forms) we can find a more approachable expression for D_∇ acting on $X \otimes \alpha \in \mathcal{W} \otimes \Lambda$ [120]:

$$D_\nabla(X \otimes \alpha) = \nabla_{\partial_i} X \otimes \mathrm{d}x^i \wedge \alpha + X \otimes \mathrm{d}x^i \wedge \nabla_{\partial_i}\alpha = \nabla_{\partial_i} X \otimes \mathrm{d}x^i \wedge \alpha + X \otimes \mathrm{d}\alpha.$$

Using the curvature tensor $\hat{R}$ of ∇_{∂_i} defined by

$$\hat{R}(u,v)w := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

for any vector fields $z, u, v, w \in \Gamma^\infty(TM)$ and its contraction R with the symplectic form ω

$$R(z,u,v,w) := \omega\Big(z, \hat{R}(u,v)w\Big),$$

which is an element of $\Gamma^\infty(\mathrm{S}^2 T^*M \otimes \Lambda^2 T^*M) \subset \mathcal{W} \otimes \Lambda$, we will cite the essential properties of D_∇ from [120, Prop. 6.4.10]:

Proposition 3.1.5 *D_∇ is a $\deg_{\mathrm{a}}$-graded derivation of $\circ_{\mathrm{F}}$:*

$$D_\nabla(a \circ_{\mathrm{F}} b) = (D_\nabla a) \circ_{\mathrm{F}} b + (-1)^{\deg_{\mathrm{a}} a} a \circ_{\mathrm{F}} D_\nabla b \qquad \textit{for all } a \in \mathcal{W} \otimes \Lambda^k, b \in \mathcal{W} \otimes \Lambda$$

Furthermore, the following equations hold

$$\delta R = D_\nabla R = 0 \qquad\qquad [\delta, D_\nabla] = 0 \qquad\qquad D_\nabla^2 = \frac{1}{2}[D_\nabla, D_\nabla] = -\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(R)$$

where $[\ ,\ ]$ denotes the $\deg_{\mathrm{a}}$-graded commutator.

The central idea of Fedosov [45] was then to use δ and D_∇ to construct a new derivation on $\mathcal{W} \otimes \Lambda$ that is a differential. We will later recognize that its kernel is isomorphic (as a $\mathbb{C}[\![\nu]\!]$-module) to $\mathscr{C}^\infty[\![\nu]\!]$ which will allow us to pull back $\circ_{\mathrm{F}}$ to $\mathscr{C}^\infty[\![\nu]\!]$ (since kernels of derivations are subalgebras). For the actual theorem, we will need the following filtration on $\mathcal{W} \otimes \Lambda$: define $\mathcal{W}_k := \{a \in \mathcal{W} \mid \operatorname{Deg} a \geq k\}$. Then we have

$$\mathcal{W} = \mathcal{W}_0 \supseteq \mathcal{W}_1 \supseteq \cdots \supseteq \{0\} \qquad \text{and} \qquad \bigcap_{k=0}^{\infty} \mathcal{W}_k = \{0\}$$

$$\mathcal{W} \otimes \Lambda = \mathcal{W}_0 \otimes \Lambda \supseteq \mathcal{W}_1 \otimes \Lambda \supseteq \cdots \supseteq \{0\} \qquad \text{and} \qquad \bigcap_{k=0}^{\infty} \mathcal{W}_k \otimes \Lambda = \{0\}.$$

Theorem 3.1.6 (Fedosov) *Let $\Omega \in \nu Z^2[\![\nu]\!]$ be a series of closed two forms. Then there exists a unique $r \in \mathcal{W}_2 \otimes \Lambda^1$ with*

$$r = \delta^{-1}\bigg(D_\nabla r - \frac{1}{\nu} r \circ_{\mathrm{F}} r + R + 1 \otimes \Omega\bigg). \tag{3.1.4}$$

The Fedosov derivation $\mathfrak{D}$ defined as

$$\mathfrak{D} = -\delta + D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)$$

is a differential, that is $\mathfrak{D}^2 = 0$, and has antisymmetric degree 1.

PROOF: We will sketch the proof from [37, Thm. 2]. First, note that $\mathcal{W} \otimes \Lambda$ can be viewed as a metric space, if equipped with the valuation

$$\|a\| := \begin{cases} 2^{-\max_{k \in \mathbb{N}_0}\{a \in \mathcal{W}_k \otimes \Lambda\}} & \text{for } a \neq 0 \\ 0 & \text{for } a = 0 \end{cases} \tag{3.1.5}$$

and the induced metric $d(a,b) := \|b-a\|$. By counting the involved degrees, one then finds that L defined as

$$L := \delta^{-1}\left(D_\nabla \bullet - \frac{1}{\nu}(\bullet \circ_{\mathrm{F}} \bullet) + R + 1 \otimes \Omega\right) \colon \mathcal{W}_2 \otimes \Lambda^1 \longrightarrow \mathcal{W}_2 \otimes \Lambda^1$$

is a contracting map with respect to d, what enables us to employ the Banach fixed point theorem [6] to obtain a unique solution of $La = a$, which proves the first part. For the second part, we have to calculate $\mathfrak{D}^2$:

$$\begin{aligned}
2\mathfrak{D}^2 &= [\mathfrak{D}, \mathfrak{D}] = \\
&= -[\delta, D_\nabla] + \left[\delta, \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right] + [D_\nabla, D_\nabla] - \left[D_\nabla, \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right] + \left[\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r), \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right] \\
&\quad - [D_\nabla, \delta] + \left[\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r), \delta\right] - \left[\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r), D_\nabla\right] \\
&= \frac{1}{\nu}\left(\operatorname{ad}_{\circ_{\mathrm{F}}}(2\delta r) - \operatorname{ad}_{\circ_{\mathrm{F}}}(2D_\nabla r) + 2\operatorname{ad}_{\circ_{\mathrm{F}}}(R) - \operatorname{ad}_{\circ_{\mathrm{F}}}\left(\frac{1}{\nu}[r,r]\right)\right) \\
&= 2\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}\left(\delta r - D_\nabla r + R - \frac{1}{\nu} r \circ_{\mathrm{F}} r\right) \\
&= 2\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}\left(\delta r - D_\nabla r + R - \frac{1}{\nu} r \circ_{\mathrm{F}} r + 1 \otimes \Omega\right).
\end{aligned} \tag{3.1.6}$$

Here we used that $1 \otimes \Omega$ is $\circ_{\mathrm{F}}$-central. Now consider $C := \delta r - D_\nabla r + R - \frac{1}{\nu} r \circ_{\mathrm{F}} r + 1 \otimes \Omega$. Clearly $\sigma(C) = 0$ (all parts have antisymmetric degree of at least 1), and hence we have from Lemma 3.1.3

$$C = \left(\delta\delta^{-1} + \delta^{-1}\delta\right)C = \delta^{-1}\delta C, \tag{3.1.7}$$

since we can calculate with $\delta^{-1} r = 0$, that

$$\begin{aligned}
\delta^{-1} C &= \delta^{-1}\left(\delta r - D_\nabla r + R - \frac{1}{\nu} r \circ_{\mathrm{F}} r + 1 \otimes \Omega\right) \\
&= \delta\delta^{-1} r - r + \delta^{-1}\left(D_\nabla r + R - \frac{1}{\nu} r \circ_{\mathrm{F}} r + 1 \otimes \Omega\right) = 0.
\end{aligned}$$

Finally, applying δ to C yields

$$\begin{aligned}
\delta C &= \delta^2 r - \delta D_\nabla r + \delta R - \frac{1}{\nu}\delta(r \circ_{\mathrm{F}} r) + \delta(1 \otimes \Omega) = \left(D_\nabla + \frac{1}{\nu}\operatorname{ad}(r)\right)\delta r \\
&= \left(D_\nabla + \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\left(C + D_\nabla r - R + \frac{1}{\nu} r \circ_{\mathrm{F}} r - 1 \otimes \Omega\right).
\end{aligned}$$

In other words, δ acts on C just as the operator

$$K \colon \mathcal{W}_1 \otimes \Lambda \longrightarrow \mathcal{W}_1 \otimes \Lambda \colon a \longmapsto \left(D_\nabla + \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\left(a + D_\nabla r - R + \frac{1}{\nu} r \circ_{\mathrm{F}} r - 1 \otimes \Omega\right).$$

Consequently we can write (3.1.7) as

$$C = \delta^{-1} K C.$$

The significance thereof is that δ^{-1} has total degree 1 while D_∇ and $\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)$ do not decrease the total degree. Hence $\delta^{-1}K$ is contracting with respect to d and we can once again use the Banach fixed point theorem to conclude that (3.1.7) has a unique solution. Of course, we would like $C = 0$ to be the unique solution, as we could then conclude that $\mathfrak{D}^2 = \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(0) = 0$. To that end we calculate with the help of Proposition 3.1.5 (see also [120])

$$\begin{aligned}K(0) &= \left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\left(D_\nabla r - R + \frac{1}{\nu} r \circ_{\mathrm{F}} r - 1 \otimes \Omega\right)\\ &= -\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(R)r - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)R + \frac{1}{\nu}D_\nabla(r \circ_{\mathrm{F}} r) + \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)D_\nabla r + \frac{1}{\nu^2}[r, r \circ_{\mathrm{F}} r] = 0\end{aligned}$$

where we used that R has antisymmetric degree 2, r has antisymmetric degree 1 and D_∇ increases the antisymmetric degree by one. □

Similar to the situation in Lemma 3.1.3, we again have a chain complex $(\mathcal{W} \otimes \Lambda^\bullet, \mathfrak{D})$ whose cohomology we can compute (compare also [37, Thm. 3], [38]):

Lemma 3.1.7 *The cohomology* $\mathrm{H}(\mathcal{W} \otimes \Lambda^\bullet, \mathfrak{D})$ *of the complex* $(\mathcal{W} \otimes \Lambda^\bullet, \mathfrak{D})$ *is concentrated in degree* 0*, that is*

$$\mathrm{H}^l(\mathcal{W} \otimes \Lambda^\bullet, \mathfrak{D}) = 0 \quad \textit{for } l \geq 1.$$

Moreover we have

$$\mathfrak{D}\mathfrak{D}^{-1}a + \mathfrak{D}^{-1}\mathfrak{D}a = a$$

for all $a \in \mathcal{W} \otimes \Lambda^l$ *with* $l \geq 1$*. Here* $\mathfrak{D}^{-1}(a)$ *is defined as the unique solution of*

$$\mathfrak{D}^{-1}(a) = \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\mathfrak{D}^{-1}(a) - \delta^{-1}a. \tag{3.1.8}$$

Proof: Let $a \in \mathcal{W} \otimes \Lambda$ with $\mathfrak{D}a = 0$, define $b := \mathfrak{D}^{-1}(a)$ for brevity and note that $\delta^{-1}b = 0$, since $(\delta^{-1})^2 = 0$, and $\sigma(b) = 0$ since δ^{-1} increases the symmetric degree. To see that $a = \mathfrak{D}b$ we define $C := a - \mathfrak{D}b$. With the definition of $\mathfrak{D}$ from Theorem 3.1.6 and Lemma 3.1.3 we calculate

$$\begin{aligned}\delta^{-1}C &= \delta^{-1}\left(a - D_\nabla b + \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)b + \delta b\right)\\ &= \delta^{-1}a - \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)b + b = 0.\end{aligned}$$

Furthermore, we clearly have $\sigma(C) = 0$ and, from $\mathfrak{D}C = \mathfrak{D}a - \mathfrak{D}^2 b = 0$, also $\delta C = \big(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\big)C$, showing that C is the unique solution of

$$C = \delta^{-1}\delta C = \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)C,$$

which obviously is $C = 0$. Now for any $A \in \mathcal{W} \otimes \Lambda^k$, $\mathfrak{D}(A)$ is clearly $\mathfrak{D}$-closed with $\deg_{\mathrm{a}} A \geq 1$ and thus $\mathfrak{D}(A) = (\mathfrak{D} \circ \mathfrak{D}^{-1} \circ \mathfrak{D})(A)$ holds. Finally define $B := A - \mathfrak{D}\mathfrak{D}^{-1}(A) - \mathfrak{D}^{-1}\mathfrak{D}(A)$. Note that

$$\begin{aligned}\left(\mathfrak{D}^{-1}\right)^2(\mathfrak{D}A) &= \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\left(\mathfrak{D}^{-1}\right)^2(\mathfrak{D}A)\\ &\quad \underbrace{- \delta^{-1}\left(\delta^{-1}\left(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right)\mathfrak{D}^{-1}(\mathfrak{D}A) - \delta^{-1}(\mathfrak{D}A)\right)}_{0}\end{aligned}$$

showing that $(\mathfrak{D}^{-1})^2(\mathfrak{D}A)$ is a fixed point of $\delta^{-1}\big(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\big)$ of which 0 is the unique one. Also use $\mathfrak{D}^2 = 0$ from Theorem 3.1.6 to show that B is $\mathfrak{D}$-closed:

$$\mathfrak{D}(B) = \mathfrak{D}(A) - \mathfrak{D}^2\mathfrak{D}^{-1}(A) - \mathfrak{D}\mathfrak{D}^{-1}\mathfrak{D}(A) = \mathfrak{D}(A) - \mathfrak{D}(A) = 0.$$

But then we can immediately conclude that $B = \mathfrak{D}\mathfrak{D}^{-1}(B)$ which entails

$$\begin{aligned} B &= \mathfrak{D}\mathfrak{D}^{-1}(B) = \mathfrak{D}\mathfrak{D}^{-1}\big(A - \mathfrak{D}\mathfrak{D}^{-1}(A) - \mathfrak{D}^{-1}\mathfrak{D}(A)\big) \\ &= \mathfrak{D}\mathfrak{D}^{-1}(A) - \mathfrak{D}\mathfrak{D}^{-1}(A) = 0 \end{aligned}$$

and thus $\mathsf{id}_{\mathcal{W}\otimes\Lambda} = \mathfrak{D}^{-1}\mathfrak{D} + \mathfrak{D}\mathfrak{D}^{-1}$ on $\mathcal{W}\otimes\Lambda^l$ with $l \geq 1$. Hence $\mathfrak{D}^{-1}$ is a contraction of $\mathcal{W}\otimes\Lambda^\bullet$ everywhere except in degree 0 and $\mathrm{H}^l(\mathcal{W}\otimes\Lambda^\bullet, \mathfrak{D}) = 0$ for ≥ 1. □

Remark 3.1.8 Upon rearranging (3.1.8) slightly and noticing that $\mathsf{id} - \delta^{-1}\big(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\big)$ is invertible, since δ^{-1} increases the symmetric and hence the total degree by one and $D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)$ does not decrease the total degree, the homotopy operator $\mathfrak{D}^{-1}$ can be written as the following geometric series

$$\mathfrak{D}^{-1} = \frac{1}{\mathsf{id} - \delta^{-1}\big(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\big)}\delta^{-1}.$$

Theorem 3.1.9 (Fedosov) *The Fedosov-Taylor series*

$$\tau\colon \mathscr{C}^\infty[\![\nu]\!] \longrightarrow \mathrm{H}^0(\mathcal{W}\otimes\Lambda^\bullet, \mathfrak{D})\colon f \longmapsto f - \mathfrak{D}^{-1}(1\otimes \mathrm{d}f)$$

is an isomorphism of $\mathbb{C}[\![\nu]\!]$*-modules with inverse* σ.

PROOF: See [37, Thm. 3]. We first note, that, for any $f \in \mathscr{C}^\infty[\![\nu]\!]$, $\tau(f)$ clearly is the unique solution of

$$\tau(f) = \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\right)\tau(f) + f. \tag{3.1.9}$$

This shows immediately that $\sigma(\tau(f)) = f$ since δ^{-1} increases the symmetric degree. Furthermore, we can directly observe that τ is linear, since for $f, g \in \mathscr{C}^\infty[\![\nu]\!]$ and $\lambda \in \mathbb{C}$ we have

$$\tau(f) + \lambda\tau(g) = \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\right)(\tau(f) + \lambda\tau(g)) + (f + \lambda g)$$

and thus $\tau(f) + \lambda\tau(g)$ satisfies the defining fixed point equation of $\tau(f + \lambda g)$. Defining $a \coloneqq \mathfrak{D}\tau(f)$ we can use $\sigma(a) = 0$, $\mathfrak{D}a = 0$, or equivalently $\delta a = D_\nabla a - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)a$, and

$$\delta^{-1}a = \delta^{-1}\left(D_\nabla - \delta - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\right)\tau(f) = \delta^{-1}(D_\nabla - \mathrm{ad}_{\circ_{\mathrm{F}}}(r))\tau(f) - \tau(f) + f = 0,$$

where we again made use of Lemma 3.1.3, to show

$$a = \big(\delta^{-1}\delta + \delta\delta^{-1}\big)f = \delta^{-1}\delta f = \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\right),$$

which once again has the unique solution $a = 0$. Hence we have $\tau(f) \in \ker\mathfrak{D}\cap\mathcal{W} = \mathrm{H}^0(\mathcal{W}\otimes\Lambda^\bullet, \mathfrak{D})$. To conclude the proof, let $a \in \ker\mathfrak{D}\cap\mathcal{W}$. Using $\mathfrak{D}a = 0$ and Lemma 3.1.3

we show

$$\begin{aligned}
\mathfrak{D}a = 0 \quad &\Leftrightarrow \quad \delta a = \left(D_\nabla - \frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(r) \right) a \\
&\Rightarrow \quad \delta^{-1}\delta a = \delta^{-1} \left(D_\nabla - \frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(r) \right) a \\
&\Leftrightarrow \quad a = \sigma(a) + \delta^{-1} \left(D_\nabla - \frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(r) \right) \\
&\Leftrightarrow \quad a = \tau(\sigma(a))
\end{aligned}$$

and thus σ is indeed the inverse of τ. □

Remark 3.1.10 Of course, τ depends on the choice of a connection and on the series of closed two forms $\Omega \in \nu Z^2[\![\nu]\!]$, just as r and $\mathfrak{D}$ from Theorem 3.1.6 do. We will again assume fixed choices throughout. If necessary, we will denote the dependency explicitly by τ_Ω.

Remark 3.1.11 One could have obtained the homotopy operator $\mathfrak{D}^{-1}$ from Lemma 3.1.7 as well as the Fedosov-Taylor series τ from Theorem 3.1.9 by means of the homological perturbation lemma, see Lemma 2.4.5. However, since we will be needing the internal details of the Fedosov construction later on, it is shown explicitly.

Remark 3.1.12 Using the geometric series expression of $\mathfrak{D}^{-1}$ according to Remark 3.1.8, a small calculation shows that the Fedosov-Taylor series can be similarly written as

$$\tau(f) = \frac{1}{\mathsf{id} - \left[\delta^{-1}, D_\nabla - \frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(r)\right]} f$$

for all $f \in \mathscr{C}^\infty[\![\nu]\!]$.

