Convergent Star Products and Abstract O^*-Algebras

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Mai 2018

Dissertation zur Erlangung des Akademischen Grades
eines Doktors der Naturwissenschaften

Betreuer: Stefan Waldmann

To my family, for always supporting me,
to my friends, for keeping me sane,
to my students, for driving me insane,
to my teachers, for doing more than just their job,
and to Su, for running up a mountain.

"The Guide is definitive. Reality is frequently inaccurate."
– The Restaurant at the End of the Universe, Douglas Adams

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Chapter 1

Overview

1.1 Introduction

When Max Planck in the year 1900 introduced the assumption that the energy of electromagnetic radiation inside a chamber surrounded by reflecting walls is stored in packages, or "quanta", having energy $\hbar\omega$, with ω the angular frequency of the radiation and $\hbar$ a new fundamental constant of nature [68], he revolutionized physics:

Not only could this assumption help him explain the spectrum of a black body, an object that only absorbs, but not reflects light, it also subsequently lead Einstein to the explanation of the photoelectric effect [32] and Bohr to a first quantum model of the hydrogen atom [13]. In the following years, these ideas solidified into what today is known as quantum physics, and what allows us to describe nature on atomic and sub-atomic scales with unprecedented precision.

However, this new insight comes at a price: The rules of quantum physics seem to be unintuitive, sometimes even contradict what once were assumed to be fundamental principles of nature. For instance, in classical (i.e. pre-quantum) physics, a particle moving in our three-dimensional world can always be described completely by a point in real three-dimensional space (its position) and a three-dimensional real vector (its velocity, or equivalently, its momentum). But in quantum physics it turns out that it is impossible to assign at the same time to a particle a certain position as well as a certain velocity or momentum! Instead, the particle is described by a wavefunction, which is a normalized vector in a complex vector space of infinite dimension.

As a consequence, it is not clear in general how to *quantize* a given classical system, i.e. how to construct the correct quantum theory describing it on small scales. One attempt at a precise mathematical formulation of this problem was given and examined by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in [3–5], and is now known as *(formal) deformation quantization.* But it should be noted that this idea has been "in the back of the mind of many physicists" for a long time [79]:

Instead of focusing on how to move from the description of the classical states (e.g. tuple of a point in space and a vector of momentum) to the description of the quantum states (e.g. given by a wavefunction), they focus on the algebraic structure of the observables (the quantities that can actually be measured). In both cases, the observables have the structure of an algebra, and the algebra of quantum observables is just a "deformation" of its classical counterpart, which it reproduces when taking the limit $\hbar \to 0$. It then becomes clear that the two descriptions of classical and quantum states seem to be so different only because they are influenced by our different interpretation of classical and quantum physics, or, more precisely, are results of different representations of the classical and quantum algebra. In the framework of deformation quantization, it now makes sense to rigorously ask (and solve) questions like "Does every classical system allow a quantization?" or "Are all quantizations of a given classical system equivalent?"

This thesis discusses and proposes a solution for one problem arising from deformation quantization: Having constructed the quantization of a classical system, one would like to understand its mathematical properties (of both the classical and quantum system). Especially if both systems are described by *-algebras over the field of complex numbers, this means to understand the properties of certain *-algebras: What are their representations? What are the properties of these representations? How can the states be described in these representations? How can the spectrum of the observables be described?

In order to allow for a sufficiently general treatment of these questions, the concept of abstract O^*-algebras is introduced. Roughly speaking, these are *-algebras together with a cone of positive linear functionals on them (e.g. the continuous ones if one starts with a *-algebra that is endowed with a well-behaved topology). This language is then applied to two examples from deformation quantization, which will be studied in great detail.

After a short discussion of notation and basic mathematical preliminaries in the next Section 1.2, the concept of observables and states in classical and quantum physics as well as the different versions of deformation quantization will be outlined in Chapter 2. This will also serve to pose the questions that arise from the mathematical point of view, some of which will be answered or at least examined in more detail later on. Chapter 3 then introduces the concept of abstract O^*-algebras as a framework in which the algebras of observables can be studied effectively. Besides clarifying some general aspects concerning the relation of abstract O^*-algebras to other types of *-algebras, especially locally convex ones like C^*-algebras, the main result here is a sufficient and quite general condition for pure states of abstract O^*-algebras to be characters, which is important for the interpretation of the classical observable algebras and their states. This is also the main result of the preprint [76].

The next two chapters then examine two examples from deformation quantization in more detail, apply the previous results from Chapter 3 and answer most of the questions from Chapter 2. The example of Chapter 4 deals with a system with arbitrary (finitely or infinitely many) degrees of freedom in flat space. This continues [83] and has been published in [77]. The second example in Chapter 5 is for a physical system modeled on the hyperbolic disc of finite dimension. Again, this model has already been examined earlier in [7], and the new results presented here are also accessible as the preprint [52].

Finally, the main part of this thesis closes with a short conclusion and outlook in Chapter 6, and the Appendix A gives short introductions to some of the mathematical concepts used.

Acknowledgements:

This thesis would not have been possible without the support of many people: First and foremost, I would like to thank my supervisor Stefan Waldmann for many years of guidance, discussions and a generally wonderful time in Würzburg and Forchheim. Many thanks also go to my other collaborators Daniela Kraus and Oliver Roth and to all those who I had the pleasure of having many fruitful discussions with, who helped proofreading this thesis or other manuscripts, or with life in general and bureaucracy in particular: These are especially the other members of ChairX, Knut Hüper, Chiara Esposito, Maximilian Hanusch, Thorsten Reichert and Heike Kus, as well as all the former and present master students without whom this chair could never be considered complete, especially Benedikt Hurle, Paul Stapor, Thomas Weber, Jonas Schnitzer, Bastian Seifert, Philipp Schmitt, Marvin Dippell, Thomas Bendokat, Andreas Kraft, Lukas Miaskiwskyi, Tobias Schmude and Stefan Franz. But I also enjoyed some visits in Louvain, Brussels, Rome and Mulhouse and would like to thank Pierre Bieliavsky, Axel de Goursac, Simone Gutt, Michel Cahen, Luca Tomassini and Martin Bordemann for their hospitality and valuable input.

1.2 Notation and Preliminaries

The *natural numbers* will be denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Similarly, $\mathbb{R}$ and $\mathbb{C}$ are the *real* and *complex numbers*, respectively. The complex conjugation on $\mathbb{C}$ is denoted by $\mathbb{C} \ni z \mapsto \overline{z} \in \mathbb{C}$ and $\mathrm{i} \in \mathbb{C}$ is the imaginary unit. If V is a vector space over a field $\mathbb{F}$, then V^* denotes its *dual vector space*, i.e. the set of all linear maps from V to $\mathbb{F}$ (also called *linear functionals*) with the pointwise addition and scalar multiplication. The evaluation of a linear functional $\omega \in V^*$ on a vector $v \in V$ will be denoted by $\langle\, \omega \,,\, v \,\rangle \in \mathbb{F}$, so that $\langle\, \cdot \,,\, \cdot \,\rangle \colon V^* \times V \to \mathbb{F}$ is a bilinear map. The linear subspace generated by an arbitrary subset S of a vector space V is written as $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{lin}} \subseteq V$, and if V is a real or complex vector space, then the convex set and the convex cone generated by S are $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{conv}} \subseteq V$ and $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{cone}} \subseteq V$. These are examples of *hull operators*, see the Appendix A.3 for details.

In the following let V and W be complex vector space. Besides the usual linear maps from V to W there are also the *antilinear* maps which will be of interest, i.e. maps $A \colon V \to W$ that fulfil $A(\lambda v + \lambda' v') = \overline{\lambda} A(v) + \overline{\lambda'} A(v')$ for all $v, v' \in V$ and $\lambda, \lambda' \in \mathbb{C}$. A special case of this are *antilinear involutions* of V, i.e. antilinear maps from V to V which are their own inverse. Such antilinear involutions are usually denoted by $\,\cdot^* \colon V \to V$ or $\overline{\,\cdot\,} \colon V \to V$, and their fixed points form a real linear subspace of V, called the subspace of *Hermitian* elements and denoted by V_{H}, and $V = V_{\mathrm{H}} \oplus \mathrm{i} V_{\mathrm{H}}$, because every $v \in V$ can uniquely be decomposed in $v = \mathsf{Re}(v) + \mathrm{i}\,\mathsf{Im}(v)$ with

$$\mathsf{Re}(v) = \frac{v + \overline{v}}{2} \in V_{\mathrm{H}} \qquad \text{and} \qquad \mathsf{Im}(v) = \frac{v - \overline{v}}{2\mathrm{i}} \in V_{\mathrm{H}} \,. \tag{1.2.1}$$

As an example, the complex conjugation on $\mathbb{C}$ is an antilinear involution, and the corresponding Hermitian elements are the real numbers, $\mathbb{C}_{\mathrm{H}} \cong \mathbb{R}$.

A *sesquilinear map* from V to W is a map $S \colon V \times V \to W$, which is antilinear in the first argument and linear in the second one. Note that such a map fulfils the *polarization identity*

$$S(v, w) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^{-k} S\big(v + \mathrm{i}^k w, v + \mathrm{i}^k w\big) \tag{1.2.2}$$

for all $v, w \in V$, which is often helpful as it shows that S is completely determined by the values $S(v, v)$ with $v \in V$. A *sesquilinear form* on V is a sesquilinear map $S \colon V \times V \to \mathbb{C}$, and in this case $\overline{S} := \overline{\,\cdot\,} \circ S \colon V \times V \to \mathbb{C}$ is again a sesquilinear map ($\circ$ is the composition of maps). The set of sesquilinear forms on V is a complex vector space with the pointwise addition and scalar multiplication, and $\overline{\,\cdot\,}$ is an antilinear involution on it. The corresponding Hermitian elements are called *Hermitian forms*, and are the sesquilinear forms S on V fulfilling $S = \overline{S}$, which is the case if and only if $S(v,v) \in \mathbb{R}$ for all $v \in V$ due to the polarization identity. A Hermitian form S on V is said to be *positive* if $S(v,v) \geq 0$ holds for all $v \in V$, and is usually written as $\langle \,\cdot\, | \,\cdot\, \rangle$. For such a positive Hermitian form $\langle \,\cdot\, | \,\cdot\, \rangle$, the *Cauchy Schwarz inequality*

$$|\langle\, v \,|\, w \,\rangle|^2 \leq \langle\, v \,|\, v \,\rangle \langle\, w \,|\, w \,\rangle \tag{1.2.3}$$

holds for all $v, w \in V$. This inequality will be used many times throughout this thesis, and will be indicated by putting CS over the inequality sign. Note that the positive Hermitian forms are a convex cone in the real vector space of Hermitian forms with the special property that if $\langle \,\cdot\, | \,\cdot\, \rangle$ and $-\langle \,\cdot\, | \,\cdot\, \rangle$ are both positive Hermitian forms, then $\langle\, v \,|\, v \,\rangle = 0$ for all $v \in V$, i.e. $\langle \,\cdot\, | \,\cdot\, \rangle = 0$ by the polarization identity. Because of this, the Hermitian forms on V can be seen as an ordered vector space whose convex cone of positive elements are the positive Hermitian forms, see the Appendix A.2 for details about ordered and quasi-ordered vector spaces.

For every positive Hermitian form $\langle \,\cdot\, | \,\cdot\, \rangle$ on a complex vector space V, the map $\|\cdot\| \colon V \to [0,\infty[$, $\|v\| \coloneqq \sqrt{\langle\, v \,|\, v \,\rangle}$ is a seminorm on V, called the *Hilbert seminorm* corresponding to $\langle \,\cdot\, | \,\cdot\, \rangle$. Note that, due to the monotony of the square root function, the order on the positive Hermitian forms is the same as the pointwise order on the corresponding Hilbert seminorms. Every Hilbert seminorm $\|\cdot\|$ on V fulfils the *parallelogram identity*

$$2\|v\|^2 + 2\|w\|^2 = \|v+w\|^2 + \|v-w\|^2 \tag{1.2.4}$$

for all $v, w \in V$ and corresponds to a unique positive Hermitian form $\langle \,\cdot\, | \,\cdot\, \rangle$, determined by the polarization identity. It is an interesting and non-trivial fact that every seminorm $\|\cdot\|$ on V fulfilling the parallelogram identity is actually a Hilbert seminorm [47]. Finally, a *pre-Hilbert space* is a complex vector space, usually denoted by $\mathcal{D}$, endowed with a positive Hermitian form $\langle \,\cdot\, | \,\cdot\, \rangle$, called *inner product* such that the corresponding Hilbert seminorm $\|\cdot\|$ is even a norm, which is the case if and only if $\langle \,\cdot\, | \,\cdot\, \rangle$ is *non-degenerate*, i.e. if and only if $\langle\, \phi \,|\, \psi \,\rangle = 0$ for one $\phi \in \mathcal{D}$ and all $\psi \in \mathcal{D}$ implies $\phi = 0$. If not explicitly stated differently, all topological notions (convergence of sequences; open, closed and compact sets) on a pre-Hilbert space will always refer to this Hilbert norm $\|\cdot\|$. A pre-Hilbert space which is complete with respect to this norm is called a *Hilbert space*.

1.2.1 *-Algebras

The concepts of (unital associative) *-algebras, as well as (quasi-)ordered *-algebras and O^*-algebras will be extremely important in the following. While the definition of *-algebras and O^*-algebras are classic, the definition of (quasi-)ordered *-algebras that is used here is from [76], and is so general that it will not lead to interesting results without additional assumptions. The discussion of these algebras follows closely the one in [76].

Definition 1.2.1 *A *-algebra is a complex vector space $\mathcal{A}$ which is endowed with an antilinear involution $\cdot^* \colon \mathcal{A} \to \mathcal{A}$ and a bilinear associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, such that $(ab)^* = b^* a^*$ holds for all $a, b \in \mathcal{A}$ and such that there is a unit, i.e. neutral element with respect to the product, which will be denoted by $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$, or simply by $\mathbb{1} \in \mathcal{A}$.*

If $\mathcal{A}$ is a *-algebra, then its convex cone of algebraically positive *elements (which, in general, might contain a non-trivial linear subspace) is*

$$\mathcal{A}_{\mathrm{H}}^{++} := \Big\{ \sum_{n=1}^{N} a_n^* a_n \;\Big|\; N \in \mathbb{N};\, a_1, \dots, a_N \in \mathcal{A} \Big\} .$$

Moreover, given a linear functional $\omega \in \mathcal{A}^*$, then ω is said to be algebraically positive *if $\langle\, \omega\,,\, a^* a \,\rangle \geq 0$ for all $a \in \mathcal{A}$, or equivalently if $\langle\, \omega\,,\, a \,\rangle \geq 0$ for all $a \in \mathcal{A}_{\mathrm{H}}^{++}$, and is said to be an* algebraic state *if it is algebraically positive and normalized to $\langle\, \omega\,,\, \mathbb{1} \,\rangle = 1$. The set of algebraically positive linear functionals on $\mathcal{A}$ will be denoted by $\mathcal{A}_{\mathrm{H}}^{*,++}$.*

Note that the unit of a *-algebra is necessarily unique and automatically Hermitian and that the convex cone of algebraically positive elements is indeed closed under addition and multiplication with non-negative reals. A *unital *-subalgebra $\mathcal{B}$* of a *-algebra $\mathcal{A}$ is a linear subspace containing the unit, closed under multiplication and stable under the *-involution, i.e. $b^* \in \mathcal{B}$ for all $b \in \mathcal{B}$. As before, the *-involution on a *-algebra $\mathcal{A}$ allows to define its real linear subspace $\mathcal{A}_{\mathrm{H}}$ of Hermitian elements, and the real linear span of $\mathcal{A}_{\mathrm{H}}^{++}$ is $\mathcal{A}_{\mathrm{H}}$ because $4a = (a+\mathbb{1})^2 - (a-\mathbb{1})^2$ for all $a \in \mathcal{A}_{\mathrm{H}}$, so the complex linear span of $\mathcal{A}_{\mathrm{H}}^{++}$ is whole $\mathcal{A}$. Moreover, the algebraic dual $\mathcal{A}^*$ of $\mathcal{A}$ also carries an antilinear involution, namely $\cdot^* \colon \mathcal{A}^* \to \mathcal{A}^*$, $\omega \mapsto \omega^*$ with

$$\langle\, \omega^*\,,\, a \,\rangle := \overline{\langle\, \omega\,,\, a^* \,\rangle} \qquad \text{for all } a \in \mathcal{A}\,, \tag{1.2.5}$$

and thus defines a real linear subspace $\mathcal{A}_{\mathrm{H}}^* \subseteq \mathcal{A}^*$ of Hermitian linear functionals, which are precisely those linear functionals on $\mathcal{A}$ that map $\mathcal{A}_{\mathrm{H}}$ to $\mathbb{R}$.

Every algebraically positive linear functional ω on a *-algebra $\mathcal{A}$ is especially Hermitian as it maps $\mathcal{A}_{\mathrm{H}} = \langle\!\langle\, \mathcal{A}_{H}^{++} \,\rangle\!\rangle_{\mathrm{lin}}$ to $\mathbb{R}$, and the Cauchy Schwarz inequality

$$\big|\langle\, \omega\,,\, a^* b \,\rangle\big|^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^* a \,\rangle \langle\, \omega\,,\, b^* b \,\rangle \tag{1.2.6}$$

holds for all $a, b \in \mathcal{A}$, because the sesquilinear form $\mathcal{A}^2 \ni (a,b) \mapsto \langle\, \omega\,,\, a^* b \,\rangle \in \mathbb{C}$ is Hermitian and positive. This also implies that an algebraically positive linear functional ω on $\mathcal{A}$ vanishes if and only if $\langle\, \omega\,,\, \mathbb{1} \,\rangle = 0$, because $0 \leq \big|\langle\, \omega\,,\, a \,\rangle\big|^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^* a \,\rangle \langle\, \omega\,,\, \mathbb{1} \,\rangle = 0$ for all $a \in \mathcal{A}$ if $\langle\, \omega\,,\, \mathbb{1} \,\rangle = 0$. So the convex cone $\mathcal{A}_{\mathrm{H}}^{*,++}$ does not contain a non-trivial linear subspace.

The above especially shows that $\mathcal{A}_{\mathrm{H}}$ can be seen as a quasi-ordered vector space with convex cone of positive elements $\mathcal{A}_{\mathrm{H}}^{++}$, and similarly, $\mathcal{A}_{\mathrm{H}}^*$ can be seen as an ordered vector space with convex cone of positive elements $\mathcal{A}_{\mathrm{H}}^{*,++}$. However, it will be more interesting to allow $\mathcal{A}_{\mathrm{H}}$, and thus also $\mathcal{A}_{\mathrm{H}}^*$, to carry different orders with more positive elements in $\mathcal{A}_{\mathrm{H}}$ than just $\mathcal{A}_{\mathrm{H}}^{++}$, and thus less positive elements in $\mathcal{A}_{\mathrm{H}}^*$ than all of $\mathcal{A}_{\mathrm{H}}^{*,++}$.

The following well-known notion of variance of an algebraic state will be very helpful later on, not so much because of any stochastic interpretation, but because of estimate (1.2.8):

Definition 1.2.2 *Let $\mathcal{A}$ be a *-algebra, ω an algebraic state on $\mathcal{A}$ and $a \in \mathcal{A}$. Then the* variance of ω on a *is defined as:*

$$\mathrm{Var}_\omega(a) := \big\langle\, \omega\,,\, \big(a - \langle\, \omega\,,\, a \,\rangle \mathbb{1}\big)^* \big(a - \langle\, \omega\,,\, a \,\rangle \mathbb{1}\big) \,\big\rangle = \langle\, \omega\,,\, a^* a \,\rangle - \big|\langle\, \omega\,,\, a \,\rangle\big|^2 . \tag{1.2.7}$$

Clearly $\mathrm{Var}_\omega(a) \geq 0$ for all $a \in \mathcal{A}$ and all algebraic states ω on $\mathcal{A}$, and the Cauchy Schwarz inequality implies that

$$\big|\langle\, \omega\,,\, b^*a\,\rangle - \overline{\langle\, \omega\,,\, b\,\rangle}\langle\, \omega\,,\, a\,\rangle\big|^2 = \big|\big\langle\, \omega\,,\, \big(b - \langle\, \omega\,,\, b\,\rangle\mathbb{1}\big)^*\big(a - \langle\, \omega\,,\, a\,\rangle\mathbb{1}\big)\,\big\rangle\big|^2 \overset{\mathrm{CS}}{\leq} \mathrm{Var}_\omega(b)\,\mathrm{Var}_\omega(a) \tag{1.2.8}$$

for all $a, b \in \mathcal{A}$, and in the special case that $\mathrm{Var}_\omega(a) = 0$, this yields

$$\langle\, \omega\,,\, ba\,\rangle = \langle\, \omega\,,\, b\,\rangle\langle\, \omega\,,\, a\,\rangle \tag{1.2.9}$$

for all $b \in \mathcal{A}$.

Besides an antilinear involution and a distinguished cone of algebraically positive linear functionals, the algebraic dual $\mathcal{A}^*$ of a *-algebra $\mathcal{A}$ has some more interesting structure:

Definition 1.2.3 *[76, Sec. 2] Let $\mathcal{A}$ be a *-algebra, then define on $\mathcal{A}^*$ the* $\mathcal{A}$-bimodule structure $\mathcal{A} \times \mathcal{A}^* \to \mathcal{A}^*$ *and* $\mathcal{A}^* \times \mathcal{A} \to \mathcal{A}^*$*,*

$$\langle\, b\cdot\omega\,,\, a\,\rangle \coloneqq \langle\, \omega\,,\, ab\,\rangle \qquad \textit{and} \qquad \langle\, \omega\cdot b\,,\, a\,\rangle \coloneqq \langle\, \omega\,,\, ba\,\rangle \qquad \textit{for all } \omega \in \mathcal{A}^* \textit{ and all } a, b \in \mathcal{A}\,, \tag{1.2.10}$$

as well as the $\mathcal{A}$-monoid action $\triangleright : \mathcal{A} \times \mathcal{A}^* \to \mathcal{A}^*$*,*

$$a \triangleright \omega \coloneqq a\cdot\omega\cdot a^* \qquad \textit{for all } \omega \in \mathcal{A}^* \textit{ and all } a \in \mathcal{A}\,. \tag{1.2.11}$$

Note that the sets of Hermitian and algebraically positive linear functionals on a *-algebra $\mathcal{A}$ are stable under the $\mathcal{A}$-monoid action of $\mathcal{A}$ on $\mathcal{A}^*$. Finally, the usual notion of morphisms between *-algebras is:

Definition 1.2.4 *Given two *-algebras $\mathcal{A}$ and $\mathcal{B}$, then a unital *-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a linear map $M\colon \mathcal{A} \to \mathcal{B}$ fulfilling $M(\mathbb{1}_\mathcal{A}) = \mathbb{1}_\mathcal{B}$, $M(a^*) = M(a)^*$ and $M(aa') = M(a)M(a')$ for all $a, a' \in \mathcal{A}$.*

We immediately see that such a unital *-homomorphism $M\colon \mathcal{A} \to \mathcal{B}$ maps Hermitian elements to Hermitian ones and algebraically positive elements to algebraically positive ones. Converse, its pullback $M^*\colon \mathcal{B}^* \to \mathcal{A}^*$, $\rho \mapsto M^*(\rho) \coloneqq \rho \circ M$ maps Hermitian linear functionals to Hermitian ones and algebraically positive linear functionals to algebraically positive ones.

1.2.2 (Quasi-)Ordered *-Algebras and O^*-Algebras

Besides ordinary *-algebras, there are a lot of different types of *-algebras endowed with additional structure. The first example is:

Definition 1.2.5 *[76, Def. 2.1] A* quasi-ordered *-algebra *is a *-algebra $\mathcal{A}$ endowed with a quasi-order $\lesssim$ (a reflexive and transitive relation) on $\mathcal{A}_\mathrm{H}$, such that the following conditions are fulfilled for all $a, b \in \mathcal{A}_\mathrm{H}$ with $a \lesssim b$, all $c \in \mathcal{A}_\mathrm{H}$ and all $d \in \mathcal{A}$:*

$$a + c \lesssim b + c\,, \qquad d^*a\,d \lesssim d^*b\,d \qquad \textit{and} \qquad 0 \lesssim \mathbb{1}\,.$$

If $\lesssim$ is even a partial order (i.e. additionally antisymmetric), then $\mathcal{A}$ is called an ordered *-algebra *and one might write $\leq$ instead of $\lesssim$. Moreover, if $\mathcal{A}$ is a quasi-ordered *-algebra, then the convex cone* *of* positive elements *in $\mathcal{A}$ is denoted by* $\mathcal{A}_{\mathrm{H}}^{+} \coloneqq \{ a \in \mathcal{A}_{\mathrm{H}} \mid a \gtrsim 0 \}$ *and we define a partial order* $\le$ *on* $\mathcal{A}_{\mathrm{H}}^{*} = \{ \omega \in \mathcal{A}^{*} \mid \omega^{*} = \omega \}$ *by*

$$\omega \le \rho \quad :\Longleftrightarrow \quad \forall_{a \in \mathcal{A}_{\mathrm{H}}^{+}} : \langle \omega \,,\, a \rangle \le \langle \rho \,,\, a \rangle$$

for all $\omega, \rho \in \mathcal{A}_{\mathrm{H}}^{*}$ *and write* $\mathcal{A}_{\mathrm{H}}^{*,+} \coloneqq \{ \omega \in \mathcal{A}_{\mathrm{H}}^{*} \mid \omega \ge 0 \}$.

Note that $\mathcal{A}_{\mathrm{H}}^{++} \subseteq \mathcal{A}_{\mathrm{H}}^{+}$ holds for every quasi-ordered *-algebra $\mathcal{A}$ and that $\mathcal{A}_{\mathrm{H}}$ is a quasi-ordered vector space with convex cone of positive elements $\mathcal{A}_{\mathrm{H}}^{+}$, and is even an ordered vector space if and only if $\mathcal{A}$ is an ordered *-algebra. Similarly, $\mathcal{A}_{\mathrm{H}}^{*,++} \supseteq \mathcal{A}_{\mathrm{H}}^{*,+}$ and $\mathcal{A}_{\mathrm{H}}^{*}$ is an ordered vector space with convex cone of positive elements $\mathcal{A}_{\mathrm{H}}^{*,+}$.

With respect to morphisms between quasi-ordered *-algebras it is easy to combine compatibility with the algebraic and order structure:

Definition 1.2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be two quasi-ordered *-algebras, then a* positive unital *-homomorphism *from $\mathcal{A}$ to $\mathcal{B}$ is a unital *-homomorphism $M \colon \mathcal{A} \to \mathcal{B}$ whose restriction to a linear map from $\mathcal{A}_{\mathrm{H}} \to \mathcal{B}_{\mathrm{H}}$ is positive, i.e. maps positive elements to positive ones.*

Of course, every *-algebra $\mathcal{A}$ yields a quasi-ordered *-algebra by choosing the order on $\mathcal{A}_{\mathrm{H}}$ as the one whose convex cone of positive elements is $\mathcal{A}_{\mathrm{H}}^{+} \coloneqq \mathcal{A}_{\mathrm{H}}^{++}$. In this case also $\mathcal{A}_{\mathrm{H}}^{*,+} = \mathcal{A}_{\mathrm{H}}^{*,++}$ and every unital *-homomorphism from $\mathcal{A}$ to an arbitrary quasi-ordered *-algebra is positive as algebraically positive elements are mapped to algebraically positive, hence positive, ones. This essentially shows (see the Appendix A.5 for some basic notions of category theory):

Proposition 1.2.7 *Assigning to every *-algebra $\mathcal{A}$ the quasi-ordered *-algebra $\mathcal{A}$ whose convex cone of positive elements is $\mathcal{A}_{\mathrm{H}}^{+} \coloneqq \mathcal{A}_{\mathrm{H}}^{++}$, and to every unital *-homomorphism between *-algebras the positive unital *-homomorphism between such quasi-ordered *-algebras, is a (covariant) functor from the category of *-algebras with unital *-homomorphisms between them to the category of quasi-ordered *-algebras with positive unital *-homomorphisms between them.*

Other important examples of ordered *-algebras are *-algebras of functions and *-algebras of operators, which are usually called O^*-algebras:

Definition 1.2.8 *Let X be a set, then write $\mathbb{C}^X$ for the ordered *-algebra of all functions from X to $\mathbb{C}$ with pointwise addition and multiplications, pointwise complex conjugation as *-involution, and the order by pointwise comparison on the Hermitian elements in $\mathbb{C}^X$, i.e. the real-valued functions on X: Given $a, b \in (\mathbb{C}^X)_{\mathrm{H}}$, then $a \le b$ if and only if $a(x) \le b(x)$ for all $x \in X$.*

It is straightforward to check that $\mathbb{C}^X$ is indeed an ordered *-algebra. The unit is the constant 1-function.

Definition 1.2.9 *Let $\mathcal{D}$ be a pre-Hilbert space, then write $\mathcal{L}^*(\mathcal{D})$ for the ordered *-algebra of all adjointable endomorphisms of $\mathcal{D}$, i.e. for the set of all (necessarily linear) $a \colon \mathcal{D} \to \mathcal{D}$ for which there exists a (necessarily unique and linear) $a^* \colon \mathcal{D} \to \mathcal{D}$ such that $\langle \phi \,|\, a(\psi) \rangle = \langle a^*(\phi) \,|\, \psi \rangle$ holds for all $\phi, \psi \in \mathcal{D}$. Addition and scalar multiplication of elements in $\mathcal{L}^*(\mathcal{D})$ are of course defined pointwise,* *multiplication of two adjointable endomorphisms is their composition and the* **-involution is the map* $a \mapsto a^*$ *to the adjoint. The Hermitian elements in* $\mathcal{L}^*(\mathcal{D})$*, i.e. the linear maps* $a\colon \mathcal{D} \to \mathcal{D}$ *for which* $\langle\, \phi \,|\, a(\phi) \,\rangle \in \mathbb{R}$ *for all* $\phi \in \mathcal{D}$*, are ordered by*

$$a \leq b \qquad :\Longleftrightarrow \qquad \forall_{\phi\in\mathcal{D}} : \big\langle\, \phi \,\big|\, a(\phi) \,\big\rangle \leq \big\langle\, \phi \,\big|\, b(\phi) \,\big\rangle \tag{1.2.12}$$

for all $a, b \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$*.*

Again, it is straightforward to show that $\mathcal{L}^*(\mathcal{D})$ is a *-algebra with unit $\mathsf{id}_{\mathcal{D}}$, $\mathsf{id}_{\mathcal{D}}(\phi) = \phi$ for all $\phi \in \mathcal{D}$. The characterization of the Hermitian elements as the linear functions $a\colon \mathcal{D} \to \mathcal{D}$ for which $\langle\, \phi \,|\, a(\phi) \,\rangle \in \mathbb{R}$ for all $\phi \in \mathcal{D}$ is due to the polarization identity for the sesquilinear functionals $\mathcal{D}^2 \ni (\phi, \psi) \mapsto \langle\, \phi \,|\, a(\psi) \,\rangle \in \mathbb{C}$ and $\mathcal{D}^2 \ni (\phi, \psi) \mapsto \langle\, a(\phi) \,|\, \psi \,\rangle \in \mathbb{C}$. It is then easy to show that $\mathcal{L}^*(\mathcal{D})$ is a quasi-ordered *-algebra, and even an ordered *-algebra due to the polarization identity again.

Definition 1.2.10 *A* *-algebra of functions *on a set* X *is a unital* **-subalgebra of* $\mathbb{C}^X$*, and a* *-algebra of operators*, or* O^*-algebra *on a pre-Hilbert space* $\mathcal{D}$ *is a unital* **-subalgebra of* $\mathcal{L}^*(\mathcal{D})$*.*

Of course, *-algebras of functions and of operators are ordered *-algebras with the order from Definitions 1.2.8 and 1.2.9. O^*-algebras have been studied for a long time, see e.g. [74]. As every normed vector space admits a completion, it is always possible to assume that a pre-Hilbert space $\mathcal{D}$ is a dense linear subspace of a Hilbert space $\mathfrak{H}$. In this case, the elements of $\mathcal{L}^*(\mathcal{D})$ can also be interpreted as (in general unbounded) operators on $\mathfrak{H}$. It is important to note that the adjoint endomorphism a^* of some $a \in \mathcal{L}^*(\mathcal{D})$ in general is not the same as the operator theoretic adjoint $a^\dagger$, even though there is a close link between the two concepts: In general, $a^\dagger\colon \mathcal{D}_{a^\dagger} \to \mathfrak{H}$ is defined on a dense linear subspace $\mathcal{D}_{a^\dagger}$ of $\mathfrak{H}$ which contains $\mathcal{D}$, and its restriction to $\mathcal{D}$ coincides with a^*, i.e. $a^\dagger$ extends a^*. See the Appendix A.4 for details about unbounded operators.

1.2.3 From Locally Convex *-Algebras to C^*-Algebras

*-Algebras that are endowed with a (locally convex) topology, e.g. C^*-algebras, have been studied for decades, see e.g. the standard textbooks like [37, 66, 73]. By demanding different features of the topology, there arise a lot of different types of topological *-algebras. So the following list of classical definitions is by no means complete, more details can be found e.g. in the above references. For the basic concepts of locally convex analysis, see the Appendix A.1.

Definition 1.2.11 *A* locally convex *-algebra *is a* **-algebra* $\mathcal{A}$ *endowed with a locally convex topology, such that the* **-involution as well as the operators* $\mathcal{A} \ni b \mapsto ab \in \mathcal{A}$ *and* $\mathcal{A} \ni b \mapsto ba \in \mathcal{A}$ *of left- and right multiplication with* a *are continuous for all* $a \in \mathcal{A}$*.*

Of course, the corresponding morphisms are the continuous unital *-homomorphisms and one is especially interested in *Hausdorff locally convex* **-algebras*, where the topology is Hausdorff. Moreover, Hausdorff locally convex *-algebras with continuous multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ have the important property that *-involution and multiplication extend continuously to the completion $\mathcal{A}^{\mathrm{cpl}}$ and turn $\mathcal{A}^{\mathrm{cpl}}$ into a new locally convex *-algebra (associativity of the product and compatibility of product and *-involution hold because they hold on a dense subset and because all involved functions are continuous).

If the locally convex topology can be defined by a set of especially well-behaved norms or seminorms, then one gets more well-behaved locally convex *-algebras:

Definition 1.2.12 *Let $\mathcal{A}$ be a *-algebra and $\|\cdot\|$ a seminorm on $\mathcal{A}$.*

- $\|\cdot\|$ *is a* *-seminorm *if $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$, and it is a* *-norm *if it is also a norm.*
- $\|\cdot\|$ *is* submultiplicative *if $\|ab\| \leq \|a\|\|b\|$ holds for all $a, b \in \mathcal{A}$.*
- $\|\cdot\|$ *is a* C^*-seminorm *if it is a submultiplicative *-seminorm and fulfils $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. It is a* C^*-norm *if it is additionally a norm.*

Note that the kernel of a submultiplicative *-seminorm is a *-ideal, i.e. an ideal that is stable under the *-involution. Thus, if $\mathcal{A}$ is a *-algebra and $\|\cdot\|$ a submultiplicative *-seminorm, then $\mathcal{A}/\ker\|\cdot\|$ is a *-algebra on which $\|\cdot\|$ remains well-defined and is even a submultiplicative *-norm.