We can briefly summarize most of what has been shown in this chapter so far by stating that σ is a quasi-isomorphism of complexes of presheaves with values in $\mathbb{C}[\![\nu]\!]$-modules and that τ is a quasi-isomorphism from the middle row to the bottom row complex inverse to σ on cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{W} \arrow[r, "\delta"] \arrow[d, "\sigma"'] & \mathcal{W} \otimes \Lambda^1 \arrow[r, "\delta"] & \dots \\
0 \arrow[r] & \mathscr{C}^\infty[\![\nu]\!] \arrow[r] \arrow[d, shift left, "\tau"] & 0 & \\
0 \arrow[r] & \mathcal{W} \arrow[r, "\mathfrak{D}"'] \arrow[u, shift left, "\sigma"] & \mathcal{W} \otimes \Lambda^1 \arrow[r, "\mathfrak{D}"'] & \dots
\end{tikzcd}$$

Using τ and σ we can then finally construct a star product on M, as the following two results show:

Corollary 3.1.13 *The following formula defines an associative product on $\mathscr{C}^\infty[\![\nu]\!]$:*

$$f \star g := \sigma(\tau(f) \circ_{\mathrm{F}} \tau(g)) \qquad \textit{for } f, g \in \mathscr{C}^\infty[\![\nu]\!]$$

PROOF: The Fedosov-Taylor series is an isomorphism $\mathscr{C}^\infty[\![\nu]\!] \longrightarrow \ker \mathfrak{D}\big|_{\mathcal{W}\otimes\Lambda^0}$ with inverse σ by Theorem 3.1.9. Since $\ker \mathfrak{D}\big|_{\mathcal{W}\otimes\Lambda^0}$ is a subalgebra of $(\mathcal{W} \otimes \Lambda, \circ_{\mathrm{F}})$, the associative product $\circ_{\mathrm{F}}$ can be pulled back to $\mathscr{C}^\infty[\![\nu]\!]$ by τ. $\square$

Lemma 3.1.14 *The product from Corollary 3.1.13 is a star product.*

PROOF: See [120, Satz 6.4.22]. We only have to show that $\star$ is, in each order of ν, given as a bidifferential operator, that $1 \star f = f \star 1 = f$ for all $f \in \mathscr{C}^\infty[\![\nu]\!]$ and that the antisymmetric part of the first order in ν is given by the Poisson bracket, according to Definition 3.0.7. First, from the Definition of τ in Theorem 3.1.9 one sees immediately that $\tau(1) = 1$. Furthermore 1 is the neutral element for the product $\circ_{\mathrm{F}}$ on $\mathcal{W} \otimes \Lambda$ and hence the second part follows. For the first part, one only has to recognize that solving all previous fixed point equations with respect to the norm (3.1.5) means solving them order by order in the total degree. Take the defining fixed point equation (3.1.9) of τ and write $L := \delta^{-1}\big(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\big) + f$. Then, by the Banach fixed point theorem [6], the unique solution of $\tau(f) = L\tau(f)$ is given by $\lim_{n\longrightarrow\infty} L^n f$. Because $\delta^{-1}\big(D_\nabla - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)\big)$ strictly increases the total degree, we can calculate all parts of $\tau(f)$ of total degree at most k by $L^k f$. Finally then, by (3.1.1), for any $f, g \in \mathscr{C}^\infty[\![\nu]\!]$ the l-th formal degree part involves only parts of total degree smaller l in both $\tau(f)$ and $\tau(g)$ where $\circ_{\mathrm{F}}$ applies at most l insertions of vector fields. Given that all constituents of L are differential operators of finite order, the l-fold application of L is again a differential operator of finite order and consequently the concatenation with finitely many insertions stays a differential operator of finite order. Hence, in each finite formal degree of $f \star g$ only a finite number of differential operators is applied. To conclude, we have to calculate $f \star g - g \star f$ for arbitrary $f, g \in \mathscr{C}^\infty[\![\nu]\!]$. By the previous discussion as well as the definition of $\circ_{\mathrm{F}}$, we need only calculate the symmetric degree 1 and formal degree 0 part of $\tau(f)$ and $\tau(g)$, which are given by $\delta^{-1}D_\nabla f = \mathrm{d}f \otimes 1$ and $\delta^{-1}D_\nabla g = \mathrm{d}g \otimes 1$ respectively (since r starts in formal degree 1). But then we clearly have

$$f \star g - g \star f = \nu\omega^{ij}\ [\mathrm{i}(\partial_i)\mathrm{d}f]\ [\mathrm{i}(\partial_j)\mathrm{d}g] + \mathcal{O}(\nu^2) = \nu\{f, g\} + \mathcal{O}(\nu^2). \qquad \square$$

Also, let us introduce some notation. First, denote by $\operatorname{Star}(M, \omega)$ the set of all star products on (M, ω). Then the Fedosov construction clearly defines a map

$$F \colon \nu Z^2(M)[\![\nu]\!] \longrightarrow \operatorname{Star}(M, \omega) \tag{3.1.10}$$

and we will occasionally denote by $\star_\Omega := F(\Omega)$ the image of a given series of closed two-forms $\Omega \in \nu Z^2(M)[\![\nu]\!]$. If we want to stress the dependency of F on a symplectic, torsion free connection ∇ on M, we will write $F_\nabla(\Omega)$. An obvious consequence of the Fedosov construction is then:

Corollary 3.1.15 *Every symplectic manifold admits a star product.*

PROOF: Let (M, ω) be a symplectic manifold. We know $0 \in \nu Z^2(M)[\![\nu]\!]$, hence $F(0)$ is a star product. $\square$

3.2 Classification of Star products

As seen in the previous section, the Fedosov construction (3.1.10) provides us with the existence of star products. However, at our current state of knowledge, it is unclear, whether or not the Fedosov construction produces even different star products for different inputs and whether or not we obtain all possible star products on a given symplectic manifold. We shall dedicate this section to finding answers to both of the previous questions. We begin by showing that the Fedosov construction is injective, that is the map (3.1.10) is injective. The proof will consist essentially in tracing how exactly the choice of a series of closed two-forms $\Omega \in \nu Z(M)[\![\nu]\!]$ propagates through the Fedosov construction, order by order in ν (see [120, Lemma 6.4.9]):

Lemma 3.2.1 *Let (M, ω) be a symplectic manifold, $\Omega \in \nu Z^2(M)[\![\nu]\!]$ and $\{C_k\}_{k\in\mathbb{N}}$ the series of bidifferential operators corresponding to the Fedosov star product $\star_\Omega$, that is*

$$f \star_\Omega g = fg + \sum_{k=0}^{\infty} \nu^k C_k(f,g) \qquad \textit{for all } f, g \in \mathscr{C}^\infty(M).$$

Denote further by Ω_k for $k \in \mathbb{N}$ the formal degree k component of Ω. Then

i) C_k is independent of Ω_n for $n \geq k$.

ii) For all $k \in \mathbb{N}$ there exists a bidifferential operator $\bar{C}_{k+1}$ independent of Ω_k such that

$$C_{k+1}(f,g) = -\frac{1}{2}\Omega_k(X_f, X_g) + \bar{C}_{k+1}(f,g)$$

for all $f, g \in \mathscr{C}^\infty(M)$.

Proof: Let us begin with the easier first part. Since all operators used during the Fedosov construction, including τ, $\mathfrak{D}$, $\mathfrak{D}^{-1}$, D_∇ and δ^{-1} are ν-homogeneous, we can show that the lowest order contribution of $\nu^k\Omega_k$ appears in formal degree k: recall the construction of $r \in \mathcal{W}_2 \otimes \Lambda^1$ from Theorem 3.1.6

$$r = \delta^{-1}\left(D_\nabla r - \frac{1}{\nu} r \circ_{\mathrm{F}} r + R + 1 \otimes \Omega\right).$$

Similar to the proof of Lemma 3.1.14, we can note that the operator $\delta^{-1}\big(D_\nabla \bullet - \frac{1}{\nu} \bullet \circ_{\mathrm{F}} \bullet\big)$ on $\mathcal{W}_2 \otimes \Lambda^1$ strictly increases the total degree, hence Ω_k only contributes to total degree $2k+1$ and higher ($\delta^{-1}(1 \otimes \nu^k\Omega_k)$ has total degree $2k+1$). Accordingly, the lowest total degree contribution in $\tau(f)$ and $\tau(g)$ for $f, g \in \mathscr{C}^\infty(M)[\![\nu]\!]$ appears in total degree $2k+1$, as seen from the defining fixed point equation (3.1.9). But then the lowest possible formal degree contributions of Ω_k in $f \star_\Omega g$ come from

$$\sigma(\mathrm{Pr}_{2k+1}\tau(f) \circ_{\mathrm{F}} \mathrm{Pr}_0\tau(g)) \qquad \text{and} \qquad \sigma(\mathrm{Pr}_0\tau(f) \circ_{\mathrm{F}} \mathrm{Pr}_{2k+1}\tau(g)). \tag{3.2.1}$$

Clearly $\mathrm{Pr}_{2k+1}\tau(f)$ is of symmetric order 1 and hence the product with $\mathrm{Pr}_0\tau(g)$ contributes at least one ν via $\circ_{\mathrm{F}}$ (see (3.1.1)), resulting in a term of formal order $k+1$. The same argument applies to $\mathrm{Pr}_0\tau(f) \circ_{\mathrm{F}} \mathrm{Pr}_{2k+1}\tau(g)$. For the second claim, note that the lowest total degree contribution of Ω_k in r is precisely given by $\delta^{-1}(1 \otimes \nu^k\Omega_k)$, that is we can write $\mathrm{Pr}_{2k+1} r = \delta^{-1}(1 \otimes \nu^k\Omega_k) + \bar{r}_{2k+1}$, where $\bar{r}_{2k+1}$ depends only on Ω_l for $l < k$. By the defining fixed point equation of τ from (3.1.9) we can extract explicitly the terms of low total degree in $\tau(f)$ for $f \in \mathscr{C}^\infty(M)$:

$$\tau(f) = f + \mathrm{d}f \otimes 1 + \delta^{-1}\left(D_\nabla - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}(r)\right)(\mathrm{d}f \otimes 1) + \dots$$

From here we can conclude that the lowest contribution of Ω_k appears as $\delta^{-1}\operatorname{ad}_{\circ_{\mathrm{F}}}(\delta^{-1}(1 \otimes \Omega_k))(\mathrm{d}f \otimes 1)$, which we can simplify by a local calculation to

$$\begin{aligned}\operatorname{ad}_{\circ_{\mathrm{F}}}(\delta^{-1}(1 \otimes \Omega_k))(\mathrm{d}f \otimes 1) &= \frac{\nu}{2}\omega^{ij}\,\mathrm{i}_{\mathrm{s}}(\partial_i)(\mathrm{d}x^n \otimes \mathrm{i}(\partial_n)\Omega_k) \cdot \mathrm{i}_{\mathrm{s}}(\partial_j)(\mathrm{d}f \otimes 1) \\ &= \frac{\nu}{2}\omega^{ij}(1 \otimes \mathrm{i}(\partial_i)\Omega_k) \cdot (\omega_{mj}(X_f)^m \otimes 1) \\ &= \frac{\nu}{2} \otimes (X_f)^i\,\mathrm{i}(\partial_i)\Omega_k = \frac{\nu}{2} \otimes \mathrm{i}(X_f)\Omega_k.\end{aligned}$$

It only remains to calculate

$$\begin{aligned}\frac{\nu^{k+1}}{2}\delta^{-1}(1 \otimes \mathrm{i}(X_f)\Omega_k) \circ_{\mathrm{F}} (\mathrm{d}g \otimes 1) &= \frac{\nu^{k+1}}{2}(\mathrm{d}x^n \otimes \mathrm{i}(\partial_n)\,\mathrm{i}(X_f)\Omega_k) \circ_{\mathrm{F}} (\mathrm{d}g \otimes 1) \\ &= \frac{\nu^{k+1}}{4}\omega^{ij}(\delta_i^n \otimes \mathrm{i}(\partial_n)\,\mathrm{i}(X_f)\Omega_k) \cdot (\mathrm{i}(\partial_j)\mathrm{d}g \otimes 1) \\ &= \frac{\nu^{k+1}}{4}\Omega_k(X_f, X_g)\end{aligned}$$

for $f, g \in \mathscr{C}^\infty(M)$ and to conclude

$$\frac{\nu^{k+1}}{2}(\mathrm{d}f \otimes 1) \circ_{\mathrm{F}} \delta^{-1}(1 \otimes \mathrm{i}(X_g)\Omega_k) = \frac{\nu^{k+1}}{4}\Omega_k(X_f, X_g).$$

Again, the previous two terms are the contributions of Ω_k to $\tau(f) \circ_{\mathrm{F}} \tau(g)$ of lowest total degree. Hence we have shown that

$$C_{k+1}(f, g) = -\frac{1}{2}\Omega_k(X_f, X_g) + \text{terms independent of } \Omega_k. \qquad \square$$

We have an immediate corollary to Lemma 3.2.1.

Corollary 3.2.2 *The Fedosov construction on a symplectic manifold* (M, ω)

$$F\colon \nu Z^2(M)[\![\nu]\!] \longrightarrow \mathrm{Star}(M, \omega)$$

is injective.

Proof: Using Lemma 3.2.1 we can reconstruct the series of two-forms $\Omega \in \nu Z^2(M)[\![\nu]\!]$ from a given Fedosov star product $\star_\Omega$. Let $C_k \in \mathrm{DiffOp}^2(\mathscr{C}^\infty(M))$ for $k \geq 1$ be the series of bidifferential operators corresponding to $\star_\Omega$ and let $\Omega^{(0)} := 0 \in \nu Z^2(M)[\![\nu]\!]$. Furthermore, denote by $C_k^{(0)}$ the series of bidifferential operators corresponding to $F\big(\Omega^{(0)}\big)$. Then by Lemma 3.2.1 we have for all $f, g \in \mathscr{C}^\infty(M)$

$$\begin{aligned}C_2(f, g) &= -\frac{1}{2}\Omega_1(X_f, X_g) + \text{terms independent of } \Omega_1 \\ C_2^{(0)}(f, g) &= 0 + \text{terms independent of } \Omega_1^{(0)}\end{aligned}$$

and hence

$$\Omega_1(X_f, X_g) = 2\Big(C_2^{(0)}(f, g) - C_2(f, g)\Big). \tag{3.2.2}$$

Since on a symplectic manifold the Hamiltonian vector fields span the tangent space T_pM at every $p \in M$, we can recover Ω_1 from (3.2.2). Next, assume that Ω_k is known for $k \leq n$ and

$n \in \mathbb{N}$. Define $\Omega^{(n)} := \sum_{k=1}^{n} \nu^k \Omega_k$, construct $F\big(\Omega^{(n)}\big)$ and denote by $C_k^{(n)}$ the corresponding bidifferential operators. Clearly then $C_k = C_k^{(n)}$ for $k \leq n+1$ since both are only dependent on $\{\Omega_k\}_{k \leq n}$ and we can recover Ω_{n+1} from

$$\Omega_{n+1}(X_f, X_g) = 2\Big(C_{n+2}^{(n)}(f,g) - C_{n+2}(f,g)\Big).$$

The above construction gives a left-inverse

$$\operatorname{im} F \longrightarrow \nu Z^2(M)[\![\nu]\!]$$

to $F\colon \nu Z^2(M)[\![\nu]\!] \longrightarrow \operatorname{im} F$, hence F must be injective. $\square$

Unfortunately, the Fedosov construction is, in general, not a bijection, that is it fails to be surjective. To construct a non-Fedosov star product, start from any Fedosov star product $\star$ with corresponding bidifferential operators C_k and define $T := \mathsf{id} + \nu D$ for any differential operator D of order at least two. Note that T is invertible as a formal power series in ν. It is then well-known, see e.g. [18, Thm. 3.4], that the order of C_k in each argument is precisely k (all Fedosov star products are Vey-type star products). However, we can obtain a new associative product $\star'$ on $\mathscr{C}^\infty(M)[\![\nu]\!]$ through $f \star' g := T^{-1}(Tf \star Tg)$ for all $f, g \in \mathscr{C}^\infty(M)$ and calculate its first order in ν as

$$\begin{aligned} T^{-1}(Tf \star Tg) &= fg + \nu[C_1(f,g) + D(f)g + fD(g) + D(fg)] + \mathcal{O}(\nu^2) \\ &=: fg + \nu C_1'(f,g) + \mathcal{O}(\nu^2) \end{aligned}$$

from which we can conclude several facts. First, $\star'$ is again a star product: clearly all orders are differential operators, 1 is a unit of $\star$ and the antisymmetric part of C_1' is identical to the antisymmetric part of C_1, hence $f \star' g - g \star' f = \nu\{f,g\} + \mathcal{O}(\nu^2)$ holds. Secondly, contrary to the original Fedosov star product, the order of differentiation of C_1' is at least two, hence $\star'$ cannot be a Fedosov star product. But despite the Fedosov construction not being surjective, we can show that it is essentially surjective, meaning for every star product $\star$ on a symplectic manifold there exists a Fedosov star product $\star_\Omega$ such that $\star$ and $\star_\Omega$ are equivalent in the following sense:

Definition 3.2.3 (Equivalence of star products) *Let (M, ω) be a symplectic manifold and $\star$, $\star'$ star products on M. We say that $\star$ and $\star'$ are equivalent if there is a formal series of differential operators*

$$T = \mathsf{id}_{\mathscr{C}^\infty(M)} + \sum_{k=1}^{\infty} \nu^k T_k$$

such that the following equation holds for all $f, g \in \mathscr{C}^\infty(M)$:

$$T(f) \star' T(g) = T(f \star g).$$

Definition 3.2.3 implies that T is an isomorphism of algebras

$$T\colon (\mathscr{C}^\infty(M)[\![\nu]\!], \star) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], \star'),$$

since T is clearly invertible as a formal power series and its inverse will again start with id in formal degree 0, as shown by the von Neumann series

$$T^{-1} = (\mathsf{id} - (\mathsf{id} - T))^{-1} = \sum_{k=0}^{\infty} (\mathsf{id} - T)^k.$$

Using the notion of equivalence of star products from Definition 3.2.3, our first goal will be to prove the following proposition from [10] [12, Prop. 5.7]:

Proposition 3.2.4 *Every star product on a symplectic manifold is equivalent to a Fedosov star product.*

PROOF: First, let $\star$ and $\star'$ be arbitrary star products on a symplectic manifold (M, ω) and denote by $\{C_k\}_{k\in\mathbb{N}}$ and $\{C'_k\}_{k\in\mathbb{N}}$ the corresponding series of bidifferential operators. By Definition 3.0.7, $\star$ is an associative product on $\mathscr{C}^\infty(M)[\![\nu]\!]$ and we can expand the equation $(f \star g) \star h = f \star (g \star h)$ in terms of C_k. One can easily check that the resulting equation is

$$(\partial C_k)(f, g, h) = \sum_{\substack{r,s>0\\ r+s=k}} [C_r(C_s(f, g), h) - C_r(f, C_s(g, h))] \tag{3.2.3}$$

where ∂ denotes the Hochschild differential. Now assume that $\star$ and $\star'$ coincide up to order n, that is for all $k \leq n$ we have $C_k = C'_k$. Then by associativity (3.2.3) we see that $C_{n+1} - C'_{n+1}$ is a differential Hochschild cocycle, that is

$$\partial\big(C_{n+1} - C'_{n+1}\big) = 0$$

since the right hand side of (3.2.3) only depends on C_k for $k \leq n$. Using Corollary 2.2.5, there exist $c \in \mathrm{HC}^1_{\mathrm{diff}}(\mathscr{C}^\infty(M))$ and $B \in \mathfrak{X}^2(M)$ with

$$\big(C_{n+1} - C'_{n+1}\big)(f, g) = (\partial c)(f, g) + B(\mathrm{d}f, \mathrm{d}g) \tag{3.2.4}$$

for all $f, g \in \mathscr{C}^\infty(M)$. Next, consider the commutator $[\ ,\ ]_\star$ on $(\mathscr{C}^\infty(M)[\![\nu]\!], \star)$ with respect to the product $\star$. Since $\star$ is associative, the commutator satisfies the Jacobi identity, which we can expand in terms of C_k

$$\mathrm{Cyc}_{f,g,h}[\{\mathcal{A}C_k(f, g), h\} + \mathcal{A}C_k(\{f, g\}, h)] = -\mathrm{Cyc}_{f,g,h}\left[\sum_{r,s>0}^{r+s=k+1} \mathcal{A}C_r(\mathcal{A}C_s(f, g), h)\right],$$

where we denoted by $\mathrm{Cyc}_{f,g,h}$ the sum over all cyclic permutations in f, g, h of the argument and by $\mathcal{A}C_k(f, g) := \frac{1}{2}(C_k(f, g) - C_k(g, f))$ the antisymmetrizer. The commutator with respect to $\star'$ also satisfies the Jacobi identity and by subtracting both equations for $k = n+1$ we can conclude that

$$\mathrm{Cyc}_{f,g,h}[\{f, B(\mathrm{d}g, \mathrm{d}h)\} - B(\mathrm{d}\{f, g\}, \mathrm{d}h)] = 0. \tag{3.2.5}$$