Definition 1.2.13 *A* locally multiplicatively convex *-algebra*, or* lmc-*-algebra *for short, is a Hausdorff locally convex *-algebra whose topology can be defined by a set of submultiplicative *-seminorms.*

Definition 1.2.14 *A* normed *-algebra *is a Hausdorff locally convex *-algebra whose topology can be defined by a submultiplicative *-norm, and a* Banach *-algebra *is a complete normed *-algebra.*

Due to submultiplicativity, the multiplication of lmc- and normed *-algebras is continuous. An lmc-*-algebra $\mathcal{A}$ is especially well-behaved as it can be reconstructed out of the normed *-algebras $\mathcal{A}/\ker\|\cdot\|$ with $\|\cdot\|$ running over all continuous submultiplicative *-seminorms on $\mathcal{A}$. However, Hausdorff locally convex *-algebras with continuous multiplication do not need to have any non-trivial submultiplicative seminorms as Proposition 4.0.1 and the example in Chapter 4 show. One important consequence of submultiplicativity is the possibility to construct holomorphic functions of algebra elements:

If $(f_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers such that the power series $\sum_{n=0}^{\infty} f_n z^n$ in z has infinite radius of convergence, then $\sum_{n=0}^{\infty} f_n a^n$ converges for every $a \in \mathcal{A}$ if $\mathcal{A}$ is a complete lmc-*-algebra. If $\mathcal{A}$ is even a Banach *-algebra, then it is sufficient for the power series $\sum_{n=0}^{\infty} f_n z^n$ in z to have a radius of convergence strictly larger than $\|a\|$, see [66, Thm. 3.3.7].

Definition 1.2.15 *A* pro-C^*-algebra *is a Hausdorff locally convex *-algebra whose topology can be defined by a set of C^*-seminorms.*

Definition 1.2.16 *A C^*-algebra is a complete Hausdorff locally convex *-algebra whose topology can be defined by a C^*-norm.*

The consequences of the C^*-property of seminorms are enormous, but less obvious. While submultiplicativity of a *-seminorm implies $\|a^*a\| \leq \|a\|^2$ for all algebra elements a, the C^*-property adds the converse estimate $\|a^*a\| \geq \|a\|^2$. Ultimately, this leads to many interesting properties, e.g.:

Lemma 1.2.17 *Let $\mathcal{A}$ be a *-algebra and $\|\cdot\|$ a C^*-seminorm on $\mathcal{A}$, then for every $a \in \mathcal{A}$ there exists an algebraic state ω on $\mathcal{A}$ fulfilling*

$$\langle\, \omega\,,\, a^*a \,\rangle = \|a^*a\| \qquad \textit{and} \qquad |\langle\, \omega\,,\, b \,\rangle| \leq \|b\| \qquad \textit{for all } b \in \mathcal{A}\,. \tag{1.2.13}$$

PROOF: See [73, Prop. 1.5.1 and 1.5.4]. □

Chapter 2

Classical and Quantum Physics

In order to develop a mathematically rigorous description of physical systems, the concept of observables and states have proven to be helpful. After presenting the general idea, it will be discussed how this applies to classical physics and quantum physics. In both cases, the observables have a very similar mathematical structure, which then motivates the definition of a Poisson *-algebra. Finally, the relation between the classical and quantum Poisson *-algebras describing a system is formulated using the language of deformation quantization. This chapter is only intended to give the heuristic motivation for the rest of the thesis and discuss some earlier work, not to present new results.

2.1 Observables and States

One approach to mathematically model a physical system is via the duality of observables and states, an idea that has permeated mathematical physics since decades and can already be found in von Neumann's treatise on the mathematical principles of quantum mechanics [80]. A more recent occurence of these ideas, to name just one, is in the Haag-Kastler axioms of algebraic quantum field theory [43]. The following exposition provides a rather general framework that should fit most theories of interest.

An *observable* describes a property of the system that can – at least in principle – be measured, while a *state* contains all our knowledge about the system, seen as a statistical ensemble. For now let $\mathfrak{O}$ be the set of all observables and $\mathcal{S}$ the set all possible states of the system. In this general case, the only requirement on the set of observables is that there is one observable $\mathbb{1}$, which describes the trivial measurement that will always give the result 1. But the set of states has to be a non-empty, convex set. The interpretation of convex combinations is that $\lambda\omega + (1-\lambda)\omega' \in \mathcal{S}$ for $\omega, \omega' \in \mathcal{S}$ and $\lambda \in [0,1]$ describes a system about which it is known that it is ω with probability λ and ω' with probability $1-\lambda$. A *pure state* then is defined as an extreme point of $\mathcal{S}$, i.e. as a state $\rho \in \mathcal{S}$ which cannot be expressed as a non-trivial convex combination $\rho = \lambda\omega + (1-\lambda)\omega'$ with $\lambda \in \,]0,1[\,$ and $\omega, \omega' \in \mathcal{S}$ such that $\omega \neq \omega'$. The subset of pure states will be denoted by $\mathcal{S}_{\mathrm{p}} \subseteq \mathcal{S}$. Pure states thus describe a system of which our knowledge is complete (not just a non-trivial probability distribution over other states). At this point it should be noted that it is easy to construct convex sets without extreme points, e.g. an open interval in $\mathbb{R}$, and that it is thus unclear in general whether every state can be represented as a convex combination (or a limit of convex combinations) of pure states. A sufficient, but not always applicable, condition for this to be true is given by the famous Theorem of Krein and Milman [72, Thm. 3.23].

In addition to the sets of observables and states, a model of a physical system should provide a pairing $\langle \cdot\,, \cdot \rangle \colon \mathcal{S} \times \mathfrak{O} \to \mathbb{R}$, such that $\langle \omega\,, a \rangle \in \mathbb{R}$ describes the expectation value of the measurement of property $a \in \mathfrak{O}$ on the system described by $\omega \in \mathcal{S}$. As a measurement of $\mathbb{1} \in \mathfrak{O}$ always gives the result 1 by definition, the pairing $\langle \cdot\,, \cdot \rangle$ has to fulfil

$$\langle \omega\,, \mathbb{1} \rangle = 1 \tag{P1}$$

for all states $\omega \in \mathcal{S}$. In order to be compatible with the stochastic interpretation of convex combinations of states, we also want that

$$\langle \lambda\omega + (1-\lambda)\omega'\,, a \rangle = \lambda\langle \omega\,, a \rangle + (1-\lambda)\langle \omega'\,, a \rangle \tag{P2}$$

holds for all $\lambda \in [0,1]$, $\omega, \omega' \in \mathcal{S}$ and $a \in \mathfrak{O}$. Moreover, if two states coincide on all observables, they should be considered identical because they cannot be distinguished by any measureable property. Similarly, if two observables coincide on all states, they should be considered to be identical as well. This leads to the requirements that

$$\forall_{a \in \mathfrak{O}} : \langle \omega\,, a \rangle = \langle \rho\,, a \rangle \qquad \Longrightarrow \qquad \omega = \rho \tag{P3}$$

holds for all $\omega, \rho \in \mathcal{S}$ and that

$$\forall_{\omega \in \mathcal{S}} : \langle \omega\,, a \rangle = \langle \omega\,, b \rangle \qquad \Longrightarrow \qquad a = b \tag{P4}$$

holds for all $a, b \in \mathfrak{O}$. Note that besides providing a criterium for two states or two observables to coincides, this also gives a natural notion of "nearness" of different states or observables: Two states can be considered "close" if they give "similar" results when paired with arbitrary observables (analogous with switched roles of observables and states). This can be made rigorous by constructing a topology or uniform structure on the sets of states and observables, and will become important later on, e.g. for the weak topology in Definition 3.1.13.

In addition to just predicting the expectation value $\langle \omega\,, a \rangle$ of the measurement of an observable $a \in \mathfrak{O}$ on a state $\omega \in \mathcal{S}$, it would also be desireable to predict the probability distribution of this measurement: Let $\operatorname{spec}(a) \subseteq \mathbb{R}$, the *spectrum* of a, be the closure of the set of possible measurement results. Then one would like to predict for every state ω the probability measure $\mu_{\omega,a}$ on $\mathbb{R}$ with support $\operatorname{spec}(a)$ for which $\int_A \mathrm{d}\mu_{\omega,a}$ for every measurable subset A of $\mathbb{R}$ describes the probability that the measurement of a on ω gives a result in A. In this case, the expectation value of the measurement is

$$\langle \omega\,, a \rangle = \int_{\operatorname{spec}(a)} \mathsf{id}_{\mathbb{R}} \, \mathrm{d}\mu_{\omega,a}\,. \tag{2.1.1}$$

In the next sections, we will discuss some additional structure that the sets of observables and states have in typical examples. However, there is one structure that can be defined already in this extremely general setting, namely a partial order $\leq$ on the observables: For all $a, b \in \mathfrak{O}$ we define

$$a \leq b \qquad :\Longleftrightarrow \qquad \forall_{\omega \in \mathcal{S}} : \langle \omega\,, a \rangle \leq \langle \omega\,, b \rangle\,. \tag{2.1.2}$$

The interpretation of this order is of course that $a \leq b$ if and only if the measurement of property a always has a smaller or equal expectation value than the measurement of property b. It follows immediately from the definition that $\leq$ is a reflexive and transitive relation on $\mathfrak{O}$, and it is even antisymmetric, hence a partial order, because of the requirement (P4) for the pairing of observables and states. Note that a similar order could also be defined on the states, however, this would usually result only in the trivial partial order by equality: If for every observable $a \in \mathfrak{O}$ there also exists a $-a \in \mathfrak{O}$ such that $\langle\, \omega \,,\, -a \,\rangle = -\langle\, \omega \,,\, a \,\rangle$ holds for all $\omega \in \mathcal{S}$ (which typically is the case), then $\langle\, \omega \,,\, a \,\rangle \leq \langle\, \rho \,,\, a \,\rangle$ for all $a \in \mathfrak{O}$ and $\omega, \rho \in \mathcal{S}$ implies $\langle\, \omega \,,\, a \,\rangle = \langle\, \rho \,,\, a \,\rangle$, hence $\omega = \rho$ by (P3).

Finally, a model of a physical system should provide equations of motion that describe the dynamics of the system, i.e. how the expectation value $\langle\, \omega \,,\, a \,\rangle$ of the measurement of the observable a on the state ω changes in time. This is also true for relativistic systems where no distinguished time direction exists: in this case the model should provide a description of how expectation values of measurements change with respect to all admissible time directions, and of the (Lorentz- or Poincaré-) transformations between them.

The most natural approach to describing the change of measurement results in time certainly is to treat the states as time-dependent, so that

$$\mathbb{R} \ni t \mapsto \langle\, \omega(t) \,,\, a \,\rangle \in \mathbb{R}$$

with a curve $\omega \colon \mathbb{R} \to \mathcal{S}$ describes the change of the expectation value of all $a \in \mathfrak{O}$ in time. In this case, the model should provide a way to describe the curve $\omega \colon \mathbb{R} \to \mathcal{S}$, typically via a differential equation (this is sometimes called the *Schrödinger picture*). However, one could also treat observables as time-dependent instead, so that

$$\mathbb{R} \ni t \mapsto \langle\, \omega \,,\, a(t) \,\rangle \in \mathbb{R}$$

with curves $a \colon \mathbb{R} \to \mathfrak{O}$ describes the change of the expectation value in time, and in this case would need a description of the curve $a \colon \mathbb{R} \to \mathfrak{O}$ (this is the *Heisenberg picture*). It is not even uncommon to have models where both observables and states may depend on time. This can even be necessary in theories where there does not exist a distinguished frame of reference with respect to which observables or states could be constant (e.g. relativistic theories), i.e. where observables and states that are constant with respect to one frame of reference are not necessarily constant with respect to another, equally valid, one.

2.2 Classical Physics

2.2.1 General Considerations

In classical physics, the pure states are already sufficient to separate the observables, i.e. the pairing has the property

$$\forall_{\omega \in \mathcal{S}_{\mathrm{p}}} : \langle\, \omega \,,\, a \,\rangle = \langle\, \omega \,,\, b \,\rangle \qquad \Longrightarrow \qquad a = b \tag{P4'}$$

which is clearly stronger than (P4). Moreover, the measurement of any observable $a \in \mathfrak{O}$ on a pure state $\omega \in \mathcal{S}_{\mathrm{p}}$ always gives a certain result, namely $\langle \omega , a \rangle$. Because of this it makes sense to treat the observables as a set of real-valued functions on essentially the set of pure states M. General states can then usually be expressed as probability measures on M and the pure states as the probability measures ω_p on M that are concentrated in one point $p \in M$ (which therefore are in one-to-one correspondence to the points of M). In this case, the order on observables is the pointwise one, and the spectrum of an observable is its image in $\mathbb{R}$, so $\operatorname{spec}(a) = \operatorname{image}(a)$ for all $a \in \mathfrak{O}$.

In classical mechanics, for example, M is typically a smooth manifold, the observables a set of real-valued functions on M, e.g. the smooth ones $\mathfrak{O} = \mathscr{C}^\infty(M, \mathbb{R})$, and the set of states $\mathcal{S}$ can be chosen as the set of all Radon probability measures on M with compact support. The pairing of states and observables then is simply the integral

$$\langle \omega , a \rangle = \int a \,\mathrm{d}\omega \tag{2.2.1}$$

for all $a \in \mathfrak{O}$ and $\omega \in \mathcal{S}$, and the probability measure $\mu_{\omega,a}$ on $\operatorname{spec}(a)$ describing the probability distribution of measuring a on ω is the push-forward measure $a_*(\omega)$. In the case of classical field theories, M is usually chosen to be a set of sections of a fibre bundle. However, finding good choices for the sets of observables and states is more complicated (see e.g. the series of preprints [40, 41]).

2.2.2 Functional Calculus

Now assume that a finite set of observables $a_1, \dots, a_N \in \mathfrak{O} \subseteq \{ a \colon M \to \mathbb{R} \}$ of a classical system are given, as well as a function $f \colon \operatorname{spec}(a_1) \times \dots \times \operatorname{spec}(a_N) \to \mathbb{R}$. Then the function $f(a_1, \dots, a_N) \colon M \to \mathbb{R}$, $p \mapsto f\big(a_1(p), \dots, a_N(p)\big)$ can be interpreted as a new observable, which can be measured by performing simultaneously the measurements corresponding to the observables $a_1, \dots, a_N$ and applying the function f to the result. However, whether $f(a_1, \dots, a_N)$ actually is an element of $\mathfrak{O}$ depends on the details of the model. Some typical requirements for $f(a_1, \dots, a_N) \in \mathfrak{O}$ to hold are the following:

- If the pairing $\langle \cdot , \cdot \rangle$ of observables and states is given by integration like in (2.2.1), then all functions in $\mathfrak{O}$ have to be integrable over all measures in $\mathcal{S}$, so f should at least be a measureable function in order to assure that $f(a_1, \dots, a_N)$ is again measurable.

- As all measurements are subject to errors, it is typical to require the observables to be at least continuous with respect to some topology on M, so that f should also be continuous. This then implies that sufficiently small errors in the measurement of $a_1, \dots, a_N$ do lead to arbitrary small errors of $f(a_1, \dots, a_N)$.

- In order to solve equations of motion, which usually are differential equations, some form of smoothness of the observables is helpful (like in the example of classical mechanics above). In this case, f should again be smooth.

- Of course, depending on the details of every individual model, further restrictions might occur, e.g. analyticity or restrictions on the assymptotic growth.

While in general it depends highly on the model at hand whether or not $f(a_1, \dots, a_N)$ is again an element of $\mathfrak{O}$, there are some functions f under which we can assume $\mathfrak{O}$ to be closed without loosing too many relevant examples. This then leads to helpful algebraic structures on the set of observables:

Consider the functions $+ \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $m_\lambda \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda x$ for all $\lambda \in \mathbb{R}$, which fulfil all the properties mentioned above. If they can be applied to all elements (or pairs of elements) in $\mathfrak{O}$, then they turn $\mathfrak{O}$ into a real vector space with addition and scalar multiplication given by the pointwise operations. Moreover, due to the linearity of the expectation value, the map $\mathfrak{O} \ni a \mapsto \langle \omega , a \rangle \in \mathbb{R}$ is linear for all states $\omega \in \mathcal{S}$, which allows us to identify states with certain linear functionals on $\mathfrak{O}$. Moreover, linearity of $\langle \omega , \cdot \rangle$ immediately implies that the order on $\mathfrak{O}$ is compatible with the vector space structure in so far as

$$a + c \leq b + c \qquad \text{and} \qquad \lambda a \leq \lambda b$$

hold for all $a, b, c \in \mathfrak{O}$ with $a \leq b$ and all $\lambda \in [0, \infty[$, so $\mathfrak{O}$ is even an ordered vector space and the $\langle \omega , \cdot \rangle$ are positive linear functionals on it (again, see the Appendix A.2 for the definitions). As models in which the observables are not closed under addition and scalar multiplication are at least uncommon, we will assume for the rest of this section that $\mathfrak{O}$ is an ordered real vector space of functions.

If $\mathfrak{O}$ is closed under the application of even more functions, then this induces additional algebraic structures. Two cases will be especially important: If $\mathfrak{O}$ is closed under pointwise multiplication of two functions, then it becomes a real associative unital algebra with unit $\mathbb{1}$, and the multiplication is in so far compatible with the order on $\mathfrak{O}$ as all squares are positive (because they map to the non-negative reals). Similarly, if $\mathfrak{O}$ is closed under taking the pointwise absolute value, then it becomes a Riesz space (Appendix A.2). Note that the multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth, even analytic, but e.g. not uniformly continuous. The absolute value $|\cdot| \colon \mathbb{R} \to \mathbb{R}$, in contrast, is uniformly continuous but not differentiable. Because of this, the assumption that $\mathfrak{O}$ forms an algebra or a Riesz space is a less trivial restriction than that $\mathfrak{O}$ forms a vector space.

2.2.3 Equations of Motion

For the discussion of the equations of motion we stay with the example of classical mechanics for simplicity. The determinism of classical mechanics is reflected by the earlier mentioned fact that the measurement of an observable $a \in \mathfrak{O}$ on a pure state $\omega \in \mathcal{S}_\mathrm{p}$ gives the *certain* result $\langle \omega , a \rangle$, together with the facts that pure states remain pure over time and that arbitrary states are just probability distributions over pure states. So the equations of motion can first be formulated for pure states and then be transfered to arbitrary states and observables:

Given a system which, at time $t = 0$, is in a pure state $\omega_p \in \mathcal{S}_\mathrm{p}$ with $p \in M$ (i.e. ω_p is the probability measure on M concentrated in p), then the time evolution of the result of an observable $a \in \mathfrak{O}$ on the system can be described by the function

$$\mathbb{R} \ni t \mapsto \big\langle \omega_{\Phi(p,t)} , a \big\rangle = a\big(\Phi(p,t)\big) \in \mathbb{R} , \tag{2.2.2}$$

with $\Phi \colon M \times \mathbb{R} \to M$ describing the curves in M through $\Phi(p, 0) = p$ that obey the equations of motion, which form a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_t \Phi(p,\tau) = X(t)\big|_{\Phi(p,t)} \tag{2.2.3}$$

for all $t \in \mathbb{R}$ and $p \in M$ with a possibly time-dependent vector field $X(t)$ on M. In well-behaved cases, this map Φ exists and is smooth. Then write $\Phi_t\colon M \to M$ for the smooth map $M \ni p \mapsto \Phi_t(p) := \Phi(p,t) \in M$. This curve of smooth maps can be used to describe the time evolution of the system in the Schrödinger picture as well as in the Heisenberg picture:

$$\langle\, \omega_{\Phi(p,t)}\,,\, a \,\rangle = \langle\, \omega_{\Phi_t(p)}\,,\, a \,\rangle = a\big(\Phi_t(p)\big) = \langle\, \omega_p\,,\, a\circ\Phi_t \,\rangle \tag{2.2.4}$$

So the evolution of a pure state $\omega_p \in \mathcal{S}_\mathrm{p}$ in the Schrödinger picture is given by $t \mapsto \omega_{\Phi_t(p)}$, the evolution of an observable $a \in \mathfrak{O}$ in the Heisenberg picture by the pullback $t \mapsto \Phi_t^*(a) := a\circ\Phi_t$, and the evolution of an arbitrary state $\omega \in \mathcal{S}$ then by the pushforward of measures $t \mapsto \Phi_{t,*}(\omega)$ because

$$\big\langle\, \omega\,,\, \Phi_t^*(a) \,\big\rangle = \int \Phi_t^*(a)\;\mathrm{d}\omega = \int a\;\mathrm{d}\Phi_{t,*}(\omega) = \big\langle\, \Phi_{t,*}(\omega)\,,\, a \,\big\rangle\,.$$

A case of special interest is the one of time-independent equations of motion, i.e. if the vector field $X(t)$ in (2.2.3) is independent of $t \in \mathbb{R}$. Then time evolution can be interpreted as the action of the Lie group $\mathbb{R}$ on M, $\mathfrak{O}$ and $\mathcal{S}$, which allows to put time evolution into the larger context of group actions on algebras:

Definition 2.2.1 *[82, Def. 1.3.10] Let $\mathcal{A}$ be an associative algebra, then a* Poisson bracket *on $\mathcal{A}$ is a bilinear map $\{\,\cdot\,,\,\cdot\,\}\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ fulfilling*

$$\{\,a\,,\,b\,\} = -\{\,b\,,\,a\,\} \qquad \textit{(antisymmetry)} \tag{2.2.5}$$

$$\{\,a\,,\,bc\,\} = \{\,a\,,\,b\,\}\,c + b\,\{\,a\,,\,c\,\} \qquad \textit{(Leibniz rule)} \tag{2.2.6}$$

$$\big\{\,a\,,\,\{\,b\,,\,c\,\}\,\big\} = \big\{\,\{\,a\,,\,b\,\}\,,\,c\,\big\} + \big\{\,b\,,\,\{\,a\,,\,c\,\}\,\big\} \qquad \textit{(Jacobi identity)} \tag{2.2.7}$$

for all $a,b,c \in \mathcal{A}$.

Definition 2.2.2 *[82, Def. 3.3.36 and 3.3.49] Let $\big(M,\{\,\cdot\,,\,\cdot\,\}\big)$ be a Poisson manifold, i.e. a smooth manifold M together with a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the associative algebra $\mathscr{C}^\infty(M,\mathbb{R})$ of smooth real-valued functions on M, G a connected Lie group and $\triangleright\colon G\times M\to M$ a smooth left action of G on M. Let $\triangleleft\colon \mathscr{C}^\infty(M,\mathbb{R})\times G\to\mathscr{C}^\infty(M,\mathbb{R})$ be the induced right action of G on smooth functions on M via pullback and $\triangleleft\colon \mathscr{C}^\infty(M,\mathbb{R})\times\mathfrak{g}\to\mathscr{C}^\infty(M,\mathbb{R})$ the corresponding infinitesimal right action of the Lie algebra $\mathfrak{g}$. An* ad^*-equivariant moment map *for this action of G is a smooth map $J\colon M\to\mathfrak{g}^*$ for which*

$$\big\{\,a\,,\,\langle\,J\,,\,\xi\,\rangle\,\big\} = a\triangleleft\xi \qquad \textit{and} \qquad \big\{\,\langle\,J\,,\,\xi\,\rangle\,,\,\langle\,J\,,\,\eta\,\rangle\,\big\} = \langle\,J\,,\,[\,\xi\,,\,\eta\,]\,\rangle \tag{2.2.8}$$

holds for all $a \in \mathscr{C}^\infty(M,\mathbb{R})$ and all $\xi,\eta\in\mathfrak{g}$. Here $\langle\,J\,,\,\xi\,\rangle \in \mathscr{C}^\infty(M,\mathbb{R})$ denotes the smooth map $M\ni p\mapsto\langle\,J(p)\,,\,\xi\,\rangle\in\mathbb{R}$. An action of G on M together with an ad^*-equivariant moment map J is called a Hamiltonian action*.*

Note that the existence of an ad^*-equivariant moment map implies that the action of the Lie algebra $\mathfrak{g}$ is compatible with the Poisson bracket in so far as

$$\{ a , b \} \triangleleft \xi = \{ \{ a , b \} , \langle J , \xi \rangle \} = \{ \{ a , \langle J , \xi \rangle \} , b \} + \{ a , \{ b , \langle J , \xi \rangle \} \} = \{ a \triangleleft \xi , b \} + \{ a , b \triangleleft \xi \}$$

holds for all $a, b \in \mathscr{C}^\infty(M, \mathbb{R})$ and all $\xi \in \mathfrak{g}$. As G was assumed to be connected, this also implies that

$$\{ a , b \} \triangleleft g = \{ a \triangleleft g , b \triangleleft g \}$$

for all $a, b \in \mathscr{C}^\infty(M, \mathbb{R})$ and $g \in G$.

In many examples from classical mechanics, the manifold of states M is indeed equipped with a Poisson bracket $\{ \,\cdot\, , \,\cdot\, \}$ and the time evolution can indeed be interpreted as a Hamiltonian action of $\mathbb{R}$ with moment map $H \colon M \to \mathbb{R}$. In this case we call $H = \langle H , 1 \rangle \in \mathscr{C}^\infty(M, \mathbb{R})$ the *Hamiltonian* of the system, and the time evolution of an observable $a \in \mathfrak{O}$ in the Heisenberg picture, $\Phi_t^*(a) = a \triangleleft t$, can be described by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_t \big(a \triangleleft \tau \big) = \{ a \triangleleft t , H \} . \tag{2.2.9}$$

It is also not uncommon that the group acting on M is larger than just $\mathbb{R}$ and includes more symmetries of the system, like translation or rotations in space.

2.2.4 Example: The Harmonic Oscillator

As an example, consider the one-dimensional harmonic oscillator: Here $M = \mathbb{T}^*\mathbb{R} \cong \mathbb{R}^2$ with the standard coordinates $q, p \colon M \to \mathbb{R}$ and as set of observables we can take the algebra $\mathfrak{O} = \mathscr{C}^\infty(M, \mathbb{R})$ which is equipped with the canonical Poisson bracket

$$\{ a , b \} = \langle \omega^{-1} , \mathrm{d}a \otimes \mathrm{d}b \rangle = \frac{\partial a}{\partial q}\frac{\partial b}{\partial p} - \frac{\partial b}{\partial q}\frac{\partial a}{\partial p} \tag{2.2.10}$$

for all $a, b \in \mathscr{C}^\infty(M, \mathbb{R})$, which is constructed out of the canonical symplectic form $\omega = -\,\mathrm{d}\theta = \mathrm{d}q \wedge \mathrm{d}p$, where $\theta = p\,\mathrm{d}q$ is the tautological field of 1-forms on the cotangent bundle $M = \mathbb{T}^*\mathbb{R}$, i.e. the one that maps every 1-form in $\mathbb{T}^*\mathbb{R}$ to itself. Note that the *canonical commutation relations*

$$\{ q , p \} = 1 \tag{2.2.11}$$

hold. As the set of states $\mathcal{S}$ we can again choose the compactly supported Radon probability measures on M. The dynamics is then given by the Hamiltonian of the harmonic oscillator

$$H = p^2 + q^2 \tag{2.2.12}$$

and (2.2.9) in the Heisenberg picture, or by (2.2.3) with time-independent vector field

$$X = \frac{\partial H}{\partial p}\partial_q - \frac{\partial H}{\partial q}\partial_p = 2p\partial_q - 2q\partial_p . \tag{2.2.13}$$

However, admitting only compactly supported Radon probability measures as states might be disadvantageous for some applications as it excludes the important Gaussian probability measures. These can be included only if the observables are restricted, e.g. to the subalgebra

$$\mathscr{C}^\infty_g(M,\mathbb{R}) := \left\{ a \in \mathscr{C}^\infty(M,\mathbb{R}) \;\middle|\; \sup_{(q,p)\in M} \left|\exp\big(-\epsilon(q^2+p^2)\big)a(q,p)\right| < \infty \text{ for all } \epsilon > 0 \right\} \tag{2.2.14}$$

of $\mathscr{C}^\infty(M,\mathbb{R})$ or even to the subalgebra of all $a \in \mathscr{C}^\infty(M,\mathbb{R})$ for which a and all arbitrarily high partial derivatives of a in q- and p-direction are in $\mathscr{C}^\infty_g(M)$. The latter choice would have the advantage that it is also closed under the Poisson bracket. This shows that already in the most standard example there is more than just one reasonable choice for the sets of observables and states.

2.3 Quantum Physics

2.3.1 General Considerations

In quantum physics, like in classical physics, the pure states separate the observables, so that again

$$\forall_{\omega\in\mathcal{S}_{\mathrm{p}}} : \langle\, \omega\,,\, a \,\rangle = \langle\, \omega\,,\, b \,\rangle \qquad \Longrightarrow \qquad a = b \tag{P4'}$$

is fulfilled. However, the result of the measurement of a quantum observable $a \in \mathfrak{O}$ on a pure state $\omega \in \mathcal{S}_{\mathrm{p}}$ need not give a certain result, and $\langle\, \omega\,,\, a \,\rangle$ still only describes the expectation value of the measurement. Note that there is good reason to see this not as a mere "incompleteness" of our understanding of quantum mechanics, but as a fundamental feature:

This was famously discussed in depth especially by Bohr and Einstein in the early days of quantum mechanics [14, 44], with the perhaps strongest argument for an incompleteness of quantum mechanics given by Einstein, Podolsky and Rosen in [33]. However, it was later shown by Bell, building upon this argument, that a theory using local hidden variables to achieve a more complete description of reality will contradict quantum mechanics [8], and by now there is sufficient experimental evidence (starting with [38]) that supports the predictions of quantum mechanics in these cases. Another point against models using hidden variables was also brought forward by Kochen and Specker [50].

This surprising behaviour of quantum mechanics is usually modelled as follows: The observables are chosen as a set of Hermitian endomorphisms $\mathfrak{O} \subseteq \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ of a pre-Hilbert space $\mathcal{D}$, which, when viewed as operators on the Hilbert space completion $\mathfrak{H}$ of $\mathcal{D}$, are essentially self-adjoint (again, see the Appendix A.4 for details about operators on Hilbert spaces). The set of states $\mathcal{S}$ is the convex hull of Hermitian projectors in $\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ on one-dimensional subspaces of $\mathcal{D}$ (or a suitable closure thereof) such that the pairing

$$\langle\, \omega\,,\, a \,\rangle = \mathsf{tr}\big(\omega a\big) \tag{2.3.1}$$

is well-defined for all $\omega \in \mathcal{S}$ and $a \in \mathfrak{O}$ and such that the pure states are the projectors $\chi_\phi \in \mathcal{S} \subseteq \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ on the one dimensional subspaces generated by vectors $\phi \in \mathcal{D}$ with $\|\phi\| = 1$, i.e. $\chi_\phi(\psi) := \phi\langle\, \phi \,|\, \psi \,\rangle$ for all $\psi \in \mathcal{D}$. Such a state is usually called a *vector state*. These pure states are often also identified with the normed vectors in $\mathcal{D}$, or rather the equivalence classes of such vectors up to a complex phase, and the pairing of observables and pure states can then be expressed alternatively as

$$\langle \chi_\phi \,,\, a \rangle = \mathsf{tr}\big(\chi_\phi a\big) = \langle\, \phi \,|\, a(\phi) \,\rangle \tag{2.3.2}$$

for all $a \in \mathfrak{O}$ and all $\phi \in \mathcal{D}$ with $\|\phi\| = 1$. The order on observables thus is the usual order of Hermitian endomorphisms on a pre-Hilbert space like in Definition 1.2.9.

Moreover, the spectral theorem for (unbounded) self-adjoint operators on a Hilbert space [36, Thm. 12.1] allows to represent every observable $a \in \mathfrak{O}$ as an integral over a projection-valued measure μ_a on the operator theoretic spectrum of a,

$$a = \int_{\mathrm{spec}(a)} \mathsf{id}_{\mathbb{R}} \,\mathrm{d}\mu_a \,. \tag{2.3.3}$$

Then the pairing $\langle\, \omega \,,\, a \,\rangle$ of a state $\omega \in \mathcal{S}$ with a is the expectation value of a probability measure $\mathsf{tr}(\omega\mu_a)$ on this spectrum,

$$\langle\, \omega \,,\, a \,\rangle = \mathsf{tr}\big(\omega a\big) = \int_{\mathrm{spec}(a)} \mathsf{id}_{\mathbb{R}} \,\mathrm{d}\,\mathsf{tr}\big(\omega\mu_a\big) \,. \tag{2.3.4}$$

The interpretation thereof is that the operator theoretic spectrum of a indeed is the pysical spectrum of the observable a, i.e. the set of all possible results of measurements of a on arbitrary states, and that the probability measure associated to pairing ω with a describes the actual probability distribution of measurement results.

2.3.2 Functional Calculus

Like for classical physics, we could now again discuss which functions of observables might again be observables. In addition to the restrictions encountered already in this case, there are new ones that are inherent to quantum physics: First of all, the interpretation of $f\big(a_1, \dots, a_N\big)$, a function $f\colon \mathrm{spec}(a_1) \times \dots \times \mathrm{spec}(a_N) \to \mathbb{R}$ applied to observables $a_1, \dots, a_N \in \mathfrak{O}$, as the observable which can be measured by simultaneously measuring $a_1, \dots, a_N$ and applying f to the results, is only possible if $a_1, \dots, a_N$ are commuting observables which can actually be measured simultaneously. But even if this is the case and if $f\big(a_1, \dots, a_N\big)$ exists as an unbounded operator on $\mathfrak{H}$, it is not clear in general whether this operator restricts to an endomorphisms of $\mathcal{D}$ (except in the case that f is a polynomial function).

2.3.3 Equations of Motion

Like in classical mechanics, a pure state remains pure over time in quantum mechanics and the time evolution of pure states already fixes the time evolution of all states, or of observables (in the Heisenberg picture). So the non-determinism of quantum mechanics is only due to pure states not always giving uniquely determined measurement results.

Given a system which, at time $t = 0$, is in a pure state $\chi_\phi \in \mathcal{S}_{\mathrm{p}}$ with normalized $\phi \in \mathcal{D}$, then the time evolution of the result of an observable $a \in \mathfrak{O}$ on the system can be described by the function

$$\mathbb{R} \ni t \mapsto \left\langle \chi_{U(\phi,t)} \,,\, a \right\rangle = a\big(U(\phi,t)\big) \in \mathbb{R}\,, \tag{2.3.5}$$

with $U\colon \mathcal{D}\times\mathbb{R}\to\mathcal{D}$ describing the curves in $\mathcal{D}$ through $U(\phi,0)=\phi$ that obey the linear Schrödinger equation

$$\mathrm{i}\hbar \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_t U(\phi,\tau) = H(t)U(\phi,t) \tag{2.3.6}$$

for all $t\in\mathbb{R}$ and $\phi\in\mathcal{D}$ with a possibly time-dependent Hermitian endomorphism $H(t)\in\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$, the *Hamiltonian operator*. Here $\hbar$ again denotes Planck's constant. In well-behaved cases, this map U exists and is, for every $t\in\mathbb{R}$, a unitary endomorphism of $\mathcal{D}$. In this case, write $U_t\in\mathcal{L}^*(\mathcal{D})$ for this endomorphism $\mathcal{D}\ni\phi\mapsto U_t(\phi)\coloneqq U(\phi,t)\in\mathcal{D}$. Again, this curve of unitaries can be used to describe the time evolution of the system in the Schrödinger picture as well as in the Heisenberg picture:

$$\left\langle \chi_{U(\phi,t)} \,,\, a \right\rangle = \left\langle \chi_{U_t(\phi)} \,,\, a \right\rangle = \left\langle U_t(\phi) \,\middle|\, a\big(U_t(\phi)\big) \right\rangle = \left\langle \chi_\phi \,,\, U_t^*\, a\, U_t \right\rangle \tag{2.3.7}$$

So the evolution of a pure state $\chi_\phi\in\mathcal{S}_{\mathrm{p}}$ in the Schrödinger picture is given by $t\mapsto\chi_{U_t(\phi)}$, the evolution of an observable $a\in\mathfrak{O}$ in the Heisenberg picture by the conjugation $t\mapsto U_t^* a\, U_t$ and the evolution of an arbitrary state $\omega\in\mathcal{S}$ then by the conjugation $t\mapsto U_t\,\omega\,U_t^*$ because

$$\left\langle \omega \,,\, U_t^* a\, U_t \right\rangle = \mathsf{tr}\left(\omega\, U_t^* a\, U_t\right) = \left\langle U_t\,\omega\,U_t^* \,,\, a \right\rangle.$$

Again, if $H(t)$ is independent of t, then the time evolution can be interpreted as the actions of the Lie group $\mathbb{R}$ on $\mathcal{D}$, $\mathfrak{O}$ and $\mathcal{S}$, which – in well-behaved cases – are fixed by the infinitesimal action of the corresponding Lie algebra $\mathbb{R}$ on $\mathfrak{O}$ via the commutator with H:

$$\mathrm{i}\hbar \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_t U_\tau^*\, a\, U_\tau = \left[U_t^*\, a\, U_t \,,\, H \right]. \tag{2.3.8}$$

2.3.4 Example: The Harmonic Oscillator

Finally, let us again consider the example of the one-dimensional harmonic oscillator: Now a suitable choice is $\mathcal{D}=\mathscr{S}(\mathbb{R})$, the Schwartz space on $\mathbb{R}$ defined as the subspace of $\mathscr{C}^\infty(\mathbb{R})$ consisting of rapidly decreasing functions, i.e. of those $\phi\in\mathscr{C}^\infty(\mathbb{R})$ for which

$$\sup_{x\in\mathbb{R}} \left| x^m \phi^{(n)}(x) \right| < \infty \tag{2.3.9}$$

holds for all $m,n\in\mathbb{N}_0$, where $\phi^{(n)}$ denotes the n-th derivative of ϕ. The inner product on $\mathscr{S}(\mathbb{R})$ is the usual one, i.e.

$$\left\langle \phi \,\middle|\, \psi \right\rangle \coloneqq \int \overline{\phi(x)}\psi(x)\,\mathrm{d}x\,. \tag{2.3.10}$$

On $\phi\in\mathcal{D}$, the position and momentum operators $Q,P\in\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ are defined as

$$\big(Q\phi\big)(x) \coloneqq x\,\phi(x) \qquad \text{for all } x\in\mathbb{R}, \qquad \text{and} \qquad P\phi \coloneqq \frac{\hbar}{\mathrm{i}}\phi^{(1)}\,, \tag{2.3.11}$$

and fulfil again the canonical commutation relations

$$\frac{1}{\mathrm{i}\hbar}[\,Q\,,\,P\,] = 1\,. \tag{2.3.12}$$

When viewed as unbounded operators on the Hilbert space completion $\mathrm{L}^2(\mathbb{R})$ of $\mathcal{D}$, then P and Q are essentially self-adjoint, because $Q \pm \mathrm{i}\mathbb{1}$ and $P \pm \mathrm{i}\mathbb{1}$ are invertible elements of $\mathcal{L}^*(\mathcal{D})$, thus have dense range (this is immediately clear for Q and is similarly clear for P after Fourier transformation). As set of observables one can e.g. choose all elements of $\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ which are essentially self-adjoint when viewed as unbounded operators on $\mathrm{L}^2(\mathbb{R})$, and as set of states the convex hull of the projectors on one-dimensional subspaces of $\mathcal{D}$. Of course, like in the classical case, one could also want to restrict the set of observables and allow for a larger set of states instead. The equations of motions are then given by (2.3.6) or (2.3.8) with the Hamiltonian operator of the harmonic oscillator

$$H \coloneqq P^2 + Q^2\,. \tag{2.3.13}$$

2.4 Poisson *-Algebras

Comparison of the description of classical and quantum physics, especially classical and quantum mechanics, shows some glaring differences, but also very clear similarities:

When focusing on the pure states and their representations as points of a manifold or (equivalence classes of) normalized vectors of a pre-Hilbert space, the differences are the most apparent: Not only is it hard to see any relation at all between these different sets of pure states, but also the observables seem to be constructed in a completely different way, once as real-valued functions and once as Hermitian endomorphisms of a vector space. Consequently, the pairings of observables and pure states seem to be very different constructions in these two cases. This is especially odd when considering that classical mechanics is a reasonably good description of nature on macroscopic scales, on which it therefore should coincide with the quantum mechanical description.

However, from a more abstract point of view, both descriptions have many structures in common. In both cases the observables can be treated at least as a subset of the Hermitian elements of a *-algebra $\mathcal{A}$: In classical mechanics, it is a not too restrictive assumption that the observables $\mathfrak{O}$ are a real unital associative and commutative algebra, which is naturally isomorphic to the real subalgebra of Hermitian elements of the complexification of $\mathfrak{O}$ to a commutative *-algebra, $\mathfrak{O} \cong (\mathfrak{O} \otimes \mathbb{C})_{\mathrm{H}}$. In quantum mechanics, the observables have already been constructed as certain Hermitian elements of the *-algebra $\mathcal{L}^*(\mathcal{D})$ of adjointable endomorphisms of a pre-Hilbert space $\mathcal{D}$, or of a *-subalgebra thereof, i.e. of an O^*-algebra on $\mathcal{D}$. Moreover, in both cases there is a possibility to describe well-behaved actions of Lie algebras on the observables by an internal action using the Poisson bracket (in classical mechanics) or the commutator of the algebra (in quantum mechanics). This leads to:

Definition 2.4.1 *[82, Def. 1.3.10] A* Poisson *-algebra *is a tuple* $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}\big)$ *consisting of a (unital associative) *-algebra $\mathcal{A}$ and a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathcal{A}$ which is* real*, i.e. which fulfils*

$$\{\,a\,,\,b\,\}^* = \{\,a^*\,,\,b^*\,\} \qquad \textit{for all } a, b \in \mathcal{A}\,. \tag{2.4.1}$$

In classical mechanics, this real Poisson bracket is simply the $\mathbb{C}$-bilinear extension of a Poisson bracket on the real commutative algebra $\mathfrak{O}$ to $\mathcal{A} := \mathfrak{O} \otimes \mathbb{C}$. In quantum mechanics, the Poisson bracket is the commutator up to a complex factor,

$$\{ \,\cdot\, , \,\cdot\, \} = \frac{1}{\mathrm{i}\hbar} [\,\cdot\, , \,\cdot\,] , \tag{2.4.2}$$

which assures that $\{ \,\cdot\, , \,\cdot\, \}$ is indeed real and that the equations of motion in the Heisenberg picture, (2.2.9) and (2.3.8), can be formulated both as

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_t a(\tau) = \{ a(t) \, , \, H \} \tag{2.4.3}$$

with a Hamiltonian $H \in \mathfrak{O}$.

The states now can be seen simply as certain linear functionals on $\mathcal{A}$, more precisely as certain algebraically positive linear functionals ω on $\mathcal{A}$ which are normalized to $\langle \omega \, , \, \mathbb{1} \rangle = 1$, i.e. as algebraic states. Equivalent to fixing a convex set of algebraic states $\mathcal{S}$ is fixing the convex cone of algebraically positive linear functionals Ω^+_{H} generated by $\mathcal{S}$, which determines the set of states as

$$\mathcal{S} = \left\{ \omega \in \Omega^+_{\mathrm{H}} \;\middle|\; \langle \omega \, , \, \mathbb{1} \rangle = 1 \right\} . \tag{2.4.4}$$

Are there additional requirements this cone Ω^+_{H} should fulfil? It should be sufficiently large so that there are enough states to separate the observables like in (P4). Moreover, it is reasonable to demand that the algebraically positive linear functional $a \triangleright \omega$ (see Definition 1.2.3) remains in Ω^+_{H} for $a \in \mathcal{A}$ and $\omega \in \Omega^+_{\mathrm{H}}$, and also that Ω^+_{H} is at least weak-*-closed (see Appendix A.1.4) in its $\mathbb{C}$-linear span $\Omega := \langle\!\langle \, \Omega^+_{\mathrm{H}} \, \rangle\!\rangle_{\mathrm{lin}}$. We have thus arrived at a rather general algebraic description of a physical system:

Definition 2.4.2 *A* physical system *is a triple* $\big(\mathcal{A}, \{ \,\cdot\, , \,\cdot\, \}, \Omega^+_{\mathrm{H}}\big)$ *consisting of a* **-algebra of* observables $\mathcal{A}$*, a* Poisson bracket $\{ \,\cdot\, , \,\cdot\, \} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *such that* $\big(\mathcal{A}, \{ \,\cdot\, , \,\cdot\, \}\big)$ *is a Poisson* **-algebra, and a set of* positive linear functionals $\Omega^+_{\mathrm{H}} \subseteq \mathcal{A}^*$ *with the following properties:*

i.) Ω^+_{H} *is a convex cone, i.e.* $\lambda\omega + \mu\rho \in \Omega^+_{\mathrm{H}}$ *for all* $\lambda, \mu \in [0, \infty[$ *and* $\omega, \rho \in \Omega^+_{\mathrm{H}}$.

ii.) Every $\omega \in \Omega^+_{\mathrm{H}}$ *is algebraically positive, i.e.* $\langle \omega \, , \, a^* a \rangle \geq 0$ *for all* $a \in \mathcal{A}$.

iii.) Ω^+_{H} *is stable under the monoid action* $\triangleright$ *of* $\mathcal{A}$ *on* $\mathcal{A}^*$*, i.e.* $a \triangleright \omega \in \Omega^+_{\mathrm{H}}$ *for all* $a \in \mathcal{A}$ *and all* $\omega \in \Omega^+_{\mathrm{H}}$.

iv.) Write $\Omega := \langle\!\langle \, \Omega^+_{\mathrm{H}} \, \rangle\!\rangle_{\mathrm{lin}}$ *for the linear span of* Ω^+_{H}*, then* Ω^+_{H} *is weak-***-closed in* Ω.

v.) Ω^+_{H} *separates elements of* $\mathcal{A}$*, i.e. if* $\langle \omega \, , \, a \rangle = 0$ *for all* $\omega \in \Omega^+_{\mathrm{H}}$ *and one* $a \in \mathcal{A}$*, then* $a = 0$.

The actual observables are then the Hermitian elements $\mathfrak{O} = \mathcal{A}_{\mathrm{H}} \subseteq \mathcal{A}$ (or a subset thereof) and the states are like in (2.4.4). It is then clear that $\mathcal{S}$ is a convex set and that the pair of $\mathfrak{O}$ and $\mathcal{S}$ together with the usual dual pairing $\langle \,\cdot\, , \,\cdot\, \rangle$ fulfil (P1) - (P4). The ordering on observables from (2.1.2) yields a partial order on $\mathcal{A}_{\mathrm{H}}$,

$$a \leq b \qquad :\Longleftrightarrow \qquad \forall_{\omega \in \Omega^+_{\mathrm{H}}} : \langle \omega \, , \, a \rangle \leq \langle \omega \, , \, b \rangle \tag{2.4.5}$$

which turns $\mathcal{A}$ into an ordered *-algebra. Finally, it is also noteworthy that the assumption that the observables form an algebra allows to define the variance $\mathrm{Var}_\omega(a)$ of a state ω on an observable a like in Definition 1.2.2. Due to the interpretation of the pairing $\langle \,\cdot\,,\,\cdot\,\rangle$ of states and observables as the expectation value of a measurement, this variance actually describes the variance of the measurement of a on ω.

2.5 Deformation Quantization

We have seen that physical systems – classical as well as quantum ones – can be modelled by a Poisson *-algebra describing the observables together with a cone of algebraically positive linear functionals describing the states. However, the precise connection between the classical and the quantum description of the same system is still open.

Ideally, the observables should be the same set in both descriptions, because the measurable properties of the system should be the same, only the relations between them have to be adapted (e.g. in both examples of the harmonic oscillator in one dimension, the classical and the quantum one, the most interesting observables are position q, momentum p and energy, which is described by the Hamiltonian H). Clearly, the product and the Poisson bracket differ in both descriptions, as the quantum product is non-commutative and its commutator yields the Poisson bracket via (2.4.2), while the classical product is commutative and thus the classical Poisson bracket needs to have a different origin. Moreover, it is a well-known observation in physics that a system can be described classically with reasonable accuracy as long as the typical actions (products of typical distance- and momentum scales, or of typical time and energy scales) are significantly larger than Planck's constant $\hbar$. Put another way, when letting $\hbar$ tend to 0 in a quantum mechanical model, the result should be the classical model. These considerations lead to the following heuristic idea of the connection between the observable algebras in classical and quantum models, which is made rigorous in the different variants of deformation quantization:

The classical and quantum observables are modelled by the Hermitian elements of the same complex vector space $\mathcal{A}$ with antilinear involution $\,\cdot^*$ and with one distinguished $\mathbb{1} \in \mathcal{A}$. Moreover, $\mathcal{A}$ is endowed with an $\hbar$-dependent product $\star_\hbar$, such that $(\mathcal{A}, \star_\hbar)$ is a *-algebra with unit $\mathbb{1}$ for all $\hbar \geq 0$, and commutative for $\hbar = 0$. The quantum description is given by the typically non-commutative *-algebra $(\mathcal{A}, \star_\hbar)$ for $\hbar > 0$ with Poisson bracket $[\,\cdot\,,\,\cdot\,]/(\mathrm{i}\hbar)$, and in the limit $\hbar \to 0$, the non-commutative product $\star_\hbar$ becomes the commutative product $\star_0$ of the classical observable algebra, and $[\,\cdot\,,\,\cdot\,]/(\mathrm{i}\hbar)$ becomes the classical Poisson bracket.

2.5.1 Formal Star Product on a Poisson Manifold

The heuristic idea that the quantum description of a system should yield the classical one in the limit $\hbar \to 0$ has inspired physics since the early days of quantum mechanics, but one of the clearest formulations of it has been given by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer who in [3–5] applied Gerstenhaber's theory of formal deformations of rings and algebras [39] to the algebra of smooth functions over a symplectic or Poisson manifold. The central definition here is the one of a formal star product, which is, following [82, Def. 6.1.1]:

Definition 2.5.1 (Formal deformation quantization) *A* formal star product *on a Poisson manifold* $\big(M, \{\,\cdot\,,\,\cdot\,\}\big)$ *is a* $\mathbb{C}[\![\hbar]\!]$*-bilinear associative multiplication on* $\mathscr{C}^\infty(M,\mathbb{C})[\![\hbar]\!]$ *such that*

$$a \star b = ab + \mathcal{O}(\hbar) \qquad \textit{and} \qquad \frac{1}{\mathrm{i}\hbar}[\,a\,,\,b\,]_\star = \{\,a\,,\,b\,\} + \mathcal{O}(\hbar)$$

holds for all $a, b \in \mathscr{C}^\infty(M,\mathbb{C})$ *and such that the constant* 1*-function remains a unit with respect to* $\star$*. Here* $[\![\hbar]\!]$ *denotes formal power series in a variable* $\hbar$ *and* $[\,\cdot\,,\,\cdot\,]_\star$ *the* $\star$*-commutator.*

If one also wants compatibility with the *-involution, i.e. pointwise complex conjugation, then one demands that a formal star product is *Hermitian*, i.e. that $\overline{a \star b} = \overline{b} \star \overline{a}$ holds for all $a, b \in \mathscr{C}^\infty(M)[\![\hbar]\!]$. This idea of *formal* deformation quantization has been very fruitful and lead to deep mathematical insights on the way to a classification of all star products on a given Poisson manifold:

In the symplectic case, existence of star products on cotangent bundles of parallelizable smooth manifolds was shown by Cahen and Gutt [22], then on general cotangent bundles by DeWilde and Lecomte [28]. On another special case of symplectic manifolds, namely Kähler manifolds, star products have been constructed e.g. by Moreno [56] on nonexceptional Kähler symmetric spaces and by Cahen, Gutt and Rawnsley [24] on compact homogeneous Kähler manifolds, and Karabegov showed that there always exist star products with separation of variables [48], i.e. where left multiplication with holomorphic, and right multiplication with antiholomorphic functions are just the pointwise multiplications. The general existence of formal star products on symplectic manifolds has been established by various authors using different methods: By DeWilde and Lecomte [27]; Omori, Maeda and Yoshioka [64]; and by Fedosov [35]. A classification of star products in the sympectic case could be achieved by Nest and Tsygan [59, 60]; Deligne [29]; Bertelson, Cahen and Gutt [9] as well as Weinstein and Xu [84].

For Poisson manifolds, the first non-trivial example of a star product has been given by Gutt for the dual of a Lie algebra [42] (and then extended to a star product on the cotangent bundle of the Lie group), but the general theory has proven to be much more difficult than in the sympectic case. In the end, the existence and classification result has finally been given by Kontesevich [51] by proving his formality conjecture.

In order to be able to examine states and representations of the algebras from deformation quantization, Bordemann and Waldmann have introduced the notion of formally positive linear functionals on $\mathbb{C}[\![\hbar]\!]$-algebras [18], see also the subsequent works by Bursztyn and Waldmann [20, 21]. This e.g. allows to construct formal representations on formal pre-Hilbert spaces, and they proved that such formally positive linear functionals exist on a large class of examples.

Despite all these results, formal deformation quantization still falls one important step short of providing a complete theory of quantization: As long as $\hbar$ is treated only as a formal parameter, a formal star product can only be evaluated at $\hbar = 0$ (where it yields the classical pointwise product), but not at $\hbar > 0$ (where it, heuristically, should yield the product of the quantum observable algebra). Nevertheless, there are many examples of formal star products, which, when restricted to a certain subalgebra of smooth functions, converge also for $\hbar > 0$. One class of such examples are the well-known exponential star products (see e.g. [82, chap. 5.2.4]), which will also be studied in much greater detail in Chapter 4:

For $m \in \mathbb{N}$, $M = \mathbb{R}^m$ with standard coordinates $x^1, \ldots, x^m$ and $\mu_0 \colon \mathscr{C}^\infty(M, \mathbb{C}) \otimes \mathscr{C}^\infty(M, \mathbb{C}) \to \mathscr{C}^\infty(M, \mathbb{C})$ the classical pointwise product, the exponential star products are

$$\begin{aligned} a \star b &:= \mu_0\Big(\exp\big(\hbar \Lambda^{ij} \partial_{x^i} \otimes \partial_{x^j}\big)(a \otimes b)\Big) \\ &= \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{\substack{i_1,\ldots,i_r=1 \\ j_1,\ldots,j_r=1}}^{m} \Lambda^{i_1 j_1} \cdots \Lambda^{i_r j_r} \frac{\partial^r a}{\partial x^{i_1} \cdots \partial x^{i_r}} \frac{\partial^r b}{\partial x^{j_1} \cdots \partial x^{j_r}} \end{aligned} \tag{2.5.1}$$

with a complex matrix $\Lambda \in \mathbb{C}^{m \times m}$. For $a, b \in \mathscr{C}^\infty(M, \mathbb{C})$, the product $f \star g$ can in general only be interpreted as a formal power series in $\hbar$. By $\mathbb{C}[\![\hbar]\!]$-bilinear extension, (2.5.1) defines a formal star product on M with Poisson bracket

$$\{\, a \,,\, b \,\} = \mathrm{i}\big(\Lambda^{ji} - \Lambda^{ij}\big) \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial x^j} \tag{2.5.2}$$

for $f, g \in \mathscr{C}^\infty(M, \mathbb{C})$, i.e. the Poisson tensor is twice the antisymmetric part of $-\mathrm{i}\Lambda$. Especially for $m = 2$ and

$$-\mathrm{i}\Lambda_{\text{asym}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{e.g.} \quad \Lambda = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad \Lambda = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}, \tag{2.5.3}$$

(yielding the Weyl and Wick product), this is just the Poisson bracket from example (2.2.10) with coordinates $x^1 = q$ and $x^2 = p$. One can check that this formal star product is Hermitian if and only if Λ is Hermitian.

However, if a and b are polynomial functions, then (2.5.1) actualy converges for all $\hbar \in \mathbb{C}$ because the infinite sum in r simply terminates after a finite number of terms. Then the polynomials on $\mathbb{R}^m$ with the product $\star$ for a Hermitian matrix Λ and $\hbar \in \mathbb{R}$ actually form a *-algebra with *-involution given by pointwise complex conjugation. Again in the case $m = 2$ with $x^1 = Q$ and $x^2 = P$ and with a choice of Λ as before, this algebra fulfils the canonical commutation relations of example (2.3.12) for $\hbar \neq 0$ and thus interpolates perfectly between the classical and quantum versions of the observable algebras.

2.5.2 Deformation Quantization of C^*-Algebras

The problems linked to treating $\hbar$ only as a formal parameter are overcome in *non-formal* versions of deformation quantization, most notably the deformation quantization by means of C^*-algebras as defined in [69] by Rieffel:

Definition 2.5.2 (Strict deformation quantization) *A strict deformation quantization of a Poisson manifold (M, π) is given by a *-subalgebra $\mathcal{A}$ of $\mathscr{C}^\infty_\infty(M)$, the smooth functions on M vanishing at infinity, which contains all smooth functions on M with compact support, an open interval $I \subseteq \mathbb{R}$ containing 0 and for every $\hbar \in I$ an associative product $\star_\hbar$, an antilinear involution $\cdot^{*_\hbar}$ and a C^*-norm $\|\cdot\|_\hbar$ on $\mathcal{A}$, such that $\star_0$ is the pointwise product, $\cdot^{*_0}$ pointwise complex conjugation and $\|\cdot\|_0$ the usual* *supremums norm, and such that $I \ni \hbar \mapsto \|a\|_\hbar \in \mathbb{R}$ is continuous for every $a \in \mathcal{A}$ and*

$$\lim_{\hbar \to 0} \left\| \frac{1}{\mathrm{i}\hbar} [\, a \,,\, b \,]_{\star_\hbar} - \{\, a \,,\, b \,\} \right\|_\hbar = 0$$

for all $a, b \in \mathcal{A}$.

Unfortunately, up to now there are no comparable results on the existence and classification of non-formal star products, only some examples have been constructed: Rieffel has treated e.g. the case of $\mathbb{R}^m$ with the constant Poisson structure that has already been discussed in the context of formal deformation quantization, and some similar constructions like the torus, as well as the linear Poisson structures on the dual of a nilpotent Lie algebra [70, 71]. Bieliavsky and Gayral have constructed a universal deformation formula for the action of Kähler Lie groups [11].

Especially concerning the deformation quantization of C^*-algebras, one might also ask whether the C^*-setting is too restrictive: Unital C^*-algebras correspond to *compact* Hausdorff spaces (in the commutative case) or algebras of *bounded* operators on a Hilbert space, but many examples from physics are not compact or bounded, respectively.

2.5.3 Deformation Quantization of Locally Convex *-Algebras

The above considerations suggest to generalize Rieffel's strict deformation quantization via C^*-algebras to locally convex *-algebras. This way, one should be able to construct more examples and to keep a close relation to formal deformation quantization with all its well-established technology, e.g. by simply restricting a formal star product to a unital *-subalgebra of $\mathscr{C}^\infty(M)$ on which the formal power series of the star product actually converges. A suitable definition of a deformation quantization of locally convex *-algebras that essentially applies to the previous work of Beiser, Esposito, Stapor and Waldmann, e.g. [7, 34, 83] is:

Definition 2.5.3 (Non-formal deformation quantization) *A deformation of a commutative locally convex *-algebra $\mathcal{A}$ with continuous product and *-involution is given by an open interval $I \subseteq \mathbb{R}$ having 0 in its closure, and for every $\hbar \in I$ a continuous product $\star_\hbar$ with respect to which $\mathcal{A}$ is again a *-algebra, such that $I \ni \hbar \mapsto a \star_\hbar b \in \mathcal{A}$ is continuous for every $a, b \in \mathcal{A}$ with $\lim_{\hbar \to 0} a \star_\hbar b = ab$ the original product in $\mathcal{A}$, and such that*

$$\{\, a \,,\, b \,\} \coloneqq \lim_{\hbar \to 0} \frac{1}{\mathrm{i}\hbar} [\, a \,,\, b \,]_{\star_\hbar}$$

for $a, b \in \mathcal{A}$ is a well-defined Poisson bracket for the original product in $\mathcal{A}$.

Examples of such non-formal deformation quantizations have essentially (sometimes without treating the *-involution) been constructed for the hyperbolic disc [7], for constant Poisson brackets on vector spaces of arbitrary dimension [83] and for the linear Poisson brackets on the dual of a Lie algebra in finite dimensions and for some examples of infinitely dimensional Lie algebras as well [34]. The first two of these will be revisited in Chapters 4 and 5.

2.6 Open Questions

Before closing this chapter, we should discuss which mathematical questions arise when dealing with deformation quantizations of locally convex *-algebras:

First, of course, there is the question of how to construct examples. One promising approach is to start with the well-understood formal deformation quantizations and examine, under which conditions the formal power series in $\hbar$ actually can be made convergent. This is indeed how the examples [34,83] have been constructed, and how we will proceed in Chapter 4. Another possibility is to start with an example of an already well-understood non-formal star product and construct a new one by means of symmetry reduction like in [7] and Chapter 5.

As a deformation of locally convex *-algebras yields a Poisson *-algebra $(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\})$ for every admissible $\hbar$ (including $\hbar = 0$), the next question should be what cone of positive linear functionals $\Omega^+_{\mathrm{H}} \subseteq \mathcal{A}^*$ to choose such that $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}, \Omega^+_{\mathrm{H}}\big)$ actually becomes a physical system. The canonical choice here is to take Ω^+_{H} as the cone of all continuous algebraically positive linear functionals, but it still needs to be shown that there exist sufficiently many such functionals, which is done for the examples discussed later in Sections 4.2.3 and 5.2.2. However, we will see in Section 5.2.1 that this canonical choice might not always be the ideal one.

Having constructed a physical system for every admissible $\hbar$, there are a lot of questions concerning the interpretation and representation of observables and states:

If $\hbar = 0$, i.e. for the classical system, one still should show that there exist sufficiently many pure states so that condition (P4') holds, that these pure states are actually characters, i.e. unital *-homomorphisms to $\mathbb{C}$, (which allows a faithful representation of the observables as an algebra of functions on the characters by Gelfand transformation, see Section 3.2.1), and then that in this representation, all states can be expressed as integrals over probability measures. The problem of existence of characters is rather trivial in those cases where the classical algebra is constructed directly as an algebra of functions. A sufficient condition for the pure states to be characters is given in Section 3.5 in a very general context, and applied in an example in Section 4.2.4. Finally, the question of representation of states on algebras of functions as integrals over probability measures is an old problem that has been solved in many special cases: The well-known Riesz–Markov representation theorem treats various cases where the algebra of functions is an algebra of continuous functions on a locally compact Hausdorff space. An adaptation to spaces of uniformly continuous functions on a complete metric space has been proven e.g. in [65, Theorem 5.28], and finally, the solutions of the Hamburger moment problem and its generalizations like [74, Thm. 12.5.2] treat states on algebras of polynomials.

For quantum systems, it follows from the well-known GNS construction that the existence of sufficiently many states such that (P4) holds, is equivalent to the existence of a faithful representation of the observable algebra as an O^*-algebra, see also Section 3.2.2. The more complicated problem is to understand under which conditions the states can actually be represented by traces with certain bounded operators like in (2.3.1). Moreover, the problem of essential self-adjointness of observables in such representations has to be addressed (one possible solution is provided in Section 3.4.3 and applied in Sections 4.2.4 and 5.2.2). This is closely related to a well-behaved spectral theory in representations. A step towards a spectral theorem independent of representations is made in Section 3.6.

Chapter 3

Abstract O^*-Algebras

Many of the questions concerning the representation of observables and states that were discussed at the end of the previous chapter do not actually concern the Poisson bracket. Because of this, a useful first step towards understanding physical systems as defined in Definition 2.4.2 is to examine tuples of a *-algebra and a distinguished cone of algebraically positive linear functionals on it. This is the essential idea behind the definition of an abstract O^*-algebra. However, we will see that abstract O^*-algebras are also interesting from the purely mathematical point of view as they allow to describe some properties (esspecially those that are related to representations) of different types of *-algebras.

Most of the content of this chapter, especially the definition of abstract O^*-algebras and their basic properties in Section 3.1 and the discussion of characters and pure states in Section 3.5 is from the author's preprint [76] and has only been slightly revised. The constructions of abstract O^*-algebras out of locally convex or quasi-ordered *-algebras and the discussion of boundedness and essential self-adjointness in representations, i.e. Sections 3.3 and 3.4, has also essentially appeared in [76], but is structured differently here. The definition and examination of representations (Section 3.2) and the application of Freudenthal's spectral theorem to abstract O^*-algebras (Section 3.6) is new. Finally, the examples and counterexamples in Section 3.7 can be considered to be well-known.

3.1 Definition and first Properties

3.1.1 Definition

Recall that an O^*-algebra on a pre-Hilbert space $\mathcal{D}$ is a unital *-subalgebra $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$, hence an ordered *-algebra with respect to the usual order on $\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ like in Definition 1.2.9. An important property of this order is that it is determined by a set of algebraically positive linear functionals on $\mathcal{A}$, namely the $\mathcal{A} \ni a \mapsto \langle\, \phi \,|\, a(\phi) \,\rangle \in \mathbb{C}$ with $\phi \in \mathcal{D}$. This idea can be generalized, which leads to the definition of abstract O^*-algebras:

Definition 3.1.1 *[76, Def. 2.3] An* abstract O^*-algebra *is a tuple $(\mathcal{A}, \Omega)$ of a quasi-ordered *-algebra $\mathcal{A}$ and a linear subspace and $\mathcal{A}$-subbimodule $\Omega \subseteq \mathcal{A}^*$ (i.e. $a \cdot \omega, \omega \cdot a \in \Omega$ for all $a \in \mathcal{A}$, $\omega \in \Omega$) that is stable under the antilinear involution $\cdot^*$, and is compatible with the order on $\mathcal{A}$ in the following way:*

Define the real linear subspace $\Omega_{\mathrm{H}} := \Omega \cap \mathcal{A}^*_{\mathrm{H}}$ of $\mathcal{A}^*$ and the convex cone $\Omega^+_{\mathrm{H}} := \Omega \cap \mathcal{A}^{*,+}_{\mathrm{H}}$, where $\mathcal{A}^*_{\mathrm{H}}$ and $\mathcal{A}^{*,+}_{\mathrm{H}}$ are like in Definition 1.2.5. Then the elements in Ω^+_{H} are simply called the positive linear functionals *of* $(\mathcal{A},\Omega)$ *and it is required that* Ω *is the linear span of* Ω^+_{H} *and that*

$$\mathcal{A}^+_{\mathrm{H}} = \big\{ a \in \mathcal{A}_{\mathrm{H}} \, \big| \, \forall_{\omega \in \Omega^+_{\mathrm{H}}} : \langle \omega \,,\, a \rangle \geq 0 \big\} .$$

Moreover, an abstract O^**-algebra* $(\mathcal{A},\Omega)$ *is said to be* Hausdorff *if* Ω *separates elements of* $\mathcal{A}$*, i.e. if* $\langle \omega \,,\, a \rangle = 0$ *for one* $a \in \mathcal{A}$ *and all* $\omega \in \Omega$ *implies* $a = 0$*.*

So every positive linear functional ω of an abstract O^*-algebra $(\mathcal{A},\Omega)$ is also algebraically positive because $\Omega^+_{\mathrm{H}} \subseteq \mathcal{A}^{*,+}_{\mathrm{H}} \subseteq \mathcal{A}^{*,++}_{\mathrm{H}}$, but conversely, an algebraically positive linear functional ω on $\mathcal{A}$ need not be in Ω^+_{H}, as neither $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$ nor $\omega \in \Omega$ are guaranteed in general.

It is important to note that the convex cones $\mathcal{A}^+_{\mathrm{H}}$ and Ω^+_{H} in $\mathcal{A}_{\mathrm{H}}$ and Ω_{H} determine each other. More precisely, using the Galois connection $\,\cdot^{\uparrow}$ from Appendix A.3.4 for the real vector spaces Ω_{H} and $\mathcal{A}_{\mathrm{H}}$ and the usual dual pairing $\langle\, \cdot \,,\, \cdot \,\rangle \colon \Omega_{\mathrm{H}} \times \mathcal{A}_{\mathrm{H}} \to \mathbb{R}$, then

$$\mathcal{A}^+_{\mathrm{H}} = \big(\Omega^+_{\mathrm{H}}\big)^{\uparrow} \qquad \text{and} \qquad \Omega^+_{\mathrm{H}} = \big(\mathcal{A}^+_{\mathrm{H}}\big)^{\uparrow} . \tag{3.1.1}$$

Having a distinguished space of linear functionals on an abstract O^*-algebra allows us to define sets of states, pure states and characters, and render some ideas of Chapter 2 more precise:

Definition 3.1.2 *[76, Def. 2.7] Let* $(\mathcal{A},\Omega)$ *be an abstract* O^**-algebra. Then*

$$\begin{aligned} &\mathcal{S}(\mathcal{A},\Omega) := \big\{ \omega \in \Omega^+_{\mathrm{H}} \,\big|\, \langle \omega \,,\, \mathbb{1} \rangle = 1 \big\} \\ \textit{and} \qquad &\mathcal{S}_{\mathrm{p}}(\mathcal{A},\Omega) := \big\{ \omega \in \mathcal{S}(\mathcal{A},\Omega) \,\big|\, \omega \textit{ is an extreme point of } \mathcal{S}(\mathcal{A},\Omega) \big\} \end{aligned}$$

are the sets of states *and* pure states *of* $(\mathcal{A},\Omega)$*, respectively, and*

$$\mathcal{M}(\mathcal{A},\Omega) := \big\{ \omega \in \mathcal{S}(\mathcal{A},\Omega) \,\big|\, \omega \textit{ is multiplicative, i.e. } \langle \omega \,,\, ab \rangle = \langle \omega \,,\, a \rangle \langle \omega \,,\, b \rangle \textit{ for all } a,b \in \mathcal{A} \big\}$$

is the set of characters *of* $(\mathcal{A},\Omega)$*.*

It is important to note that, while every state and every character of an abstract O^*-algebra is also an algebraic state or algebraic character, a pure state is only an extreme point in the set of states, not in the possibly larger set of all algebraic states! Moreover, as $\langle \omega \,,\, \mathbb{1} \rangle \neq 0$ for every non-zero algebraically positive linear functional ω on a *-algebra $\mathcal{A}$, the convex cone of positive linear functionals Ω^+_{H} of an abstract O^*-algebra $(\mathcal{A},\Omega)$ is generated by the convex set $\mathcal{S}(\mathcal{A},\Omega)$ of states, because every non-zero positive linear functional can be rescaled to a state. Similarly, the linear hull of $\mathcal{S}(\mathcal{A},\Omega)$ is whole Ω. As a consequence, it is e.g. sufficient to test positivity of an Hermitian algebra element on all states, rather than on all positive linear functionals: Given $a \in \mathcal{A}_{\mathrm{H}}$, then $a \gtrsim 0$ if and only if $\langle \omega \,,\, a \rangle \geq 0$ for all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$. For the Hausdorff property of abstract O^*-algebras we now get the following characterization:

Lemma 3.1.3 *Let* $(\mathcal{A},\Omega)$ *be an abstract* O^**-algebra, then the following is equivalent:*

- $(\mathcal{A},\Omega)$ *is Hausdorff.*
- $\mathcal{A}$ *is an ordered* **-algebra.*

- *If $\langle\, \omega\,,\, a^*a\,\rangle = 0$ holds for one $a \in \mathcal{A}$ and all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$, then $a = 0$.*

Proof: The first and second point are equivalent: If $(\mathcal{A},\Omega)$ is Hausdorff and $a \in \mathcal{A}_{\mathrm{H}}$ fulfils $0 \lesssim a \lesssim 0$, then $0 \leq \langle\, \omega\,,\, a\,\rangle \leq 0$ for all $\omega \in \Omega^+_{\mathrm{H}}$. As Ω is the linear span of Ω^+_{H}, this implies $\langle\, \omega\,,\, a\,\rangle = 0$ for all $\omega \in \Omega$, so $a = 0$ as $(\mathcal{A},\Omega)$ is Hausdorff. Conversely, if $\mathcal{A}$ is an ordered *-algebra and $a \in \mathcal{A}$ fulfils $\langle\, \omega\,,\, a\,\rangle = 0$ for all $\omega \in \Omega$, then also $\langle\, \omega\,,\, a^*\,\rangle = \overline{\langle\, \omega^*\,,\, a\,\rangle} = 0$ for all $\omega \in \Omega$ and especially $\langle\, \omega\,,\, \mathsf{Re}(a)\,\rangle = 0 = \langle\, \omega\,,\, \mathsf{Im}(a)\,\rangle$ for all $\omega \in \Omega^+_{\mathrm{H}}$. As $\mathsf{Re}(a)$ and $\mathsf{Im}(a)$ are both Hermitian, this implies $0 \leq \mathsf{Re}(a) \leq 0$ and $0 \leq \mathsf{Im}(a) \leq 0$, and thus $a = \mathsf{Re}(a) + \mathrm{i}\,\mathsf{Im}(a) = 0$.