Defining the two-form $B^\flat$ by $B^\flat(X_f, X_g) := B(\mathrm{d}f, \mathrm{d}g)$, one can easily recognize (3.2.5) to be equivalent to $\mathrm{d}B^\flat = 0$. Further, define an equivalence T through $T := \mathsf{id} + \nu^{n+1}c$ with c from (3.2.4) and a star product $\star''$ by $f \star'' g := T^{-1}(T(f) \star' T(g))$. Denoting by $\{C''_k\}$ the corresponding bidifferential operators, we see that $C_k = C'_k = C''_k$ for all $k \leq n$ and

$$C''_{n+1}(f, g) = (\partial c)(f, g) + C'_{n+1}(f, g) = C_{n+1}(f, g) - B^\flat(X_f, X_g).$$

Let us briefly summarize, what we have shown so far: for all star products $\star$ and $\star'$ given by $\{C_k\}_{k\in\mathbb{N}}$ and $\{C'_k\}_{k\in\mathbb{N}}$ such that $C_k = C'_k$ for all $k \leq n$ for a given $n \in \mathbb{N}$, there exists a star product $\star''$ equivalent to $\star'$ given by $\{C''_k\}_{k\in\mathbb{N}}$ such that $C_k = C'_k = C''_k$ for all $k \leq n$ and $C_{n+1}(f, g) = C''_{n+1}(f, g) - B^\flat(X_f, X_g)$ for a d-closed two-form $B^\flat$. All that is left is to combine this result with the Fedosov construction and especially Lemma 3.2.1. Let $\star^{(0)}$ be any star product given by $\left\{C^{(0)}_k\right\}_{k\in\mathbb{N}}$ and let $\star_0 := F(0)$ given by $\{C_{0,k}\}_{k\in\mathbb{N}}$. $\star^{(0)}$ and $\star_0$

coincide up to formal order 0 (all star products do) and hence we can find a star product $\star^{(1)}$ equivalent to $\star^{(0)}$, given by $\left\{C_k^{(1)}\right\}_{k\in\mathbb{N}}$, and a two-form $B_1^\flat$ such that

$$C_1^{(1)}(f,g) = C_{0,1}(f,g) - B_1^\flat(X_f, X_g).$$

Next define $\Omega_1 := 2\nu B_1^\flat$ and $\star_{\Omega_1} := F(\Omega_1)$. But then we clearly have by Lemma 3.2.1

$$C_1^{(1)} = C_{\Omega_1,1},$$

so in conclusion we have found a star product $\star^{(1)}$ equivalent to $\star^{(0)}$ such that the $C^{(1)}$ coincide up to formal degree 1 with a Fedosov star product. Clearly we can iterate this process to obtain a star product $\star^{(\infty)}$ equivalent to $\star^{(0)}$ that coincides in all orders with a Fedosov star product. □

Now that we have recognized that every star product is at least equivalent to a Fedosov star product, the obvious remaining question would ask if we can classify equivalence classes of star products. This question has been asked and answered already conclusively in the very beginnings of deformation quantization, as seen in [12,31,32,46,59,92,93,122]. We split the proof of the main theorem into three parts, beginning with an explicit construction from [94] that will also be necessary later on.

Lemma 3.2.5 *Let (M,ω) be a symplectic manifold and $\Omega, \Psi \in \nu Z^2(M)[\![\nu]\!]$. If $[\Omega - \Psi] = 0 \in \nu\mathrm{H}_{\mathrm{dR}}(M)[\![\nu]\!]$, then $F(\Omega)$ and $F(\Psi)$ are equivalent.*

Proof: We will throughout the proof assume that we have chosen one $C \in \nu\Omega^1(M)[\![\nu]\!]$ with $\Omega = \Psi + \mathrm{d}C$. Our strategy for the proof is to use C to find a unique solution $h \in \mathcal{W}_3$ for the fixed point equation

$$h = C \otimes 1 + \delta^{-1}\left(D_\nabla h - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)h - \frac{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)}{\exp\left\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\right\} - \mathsf{id}}(r' - r)\right) \tag{3.2.6}$$

with $\sigma(h) = 0$, where r and r' are the unique solutions from Theorem 3.1.6 constructed from Ω and Ψ respectively. Similarly, we will denote by $\mathfrak{D}$, $\mathfrak{D}'$, τ and τ' the corresponding Fedosov derivations and Fedosov-Taylor series. Note that $\sigma = \sigma'$, hence no distinction will be necessary here. Existence and uniqueness of h can again be shown by counting involved degrees. We will then show that

$$S_C := \sigma \circ \mathcal{A}_{h(C)} \circ \tau \qquad \text{with} \qquad \mathcal{A}_h := \exp\left\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\right\} \tag{3.2.7}$$

is an equivalence between $F(\Omega)$ and $F(\Psi)$. This construction has been adapted from [94, Lemma 3.5.1], where it was used to construct equivalences between Wick-type star products, to the current setting in [104]. To begin, note that S_C is well-defined, since $h \in \mathcal{W}_3$ and thus $\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)$ strictly increases the total degree. Also, $\mathcal{A}_h$ is clearly invertible with inverse $\mathcal{A}_{-h}$. Finally, $\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)$ is a graded derivation of $\circ_{\mathrm{F}}$, hence $\mathcal{A}_h$ is an automorphism of $(\mathcal{W}\otimes\Lambda, \circ_{\mathrm{F}})$, that is $\mathcal{A}_h(a \circ_{\mathrm{F}} b) = \mathcal{A}_h(a) \circ_{\mathrm{F}} \mathcal{A}_h(b)$ for all $a, b \in \mathcal{W}\otimes\Lambda$. The bulk of the proof will now be to show that $\mathcal{A}_h$ satisfies the following equation:

$$\mathfrak{D}' = \mathcal{A}_h \circ \mathfrak{D} \circ \mathcal{A}_{-h}. \tag{3.2.8}$$

First, let us rewrite (3.2.8) using the explicit form of $\mathcal{A}_h$ from (3.2.7) to obtain

$$\mathfrak{D}' = \mathfrak{D} - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}\left(\frac{\exp\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}}{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)}(\mathfrak{D}h)\right), \tag{3.2.9}$$

where we used the fact that $[\operatorname{ad}_{\circ_{\mathrm{F}}}(h), \mathfrak{D}] = -\operatorname{ad}_{\circ_{\mathrm{F}}}(\mathfrak{D}h)$ holds as $\operatorname{ad}_{\circ_{\mathrm{F}}}(h)$ and $\mathfrak{D}$ are graded derivations of $\circ_{\mathrm{F}}$. From the construction of $\mathfrak{D}$ and $\mathfrak{D}'$ in Theorem 3.1.6 one can immediately recognize that (3.2.9) holds if $r' - r - \frac{\exp\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}}{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)}(\mathfrak{D}h)$ is $\circ_{\mathrm{F}}$-central. A sufficient condition therefor is

$$r' - r - \frac{\exp\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}}{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)}(\mathfrak{D}h) = 1 \otimes C \tag{3.2.10}$$

and we will show that the unique solution from (3.2.6) with $\sigma(h) = 0$ satisfies this equation. To do so, we define

$$B := \frac{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)}{\exp\{\frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}}(r' - r) - \mathfrak{D}h - 1 \otimes C \tag{3.2.11}$$

and will show that B satisfies a fixed point equation whose unique fixed point is 0. Note here that $B = 0$ is clearly equivalent to (3.2.10). At this point we will employ a technical result from [94, Sect. 3.5.1.1] stating that there exists a linear operator $R_{h,r,r'}$ on $\mathcal{W} \otimes \Lambda$ that does not decrease the total degree and satisfies $\mathfrak{D}B = R_{h,r,r'}B$. It is in this calculation, where $\Omega = \Psi + \mathrm{d}C$ is necessary. Expanding the last equation and using Lemma 3.1.3 as well as the definition of $\mathfrak{D}$ from Theorem 3.1.6, we arrive at

$$B = \delta^{-1}\left(D_{\nabla}B - \frac{1}{\nu}\operatorname{ad}_{\circ_{\mathrm{F}}}(r)B - R_{h,r,r'}B\right).$$

By counting degrees, one can clearly see that this fixed point equation has the unique solution $B = 0$. To conclude the proof, note that τ maps $\mathscr{C}^\infty(M)[\![\nu]\!]$ into $\ker\mathfrak{D}$, hence $\mathfrak{D}\tau(f) = 0$ for all $f \in \mathscr{C}^\infty(M)[\![\nu]\!]$. Consequently, using (3.2.8), we see that $\mathfrak{D}'\mathcal{A}_h\tau(f) = \mathcal{A}_h\mathfrak{D}\tau(f) = 0$ and, furthermore, $\mathcal{A}_h\tau(f) = (\tau' \circ \sigma)(\mathcal{A}_h\tau(f)) = \tau' S_h(f)$. But then we calculate

$$S_C(f \star_\Omega g) = \sigma(\mathcal{A}_h\tau(f) \circ_{\mathrm{F}} \mathcal{A}_h\tau(g)) = \sigma(\tau' S_C(f) \circ_{\mathrm{F}} \tau' S_C(g)) = S_C(f) \star_\Psi S_C(g)$$

for all $f, g \in \mathscr{C}^\infty(M)[\![\nu]\!]$. □

Remark 3.2.6 Since in Lemma 3.2.5 we were free to choose a one-form $C \in \nu\Omega^1(M)[\![\nu]\!]$ with $\Omega - \Psi = \mathrm{d}C$, Lemma 3.2.5 actually gives, for any pair of equivalent Fedosov star products $F(\Omega)$ and $F(\Psi)$, a map from $C + \nu Z^1(M)[\![\nu]\!]$ to the set of equivalences between $F(\Omega)$ and $F(\Psi)$.

The second lemma is an adaption from a more general result in [12].

Lemma 3.2.7 *Let (M, ω) be a symplectic manifold and $\Omega, \Psi \in \nu Z^2(M)[\![\nu]\!]$. If $F(\Omega)$ and $F(\Psi)$ are equivalent, then $[\Omega - \Psi] = 0 \in \nu\mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$.*

PROOF: We will again use an inductive proof in the formal degree. So assume that $\star_\Omega = F(\Omega)$ and $\star_\Psi = F(\Psi)$ are given by $\{C_{\Omega,k}\}_{k\in\mathbb{N}}$ and $\{C_{\Psi,k}\}_{k\in\mathbb{N}}$ respectively and that $C_{\Omega,r} = C_{\Psi,r}$ for all $r \leq n$. Additionally, let T be an equivalence between $F(\Omega)$ and $F(\Psi)$, that is

$$T(f \star_\Omega g) = T(f) \star_\Psi T(g) \qquad \text{for all } f, g \in \mathscr{C}^\infty(M) \tag{3.2.12}$$

and assume $T = \mathsf{id} + \nu^s T_s + \mathcal{O}(\nu^{s+1})$ for some $1 \leq s \leq n$. We will begin by showing that there exists another equivalence $\tilde{T}$ between $F(\Omega)$ and $F(\Psi)$ such that $\tilde{T} = \mathsf{id} + \nu^n \tilde{T}_n + \mathcal{O}(\nu^{n+1})$. If $s = n$ then this is already the case, so assume $s < n$. Now evaluate (3.2.12) in formal degree s, to obtain

$$T_s(fg) + C_{\Omega,s}(f,g) = T_s(f)g + fT_s(g) + C_{\Psi,s}(f,g) \qquad \text{for all } f,g \in \mathscr{C}^\infty(M).$$

Since $C_{\Omega,s} = C_{\Psi,s}$, we have $\partial T_s = 0$ and can conclude that T_s is a derivation of $\mathscr{C}^\infty(M)$, hence a vector field on M. In turn, evaluating (3.2.12) in formal degree $s+1$, we obtain

$$\begin{aligned} &T_{s+1}(fg) + T_s(C_{\Omega,1}(f,g)) + C_{\Omega,s+1}(f,g) \\ &\quad = T_{s+1}(f)g + fT_{s+1}(g) + C_{\Psi,1}(T_s(f),g) + C_{\Psi,1}(f,T_s(g)) + C_{\Psi,s+1}(f,g) \end{aligned}$$

or, taking only the antisymmetric part in f and g and using $C_{\Omega,s+1} = C_{\Psi,s+1}$:

$$T_s\{f,g\} = \{T_s(f),g\} + \{f,T_s(g)\}.$$

In other words, T_s is even a symplectic vector field on (M,ω). And, since symplectic vector fields are locally Hamiltonian, we can find an open cover $\mathfrak{U}$ of M and functions $\{t_U \in \mathscr{C}^\infty(U)\}_{U\in\mathfrak{U}}$ such that

$$T_s f\big|_U = \big\{t_U, f\big|_U\big\} \qquad \text{for all } f \in \mathscr{C}^\infty(M) \text{ and } U \in \mathfrak{U}.$$

Since $t_U\big|_{U\cap V} - t_V\big|_{U\cap V}$ is constant on connected components of $U \cap V$ for all $U,V \in \mathfrak{U}$ and all $C_{\Omega,k}$ vanish on constants, the following maps on all $U \in \mathfrak{U}$

$$D_s^U \colon \mathscr{C}^\infty(U)[\![\nu]\!] \longrightarrow \mathscr{C}^\infty(U)[\![\nu]\!] \colon f \longmapsto \frac{1}{2\nu}(t_U \star_\Omega f - f \star_\Omega t_U)$$

yield a well defined global map $D_s \colon \mathscr{C}^\infty(M)[\![\nu]\!] \longrightarrow \mathscr{C}^\infty(M)[\![\nu]\!]$ with $D_s\big|_U = D_s^U$. We can show that D_s is a derivation of $(\mathscr{C}^\infty(M)[\![\nu]\!], \star_\Omega)$ by using the associativity of $\star_\Omega$ and calculating locally for $f,g \in \mathscr{C}^\infty(U)$:

$$\begin{aligned} D_s^U(f \star_\Omega g) &= \frac{1}{2\nu}(t_U \star_\Omega (f \star_\Omega g) - f \star_\Omega t_U \star_\Omega g + f \star_\Omega t_U \star_\Omega g - (f \star_\Omega g) \star_\Omega t_U) \\ &= D_s^U(f) \star_\Omega g + f \star_\Omega D_s^U(g). \end{aligned}$$

But then we have automatically that $S_1 := \exp\{\nu^s D_s\}$ is an equivalence from $\star_\Omega$ to itself and consequently $T^1 := S_1^{-1} \circ T$ is still an equivalence between $\star_\Omega$ and $\star_\Psi$. The crucial part of T^1 is then

$$\begin{aligned} T^1 &= \exp\{-\nu^s D_s\} \circ \big(\mathsf{id} + \nu^s + \mathcal{O}(\nu^{s+1})\big) = \mathsf{id} - \nu^s D_s + \nu^s T_s + \mathcal{O}(\nu^{s+1}) \\ &= \mathsf{id} + \mathcal{O}(\nu^{s+1}), \end{aligned}$$

since for $f \in \mathscr{C}^\infty(U)$

$$T_s\big|_U(f) = \{t_U, f\} = D_s^U(f) + \mathcal{O}(\nu).$$

Clearly, we can iterate the above argument while $s < n$ to finally obtain derivations $D_s, \ldots, D_{n-1}$ which we use to construct equivalences $S_1, \ldots, S_{n-s}$ from $\star_\Omega$ to itself. We then assemble all those into an equivalence $\tilde{T} := S_{n-s}^{-1} \circ \ldots \circ S_1^{-1} \circ T = \mathsf{id} + \nu^n \tilde{T}_n + \mathcal{O}(\nu^{n+1})$ between $\star_\Omega$ and $\star_\Psi$. Next, we evaluate

$$\tilde{T}(f \star_\Omega g) = \tilde{T}(f) \star_\Psi \tilde{T}(g) \tag{3.2.13}$$

in formal degree n to see that $\tilde{T}_n$ is a vector field. This allows us to define a one-form $A^\flat$ through $A^\flat(X_f) := \tilde{T}_n(f)$ for all $f \in \mathscr{C}^\infty(M)$. Evaluating (3.2.13) in formal degree $n-1$ and taking the antisymmetric part finally yields

$$\begin{aligned}\Psi_n(X_f, X_g) - \Omega_n(X_f, X_g) &= -\tilde{T}_n(\{f, g\}) + \left\{\tilde{T}_n(f), g\right\} + \left\{f, \tilde{T}_n(g)\right\} \\ &= -\mathrm{d}A^\flat(X_f, X_g)\end{aligned} \tag{3.2.14}$$

where we used the formal degree expansions $\Omega = \sum_{k=1}^\infty \nu^k \Omega_k$ and $\Psi = \sum_{k=1}^\infty \nu^k \Psi_k$. To briefly summarize, we have shown so far that if two Fedosov star products are equivalent and coincide up to formal degree n, then the difference of the respective series of closed two-forms is exact in order $n+1$. To conclude the proof, write $\Xi^{(0)} = \sum_{k=1}^\infty \nu^k \Xi_k^{(0)} := \Psi$ and $\Xi^{(\infty)} = \sum_{k=0}^\infty \nu^k \Xi_k^{(\infty)} := \Omega$. Then $F(\Xi^{(0)})$ and $F(\Xi^{(\infty)})$ coincide up to formal degree 1, hence there exists a one-form $A_1^\flat$ such that

$$\Xi_1^{(\infty)} - \Xi_1^{(0)} = \mathrm{d}A_1^\flat.$$

We can now define $\Xi^{(1)} := \Xi^{(0)} + \mathrm{d}A_1^\flat$ and recognize that $\Xi^{(\infty)}$ and $\Xi^{(1)}$ coincide up to formal degree 1 (and hence $F(\Xi^{(\infty)})$ and $F(\Xi^{(1)})$ coincide up to formal degree 2) and that $F(\Xi^{(0)})$ and $F(\Xi^{(1)})$ are equivalent by Lemma 3.2.5. Thus $F(\Xi^{(\infty)})$ and $F(\Xi^{(1)})$ are also equivalent and by iterating the above argument to obtain intermediate $\Xi^{(n+1)} := \Xi^{(n)} + \mathrm{d}A_n^\flat$, we see that the difference $\Omega - \Psi$ is exact in all formal degrees, hence $[\Omega - \Psi] = 0 \in \nu \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$.□

With Lemma 3.2.5 and Lemma 3.2.7 in place, we are prepared to state the full classification theorem for star products. As we will see in Section 3.3 and Section 3.4, this theorem will be an essential basis for more specific classification theorems in settings with symmetry. Furthermore, it will be of central importance in Section 4.3, when we classify star products obtained by Marsden-Weinstein reduction. The basic message of all classification theorems presented in this thesis will be that we have a certain subset of the set of all star products on a symplectic manifold (M, ω) and a suitable notion of equivalence inside that subset. We will then show in three cases that the set of equivalence classes is in bijection with a certain cohomology group, that only depends on the topology of the underlying symplectic manifold (and, if present, a group action on said manifold). In this basic case, we consider the set of all star products $\mathrm{Star}(M, \omega)$ on (M, ω), the equivalence classes under the notion of equivalence specified in Definition 3.2.3, which is denoted by $\mathrm{Def}(M, \omega)$. Then one can show that $\mathrm{Def}(M, \omega) \cong \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$ [12, Thm 5.8], [59, Thm. 6.4], [95, Thm. 4.4]:

Theorem 3.2.8 (Characteristic class) *Let (M, ω) be a symplectic manifold. There exists a unique map*

$$c\colon \mathrm{Star}(M, \omega) \longrightarrow \frac{[\omega]}{\nu} + \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$$

which descends to a bijection

$$c\colon \mathrm{Def}(M, \omega) \longrightarrow \frac{[\omega]}{\nu} + \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$$

and is, on all Fedosov star products $F(\Omega)$ for $\Omega \in \nu Z^2(M)[\![\nu]\!]$, given by

$$c(F(\Omega)) = \frac{1}{\nu}[\omega + \Omega]. \tag{3.2.15}$$

Here we denoted by $[\omega] \in \mathrm{H}^2_{\mathrm{dR}}(M)$ the de Rham class of ω.

Proof: We take (3.2.15) as a definition and extend it to all star products, using Proposition 3.2.4. So let us begin by noting that c is well-defined on $\operatorname{im} F$. Let $\Omega, \Psi \in \nu Z^2(M)[\![\nu]\!]$, then, by Lemma 3.2.5 and Lemma 3.2.7, $F(\Omega)$ and $F(\Psi)$ are equivalent if and only if $[\Omega - \Psi] = [\Omega] - [\Psi] = 0 \in \nu \mathrm{H}^2_{\mathrm{dR}}(M)[\![\nu]\!]$ or, equivalently, $[\Omega] = [\Psi]$, hence c is well-defined on $\operatorname{im} F$. Next, let $\star$ be any star product, choose (through Proposition 3.2.4) a Fedosov star product $F(\Omega)$ equivalent to $\star$ and define

$$c(\star) := c(F(\Omega)) = \frac{1}{\nu}[\omega + \Omega].$$

To show that $c(\star)$ is well-defined, let $F(\Psi)$ be another Fedosov star product equivalent to $\star$. Clearly, by transitivity, $F(\Omega)$ is equivalent to $F(\Psi)$ and hence $\Omega - \Psi$ is exact by Lemma 3.2.7. But then we immediately have $c(F(\Omega)) = c(F(\Psi))$. This argument also shows the uniqueness of c. Finally, c is injective by Lemma 3.2.5 and surjective by construction.□

Remark 3.2.9 The result of Theorem 3.2.8 has been proven independently by Bertelson-Cahen-Gutt [12], Deligne [31], Nest-Tsygan [92, 93] and Weinstein-Xu [122]. The characteristic class of Fedosov star products has been calculated explicitly in [95].

Remark 3.2.10 The terminology characteristic class of star products for c is justified by [31] [59, Thm. 6.4]: given a star product $\star$ on M and any diffeomorphism $\phi\colon M \longrightarrow M'$, then $f \star' g := (\phi^{-1})^*(\phi^* f \star \phi^* g)$ defines a star product on M'. Denoting $\star' = (\phi^{-1})^*\star$, the characteristic classes of $\star$ and $\star'$ then satisfy

$$c\big((\phi^{-1})^*\star\big) = (\phi^{-1})^* c(\star).$$

Remark 3.2.11 The reason for c starting in formal degree -1 can be found in [31] [59, Thm. 6.4]: let $\star$ be a star product given by bidifferential operators C_k and consider a change of formal parameter

$$r\colon \nu \longmapsto \nu \sum_{k=2}^{\infty} \nu^k f_k$$

for any $f_k \in \mathbb{R}$ and define a new star product

$$f \star' g := fg + \sum_{k=1}^{\infty} [r(\nu)]^k C_k(f, g).$$

Then one has $c(\star) = c(\star')$ for the choice of $\frac{[\omega]}{\nu}$ as the formal degree -1 part of c, that is the characteristic class of star products is equivariant with respect to such changes of formal parameter. Writing the dependency of c explicitly, one has

$$c(\star')(\nu) = c(\star)(r(\nu)).$$

3.3 Classification of Invariant Star Products

In physics, it is often the case, that a classical system has gauge symmetries such as the Galilei symmetry in Newtonian mechanics. One class of gauge symmetries is modelled mathematically by equipping the symplectic manifold (M, ω), which models the system itself, with a smooth action of a Lie group $G \times M \longrightarrow M$ that respects the symplectic structure, turning (M, ω, G) into a symplectic G-space, see Definition 2.3.6. We would certainly wish to reflect classical symmetries in quantizations of that system. In deformation quantization, one approach is to consider so-called invariant star products: recall that, given a star product $\star$ on M, the observable algebra of the corresponding quantum theory is $(\mathscr{C}^\infty(M)[\![\nu]\!], \star)$. Viewed as sets, this is nothing but power series in the classical observable algebra $\mathscr{C}^\infty(M)$. This allows to immediately transfer the induced pullback action

$$G \times \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M) \colon (g, f) \longmapsto (g^{-1})^* f$$

to $\mathscr{C}^\infty(M)[\![\nu]\!]$ by

$$(g^{-1})^* \left(\sum_{k=0}^\infty \nu^k f_k \right) := \sum_{k=0}^\infty \nu^k \big((g^{-1})^* f_k \big)$$

for all $g \in G$ and $f_k \in \mathscr{C}^\infty(M)$. Classically, the pullback action respects the pointwise product of $\mathscr{C}^\infty(M)$ and hence the natural idea is to define invariant star products as those star products, for which G acts by automorphisms [11]:

Definition 3.3.1 (G-invariant star product) *Let G be a Lie group and (M, ω, G) a symplectic G-space. A star product $\star$ on M is called G-invariant if, for all $g \in G$ and $f_1, f_2 \in \mathscr{C}^\infty(M)[\![\nu]\!]$, the following equation holds:*

$$g^*(f_1 \star f_2) = (g^* f_1) \star (g^* f_2).$$

Also recall, that for any Lie group action on M, there is an associated derived action of the Lie algebra $\mathfrak{g}$ of G by fundamental vector fields, see Definition 2.3.4 and (2.3.3). Standard arguments from the theory of Lie groups then show that if $\star$ is a G-invariant star product, then

$$\mathscr{L}_\xi(f_1 \star f_2) = \mathscr{L}_\xi f_1 \star f_2 + f_1 \star \mathscr{L}_\xi f_2 \tag{3.3.1}$$

holds for all $\xi \in \mathfrak{g}$ and $f_1, f_2 \in \mathscr{C}^\infty(M)[\![\nu]\!]$. In other words, the Lie derivative is a derivation of $\star$. However, we might also wish to consider Lie algebra actions independently without referring to any global action of a Lie group, that is we consider symplectic $\mathfrak{g}$-spaces, see Definition 2.3.7. Imitating (3.3.1), we can then define $\mathfrak{g}$-invariant star products [3]:

Definition 3.3.2 ($\mathfrak{g}$-invariant star products) *Let $\mathfrak{g}$ be a Lie algebra and $(M, \omega, \mathfrak{g})$ a symplectic $\mathfrak{g}$-space. A star product $\star$ on M is called $\mathfrak{g}$-invariant if, for all $\xi \in \mathfrak{g}$ and $f_1, f_2 \in \mathscr{C}^\infty(M)[\![\nu]\!]$, the following equation holds:*

$$\mathscr{L}_\xi(f_1 \star f_2) = \mathscr{L}_\xi f_1 \star f_2 + f_1 \star \mathscr{L}_\xi f_2.$$

Remark 3.3.3 For any connected Lie group G, symplectic G-space (M, ω, G) and star product $\star$ on M, $\star$ is G-invariant if and only if $\star$ is $\mathfrak{g}$-invariant with respect to $X_\bullet$. In this case we will say that $\star$ is an invariant star product. We will mostly be interested in $\mathfrak{g}$-invariant star products throughout.

With this new subclass of $\mathfrak{g}$-invariant star products in place, naturally the same questions as in Section 3.2 arise: Do $\mathfrak{g}$-invariant star products exist? And can we obtain a classification of $\mathfrak{g}$-invariant star products similar to the previous results? Fortunately the answer to both questions turns out to be positive and we shall give a brief summary of the main results in this section. Of course, we will make extensive use of the Fedosov construction, hence the first intermediate lemma will concern $\mathfrak{g}$-invariant Fedosov star products: [60, 90]

Lemma 3.3.4 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, ∇ a torsion-free, symplectic connection on M and $\Omega \in \nu Z^2(M)[\![\nu]\!]$. $F_\nabla(\Omega)$ is $\mathfrak{g}$-invariant if and only if*

$$[D_\nabla, \mathscr{L}_\xi] = 0 \qquad \text{and} \qquad \mathscr{L}_\xi \Omega = 0$$

for all $\xi \in \mathfrak{g}$.

The proof of Lemma 3.3.4 in [90] is based on a deformation of the usual Cartan-formula $\mathscr{L}_\xi = \mathrm{d}\,\mathrm{i}_\xi + \mathrm{i}_\xi\,\mathrm{d}$, that relates the Lie derivative in the direction of symplectic vector fields with the Fedosov derivation corresponding to Ω: [90]

Lemma 3.3.5 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, $\Omega \in \nu Z^2(M)[\![\nu]\!]$, r the unique solution of (3.1.4), $\mathfrak{D}$ the corresponding Fedosov derivation, $X \in \mathfrak{X}_{\mathrm{sympl}}(M)$, $\theta_X := \mathrm{i}_X\,\omega$ and $D^\nabla := (\mathrm{d}x^i \otimes 1)\nabla_{\partial_i}$. Then*

$$\mathscr{L}_X = \mathfrak{D}\,\mathrm{i}_{\mathrm{a}}(X) + \mathrm{i}_{\mathrm{a}}(X)\mathfrak{D} - \frac{1}{\nu}\,\mathrm{ad}_{\circ_{\mathrm{F}}}\left(\theta_X \otimes 1 + \frac{1}{2}D^\nabla\theta_X \otimes 1 - \mathrm{i}_{\mathrm{a}}(X)r\right)$$

holds on $\mathcal{W} \otimes \Lambda(M)$.

Using Lemma 3.3.4, one can immediately observe that the existence of $\mathfrak{g}$-invariant star products revolves solely around the existence of invariant connections (those, that satisfy $[D_\nabla, \mathscr{L}_\xi] = 0$ for all $\xi \in \mathfrak{g}$). If one is able to obtain such, then $F(0)$ clearly is a $\mathfrak{g}$-invariant star product. Hence we will, in the following, always assume the existence of an invariant connection and choose one such connection once and for all. So all that remains is the classification of $\mathfrak{g}$-invariant star products. For this we will also use an adapted notion of equivalence, namely $\mathfrak{g}$-invariant equivalence [11]:

Definition 3.3.6 ($\mathfrak{g}$-invariant equivalence) *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, $\star$ and $\star'$ $\mathfrak{g}$-invariant star products. $\star$ and $\star'$ are said to be $\mathfrak{g}$-invariantly equivalent, if there exists an equivalence T between $\star$ and $\star'$ such that*

$$\mathscr{L}_\xi \circ T = T \circ \mathscr{L}_\xi$$

holds for all $\xi \in \mathfrak{g}$.

The following three results are then the direct analogues of Proposition 3.2.4, Lemma 3.2.5 and Lemma 3.2.7. The proofs are essentially the same, one only has to take care of the $\mathfrak{g}$-invariance. For the G-invariant case, proofs can also be found in [11] (to obtain the corresponding results replace all occurrences of $\mathfrak{g}$-invariance by G-invariance and the $\mathfrak{g}$-invariant de Rham cohomology $\mathrm{H}^{\mathfrak{g}}_{\mathrm{dR}}(M)$ by its G-invariant analogon $\mathrm{H}^{G}_{\mathrm{dR}}(M)$).

Proposition 3.3.7 *Every $\mathfrak{g}$-invariant star product on a symplectic $\mathfrak{g}$-space is $\mathfrak{g}$-invariantly equivalent to a $\mathfrak{g}$-invariant Fedosov star product.*

Lemma 3.3.8 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space and $\Omega, \Psi \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$. If $[\Omega - \Psi]^{\mathfrak{g}} = 0 \in \nu\mathrm{H}^{\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!]$, then $F(\Omega)$ and $F(\Psi)$ are $\mathfrak{g}$-invariantly equivalent.*

Lemma 3.3.9 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space and $\Omega, \Psi \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$. If $F(\Omega)$ and $F(\Psi)$ are $\mathfrak{g}$-invariantly equivalent, then $[\Omega - \Psi]^{\mathfrak{g}} = 0 \in \nu\mathrm{H}^{\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!]$.*

It is noteworthy, that the equivalence S_C constructed in Lemma 3.2.5 is $\mathfrak{g}$-invariant if the one-form C is $\mathfrak{g}$-invariant [104]:

Corollary 3.3.10 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, $\Omega, \Psi \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ with $\Psi - \Omega = \mathrm{d}C$ for $C \in \nu\Omega^1(M)^{\mathfrak{g}}[\![\nu]\!]$ and h the unique solution of* (3.2.6)*. Then*

$$\mathscr{L}_\xi h = 0$$

and

$$\mathscr{L}_\xi \circ S_C = S_C \circ \mathscr{L}_\xi,$$

that is S_C is $\mathfrak{g}$-invariant, for all $\xi \in \mathfrak{g}$.

Finally then, we can construct a characteristic class of $\mathfrak{g}$-invariant star products via the Fedosov construction, similarly to Theorem 3.2.8. We will denote the set of $\mathfrak{g}$-invariant star products on a symplectic $\mathfrak{g}$-space by $\mathrm{Star}^{\mathfrak{g}}(M, \omega)$ and accordingly the $\mathfrak{g}$-invariant equivalence classes by $\mathrm{Def}^{\mathfrak{g}}(M, \omega)$. The corresponding classification theorem, the analogue to Theorem 3.2.8, reads [11, Thm. 4.1]:

Theorem 3.3.11 (Invariant characteristic class) *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space. There exists a unique map*

$$c^{\mathfrak{g}} \colon \mathrm{Star}^{\mathfrak{g}}(M, \omega) \longrightarrow \frac{[\omega]^{\mathfrak{g}}}{\nu} + \mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!]$$

which descends to a bijection

$$c^{\mathfrak{g}} \colon \mathrm{Def}^{\mathfrak{g}}(M, \omega) \longrightarrow \frac{[\omega]^{\mathfrak{g}}}{\nu} + \mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!]$$

and is, on all $\mathfrak{g}$-invariant Fedosov star products $F(\Omega)$ for $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$, given by

$$c^{\mathfrak{g}}(F(\Omega)) = \frac{1}{\nu}[\omega + \Omega]^{\mathfrak{g}}.$$

Here we denote by $[\omega]^{\mathfrak{g}} \in \mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M)$ the $\mathfrak{g}$-invariant de Rham class of ω.

Of course, we can again obtain the analogue result for G-invariant star products, see [11]. As a last corollary, note that, since $\mathfrak{g}$-invariant star products are star products and $\mathfrak{g}$-invariant equivalences are equivalences, we have a map $\mathrm{Star}^{\mathfrak{g}}(M, \omega) \longrightarrow \mathrm{Star}(M, \omega)$ that descends to a map $\mathrm{Def}^{\mathfrak{g}}(M, \omega) \longrightarrow \mathrm{Def}(M, \omega)$. On the other hand, we have the inclusion of $\mathfrak{g}$-invariant differential forms into all differential forms $\Omega^{\mathfrak{g}}(M) \longrightarrow \Omega(M)$ that descends to a map $\mathrm{H}^{\mathfrak{g}}_{\mathrm{dR}}(M) \longrightarrow \mathrm{H}_{\mathrm{dR}}(M)$. This leads to:

Corollary 3.3.12 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space. Then the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Def}^{\mathfrak{g}}(M, \omega) & \longrightarrow & \mathrm{Def}(M, \omega) \\
{\scriptstyle c^{\mathfrak{g}}}\big\downarrow & & \big\downarrow{\scriptstyle c} \\
\frac{1}{\nu}\mathrm{H}^{\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!] & \longrightarrow & \frac{1}{\nu}\mathrm{H}_{\mathrm{dR}}(M)[\![\nu]\!]
\end{array}$$

PROOF: By construction of c and $c^{\mathfrak{g}}$ the diagram commutes on Fedosov star products and hence everywhere. $\square$

3.4 Classification of Equivariant Star products

The last subclass of star products we wish to discuss, is that of equivariant star products. The central classification result here has recently been published by Waldmann and the author in [104], from where we will cite throughout. Once again, the motivational background comes from classical mechanics. Consider a system with symmetry, modelled by a symplectic $\mathfrak{g}$-space $(M, \omega, \mathfrak{g}, \rho)$ for any Lie algebra $\mathfrak{g}$ and $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}_{\mathrm{sympl}}(M)$. Then one can sometimes find a momentum map corresponding to ρ, that is a linear map $J\colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)$ with

$$\rho(\xi) = X_{J(\xi)} \qquad \text{and} \qquad J([\xi, \eta]) = \{J(\xi), J(\eta)\} \tag{3.4.1}$$

for all $\xi, \eta \in \mathfrak{g}$. That is, the images of ρ are not only symplectic vector fields, but Hamiltonian ones (J chooses a Hamiltonian for each element ξ) and J is a Lie algebra homomorphism from $\mathfrak{g}$ to $(\mathscr{C}^\infty(M), \{\ ,\ \})$. If only the first equality in (3.4.1) is satisfied, we will say that J is a Hamiltonian. Furthermore, we will call the quadruple $(M, \omega, \mathfrak{g}, J)$ a Hamiltonian $\mathfrak{g}$-space if J is a momentum map. As an immediate consequence, one can see that for any $f \in \mathscr{C}^\infty(M)$ we have

$$\mathscr{L}_\xi f = -\{J(\xi), f\}. \tag{3.4.2}$$

Remembering back that the principal idea of quantization was to replace the Poisson bracket with a commutator in some algebra, one is lead to an analogue of Hamiltonians in the setting of deformation quantization, namely quantum Hamiltonians, by replacing the Poisson bracket in (3.4.2) by the commutator $\frac{1}{\nu}[\ ,\ ]_\star$ with respect to any $\mathfrak{g}$-invariant star product $\star$ on M. Additionally, in complete analogy to momentum maps, quantum momentum maps are required to be Lie algebra homomorphisms

$$\mathfrak{g} \longrightarrow \left(\mathscr{C}^\infty(M)[\![\nu]\!], \frac{1}{\nu}[\ ,\ ]_\star\right).$$

Writing $\operatorname{ad}_\star(f)g := [f, g]_\star$ for any $f, g \in \mathscr{C}^\infty(M)[\![\nu]\!]$, one then arrives at the following definition, which by now is completely standard [90, 123]:

Definition 3.4.1 (Quantum momentum map) *Let $\star$ a star product on a symplectic $\mathfrak{g}$-space $(M, \omega, \mathfrak{g})$. A linear map $\mathbf{J}\colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[\![\nu]\!]$ is called a quantum momentum map of $\star$ if*

$$\mathscr{L}_\xi = -\frac{1}{\nu}\operatorname{ad}_\star(\mathbf{J}(\xi)) \qquad \textit{and} \qquad \mathbf{J}([\xi, \eta]) = \frac{1}{\nu}[\mathbf{J}(\xi), \mathbf{J}(\eta)]_\star \tag{3.4.3}$$

hold for all $\xi, \eta \in \mathfrak{g}$. If only the first equality is satisfied, we will call $\mathbf{J}$ a quantum Hamiltonian.

Note that if a star product $\star$ on $(M, \omega, \mathfrak{g})$ admits a quantum momentum map, it is automatically $\mathfrak{g}$-invariant, since $\mathscr{L}_\xi = \frac{1}{\nu}\operatorname{ad}_\star(\mathbf{J}(\xi))$ is a derivation of $\star$.