Moreover, the first and third point are also equivalent: Assume that $\langle\, \omega\,,\, a^*a\,\rangle = 0$ for one $a \in \mathcal{A}$ and all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ implies $a = 0$, and let $a \in \mathcal{A}$ be given such that $\langle\, \omega\,,\, a\,\rangle = 0$ for all $\omega \in \Omega$. Then the polarization identity shows that especially

$$\langle\, \omega\,,\, a^*a\,\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^{-k} \langle\, \omega\,,\, (a+\mathrm{i}^k\mathbb{1})^* a (a+\mathrm{i}^k\mathbb{1})\,\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^{-k} \langle\, (a+\mathrm{i}^k\mathbb{1}) \triangleright \omega\,,\, a\,\rangle = 0$$

for all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$, thus $a = 0$. Conversely, if $(\mathcal{A},\Omega)$ is Hausdorff and $\langle\, \omega\,,\, a^*a\,\rangle = 0$ for one $a \in \mathcal{A}$ and all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$, then $0 \leq \lvert\langle\, \omega\,,\, a\,\rangle\rvert \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^*a\,\rangle^{1/2} = 0$ for all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ shows $\langle\, \omega\,,\, a\,\rangle = 0$ for all $\omega \in \Omega$ as Ω is the linear hull of $\mathcal{S}(\mathcal{A},\Omega)$, so $a = 0$. □

A very easy first example of abstract O^*-algebras is given by complex quadratic matrices, more examples will follow later:

Example 3.1.4 *Let $n \in \mathbb{N}$ and $\mathcal{A} = \mathbb{C}^{n\times n}$ with matrix multiplication as product and the combination of transposition and elementwise complex conjugation as *-involution. Then it follows from basic linear algebra that $\mathcal{A}^*$ is, as a complex vector space, isomorphic to $\mathcal{A}$ via $\mathbb{C}^{n\times n} \ni b \mapsto \mathsf{tr}(b\,\cdot\,) \in \mathcal{A}^*$. Now there are exactly two possibilities to turn $\mathcal{A}$ into an abstract O^*-algebra:*

The first one is by defining an order on $\mathcal{A}_{\mathrm{H}}$ whose cone of positive elements is the minimal one which turns $\mathcal{A}$ into a quasi-ordered *-algebra, i.e. $\mathcal{A}^+_{\mathrm{H}} := \mathcal{A}^{++}_{\mathrm{H}}$. For this order it is well-known that $\mathcal{A}$ actually becomes an ordered *-algebra and that a Hermitian element $a \in \mathcal{A}_{\mathrm{H}}$ is positive if and only if $\mathsf{tr}(ba) \geq 0$ for all $b \in \mathcal{A}^{++}_{\mathrm{H}}$. This also implies that, conversely, a linear functional $\mathsf{tr}(b\,\cdot\,) \in \mathcal{A}^*$ comes from some $b \in \mathcal{A}^{++}_{\mathrm{H}}$ if and only if $\mathsf{tr}(ba) \geq 0$ for all $a \in \mathcal{A}^+_{\mathrm{H}}$, so $(\mathcal{A},\mathcal{A}^*)$ is a Hausdorff abstract O^*-algebra and $(\mathcal{A}^*)^+_{\mathrm{H}} = \{\, \mathsf{tr}(b\,\cdot\,) \mid b \in \mathcal{A}^{++}_{\mathrm{H}} \,\}$. This should probably be considered the canonical abstract O^*-algebra of $\mathbb{C}^{n\times n}$.

However, there is another possibility: One can turn $\mathcal{A}$ into a quasi-ordered *-algebra by defining an order on $\mathcal{A}_{\mathrm{H}}$ such that $\mathcal{A}^+_{\mathrm{H}} := \mathcal{A}_{\mathrm{H}}$. Then, of course, $\mathcal{A}$ is only a quasi-ordered, but not ordered *-algebra. Nevertheless, its order can be defined by means of algebraically positive linear functionals: It is easy to check that $(\mathcal{A},\{0\})$ is an abstract O^*-algebra, but a trivial one.

Proof: One can verify that these constructions indeed yields abstract O^*-algebras, but there are indeed no other possibilities to turn the *-algebra $\mathcal{A}$ into an abstract O^*-algebra:

Assume that $\lesssim$ is an order on $\mathcal{A}_{\mathrm{H}}$ and $\Omega \subseteq \mathcal{A}^*$ such that $(\mathcal{A},\Omega)$ is an abstract O^*-algebra. If $\Omega = \{0\}$, then this gives the trivial, second version. But if there is one $\mathsf{tr}(b\,\cdot\,) \in \Omega\backslash\{0\}$, then already $\mathcal{A}^* = \Omega$ as Ω has to be an $\mathcal{A}$-subbimodule of $\mathcal{A}^*$. So there also exists a $\mathsf{tr}(b\,\cdot\,) \in \Omega^+_{\mathrm{H}}\backslash\{0\}$ which has to fulfil $\mathsf{tr}(ba) \geq 0$ for all $a \in \mathcal{A}^{++}_{\mathrm{H}}$, so $b \in \mathcal{A}^{++}_{\mathrm{H}}$. Using that Ω^+_{H} is stable under the $\mathcal{A}$-monoid action $\triangleright$, some linear algebra now shows that $\mathsf{tr}(b\,\cdot\,) \in \Omega^+_{\mathrm{H}}$ for all $b \in \mathcal{A}^{++}_{\mathrm{H}}$. This gives an upper bound on the cone of positive algebra elements $\mathcal{A}^+_{\mathrm{H}}$, namely $\mathcal{A}^+_{\mathrm{H}} \subseteq \mathcal{A}^{++}_{\mathrm{H}}$. As always $\mathcal{A}^+_{\mathrm{H}} \supseteq \mathcal{A}^{++}_{\mathrm{H}}$, the only possibility is $\mathcal{A}^+_{\mathrm{H}} = \mathcal{A}^{++}_{\mathrm{H}}$, i.e. the first, canonical version. □

We have already seen that the order on $\mathcal{A}_{\mathrm{H}}$ determines the order on Ω_{H} and vice versa. Because of this, the construction of an abstract O^*-algebra is rather easy provided one has a distinguished set of algebraically positive linear functionals:

Proposition 3.1.5 *[76, Prop. 2.4] Let $\mathcal{A}$ be a *-algebra and $P^+_{\mathrm{H}} \subseteq \mathcal{A}^*_{\mathrm{H}}$ a set of algebraically positive linear functionals that is stable under the monoid action $\triangleright$ of $\mathcal{A}$, i.e. $a \triangleright \omega \in P^+_{\mathrm{H}}$ for all $a \in \mathcal{A}$ and all $\omega \in P^+_{\mathrm{H}}$. Define a relation $\lesssim$ on $\mathcal{A}_{\mathrm{H}}$ by*

$$a \lesssim b \quad :\Longleftrightarrow \quad \forall_{\omega \in P^+_{\mathrm{H}}} : \langle\, \omega\,,\, a \,\rangle \leq \langle\, \omega\,,\, b \,\rangle\,,$$

then $\lesssim$ is a quasi-order that turns $\mathcal{A}$ into a quasi-ordered *-algebra. Moreover, let Ω be the linear subspace of $\mathcal{A}^*$ generated by P^+_{H}, then $(\mathcal{A}, \Omega)$ is an abstract O^*-algebra and Ω^+_{H} is the weak-*-closure in Ω of $\langle\!\langle\, P^+_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{cone}} = \big\{ \sum_{n=1}^N \omega_n \bigm| N \in \mathbb{N}_0;\, \omega_1, \dots, \omega_N \in P^+_{\mathrm{H}} \big\}$, the convex cone generated by P^+_{H}.

PROOF: It is clear that $\mathcal{A}$ with $\lesssim$ is a quasi-ordered *-algebra, and $(\mathcal{A}, \Omega)$ is then an abstract O^*-algebra: The polarization identity

$$a \cdot \omega \cdot b^* = \frac{1}{4} \sum\nolimits_{k=0}^{3} \mathrm{i}^k \big((a + \mathrm{i}^k b) \triangleright \omega \big) \in \Omega$$

holds for all $a, b \in \mathcal{A}$ and all $\omega \in \Omega$ and proves that Ω is an $\mathcal{A}$-subbimodule. Ω is stable under $\cdot^*$ because P^+_{H} is, and it determines $\mathcal{A}^+_{\mathrm{H}}$: From $\mathcal{A}^+_{\mathrm{H}} = (P^+_{\mathrm{H}})^\uparrow$ with $\cdot^\uparrow$ like in (3.1.1), it follows that $\Omega^+_{\mathrm{H}} = (\mathcal{A}^+_{\mathrm{H}})^\uparrow = (P^+_{\mathrm{H}})^{\uparrow\uparrow}$ and thus $(\Omega^+_{\mathrm{H}})^\uparrow = (P^+_{\mathrm{H}})^{\uparrow\uparrow\uparrow} = (P^+_{\mathrm{H}})^\uparrow = \mathcal{A}^+_{\mathrm{H}}$ by the general properties of Galois connections. Moreover, $\Omega^+_{\mathrm{H}} = (\mathcal{A}^+_{\mathrm{H}})^\uparrow = (P^+_{\mathrm{H}})^{\uparrow\uparrow} = \langle\!\langle\, \langle\!\langle\, P^+_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{cone}} \,\rangle\!\rangle_{\mathrm{cl}}$ by Proposition A.3.15, where $\langle\!\langle\, \cdot \,\rangle\!\rangle_{\mathrm{cl}}$ denotes the closure with respect to the weak topology defined by $\mathcal{A}_{\mathrm{H}}$ on Ω_{H}. As this topology coincides with the weak-*-topology on Ω_{H}, the weak-*-closure of $\langle\!\langle\, P^+_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{cone}}$ in Ω_{H} is indeed Ω^+_{H}. □

Conversely, we have:

Proposition 3.1.6 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then Ω^+_{H} is weak-*-closed in Ω.*

PROOF: As $\Omega^+_{\mathrm{H}} = \bigcap_{a \in \mathcal{A}^+_{\mathrm{H}}} \{\, \omega \in \Omega \mid \langle\, \omega\,,\, a \,\rangle \in [0, \infty[\,\}$, Ω^+_{H} is weak-*-closed as the intersection of the preimages of the closed interval $[0, \infty[$ under the weak-*-continuous maps $\Omega \ni \omega \mapsto \langle\, \omega\,,\, a \,\rangle \in \mathbb{C}$ with $a \in \mathcal{A}^+_{\mathrm{H}}$. □

So the two previous Propositions 3.1.5 and 3.1.6 together show that an abstract O^*-algebra is equivalently determined by a tuple $(\mathcal{A}, \Omega^+_{\mathrm{H}})$ of a *-algebra $\mathcal{A}$ and a convex cone Ω^+_{H} of algebraically positive linear functionals, that is stable under the monoid action of $\mathcal{A}$ and weak-*-closed in its linear hull Ω. This allows to rephrase Definition 2.4.2 of a physical system as a triple $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}, \Omega^+_{\mathrm{H}}\big)$, such that $(\mathcal{A}, \Omega^+_{\mathrm{H}})$ describes a Hausdorff abstract O^*-algebra and $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}\big)$ a Poisson-*-algebra. Many of the questions raised in Section 2.6 can thus be seen as questions concerning abstract O^*-algebras.

With the following notion of morphisms of abstract O^*-algebras, the class of abstract O^*-algebras together with the morphisms between them clearly becomes a category (see also Appendix A.5):

Definition 3.1.7 *Let $(\mathcal{A}, \Omega)$ and $(\mathcal{B}, \mathcal{R})$ be two abstract O^*-algebras, then a* morphism *from $(\mathcal{A}, \Omega)$ to $(\mathcal{B}, \mathcal{R})$ is a unital *-homomorphism $M : \mathcal{A} \to \mathcal{B}$ with the additional property that $M^*(\psi) \in \Omega^+_{\mathrm{H}}$ for all $\psi \in \mathcal{R}^+_{\mathrm{H}}$, where $M^* : \mathcal{B}^* \to \mathcal{A}^*$ is the pullback with M. Moreover, M is called an* order embedding *if its restriction to the Hermitian elements is an order embedding, i.e. if $a \in \mathcal{A}_{\mathrm{H}}$ and $M(a) \geq 0$ imply $a \in \mathcal{A}^+_{\mathrm{H}}$.*

Note that such a morphism of abstract O^*-algebras $M : (\mathcal{A}, \Omega) \to (\mathcal{B}, \mathcal{R})$ is automatically a positive unital *-homomorphism, because for every $a \in \mathcal{A}^+_{\mathrm{H}}$ the inequality $\langle\, \psi \,,\, M(a) \,\rangle = \langle\, M^*(\psi) \,,\, a \,\rangle \geq 0$ holds for all $\psi \in \mathcal{R}^+_{\mathrm{H}}$.

Lemma 3.1.8 *Let $(\mathcal{A}, \Omega)$ be a Hausdorff abstract O^*-algebra. Then every order embedding morphism of abstract O^*-algebras M from $(\mathcal{A}, \Omega)$ to another abstract O^*-algebra $(\mathcal{B}, \mathcal{R})$ is injective.*

PROOF: Given $a \in \mathcal{A}_{\mathrm{H}}$ with $M(a) = 0$ then $a \in \mathcal{A}^+_{\mathrm{H}}$ and $-a \in \mathcal{A}^+_{\mathrm{H}}$ because $M(a) = M(-a) = 0 \geq 0$. As $(\mathcal{A}, \Omega)$ was assumed to be Hausdorff, $a = 0$. □

3.1.2 Special Types of Abstract O^*-Algebras

Definition 3.1.9 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra.*

- *$(\mathcal{A}, \Omega)$ is said to be* regular*, if every $\omega \in \Omega_{\mathrm{H}}$ that is algebraically positive, is positive, i.e. if $\Omega^+_{\mathrm{H}} = \Omega_{\mathrm{H}} \cap \mathcal{A}^{*,++}_{\mathrm{H}}$ (note that the inclusion $\Omega^+_{\mathrm{H}} = \Omega_{\mathrm{H}} \cap \mathcal{A}^{*,+}_{\mathrm{H}} \subseteq \Omega_{\mathrm{H}} \cap \mathcal{A}^{*,++}_{\mathrm{H}}$ is always true).*
- *$(\mathcal{A}, \Omega)$ is said to be* hyper-regular*, if every $\omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$, for which there exists a $\rho \in \Omega^+_{\mathrm{H}}$ fulfilling $\rho - \omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$, is in Ω^+_{H}.*
- *$(\mathcal{A}, \Omega)$ is said to be* downwards closed *if every $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$, for which there exists a $\rho \in \Omega^+_{\mathrm{H}}$ fulfilling $\rho - \omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$, is in Ω^+_{H}.*
- *$(\mathcal{A}, \Omega)$ is said to be* closed *if Ω^+_{H} is weak-*-closed in $\mathcal{A}^*$.*

Regular and downwards closed abstract O^*-algebras have already been defined in Definitions 4.4 and 5.14 of [76]. The properties of regularity and hyper-regularity link the ordering of $\mathcal{A}_{\mathrm{H}}$ and Ω_{H} to the canonical ones by algebraic positivity. Because of this, they are typically fulfilled in examples where the ordering of $\mathcal{A}_{\mathrm{H}}$ and Ω_{H} is constructed in some way out of this canonical one. The other two properties are in no way related to algebraic positivity. Nevertheless, Proposition 3.5.19 will give a sufficient condition for a downwards-closed abstract O^*-algebra to be hyper-regular. By now, we can already say the following:

Proposition 3.1.10 *Every hyper-regular abstract O^*-algebra is regular and downwards closed.*

PROOF: Let $(\mathcal{A}, \Omega)$ be a hyper-regular abstract O^*-algebra and $\omega \in \Omega_{\mathrm{H}} \cap \mathcal{A}^{*,++}_{\mathrm{H}}$. Then there exist $\rho_1, \rho_2 \in \Omega^+_{\mathrm{H}}$ such that $\omega = \rho_1 - \rho_2$ because Ω_{H} is the $\mathbb{R}$-linear hull of the convex cone Ω^+_{H}, and thus $\omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ and $\rho_2 = \rho_1 - \omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ as well as $\rho_1 \in \Omega^+_{\mathrm{H}}$ hold. By hyper-regularity, $\omega \in \Omega^+_{\mathrm{H}}$, so $(\mathcal{A}, \Omega)$ is regular. By using that $\mathcal{A}^{++}_{\mathrm{H}} \subseteq \mathcal{A}^+_{\mathrm{H}}$, hence $\mathcal{A}^{*,++}_{\mathrm{H}} \supseteq \mathcal{A}^{*,+}_{\mathrm{H}}$, it is also easy to see that $(\mathcal{A}, \Omega)$ is downwards closed. □

Proposition 3.1.11 *An abstract O^*-algebra $(\mathcal{A}, \Omega)$ is closed if and only if $\Omega^+_{\mathrm{H}} = \mathcal{A}^{*,+}_{\mathrm{H}}$. Consequently, every closed abstract O^*-algebra is also downwards closed.*

PROOF: If $\Omega^+_{\mathrm{H}} = \mathcal{A}^{*,+}_{\mathrm{H}}$ then $\Omega^+_{\mathrm{H}} = \bigcap_{a \in \mathcal{A}^+} \{\, \omega \in \mathcal{A}^* \mid \langle\, \omega \,,\, a \,\rangle \geq 0 \,\}$ is weak-*-closed in $\mathcal{A}^*$, because it is the intersection of the preimages of the closed interval $[0, \infty[$ under the weak-*-continuous maps $\mathcal{A}^* \ni \omega \mapsto \langle\, \omega \,,\, a \,\rangle \in \mathbb{C}$ with $a \in \mathcal{A}^+_{\mathrm{H}}$.

Conversely, consider the Galois connection $\,\cdot^{\uparrow}$, but this time for the real vector spaces $\mathcal{A}^*_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{H}}$ together with their dual pairing $\langle\, \cdot \,,\, \cdot \,\rangle \colon \mathcal{A}^*_{\mathrm{H}} \times \mathcal{A}_{\mathrm{H}} \to \mathbb{R}$. If $(\mathcal{A}, \Omega)$ is closed, then Ω^+_{H} is closed in $\mathcal{A}^*_{\mathrm{H}}$ with respect to the weak topology induced by $\mathcal{A}_{\mathrm{H}}$. Thus $\Omega^+_{\mathrm{H}} = (\Omega^+_{\mathrm{H}})^{\uparrow\uparrow} = (\mathcal{A}^+_{\mathrm{H}})^{\uparrow} = \mathcal{A}^{*,+}_{\mathrm{H}}$ by Proposition A.3.15. □

The important property of closed abstract O^*-algebras is:

Corollary 3.1.12 *If $(\mathcal{A}, \Omega)$ is a closed abstract O^*-algebra, then every positive unital *-homomorphism $M \colon \mathcal{A} \to \mathcal{B}$ to another abstract O^*-algebra $(\mathcal{B}, \mathcal{R})$ is a morphism of abstract O^*-algebras.*

PROOF: Given $\rho \in \mathcal{R}^+_{\mathrm{H}}$, then $\langle\, M^*(\rho) \,,\, a \,\rangle = \langle\, \rho \,,\, M(a) \,\rangle \geq 0$ for all $a \in \mathcal{A}^+_{\mathrm{H}}$, so $M^*(\rho) \in \mathcal{A}^{*,+}_{\mathrm{H}} = \Omega^+_{\mathrm{H}}$ by the previous Proposition 3.1.11. □

3.1.3 Topologies on Abstract O^*-Algebras

The weak and strong operator topologies are a well-known tool for examinining O^*-algebras. They can also be adapted to abstract O^*-algebras in a straightforward way, see Definitions 4.5 and 5.12 of [76]:

Definition 3.1.13 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then define the seminorms $\|\cdot\|_{\omega,\mathrm{wk}}$ and $\|\cdot\|_{\rho,\mathrm{st}}$ for all $\omega \in \Omega$ or all $\rho \in \Omega^+_{\mathrm{H}}$, respectively, by*

$$\|a\|_{\omega,\mathrm{wk}} := |\langle\, \omega \,,\, a \,\rangle| \qquad \textit{and} \qquad \|a\|_{\rho,\mathrm{st}} := \langle\, \rho \,,\, a^* a \,\rangle^{1/2}$$

for all $a \in \mathcal{A}$. The locally convex topology on $\mathcal{A}$ defined by the seminorms $\|\cdot\|_{\omega,\mathrm{wk}}$ for all $\omega \in \Omega$ is called the weak topology *and the one defined by the seminorms $\|\cdot\|_{\rho,\mathrm{st}}$ for all $\rho \in \Omega^+_{\mathrm{H}}$ is the* strong topology*.*

The weak topology is rather well-behaved with respect to the algebraic structures on an abstract O^*-algebra:

Proposition 3.1.14 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then $\mathcal{A}$ with the weak topology is a locally convex *-algebra. This topology on $\mathcal{A}$ is Hausdorff if and only if $(\mathcal{A}, \Omega)$ is a Hausdorff abstract O^*-algebra. Moreover, the convex cone of positive elements $\mathcal{A}^+_{\mathrm{H}}$ is weakly closed in $\mathcal{A}_{\mathrm{H}}$.*

PROOF: Let $\omega \in \Omega$ and $a \in \mathcal{A}$ be given, then $\|ab\|_{\omega,\mathrm{wk}} = \|b\|_{\omega\cdot a,\mathrm{wk}}$, $\|ba\|_{\omega,\mathrm{wk}} = \|b\|_{a\cdot\omega,\mathrm{wk}}$ and $\|b^*\|_{\omega,\mathrm{wk}} = \|b\|_{\omega^*,\mathrm{wk}}$ hold for all $b \in \mathcal{A}$, which proves the weak continuity of left and right multiplication and of the *-involution. So $\mathcal{A}$ with the weak topology is a locally convex *-algebra.

The weak topology on $\mathcal{A}$ is Hausdorff if and only if for every $a \in \mathcal{A}\backslash\{0\}$ there exists an $\omega \in \Omega$ such that $\|a\|_{\omega,\mathrm{wk}} \neq 0$, i.e. such that $\langle\, \omega \,,\, a \,\rangle \neq 0$, which is equivalent to $(\mathcal{A}, \Omega)$ being Hausdorff as an abstract O^*-algebra.

Finally, $\mathcal{A}^+_{\mathrm{H}} = \bigcap_{\omega\in\Omega^+_{\mathrm{H}}}\{\, a \in \mathcal{A} \mid \langle\, \omega\,,\, a\,\rangle \geq 0 \,\} \cap \mathcal{A}_{\mathrm{H}}$, so $\mathcal{A}^+_{\mathrm{H}}$ is weakly closed in $\mathcal{A}_{\mathrm{H}}$ because it is the intersection of the preimages in $\mathcal{A}_{\mathrm{H}}$ of the closed set $[0,\infty[$ under the weakly continuous $\omega \in \Omega^+_{\mathrm{H}}$. □

Proposition 3.1.15 *[76, Prop. 4.6] Let $(\mathcal{A},\Omega)$ be an abstract O^*-algebra, then $(\mathcal{A},\Omega)$ is regular if and only if the weak closure of $\mathcal{A}^{++}_{\mathrm{H}}$ in $\mathcal{A}_{\mathrm{H}}$ is $\mathcal{A}^+_{\mathrm{H}}$.*

PROOF: Consider again the Galois connection $\,\cdot^{\uparrow}$ for the dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle\colon \Omega_{\mathrm{H}} \times \mathcal{A}_{\mathrm{H}} \to \mathbb{R}$, for which $\Omega^+_{\mathrm{H}} = (\mathcal{A}^+_{\mathrm{H}})^{\uparrow}$. Then $(\mathcal{A},\Omega)$ is regular if and only if $\Omega^+_{\mathrm{H}} = (\mathcal{A}^{++}_{\mathrm{H}})^{\uparrow}$, i.e. if and only if $(\mathcal{A}^+_{\mathrm{H}})^{\uparrow} = (\mathcal{A}^{++}_{\mathrm{H}})^{\uparrow}$. By the general properties of Galois connections, this is equivalent to $(\mathcal{A}^+_{\mathrm{H}})^{\uparrow\uparrow} = (\mathcal{A}^{++}_{\mathrm{H}})^{\uparrow\uparrow}$, i.e. to $\langle\!\langle\, \mathcal{A}^+_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{cl}} = \langle\!\langle\, \mathcal{A}^{++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{cl}}$ by Proposition A.3.15, where $\langle\!\langle\, \cdot \,\rangle\!\rangle_{\mathrm{cl}}$ denotes the weak closure in $\mathcal{A}_{\mathrm{H}}$. As $\mathcal{A}^+_{\mathrm{H}}$ is already weakly closed in $\mathcal{A}_{\mathrm{H}}$ by the previous Proposition 3.1.14, $(\mathcal{A},\Omega)$ is regular if and only if the weak closure of $\mathcal{A}^{++}_{\mathrm{H}}$ in $\mathcal{A}_{\mathrm{H}}$ is $\mathcal{A}^+_{\mathrm{H}}$. □

The strong topology, however, is less well-behaved, except for commutative abstract O^*-algebras:

Proposition 3.1.16 *Let $(\mathcal{A},\Omega)$ be a commutative abstract O^*-algebra, then $\mathcal{A}$ with the strong topology is a locally convex *-algebra.*

PROOF: Let $\omega \in \Omega^+_{\mathrm{H}}$ and $a \in \mathcal{A}$ be given, then $\|ab\|_{\omega,\mathrm{st}} = \|ba\|_{\omega,\mathrm{st}} = \|b\|_{a\triangleright\omega,\mathrm{st}}$ and $\|b^*\|_{\omega,\mathrm{st}} = \|b\|_{\omega,\mathrm{st}}$ hold for all $b \in \mathcal{A}$, which proves the strong continuity of left and right multiplication and of the *-involution. □

In Section 3.7, an example of an abstract O^*-algebra with not strongly continuous left-multiplication will be constructed.

3.1.4 Growth of Powers

The following lemma about the growth of positive linear functionals on powers of positive elements of a quasi-ordered *-algebra will be especially helpful later on when applied to abstract O^*-algebras:

Lemma 3.1.17 *[76, Lemma 4.2] Let $\mathcal{A}$ be a quasi-ordered *-algebra, $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$ and $a \in \mathcal{A}^+_{\mathrm{H}}$. Then $\langle\, \omega\,,\, a^n \,\rangle = 0$ for one $n \in \mathbb{N}$ implies $\langle\, \omega\,,\, a^n \,\rangle = 0$ for all $n \in \mathbb{N}$ and also $\mathrm{Var}_\omega(a) = 0$. Otherwise, $\langle\, \omega\,,\, a^n \,\rangle > 0$ and*

$$\frac{\langle\, \omega\,,\, a^n \,\rangle}{\langle\, \omega\,,\, a^{n-1} \,\rangle} \leq \frac{\langle\, \omega\,,\, a^{n+1} \,\rangle}{\langle\, \omega\,,\, a^n \,\rangle}$$

$$\text{as well as} \qquad \big\langle\, \omega\,,\, a^n \,\big\rangle^{\frac{1}{n}} \leq \big\langle\, \omega\,,\, a^{n+1} \,\big\rangle^{\frac{1}{n+1}}$$

hold for all $n \in \mathbb{N}$.

PROOF: The sesquilinear form $\mathcal{A}^2 \ni (b,c) \mapsto \langle\, \omega\,,\, b^* a\, c \,\rangle \in \mathbb{C}$ is positive because $a \in \mathcal{A}^+_{\mathrm{H}}$, which yields $|\langle\, \omega\,,\, b^* a\, c \,\rangle|^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, b^* a\, b \,\rangle \langle\, \omega\,,\, c^* a\, c \,\rangle$ for all $b, c \in \mathcal{A}$. So $\langle\, \omega\,,\, a^{m-1} a^m \,\rangle^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^{2m-2} \,\rangle \langle\, \omega\,,\, a^{2m} \,\rangle$ and $\langle\, \omega\,,\, a^{m-1} a\, a^m \,\rangle^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^{2m-1} \,\rangle \langle\, \omega\,,\, a^{2m+1} \,\rangle$ hold for all $m \in \mathbb{N}$ and show that, for all odd and all even $n \in \mathbb{N}$, the estimate $\langle\, \omega\,,\, a^n \,\rangle^2 \leq \langle\, \omega\,,\, a^{n-1} \,\rangle \langle\, \omega\,,\, a^{n+1} \,\rangle$ holds. Especially if $\langle\, \omega\,,\, a^{n-1} \,\rangle = 0$ or $\langle\, \omega\,,\, a^{n+1} \,\rangle = 0$, then also $\langle\, \omega\,,\, a^n \,\rangle = 0$. By induction it follows that $\langle\, \omega\,,\, a^n \,\rangle = 0$ for one $n \in \mathbb{N}$ implies $\langle\, \omega\,,\, a^n \,\rangle = 0$ for all $n \in \mathbb{N}$, and then also $\mathrm{Var}_\omega(a) = 0$.

Otherwise $\langle \omega , a^n \rangle > 0$ for all $n \in \mathbb{N}$, because $a^{2m} = (a^m)^*(a^m)$ and $a^{2m+1} = (a^m)^* a\,(a^m)$ are positive for all $m \in \mathbb{N}_0$. The estimate for quotients has already been proven, the one for roots is surely true if $n = 1$, in which case it is just the Cauchy Schwarz inequality again. Now assume that it holds for one $n \in \mathbb{N}$, then

$$\big\langle \omega , a^{n+1} \big\rangle^{\frac{1}{n+1}} \leq \frac{\langle \omega , a^{n+1} \rangle}{\langle \omega , a^n \rangle} \leq \frac{\langle \omega , a^{n+2} \rangle}{\langle \omega , a^{n+1} \rangle} ,$$

which then implies $\langle \omega , a^{n+1} \rangle^{1/(n+1)} \leq \langle \omega , a^{n+2} \rangle^{1/(n+2)}$. □

3.2 Representations

There are two important examples of *-algebras, namely (commutative) *-algebras of functions on a set with the pointwise operations, and (in general non-commutative) *-algebras of operators, i.e. O^*-algebras. Consequently, one can discuss two different types of representations of *-algebras, representations as functions and representations as operators. Understanding under which conditions such representations exist is the first step to answering the representation-related questions from Section 2.6.

3.2.1 Representations as Functions

Representations as functions are of course especially interesting for commutative abstract O^*-algebras, but their definition can be formulated also for non-commutative ones.

Definition 3.2.1 *Let X be a non-empty set and $\mathbb{C}^X$ the commutative ordered *-algebra of all functions $X \to \mathbb{C}$ with the pointwise operations and the pointwise order on the Hermitian elements like in Definition 1.2.8. For $x \in X$, define the* evaluation functional *$\delta_x \colon \mathbb{C}^X \to \mathbb{C}$ by $\langle \delta_x , f \rangle := f(x)$ for all $f \in \mathbb{C}^X$ and the* abstract O^*-algebra of functions *on X as $\big(\mathbb{C}^X, \Delta(X)\big)$ with $\Delta(X)$ the linear hull of $\{ \delta_x \mid x \in X \}$.*

It is easy to check that $\Delta(X)$ is not only a linear subspace, but also an $\mathcal{A}$-subbimodule of $\mathcal{A}^*$ and stable under the antilinear involution $\,\cdot^*$. Moreover, as the order on the Hermitian elements of $\mathbb{C}^X$ comes from a set of positive linear functionals in $\Delta(X)$, namely the evaluation functionals, it follows from Proposition 3.1.5 that $\big(\mathbb{C}^X, \Delta(X)\big)$ is indeed an abstract O^*-algebra. It is Hausdorff because the evaluation functionals clearly separate points of $\mathbb{C}^X$.

Note that, for a given set X, the abstract O^*-algebra $\big(\mathbb{C}^X, \Delta(X)\big)$ is usually not the most interesting way to turn functions on X with the pointwise comparison into an abstract O^*-algebra: One would probably like to have also some positive linear functionals that come from integration over more general (positive) measures on X. The above version $\big(\mathbb{C}^X, \Delta(X)\big)$ is a minimal one that is compatible with the pointwise comparison and thus gives rather mild restrictions on morphisms of abstract O^*-algebras into $\big(\mathbb{C}^X, \Delta(X)\big)$, which will be important for representations:

Definition 3.2.2 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then a* representation as functions *of $(\mathcal{A}, \Omega)$ is a tuple (X, π) consisting of a non-empty set X and a morphism of abstract O^*-algebras $\pi \colon (\mathcal{A}, \Omega) \to \big(\mathbb{C}^X, \Delta(X)\big)$. Such a representation as functions is called quasi-faithful if π is injective and faithful if π is injective and an order embedding.*

The existence of representations as functions is closely related to the existence of characters: If (X, π) is a representation as functions of an abstract O^*-algebra $(\mathcal{A}, \Omega)$, then $\pi^*(\delta_x) \in \mathcal{M}(\mathcal{A}, \Omega)$ for every $x \in X$ because $\pi^*(\delta_x) \in \Omega_{\mathrm{H}}^+$ holds by definition of a representation and of morphisms of abstract O^*-algebras, and because π and δ_x are unital-*-homomorphisms. Conversely, the well-known Gelfand transformation allows to construct such a representation as functions on the set of characters:

Definition 3.2.3 *Let (A, Ω) be an abstract O^*-algebra with non-empty set of characters, then define the* Gelfand transformation $\pi_{\mathrm{Gelfand}} \colon \mathcal{A} \to \mathbb{C}^{\mathcal{M}(\mathcal{A},\Omega)}$ *by*

$$\pi_{\mathrm{Gelfand}}(a)(\omega) := \langle\, \omega\, ,\, a\, \rangle \tag{3.2.1}$$

for all $a \in \mathcal{A}$ and all $\omega \in \mathcal{M}(\mathcal{A}, \Omega)$.

Proposition 3.2.4 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra with non-empty set of characters, then $\big(\mathcal{M}(\mathcal{A}, \Omega), \pi_{\mathrm{Gelfand}}\big)$ is a representation as functions of $(\mathcal{A}, \Omega)$.*

PROOF: It follows immediately from the definition of π_{Gelfand} and the algebraic properties of characters that π_{Gelfand} is a unital *-homomorphism from $\mathcal{A}$ to $\mathbb{C}^{\mathcal{M}(\mathcal{A},\Omega)}$. As $\langle\, \delta_\omega\, ,\, \pi_{\mathrm{Gelfand}}(a)\, \rangle = \langle\, \omega\, ,\, a\, \rangle$ for all $\omega \in \mathcal{M}(\mathcal{A}, \Omega)$ and all $a \in \mathcal{A}$ we see that $\pi^*_{\mathrm{Gelfand}}(\delta_\omega) = \omega \in \Omega_{\mathrm{H}}^+$, so π_{Gelfand} is a morphism of abstract O^*-algebras from $(\mathcal{A}, \Omega)$ to $\big(\mathbb{C}^{\mathcal{M}(\mathcal{A},\Omega)}, \Delta(\mathcal{M}(\mathcal{A}, \Omega))\big)$. □

Theorem 3.2.5 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra. There exists a quasi-faithful representation as functions of $(\mathcal{A}, \Omega)$ if and only if the set $\mathcal{M}(\mathcal{A}, \Omega)$ separates elements of $\mathcal{A}$, i.e. if and only if $\langle\, \omega\, ,\, a\, \rangle = 0$ for one $a \in \mathcal{A}$ and all $\omega \in \mathcal{M}(\mathcal{A}, \Omega)$ implies $a = 0$. Similarly, there exists a faithful representation as functions of $(\mathcal{A}, \Omega)$ if and only if $(\mathcal{A}, \Omega)$ is Hausdorff and the convex hull of $\mathcal{M}(\mathcal{A}, \Omega)$ is weak-*-dense in $\mathcal{S}(\mathcal{A}, \Omega)$. Moreover, if a quasi-faithful or faithful representation exists, then the Gelfand transformation is quasi-faithful or faithful, respectively.*

PROOF: First assume that there exists a quasi-faithful representation as functions (X, π) of $(\mathcal{A}, \Omega)$ and let an element $a \in \mathcal{A}$ be given which fulfils $\langle\, \omega\, ,\, a\, \rangle = 0$ for all $\omega \in \mathcal{M}(\mathcal{A}, \Omega)$. Then $\langle\, \delta_x\, ,\, \pi(a)\, \rangle = \langle\, \pi^*(\delta_x)\, ,\, a\, \rangle = 0$ for all $x \in X$ because $\pi^*(\delta_x) \in \mathcal{M}(\mathcal{A}, \Omega)$, so $\pi(a) = 0$ and thus $a = 0$ as (X, π) is quasi-faithful. This shows that $\mathcal{M}(\mathcal{A}, \Omega)$ separates elements of $\mathcal{A}$, and especially that $(\mathcal{A}, \Omega)$ is Hausdorff.

Next assume that there even exists a faithful representation as functions (X, π) of $(\mathcal{A}, \Omega)$. As (X, π) is also quasi-faithful, $(\mathcal{A}, \Omega)$ must be Hausdorff. Moreover, every $a \in \mathcal{A}_{\mathrm{H}}$ which fulfils $\langle\, \omega\, ,\, a\, \rangle \geq 0$ for all $\omega \in \mathcal{M}(\mathcal{A}, \Omega)$ is already positive, i.e. in $\mathcal{A}_{\mathrm{H}}^+$: This is because $\langle\, \delta_x\, ,\, \pi(a)\, \rangle = \langle\, \pi^*(\delta_x)\, ,\, a\, \rangle \geq 0$ for all $x \in X$ by using again that $\pi^*(\delta_x) \in \mathcal{M}(\mathcal{A}, \Omega)$, so $\pi(a) \geq 0$ and thus $a \geq 0$ as (X, π) is faithful. As a consequence, the convex hull of $\mathcal{M}(\mathcal{A}, \Omega)$ is weak-*-dense in $\mathcal{S}(\mathcal{A}, \Omega)$: Consider the dual pairing $\langle\, \cdot\, ,\, \cdot\, \rangle \colon \Omega_{\mathrm{H}} \times \mathcal{A}_{\mathrm{H}} \to \mathbb{R}$, then the weak topology defined by $\mathcal{A}_{\mathrm{H}}$ on Ω_{H} is just the weak-*-topology. Given an element $\omega \in \Omega_{\mathrm{H}}$ which is not in the weak-*-closure of the convex hull of $\mathcal{M}(\mathcal{A}, \Omega)$, then Lemma A.1.13 shows that there exists an $a \in \mathcal{A}_{\mathrm{H}}$ fulfilling $\langle\, \rho\, ,\, a\, \rangle \geq \langle\, \omega\, ,\, a\, \rangle + 1$ for all $\rho \in \langle\!\langle\, \mathcal{M}(\mathcal{A}, \Omega)\, \rangle\!\rangle_{\mathrm{conv}}$, especially for all $\rho \in \mathcal{M}(\mathcal{A}, \Omega)$. By adding a suitable multiple of $\mathbb{1}$ to a we can assume that $\langle\, \omega\, ,\, a\, \rangle = -1/2$ and $\langle\, \rho\, ,\, a\, \rangle \geq 1/2$ for all $\rho \in \mathcal{M}(\mathcal{A}, \Omega)$, so $a \in \mathcal{A}_{\mathrm{H}}^+$ and ω cannot be a state as it is not even positive.

Conversely, if the set of characters separates elements of $(\mathcal{A},\Omega)$, then it is certainly non-empty and the Gelfand transformation is quasi-faithful: Given $a \in \mathcal{A}$ with $\pi_{\mathrm{Gelfand}}(a) = 0$, then $\langle\, \omega\,,\, a \,\rangle = \langle\, \delta_\omega\,,\, \pi_{\mathrm{Gelfand}}(a) \,\rangle = 0$ for all $\omega \in \mathcal{M}(\mathcal{A},\Omega)$, so $a = 0$.

Finally, if $(\mathcal{A},\Omega)$ is Hausdorff and the convex hull of $\mathcal{M}(\mathcal{A},\Omega)$ is weak-*-dense in $\mathcal{S}(\mathcal{A},\Omega)$, then again $\mathcal{M}(\mathcal{A},\Omega) \neq \emptyset$ and the Gelfand transformation is faithful: Given $a \in \mathcal{A}_{\mathrm{H}} \backslash \mathcal{A}^+_{\mathrm{H}}$, then there exists an $\omega \in \Omega^+_{\mathrm{H}}$ with $\langle\, \omega\,,\, a \,\rangle < 0$, and by rescaling ω we can even assume that $\omega \in \mathcal{S}(\mathcal{A},\Omega)$. As the convex hull of $\mathcal{M}(\mathcal{A},\Omega)$ is weak-*-dense in $\mathcal{S}(\mathcal{A},\Omega)$ by assumption, there exists a character $\rho \in \mathcal{M}(\mathcal{A},\Omega)$ with $\langle\, \rho\,,\, a \,\rangle < 0$, hence $\pi_{\mathrm{Gelfand}}(a)(\rho) < 0$ and thus $\pi_{\mathrm{Gelfand}}(a)$ is Hermitian, but not positive. So π_{Gelfand} is an order embedding, hence also injective by Lemma 3.1.8. □

The Gelfand transformation also has some interesting properties from a more general point of view (see Appendix A.5 for the definitions related to category-theory):

In addition to constructing out of every non-empty set X an abstract O^*-algebra $\mathscr{F}(X) \coloneqq \big(\mathbb{C}^X, \Delta(X)\big)$ of functions on X, one can also construct out of every map $\phi\colon X \to Y$ between two sets X, Y a morphism of abstract O^*-algebras $\mathscr{F}(\phi) \coloneqq \phi^*\colon \big(\mathbb{C}^Y, \Delta(Y)\big) \to \big(\mathbb{C}^X, \Delta(X)\big)$ by pulling back functions on Y to functions on X using ϕ, i.e. $\phi^*(g)(x) = g\big(\phi(x)\big)$ for all $x \in X$ and all $g \in \mathbb{C}^Y$. It is easy to check that this yields a contravariant functor $\mathscr{F}$ from the category of non-empty sets to the category of (commutative) abstract O^*-algebras with non-empty set of characters.

Conversely, for every morphism of abstract O^*-algebras $M\colon (\mathcal{A},\Omega) \to (\mathcal{B},\mathcal{R})$, the pullback $M^*\colon \mathcal{R} \to \Omega$ restricts to a map $\mathscr{G}(M)$ from $\mathcal{M}(\mathcal{B},\mathcal{R})$ to $\mathcal{M}(\mathcal{A},\Omega)$. This yields a contravariant functor $\mathscr{G}$ from the category of abstract O^*-algebras with non-empty set of characters to the category of non-empty sets mapping every such abstract O^*-algebra $(\mathcal{A},\Omega)$ to the set $\mathscr{G}(\mathcal{A},\Omega) \coloneqq \mathcal{M}(\mathcal{A},\Omega)$.

Evaluation functionals $\delta\colon X \to \mathcal{M}\big(\mathbb{C}^X, \Delta(X)\big)$ and Gelfand transformations $\pi_{\mathrm{Gelfand}}\colon (\mathcal{A},\Omega) \to \big(\mathbb{C}^{\mathcal{M}(\mathcal{A},\Omega)}, \Delta(\mathcal{M}(\mathcal{A},\Omega))\big)$ for arbitrary non-empty sets X and arbitrary abstract O^*-algebras $(\mathcal{A},\Omega)$ with non-empty set of characters now describe natural transformations from the identity functor $\mathsf{id}_{\mathrm{set}}$ to $\mathscr{G} \circ \mathscr{F}$ and from $\mathsf{id}_{\mathrm{abs}\,O^*}$ to $\mathscr{F} \circ \mathscr{G}$, respectively. This means that $\delta \circ \phi = \mathscr{G}\big(\mathscr{F}(\phi)\big) \circ \delta$ and $\pi_{\mathrm{Gelfand}} \circ M = \mathscr{F}\big(\mathscr{G}(M)\big) \circ \pi_{\mathrm{Gelfand}}$ hold for every map $\phi\colon X \to Y$ and every morphism of abstract O^*-algebras $M\colon (A,\Omega) \to (\mathcal{B},\mathcal{R})$.

Finally, one can check that all the $\delta\colon X \to \mathcal{M}\big(\mathbb{C}^X, \Delta(X)\big)$ for all non-empty sets X are actually bijections, i.e. that $\big(\mathbb{C}^X, \Delta(X)\big)$ does not have more characters than the evaluation functionals: By definition, every element in $\Delta(X)$, and especially every character ω, is a linear combination of evaluation functionals $\omega = \sum_{n=1}^N \lambda_n \delta_{x_n}$ with $N \in \mathbb{N}$ and $\lambda_n \in \mathbb{C}$, $x_n \in X$ for all $n \in \{1,\dots,N\}$. But by evaluating on $\mathbb{1}$ and on indicator functions of points in x, i.e. functions $e_x\colon X \to \{0,1\}$, $e_x(x') = 1$ if and only if $x = x'$, it becomes clear that $\omega = \delta_{x_n}$ for one $n \in \{1,\dots,N\}$ as $\langle\, \omega\,,\, e_x \,\rangle = \langle\, \omega\,,\, (e_x)^2 \,\rangle = \langle\, \omega\,,\, e_x \,\rangle^2$, hence $\langle\, \omega\,,\, e_x \,\rangle \in \{0,1\}$ for all $x \in X$.

3.2.2 Representations as Operators

Definition 3.2.6 *Let $\mathcal{D}$ be a pre-Hilbert space and $\mathcal{L}^*(\mathcal{D})$ the ordered *-algebra of all adjointable endomorphisms of $\mathcal{D}$ with the usual order on the Hermitian elements like in Definition 1.2.9. For all $\phi \in \mathcal{D}$ define the* vector functional $\chi_\phi\colon \mathcal{L}^*(\mathcal{D}) \to \mathbb{C}$ *by $\chi_\phi(a) \coloneqq \langle\, \phi \,|\, a(\phi) \,\rangle$ for all $a \in \mathcal{L}^*(\mathcal{D})$ and the* abstract O^*-algebra of operators *on $\mathcal{D}$ as $\big(\mathcal{L}^*(\mathcal{D}), \mathcal{X}(\mathcal{D})\big)$ with $\mathcal{X}(\mathcal{D}) \coloneqq \big\langle\!\!\big\langle \{\, \chi_\phi \mid \phi \in \mathcal{D} \,\} \big\rangle\!\!\big\rangle_{\mathrm{lin}}$.*

Again one can check that $\mathcal{X}(\mathcal{D})$ is not only a linear subspace, but also stable under the antilinear involution $\cdot^*$ because all χ_ϕ with $\phi \in \mathcal{D}$ are Hermitian, and it is an $\mathcal{A}$-subbimodule of $\mathcal{A}^*$ because $a \triangleright \chi_\phi = \chi_{a(\phi)}$ holds for all $a \in \mathcal{A}$ and $\phi \in \mathcal{D}$ and because of the polarization-identity

$$a \cdot \chi_\phi \cdot b^* = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k (a + \mathrm{i}^k b) \triangleright \chi_\phi \in \mathcal{X}(\mathcal{D})$$

for all $a, b \in \mathcal{A}$ and $\phi \in \mathcal{D}$ (note that $a^* \cdot \chi_\phi = a^* \cdot \chi_\phi \cdot \mathbb{1} \in \mathcal{X}(\mathcal{D})$ and $\chi_\phi \cdot b = \mathbb{1} \cdot \chi_\phi \cdot b \in \mathcal{X}(\mathcal{D})$). So it follows from Proposition 3.1.5 again that $\big(\mathcal{L}^*(\mathcal{D}), \mathcal{X}(\mathcal{D})\big)$ is indeed an abstract O^*-algebra and it is Hausdorff because $\mathcal{L}^*(\mathcal{D})$ is an ordered *-algebra by the discussion under the Definition 1.2.9. Like before, the choice of positive linear functionals here is a minimal one that gives the correct order.

Definition 3.2.7 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then a* representation as operators *of $(\mathcal{A}, \Omega)$ is a tuple $(\mathcal{D}, \pi)$ consisting of a pre-Hilbert space $\mathcal{D}$ and a morphism of abstract O^*-algebras $\pi \colon (\mathcal{A}, \Omega) \to \big(\mathcal{L}^*(\mathcal{D}), \mathcal{X}(\mathcal{D})\big)$. Such a representation as operators is called quasi-faithful if π is injective and faithful if π is injective and an order embedding.*

Representations as operators are closely linked to the positive linear functionals by the well-known GNS construction, named after Gelfand, Naimark and Segal:

Lemma 3.2.8 *Let $\mathcal{A}$ be a *-algebra and ω an algebraically positive linear functional on $\mathcal{A}$. Then the* Gelfand ideal $\mathcal{G}_\omega := \{\, a \in \mathcal{A} \mid \langle\, \omega \,,\, a^*a \,\rangle = 0 \,\}$ *is a left ideal of $\mathcal{A}$. We will denote the equivalence class of an $a \in \mathcal{A}$ in the quotient vector space $\mathcal{A}/\mathcal{G}_\omega$ by $[a]_\omega$. Then the sesquilinear form $\langle\, \cdot \mid \cdot \,\rangle_\omega$ on $\mathcal{A}/\mathcal{G}_\omega$,*

$$\big\langle\, [a]_\omega \,\big|\, [b]_\omega \,\big\rangle_\omega := \langle\, \omega \,,\, a^*b \,\rangle \qquad \text{for all } [a]_\omega, [b]_\omega \in \mathcal{A}/\mathcal{G}_\omega \,, \tag{3.2.2}$$

is well-defined, Hermitian, positive and non-degenerate and thus turns $\mathcal{A}/\mathcal{G}_\omega$ into a pre-Hilbert space. Finally, the map $\pi_{\mathrm{GNS},\omega} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{A}/\mathcal{G}_\omega)$,

$$\pi_{\mathrm{GNS},\omega}(a)([b]_\omega) := [ab]_\omega \qquad \text{for all } a \in \mathcal{A}, [b]_\omega \in \mathcal{A}/\mathcal{G}_\omega \,, \tag{3.2.3}$$

is a well-defined unital *-homomorphism.

Moreover, if $(\mathcal{A}, \Omega)$ is an abstract O^*-algebra and $\omega \in \Omega^+_{\mathrm{H}}$, then $\pi_{\mathrm{GNS},\omega}$ is even a morphism of abstract O^*-algebras from $(\mathcal{A}, \Omega)$ to $\big(\mathcal{A}/\mathcal{G}_\omega, \mathcal{X}(\mathcal{A}/\mathcal{G}_\omega)\big)$, so $\big(\mathcal{A}/\mathcal{G}_\omega, \pi_{\mathrm{GNS},\omega}\big)$ is a representation as operators of $\mathcal{A}$.

PROOF: While this is a classical construction by now, the proof is given here for convenience: If $a \in \mathcal{G}_\omega$, then $0 \leq |\langle\, \omega \,,\, a \,\rangle|^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega \,,\, a^*a \,\rangle = 0$ implies $\langle\, \omega \,,\, a \,\rangle = 0$ and thus also $\mathrm{Var}_\omega(a) = 0$. Using this it follows easily from (1.2.9) that $\mathcal{G}_\omega$ is a left ideal and that $\langle\, \cdot \mid \cdot \,\rangle_\omega$ is well-defined. Moreover, $\langle\, \cdot \mid \cdot \,\rangle_\omega$ is clearly a sesquilinear form and is Hermitian because

$$\overline{\big\langle\, [a]_\omega \,\big|\, [b]_\omega \,\big\rangle_\omega} = \overline{\langle\, \omega \,,\, a^*b \,\rangle} = \langle\, \omega \,,\, b^*a \,\rangle = \big\langle\, [b]_\omega \,\big|\, [a]_\omega \,\big\rangle_\omega$$

for all $a, b \in \mathcal{A}$, certainly positive and even non-degenerate as $0 = \langle\, [a]_\omega \mid [a]_\omega \,\rangle_\omega = \langle\, \omega \,,\, a^*a \,\rangle$ implies that $a \in \mathcal{G}_\omega$, i.e. $[a]_\omega = [0]_\omega$.

Now let $a \in \mathcal{A}$ be given and consider $\pi_{\mathrm{GNS},\omega}(a)$: This is a well-defined map because $\mathcal{G}_\omega$ is a left ideal so that $[ab]_\omega = [ab']_\omega$ whenever $b, b' \in \mathcal{A}$ differ only by $b - b' \in \mathcal{G}_\omega$. It is adjointable with adjoint $\pi_{\mathrm{GNS},\omega}(a^*)$ because

$$\left\langle [b]_\omega \,\middle|\, \pi_{\mathrm{GNS},\omega}(a)\big([c]_\omega\big) \right\rangle_\omega = \langle \omega \,,\, b^* a\, c \rangle = \langle \omega \,,\, (a^* b)^* c \rangle = \left\langle \pi_{\mathrm{GNS},\omega}(a^*)\big([b]_\omega\big) \,\middle|\, [c]_\omega \right\rangle_\omega$$

for all $a \in \mathcal{A}$ and $[b]_\omega, [c]_\omega \in \mathcal{A}/\mathcal{G}_\omega$. Linearity and multiplicativity of $\pi_{\mathrm{GNS},\omega}$ are now easy to see, compatibility with the *-involution has already been shown and $\pi_{\mathrm{GNS},\omega}(\mathbb{1}) = \mathsf{id}_{\mathcal{A}/\mathcal{G}_\omega}$ is also clear. So $\pi_{\mathrm{GNS},\omega}$ is a unital *-homomorphism.

Finally, if $(\mathcal{A}, \Omega)$ is an abstract O^*-algebra and $\omega \in \Omega^+_{\mathrm{H}}$, then $(\pi_{\mathrm{GNS},\omega})^*(\chi_{[a]_\omega}) = a \triangleright \omega$ holds for all $a \in \mathcal{A}$ due to the identity

$$\left\langle (\pi_{\mathrm{GNS},\omega})^*(\chi_{[a]_\omega}) \,,\, b \right\rangle = \left\langle \chi_{[a]_\omega} \,,\, \pi_{\mathrm{GNS},\omega}(b) \right\rangle = \left\langle [a]_\omega \,\middle|\, \pi_{\mathrm{GNS},\omega}(b)\big([a]_\omega\big) \right\rangle_\omega = \langle \omega \,,\, a^* b\, a \rangle = \langle a \triangleright \omega \,,\, b \rangle$$

for all $b \in \mathcal{A}$. So $\pi_{\mathrm{GNS},\omega}$ is a morphism of abstract O^*-algebras from $(\mathcal{A}, \Omega)$ to $\big(\mathcal{A}/\mathcal{G}_\omega, \mathcal{X}(\mathcal{A}/\mathcal{G}_\omega)\big)$. $\square$

This lemma especially shows that every positive linear functional of an abstract O^*-algebra yields a representation as operators:

Definition 3.2.9 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $\omega \in \Omega^+_{\mathrm{H}}$, then the representation as operators $\big(\mathcal{A}/\mathcal{G}_\omega, \pi_{\mathrm{GNS},\omega}\big)$ of the previous Lemma 3.2.8 is called the* GNS representation *associated to ω.*

Lemma 3.2.10 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $I \subseteq \Omega^+_{\mathrm{H}}$ a non-empty subset. Then the direct sum $(\mathcal{D}_I, \pi_I)$ of GNS representations, i.e.*

$$\mathcal{D}_I := \bigoplus_{\omega \in I} \mathcal{A}/\mathcal{G}_\omega \qquad \text{and} \qquad \pi_I := \bigoplus_{\omega \in I} \pi_{\mathrm{GNS},\omega} \tag{3.2.4}$$

with inner product on $\mathcal{D}_I$ defined as

$$\left\langle \sum\nolimits_{\omega \in I} \phi_\omega \,\middle|\, \sum\nolimits_{\omega \in I} \psi_\omega \right\rangle := \sum_{\omega \in I} \langle \phi_\omega \,|\, \psi_\omega \rangle_\omega \qquad \text{for all } \sum_{\omega \in I} \phi_\omega, \sum_{\omega \in I} \psi_\omega \in \mathcal{D}_I \tag{3.2.5}$$

is again a representation as operators of $(\mathcal{A}, \Omega)$.

Proof: The inner product on $\mathcal{D}_I$ is well-defined because only finitely many terms in the sums are unequal 0, and it is clearly Hermitian, positive and non-degenerate. Moreover, π_I is linear by construction and even a unital *-homomorphism, because the $\pi_{\mathrm{GNS},\omega} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{A}/\mathcal{G}_\omega)$ are. Finally, given $\phi = \sum_{\omega \in I} \phi_\omega \in \mathcal{D}$ with $\phi_\omega \in \mathcal{A}/\mathcal{G}_\omega$ for all $\omega \in I$, then

$$\left\langle \pi_I^*(\chi_\phi) \,,\, a \right\rangle = \left\langle \chi_\phi \,,\, \pi_I(a) \right\rangle = \left\langle \phi \,\middle|\, \pi_I(a)(\phi) \right\rangle = \sum_{\omega \in I} \left\langle \phi_\omega \,\middle|\, \pi_{\mathrm{GNS},\omega}(a)(\phi_\omega) \right\rangle_\omega = \sum_{\omega \in I} \left\langle \pi^*_{\mathrm{GNS},\omega}(\chi_{\phi_\omega}) \,,\, a \right\rangle$$

for all $a \in \mathcal{A}$ shows that $\pi_I^*(\chi_\phi) = \sum_{\omega \in I} \pi^*_{\mathrm{GNS},\omega}(\chi_{\phi_\omega}) \in \Omega^+_{\mathrm{H}}$ because $\pi^*_{\mathrm{GNS},\omega}(\chi_{\phi_\omega}) \in \Omega^+_{\mathrm{H}}$ for all $\omega \in I$ by Lemma 3.2.8. So π_I is even a morphism of abstract O^*-algebras from $(\mathcal{A}, \Omega)$ to $\big(\mathcal{L}^*(\mathcal{D}_I), \mathcal{X}(\mathcal{D}_I)\big)$, and thus $(\mathcal{D}_I, \pi_I)$ is a representation as operators of $(\mathcal{A}, \Omega)$. $\square$

Theorem 3.2.11 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, then the following is equivalent:*

- *There exists a faithful representation as operators of $(\mathcal{A}, \Omega)$.*
- *There exists a quasi-faithful representation as operators of $(\mathcal{A}, \Omega)$.*
- *$(\mathcal{A}, \Omega)$ is Hausdorff.*

PROOF: Of course, if there exists a faithful representation then there also exists a quasi-faithful one, and if there exists a quasi-faithful representation $(\mathcal{D}, \pi)$, let $a \in \mathcal{A}$ with $\langle\, \omega\,,\, a\,\rangle = 0$ for all $\omega \in \Omega$ be given. Then especially $\langle\, \phi \,|\, \pi(a)(\phi)\,\rangle = \langle\, \chi_\phi\,,\, \pi(a)\,\rangle = \langle\, \pi^*(\chi_\phi)\,,\, a\,\rangle = 0$ for all $\phi \in \mathcal{D}$, thus $\pi(a) = 0$ and therefore $a = 0$ because π is injective by assumption. So existence of a quasi-faithful representation as operators implies that $(\mathcal{A}, \Omega)$ is Hausdorff.

It only remains to show that every Hausdorff abstract O^*-algebra $(\mathcal{A}, \Omega)$ has a faithful representation: Construct the representation as operators $(\mathcal{D}_I, \pi_I)$ like in the previous Lemma 3.2.10 with $I := \Omega^+_{\mathrm{H}}$. Then π_I is even an order embedding, because given $a \in \mathcal{A}_{\mathrm{H}}$ with $\pi_I(a) \geq 0$, then $\langle\, \omega\,,\, a\,\rangle = \big\langle\, [\mathbb{1}]_\omega \,\big|\, \pi_{\mathrm{GNS},\omega}(a)\big([\mathbb{1}]_\omega\big)\,\big\rangle_\omega \geq 0$ holds for all $\omega \in \Omega^+_{\mathrm{H}}$, so $a \in \mathcal{A}^+_{\mathrm{H}}$. As $(\mathcal{A}, \Omega)$ was assumed to be Hausdorff, Lemma 3.1.8 shows that π_I is also injective. So the representation as operators $(\mathcal{D}_I, \pi_I)$ is faithful. □

Unfortunately, the above construction of representations as operators does not enjoy similar nice properties like the Gelfand transformation for the construction of representations as functions:

Consider the example of the Hilbert space $\mathbb{C}^2$ with the standard inner product, then $\mathcal{L}^*(\mathbb{C}^2) \cong \mathbb{C}^{2\times 2}$ and the linear functionals in $\mathcal{X}(\mathbb{C}^2)$ are the trace-functionals $\mathcal{L}^*(\mathbb{C}^2) \ni a \mapsto \mathsf{tr}(\omega a) \in \mathbb{C}$ with $\omega \in \mathbb{C}^{2\times 2}$. The positive linear functionals thus are the traces with positive Hermitian matrices $\omega \in \mathbb{C}^{2\times 2}$, and it becomes clear that the pre-Hilbert space $\mathcal{D}_I$ constructed in the proof of the above Theorem 3.2.11 has the positive Hermitian matrices in $\mathbb{C}^{2\times 2}$ as a basis and certainly cannot be identified with $\mathbb{C}^2$.

3.3 Constructions of Abstract O^*-Algebras

Proposition 3.1.5 already shows how abstract O^*-algebras can be constructed if a *-algebra and a suitable set of algebraically positive linear functionals on it are given. In this section, this will be applied to quasi-ordered *-algebras and the set of all positive linear functionals (including ordinary *-algebras and the set of all algebraically positive linear functionals as a special case), as well as locally convex *-algebras and certain sets of continuous algebraically positive linear functionals on them.

3.3.1 From Quasi-Ordered *-Algebras

Lemma 3.3.1 *Let $\mathcal{A}$ be a quasi-ordered *-algebra, then the set $\mathcal{A}^{*,+}_{\mathrm{H}}$ of all positive linear functionals on $\mathcal{A}$ is a weak-*-closed convex cone in $\mathcal{A}^*$ and stable under the monoid action $\triangleright$ of $\mathcal{A}$ on $\mathcal{A}^*$.*

PROOF: It is clear that $\mathcal{A}^{*,+}_{\mathrm{H}}$ is a convex cone in $\mathcal{A}^*$ and as $\mathcal{A}^{*,+}_{\mathrm{H}} = \bigcap_{a\in\mathcal{A}^+_{\mathrm{H}}}\{\, \omega \in \mathcal{A}^* \mid \langle\, \omega\,,\, a\,\rangle \geq 0\,\}$ is the intersection of the preimages of the closed interval $[0, \infty[$ under the weak-*-continuous maps $\mathcal{A}^* \ni \omega \mapsto \langle\, \omega\,,\, a\,\rangle \in \mathbb{C}$, it is weak-*-closed. Moreover, given $a \in \mathcal{A}$ and $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$, then $\langle\, a \triangleright \omega\,,\, b\,\rangle = \langle\, \omega\,,\, a^* b\, a\,\rangle \geq 0$ for all $b \in \mathcal{A}^+_{\mathrm{H}}$, hence $a \triangleright \omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$. □

The above Lemma 3.3.1 together with Proposition 3.1.5 show that $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is an abstract O^*-algebra with convex cone of positive linear functionals $(\langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}})^{+}_{\mathrm{H}} = \mathcal{A}^{*,+}_{\mathrm{H}}$. However, it is important to note that the order on $\mathcal{A}$ in the abstract O^*-algebra sense, i.e. the order on $\mathcal{A}$ induced by the positive linear functionals in $\mathcal{A}^{*,+}_{\mathrm{H}}$, may in general be different from the original one if $\mathcal{A}^{+}_{\mathrm{H}}$ is not closed in $\mathcal{A}_{\mathrm{H}}$ with respect to the weak topology defined by $\mathcal{A}^{*}_{\mathrm{H}}$.

Lemma 3.3.2 *Let $\mathcal{A}$ be a quasi-ordered *-algebra, $(\mathcal{B}, \mathcal{R})$ an abstract O^*-algebra and $M\colon \mathcal{A} \to \mathcal{B}$ a positive unital *-homomorphism, then M is a morphism of abstract O^*-algebras from $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ to $(\mathcal{B}, \mathcal{R})$.*

PROOF: Given $\rho \in \mathcal{R}^{+}_{\mathrm{H}}$, then $M^*(\rho) \in \mathcal{A}^{*,+}_{\mathrm{H}}$ as $\langle\, M^*(\rho)\,,\, a \,\rangle = \langle\, \rho\,,\, M(a) \,\rangle \geq 0$ for all $a \in \mathcal{A}^{+}_{\mathrm{H}}$. □

Combining the two Lemmas 3.3.1 and 3.3.2, we immediately get:

Proposition 3.3.3 *Assigning to all quasi-ordered *-algebras $\mathcal{A}$ the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ and to every positive unital *-homomorphism M between two quasi-orderd *-algebras $\mathcal{A}$ and $\mathcal{B}$ the morphism of abstract O^*-algebras $M\colon \big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big) \to \big(\mathcal{B}, \langle\!\langle\, \mathcal{B}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is a (covariant) functor from the category of quasi-ordered *-algebras with the positive unital *-homomorphisms between them to the category of abstract O^*-algebras with the morphisms of abstract O^*-algebras between them.*

With respect to the properties of this construction we just note the following:

Proposition 3.3.4 *Let $\mathcal{A}$ be a quasi-ordered *-algebra, then the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is closed and downwards closed.*

PROOF: We have already seen in Lemma 3.3.1 that $\mathcal{A}^{*,+}_{\mathrm{H}}$ is weak-*-closed in $\mathcal{A}^*$, so $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,+}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is closed. Proposition 3.1.11 then shows that it is also downwards closed. □

Given a *-algebra $\mathcal{A}$, then we can apply this construction to the quasi-ordered *-algebra $\mathcal{A}$ with positive elements $\mathcal{A}^{+}_{\mathrm{H}} := \mathcal{A}^{++}_{\mathrm{H}}$, which yields the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$:

Lemma 3.3.5 *Let $\mathcal{A}$ be a *-algebra, $(\mathcal{B}, \mathcal{R})$ an abstract O^*-algebra and $M\colon \mathcal{A} \to \mathcal{B}$ a unital *-homomorphism, then M is a morphism of abstract O^*-algebras from $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ to $(\mathcal{B}, \mathcal{R})$.*

PROOF: Given $\rho \in \mathcal{R}^{+}_{\mathrm{H}}$, then $M^*(\rho) \in \mathcal{A}^{*,++}_{\mathrm{H}}$ because $\langle\, M^*(\rho)\,,\, a^*a \,\rangle = \langle\, \rho\,,\, M(a)^*M(a) \,\rangle \geq 0$ for all $a \in \mathcal{A}$. □

We thus conclude:

Proposition 3.3.6 *Assigning to every *-algebra $\mathcal{A}$ the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ and to every unital *-homomorphism M between two *-algebras $\mathcal{A}$ and $\mathcal{B}$ the morphism of abstract O^*-algebras $M\colon \big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big) \to \big(\mathcal{B}, \langle\!\langle\, \mathcal{B}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is a (covariant) functor from the category of *-algebras with the unital *-homomorphisms between them to the category of abstract O^*-algebras with the morphisms of abstract O^*-algebras between them.*

Proposition 3.3.7 *Let $\mathcal{A}$ be a *-algebra, then the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}^{*,++}_{\mathrm{H}} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is hyperregular, regular, closed and downwards closed.*

Proof: Proposition 3.3.4 shows that $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}_{\mathrm{H}}^{*,++} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ is closed and downwards closed, and it is clearly hyper-regular and regular because its convex cone of positive linear functionals is the convex cone $\mathcal{A}_{\mathrm{H}}^{*,++}$ of algebraically positive ones. $\square$

With respect to representations as operators it is worthwhile to mention that every positive unital *-homomorphism π from a quasi-ordered *-algebra $\mathcal{A}$ to an O^*-algebra $\mathcal{B} \subseteq \mathcal{L}^*(\mathcal{D})$ on a pre-Hilbert space $\mathcal{D}$ gives a representation as operators $(\mathcal{D}, \pi)$ of $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}_{\mathrm{H}}^{*,+} \,\rangle\!\rangle_{\mathrm{lin}}\big)$, and conversely, in every such representation as operators $(\mathcal{D}, \pi)$, the map π is of course a positive unital *-homomorphism from $\mathcal{A}$ to $\mathcal{L}^*(\mathcal{D})$ by definition. In this sense, the abstract O^*-algebra $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}_{\mathrm{H}}^{*,+} \,\rangle\!\rangle_{\mathrm{lin}}\big)$ has the same representations as the quasi-ordered *-algebra $\mathcal{A}$.

An analogous relation also exists between positive unital *-homomorphisms from $\mathcal{A}$ to an ordered *-algebra of functions and the representations as functions of $\big(\mathcal{A}, \langle\!\langle\, \mathcal{A}_{\mathrm{H}}^{*,+} \,\rangle\!\rangle_{\mathrm{lin}}\big)$. By treating general *-algebras as special quasi-ordered *-algebras, this also applies to representations of general *-algebras.