Definition 3.4.2 (Equivariant star product) *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, $\star$ a star product on M and $\mathbf{J}$ a quantum momentum map of $\star$. The tuple $(\star, \mathbf{J})$ is then called an equivariant star product.*

If we denote by $\mathbf{J}_0 := \mathbf{J}\big|_{\nu=0}$ the 0th formal degree of $\mathbf{J}$, we can evaluate (3.4.3) in 0th formal degree and recognize that $\mathbf{J}_0$ is a classical momentum map (classical Hamiltonian respectively). Conversely, given any classical momentum map J, we say that a quantum momentum map $\mathbf{J}$ deforms J if $\mathbf{J}_0 = J$. It remains to show if, and under which conditions, star products admit quantum momentum maps, a question that has been answered fully in [60, 62, 90] for Fedosov star products

Lemma 3.4.3 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space and $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$. Then $F(\Omega)$ admits a quantum momentum map if and only if there exists a linear map $\mathbf{J}\colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[\![\nu]\!]$ such that*

$$\mathrm{i}_\xi(\omega + \Omega) = \mathrm{d}\mathbf{J}(\xi) \qquad \text{and} \qquad (\omega+\Omega)(X_\xi, X_\eta) = \mathbf{J}([\xi,\eta]) \tag{3.4.4}$$

holds for all $\xi, \eta \in \mathfrak{g}$.

With the additional small calculation

$$\mathbf{J}([\xi,\eta]) = \frac{1}{\nu}\operatorname{ad}_{F(\Omega)}(\mathbf{J}(\xi))\mathbf{J}(\eta) = -\,\mathrm{i}_\xi\,\mathrm{d}\mathbf{J}(\eta) = (\omega+\Omega)(X_\xi, X_\eta)$$

we can show that any quantum momentum map $\mathbf{J}$ satisfies (3.4.4) and conversely, that any $\mathbf{J}$ satisfying (3.4.4) is in fact a quantum momentum map of $F(\Omega)$. It is here that one can recognize a useful interpretation of (3.4.4), namely that these two equations are precisely the conditions for $\omega + \Omega - \mathbf{J}$ to be a closed element of the Lie algebraic Cartan complex $\Omega^2_{\mathfrak{g}}(M)[\![\nu]\!]$ as defined in Definition 2.3.11. First, to show that $\omega + \Omega - J$ is an element of $\Omega^2_{\mathfrak{g}}(M)[\![\nu]\!]$, we have to demonstrate that

$$\mathbf{J}([\xi,\eta]) = -\,\mathrm{i}_\xi\,\mathrm{d}\mathbf{J}(\eta) \qquad \text{and} \qquad \mathscr{L}_\xi(\omega+\Omega) = 0 \tag{3.4.5}$$

hold. The first equation however is satisfied since $\mathbf{J}$ is a quantum momentum map and the second one by the assumption that $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ and $X_\xi \in \mathfrak{X}_{\mathrm{sympl}}(M)$. By the definition of the differential $\mathrm{d}_{\mathfrak{g}} = \mathrm{d} + \mathrm{i}_\bullet$ of the Cartan complex we see then that $\omega + \Omega - \mathbf{J}$ is $\mathrm{d}_{\mathfrak{g}}$-closed if

$$\mathrm{d}(\omega+\Omega) = 0 \qquad \text{and} \qquad \mathrm{i}_\xi(\omega+\Omega) = \mathrm{d}\mathbf{J}(\xi) \qquad \text{and} \qquad \mathrm{i}_\xi\,\mathbf{J} = 0$$

hold. Here the first equation is satisfied by assumption (ω and Ω are d-closed), the second equation is precisely the first equation in (3.4.4) and the third is trivial, since $\mathrm{i}_\bullet$ vanishes on 0-forms. Hence we can restate Lemma 3.4.3 in the following form:

Corollary 3.4.4 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space and $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$. Then $F(\Omega)$ admits a quantum momentum map, if there exists a linear map $\mathbf{J}\colon \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[\![\nu]\!]$ with*

$$\omega + \Omega - \mathbf{J} \in \Omega^2_{\mathfrak{g}}(M)[\![\nu]\!] \qquad \text{and} \qquad \mathrm{d}_{\mathfrak{g}}(\omega + \Omega - \mathbf{J}) = 0.$$

With these preliminary results at hand, we follow the example of Section 3.2 and Section 3.3 and aim to define a suitable notion of equivalence for equivariant star products, to show that every equivariant star product is equivalent to an equivariant Fedosov star product and finally, to construct an equivariant characteristic class with values in a suitable cohomology, which will turn out to be the Lie algebraic equivariant cohomology. We begin with the definition of equivariant equivalences [104]:

Definition 3.4.5 (Equivariant equivalence) *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, and $(\star, \mathbf{J})$ and $(\star', \mathbf{J}')$ equivariant star products on M. A $\mathfrak{g}$-invariant equivalence*

$$T\colon (\mathscr{C}^\infty(M)[\![\nu]\!], \star) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], \star')$$

is called equivariant if

$$T \circ \mathbf{J} = \mathbf{J}'$$

holds.

With this definition, we can immediately give a positive answer to the question of whether every equivariant star product is equivalent to an equivariant Fedosov one. Unlike Proposition 3.2.4, where we had to expend a significant amount of work for the similar result, we are able to use the invariant version Proposition 3.3.7 to directly write down the correct equivariant Fedosov star product.

Corollary 3.4.6 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space. Every equivariant star product on M is equivariantly equivalent to an equivariant Fedosov star product.*

PROOF: Let $(\star, \mathbf{J})$ be any equivariant star product. Then by Proposition 3.3.7, there exists $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ and a $\mathfrak{g}$-invariant equivalence

$$T\colon (\mathscr{C}^\infty(M)[\![\nu]\!], \star) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Omega))$$

and we claim that $T \circ \mathbf{J}$ is a quantum momentum map of $F(\Omega)$. We only have to check that (3.4.3) holds. So let $\xi, \eta \in \mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)[\![\nu]\!]$, then

$$\mathrm{ad}_{F(\Omega)}(T\mathbf{J}(\xi))f = \big[T\mathbf{J}(\xi), TT^{-1}f\big]_{F(\Omega)} = T\big[\mathbf{J}(\xi), T^{-1}f\big]_\star = -\nu T\mathscr{L}_\xi T^{-1}f = -\nu\mathscr{L}_\xi f$$

and

$$(T \circ \mathbf{J})([\xi, \eta]) = T\left(\frac{1}{\nu}[\mathbf{J}(\xi), \mathbf{J}(\eta)]_\star\right) = [T\mathbf{J}(\xi), T\mathbf{J}(\eta)]_{F(\Omega)}. \qquad \square$$

The goal of the remainder of this section will then be the full classification of equivariant star products. We will prove the final result in three steps, following [104]. First, we shall consider the special case of two different quantum momentum maps for the same star product and investigate under which conditions equivariant equivalences exist here. Secondly, we extend this result to arbitrary equivariant Fedosov star products and finally, we will define an equivariant characteristic class similar to Theorem 3.2.8 and Theorem 3.3.11. Throughout we will assume that all manifolds are connected. Otherwise we can just repeat arguments on each connected component to obtain similar results.

Lemma 3.4.7 *Let $(M, \omega, \mathfrak{g}, J)$ be a connected Hamiltonian $\mathfrak{g}$-space, $\star$ a $\mathfrak{g}$-invariant star product, and $\mathbf{J}$ and $\mathbf{J}'$ quantum momentum maps of $\star$ deforming J. Then there exists an equivariant equivalence between $(\star, \mathbf{J})$ and $(\star, \mathbf{J}')$ if and only if $\mathbf{J}' - \mathbf{J} \in \Omega^2_{\mathfrak{g}}(M)[\![\nu]\!]$ is $\mathrm{d}_{\mathfrak{g}}$-exact.*

PROOF: The statement of the lemma contains an implicit assumption, namely that $\mathbf{J}' - \mathbf{J} \in \Omega^2_{\mathfrak{g}}(M)[\![\nu]\!]$, which follows directly from (3.4.5). Also note that, since $\mathrm{ad}_\star(\mathbf{J}(\xi)) = -\nu\mathscr{L}_\xi = \mathrm{ad}_\star(\mathbf{J}'(\xi))$ for all $\xi \in \mathfrak{g}$, we have $\mathrm{ad}_\star(\mathbf{J}'(\xi) - \mathbf{J}(\xi)) = 0$ and hence $j := \mathbf{J}' - \mathbf{J}$ must be a map into constant functions on M, that is $j\colon \mathfrak{g} \longrightarrow \mathbb{C}[\![\nu]\!]$. This however implies that j is $\mathrm{d}_{\mathfrak{g}}$-closed. We begin the actual proof by assuming that j is $\mathrm{d}_{\mathfrak{g}}$-exact, that is there exists a $\theta \in \Omega^1_{\mathfrak{g}}(M)[\![\nu]\!]$ such that $j = \mathrm{d}_{\mathfrak{g}}\theta = \mathrm{d}\theta + \mathrm{i}_\bullet\,\theta$. Since $\Omega^2_{\mathfrak{g}}(M)$ is the direct sum of invariant two forms and invariant polynomials on $\mathfrak{g}$ in one variable, $j = \mathrm{d}_{\mathfrak{g}}\theta$ is equivalent to the two separate equations $j = \mathrm{i}_\bullet\,\theta$ and $\mathrm{d}\theta = 0$. Use then that all closed differential forms are locally exact to obtain an open cover $\mathfrak{U}$ of M and functions $\{t_U \in \mathscr{C}^\infty(U)[\![\nu]\!]\}_{U \in \mathfrak{U}}$ with $\mathrm{d}t_U = \theta\big|_U$ for all $U \in \mathfrak{U}$. Since all $\mathrm{d}t_U$ must agree on intersections of elements in $\mathfrak{U}$, we have

$$t_U\big|_{U\cap V} - t_V\big|_{U\cap V} \in \mathbb{C}[\![\nu]\!] \qquad \text{for all } U, V \in \mathfrak{U},$$

where we assumed that $U \cap V$ is connected. Otherwise the above equality holds on all connected components of $U \cap V$. Hence the operators $\mathrm{ad}_\star(t_U)$ agree on overlaps and

$$\mathrm{ad}_\star(\theta)\big|_U := \mathrm{ad}_\star(t_U) \qquad \text{for all } U \in \mathfrak{U}.$$

defines a differential operator on $\mathscr{C}^\infty(M)[\![\nu]\!]$. By the very nature of $\operatorname{ad}_\star$, $\operatorname{ad}_\star(\theta)$ is also a derivation of $\star$ and thus

$$A := \exp\left\{\frac{1}{\nu}\operatorname{ad}_\star(\theta)\right\}$$

is a $\mathfrak{g}$-invariant automorphism of $\star$. Our goal will be to directly calculate $A\mathbf{J}$. We start with the following auxiliary calculation

$$\frac{1}{\nu}\operatorname{ad}_\star(t_U)\mathbf{J}(\xi)\big|_U = \mathscr{L}_\xi t_U = \mathrm{i}_\xi \,\mathrm{d}t_U = \mathrm{i}_\xi\,\theta\big|_U = j(\xi)\big|_U,$$

which can be used to show

$$A\mathbf{J}(\xi)\big|_U = \exp\left\{\frac{1}{\nu}\operatorname{ad}_\star(t_U)\right\}\mathbf{J}(\xi)\big|_U = \mathbf{J}(\xi)\big|_U + j(\xi)\big|_U + 0 = \mathbf{J}'(\xi)\big|_U. \tag{3.4.6}$$

Conversely, assume that we are given an equivariant equivalence

$$A\colon (\mathscr{C}^\infty(M)[\![\nu]\!], (\star, \mathbf{J})) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], (\star, \mathbf{J}')).$$

Then there exists a d-closed, $\mathfrak{g}$-invariant one-form $\theta \in Z^1(M)^{\mathfrak{g}}[\![\nu]\!]$ such that

$$A = \exp\left\{\frac{1}{\nu}\operatorname{ad}_\star(\theta)\right\},$$

where $\operatorname{ad}_\star(\theta)$ is defined locally as $\operatorname{ad}_\star(\theta)\big|_U = \operatorname{ad}_\star(t_U)$ for any t_U with $\theta\big|_U = \mathrm{d}t_U$, as seen e.g. in [120, Thm. 6.3.18]. We can now rewrite j locally as

$$j(\xi)\big|_U = (A\mathbf{J}(\xi) - \mathbf{J}(\xi))\big|_U = \underbrace{\left(\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{1}{\nu}\operatorname{ad}_\star(t_U)\right)^{k-1}\right)}_{T}\frac{1}{\nu}\operatorname{ad}_\star(\mathbf{J}(\xi)\big|_U)t_U. \tag{3.4.7}$$

Note that T is a series of differential operators starting with id, thus T is invertible. We can hence apply T^{-1} to both sides of (3.4.7) while using that $j(\xi)$ is constant:

$$j(\xi)\big|_U = \frac{1}{\nu}\operatorname{ad}_\star(\mathbf{J}(\xi)\big|_U)t_U = \mathscr{L}_\xi t_U = (\mathrm{d}_{\mathfrak{g}}\theta)(\xi)\big|_U. \qquad \square$$

By applying Lemma 3.4.7 to Fedosov star products, we can already hint at the full classification result. Indeed, let $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ and assume that $F(\Omega)$ admits a quantum momentum map $\mathbf{J}$, that is, as we have seen, $\omega + \Omega - \mathbf{J} \in \Omega^2_{\mathfrak{g}}(M)[\![\nu]\!]$ is $\mathrm{d}_{\mathfrak{g}}$-closed. Now assume that $F(\Omega)$ admits a second quantum momentum map $\mathbf{J}'$. Then $(F(\Omega), \mathbf{J})$ and $(F(\Omega), \mathbf{J}')$ are equivariantly equivalent if and only if $\mathbf{J}' - \mathbf{J}$ is $\mathrm{d}_{\mathfrak{g}}$-exact, what we can rewrite as

$$[\mathbf{J}' - \mathbf{J}]_{\mathfrak{g}} = [\omega + \Omega - \mathbf{J}]_{\mathfrak{g}} - [\omega + \Omega - \mathbf{J}']_{\mathfrak{g}} = 0 \in \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!].$$

From there it seems natural to take the educated guess that in general two equivariant Fedosov star products $(F(\Omega), \mathbf{J})$ and $(F(\Psi), \mathbf{J}')$ are equivariantly equivalent if and only if

$$[\omega + \Omega - \mathbf{J}]_{\mathfrak{g}} - [\omega + \Psi - \mathbf{J}']_{\mathfrak{g}} = 0 \in \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!],$$

which will turn out to be true. However, we will be requiring a fair bit of preparation. First, we cite a result from [90]:

Lemma 3.4.8 *Let $(M, \omega, \mathfrak{g}, J)$ be a Hamiltonian $\mathfrak{g}$-space, $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ and $(F(\Omega), \mathbf{J})$ an equivariant star product. Then the Fedosov-Taylor series of $\mathbf{J}$ is given by*

$$\tau(\mathbf{J}(\xi)) = \mathbf{J}(\xi) + \theta_\xi \otimes 1 + \frac{1}{2} D^\nabla \theta_\xi \otimes 1 + \mathrm{i}_a(\xi) r$$

for all $\xi \in \mathfrak{g}$, where $\theta_\xi := \mathrm{i}_\xi \omega$, $D^\nabla = (\mathrm{d}x^i \otimes 1)\nabla_{\partial_i}$ as in Lemma 3.3.5, and r is the unique solution of (3.1.4).

Lemma 3.4.8 is surprising in the following sense: usually, one has no good handle on explicit expressions of Fedosov-Taylor series for general functions. However, the Fedosov-Taylor series of quantum momentum maps turn out to be almost completely determined by the Fedosov star product alone. The actual quantum momentum map only determines the part of simultaneous symmetric and antisymmetric degree 0, which is given by the quantum momentum map itself. This very special behaviour allows us to calculate the effect of the equivalences S_C from Lemma 3.2.5 on quantum momentum maps, see [104]:

Lemma 3.4.9 *Let $(M, \omega, \mathfrak{g})$ be a symplectic $\mathfrak{g}$-space, $\Omega \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$, $C \in \nu\Omega^1(M)^{\mathfrak{g}}[\![\nu]\!]$ and let $(F(\Omega), \mathbf{J})$ be an equivariant star product. Then*

$$S_C \mathbf{J}(\xi) = \mathbf{J}(\xi) + \mathrm{i}_\xi C$$

holds for all $\xi \in \mathfrak{g}$, where S_C is the $\mathfrak{g}$-invariant equivalence constructed from C in Corollary 3.3.10.

Proof: All ingredients are given explicitly enough to calculate directly, using Lemma 3.3.5, Lemma 3.4.8, (3.2.11), Corollary 3.3.10 as well as $\mathrm{i}_a(\xi) h = 0$ since $\deg_{\mathrm{a}} h = 0$:

$$\begin{aligned}
\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h) \tau(\mathbf{J}(\xi)) &= -\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}} \left(\mathbf{J}(\xi) + \theta_\xi \otimes 1 + \frac{1}{2} D^\nabla \theta_\xi \otimes 1 - \mathrm{i}_{\mathrm{a}}(\xi) r \right) h \\
&= (\mathscr{L}_\xi - \mathfrak{D}\, \mathrm{i}_{\mathrm{a}}(\xi) - \mathrm{i}_{\mathrm{a}}(\xi) \mathfrak{D}) h \\
&= -\, \mathrm{i}_{\mathrm{a}}(\xi) \mathfrak{D} h \\
&= \mathrm{i}_{\mathrm{a}}(\xi) \left(1 \otimes C - \frac{\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h)}{\exp\{\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}} (r' - r) \right) \\
&= 1 \otimes \mathrm{i}_\xi C - \underbrace{\frac{\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h)}{\exp\{\frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h)\} - \mathsf{id}}}_{T} (\mathbf{J}'(\xi) - \tau'(\mathbf{J}'(\xi)) - \mathbf{J}(\xi) + \tau(\mathbf{J}(\xi))),
\end{aligned}$$

where $\mathbf{J}'$ is any quantum momentum map of $F(\Omega - \mathrm{d}C)$ (e.g. $S_C \mathbf{J}$) and τ' the Fedosov-Taylor series corresponding to $\Omega - \mathrm{d}C$. We first apply T^{-1} to both sides to obtain

$$\left(\exp\left\{ \frac{1}{\nu} \operatorname{ad}_{\circ_{\mathrm{F}}}(h) \right\} - \mathsf{id} \right) \tau(\mathbf{J}(\xi)) - 1 \otimes \mathrm{i}_\xi C = \mathbf{J}(\xi) - \tau(\mathbf{J}(\xi)) - \mathbf{J}'(\xi) + \tau'(\mathbf{J}'(\xi))$$

and secondly σ and observe that, with $\sigma(\mathbf{J}(\xi)) = \mathbf{J}(\xi)$ and $\sigma\tau(\mathbf{J}(\xi)) = \mathbf{J}(\xi)$, the right hand side vanishes entirely and we are left with

$$S_C \mathbf{J}(\xi) - \mathbf{J}(\xi) - 1 \otimes \mathrm{i}_\xi C = 0. \qquad \square$$

With Lemma 3.4.9 we are finally in a position to prove the classification result for Fedosov star products we hinted at earlier:

Proposition 3.4.10 *Let $(F(\Omega), \mathbf{J}_\Omega)$ and $(F(\Psi), \mathbf{J}_\Psi)$ be equivariant Fedosov star products on a Hamiltonian $\mathfrak{g}$-space $(M, \omega, \mathfrak{g}, J)$ with $\Omega, \Psi \in \nu Z^2(M)^{\mathfrak{g}}[\![\nu]\!]$ such that $\mathbf{J}_\Omega$ and $\mathbf{J}_\Psi$ both deform J. Then $(F(\Omega), \mathbf{J}_\Omega)$ and $(F(\Psi), \mathbf{J}_\Psi)$ are equivariantly equivalent if and only if*

$$c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega) - c_{\mathfrak{g}}(F(\Psi), \mathbf{J}_\Psi) = 0 \in \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!]$$

holds, where $c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega)$ is defined as

$$c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega) \coloneqq \frac{1}{\nu}[\omega + \Omega - \mathbf{J}_\Omega]_{\mathfrak{g}} \in \frac{[\omega - J]_{\mathfrak{g}}}{\nu} + \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!]$$

Proof: We begin by assuming that $(F(\Omega), \mathbf{J}_\Omega)$ and $(F(\Psi), \mathbf{J}_\Psi)$ are equivariantly equivalent, that is there is a $\mathfrak{g}$-invariant equivalence

$$T\colon (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Omega)) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Psi)) \qquad \text{with} \qquad T\mathbf{J}_\Omega = \mathbf{J}_\Psi.$$

But then we already know from Lemma 3.3.9 that $[\Omega - \Psi]^{\mathfrak{g}} = 0 \in \nu\mathrm{H}^{2;\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!]$ and hence we can choose a $C \in \nu\Omega^1(M)^{\mathfrak{g}}[\![\nu]\!]$ with $\Omega = \Psi + \mathrm{d}C$. Using C, we can construct another $\mathfrak{g}$-invariant equivalence S_C parallel to T from Lemma 3.2.5 and we can obtain another equivariant star product $((F(\Psi), S_C\mathbf{J}_\Omega)$. Notice, that $((F(\Psi), \mathbf{J}_\Psi)$ and $(F(\Psi), S_C\mathbf{J})$ are equivariantly equivalent by $S_C \circ T^{-1}$, thus we can apply Lemma 3.4.7 to conclude

$$[T\mathbf{J} - S_C\mathbf{J}]_{\mathfrak{g}} = 0 \in \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!].$$

From here we use Lemma 3.4.9 to calculate

$$\begin{aligned}\nu c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega) - \nu c_{\mathfrak{g}}(F(\Psi), \mathbf{J}_\Psi) &= [\Omega - \Psi + \mathbf{J}_\Psi - \mathbf{J}_\Omega]_{\mathfrak{g}} = [\mathrm{d}C + S_C\mathbf{J}_\Omega - \mathbf{J}_\Omega]_{\mathfrak{g}} \\ &= [\mathrm{d}C + \mathrm{i}_\bullet C]_{\mathfrak{g}} = [\mathrm{d}_{\mathfrak{g}}C]_{\mathfrak{g}} = 0.\end{aligned}$$

Conversely, assume that

$$c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega) - c_{\mathfrak{g}}(F(\Psi), \mathbf{J}_\Psi) = 0$$

holds. By Definition 2.3.11, this is equivalent to the existence of $\theta \in \nu\Omega^1_{\mathfrak{g}}(M)[\![\nu]\!] = \nu\Omega^1(M)^{\mathfrak{g}}[\![\nu]\!]$ such that

$$\mathrm{d}\theta = \Omega - \Psi \qquad \text{and} \qquad \mathrm{i}_\xi\,\theta = \mathbf{J}_\Psi - \mathbf{J}_\Omega.$$

Consequently, we can, again from Lemma 3.3.8, obtain a $\mathfrak{g}$-invariant equivalence S_θ

$$S_\theta\colon (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Omega)) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Psi)).$$

Subsequently, Lemma 3.4.9 can be used to calculate

$$[\mathbf{J}_\Psi - S_\theta\mathbf{J}_\Omega] = [\Omega - \mathrm{d}\theta - \Psi + \mathbf{J}_\Psi - \mathbf{J}_\Omega - \mathrm{i}_\bullet\,\theta] = \nu c_{\mathfrak{g}}(F(\Omega), \mathbf{J}_\Omega) - \nu c_{\mathfrak{g}}(F(\Psi), \mathbf{J}_\Psi) = 0.$$

This shows, together with Lemma 3.4.7, that there exists an equivariant equivalence A from $(F(\Psi), S_\theta\mathbf{J}_\Omega)$ to $(F(\Psi), \mathbf{J}_\Psi)$ and hence $(F(\Omega), \mathbf{J}_\Omega)$ and $(F(\Psi), \mathbf{J}_\Psi)$ are equivariantly equivalent through $A \circ S_\theta$. □

From here, it is an easy task to extend the above result to all equivariant star products, similar to Theorem 3.2.8 and Theorem 3.3.11. For this we will denote by $\mathrm{Star}_{\mathfrak{g}}(M, \omega, J)$ the set of equivariant star products on (M, ω) that deform a given classical momentum map J and by $\mathrm{Def}(M, \omega, J)$ the equivalence classes under equivariant equivalence.

Theorem 3.4.11 *Let $(M, \omega, \mathfrak{g}, J)$ be a Hamiltonian $\mathfrak{g}$-space. There exists a unique map*

$$c_{\mathfrak{g}} \colon \mathrm{Star}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \frac{[\omega - J]_{\mathfrak{g}}}{\nu} + \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!]$$

that descends to a bijection

$$c_{\mathfrak{g}} \colon \mathrm{Def}(M, \omega, J) \longrightarrow \frac{[\omega - J]_{\mathfrak{g}}}{\nu} + \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!]$$

and is, on all equivariant Fedosov star products $(F(\Omega), \mathbf{J})$ for $\Omega \in \nu Z^{2,\mathfrak{g}}(M)[\![\nu]\!]$ given by

$$c_{\mathfrak{g}}(F(\Omega), \mathbf{J}) = \frac{1}{\nu}[\omega + \Omega - \mathbf{J}]_{\mathfrak{g}}.$$

Proof: Let $(\star, \mathbf{J})$ be any equivariant star product. By Proposition 3.3.7, there exists a $\mathfrak{g}$-invariant Fedosov star product $F(\Omega)$ and a $\mathfrak{g}$-invariant equivalence

$$T \colon (\mathscr{C}^\infty(M)[\![\nu]\!], \star) \longrightarrow (\mathscr{C}^\infty(M)[\![\nu]\!], F(\Omega)).$$

But then $(F(\Omega), T\mathbf{J})$ is an equivariant Fedosov star product and clearly equivariantly equivalent to $(\star, \mathbf{J})$. Hence we define

$$c_{\mathfrak{g}}(\star, \mathbf{J}) := c_{\mathfrak{g}}(F(\Omega), T\mathbf{J}) = \frac{1}{\nu}[\omega + \Omega - T\mathbf{J}]_{\mathfrak{g}}.$$

To show that $c_{\mathfrak{g}}(\star, \mathbf{J})$ is well-defined, assume that we chose a different equivariant equivalence T' to a different equivariant Fedosov star product $(F(\Omega'), T'\mathbf{J})$. Obviously, $(F(\Omega), T\mathbf{J})$ and $(F(\Omega'), T'\mathbf{J})$ are equivariantly equivalent and thus $c_{\mathfrak{g}}(F(\Omega), T\mathbf{J}) = c_{\mathfrak{g}}(F(\Omega'), T'\mathbf{J})$ holds by Proposition 3.4.10. □

Finally, note that, since every equivariant star product $(\star, \mathbf{J})$ is in particular a $\mathfrak{g}$-invariant star product, we have a map $\mathrm{Star}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \mathrm{Star}^{\mathfrak{g}}(M, \omega)$. Since every equivariant equivalence is also a $\mathfrak{g}$-invariant equivalence of the underlying $\mathfrak{g}$-invariant star products, the previous map descends to a map $\mathrm{Def}_{\mathfrak{g}}(M, \omega, J) \longrightarrow \mathrm{Def}^{\mathfrak{g}}(M, \omega)$. Simultaneously, by Definition 2.3.11, there is the projection

$$\Omega^2_{\mathfrak{g}}(M) = \mathrm{S}^1(\mathfrak{g}^*)^{\mathfrak{g}} \oplus \Omega^2(M)^{\mathfrak{g}} \longrightarrow \Omega^2(M)^{\mathfrak{g}}$$

onto the second summand, that descends to cohomology:

$$\mathrm{H}^2_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M).$$

We can state the following corollary [104]:

Corollary 3.4.12 *Let $(M, \omega, \mathfrak{g}, J)$ be a Hamiltonian $\mathfrak{g}$-space. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Def}_{\mathfrak{g}}(M, \omega, J) \arrow[r] \arrow[d, "c_{\mathfrak{g}}"'] & \mathrm{Def}^{\mathfrak{g}}(M, \omega) \arrow[r] \arrow[d, "c^{\mathfrak{g}}"'] & \mathrm{Def}(M, \omega) \arrow[d, "c"] \\
\frac{1}{\nu}\mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!] \arrow[r] & \frac{1}{\nu}\mathrm{H}^{2,\mathfrak{g}}_{\mathrm{dR}}(M)[\![\nu]\!] \arrow[r] & \frac{1}{\nu}\mathrm{H}_{\mathrm{dR}}(M)[\![\nu]\!]
\end{tikzcd}$$

PROOF: The right rectangle commutes by Corollary 3.3.12. As for the left rectangle, it commutes by construction on Fedosov star products and hence everywhere. $\square$

Remark 3.4.13 Together with Lemma 3.4.3, the map $\mathrm{pr}_2 \colon \frac{1}{\nu}\mathrm{H}^2_{\mathfrak{g}}(M)[[\nu]] \longrightarrow \frac{1}{\nu}\mathrm{H}^{2;\mathfrak{g}}_{\mathrm{dR}}(M)[[\nu]]$ allows for the following interpretation: any $\mathfrak{g}$-invariant star product $\star$ on a Hamiltonian $\mathfrak{g}$-space admits a quantum momentum map if and only if $c^{\mathfrak{g}}(\star)$ lies in its image. Note that $\star$ admits a quantum momentum map if and only if there exists an invariantly equivalent Fedosov star product $\star_\Omega$ which admits a quantum momentum map. But then Lemma 3.4.3 immediately states that this is the case if any only if Ω can be extended to an equivariant differential form, that is $c^{\mathfrak{g}}(\star) = \frac{1}{\nu}[\omega + \Omega]^{\mathfrak{g}} \in \operatorname{im} \mathrm{pr}_2$.

Chapter 4

Symmetry Reduction of Star Products

In classical mechanics, whenever a system has gauge symmetries, modelled as a Lie group action on the underlying symplectic manifold (M, ω), one is actually describing the system with more information than is strictly necessary, that is some parts of M and G are redundant. As M describes the possible states of the system, if we take any of them, say $p \in M$, then for any gauge transformation $g \in G$, the intuition is that p and $g \triangleright p$ describe the same state of the system. So while M certainly contains all possible states, some of the points of M represent identical physical states. Of course, one can still work with M and G despite that drawback. However, it is then imperative to keep in mind to correctly identify states. Another approach would be searching for a truly non-redundant description of the system. The process of obtaining this description from the initial data (M, ω) and $(G, \triangleright)$ is called symmetry-reduction and we will dedicate this section to it. The naive approach here is, of course, to take the orbit space M/G to get rid of excess information. However, the main problem one encounters is that this quotient may be very singular. This can occur even in very simple examples, such as the 2-sphere with radius 1 acted upon by $U(1)$ by rotation around any axis. The resulting quotient is then easily seen to be the closed interval $[-1; 1]$, which is not a symplectic manifold anymore. So we obtain a unique description of the physical states, but leave the framework of symplectic geometry. As usual, one can apply some more sophisticated technology to the problem at hand: the symplectic manifold M and the action of G on M can be combined into what is called the action-groupoid $M \rtimes G$ (see e.g. [65]). It is the category that has all points of M as objects and for any two points $p, q \in M$ the set of morphisms from p to q is given by

$$\mathsf{Hom}_{M \rtimes G}(p, q) = \{g \in G \mid g \triangleright p = q\}.$$

From $M \rtimes G$ then, one can construct a differentiable stack [9], the quotient stack $[M/G]$, which still admits a lot of geometry (we are being deliberately vague here). However, ultimately, our goal would be to do deformation quantization on $[M/G]$ and while there are certainly advances in quantizing more algebraic objects e.g. in [21] or [124], in this thesis we aspire to strictly stay in the setting of symplectic manifolds. Accordingly, our treatise of the matter will be restricted to certain special cases, namely, we require G to be connected and the action $\triangleright$ to be Hamiltonian, that is it admits a momentum map. In this case, there is an easier method of symmetry reduction, called Marsden-Weinstein reduction [85]. We shall give a short overview on it in Section 4.1, followed by an exposition of a quantized version of Marsden-Weinstein reduction from [16, 61] based on BRST cohomology in Section 4.2 and finally present a classification of reduced star products by the author [104] in Section 4.3.

4.1 Marsden-Weinstein Reduction

Marsden-Weinstein reduction applies to any Lie group G that acts on a symplectic manifold (M, ω) and admits a momentum map J, that is it applies to any Hamiltonian G-space, see (3.4.1) and (3.4.2). An equivalent view of momentum maps, one that is more suitable to Marsden-Weinstein reduction, sees momentum maps as maps

$$\mu\colon M \longrightarrow \mathfrak{g}^* \qquad \text{with} \qquad X_{\langle\mu,\xi\rangle} = X_\xi \qquad \text{and} \qquad \mu(g \triangleright p) = \mathrm{Ad}^*_{g^{-1}}\,\mu(p) \tag{4.1.1}$$

for all $\xi \in \mathfrak{g}$, $g \in G$ and $p \in M$ together with the definition $\langle\mu, \xi\rangle(p) := \mu(p)\xi$ and the Hamiltonian vector field $X_{\langle\mu,\xi\rangle}$ of the function $\langle\mu, \xi\rangle$. The connection between both points of view is made through the equation

$$J(\xi)(p) = \mu(p)\xi. \tag{4.1.2}$$

An easy calculation shows, if μ satisfies (4.1.1), then J defined via (4.1.2) satisfies (3.4.1) and vice versa. We will also refer to any symplectic G-space together with a momentum map in the sense of (4.1.1) as a Hamiltonian G-space and denote it by the quadruple (M, ω, G, μ).

Remark 4.1.1 Note that there exist various conventions about the terminology. Sometimes the distinction between whether or not μ satisfies the right equality in (4.1.1) is made, and μ is called an Ad^*-equivariant momentum map, if it does. Since this feature is essential for Marsden-Weinstein reduction, we agree that all our momentum maps shall be Ad^*-equivariant.

Remark 4.1.2 Since we will be switching frequently between both points of view (3.4.1) and (4.1.1), we shall denote, given μ satisfying (4.1.1), by J_μ the dual version defined via (4.1.2). And, analogously, denote by μ_J the dual version of any J that satisfies (3.4.1).

Let us further assume that 0 is a value and regular value of μ, then $C := \mu^{-1}(\{0\})$ is a submanifold of M such that all physical motion (subject to the gauge symmetry conveyed by G) that begins on C, stays constrained to C. Also, by (4.1.1), the action of G on M restricts to C (we will, for convenience, also denote the action there by $\triangleright$). Marsden-Weinstein reduction is then the following two-step process: first, pass to the momentum level C and, second, take the quotient C/G, which, under suitable conditions, is guaranteed to be a symplectic manifold. Since we will not be concerned with the intricacies of Marsden-Weinstein reduction itself, we shall only cite the relevant result. For more detailed expositions consider i.e. [86] or [120]. The original result is, of course, due to [85].

Theorem 4.1.3 *Let (M, ω, G, μ) be a Hamiltonian G-space and let 0 be a value and regular value of μ. If the action of G on $C := \mu^{-1}(\{0\})$ is free and proper then the quotient $M_{\mathrm{red}} := C/G$ is a symplectic manifold with symplectic structure ω_{red} uniquely defined by*

$$\pi^*\omega_{\mathrm{red}} = \iota^*\omega,$$

where $\pi\colon C \twoheadrightarrow M_{\mathrm{red}}$ is the quotient projection and ι the inclusion of C in M.

We will frequently summarize the complete setting as the following diagram

$$M \xhookleftarrow{\iota} C \xrightarrow{\pi} M_{\mathrm{red}}$$

and say that M_{red} is Marsden-Weinstein reduced from M via C by G.

Remark 4.1.4 It is also possible to perform Marsden-Weinstein reduction with respect to different momentum-level-sets $C_\xi := \mu^{-1}(\{\xi\})$ for any $\xi \in \mathfrak{g}^*$. However, one then has to take the quotient with respect to the isotropy group $G_\xi = \{g \in G \mid \mathrm{Ad}_g^* \xi = \xi\}$ instead of the full group G. The reduced manifold is then $M_{\mathrm{red}} = C_\xi / G_\xi$. Marsden-Weinstein reduction as described above is then the special case $\xi = 0$, for which one has trivially $G_0 = G$.

4.2 The Reduction Scheme

Marsden-Weinstein reduction enables us to obtain a redundancy-free description of a given classical system, if certain restrictions are met. So let us consider a symplectic manifold $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ reduced from (M, ω). We can then certainly consider star products on M and M_{red} separately. However, as we have seen in Theorem 3.2.8, Theorem 3.3.11 and Theorem 3.4.11, depending on the topology of M and M_{red}, there may be a multitude of nonequivalent star products on either. Since every star product is thought of as a quantization of any given classical system, nonequivalent star products represent different quantizations of said system. But then one immediately recognizes that an independent choice of $\star$ on M and $\star_{\mathrm{red}}$ on M_{red} introduces too much ambiguity: just as the classical system is determined completely by either M or M_{red}, at the moment, we should be able to fix a quantization of the system by specifying either $\star$ or $\star_{\mathrm{red}}$ to completely specify a choice of quantization. In short, we need an analogue to symmetry reduction for star products, which is commonly referred to as quantum reduction. Quantum reduction should allow us to construct $\star_{\mathrm{red}}$ from a given choice of $\star$ on M. To summarize, a first proposal for quantum reduction would be, given

$$M \hookrightarrow C \twoheadrightarrow M_{\mathrm{red}},$$

we wish to construct a map

$$_{\text{naive red}}\colon \mathrm{Star}(M, \omega) \longrightarrow \mathrm{Star}(M_{\mathrm{red}}, \omega_{\mathrm{red}}).$$

However, using the set of all star products on M seems unnatural since there might be ones that do not respect the given gauge symmetry on M and one would not expect to obtain meaningful results on M_{red} from those. Since furthermore M already comes equipped with a momentum map (in order to perform Marsden-Weinstein reduction at all), it seems more natural to only use star products compatible with the given structures. In this case, given the Hamiltonian G-space (M, ω, G, μ) and the Lie algebra $\mathfrak{g}$ of G, we are lead to use equivariant star products on M that deform the momentum map J_μ (see Remark 4.1.2). Consequently, quantum reduction should be a map

$$_{\mathrm{red}}\colon \mathrm{Star}_{\mathfrak{g}}(M, \omega, J_\mu) \longrightarrow \mathrm{Star}(M_{\mathrm{red}}, \omega_{\mathrm{red}}).$$

Since, in general, M_{red} need not have any further gauge symmetry, we can only map into the set of all star products on M_{red}. And indeed, such a map has been constructed in [16], which we will present here, following largely the more compact exposition in [61]. But before, let us fix some conventions. Throughout, we will assume that (M, ω, G, μ) is a Hamiltonian G-space for a connected, finite-dimensional Lie group G and that the action $\triangleright$ of G on M is proper. Furthermore, we shall assume that the restricted action of G on $C := \mu^{-1}(\{0\})$ is free and proper (note that properness is not a restriction here, as all restrictions of proper actions are again proper). Finally, we agree to reserve the symbols ι for the inclusion $\iota\colon C \hookrightarrow M$ and π for the surjective submersion $\pi\colon C \twoheadrightarrow M_{\mathrm{red}}$. The reason for working with connected Lie groups exclusively is to use Corollary 2.3.20 and Theorem 2.3.23 later on. Contrarily, we will from the very beginning use the properness of $\triangleright$ to apply the following result [61], [16, Lemma 3]:

Lemma 4.2.1 *Let (M, ω, G, μ) be a Hamiltonian G-space with proper action of G on M, assume that 0 is a regular value and value of μ, and define $C := \mu^{-1}(\{0\})$. Then there exists an open neighbourhood $\tilde{M} \subseteq M$ of C and an open neighbourhood $\tilde{U} \subseteq C \times \mathfrak{g}^*$ of $C \times \{0\}$ such that:*

i) There exists a G-equivariant diffeomorphism

$$\Phi\colon \tilde{M} \longrightarrow \tilde{U}$$

where the G action on $\tilde{U}$ is the product $\triangleright\big|_C \times \mathrm{Ad}^*$ of the restriction of $\triangleright$ on C and the coadjoint action on $\mathfrak{g}^*$.

ii) For each $p \in C$ the set $\tilde{M} \cap (\{p\} \times \mathfrak{g}^*)$ is star shaped around $\{p\} \times \{0\}$.

iii) The restriction of μ on $\tilde{M}$ is given by the projection onto the second factor of $C \times \mathfrak{g}^*$:

$$\mu\big|_{\tilde{M}} = \mathrm{pr}_2 \circ \Phi$$

Since we will ultimately be interested in the quotient C/G and since star products (at least the types considered in this thesis, see Definition 3.0.7), are inherently local objects on M, Lemma 4.2.1 allows us to assume that $M = \tilde{M}$. Now one central object in this particular quantum reduction scheme is the so-called Koszul complex, which comes in two variations: a classical version and a quantum version. We begin by introducing the classical one.