3.3.2 From Locally Convex *-Algebras

If $\mathcal{A}$ is a locally convex *-algebra, then there are similar constructions like those above which are linked to weakly- and strongly continuous representations of locally convex *-algebras:

Definition 3.3.8 *Let $\mathcal{A}$ be a locally convex *-algebra, then a* weakly continuous representation as operators *of $\mathcal{A}$ is a tuple $(\mathcal{D}, \pi)$ of a pre-Hilbert space $\mathcal{D}$ and a unital *-homomorphism $\pi \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$ which is weakly continuous, i.e. continuous with respect to the weak topology on $\mathcal{L}^*(\mathcal{D})$, which is the locally convex one defined by the seminorms $\mathcal{L}^*(\mathcal{D}) \ni a \mapsto \|a\|_{\phi,\psi,\mathrm{wk}} := |\langle\, \phi \,|\, a(\psi) \,\rangle|$ for all $\phi, \psi \in \mathcal{D}$.*

Similarly, a strongly continuous representation as operators *of $\mathcal{A}$ is a weakly continuous representation $(\mathcal{D}, \pi)$ for which π is even strongly continuous, i.e. continuous with respect to the strong topology on $\mathcal{L}^*(\mathcal{D})$, which is the locally convex one defined by the seminorms $\mathcal{L}^*(\mathcal{D}) \ni a \mapsto \|a\|_{\phi,\mathrm{st}} := \|a(\phi)\|$ for all $\phi \in \mathcal{D}$, where $\|\cdot\|$ is the usual norm on $\mathcal{D}$ induced by the inner product.*

Of course, the strong topology on $\mathcal{L}^*(\mathcal{D})$ is stronger than the weak one due to the Cauchy Schwarz inequality: $\|a\|_{\phi,\psi,\mathrm{wk}} = |\langle\, \phi \,|\, a(\psi) \,\rangle| \overset{\mathrm{CS}}{\leq} \|\phi\| \|a(\psi)\| = \|\phi\| \|a\|_{\psi,\mathrm{st}}$ holds for all $\phi, \psi \in \mathcal{D}$, $a \in \mathcal{L}^*(\mathcal{D})$. The relation between weakly and strongly continuous representations are quite clear and well-known:

Proposition 3.3.9 *Let $\mathcal{A}$ be a locally convex *-algebra, then every strongly continuous representation as operators of $\mathcal{A}$ is also a weakly continuous one. Moreover, if the product on $\mathcal{A}$ is continuous, then the converse is true as well, i.e. in this case the weakly and strongly continuous representations coincide.*

Proof: As the strong topology on a *-algebra of operators $\mathcal{L}^*(\mathcal{D})$ is stronger than the weak one, every strongly continuous representation of $\mathcal{A}$ is also weakly continuous. Conversely, if the product on $\mathcal{A}$ is continuous and $(\mathcal{D}, \pi)$ is a weakly continuous representation of $\mathcal{A}$, then $\mathcal{A} \ni a \mapsto \|\pi(a)\|_{\phi,\mathrm{st}} = \langle\, \pi(a)(\phi) \,|\, \pi(a)(\phi) \,\rangle^{1/2} = \langle\, \phi \,|\, \pi(a^* a)(\phi) \,\rangle^{1/2} = \|\pi(a^* a)\|_{\phi,\phi,\mathrm{wk}}^{1/2}$ is a continuous seminorm on $\mathcal{A}$ for every $\phi \in \mathcal{D}$, hence π is strongly continuous. $\square$

Because of this there is no reason to distinguish weakly and strongly continuous representations of locally convex *-algebras with continuous product:

Definition 3.3.10 *Let $\mathcal{A}$ be a locally convex *-algebra with continuous product, then a* continuous representation as operators *of $\mathcal{A}$ is a weakly, or equivalently strongly, continuous representation.*

For representations as functions there is generally no need to distinguish between weakly and strongly continuous ones:

Definition 3.3.11 *Let $\mathcal{A}$ be a locally convex *-algebra, then a* continuous representation as functions *of $\mathcal{A}$ is defined as a tuple (X, π) consisting of a set X and a unital *-homomorphism π from $\mathcal{A}$ to the *-algebra $\mathbb{C}^X$ of all complex-valued functions on X with the pointwise operations, which is continuous with respect to the topology of pointwise convergence on $\mathbb{C}^X$, i.e. the locally convex topology defined by the seminorms $\mathbb{C}^X \ni a \mapsto \|a\|_x \coloneqq |a(x)|$ for all $x \in X$.*

It is now possible to construct out of a locally convex *-algebra $\mathcal{A}$ an abstract O^*-algebra whose representations as operators are exactly the weakly- or strongly continuous ones:

Proposition 3.3.12 *Let $\mathcal{A}$ be a locally convex *-algebra, then the set*

$$\mathcal{T}^+_{\mathrm{wk,H}} \coloneqq \big\{\, \omega \in \mathcal{A}^{*,++}_{\mathrm{H}} \;\big|\; \omega \textit{ is continuous} \,\big\} \tag{3.3.1}$$

is stable under the monoid action $\triangleright$ of $\mathcal{A}$ on $\mathcal{A}^*$, hence, by the construction from Proposition 3.1.5, yields an abstract O^*-algebras $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ with $\mathcal{T}_{\mathrm{wk}} \coloneqq \langle\!\langle\, \mathcal{T}^+_{\mathrm{wk,H}} \,\rangle\!\rangle_{\mathrm{lin}}$ and order on $\mathcal{A}_{\mathrm{H}}$ given by

$$a \le b \qquad :\Longleftrightarrow \qquad \forall_{\omega \in \mathcal{T}^+_{\mathrm{wk,H}}} : \langle\, \omega \,,\, a \,\rangle \le \langle\, \omega \,,\, b \,\rangle \,. \tag{3.3.2}$$

Moreover, the identity

$$\mathcal{T}^+_{\mathrm{wk,H}} = \big\{\, \omega \in \mathcal{T}_{\mathrm{wk}} \;\big|\; \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega \,,\, a \,\rangle \ge 0 \,\big\} = \big\{\, \omega \in \mathcal{T}_{\mathrm{wk}} \;\big|\; \forall_{a \in \mathcal{A}} : \langle\, \omega \,,\, a^* a \,\rangle \ge 0 \,\big\} \tag{3.3.3}$$

holds, so $\mathcal{T}^+_{\mathrm{wk,H}}$, defined like above, is indeed the convex cone of positive linear functionals of the abstract O^*-algebra $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$.

PROOF: Given a continuous $\omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ and $a \in \mathcal{A}$, then $a \triangleright \omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ is continuous because $\mathcal{A} \ni b \mapsto \langle\, a \triangleright \omega \,,\, b \,\rangle = \langle\, \omega \,,\, a^* b\, a \,\rangle$ is continuous as the composition of the continuous left and right multiplication with a^* and a, respectively, and the continuous ω. So Proposition 3.1.5 can be applied to $\mathcal{A}$ and the set $\mathcal{T}^+_{\mathrm{wk,H}}$ of continuous algebraically positive linear functionals on $\mathcal{A}$ and yields the abstract O^*-algebra described above. The inclusion $\mathcal{T}^+_{\mathrm{wk,H}} \subseteq \{\, \omega \in \mathcal{T}_{\mathrm{wk}} \mid \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega \,,\, a \,\rangle \ge 0 \,\}$ is part of Proposition 3.1.5, the inclusion $\{\, \omega \in \mathcal{T}_{\mathrm{wk}} \mid \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega \,,\, a \,\rangle \ge 0 \,\} \subseteq \{\, \omega \in \mathcal{T}_{\mathrm{wk}} \mid \forall_{a \in \mathcal{A}} : \langle\, \omega \,,\, a^* a \,\rangle \ge 0 \,\}$ is true in general because $\mathcal{A}^+_{\mathrm{H}} \supseteq \mathcal{A}^{++}_{\mathrm{H}}$. Conversely, as $\mathcal{T}_{\mathrm{wk}}$ is generated by continuous linear functionals on $\mathcal{A}$, all linear functionals in $\mathcal{T}_{\mathrm{wk}}$ are continuous. So every $\omega \in \mathcal{T}_{\mathrm{wk}}$ which fulfils $\langle\, \omega \,,\, a^* a \,\rangle \ge 0$ for all $a \in \mathcal{A}$ is continuous and algebraically positive, hence in $\mathcal{T}^+_{\mathrm{wk,H}}$, which proves (3.3.3). □

Proposition 3.3.13 *Let $\mathcal{A}$ be a locally convex *-algebra, then the set*

$$\mathcal{T}^+_{\mathrm{st,H}} \coloneqq \big\{\, \omega \in \mathcal{A}^{*,++}_{\mathrm{H}} \;\big|\; \mathcal{A} \ni a \mapsto \langle\, \omega \,,\, a^* a \,\rangle \in \mathbb{R} \textit{ is continuous} \,\big\} \tag{3.3.4}$$

is stable under the monoid action $\triangleright$ of $\mathcal{A}$ on $\mathcal{A}^*$, hence, by the construction from Proposition 3.1.5, yields an abstract O^*-algebra $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ with $\mathcal{T}_{\mathrm{st}} := \langle\!\langle\, \mathcal{T}^+_{\mathrm{st,H}} \,\rangle\!\rangle_{\mathrm{lin}}$ and order on $\mathcal{A}_{\mathrm{H}}$ given by

$$a \leq b \qquad :\Longleftrightarrow \qquad \forall_{\omega \in \mathcal{T}^+_{\mathrm{st,H}}} : \langle\, \omega\,,\, a \,\rangle \leq \langle\, \omega\,,\, b \,\rangle\,. \tag{3.3.5}$$

Moreover, the identity

$$\mathcal{T}^+_{\mathrm{st,H}} = \big\{\, \omega \in \mathcal{T}_{\mathrm{st}} \;\big|\; \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega\,,\, a \,\rangle \geq 0 \,\big\} = \big\{\, \omega \in \mathcal{T}_{\mathrm{st}} \;\big|\; \forall_{a \in \mathcal{A}} : \langle\, \omega\,,\, a^* a \,\rangle \geq 0 \,\big\} \tag{3.3.6}$$

holds, so $\mathcal{T}^+_{\mathrm{st,H}}$, defined like above, is indeed the convex cone of positive linear functionals of the abstract O^*-algebra $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$.

PROOF: Given $\omega \in \mathcal{T}^+_{\mathrm{st,H}}$ and $b \in \mathcal{A}$, then continuity of the right multiplication $\mathcal{A} \ni a \mapsto ab \in \mathcal{A}$ guarantees that $\mathcal{A} \ni a \mapsto \|a\|_{b \triangleright \omega, \mathrm{st}} = \|ab\|_{\omega,\mathrm{st}} \in \mathbb{R}$ is continuous. So $\mathcal{T}^+_{\mathrm{st,H}}$ is stable under the monoid action $\triangleright$ of $\mathcal{A}$ on $\mathcal{A}^*$ and Proposition 3.1.5 can be applied to $\mathcal{A}$ and $\mathcal{T}_{\mathrm{st}}$ and yields the abstract O^*-algebra described above. As before, the inclusion $\mathcal{T}^+_{\mathrm{st,H}} \subseteq \{\, \omega \in \mathcal{T}_{\mathrm{st}} \mid \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega\,,\, a \,\rangle \geq 0 \,\}$ is part of Proposition 3.1.5 and the inclusion $\{\, \omega \in \mathcal{T}_{\mathrm{st}} \mid \forall_{a \in \mathcal{A}^+_{\mathrm{H}}} : \langle\, \omega\,,\, a \,\rangle \geq 0 \,\} \subseteq \{\, \omega \in \mathcal{T}_{\mathrm{st}} \mid \forall_{a \in \mathcal{A}} : \langle\, \omega\,,\, a^* a \,\rangle \geq 0 \,\}$ is true in general. Now let $\omega \in \mathcal{T}_{\mathrm{st}}$ be given such that $\langle\, \omega\,,\, a^* a \,\rangle \geq 0$ holds for all $a \in \mathcal{A}$. Then ω is Hermitian and algebraically positive and it is to show that $\|\cdot\|_{\omega,\mathrm{st}}$ is a continuous seminorm on $\mathcal{A}$. By construction of $\mathcal{T}_{\mathrm{st}}$ as the $\mathbb{C}$-linear span of $\mathcal{T}^+_{\mathrm{st,H}}$, the Hermitian ω can be expressed as a difference $\omega = \rho - \rho'$ with $\rho, \rho' \in \mathcal{T}^+_{\mathrm{st,H}}$, so $\langle\, \omega\,,\, b^* b \,\rangle \leq \langle\, \rho\,,\, b^* b \,\rangle$ holds for all $b \in \mathcal{A}$, thus $\|\cdot\|_{\omega,\mathrm{st}} \leq \|\cdot\|_{\rho,\mathrm{st}}$ and $\|\cdot\|_{\omega,\mathrm{st}}$ is indeed continuous. □

Proposition 3.3.14 *Let $\mathcal{A}$ be a locally convex *-algebra, then, with the notation of the previous two Propositions 3.3.12, $\mathcal{T}^+_{\mathrm{wk,H}} \supseteq \mathcal{T}^+_{\mathrm{st,H}}$ holds. Moreover, if the product on $\mathcal{A}$ is continuous, then the converse inclusion is also true, hence $\mathcal{T}^+_{\mathrm{wk,H}} = \mathcal{T}^+_{\mathrm{st,H}}$ and the two abstract O^*-algebras $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ and $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ coincide.*

PROOF: Given $\omega \in \mathcal{T}^+_{\mathrm{st,H}}$, then ω is continuous because $|\langle\, \omega\,,\, a \,\rangle| \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, a^* a \,\rangle^{1/2} = \|a\|_{\omega,\mathrm{st}}$ holds for all $a \in \mathcal{A}$ and $\|\cdot\|_{\omega,\mathrm{st}}$ is a continuous seminorm on $\mathcal{A}$ by assumption; so $\omega \in \mathcal{T}^+_{\mathrm{wk,H}}$.

Conversely, if the multiplication on $\mathcal{A}$ is continuous and $\omega \in \mathcal{T}^+_{\mathrm{wk,H}}$, then the seminorm $\|\cdot\|_{\omega,\mathrm{st}}$ is continuous because $\mathcal{A} \ni b \mapsto \|b\|_{\omega,\mathrm{st}} = \langle\, \omega\,,\, b^* b \,\rangle^{1/2} \in \mathbb{R}$ is continuous as the composition of continuous maps. □

Definition 3.3.15 *Let $\mathcal{A}$ be a locally convex *-algebra, then define the abstract O^*-algebras $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ and $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ like in Propositions 3.3.12 and 3.3.13. Moreover, if the product on $\mathcal{A}$ is continuous and thus these two abstract O^*-algebras coincide, then we simply write $(\mathcal{A}, \mathcal{T})$. If there is danger of confusion, we might also write $\mathcal{T}_{\mathrm{wk}}(\mathcal{A}), \mathcal{T}_{\mathrm{st}}(\mathcal{A})$ and $\mathcal{T}(\mathcal{A})$ in order to make the dependence of the space of linear functionals on the locally convex *-algebra explicit.*

Proposition 3.3.16 *Let $\mathcal{A}$ be a locally convex *-algebra, $(\mathcal{B}, \mathcal{R})$ an abstract O^*-algebra and $M\colon \mathcal{A} \to \mathcal{B}$ a unital *-homomorphism. Then M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ to $(\mathcal{B}, \mathcal{R})$ if and only if M is continuous with respect to the weak topology on $\mathcal{B}$. Similarly, M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ to $(\mathcal{B}, \mathcal{R})$ if and only if M is continuous with respect to the strong topology on $\mathcal{B}$.*

PROOF: First assume that M is continuous with respect to the weak topology on $\mathcal{B}$. Then $M^*(\psi)$ is a continuous algebraically positive linear functional for all $\psi \in \mathcal{R}^+_{\mathrm{H}}$, because ψ is weakly continuous and algebraically positive. Because of this, $M^*(\psi) \in \mathcal{T}^+_{\mathrm{wk,H}}$ and M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ to $(\mathcal{B}, \mathcal{R})$.

Next assume that M is continuous with respect to the strong topology on $\mathcal{B}$. Given $\psi \in \mathcal{R}^+_{\mathrm{H}}$, then the identity $\|\cdot\|_{\psi,\mathrm{st}} \circ M = \|\cdot\|_{M^*(\psi),\mathrm{st}}$ holds because $\|M(a)\|^2_{\psi,\mathrm{st}} = \langle\, \psi \,,\, M(a^*a)\,\rangle = \langle\, M^*(\psi) \,,\, a^*a\,\rangle = \|a\|^2_{M^*(\psi),\mathrm{st}}$ for all $a \in \mathcal{A}$. As $\|\cdot\|_{\psi,\mathrm{st}}$ is a strongly continuous seminorm on $\mathcal{B}$ and M continuous with respect to this topology, this identity shows that $\|\cdot\|_{M^*(\psi),\mathrm{st}}$ is a continuous seminorm on $\mathcal{A}$. So $M^*(\psi) \in \mathcal{T}^+_{\mathrm{st,H}}$ and M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ to $(\mathcal{B}, \mathcal{R})$.

Conversely, if M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ to $(\mathcal{B}, \mathcal{R})$, then $M^*(\psi) \in \mathcal{T}_{\mathrm{wk}}$ for all $\psi \in \mathcal{R}$, hence $\|\cdot\|_{\psi,\mathrm{wk}} \circ M = \|\cdot\|_{M^*(\psi),\mathrm{wk}}$ is a continuous seminorm on $\mathcal{A}$ and $M\colon \mathcal{A} \to \mathcal{B}$ is continuous with respect to the weak topology on $\mathcal{B}$.

Finally, if M is a morphism of abstract O^*-algebras from $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ to $(\mathcal{B}, \mathcal{R})$ and $\psi \in \mathcal{R}^+_{\mathrm{H}}$, then $M^*(\psi) \in \mathcal{T}^+_{\mathrm{st,H}}$, hence $\|\cdot\|_{\psi,\mathrm{st}} \circ M = \|\cdot\|_{M^*(\psi),\mathrm{st}}$ is a continuous seminorm on $\mathcal{A}$. So M is continuous with respect to the strong topology on $\mathcal{B}$. □

Corollary 3.3.17 *If $\mathcal{A}$ and $\mathcal{B}$ are two locally convex *-algebras and $M\colon \mathcal{A} \to \mathcal{B}$ a continuous unital *-homomorphism, then M is a morphism of abstract O^*-algebras from $\big(\mathcal{A}, \mathcal{T}_{\mathrm{wk}}(\mathcal{A})\big)$ to $\big(\mathcal{B}, \mathcal{T}_{\mathrm{wk}}(\mathcal{B})\big)$, as well as from $\big(\mathcal{A}, \mathcal{T}_{\mathrm{st}}(\mathcal{A})\big)$ to $\big(\mathcal{B}, \mathcal{T}_{\mathrm{st}}(\mathcal{B})\big)$.*

PROOF: This follows from the previous Proposition 3.3.16 by construction of $\mathcal{T}_{\mathrm{wk}}(\mathcal{B})$ and $\mathcal{T}_{\mathrm{st}}(\mathcal{B})$, because the weak and strong topology on $\big(\mathcal{B}, \mathcal{T}_{\mathrm{wk}}(\mathcal{B})\big)$ and $\big(\mathcal{B}, \mathcal{T}_{\mathrm{st}}(\mathcal{B})\big)$, respectively, are weaker than the given one on $\mathcal{B}$, hence M is also continuous with respect to these. □

This yields:

Proposition 3.3.18 *Assigning to every locally convex *-algebra $\mathcal{A}$ the abstract O^*-algebra $\big(\mathcal{A}, \mathcal{T}_{\mathrm{wk}}(\mathcal{A})\big)$ and to every continuous unital *-homomorphism M between two locally convex *-algebras $\mathcal{A}$ and $\mathcal{B}$ the morphism of abstract O^*-algebras $M\colon \big(\mathcal{A}, \mathcal{T}_{\mathrm{wk}}(\mathcal{A})\big) \to \big(\mathcal{B}, \mathcal{T}_{\mathrm{wk}}(\mathcal{B})\big)$ is a covariant functor from the category of locally convex *-algebras with continuous unital *-homomorphisms between them to the category of abstract O^*-algebras with the morphisms of abstract O^*-algebras between them.*

The same is true for assigning to every locally convex *-algebra $\mathcal{A}$ the abstract O^*-algebra $\big(\mathcal{A}, \mathcal{T}_{\mathrm{st}}(\mathcal{A})\big)$ and to every continuous unital *-homomorphism M between two locally convex *-algebras $\mathcal{A}$ and $\mathcal{B}$ the morphism of abstract O^*-algebras $M\colon \big(\mathcal{A}, \mathcal{T}_{\mathrm{st}}(\mathcal{A})\big) \to \big(\mathcal{B}, \mathcal{T}_{\mathrm{st}}(\mathcal{B})\big)$.

Some general properties of these abstract O^*-algebras are:

Proposition 3.3.19 *Let $\mathcal{A}$ be a locally convex *-algebra, then $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ is regular and $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ is hyper-regular (hence also regular and downwards closed).*

Moreover, if $\mathcal{A}$ is even a Fréchet *-algebra, i.e. if its underlying locally convex space is a Fréchet space (see Appendix A.1.6), then its multiplication as well as every algebraically positive linear functional on $\mathcal{A}$ is automatically continuous. Consequently, $\mathcal{T}^+_{\mathrm{wk,H}} = \mathcal{T}^+_{\mathrm{st,H}} = \mathcal{A}^{*,++}_{\mathrm{H}}$ and $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}}) = (\mathcal{A}, \mathcal{T}_{\mathrm{st}}) = (\mathcal{A}, \mathcal{T})$ is a closed abstract O^*-algebra.

PROOF: If $\mathcal{A}$ is an arbitrary locally convex *-algebra, and $\omega \in \mathcal{T}_{\mathrm{wk}}$, then ω is a continuous linear functional. So if ω is also algebraically positive, then $\omega \in \mathcal{T}^+_{\mathrm{wk,H}}$. This shows that $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ is regular. Similarly, if $\omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ and if there exists a $\rho \in \mathcal{T}^+_{\mathrm{st,H}}$ such that $\rho - \omega$ is algebraically positive, then $\|a\|^2_{\omega,\mathrm{st}} = \langle\, \omega\,,\, a^*a\,\rangle \leq \langle\, \rho\,,\, a^*a\,\rangle = \|a\|^2_{\rho,\mathrm{st}}$ for all $a \in \mathcal{A}$, showing that $\|\cdot\|_{\omega,\mathrm{st}}$ is continuous on $\mathcal{A}$. So $\omega \in \mathcal{T}^+_{\mathrm{st,H}}$ and $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ is hyper-regular. By Proposition 3.1.10, $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ is also regular and downwards closed.

Moreover, if $\mathcal{A}$ is even a Fréchet *-algebra, then it follows from Proposition A.1.20 that the product on $\mathcal{A}$ is continuous, so $\mathcal{T}^+_{\mathrm{wk,H}} = \mathcal{T}^+_{\mathrm{st,H}}$ by Proposition 3.3.14. A theorem by Xia, later generalized by Ng and Warner [61] (see also [74, Thm. 3.6.1]) shows that every algebraically positive linear functional on $\mathcal{A}$ is continuous because $\mathcal{A}$ is a Fréchet-*-algebra, so $\mathcal{T}^+_{\mathrm{H}} = \mathcal{A}^{*,++}_{\mathrm{H}}$. As $\mathcal{A}^{*,++}_{\mathrm{H}}$ is weak-*-closed by Lemma 3.3.1, this shows that $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}}) = (\mathcal{A}, \mathcal{T}_{\mathrm{st}}) = (\mathcal{A}, \mathcal{T})$ is a closed abstract O^*-algebra. □

With respect to representations, the above shows that every unital-*-homomorphism from a Fréchet-*-algebra to a *-algebra of operators on a pre-Hilbert space is automatically weakly and strongly continuous, hence a continuous representation as operators.

As mentioned before, the representations as functions of a locally convex *-algebra do not require to distinguish between weakly and strongly continuous ones. Similarly, $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$ and $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$ have the same representations as functions, which are precisely the continuous ones of $\mathcal{A}$:

Theorem 3.3.20 *Let $\mathcal{A}$ be a locally convex *-algebra and (X, π) a tuple of a non-empty set X and a unital *-homomorphism $\pi\colon \mathcal{A} \to \mathbb{C}^X$. Then the following is equivalent:*

- *(X, π) is a representation as functions of the abstract O^*-algebra $(\mathcal{A}, \mathcal{T}_{\mathrm{st}})$.*
- *(X, π) is a representation as functions of the abstract O^*-algebra $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$.*
- *(X, π) is a continuous representation as functions of the locally convex *-algebra $\mathcal{A}$.*

PROOF: This is a direct consequence of Proposition 3.3.16 because on $\mathbb{C}^X$, the strong and the weak topology of the abstract O^*-algebra $\big(\mathbb{C}^X, \Delta(X)\big)$ both coincide with the topology of pointwise convergence:

This is immediately clear for the weak topology as $\Delta(X)$ is the linear span of all the evaluation functionals δ_x with $x \in X$. The strong topology is certainly stronger than the weak one. Conversely, for all $x \in X$ the identity $\|b\|_{\delta_x,\mathrm{st}} = \langle\, \delta_x\,,\, b^*b\,\rangle^{1/2} = |\langle\, \delta_x\,,\, b\,\rangle| = \|b\|_{\delta_x,\mathrm{wk}}$ for all $b \in \mathbb{C}^X$ shows that $\|\cdot\|_{\delta_x,\mathrm{st}} = \|\cdot\|_{\delta_x,\mathrm{wk}}$. As $\Delta(X)^+_{\mathrm{H}}$ is the convex cone generated by all the evaluation functionals δ_x with $x \in X$, these seminorms $\|\cdot\|_{\delta_x,\mathrm{st}}$ already define the strong topology on $\mathbb{C}^X$, which thus coincides with the weak one. □

For representations of operators, a similar result holds. But this time, we have to distinguish between weakly and strongly continuous representations:

Theorem 3.3.21 *Let $\mathcal{A}$ be a locally convex *-algebra and $(\mathcal{D}, \pi)$ a tuple of a pre-Hilbert space $\mathcal{D}$ and a unital *-homomorphism $\pi\colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$. Then $(\mathcal{D}, \pi)$ is a weakly continuous representation as operators of $\mathcal{A}$ if and only if $(\mathcal{D}, \pi)$ is a representation as operators of $(\mathcal{A}, \mathcal{T}_{\mathrm{wk}})$.*

PROOF: $(\mathcal{D},\pi)$ is a weakly continuous representation as operators of $\mathcal{A}$ if and only if π is continuous with respect to the locally convex topology on $\mathcal{L}^*(\mathcal{D})$ defined by the seminorms $\|\cdot\|_{\phi,\psi,\mathrm{wk}}$ for all $\phi,\psi\in\mathcal{D}$. From the polarization identity

$$\langle\,\phi\,|\,a(\psi)\,\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^{-k}\langle\,\phi+\mathrm{i}^k\psi\,|\,a(\phi+\mathrm{i}^k\psi)\,\rangle$$

it follows that the linear span of the set of linear functionals $\mathcal{L}^*(\mathcal{D})\ni a\mapsto\langle\,\phi\,|\,a(\psi)\,\rangle\in\mathbb{C}$ with $\phi,\psi\in\mathcal{D}$ coincides with the linear span of the set of linear functionals $\mathcal{L}^*(\mathcal{D})\ni a\mapsto\langle\,\phi\,|\,a(\phi)\,\rangle\in\mathbb{C}$ for all $\phi\in\mathcal{D}$, which is $\mathcal{X}(\mathcal{D})$. So $(\mathcal{D},\pi)$ is a weakly continuous representation as operators of $\mathcal{A}$ if and only if π is continuous with respect to the weak topology on $\mathcal{L}^*(\mathcal{D})$ coming from the abstract O^*-algebra $\big(\mathcal{L}^*(\mathcal{D}),\mathcal{X}(\mathcal{D})\big)$. Using Proposition 3.3.16, this is the case if and only if $(\mathcal{D},\pi)$ is a representation as operators of $(\mathcal{A},\mathcal{T}_{\mathrm{wk}})$. □

Theorem 3.3.22 *Let $\mathcal{A}$ be a locally convex *-algebra and $(\mathcal{D},\pi)$ a tuple of a pre-Hilbert space $\mathcal{D}$ and a unital *-homomorphism $\pi\colon\mathcal{A}\to\mathcal{L}^*(\mathcal{D})$. Then $(\mathcal{D},\pi)$ is a strongly continuous representation as operators of $\mathcal{A}$ if and only if $(\mathcal{D},\pi)$ is a representation as operators of $(\mathcal{A},\mathcal{T}_{\mathrm{st}})$.*

PROOF: By Proposition 3.3.16, $(\mathcal{D},\pi)$ is a representation as operators of $(\mathcal{A},\mathcal{T}_{\mathrm{st}})$ if and only if π is continuous with respect to the strong topology on $\mathcal{L}^*(\mathcal{D})$ coming from the abstract O^*-algebra $\big(\mathcal{L}^*(\mathcal{D}),\mathcal{X}(\mathcal{D})\big)$. As $\mathcal{X}(\mathcal{D})$ is the linear hull of all the vector functionals χ_ϕ with $\phi\in\mathcal{D}$, this strong topology is the locally convex topology on $\mathcal{L}^*(\mathcal{D})$ defined by the seminorms $\|\cdot\|_{\chi_\phi,\mathrm{st}}$ for all $\phi\in\mathcal{D}$, because every $\rho\in\mathcal{X}(\mathcal{D})_{\mathrm{H}}$ is a difference $\rho=\sum_{n=1}^{N}\chi_{\phi_n}-\sum_{n'=1}^{N'}\chi_{\phi'_{n'}}$ with $N,N'\in\mathbb{N}_0$ and $\phi_1,\dots,\phi_N,\phi'_1,\dots,\phi'_{N'}\in\mathcal{D}$, hence $\rho\le\sum_{n=1}^{N}\chi_{\phi_n}$ and $\|\cdot\|_{\rho,\mathrm{st}}\le\sum_{n=1}^{N}\|\cdot\|_{\chi_{\phi_n},\mathrm{st}}$. But as $\|b\|^2_{\chi_\phi,\mathrm{st}}=\langle\,\chi_\phi\,,\,b^*b\,\rangle=\langle\,b(\phi)\,|\,b(\phi)\,\rangle=\|b\|^2_{\phi,\mathrm{st}}$, for all $b\in\mathcal{A}$ and $\phi\in\mathcal{D}$, i.e. $\|\cdot\|_{\chi_\phi,\mathrm{st}}=\|\cdot\|_{\phi,\mathrm{st}}$, it follows that $(\mathcal{D},\pi)$ is a representation as operators of $(\mathcal{A},\mathcal{T}_{\mathrm{st}})$ if and only if $(\mathcal{D},\pi)$ is a strongly continuous representation of $\mathcal{A}$. □

This shows that the representation theory (as functions or as operators) of locally convex *-algebras can be studied through the representation theory of their corresponding abstract O^*-algebras. As an easy example, the result of Proposition 3.3.14 that $\mathcal{T}^+_{\mathrm{wk,H}}=\mathcal{T}^+_{\mathrm{st,H}}$ for locally convex *-algebras with continuous multiplication shows again that the weakly and strongly continuous representations as operators of such algebras coincide (which was shown directly in Proposition 3.3.9).

3.4 Representations by Essentially Self-Adjoint Operators

One of the problems from Section 2.6 was to give applicable sufficient conditions so that Hermitian elements of certain *-algebras can be represented by essentially self-adjoint operators (see also Appendix A.4). This can be achieved in multiple ways: If all representations are automatically by bounded operators, then the Hermitian ones are certainly essentially self-adjoint. However, this condition is much too restrictive for many applications. A less restrictive condition can be formulated by exploiting Nelson's criterium for essential self-adjointness (or a variant thereof).

We start with a well-known construction of C^*-seminorms on *-algebras, which is essentially the operator norm in a GNS representation or in a direct sum of GNS representations:

Lemma 3.4.1 *Let $\mathcal{A}$ be a *-algebra and S a non-empty set of algebraic states on $\mathcal{A}$ such that $a \triangleright \omega \in S$ for all $\omega \in S$ and all $a \in \mathcal{A}$ with $\langle \omega , a^* a \rangle = 1$. Define*

$$\|a\|_{S,\infty} := \sup_{\omega \in S} \sqrt{\langle \omega , a^* a \rangle} \in [0, \infty] \tag{3.4.1}$$

for all $a \in \mathcal{A}$. Then

$$\mathcal{B}(\mathcal{A}, S) := \left\{ a \in \mathcal{A} \;\middle|\; \|a\|_{S,\infty} < \infty \right\} \tag{3.4.2}$$

is a unital *-subalgebra of $\mathcal{A}$ and $\|\cdot\|_{S,\infty}$ is a C^*-seminorm on $\mathcal{B}(\mathcal{A}, S)$.

Proof: A proof is given for convenience of the reader: As $\|\cdot\|_{S,\infty} = \sup_{\omega \in S} \|\cdot\|_{\omega,\mathrm{st}}$ (with the point-wise supremum), it is not hard to see that $\mathcal{B}(\mathcal{A}, S)$ is indeed a linear subspace of $\mathcal{A}$ on which $\|\cdot\|_{S,\infty}$ is a seminorm. It is also clear that $\|\mathbb{1}\|_{S,\infty} = 1$ and thus $\mathbb{1} \in \mathcal{B}(\mathcal{A}, S)$.

Next let $a, b \in \mathcal{B}(\mathcal{A}, S)$ be given. If $\langle \omega , b^* b \rangle = 0$ then $\langle \omega , b \rangle = \mathrm{Var}_\omega(b) = 0$ by the Cauchy Schwarz inequality, so $\langle \omega , b^* a^* a\, b \rangle^{1/2} = \langle \omega , b^* a^* a \rangle^{1/2} \langle \omega , b \rangle^{1/2} = 0 \le \|a\|_{S,\infty} \|b\|_{S,\infty}$ due to (1.2.9). Otherwise

$$\langle \omega , b^* a^* a\, b \rangle^{1/2} = \langle \omega' , a^* a \rangle^{1/2} \langle \omega , b^* b \rangle^{1/2} \le \|a\|_{S,\infty} \|b\|_{S,\infty}$$

with $\omega' = \big(b / \langle \omega , b^* b \rangle^{1/2}\big) \triangleright \omega \in S$. This shows that $\|ab\|_{S,\infty} \le \|a\|_{S,\infty} \|b\|_{S,\infty}$ and we conclude that $\mathcal{B}(\mathcal{A}, S)$ is closed under multiplication and that $\|\cdot\|_{S,\infty}$ is submultiplicative on it.

Now it only remains to show that $\|a^*\|_{S,\infty} = \|a\|_{S,\infty}$ and $\|a^* a\|_{S,\infty} = \|a\|_{S,\infty}^2$ for all $a \in \mathcal{B}(\mathcal{A}, S)$. The estimate

$$\|a\|_{S,\infty}^2 = \sup_{\omega \in S} \langle \omega , a^* a \rangle \overset{\mathrm{CS}}{\le} \sup_{\omega \in S} \langle \omega , a^* a\, a^* a \rangle^{1/2} = \|a^* a\|_{S,\infty} \le \|a^*\|_{S,\infty} \|a\|_{S,\infty}$$

holds for all $a \in \mathcal{A}$, which first of all shows $\|a\|_{S,\infty} \le \|a^*\|_{S,\infty}$, hence also $\|a^*\|_{S,\infty} \le \|a\|_{S,\infty}$ for all $a \in \mathcal{A}$ because $\cdot^*$ is involutive. But then the above estimate also yields $\|a\|_{S,\infty}^2 \le \|a^* a\|_{S,\infty} \le \|a\|_{S,\infty}^2$, hence $\|a^* a\|_{S,\infty} = \|a\|_{S,\infty}^2$ for all $a \in \mathcal{A}$. □

The following is to some extend the converse construction:

Lemma 3.4.2 *Let $\mathcal{A}$ be a *-algebra and $\|\cdot\|$ a submultiplicative *-seminorm on $\mathcal{A}$. Construct S as the set of all algebraic states ω on $\mathcal{A}$ which are continuous with respect to $\|\cdot\|$, i.e. for which there exists a $C \in \mathbb{R}^+$ such that $|\langle \omega , a \rangle| \le C\|a\|$ holds for all $a \in A$. Then even $|\langle \omega , a \rangle| \le \|a\|$ holds for all $a \in A$ and $\omega \in S$. Moreover, if $\omega \in S$ and $a \in \mathcal{A}$ fulfil $\langle \omega , a^* a \rangle = 1$, then also $a \triangleright \omega \in S$.*

Proof: If $\omega \in S$ and $C \in \mathbb{R}^+$ are such that $|\langle \omega , a \rangle| \le C\|a\|$ for all $a \in \mathcal{A}$, then

$$|\langle \omega , a \rangle|^2 \le \langle \omega , a^* a \rangle \le \langle \omega , (a^* a)^n \rangle^{\frac{1}{n}} \le \big(C \|(a^* a)^n\|\big)^{\frac{1}{n}} \le C^{\frac{1}{n}} \|a^* a\| \le C^{\frac{1}{n}} \|a\|^2$$

holds for all $n \in \mathbb{N}$ by Lemma 3.1.17. In the limit $n \to \infty$, this shows that $|\langle \omega , a \rangle| \le \|a\|$. Moreover, if $a \in \mathcal{A}$ fulfils $\langle \omega , a^* a \rangle = 1$, then $a \triangleright \omega$ is an algebraically positive state, and it is continuous with respect to $\|\cdot\|$ because $|\langle a \triangleright \omega , b \rangle| = |\langle \omega , a^* b\, a \rangle| \le \|a^* b\, a\| \le \|a^*\| \|b\| \|a\|$ for all $b \in \mathcal{A}$. As a consequence, $a \triangleright \omega \in S$. □

3.4.1 Uniform Boundedness

Elements of a *-algebra on which such a seminorm $\|\cdot\|_{S,\infty}$ gives a finite result are in some way "bounded". The first example are the uniform bounded elements:

Definition 3.4.3 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra such that $\mathcal{S}(\mathcal{A}, \Omega) \neq \emptyset$, then define the map $\|\cdot\|_\infty \colon \mathcal{A} \to [0, \infty]$ by*

$$\|a\|_\infty := \sup_{\omega \in \mathcal{S}(\mathcal{A},\Omega)} \sqrt{\langle\, \omega \,,\, a^*a \,\rangle} \tag{3.4.3}$$

for all $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ for which $\|a\|_\infty < \infty$ holds, will be called uniformly bounded*, and the set of all uniformly bounded elements of $\mathcal{A}$ will be denoted by $\mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega)$. If $\mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega) = \mathcal{A}$, then $(\mathcal{A}, \Omega)$ is said to be* uniformly bounded*.*

Proposition 3.4.4 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra such that $\mathcal{S}(\mathcal{A}, \Omega) \neq \emptyset$, then the subset $\mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega)$ of all uniformly bounded elements is a unital *-subalgebra of $\mathcal{A}$. The restriction of $\|\cdot\|_\infty$ is a C^*-seminorm on $\mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega)$ and it is a C^*-norm if and only if $(\mathcal{A}, \Omega)$ is additionally Hausdorff.*

PROOF: Given $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ and $a \in \mathcal{A}$ with $\langle\, \omega \,,\, a^*a \,\rangle = 0$, then $a \triangleright \omega \in \mathcal{S}(\mathcal{A}, \Omega)$ because Ω^+_{H} is stable under the monoid action $\triangleright$ of $\mathcal{A}$ on $\mathcal{A}^*$ and because $\langle\, a \triangleright \omega \,,\, \mathbb{1} \,\rangle = \langle\, \omega \,,\, a^*a \,\rangle = 1$. So we can apply Lemma 3.4.1, which shows that $\mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega)$ is a unital *-subalgebra of $\mathcal{A}$ on which $\|\cdot\|_\infty$ is a C^*-seminorm.

If $\|\cdot\|_\infty$ is even a C^*-norm, then for every $a \in \mathcal{A}\backslash\{0\}$ there exists an $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ with $\langle\, \omega \,,\, a^*a \,\rangle > 0$. By Lemma 3.1.3, this implies that $(\mathcal{A}, \Omega)$ is Hausdorff. Conversely, if $(\mathcal{A}, \Omega)$ is Hausdorff and $a \in \mathcal{B}_{\mathrm{unif}}(\mathcal{A}, \Omega)\backslash\{0\}$, then, again by Lemma 3.1.3, there exists an $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ with $0 < \langle\, \omega \,,\, a^*a \,\rangle$ and thus $0 < \|a\|_\infty$, so $\|\cdot\|_\infty$ is a C^*-norm. □

Proposition 3.4.5 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra such that $\mathcal{S}(\mathcal{A}, \Omega) \neq \emptyset$ and $a \in \mathcal{A}$, then a is uniformly bounded if and only if a yields a bounded operator in every representation as operators of $(\mathcal{A}, \Omega)$. In this case, the operator norm of $\pi(a)$ for every representation as operators $(\mathcal{D}, \pi)$ of $(\mathcal{A}, \Omega)$ fulfils the estimate $\|\pi(a)\| \leq \|a\|_\infty$.*

PROOF: Let $(\mathcal{D}, \pi)$ be a representation as operators of $(\mathcal{A}, \Omega)$ and $\phi \in \mathcal{D}$ with $\|\phi\| = 1$, then the estimate

$$\|\pi(a)(\phi)\| = \big\langle\, \phi \,\big|\, \pi(a^*a)(\phi) \,\big\rangle^{1/2} = \big\langle\, \pi^*(\chi_\phi) \,,\, a^*a \,\big\rangle^{1/2} \leq \|a\|_\infty$$

holds. Consequently, if a is uniformly bounded, then $\pi(a)$ is bounded in every representation as operators $(\mathcal{D}, \pi)$ of $(\mathcal{A}, \Omega)$ and fulfils $\|\pi(a)\| \leq \|a\|_\infty$.

Conversely, assume that $\|a\|_\infty = \infty$. Then for every $n \in \mathbb{N}$ there exists an $\omega_n \in \mathcal{S}(\mathcal{A}, \Omega)$ such that $\langle\, \omega_n \,,\, a^*a \,\rangle \geq n$. Construct the representation as operators $(\mathcal{D}_I, \pi_I)$ of $(\mathcal{A}, \Omega)$ like in Lemma 3.2.10 with $I := \{\, \omega_n \mid n \in \mathbb{N} \,\}$, then

$$\big\|\pi_I(a)\big([\mathbb{1}]_{\omega_n}\big)\big\| = \big\|\pi_{\mathrm{GNS},\omega_n}(a)\big([\mathbb{1}]_{\omega_n}\big)\big\| = \big\langle\, [\mathbb{1}]_{\omega_n} \,\big|\, \pi_{\mathrm{GNS},\omega_n}(a^*a)\big([\mathbb{1}]_{\omega_n}\big) \,\big\rangle^{1/2} = \langle\, \omega_n \,,\, a^*a \,\rangle \geq n$$

for all $n \in \mathbb{N}$ even though $\|[\mathbb{1}]_{\omega_n}\| = \langle\, [\mathbb{1}]_{\omega_n} \,|\, [\mathbb{1}]_{\omega_n} \,\rangle^{1/2} = \langle\, \omega_n \,,\, \mathbb{1} \,\rangle = 1$. Because of this, $\pi_I(a)$ is unbounded. □

A uniformly bounded Hausdorff abstract O^*-algebra is, by Proposition 3.4.4, a normed *-algebra with C^*-norm $\|\cdot\|_\infty$. Conversely, we have:

Proposition 3.4.6 *Let $\mathcal{A}$ be a normed *-algebra with norm $\|\cdot\|$ and such that $\mathcal{S}(\mathcal{A},\mathcal{T}) \neq \emptyset$, then $(\mathcal{A},\mathcal{T})$ is a uniformly bounded abstract O^*-algebra and $\|\cdot\|_\infty \leq \|\cdot\|$ holds.*

PROOF: This is an immediate consequence of Lemma 3.4.2 as the set of states of $(\mathcal{A},\mathcal{T})$ is, by definition, the set of all $\|\cdot\|$-continuous algebraic states on $\mathcal{A}$. □

Note that $(\mathcal{A},\mathcal{T})$ need not be Hausdorff, see e.g. Section 3.7, as the existence of continuous positive linear functionals is not guaranteed. However, by using deep results from the theory of C^*-algebras, one can show:

Proposition 3.4.7 *Let $\mathcal{A}$ be a normed *-algebra with norm $\|\cdot\|$. Then $\mathcal{S}(\mathcal{A},\mathcal{T}) \neq \emptyset$ and $\|\cdot\|_\infty = \|\cdot\|$ with $\|\cdot\|_\infty$ like in the previous Proposition 3.4.6 hold if and only if $\|\cdot\|$ is a C^*-norm.*

PROOF: As $\|\cdot\|_\infty$ is a C^*-norm by Lemma 3.4.1, the condition that $\|\cdot\|$ be a C^*-norm is certainly necessary. However, it is also sufficient, because Lemma 1.2.17 shows that for every $a \in \mathcal{A}$ there exists an $\omega \in \mathcal{S}(\mathcal{A},\mathcal{T})$ such that $\langle\, \omega\,,\, a^*a\,\rangle = \|a^*a\| = \|a\|^2$, hence $\|a\|_\infty \geq \langle\, \omega\,,\, a^*a\,\rangle^{1/2} = \|a\|$. The converse $\|a\|_\infty \leq \|a\|$ has been shown in the previous Proposition 3.4.6. □

So in the special case that $\mathcal{A}$ is a C^*-algebra, the abstract O^*-algebra $(\mathcal{A},\mathcal{T})$ is always Hausdorff. The standard example here is the following:

Example 3.4.8 *Let $\mathcal{A} = \mathscr{C}\big([-1,1],\mathbb{C}\big)$ be the C^*-algebra of all continuous functions from $[-1,1]$ to $\mathbb{C}$ with the pointwise operations and the usual C^*-norm $\|a\| := \sup_{x\in[-1,1]}|a(x)|$ for $a \in \mathcal{A}$. Then $(\mathcal{A},\mathcal{T})$ is a uniformly bounded Hausdorff abstract O^*-algebra and, by the Riesz-Markov representation theorem, $\mathcal{T}$ is the set of all linear functionals on $\mathcal{A}$ that can be described by integration over a (complex) Radon measure on $[-1,1]$ and $\mathcal{T}_{\mathrm{H}}^+$ correponds to the positive Radon measures.*

3.4.2 Boundedness

We have seen that there is a close relation between uniform boundedness of abstract O^*-algebras and normed- or C^*-algebras. The next class of bounded abstract O^*-algebras will be related to lmc- and pro-C^*-algebras:

Definition 3.4.9 *[76, Def. 4.1] Let $\mathcal{A}$ be a *-algebra and ω an algebraic state on $\mathcal{A}$, then define the map $\|\cdot\|_{\omega,\infty} \colon \mathcal{A} \to [0,\infty]$ as*

$$a \mapsto \|a\|_{\omega,\infty} := \sup_{b\in\mathcal{A},\langle\, \omega\,,\, b^*b\,\rangle=1} \sqrt{\langle\, b \triangleright \omega\,,\, a^*a\,\rangle} \in [0,\infty]\,.$$

Moreover, given $a \in \mathcal{A}$, then ω is said to be bounded *on a if $\|a\|_{\omega,\infty} < \infty$. Now let $(\mathcal{A},\Omega)$ be an abstract O^*-algebra, then $a \in \mathcal{A}$ is said to be* bounded *if all $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ are bounded on a. The set of all $a \in \mathcal{A}$ on which ω is bounded will be denoted by $\mathcal{B}_\omega(\mathcal{A},\Omega)$ and the set of all bounded elements by $\mathcal{B}(\mathcal{A},\Omega) := \bigcap_{\omega\in\mathcal{S}(\mathcal{A},\Omega)} \mathcal{B}_\omega(\mathcal{A},\Omega)$. If $\mathcal{S}(\mathcal{A},\Omega)$ is empty, then this is understood as $\mathcal{B}(\mathcal{A},\Omega) := \mathcal{A}$. If all $a \in \mathcal{A}$ are bounded, then $(\mathcal{A},\Omega)$ is called a* bounded *abstract O^*-algebra.*

Analogous to Proposition 3.4.4 we get:

Proposition 3.4.10 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$, then the subset $\mathcal{B}_\omega(\mathcal{A}, \Omega)$ of all elements in $\mathcal{A}$ for which ω is bounded is a unital *-subalgebra of $\mathcal{A}$ and the restriction of $\|\cdot\|_{\omega,\infty}$ is a C^*-seminorm on $\mathcal{B}_\omega(\mathcal{A}, \Omega)$.*

Similarly, the subset $\mathcal{B}(\mathcal{A}, \Omega)$ of all bounded elements in $\mathcal{A}$ is also a unital *-subalgebra of $\mathcal{A}$ on which the restrictions of all $\|\cdot\|_{\omega,\infty}$ for all $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ are C^*-seminorms. Moreover, $\mathcal{B}(\mathcal{A}, \Omega)$ with the locally convex topology of these C^*-seminorms is Hausdorff if and only if $(\mathcal{A}, \Omega)$ is Hausdorff as an abstract O^*-algebra.

PROOF: The set $S_\omega := \{\, b \triangleright \omega \mid b \in \mathcal{A} \text{ with } \langle\, \omega\,,\, b^*b \,\rangle = 1 \,\}$ is non-empty as it contains $\omega = \mathbb{1} \triangleright \omega$, and given some $b \triangleright \omega \in S_\omega$ and $c \in \mathcal{A}$ such that $\langle\, b \triangleright \omega\,,\, c^*c \,\rangle = 1$, then $\langle\, c \triangleright b \triangleright \omega\,,\, \mathbb{1} \,\rangle = 1$ and thus $c \triangleright b \triangleright \omega \in S_\omega$. So Lemma 3.4.1 can be applied with $S = S_\omega$ and shows that $\mathcal{B}_\omega(\mathcal{A}, \Omega)$ is a unital *-subalgebra of $\mathcal{A}$ and that the restriction of $\|\cdot\|_{\omega,\infty}$ to $\mathcal{B}_\omega(\mathcal{A}, \Omega)$ is a C^*-seminorm.

Moreover, $\mathcal{B}(\mathcal{A}, \Omega) = \bigcap_{\omega \in \mathcal{S}(\mathcal{A}, \Omega)} \mathcal{B}_\omega(\mathcal{A}, \Omega)$ is clearly a unital *-subalgebra of $\mathcal{A}$ and we have already seen that all $\|\cdot\|_{\omega,\infty}$ with $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ are C^*-seminorms on *-subalgebras of $\mathcal{A}$ where they remain finite. Now $\mathcal{B}(\mathcal{A}, \Omega)$ with the locally convex topology of these C^*-seminorms is Hausdorff if and only if for every $a \in \mathcal{A}\backslash\{0\}$ there exist $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ and $b \in \mathcal{A}$ with $\langle\, \omega\,,\, b^*b \,\rangle = 1$ such that $\langle\, b \triangleright \omega\,,\, a^*a \,\rangle > 0$. As $b \triangleright \omega$ is again a state of $(\mathcal{A}, \Omega)$, this is equivalent to the existence of an $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ such that $\langle\, \omega\,,\, a^*a \,\rangle > 0$. Lemma 3.1.3 thus shows that $\mathcal{B}(\mathcal{A}, \Omega)$ with this topology is Hausdorff if and only if $(\mathcal{A}, \Omega)$ is Hausdorff as an abstract O^*-algebra. $\square$

Proposition 3.4.11 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, $a \in \mathcal{A}$ and $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$, then ω is bounded for a if and only if a yields a bounded operator in the GNS representation of $(\mathcal{A}, \Omega)$ associated to ω. More precisely, the operator norm of $\pi_{\mathrm{GNS},\omega}(a)$ on $\mathcal{D}_\omega$ fulfils $\|\pi_{\mathrm{GNS},\omega}(a)\| = \|a\|_{\omega,\infty}$.*

PROOF: This follows immediately from the definitions of the operator norm $\|\cdot\|$ and the seminorm $\|\cdot\|_{\omega,\infty}$ by using that $\big\langle\, \pi_{\mathrm{GNS},\omega}(a)\big([b]_\omega\big) \,\big|\, \pi_{\mathrm{GNS},\omega}(a)\big([b]_\omega\big) \,\big\rangle_\omega = \big\langle\, [ab]_\omega \,\big|\, [ab]_\omega \,\big\rangle_\omega = \langle\, b \triangleright \omega\,,\, a^*a \,\rangle$ for all $b \in \mathcal{A}$. $\square$

As a consequence we get:

Corollary 3.4.12 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $a \in \mathcal{A}$, then a is bounded if and only if a yields a bounded operator in all GNS representations of $(\mathcal{A}, \Omega)$ associated to all $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$.*

With respect to uniform boundedness we have the following relation:

Proposition 3.4.13 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $a \in \mathcal{A}$. If a is uniformly bounded, then a is bounded. Conversely, if a is bounded, then a is uniformly bounded if and only if the set of all $\|a\|_{\omega,\infty}$ with $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ is bounded from above. In this case, $\|a\|_\infty = \sup_{\omega \in \mathcal{S}(\mathcal{A}, \Omega)} \|a\|_{\omega,\infty}$.*

PROOF: We immediately see that

$$\|a\|_{\omega,\infty} = \sup_{b \in \mathcal{A},\, \langle\, \omega\,,\, b^*b \,\rangle = 1} \sqrt{\langle\, b \triangleright \omega\,,\, a^*a \,\rangle} \le \sup_{\rho \in \mathcal{S}(\mathcal{A}, \Omega)} \sqrt{\langle\, \rho\,,\, a^*a \,\rangle} = \|a\|_\infty$$

holds for all states ω because $b \triangleright \omega$ is again a state if $\langle\, \omega \,,\, b^*b \,\rangle = 1$. As a consequence of this, $\|a\|_\infty \geq \sup_{\omega\in\mathcal{S}(\mathcal{A},\Omega)} \|a\|_{\omega,\infty}$ and if a is uniformly bounded, then it is also bounded. Conversely, the estimate

$$\|a\|_\infty = \sup_{\omega\in\mathcal{S}(\mathcal{A},\Omega)} \sqrt{\langle\, \omega \,,\, a^*a \,\rangle} \leq \sup_{\omega\in\mathcal{S}(\mathcal{A},\Omega)} \|a\|_{\omega,\infty}$$

holds due to the very definition of $\|\cdot\|_{\omega,\infty}$, so $\|a\|_\infty = \sup_{\omega\in\mathcal{S}(\mathcal{A},\Omega)} \|a\|_{\omega,\infty}$. □

A bounded Hausdorff abstract O^*-algebra is, by Proposition 3.4.10, an lmc *-algebra with C^*-seminorms $\|\cdot\|_{\omega,\infty}$. Conversely:

Proposition 3.4.14 *Let $\mathcal{A}$ be an lmc *-algebra, then $(\mathcal{A},\mathcal{T})$ is a bounded abstract O^*-algebra, and the locally convex topology on $\mathcal{A}$ of all the C^*-seminorms $\|\cdot\|_{\omega,\infty}$ with $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ is weaker than the given one of $\mathcal{A}$.*

PROOF: This again is an immediate consequence of Lemma 3.4.2: Every $\omega \in \mathcal{S}(\mathcal{A},\mathcal{T})$ is, by definition of $\mathcal{T}$, continuous with respect to a continuous submultiplicative *-seminorm $\|\cdot\|$ on $\mathcal{A}$, hence $b \triangleright \omega$ is again continuous with respect to $\|\cdot\|$ for all $b \in \mathcal{A}$ with $\langle\, \omega \,,\, b^*b \,\rangle = 1$ and thus $\langle\, b \triangleright \omega \,,\, a^*a \,\rangle \leq \|a^*a\|$ and $\|a\|_{\omega,\infty} \leq \|a^*a\|^{1/2} \leq \|a\|$ for all $a \in \mathcal{A}$. □

Again, $(\mathcal{A},\mathcal{T})$ need not be Hausdorff in general, but, similar to Proposition 3.4.7, one can show that $(\mathcal{A},\mathcal{T})$ will be Hausdorff if $\mathcal{A}$ is an lmc-*-algebra whose topology can be defined by C^*-seminorms, i.e. a pro-C^*-algebra: In this case, if $a \in \mathcal{A}\backslash\{0\}$ is given, then there exists a C^*-seminorm $\|\cdot\|$ on $\mathcal{A}$ with $\|a\| > 0$ and thus Lemma 1.2.17 shows the existence of a continuous algebraic state ω with $\langle\, \omega \,,\, a \,\rangle \neq 0$.

One nice observation is that the seminorms $\|\cdot\|_{\omega,\infty}$ with $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ have an alternative description for commutative abstract O^*-algebras $(\mathcal{A},\Omega)$:

Proposition 3.4.15 *[76, Prop. 4.3] Let $(\mathcal{A},\Omega)$ be an abstract O^*-algebra, $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ and $a \in \mathcal{A}$ in the center of $\mathcal{A}$, i.e. such that $ab = ba$ for all $b \in \mathcal{A}$. Then*

$$\|a\|_{\omega,\infty} = \sup_{n\in\mathbb{N}} \big\langle\, \omega \,,\, (a^*a)^n \,\big\rangle^{\frac{1}{2n}} = \lim_{n\to\infty} \big\langle\, \omega \,,\, (a^*a)^n \,\big\rangle^{\frac{1}{2n}} \in [0,\infty]\,.$$

PROOF: The second identity is clear because $n \mapsto \langle\, \omega \,,\, (a^*a)^n \,\rangle^{1/(2n)}$ is a non-decreasing sequence by Lemma 3.1.17. Define the shorthand $\|a\|'_{\omega,\infty} := \sup_{n\in\mathbb{N}} \langle\, \omega \,,\, (a^*a)^n \,\rangle^{1/(2n)}$, then

$$\langle\, b \triangleright \omega \,,\, a^*a \,\rangle \leq \langle\, b \triangleright \omega \,,\, (a^*a)^m \,\rangle^{\frac{1}{m}} \overset{\text{CS}}{\leq} \langle\, \omega \,,\, (b^*b)^2 \,\rangle^{\frac{1}{2m}} \langle\, \omega \,,\, (a^*a)^{2m} \,\rangle^{\frac{1}{2m}} \leq \langle\, \omega \,,\, (b^*b)^2 \,\rangle^{\frac{1}{2m}} \big(\|a\|'_{\omega,\infty}\big)^2$$

holds for all $b \in \mathcal{A}$ with $\langle\, \omega \,,\, b^*b \,\rangle = 1$ and all $m \in \mathbb{N}$ by Lemma 3.1.17 again and by the Cauchy Schwarz inequality, thus $\|a\|_{\omega,\infty} \leq \|a\|'_{\omega,\infty}$. Conversely, if $\|a\|_{\omega,\infty} < \infty$, then $\langle\, \omega \,,\, (a^*a)^n \,\rangle \leq \|a\|^{2n}_{\omega,\infty}$ for all $n \in \mathbb{N}$, because $\|\cdot\|_{\omega,\infty}$ is a C^*-seminorm on $\mathcal{B}_\omega(\mathcal{A},\Omega)$ and because $|\langle\, \omega \,,\, c \,\rangle| \leq \langle\, \omega \,,\, c^*c \,\rangle^{1/2} \leq \|c\|_{\omega,\infty}$ for all $c \in \mathcal{B}_\omega(\mathcal{A},\Omega)$. This proves $\|a\|'_{\omega,\infty} \leq \|a\|_{\omega,\infty}$. □

A standard example here is the following:

Example 3.4.16 *Let $\mathcal{A} = \mathscr{C}(\mathbb{R},\mathbb{C})$ be the Fréchet pro-C^*-algebra of all continuous linear functions from $\mathbb{R}$ to $\mathbb{C}$ with the pointwise operations and the locally convex topology defined by the C^*-seminorms*

$\|a\|_K := \sup_{x\in K}|a(x)|$ *for* $a \in \mathcal{A}$ *and* $K \subseteq \mathbb{R}$ *compact. Then* $(\mathcal{A}, \mathcal{T})$ *is a bounded Hausdorff abstract* O^**-algebra and, by the Riesz Markov representation theorem,* $\mathcal{T}$ *is the set of all linear functionals on* $\mathcal{A}$ *that can be described by integration over a compactly supported (complex) Radon measure on* $\mathbb{R}$ *and* $\mathcal{T}^+_{\mathrm{H}}$ *correponds to the positive such Radon measures.*

3.4.3 Stieltjes States

As we will see in Proposition 4.0.1, the notions of uniformly bounded or bounded abstract O^*-algebras are too restrictive for some applications. A suitable generalization of this are positive elements for which all states fulfil the following condition, analogous to the notion of Stieltjes vectors, see e.g. [55] and the appendix, especially Definition A.4.40:

Definition 3.4.17 *[76, Def. 5.6] Let* $(\mathcal{A}, \Omega)$ *be an abstract* O^**-algebra,* $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ *and* $a \in \mathcal{A}^+_{\mathrm{H}}$*, then* ω *is said to be a* Stieltjes state for a *if*

$$\langle\, b \triangleright \omega \,,\, a \,\rangle = 0 \qquad or \qquad \sum_{n=1}^{\infty} \langle\, b \triangleright \omega \,,\, a^n \,\rangle^{-\frac{1}{2n}} = \infty$$

holds for all $b \in \mathcal{A}$ *with* $\langle\, \omega \,,\, b^* b \,\rangle = 1$*. If* $a \in \mathcal{A}^+_{\mathrm{H}}$ *and all* $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ *are Stieltjes states for* a*, then* a *is said to be a* Stieltjes element*.*

Note that Lemma 3.1.17 assures that in the above definition, $\langle\, b \triangleright \omega \,,\, a \,\rangle \neq 0$ implies that $\langle\, b \triangleright \omega \,,\, a^n \,\rangle > 0$ for all $n \in \mathbb{N}$ and that $n \mapsto \langle\, b \triangleright \omega \,,\, a^n \,\rangle^{-1/(2n)}$ is non-increasing. If ω is a bounded state for a, then either $\|a\|_{\omega,\infty} = 0$, in which case $\langle\, b \triangleright \omega \,,\, a \,\rangle = 0$, or $\langle\, b \triangleright \omega \,,\, a^n \,\rangle^{-1/(2n)} \overset{\mathrm{CS}}{\geq} \langle\, b \triangleright \omega \,,\, a^{2n} \,\rangle^{-1/(4n)} \geq \|a\|^{-1/2}_{\omega,\infty}$ for all $n \in \mathbb{N}$, so every bounded state for a is also a Stieltjes state. However, the notion of a Stieltjes state is much less restrictive. For example, if $\langle\, b \triangleright \omega \,,\, a^n \,\rangle \leq C_b (2n)^{2n}$ holds for all $n \in \mathbb{N}$ with an arbitrary $C_b \in [0, \infty[$, which may depend on b, then ω is a Stieltjes state for a.

The notion of Stieltjes states and Stieltjes elements will be interesting already in cases where an abstract O^*-algebra is generated as a unital *-algebra by Stieltjes elements (and not only if all positive algebra elements are Stieltjes elements). The importance of Stieltjes elements is that all their representations as operators have many Stieltjes vectors (see Definition A.4.40 in the appendix):

Proposition 3.4.18 *Let* $(\mathcal{A}, \Omega)$ *be an abstract* O^**-algebra and* $a \in \mathcal{A}^+_{\mathrm{H}}$*. Then* a *is a Stieltjes element if and only if for every representation as operators* $(\mathcal{D}, \pi)$ *of* $(\mathcal{A}, \Omega)$*, every vector* $\phi \in \mathcal{D}$ *is a Stieltjes vector of* $\pi(a)$*.*

PROOF: If a is a Stieltjes element, $(\mathcal{D}, \pi)$ a representation as operators of $(\mathcal{A}, \Omega)$ and $\phi \in \mathcal{D}$ with $\|\phi\| = 1$, then $\langle\, \phi \,|\, \pi(a)^n(\phi) \,\rangle = \langle\, \pi^*(\chi_\phi) \,,\, a^n \,\rangle$ for all $n \in \mathbb{N}$ with $\pi^*(\chi_\phi) \in \mathcal{S}(\mathcal{A}, \Omega)$. Lemma A.4.41 then shows that ϕ is a Stieltjes vector of $\pi(a)$ because either $\langle\, \phi \,|\, \pi(a)(\phi) \,\rangle = 0$ or

$$\sum_{n=1}^{\infty} \big\langle\, \phi \,\big|\, \pi(a)^n(\phi) \,\big\rangle^{-\frac{1}{2n}} = \sum_{n=1}^{\infty} \big\langle\, \pi^*(\chi_\phi) \,,\, a^n \,\big\rangle^{-\frac{1}{2n}} = \infty$$

hold. So all normalized vectors of $\mathcal{D}$ are Stieltjes vectors for $\pi(a)$, and it is then clear that all their scalar multiples are also Stieltjes vectors.

Conversely, if every vector $\phi \in \mathcal{D}$ for every representation as operators $(\mathcal{D}, \pi)$ of $(\mathcal{A}, \Omega)$ is a Stieltjes vector of $\pi(a)$, then a is a Stieltjes element: In fact, given $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$, then $\langle\, b \triangleright \omega\,,\, a^n \,\rangle = \langle\, [b]_\omega \,|\, \pi_{\mathrm{GNS},\omega}(a^n)\big([b]_\omega\big) \,\rangle_\omega$ for all $b \in \mathcal{A}$ with $\big(\mathcal{A}/\mathcal{G}_\omega, \pi_{\mathrm{GNS},\omega}\big)$ the GNS representation of $(\mathcal{A}, \Omega)$ associated to ω. Thus either $\langle\, b \triangleright \omega\,,\, a \,\rangle = 0$ or

$$\sum_{n=1}^{\infty} \langle\, b \triangleright \omega\,,\, a^n \,\rangle^{-\frac{1}{2n}} = \sum_{n=1}^{\infty} \langle\, [b]_\omega \,|\, \pi_{\mathrm{GNS},\omega}(a^n)\big([b]_\omega\big) \,\rangle_\omega^{-\frac{1}{2n}} = \infty$$

by Lemma A.4.41 again. □

From this and the results about Stieltjes vectors and essentially self-adjoint endomorphisms of pre-Hilbert spaces from Theorem A.4.45 and its Corollaries A.4.46 and A.4.47 in the appendix we get:

Corollary 3.4.19 *Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $(\mathcal{D}, \pi)$ a representation as operators of $(\mathcal{A}, \Omega)$.*

- *If $a \in \mathcal{A}_{\mathrm{H}}$ is semibounded, i.e. if there exists a $\lambda \in \mathbb{R}$ such that $a + \lambda\mathbb{1} \in \mathcal{A}_{\mathrm{H}}^{+}$ or $-a + \lambda\mathbb{1} \in \mathcal{A}_{\mathrm{H}}^{+}$, and if $a + \lambda\mathbb{1}$ or $a - \lambda\mathbb{1}$ is a Stieltjes element, respectively, then $\pi(a)$ is essentially self-adjoint.*
- *If $a \in \mathcal{A}_{\mathrm{H}}$ and a^2 is a Stieltjes element, then $\pi(a)$ is essentially self-adjoint.*
- *If $a \in \mathcal{A}$ and a^*a is a Stieltjes element, then $\mathcal{D}_{(\pi(a)^*)^{\mathrm{cl}}} = \mathcal{D}_{\pi(a)^\dagger}$.*

Similarly like before, certain locally convex *-algebras guarantee that at least some of their positive elements are Stieltjes elements:

Proposition 3.4.20 *Let $\mathcal{A}$ be a locally convex *-algebra, $a \in \mathcal{A}_{\mathrm{H}}^{+}$ and assume that there exists an upwards directed set of continuous seminorms $\mathcal{P}$ on $\mathcal{A}$ that defines the topology of $\mathcal{A}$ and such that every $\|\cdot\|_p \in \mathcal{P}$ fulfils $\|a^n\|_p = 0$ for one $n \in \mathbb{N}$ or $\sum_{n=1}^{\infty} \|a^n\|_p^{-1/(2n)} = \infty$, then all $\omega \in \mathcal{S}(\mathcal{A}, \mathcal{T})$ are Stieltjes states for a.*

PROOF: Let $\omega \in \mathcal{S}(\mathcal{A}, \mathcal{T})$ as well as $b \in \mathcal{A}$ be given. Then the continuity of ω and of the left and right multiplication with b imply that there exist $C \in [1, \infty[$ and $\|\cdot\|_p \in \mathcal{P}$ such that $|\langle\, b \triangleright \omega\,,\, c \,\rangle| \leq C\|c\|_p$ holds for all $c \in \mathcal{A}$. If $\|a^n\|_p = 0$ for one $n \in \mathbb{N}$, then $\langle\, b \triangleright \omega\,,\, a^n \,\rangle = 0$, hence $\langle\, b \triangleright \omega\,,\, a \,\rangle = 0$ by Lemma 3.1.17. Otherwise

$$\sum_{n=1}^{\infty} \langle\, b \triangleright \omega\,,\, a^n \,\rangle^{-\frac{1}{2n}} \geq \sum_{n=1}^{\infty} \big(C\|a^n\|_p\big)^{-\frac{1}{2n}} \geq \frac{1}{C} \sum_{n=1}^{\infty} \|a^n\|_p^{-\frac{1}{2n}} = \infty\,. \qquad □$$

Like for bounded states, the condition for being a Stieltjes state for an element a of an abstract O^*-algebra $(\mathcal{A}, \Omega)$ simplifies if a is in the center of $\mathcal{A}$:

Proposition 3.4.21 *[76, Prop. 5.7] Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ and $a \in \mathcal{A}_{\mathrm{H}}^{+}$ in the center of $\mathcal{A}$, i.e. $ab = ba$ for all $b \in \mathcal{A}$. Then ω is a Stieltjes state for a if and only if*

$$\langle\, \omega\,,\, a \,\rangle = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \langle\, \omega\,,\, a^n \,\rangle^{-\frac{1}{2n}} = \infty$$

holds.

PROOF: This is clearly necessary, but also sufficient: Given $b \in \mathcal{A}$ with $\langle \omega , b^*b \rangle = 1$, then $b \triangleright \omega$ is a state of $(\mathcal{A}, \Omega)$ and $\langle b \triangleright \omega , a \rangle^{1/2} \leq \langle b \triangleright \omega , a^n \rangle^{1/(2n)} \overset{\text{CS}}{\leq} \langle \omega , (b^*b)^2 \rangle^{1/(4n)} \langle \omega , a^{2n} \rangle^{1/(4n)}$ holds for all $n \in \mathbb{N}$ by Lemma 3.1.17 and the Cauchy Schwarz inequality. So either $\langle b \triangleright \omega , a \rangle = 0$ or

$$\sum_{n=1}^{\infty} \langle b \triangleright \omega , a^n \rangle^{-\frac{1}{2n}} \geq \sum_{n=1}^{\infty} \langle \omega , (b^*b)^2 \rangle^{-\frac{1}{4n}} \langle \omega , a^{2n} \rangle^{-\frac{1}{4n}} \geq \frac{\langle \omega , (b^*b)^2 \rangle^{-\frac{1}{4}}}{2} \sum_{n=2}^{\infty} \langle \omega , a^n \rangle^{-\frac{1}{2n}} = \infty$$

by using that $1 = \langle \omega , b^*b \rangle^2 \leq \langle \omega , (b^*b)^2 \rangle$ and that $\langle \omega , a^{2n} \rangle^{1/(4n)} \leq \langle \omega , a^{2n+1} \rangle^{1/(4n+2)}$ for all $n \in \mathbb{N}$ by Lemma 3.1.17 again. □

The example for this section should illustrate the following: An abstract O^*-algebra with many Stieltjes elements, which is not bounded, does not necessarily have elements which are in some obvious, algebraic sense "less bounded" than those of bounded abstract O^*-algebras. Admitting more positive linear functionals can be enough. Nevertheless, note that in the non-commutative case there can also arise purely algebraic obstructions to the boundedness of an abstract O^*-algebra (see Proposition 4.0.1).

Example 3.4.22 *Let $\mathcal{A}$ be the ordered *-algebra of all polynomial functions from $\mathbb{R}$ to $\mathbb{C}$ with the pointwise operations and the pointwise comparison. Using the fundamental theorem of algebra one sees that $\mathcal{A}^+_{\mathrm{H}} = \mathcal{A}^{++}_{\mathrm{H}}$ (see also Example 3.7.2 and note that this is famously not true for polynomials in more than one variable, see [45]). Let Ω^+_{H} be the cone of all algebraically positive linear functionals ω on $\mathcal{A}$ for which there exist $C, D \in [0, \infty[$ such that*

$$\big|\langle \omega , x^n \rangle\big| \leq C D^n n! \tag{3.4.4}$$

for all $n \in \mathbb{N}_0$, where $x \in \mathcal{A}$ is the polynomial function $x = \mathsf{id}_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R} \subseteq \mathbb{C}$. Then one can check that Ω^+_{H} is stable under the $\mathcal{A}$-monoid action $\triangleright$, which essentially follows from the observation that

$$\big|\langle x \triangleright \omega , x^n \rangle\big| = \big|\langle \omega , x^{n+2} \rangle\big| \leq C D^{n+2} (n+2)! = C D^{n+2} 2! \binom{n+2}{n} n! \leq (8CD^2)(2D)^n n!$$

for such $\omega \in \Omega^+_{\mathrm{H}}$ and all $n \in \mathbb{N}$. Moreover, all evaluation functionals on $\mathbb{R}$ are clearly in Ω^+_{H}, so Ω^+_{H} indeed yields the correct order on $\mathcal{A}$ and $(\mathcal{A}, \Omega)$ with $\Omega = \langle\!\langle \Omega^+_{\mathrm{H}} \rangle\!\rangle_{\mathrm{lin}}$ is a Hausdorff abstract O^*-algebra by Proposition 3.1.5. The positive linear functionals in Ω then are indeed Ω^+_{H} as defined above.