Definition 4.2.2 (Koszul complex) *Let (M, ω, G, μ) be a Hamiltonian G-space. The classical Koszul complex is defined as*

$$\mathrm{C}^\bullet_{\mathrm{Koszul}}(M, \mathfrak{g}, \mu) := \mathbb{C} \otimes \mathscr{C}^\infty(M) \otimes \Lambda^\bullet \mathfrak{g} \qquad \text{with differential} \qquad \partial = \mathrm{i}_\mu \,.$$

We will frequently abbreviate $\mathrm{C}_{\mathrm{Koszul}}(M, \mathfrak{g}, \mu)$ to $\mathrm{C}_{\mathrm{Koszul}}(M)$ if $\mathfrak{g}$ and μ are understood. We will also drop the explicit complexification in the following.

Remark 4.2.3 G acts on the Koszul complex by the tensor product of the induced action by pullbacks on $\mathscr{C}^\infty(M)$ and the adjoint representation of G on $\mathfrak{g}$. All later references to equivariance of maps on the Koszul complex mean equivariance with respect to this action.

As a first proposition, we shall consider the cohomology $\mathrm{H}^\bullet_{\mathrm{Koszul}}(M)$ of the Koszul complex and find, that the Koszul complex is acyclic and that one can even find an explicit contraction. Furthermore, by viewing $\mathscr{C}^\infty(C)$ as a $\mathscr{C}^\infty(M)$-module via the prolongation map

$$\mathrm{prol}\colon \mathscr{C}^\infty(C) \longrightarrow \mathscr{C}^\infty(M)\colon \phi \longmapsto (\mathrm{pr}_1 \circ \Phi)^* \phi \tag{4.2.1}$$

and the module structure (where one uses $\iota^* \circ \mathrm{prol} = \mathsf{id}_{\mathscr{C}^\infty(C)}$)

$$f \cdot \phi := \iota^*(f \, \mathrm{prol}(\phi)) \qquad \text{for all} \qquad f \in \mathscr{C}^\infty(M),\ \phi \in \mathscr{C}^\infty(C),$$

one can observe that the Koszul complex is a free resolution of $\mathscr{C}^\infty(C)$ as $\mathscr{C}^\infty(M)$-modules [61, Prop. 2.1], [16, Lemma 5,6], [48]:

Proposition 4.2.4 *Let (M, ω, G, μ) be a Hamiltonian G-space, $\mathfrak{g}$ the Lie algebra of G and $\{e_a\}$ a basis of $\mathfrak{g}$. Then the augmented Koszul complex with*

$$\mathrm{C}^{-1}_{\mathrm{Koszul}}(M) \coloneqq \mathscr{C}^\infty(C) \qquad \text{and} \qquad \partial_0 = \iota^* \colon \mathrm{C}^0_{\mathrm{Koszul}}(M) \longrightarrow \mathrm{C}^{-1}_{\mathrm{Koszul}}(M)$$

is exact via the G-equivariant contraction h defined (for $k \geq 0$) by

$$h_k \colon \mathrm{C}^k_{\mathrm{Koszul}}(M) \longrightarrow \mathrm{C}^{k+1}_{\mathrm{Koszul}}(M) \colon (h_k f)(p) \coloneqq e_a \wedge \int_0^1 t^k \frac{\partial (f \circ \Phi^{-1})}{\partial \xi^*_a}(c, t\xi^*)\, \mathrm{d}t$$

and

$$h_{-1} \colon \mathrm{C}^{-1}_{\mathrm{Koszul}}(M) \longrightarrow \mathrm{C}^0_{\mathrm{Koszul}}(M) \colon h_{-1} f \coloneqq \operatorname{prol} f$$

for all $p \in M$ with $\Phi(p) = (c, \xi^*) \in C \times \mathfrak{g}^*$ and $\xi^* = e^a \xi^*_a$ for the dual basis $\{e^a\}$ of $\{e_a\}$:

$$h_{k-1}\partial_k + \partial_{k+1} h_k = \mathsf{id} \qquad \text{for all} \qquad k \geq -1$$

Proof: We shall, for simplicity of notation, assume $M \subseteq C \times \mathfrak{g}^*$ and $\phi = \mathsf{id}_M$ for the duration of the proof. For $k = -1$ then, take any $f \in \mathscr{C}^\infty(C)$. Since prol is right-inverse to ι^* and $\partial^0 = 0$ we have automatically

$$(h_{-2}\partial_0 + \partial^1 h_{-1})(f) = \iota^* \operatorname{prol}(f) = f.$$

Note that, as per convention, the augmented Koszul complex and h are extended by 0 to the left. For $k = 0$ take any $f \in \mathscr{C}^\infty(M)$ and note that

$$\frac{\mathrm{d}}{\mathrm{d}\, t}[f(c, t\xi)] = \xi_a \frac{\partial f}{\partial \xi_a}(c, t\xi)$$

holds for all $(c, \xi) \in M$, which, together with $\mu(c, \xi) = \xi$, yields

$$\begin{aligned}
(\partial_1 h_0 f)(c, \xi) &= (\mathrm{i}_{\mu(c,\xi)}\, e_a) \int_0^1 \frac{\partial f}{\partial \xi_a}(c, t\xi)\, \mathrm{d}t = \int_0^1 \xi_a \frac{\partial f}{\partial \xi_a}(c, t\xi)\, \mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\, t}[f(c, t\xi)]\, \mathrm{d}t \\
&= f(c, \xi) - f(c, 0) = f(c, \xi) - (\operatorname{prol} \iota^* f)(c, \xi) = f(c, \xi) - (h_{-1}\partial_0 f)(c, \xi).
\end{aligned}$$

Finally, for $k \geq 1$, let $f \in \mathscr{C}^\infty(M)$ and $\theta \in \Lambda^k \mathfrak{g}$. By elementary calculations, one can show that

$$e_a \wedge \mathrm{i}_{e^a} \theta = k \cdot \theta \qquad \text{and} \qquad \left[\frac{\partial}{\partial \xi_a} \mathrm{i}_\mu\, \theta\right](c, t\xi) = \mathrm{i}_{e^a} \theta$$

hold. With these, we can, similar to the case $k = 0$, derive the homotopy identity

$$\begin{aligned}
&[(h_{k-1}\partial_k \; + \; \partial_k h_k) f\theta](c, \xi) \\
&\quad = e_a \wedge \int t^{k-1} \left[\frac{\partial}{\partial \xi_a}(f\, \mathrm{i}_\mu\, \theta)\right](c, t\xi)\, \mathrm{d}t + \mathrm{i}_{\mu(c,\xi)} \left[e_a \wedge \int t^k \theta \frac{\partial f}{\partial \xi_a}(c, t\xi)\, \mathrm{d}t\right] \\
&\quad = e_a \wedge \int t^{k-1}\, \mathrm{i}_{\mu(c,t\xi)}\, \theta \frac{\partial f}{\partial \xi_a}(c, t\xi) + f(c, t\xi)\, \mathrm{i}_{e^a}\, \theta\, \mathrm{d}t \\
&\qquad + \xi_a \int t^k \theta \frac{\partial f}{\partial \xi_a}(c, t\xi)\, \mathrm{d}t - e_a \wedge \int t^k\, \mathrm{i}_{\mu(c,\xi)}\, \theta \frac{\partial f}{\partial \xi_a}(c, t\xi)\, \mathrm{d}t \\
&\quad = \int t^{k-1} \left[k\theta f(c, t\xi) + t\xi_a \theta \frac{\partial f}{\partial \xi_a}(c, t\xi)\right] \mathrm{d}t \\
&\quad = \theta \int \frac{\mathrm{d}}{\mathrm{d}\, t}\left[t^k f(c, t\xi)\right] \mathrm{d}t = (f\theta)(c, \xi)
\end{aligned}$$

and conclude the proof. □

Viewed slightly differently, Proposition 4.2.4 states, that ι^* (extended by 0 to higher degrees) is a quasi-isomorphism with homotopy inverse prol

$$\begin{tikzcd}
0 & \mathscr{C}^\infty(M) \arrow[l] \arrow[d, shift right, "\iota^*"'] & \mathscr{C}^\infty(M) \otimes \Lambda^1 \mathfrak{g} \arrow[l, "\partial"'] & \dots \arrow[l, "\partial"'] \\
0 & \mathscr{C}^\infty(C) \arrow[l] \arrow[u, shift right, "\mathrm{prol}"'] & 0 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

In fact, the maps ι^*, prol and h constitute a deformation retract according to [29] (see also Definition 2.4.2):

$$\iota^* \circ \mathrm{prol} = \mathsf{id}_{\mathscr{C}^\infty(C)} \qquad \mathsf{id}_{\mathscr{C}^\infty(M)} - \mathrm{prol} \circ \iota^* = \partial h + h \partial$$

and one can additionally check that $h_0 \circ \mathrm{prol} = 0$ holds (see e.g. [61]).

Remark 4.2.5 The unique symplectic structure on M_{red} from Theorem 4.1.3 can easily be expressed using the prolongation map prol:

$$\pi^*\{f, g\}_{\mathrm{red}} = \iota^*\{\mathrm{prol}(\pi^* f), \mathrm{prol}(\pi^* g)\} \tag{4.2.2}$$

where $\{\ ,\ \}$ is the Poisson bracket on M.

Turning towards quantum reduction, we first consider the deformation retract

$$\begin{tikzcd}
(\mathscr{C}^\infty(C)[\![\nu]\!], 0) \arrow[r, shift right, "\mathrm{prol}"'] & (\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!], \partial) \arrow[l, shift right, "\iota^*"'] \arrow[loop right, "h"]
\end{tikzcd}$$

where ι^*, prol and h have been extended ν-linearly. Of course, Proposition 4.2.4 still holds order by order in ν. Now on $\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$, one can define the so-called quantized or deformed Koszul differential, see [16]:

Definition 4.2.6 (Deformed Koszul operator) *Let $(\star, \mathbf{J})$ be an equivariant star product on a Hamiltonian G-space (M, ω, G, μ) deforming J_μ and let $\kappa \in C[\![\nu]\!]$. The deformed Koszul operator $\boldsymbol{\partial}\colon \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!] \longrightarrow \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$ is defined by*

$$\boldsymbol{\partial} f := \mathrm{i}_{e^a} f \star \mathbf{J}(e_a) + \frac{\nu}{2} g_{ab}^c e_c \wedge \mathrm{i}_{e^a} \mathrm{i}_{e^b} f + \nu\kappa\, \mathrm{i}_\Delta f$$

for all $f \in \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$, where $\{e_a\}$ is any basis of $\mathfrak{g}$, $\{e^a\}$ its dual basis, $g_{ab}^c = e^c([e_a, e_b])$ are the structure constants of $\mathfrak{g}$ and

$$\Delta(\xi) = \mathsf{tr}\,\mathrm{ad}(\xi) \qquad \text{for } \xi \in \mathfrak{g}$$

is the modular one-form $\Delta \in \mathfrak{g}^*$ of $\mathfrak{g}$.

Here we extended $\star$ implicitly from $\mathscr{C}^\infty(M)[\![\nu]\!]$ to $\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$ as $(f \otimes \alpha) \star (g \otimes \beta) := (f \star g) \otimes \alpha \wedge \beta$ on factorizing tensors $(f \otimes \alpha), (f \otimes \beta) \in \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!] = (\mathscr{C}^\infty(M) \otimes \Lambda^\bullet \mathfrak{g})[\![\nu]\!]$. Since the precise value of κ in Definition 4.2.6 will not matter in the following, we will henceforth assume that we chose a fixed value once and for all and omit all further mention of it. Some basic properties of $\boldsymbol{\partial}$ can be found in [61, Lemma 3.4]:

Lemma 4.2.7 *Let $\eth$ be as in Definition 4.2.6. Then $\eth$ is a G-equivariant, left $\star$-linear differential on $\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$ with classical limit ∂, that is*

$$\eth\big|_{\nu=0} = \partial \qquad \textit{and} \qquad \eth^2 = 0.$$

Lemma 4.2.7 implies that $\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$ together with $\eth$ is a chain complex, which we will call the deformed Koszul complex. Especially the classical limit of $\eth$ will be important in the following. It shows in particular, that $\delta := \eth - \partial$ only starts in order 1 of ν, hence $(\mathsf{id} - \delta h)$ is invertible as a formal power series. This will allow us to apply the homological perturbation lemma (see Lemma 2.4.5) in order to find quasi-isomorphisms I^* and Prol, as well as a contraction H of $(\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!], \eth)$ such that

$$\begin{tikzcd}
0 & \mathscr{C}^\infty(M)[\![\nu]\!] \arrow[l] \arrow[r, shift right, "H"'] \arrow[d, shift right, "I^*"'] & (\mathscr{C}^\infty(M) \otimes \Lambda^1\mathfrak{g})[\![\nu]\!] \arrow[l, shift right, "\eth"'] \arrow[r, shift right, "H"'] & \cdots \arrow[l, shift right, "\eth"'] \\
0 & \mathscr{C}^\infty(C)[\![\nu]\!] \arrow[l] \arrow[u, shift right, "\mathrm{Prol}"'] & 0 \arrow[l] &
\end{tikzcd} \tag{4.2.3}$$

is again a deformation retract with

$$I^* = i^*\big(\mathsf{id}_{\mathscr{C}^\infty(M)[\![\nu]\!]} + \delta_1 h_0\big)^{-1} \qquad \mathrm{Prol} = \mathrm{prol} \qquad H = h\big(\mathsf{id}_{\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]} + \delta h\big)^{-1},$$

see (2.4.9). Note that I^* is not the pullback with any map I, the notation is chosen purely to remind of its origins. One important feature of I^* is the following [16, Lemma 27], [61, Lemma 3.6]:

Lemma 4.2.8 *There is a formal series of G-invariant differential operators $S = \mathsf{id} + \sum_{k=1}^{\infty} \nu^k S_k$ on M such that*

$$I^* = \iota^* \circ S.$$

Moreover, S can be chosen such that $S(1) = 1$.

This statement is insofar peculiar, since the homotopy operator h, which appears in the definition of I^*, is clearly non-local, whereas I^* is. Finally, one follows the algebraic point of view of symplectic reduction (see e.g. [72]), wherein, instead of a manifold M_{red} together with a symplectic two-form ω_{red}, the Poisson algebra of functions $(\mathscr{C}^\infty(M_{\mathrm{red}}), \{\ ,\ \}_{\mathrm{red}})$ is constructed directly via

$$\mathscr{C}^\infty(M_{\mathrm{red}}) \cong \frac{B_C}{J_C}, \tag{4.2.4}$$

where J_C denotes the vanishing ideal of the submanifold C in $\mathscr{C}^\infty(M)$ and B_C the normalizer of J_C in $\mathscr{C}^\infty(M)$:

$$\begin{aligned} J_C &= \{f \in \mathscr{C}^\infty(M) \mid \iota^* f = f\big|_C = 0\} \\ B_C &= \{f \in \mathscr{C}^\infty(M) \mid \{f, J_C\} \subseteq J_C\}. \end{aligned}$$

Let us quickly note that, by the exactness of the augmented Koszul complex, we have $J_C = \ker \iota^* = \operatorname{im} \partial_1$. This expression can then be used to obtain analogues of the vanishing ideal J_C and its normalizer B_C for the deformed Koszul complex, where we define

$$\begin{aligned} \mathcal{J}_C &:= \operatorname{im} \eth_1 \subseteq \mathscr{C}^\infty(M)[\![\nu]\!] \\ \mathcal{B}_C &:= \{f \in \mathscr{C}^\infty(M)[\![\nu]\!] \mid [f, \mathcal{J}_C]_\star \subseteq \mathcal{J}_C\}. \end{aligned}$$

Note that, since $\mathfrak{d}$ is left-$\star$-linear, we automatically have $\mathcal{J}_C \subseteq \mathcal{B}_C$. This allows us to define the following pair of maps

$$\begin{array}{ccccccc} \mathcal{B}_C/\mathcal{J}_C & \longrightarrow & \pi^*\mathscr{C}^\infty(M_{\mathrm{red}})[\![\nu]\!] & : & [f] & \longmapsto & I^*f \\ \pi^*\mathscr{C}^\infty(M_{\mathrm{red}})[\![\nu]\!] & \longrightarrow & \mathcal{B}_C/\mathcal{J}_C & : & u & \longmapsto & [\mathrm{prol}(u)] \end{array} \tag{4.2.5}$$

where the first map is well-defined since $\operatorname{im}\mathfrak{d}_1 \subseteq \ker I^*$. But even further, one can show that, as a direct consequence of (4.2.3) being a deformation retract, those maps are mutually inverse: on the one hand we have $I^*\mathrm{Prol} = \mathsf{id}$ and on the other $[\mathrm{Prol}I^*f] = [f - \mathfrak{d}_1 H_1 f] = [f]$ for $f \in \mathcal{B}_C$. Hence we have just demonstrated the quantum analogon to (4.2.4), namely

$$\mathscr{C}^\infty(M_{\mathrm{red}})[\![\nu]\!] \cong \frac{\mathcal{B}_C}{\mathcal{J}_C}. \tag{4.2.6}$$

Subsequently then, we will use this bijection to define a star product $\star_{\mathrm{red}}$ on M_{red}, similarly to (4.2.2), through

$$\pi^*(u \star_{\mathrm{red}} v) := I^*(\mathrm{Prol}(\pi^*u) \star \mathrm{Prol}(\pi^*v)) \tag{4.2.7}$$

for all $u, v \in \mathscr{C}^\infty(M_{\mathrm{red}})$, as shown in [16, 61]:

Proposition 4.2.9 *Let (M, ω, G, μ) be a Hamiltonian G-space and $(\star, \mathbf{J})$ an equivariant star product on M deforming J_μ. Then*

$$\pi^*(u \star_{\mathrm{red}} v) := I^*(\mathrm{Prol}(\pi^*u) \star \mathrm{Prol}(\pi^*v))$$

defines a star product on $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$.

The proof of Proposition 4.2.9 relies on the fact, that π^* and Prol are maps of formal degree 0 and that I^* can be expressed as the concatenation of another formal degree 0 map (namely ι^*) with an equivalence of star products, due to Lemma 4.2.8. This clarifies that $\star_{\mathrm{red}}$ is indeed a formal series of bidifferential operators, as required for star products. Using the definition of ω_{red} from (4.2.2), one can immediately check that $\star_{\mathrm{red}}$ deforms ω_{red}.

Remark 4.2.10 Of course, $\star_{\mathrm{red}}$ depends on the parameter κ from the definition of the quantized Koszul operator $\mathfrak{d}$ in Definition 4.2.6, which we omitted since then. One could stress this dependency by denoting the reduced star product as $\star_{\mathrm{red}}^{(\kappa)}$.

With Proposition 4.2.9 then we achieved the goal we set out to, namely to obtain a reduction map for star products

$$_{\mathrm{red}}\colon \mathrm{Star}(M, \omega, J_\mu) \longrightarrow \mathrm{Star}(M, \omega_{\mathrm{red}}).$$

4.3 Characteristic Classes of Reduced Star Products

Having obtained a quantum reduction scheme for equivariant star products in the previous section, we will now aim to combine symmetry reduction with the classification results of star products from Section 3.2, Section 3.3 and Section 3.4, which has been achieved by the author in [103]. The main question arising in this context is the following: As Theorem 3.4.11 gives a classification of all equivariant star products on M by the cohomology of the Cartan complex through a bijection

$$c_{\mathfrak{g}}\colon \mathrm{Star}_{\mathfrak{g}}(M, \omega, \mu) \longrightarrow \frac{[\omega - J_\mu]_{\mathfrak{g}}}{\nu} + \mathrm{H}^2_{\mathfrak{g}}(M)[\![\nu]\!]$$

and Theorem 3.2.8 gives a classification of all star products on M_{red} by the de Rham cohomology through a bijection

$$c\colon \mathrm{Star}(M_{\mathrm{red}}, \omega_{\mathrm{red}}) \longrightarrow \frac{[\omega_{\mathrm{red}}]}{\nu} + \mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}})$$

we are, at first glance, left with two characteristic classes, namely $c_{\mathfrak{g}}(\star, \mathbf{J})$ and $c(\star_{\mathrm{red}})$ for any equivariant star product $(\star, \mathbf{J})$ on M. Is there any relation between those classes? One would certainly expect there to be one, since $\star$ and $\star_{\mathrm{red}}$ are clearly not independent star products on their respective manifolds. However, for any such relation to be meaningful in the first place, we should examine, whether or not equivariantly equivalent star products on M reduce to equivalent star products on M_{red}. If this was not the case, then the classification by equivariant cohomology would be to coarse for the problem at hand and we would have to find a more fine-grained classification. Fortunately, we can use the defining formula for $\star_{\mathrm{red}}$ from Proposition 4.2.9 to explicitly give a reduced equivalence on M_{red} for any equivariant equivalence on M [103]:

Lemma 4.3.1 *Let M_{red} be Marsden-Weinstein reduced from M and let $T\colon (\star^1, \mathbf{J}_1) \longrightarrow (\star^2, \mathbf{J}_2)$ be an equivariant equivalence. Then*

$$T_{\mathrm{red}} := \big((\pi^*)^{-1} \circ I^*\big) \circ T \circ (\mathrm{prol} \circ \pi^*)$$

is an equivalence $T_{\mathrm{red}}\colon \star^1_{\mathrm{red}} \longrightarrow \star^2_{\mathrm{red}}$.

Proof: We repeat the proof from [103]. First, note that $\pi^*\colon \mathscr{C}^\infty(M_{\mathrm{red}})[\![\nu]\!] \longrightarrow \mathrm{im}(I^*)$ is well-defined and invertible (see (4.2.5) and (4.2.6)). Next, observe that T_{red}, expanded as a power series in ν, starts with $\mathsf{id}_{\mathscr{C}^\infty(M_{\mathrm{red}})}$, since T starts with $\mathsf{id}_{\mathscr{C}^\infty(M)}$, I^* starts with ι^* and prol is right-inverse to ι^*. Also, by Lemma 4.2.8, T_{red} is a series of differential operators. All that remains, is to show that T_{red} is well-defined as a map between the reduced observable algebras $\mathscr{C}^\infty(M^1_{\mathrm{red}})$ and $\mathscr{C}^\infty(M^2_{\mathrm{red}})$. To do so, denote by $\mathscr{J}^1_C$, $\mathscr{J}^2_C$, $\mathscr{B}^1_C$ and $\mathscr{B}^2_C$ the respective vanishing ideals and their normalizers (see (4.2.6)) and by $(\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!], \mathfrak{d}^1)$ and $(\mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!], \mathfrak{d}^2)$ the respective deformed Koszul complexes (see Definition 4.2.6). Note here, that the momentum-level-set C as well as the underlying sets of the deformed Koszul complexes are the same in both cases, since $\mathbf{J}_1$ and $\mathbf{J}_2$ deform the same classical momentum map. Consider then the map

$$T \otimes \mathsf{id}_{\Lambda^\bullet \mathfrak{g}}\colon \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!] \longrightarrow \mathrm{C}_{\mathrm{Koszul}}(M)[\![\nu]\!]$$

and take any $f \in \mathrm{C}_{\mathrm{Koszul}}(M)$ to calculate

$$\begin{aligned} T\mathfrak{d}^1 f &= T\Big(\mathrm{i}_{e^a} f \star^1 \mathbf{J}_1(e_a) + \frac{\nu}{2} g^c_{ab} e_c \wedge \mathrm{i}_{e_b} f + \nu\kappa\, \mathrm{i}_\Delta f\Big) \\ &= \mathrm{i}_{e^a} Tf \star^2 \mathbf{J}_2(e_a) + \frac{\nu}{2} g^c_{ab} e_c \wedge \mathrm{i}_{e_b} Tf + \nu\kappa\, \mathrm{i}_\Delta Tf \\ &= \mathfrak{d}^2 Tf, \end{aligned}$$

showing that $T \otimes \mathsf{id}$ is a chain map between the deformed Koszul complexes. As such, T can be restricted to $\mathscr{J}^1_C = \mathrm{im}\,\mathfrak{d}^1$:

$$T\big|_{\mathscr{J}^1_C}\colon \mathscr{J}^1_C \longrightarrow \mathscr{J}^2_C.$$

T being invertible then implies directly $\mathcal{J}_C^1 \cong \mathcal{J}_C^2$. And furthermore, for any $j_2 \in \mathcal{J}_C^2$, $j_1 := T^{-1}j_2$ and $f \in \mathcal{B}_C^1$ calculate

$$[Tf, j_2]_{\star^2} = [Tf, Tj_1]_{\star^2} = T[f, j_1]_{\star^1} \in T\mathcal{J}_1 = \mathcal{J}_2$$

which establishes $\mathcal{B}_C^1 \cong \mathcal{B}_C^2$. Consequently, T induces a bijection on the quotients $\mathcal{B}_C^1/\mathcal{J}_C^1 \cong \mathcal{B}_C^2/\mathcal{J}_C^2$ and we can immediately conclude, by virtue of Proposition 4.2.9 and (4.2.5), that T_{red} is an equivalence between $\star_{\mathrm{red}}^1$ and $\star_{\mathrm{red}}^2$. □

One result connecting the characteristic classes of equivariant star products and their corresponding reduced star products has already been given in [15], on which we shall elaborate:

Lemma 4.3.2 *Let M_{red} be Marsden-Weinstein reduced from M via C and with principal bundle projection $\pi\colon C \longrightarrow M_{\mathrm{red}}$ and inclusion $\iota\colon C \longrightarrow M$. Additionally, let $(\star, \mathbf{J})$ be an equivariant star product on M and $\star_{\mathrm{red}}$ the corresponding reduced star product on M_{red}. Then we have*

$$\iota^* c(\star) = \pi^* c(\star_{\mathrm{red}}).$$

Lemma 4.3.2 poses a very clear restriction to the possible combinations of values for $c_{\mathfrak{g}}(\star, \mathbf{J})$ and $c(\star_{\mathrm{red}})$, confirming our initial suspicion. Note however, that Lemma 4.3.2 inherently views equivariant star products as just star products, by using its characteristic class instead of the equivariant characteristic class. Consequently, one is not able to differentiate between star products, that have been reduced from equivalent equivariant star products which are not equivariantly equivalent. Also, depending on the topology of the momentum level set C, the condition $\iota^* c(\star) = \pi^* c(\star_{\mathrm{red}})$ may become a tautology, as exemplified by the Hopf-fibration

$$\mathbb{C}^{n+1} \setminus \{0\} \hookleftarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n.$$

Here one has $\mathrm{H}^2_{\mathrm{dR}}(S^{2n+1}) = 0$ and $\iota^*\alpha = \pi^*\beta$ is true for all $\alpha \in \mathrm{H}^2_{\mathrm{dR}}(M)$ and $\beta \in \mathrm{H}^2_{\mathrm{dR}}(M_{\mathrm{red}})$. So for the remaining part, we wish to find a refinement of Lemma 4.3.2, which, on one hand, takes equivariant classes into account and, on the other hand, circumvents possible trivial cohomologies of C. One can even obtain hints on the nature of this refinement, once we remember from Theorem 2.3.15 that, in the case of a compact, connected Lie group G acting on a compact manifold M, the cohomology of the Lie algebraic Cartan complex is actually isomorphic to the (topological) equivariant cohomology $\mathrm{HE}_G(M)$ of M. Hence equivariant star products are classified by $\mathrm{HE}^2_G(M)$. But for (topological) equivariant cohomology of a Hamiltonian G-space there is a well-known homomorphism from $\mathrm{HE}_G(M)$ to the singular cohomology of its symplectic quotient $M /\!\!/ G := \mu^{-1}(\{0\})/G$, namely the Kirwan map [74]

$$K\colon \mathrm{HE}_G(M) \longrightarrow \mathrm{H}_{\mathrm{Sing}}(M /\!\!/ G),$$

whenever 0 is a value and regular value of μ. To obtain K, it is shown that the surjection $\pi\colon \mu^{-1}(\{0\}) \longrightarrow M /\!\!/ G$ gives rise to an isomorphism on cohomology, that is

$$\pi^*\colon \mathrm{HE}_G\big(\mu^{-1}(\{0\})\big) \cong \mathrm{H}_{\mathrm{Sing}}(M /\!\!/ G).$$

K is then constructed as the concatenation of $\iota^*\colon \mathrm{HE}_G(M) \longrightarrow \mathrm{HE}_G(\mu^{-1}(\{0\}))$ with $(\pi^*)^{-1}$. For the special case, where the restricted action of G on $\mu^{-1}(\{0\})$ is free and proper, we see immediately that $M /\!\!/ G = M_{\mathrm{red}}$ holds. Furthermore, since M_{red} is a manifold, we can use the de Rham theorem (see e.g. [20]) to obtain a homomorphism

$$\begin{tikzcd}
& \mathrm{HE}_G(M) \arrow[dl, "\iota^*"'] & \cong & \mathrm{H}_{\mathfrak{g}}(M) \arrow[dd, "K"] \\
\mathrm{HE}_G(\mu^{-1}(\{0\})) \arrow[dr, "(\pi^*)^{-1}"'] & & & \\
& \mathrm{H}_{\mathrm{Sing}}(M_{\mathrm{red}}) & \cong & \mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}})
\end{tikzcd}$$

By leaving out the middle cohomologies, one could suspect, that there is a direct map $K\colon \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}})$, even in the noncompact case. Having such a map would provide us with the means to compare $c_{\mathfrak{g}}(\star, \mathbf{J})$ with $c(\star_{\mathrm{red}})$ simply by comparing $K(c_{\mathfrak{g}}(\star, \mathbf{J}))$ with $c(\star_{\mathrm{red}})$ in $\mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}})$, where we implicitly use the ν-linear extension of K. And indeed, we can define a suitable map K with the preparations done in Section 2.3.3. More specifically, let M_{red} be Marsden-Weinstein reduced from M via C by the action of a connected Lie group G. If we then use the fact, that C ist a G-principal bundle with connected fibres from Section 4.1, Corollary 2.3.20 states that $\pi^*\colon \Omega(M_{\mathrm{red}}) \longrightarrow \Omega_{\mathrm{bas}}(C)$ is a chain isomorphism. Accordingly, the induced map on cohomology

$$\pi^*\colon \mathrm{H}(\Omega(M_{\mathrm{red}})) \longrightarrow \mathrm{H}(\Omega_{\mathrm{bas}}(C))$$

is also an isomorphism. Together with Theorem 2.3.23 we conclude then that π^* can actually be viewed as an isomorphism (see also Remark 2.3.24)

$$\pi^*\colon \mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}}) \longrightarrow \mathrm{H}_{\mathfrak{g}}(C).$$

All that is left from here on, is to concatenate π^* with $\mathrm{H}_{\mathfrak{g}}(\iota)\colon \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathfrak{g}}(C)$, which we will for brevity also denote by ι^*, to obtain:

Lemma 4.3.3 *Let M_{red} be Marsden-Weinstein reduced from M via C by the action of a connected Lie group G. Then the map*

$$K = (\pi^*)^{-1} \circ \iota^*\colon \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathfrak{g}}(C) \longrightarrow \mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}})$$

is well-defined and surjective.

PROOF: In the preceeding discussion we have already demonstrated that K is well-defined. To see that K is also surjective, it is sufficient to recognize that ι^* is surjective, since π^* is already an isomorphism. For this, consider the prolongation map (4.2.1). Its very definition extends immediately to a chain map on differential forms

$$\mathrm{prol}\colon \Omega(C) \longrightarrow \Omega(M)\colon \alpha \longmapsto (\mathrm{pr}_1 \circ \Phi)^*\alpha$$

and one can readily verify that the extended prol is a right-inverse to ι^*. But then prol is also a right-inverse to ι^* on cohomology and hence ι^* must be surjective on cohomology. □

With K at hand, it is finally possible to tackle our initial problem by comparing $K(c_{\mathfrak{g}}(\star, \mathbf{J}))$ to $c(\star_{\mathrm{red}})$ in $\mathrm{H}_{\mathrm{dR}}(M_{\mathrm{red}})$ (see [103]):

Theorem 4.3.4 *Let $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be Marsden-Weinstein reduced from (M, ω, G, μ) via C by the action of a connected Lie group G. Additionally, let $(\star, \mathbf{J})$ be an equivariant star product on M and $\star_{\mathrm{red}}$ the corresponding reduced star product on M_{red}. Then we have*

$$K(c_{\mathfrak{g}}(\star, \mathbf{J})) = c(\star_{\mathrm{red}}).$$

PROOF: The formal degree -1 parts of $c_{\mathfrak{g}}(\star, \mathbf{J})$ and $c(\star_{\mathrm{red}})$ are given by $[\omega - J_\mu]_{\mathfrak{g}}$ and $[\omega_{\mathrm{red}}]$ respectively. Since $J_\mu\big|_C = 0$ by definition, we have immediately

$$\iota^*(\omega - J_\mu) = \pi^*\omega_{\mathrm{red}}$$

from Theorem 4.1.3 and hence $K(c(\star, \mathbf{J})) = c(\star_{\mathrm{red}})$ holds in formal degree -1. For the remaining degrees, we will denote terms of formal degree higher than -1 with an index $_+$, e.g. $c(\star_{\mathrm{red}})_+ := c(\star_{\mathrm{red}}) - [\omega_{\mathrm{red}}] \in \mathrm{H}_{\mathrm{dR}}(M)[\![\nu]\!]$. We start by choosing an equivariant Fedosov star product $(F(\Omega), \mathbf{J}')$ on M equivariantly equivalent to $(\star, \mathbf{J})$ (see Corollary 3.4.6, [104]) and a Fedosov star product $F(\Omega_{\mathrm{red}})$ on M_{red} equivalent to $\star_{\mathrm{red}}$ (see Proposition 3.2.4, [12]). Theorem 3.2.8 and Theorem 3.4.11 allow us then to use these Fedosov star products for all calculations. And furthermore, those theorems allow us to state

$$\begin{aligned} \iota^* c_{\mathfrak{g}}(F(\Omega), \mathbf{J}')_+ &= \frac{1}{\nu}\big[\iota^*(\Omega - \mathbf{J}'_+)\big]_{\mathfrak{g}} \\ \pi^* c(F(\Omega_{\mathrm{red}}))_+ &= \frac{1}{\nu}[\pi^*\Omega_{\mathrm{red}}]. \end{aligned}$$

Writing $\tilde{\Omega} := K(c_{\mathfrak{g}}(F(\Omega), \mathbf{J}'))$ for brevity, we have $\iota^* c_{\mathfrak{g}}(F(\Omega), \mathbf{J}') = \big[\pi^* K(c_{\mathfrak{g}}(F(\Omega), \mathbf{J}'))\big]_{\mathfrak{g}}$ from the very definition of K, which is equivalent to the existence of a $\theta \in \Omega^1(C)^{\mathfrak{g}}[\![\nu]\!]$ such that

$$\mathrm{d}_{\mathfrak{g}}\theta = \iota^*(\Omega - \mathbf{J}'_+) - \pi^*\tilde{\Omega} \quad \text{or equivalently} \quad \iota^*\Omega - \pi^*\tilde{\Omega} = \mathrm{d}\theta \quad \text{and} \quad \mathrm{i}_\bullet\,\theta = -\mathbf{J}'_+.$$

On the other hand, we know from Lemma 4.3.2 that $\iota^*\Omega - \pi^*\Omega_{\mathrm{red}} = \mathrm{d}\upsilon$ for some $\upsilon \in \Omega^1(C)[\![\nu]\!]$. From the previous two statements we can then conclude that

$$\pi^*\tilde{\Omega} - \pi^*\Omega_{\mathrm{red}} = \mathrm{d}(\upsilon - \theta)$$

holds. Analyzing this equation, we see that the left hand side is a basic differential form and that the right hand side is a d-exact differential form. Since furthermore $\Omega_{\mathrm{bas}}(C)$ is isomorphic to $\Omega(M_{\mathrm{red}})$, we can infer the existence of a $\chi \in \Omega(M_{\mathrm{red}})[\![\nu]\!]$ such that $\tilde{\Omega} - \Omega_{\mathrm{red}} = \mathrm{d}\chi$. Using Theorem 2.3.23 then shows that $[\pi^*\tilde{\Omega}]_{\mathfrak{g}} = [\pi^*\Omega_{\mathrm{red}}]_{\mathfrak{g}}$. Altogether we arrive at

$$\iota^* c_{\mathfrak{g}}(F(\Omega), \mathbf{J}')_+ = \frac{1}{\nu}[\iota^*(\Omega - \mathbf{J}')]_{\mathfrak{g}} = \frac{1}{\nu}[\pi^*\tilde{\Omega}]_{\mathfrak{g}} = \frac{1}{\nu}[\pi^*\Omega_{\mathrm{red}}]_{\mathfrak{g}} = \pi^* c(F(\Omega_{\mathrm{red}}))_+.$$

So the image of the equivariant characteristic class of $(F(\Omega), \mathbf{J}')$ under K agrees with the characteristic class of $F(\Omega_{\mathrm{red}})$. But this in turn directly implies that

$$K(c_{\mathfrak{g}}(\star, \mathbf{J})) = K(c_{\mathfrak{g}}(F(\Omega), \mathbf{J}')) = c(F(\Omega_{\mathrm{red}})) = c(\star_{\mathrm{red}}). \qquad \square$$

Theorem 4.3.4 taken together with Lemma 4.3.3 has a very appealing interpretation:

Corollary 4.3.5 *Let M_{red} be Marsden-Weinstein reduced from M by the action of a connected Lie group. Then for every star product $\tilde{\star}$ on M_{red} there exists an equivariant star product $(\star, \mathbf{J})$ on M such that $\tilde{\star}$ is equivalent to $\star_{\mathrm{red}}$.*

Or in other words, all star products on M_{red} can, up to equivalence, be obtained as a reduction of an equivariant star product on M.

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