Note that $\mathcal{A}$ is a *-subalgebra of $\mathscr{C}(\mathbb{R}, \mathbb{C})$, which, by Example 3.4.16, can be turned into a bounded abstract O^*-algebra, and that the inclusion is a morphism of abstract O^*-algebras and an order embedding. But $(\mathcal{A}, \Omega)$ is not bounded itself: Consider for example the linear functional $\omega \in \mathcal{A}^*$,

$$\langle \omega , a \rangle := \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2} a(x) \,\mathrm{d}x\,, \tag{3.4.5}$$

then it is a standard exercise in analysis to show that $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$, but that ω is not bounded on $x = \mathsf{id}_{\mathbb{R}}$. By construction of Ω^+_{H}, however, all $a \in \mathcal{A}^+_{\mathrm{H}}$ with polynomial degree at most 2 are Stieltjes elements.

The Hamburger moment problem asks, under which conditions an algebraic state ω on $\mathcal{A}$ can be represented by integration over a positive measure on $\mathbb{R}$. Carleman gave a sufficient condition for the existence of a unique solution: In the language used here, this is the case if ω is a Stieltjes state of x^2 [2]. This result is closely related to the theory of self-adjoint (unbounded) operators on a Hilbert space. As a consequence, all $\omega \in \Omega^+_{\mathrm{H}}$ can be represented by integration over a unique measure on $\mathbb{R}$.

3.5 Characters and Pure States

This section gives the first truely non-trivial application of the concept of abstract O^*-algebras: Already in Section 2.6, the question about the relation between characters and pure states of certain commutative *-algebras was posed. One inclusion is not hard to prove:

Proposition 3.5.1 *[76, Prop. 3.3] On an abstract O^*-algebra $(\mathcal{A}, \Omega)$, every character is a pure state, i.e.*

$$\mathcal{M}(\mathcal{A}, \Omega) \subseteq \mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega)\,.$$

PROOF: For all $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{A}, \Omega)$, all $\lambda \in [0,1]$ and all $a \in \mathcal{A}$, one can check that the identity

$$\mathrm{Var}_{\lambda\rho_1+(1-\lambda)\rho_2}(a) = \lambda\,\mathrm{Var}_{\rho_1}(a) + (1-\lambda)\,\mathrm{Var}_{\rho_2}(a) + \lambda(1-\lambda)\big|\langle\,\rho_1 - \rho_2\,,\,a\,\rangle\big|^2$$

holds. So if $\omega \in \mathcal{S}(\mathcal{A}, \Omega)$ is even a character of $(\mathcal{A}, \Omega)$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{A}, \Omega)$ fulfil $\omega = \lambda\rho_1 + (1-\lambda)\rho_2$ with $\lambda \in \,]0,1[\,$, then $\mathrm{Var}_\omega(a) = 0$ and $\mathrm{Var}_{\rho_1}(a), \mathrm{Var}_{\rho_2}(a) \geq 0$ for all $a \in \mathcal{A}$ imply that $\big|\langle\,\rho_1 - \rho_2\,,\,a\,\rangle\big|^2 = 0$ for all $a \in \mathcal{A}$, hence $\omega = \rho_1 = \rho_2$. We conclude that ω is an extreme point of $\mathcal{S}(\mathcal{A}, \Omega)$. □

However, as we will see in the following in Theorem 3.5.20, the converse is also true under rather general circumstances. On the way we will also prove a sufficient condition for downwards closed abstract O^*-algebras to be hyper-regular in Proposition 3.5.19.

3.5.1 Preliminaries

The observation from (1.2.9), that vanishing variance of a state ω implies that ω is multiplicative, can even be strengthened:

Lemma 3.5.2 *[76, Lemma 3.4] If $\mathcal{A}$ is a *-algebra and ω an algebraic state on $\mathcal{A}$, then ω is multiplicative if and only if $\mathrm{Var}_\omega(a^2) = 0$ holds for all $a \in \mathcal{A}_{\mathrm{H}}$.*

PROOF: This condition is clearly necessary, but also sufficient: If $\mathrm{Var}_\omega(a^2) = 0$ for all $a \in \mathcal{A}_{\mathrm{H}}$, then also $\mathrm{Var}_\omega((a \pm \mathbb{1})^2) = 0$ for all $a \in \mathcal{A}_{\mathrm{H}}$, thus

$$4\langle\,\omega\,,\,a^2\,\rangle = \langle\,\omega\,,\,a(a{+}\mathbb{1})^2\,\rangle - \langle\,\omega\,,\,a(a{-}\mathbb{1})^2\,\rangle = \langle\,\omega\,,\,a\,\rangle\langle\,\omega\,,\,(a{+}\mathbb{1})^2\,\rangle - \langle\,\omega\,,\,a\,\rangle\langle\,\omega\,,\,(a{-}\mathbb{1})^2\,\rangle = 4\langle\,\omega\,,\,a\,\rangle^2$$

due to (1.2.9), which proves $\mathrm{Var}_\omega(a) = 0$ for all $a \in \mathcal{A}_{\mathrm{H}}$. As every element of $\mathcal{A}$ can be expressed as a linear combination of Hermitian elements, it follows from (1.2.9) again that ω is multiplicative. □

The essential property of pure states that we will have to exploit is the following:

Lemma 3.5.3 *[76, Lemma 3.5] If $(\mathcal{A}, \Omega)$ is an abstract O^*-algebra, $\omega \in \mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega)$ and $\rho \in \Omega^+_{\mathrm{H}}$ such that $\rho \leq \omega$, then $\rho = \langle\,\rho\,,\,\mathbb{1}\,\rangle\,\omega$.*

PROOF: If $\langle\,\rho\,,\,\mathbb{1}\,\rangle = 0$, then $\rho = 0$ due to the Cauchy Schwarz inequality, and $\rho = \langle\,\rho\,,\,\mathbb{1}\,\rangle\,\omega$ is trivial. If $\langle\,\rho\,,\,\mathbb{1}\,\rangle = 1$, then $\langle\,\omega - \rho\,,\,\mathbb{1}\,\rangle = 0$ together with $\omega - \rho \in \Omega^+_{\mathrm{H}}$ shows that $\omega = \rho$ and again $\rho = \langle\,\rho\,,\,\mathbb{1}\,\rangle\,\omega$ is trivial. Otherwise, let $\lambda \coloneqq \langle\,\rho\,,\,\mathbb{1}\,\rangle \in \,]0,1[\,$, then $\omega = \lambda\big(\rho/\lambda\big) + (1-\lambda)\big((\omega-\rho)/(1-\lambda)\big)$ with states ρ/λ and $(\omega-\rho)/(1-\lambda)$ implies $\omega = \rho/\lambda$. □

Proposition 3.5.4 *[76, Prop. 3.6] Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra, $\omega \in \mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega)$ and $\mathcal{B} \subseteq \mathcal{A}$ a unital *-subalgebra such that for every $b \in \mathcal{B}_{\mathrm{H}}$ there exists a $C_b \in [0, \infty[$ for which $b \triangleright \omega \leq C_b\, \omega$ holds. Then ω is multiplicative on $\mathcal{B}$.*

PROOF: Given $b \in \mathcal{B}_{\mathrm{H}}$ and a corresponding $C_b \in [0, \infty[$, then we can assume without loss of generality that $C_b > 0$, in which case it follows from the previous Lemma 3.5.3 that $C_b^{-1}\,(b \triangleright \omega) = C_b^{-1}\,\langle\, b \triangleright \omega\,,\, \mathbb{1}\,\rangle\, \omega$. Evaluating this on b^2 yields $\langle\, \omega\,,\, b^4\,\rangle = \langle\, \omega\,,\, b^2\,\rangle^2$ and thus $\mathrm{Var}_\omega(b^2) = 0$. By Lemma 3.5.2, ω is multiplicative on $\mathcal{B}$. □

It might be worth mentioning that for every commutative *-algebra $\mathcal{A}$ and every algebraic state ω on $\mathcal{A}$ there exists the largest unital *-subalgebra of $\mathcal{A}$ on which ω is multiplicative, namely the set $\{\, a \in \mathcal{A} \mid \mathrm{Var}_\omega(a) = 0\,\}$, which can be confirmed to be a unital subalgebra of $\mathcal{A}$ using (1.2.9) and is even a unital *-subalgebra because $\mathrm{Var}_\omega(a) = \mathrm{Var}_\omega(a^*)$ for all $a \in \mathcal{A}$ due to the commutativity of $\mathcal{A}$.

3.5.2 Symmetric Abstract O^*-algebras

One example where all pure states are characters are symmetric commutative abstract O^*-algebra:

Definition 3.5.5 *[76, Def. 3.7] An abstract O^*-algebra $(\mathcal{A}, \Omega)$ is called* symmetric*, if for every $a \in \mathcal{A}_{\mathrm{H}}$ there exists a multiplicative inverse of $\mathbb{1} + a^2$ in $\mathcal{A}$.*

Note that an abstract O^*-algebra $(\mathcal{A}, \Omega)$ is symmetric if and only if for every $a \in \mathcal{A}_{\mathrm{H}}$ and every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exists a multiplicative inverse of $a - \mathbb{1}\lambda$. So the above definition is completely analogous to the one of a symmetric *-algebra in [74, Chap. 1.4].

Theorem 3.5.6 *[76, Thm. 3.8] Let $(\mathcal{A}, \Omega)$ be a symmetric commutative abstract O^*-algebra, then $\mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega) = \mathcal{M}(\mathcal{A}, \Omega)$.*

PROOF: Proposition 3.5.1 already shows that $\mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega) \supseteq \mathcal{M}(\mathcal{A}, \Omega)$ and it remains to show that every pure state ω of $(\mathcal{A}, \Omega)$ is multiplicative. So let $a \in \mathcal{A}_{\mathrm{H}}$ be given and write $b := (\mathbb{1} + a^2)^{-1}$, then $b^* = b^*(\mathbb{1} + a^2)\, b = \big(b\,(\mathbb{1} + a^2)\big)^* b = b$. Assume $\langle\, \omega\,,\, b^2\,\rangle = 0$, then $|\langle\, \omega\,,\, b\,\rangle|^2 \overset{\mathrm{CS}}{\leq} \langle\, \omega\,,\, b^2\,\rangle = 0$ implies $\langle\, \omega\,,\, b\,\rangle = \mathrm{Var}_\omega(b) = 0$ and $1 = \langle\, \omega\,,\, (\mathbb{1} + a^2)\, b\,\rangle = \langle\, \omega\,,\, \mathbb{1} + a^2\,\rangle \langle\, \omega\,,\, b\,\rangle = 0$ by (1.2.9) yields a contradiction, so $\langle\, \omega\,,\, b^2\,\rangle > 0$.

Now observe that $b \triangleright \omega \leq \omega$ because $\langle\, b \triangleright \omega\,,\, c\,\rangle \leq \langle\, b \triangleright \omega\,,\, c + 2a^2c + a^4c\,\rangle = \langle\, \omega\,,\, c\,\rangle$ for all $c \in \mathcal{A}_{\mathrm{H}}^+$, hence $b \triangleright \omega = \langle\, \omega\,,\, b^2\,\rangle\, \omega$ by Lemma 3.5.3. It follows that $a \triangleright \omega = \langle\, \omega\,,\, b^2\,\rangle^{-1}\,(ab \triangleright \omega) \leq \langle\, \omega\,,\, b^2\,\rangle^{-1}\, \omega$, because $\langle\, ab \triangleright \omega\,,\, c\,\rangle \leq \langle\, b \triangleright \omega\,,\, c + 2a^2c + a^4c\,\rangle = \langle\, \omega\,,\, c\,\rangle$ holds for all $c \in \mathcal{A}_{\mathrm{H}}^+$. By Proposition 3.5.4, the state ω is multiplicative on $\mathcal{A}$. □

However, the assumption of a symmetric commutative abstract O^*-algebra is a rather strong one. In the following, similar theorems for more general classes of algebras will be proven.

3.5.3 Bounded Abstract O^*-Algebras

Another example where it is rather easy to prove that all pure states are characters is given by bounded abstract O^*–algebras.

Proposition 3.5.7 *[76, Prop. 4.7] Let $(\mathcal{A}, \Omega)$ be a regular commutative abstract O^*-algebra and $\omega \in \mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega)$, then the restriction of ω to $\mathcal{B}_\omega(\mathcal{A}, \Omega)$ is multiplicative.*

PROOF: This is essentially the argument given in [19]. In our case it is sufficient to check that the unital *-subalgebra $\mathcal{B}_\omega(\mathcal{A}, \Omega)$ of $\mathcal{A}$ fulfils the condition of Proposition 3.5.4, which is clear: For every $a \in \mathcal{B}_\omega(\mathcal{A}, \Omega)_{\mathrm{H}}$ the inequality $a \triangleright \omega \leq \|a\|^2_{\omega,\infty}\, \omega$ is fulfilled, because

$$\left\langle\, \|a\|^2_{\omega,\infty}\, \omega - (a \triangleright \omega) \,,\, b^* b \,\right\rangle = \|a\|^2_{\omega,\infty} \langle\, \omega \,,\, b^* b \,\rangle - \langle\, b \triangleright \omega \,,\, a^2 \,\rangle \geq 0$$

holds for all $b \in \mathcal{A}$ by definition of $\|\cdot\|_{\omega,\infty}$ and because $(\mathcal{A}, \Omega)$ was assumed to be regular. □

As an immediate consequence of the above Proposition 3.5.7 and Proposition 3.5.1 we get:

Theorem 3.5.8 *[76, Thm. 4.8] Let $(\mathcal{A}, \Omega)$ be a commutative regular abstract O^*-algebra. If $\mathcal{B}(\mathcal{A}, \Omega)$ is weakly dense in $\mathcal{A}$, then $\mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega) = \mathcal{M}(\mathcal{A}, \Omega)$.*

PROOF: It is only left to check that an $\omega \in \mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega)$ which is multiplicative on $\mathcal{B}(\mathcal{A}, \Omega)$, is also multiplicative on $\mathcal{A}$. This is true because $\mathcal{A} \ni a \mapsto \langle\, \omega \,,\, ba \,\rangle - \langle\, \omega \,,\, b \,\rangle \langle\, \omega \,,\, a \,\rangle = \langle\, \omega \cdot b \,,\, a \,\rangle - \langle\, \omega \,,\, b \,\rangle \langle\, \omega \,,\, a \,\rangle \in \mathbb{C}$ is weakly continuous for all $b \in \mathcal{A}$ and vanishes on $\mathcal{B}(\mathcal{A}, \Omega)$ by (1.2.9) because $\mathrm{Var}_\omega(a) = 0$ for all $a \in \mathcal{B}(\mathcal{A}, \Omega)$ as a consequence of the previous Proposition 3.5.7. □

Example 3.5.9 *[76, Expl. 4.9] Let $\mathcal{A}$ be a commutative lmc *-algebra, then $\mathcal{S}_{\mathrm{p}}(\mathcal{A}, \mathcal{T}) = \mathcal{M}(\mathcal{A}, \mathcal{T})$.*

PROOF: The above Theorem 3.5.8 applies to $(\mathcal{A}, \mathcal{T})$ because $(\mathcal{A}, \mathcal{T})$ is a regular abstract O^*-algebra by Proposition 3.3.19 and bounded by Proposition 3.4.14. □

3.5.4 Abstract O^*-Algebras with many Stieltjes Elements

In order to treat cases where no a priori boundedness assumptions can be made, we will have to assume that most algebra elements are at least somehow dominated by essentially self-adjoint ones:

Definition 3.5.10 *[76, Def. 5.1] Let $\mathcal{A}$ be a quasi-ordered *-algebra. An element $q \in \mathcal{A}^+_{\mathrm{H}}$ is called* coercive *if there exists an $\epsilon > 0$ such that $q \gtrsim \epsilon \mathbb{1}$. Let $Q \subseteq \mathcal{A}_{\mathrm{H}}$ be a non-empty set of pairwise commuting elements and such that q^2 is coercive and $\lambda q \in Q$ as well as $qr \in Q$ hold for all $q, r \in Q$ and all $\lambda \in [1, \infty[$; such a set will be called* dominant. *Then define*

$$Q^{\downarrow} := \left\{\, a \in \mathcal{A} \;\middle|\; \forall_{q \in Q} \exists_{r,s \in Q} : a^* q^2 a \lesssim r^2 \text{ and } a\, q^2 a^* \lesssim s^2 \,\right\}.$$

Note that this especially implies that for every $a \in Q^{\downarrow}$ there exists an $r \in Q$ such that $a^* a \lesssim r^2$ holds, and if $aq = qa$ and $aa^* = a^* a$ hold for all $q \in Q$, especially if $\mathcal{A}$ is commutative, then $a^* a \lesssim r^2$ is even sufficient for an $a \in \mathcal{A}$ to be in $Q^{\downarrow}$.

Lemma 3.5.11 *[76, Lemma 5.2] Let $\mathcal{A}$ be a quasi-ordered *-algebra and $q, r \in \mathcal{A}_{\mathrm{H}}$ commuting elements with the property that q^2 and r^2 are coercive. Then $\lambda q^2 r^2$ is coercive for all $\lambda \in \,]0, \infty[$ and there exists a $\lambda \in [1, \infty[$ such that $q^2 + r^2 \lesssim \lambda q^2 r^2$ holds.*

PROOF: Let $2 \geq \epsilon > 0$ be given such that $q^2 \gtrsim \epsilon\mathbb{1}$ and $r^2 \gtrsim \epsilon\mathbb{1}$, then

$$(2/\epsilon)\, q^2 r^2 = q\,(r^2/\epsilon - \mathbb{1})\, q + r\,(q^2/\epsilon - \mathbb{1})\, r + q^2 + r^2 \gtrsim q^2 + r^2$$

holds. So $q^2 + r^2 \lesssim \lambda q^2 r^2$ if one chooses $\lambda \coloneqq 2/\epsilon \geq 1$ and $\lambda q^2 r^2$ is coercive for all $\lambda \in \,]0,\infty[$ because $\lambda q^2 r^2 \gtrsim (\epsilon\lambda/2)\,(q^2 + r^2) \gtrsim \epsilon^2 \lambda \mathbb{1}$. □

Proposition 3.5.12 *[76, Prop. 5.3] Let $\mathcal{A}$ be a quasi-ordered *-algebra and $Q \subseteq \mathcal{A}_{\mathrm{H}}$ dominant, then $Q^{\downarrow}$ is a unital *-subalgebra of $\mathcal{A}$, even a quasi-ordered *-algebra with the order inherited from $\mathcal{A}$, and $Q \subseteq Q^{\downarrow}$.*

PROOF: It is immediately clear that $Q^{\downarrow}$ is stable under the *-involution and under multiplication with scalars and that $\mathbb{1} \in Q^{\downarrow}$. Given $a, b \in Q^{\downarrow}$ and $q \in Q$, then there exist $r, s, t \in Q$ such that $a^* q^2 a \lesssim r^2$ and $b^* q^2 b \lesssim s^2$ as well as $b^* r^2 b \lesssim t^2$ hold, so

$$(a+b)^* q^2 (a+b) \lesssim (a+b)^* q^2 (a+b) + (a-b)^* q^2 (a-b) = 2a^* q^2 a + 2b^* q^2 b \lesssim 2\,(r^2 + s^2) \lesssim 2\lambda r^2 s^2$$

with sufficiently large $\lambda \in [1,\infty[$ by the previous Lemma 3.5.11, and $(ab)^* q^2 (ab) = b^* a^* q^2 a\, b \lesssim b^* r^2 b \lesssim t^2$. Of course, there are similar estimates for a and b replaced by a^* and b^*, and thus $a + b \in Q^{\downarrow}$ and $ab \in Q^{\downarrow}$. This shows that $Q^{\downarrow}$ is a unital *-subalgebra of $\mathcal{A}$ and it is clear that it is even a quasi-ordered *-algebra with the order inherited from $\mathcal{A}$. Finally, $Q \subseteq Q^{\downarrow}$ is an immediate consequence of the closedness of Q under multiplication and its commutativity. □

In the special case of O^*-algebras, this dominated unital *-subalgebra $Q^{\downarrow}$ has a particularly easy interpretation as a *-algebra of continuous adjointable endomorphisms. Recall the definition of the graph topology of an O^*-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{D}$, Definition A.4.51 in the appendix. This topology on $\mathcal{D}^{\mathrm{cl}}$ is defined by the seminorms $\|\cdot\|_a$ for all $a \in \mathcal{A}^+_{\mathrm{H}}$, which, restricted to $\mathcal{D}$, are just $\|\phi\|_a = \langle\, \phi \,|\, a(\phi)\,\rangle^{1/2}$ for all $\phi \in \mathcal{D}$. Note that, because of this, two elements $a, b \in \mathcal{A}^+_{\mathrm{H}}$ fulfil $a \leq b$ if and only if $\|\cdot\|_a \leq \|\cdot\|_b$.

Proposition 3.5.13 *[76, Prop. 5.4] Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $Q \subseteq \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ dominant. Then $\{\, \|\cdot\|_{q^2} \mid q \in Q \,\}$ is a cofinal subset of the set of all seminorms on $\mathcal{D}$ that are continuous with respect to the graph topology of the O^*-algebra $Q^{\downarrow}$ on $\mathcal{D}$. Moreover, given $a \in \mathcal{L}^*(\mathcal{D})$, then $a \in Q^{\downarrow}$ holds if and only if a and a^* are both continuous with respect to this graph topology.*

PROOF: In this proof, all topological notions are with respect to the graph topology of $Q^{\downarrow}$ on $\mathcal{D}$:

As $Q \subseteq Q^{\downarrow}$ by the previous Proposition 3.5.12, it is clear that all $\|\cdot\|_{q^2}$ are continuous. Conversely, the set $\{\, \|\cdot\|_a \mid a \in (Q^{\downarrow})^+_{\mathrm{H}} \,\}$ defines the graph topology and is upwards directed and closed under multiplication with non-negative scalars because $(Q^{\downarrow})^+_{\mathrm{H}}$ is. As for all $a \in (Q^{\downarrow})^+_{\mathrm{H}}$ there exists a $q \in Q$ such that $a \leq (a + \mathbb{1})^2 \leq q^2$, hence $\|\cdot\|_a \leq \|\cdot\|_{q^2}$, this shows that $\{\, \|\cdot\|_{q^2} \mid q \in Q \,\}$ is a cofinal subset of the continuous seminorms on $\mathcal{D}$.

If $a \in Q^{\downarrow}$, then a is certainly continuous, because $\|a(\phi)\|_b = \|\phi\|_{a^*ba}$ holds for all $b \in (Q^{\downarrow})^+_{\mathrm{H}}$ and all $\phi \in \mathcal{D}$. It also follows that a^* is continuous because $a^* \in Q^{\downarrow}$ as well. Conversely, given an $a \in \mathcal{L}^*(\mathcal{D})$

such that a is continuous, then for every $q \in Q$ there exists an $r \in Q$ such that $\|a(\phi)\|_{q^2} \leq \|\phi\|_{r^2}$ holds for all $\phi \in \mathcal{D}$, hence $a^* q^2 a \leq r^2$. If a^* is continuous as well, then there also exists an $s \in Q$ such that $a\, q^2 a^* \leq s^2$ and we conclude that $a \in Q^{\downarrow}$. □

If $Q^{\downarrow} \subseteq \mathcal{L}^*(\mathcal{D})$ is a closed O^*-algebra and every q^2 with $q \in Q$ is essentially self-adjoint, then $Q^{\downarrow}$ is a strictly self-adjoint O^*-algebra, see Definition A.4.55 in the appendix, and is especially well-behaved:

Lemma 3.5.14 *[76, Lemma 5.5] Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $Q \subseteq \mathcal{L}^*(\mathcal{D})$ a dominant set with the properties that $Q^{\downarrow} \subseteq \mathcal{L}^*(\mathcal{D})$ is a closed O^*-algebra on $\mathcal{D}$ and that q^2 is essentially self-adjoint for every $q \in Q$. Then every $q \in Q$ has a bounded inverse $q^{-1} \in Q^{\downarrow}$.*

PROOF: For every $q \in Q$ the coercive essentially self-adjoint q^2 is injective and its image is dense in $\mathfrak{H}$ with respect to $\|\cdot\|$. Even more, for every $r \in Q$ the image of q^2 is dense in $\mathcal{D}$ with respect to $\|\cdot\|_{r^2}$: Let $\psi \in \mathcal{D}$ and $\epsilon > 0$ be given. Then r^2 is coercive and we can assume without loss of generality that even $r^2 \geq \mathbb{1}$, hence $r^4 = r\,(r^2 - \mathbb{1})\,r + r^2 \geq r^2$. Being coercive and essentially self-adjoint, $r^2 q^2$ has dense image in $\mathfrak{H}$ with respect to $\|\cdot\|$, and so there exists a $\phi \in \mathcal{D}$ such that $\big\|\big(r^2 q^2\big)(\phi) - r^2(\psi)\big\| \leq \epsilon$ holds, hence $\|q^2(\phi) - \psi\|_{r^2} \leq \|q^2(\phi) - \psi\|_{r^4} = \big\|\big(r^2 q^2\big)(\phi) - r^2(\psi)\big\| \leq \epsilon$.

As q^2 is coercive, injective and has dense image in $\mathfrak{H}$, it follows that q^2 has a bounded Hermitian (left-)inverse $B \in \mathcal{L}^*(\mathfrak{H})_{\mathrm{H}}$, which fulfils

$$\big\langle\, B(\phi) \,\big|\, r\big(q^2(\psi)\big) \,\big\rangle = \big\langle\, \phi \,\big|\, B\big(q^2(r(\psi))\big) \,\big\rangle = \langle\, \phi \,|\, r(\psi) \,\rangle = \langle\, r(\phi) \,|\, \psi \,\rangle = \big\langle\, B\big(r(\phi)\big) \,\big|\, q^2(\psi) \,\big\rangle$$

for all $\phi, \psi \in \mathcal{D}$ and all $r \in Q$, hence $\langle\, B(\phi) \,|\, r(\psi) \,\rangle = \big\langle\, B\big(r(\phi)\big) \,\big|\, \psi \,\big\rangle$ for all $\phi, \psi \in \mathcal{D}$ and all $r \in Q$ by using that the image of q^2 is dense in $\mathcal{D}$ with respect to $\|\cdot\|$ and $\|\cdot\|_{r^2}$. By Corollary A.4.59, B restricts to a left inverse $b \in \mathcal{L}^*(\mathcal{D})$ of q^2, which commutes with q^2 and therefore is also a right inverse. Moreover, b is also continuous with respect to the graph topology defined by $Q^{\downarrow}$ on $\mathcal{D}$, so $b \in Q^{\downarrow}$. Then $q^{-1} := qb \in Q^{\downarrow}$ is the inverse of q. □

Such strictly self-adjoint O^*-algebras can be constructed out of closures of representations of abstract O^*-algebras with many Stieltjes elements. The following definition of the closure of a representation as operators of an abstract O^*-algebra is completely analogous to the one of a closure of a representation of a general *-algebra, see e.g. [74, Def. 8.1.2]:

Definition 3.5.15 *[76, similar to Def. 2.5] Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $(\pi, \mathcal{D})$ a representation as operators of $(\mathcal{A}, \Omega)$. Moreover, let $\,\cdot^{\mathrm{cl}} \colon \pi(\mathcal{A}) \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}})$ be the closure of the O^*-algebra $\pi(\mathcal{A})$ on $\mathcal{D}$ like in Definition A.4.53. Then define the* closure of the representation $(\pi, \mathcal{D})$ *as the tuple $(\pi^{\mathrm{cl}}, \mathcal{D}^{\mathrm{cl}})$ with $\pi^{\mathrm{cl}} := \,\cdot^{\mathrm{cl}} \circ \pi \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}})$.*

Lemma 3.5.16 *[76, similar to Lemma 2.6] Let $(\mathcal{A}, \omega)$ be an abstract O^*-algebra, $(\pi, \mathcal{D})$ a representation as operators of $(\mathcal{A}, \omega)$ and $(\pi^{\mathrm{cl}}, \mathcal{D}^{\mathrm{cl}})$ its closure, then π^{cl} is a positive unital *-homomorphism.*

PROOF: The map π^{cl} is defined as a composition of two unital *-homomorphisms and thus is a unital *-homomorphism itself. It is also positive: Given $a \in \mathcal{A}^+_{\mathrm{H}}$, then

$$\big\langle\, \phi \,\big|\, \pi^{\mathrm{cl}}(a)(\phi) \,\big\rangle = \big\langle\, \phi \,\big|\, \pi(a)(\phi) \,\big\rangle = \langle\, \pi^*(\chi_\phi) \,,\, a \,\rangle \geq 0$$

for all $\phi \in \mathcal{D}$, and thus even $\langle \phi \,|\, \pi^{\mathrm{cl}}(a)(\phi) \rangle \geq 0$ for all $\phi \in \mathcal{D}^{\mathrm{cl}}$ because $\mathcal{D}$ is dense in $\mathcal{D}^{\mathrm{cl}}$ with respect to the graph topology of $\pi(\mathcal{A})$ and because $\mathcal{D} \ni \phi \mapsto \langle \phi \,|\, \pi^{\mathrm{cl}}(a)(\phi) \rangle \in \mathbb{R}$ is continuous in the graph topology as a composition of continuous maps. □

Note that, in general, π^{cl} is not a morphism of abstract O^*-algebras, hence $(\pi^{\mathrm{cl}}, \mathcal{D}^{\mathrm{cl}})$ is not a representation: Take for example a non-closed O^*-algebra $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$, then $\big(\mathcal{A}, \mathcal{X}(\mathcal{D})\big)$ is an abstract O^*-algebra, which can be represented as $\mathcal{A}$ on $\mathcal{D}$. But the closure of this representation is not described by a morphism of abstract O^*-algebras as $\chi_\phi \notin \mathcal{X}(\mathcal{D})^+_{\mathrm{H}}$ for $\phi \in \mathcal{D}^{\mathrm{cl}} \backslash \mathcal{D}$.

However, if $(\mathcal{A}, \Omega)$ is a closed abstract O^*-algebra, then the closure of every representation as operators of $(\mathcal{A}, \Omega)$ is again a representation due to Corollary 3.1.12.

Lemma 3.5.17 *[76, Lemma 5.8] Let $\mathcal{A}$ be a quasi-ordered *-algebra and $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ a non-empty set of coercive and pairwise commuting elements. Then*

$$Q \coloneqq \Big\{ \lambda \prod_{n=1}^{N} q'_n \;\Big|\; \lambda \in [1, \infty[\,;\; N \in \mathbb{N};\; q'_1, \dots, q'_N \in Q' \Big\}$$

is dominant and $Q' \subseteq Q$.

PROOF: If $q' \in \mathcal{A}^+_{\mathrm{H}}$ is coercive, then $q' \gtrsim \epsilon \mathbb{1}$ and thus $(q')^2 = (q' - \epsilon\mathbb{1})^2 + 2\epsilon\,(q' - \epsilon\mathbb{1}) + \epsilon^2 \mathbb{1} \gtrsim \epsilon^2 \mathbb{1}$ hold for some $\epsilon > 0$. So $(q')^2$ is also coercive. From Lemma 3.5.11 it now follows that q^2 is coercive for all $q \in Q$. As Q is closed under the multiplications and its elements are pairwise commuting by construction, Q is dominant. Finally, $Q' \subseteq Q$ is obvious. □

Proposition 3.5.18 *[76, similar to Cor. 5.11] Let $(\mathcal{A}, \Omega)$ be an abstract O^*-algebra and $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ a non-empty set of coercive and pairwise commuting Stieltjes elements. Construct the dominant set Q out of Q' like in Lemma 3.5.17 and assume that $Q^{\downarrow} = \mathcal{A}$. Moreover, let $(\pi, \mathcal{D})$ be a representation as operators of $(\mathcal{A}, \Omega)$ and $(\pi^{\mathrm{cl}}, \mathcal{D}^{\mathrm{cl}})$ its closure, then the set $\pi^{\mathrm{cl}}(Q)$ of operators on $\mathcal{D}^{\mathrm{cl}}$ fulfils the conditions of Lemma 3.5.14, i.e. $\pi^{\mathrm{cl}}(Q)$ is a dominant set with the property that $\pi^{\mathrm{cl}}(Q)^{\downarrow}$ is a closed O^*-algebra on $\mathcal{D}^{\mathrm{cl}}$ and $\pi^{\mathrm{cl}}(q)^2$ is essentially self-adjoint for every $\pi^{\mathrm{cl}}(q) \in \pi^{\mathrm{cl}}(Q)$. In addition, this also shows that $\pi^{\mathrm{cl}}(Q)^{\downarrow}$ and $\pi(\mathcal{A})$ are strictly self-adjoint O^*-algebras.*

PROOF: Lemma 3.5.16 shows that $\pi^{\mathrm{cl}} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}})$ is a positive unital *-homomorphism. Consequently, $\pi^{\mathrm{cl}}(Q)$ is again a dominant set and $\pi^{\mathrm{cl}}(Q^{\downarrow}) \subseteq \pi^{\mathrm{cl}}(Q)^{\downarrow}$. Moreover, $\pi^{\mathrm{cl}}(Q)^{\downarrow}$ is a closed O^*-algebra on $\mathcal{D}^{\mathrm{cl}}$ because $\pi^{\mathrm{cl}}(Q^{\downarrow}) = \pi^{\mathrm{cl}}(\mathcal{A})$ is a closed O^*-algebra by construction and because the graph topologies induced by $\pi^{\mathrm{cl}}(Q)^{\downarrow}$ and $\pi^{\mathrm{cl}}(Q^{\downarrow})$ can both be defined by the system of seminorms $\{ \|\cdot\|_{\pi^{\mathrm{cl}}(q^2)} \mid q \in Q \}$ (see Proposition 3.5.13) and thus coincide. Furthermore, $\pi^{\mathrm{cl}}(q)^2$ is essentially self-adjoint for every $\pi^{\mathrm{cl}}(q) \in \pi^{\mathrm{cl}}(Q)$:

All $\phi \in \mathcal{D}$ are Stieltjes vectors for all $\pi(q')$ with $q' \in Q'$ due to Proposition 3.4.18, so all $\pi(q)^2$ with $q \in Q$ are essentially self-adjoint by Proposition A.4.26 and Corollary A.4.49. It is then clear that the $\pi^{\mathrm{cl}}(q)^2$ with $q \in Q$ are also essentially self-adjoint as the closures of $\pi(q)^2$ and of $\pi^{\mathrm{cl}}(q)^2$ coincide, and $\pi^{\mathrm{cl}}(Q)^{\downarrow}$ as well as $\pi(\mathcal{A})$ are strictly self-adjoint O^*-algebras. □

Proposition 3.5.19 *[76, Prop. 5.15] Let $(\mathcal{A}, \Omega)$ be a downwards closed abstract O^*-algebra and assume that there exists a subset $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ of coercive and pairwise commuting Stieltjes elements for* *which* $(Q^{\downarrow})^+_{\mathrm{H}}$ *is strongly dense in* $\mathcal{A}^+_{\mathrm{H}}$*, where* $Q \subseteq \mathcal{A}_{\mathrm{H}}$ *is constructed out of* Q' *like in Lemma 3.5.17. Then* $(\mathcal{A}, \Omega)$ *is hyper-regular.*

PROOF: Given $\omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$ and $\rho \in \Omega^+_{\mathrm{H}}$ such that $\rho - \omega \in \mathcal{A}^{*,++}_{\mathrm{H}}$, then it is sufficient to show that $0 \leq \langle \omega\,,\, c \rangle$ and $0 \leq \langle \rho - \omega\,,\, c \rangle$ hold for all $c \in \mathcal{A}^+_{\mathrm{H}}$, because then $\omega \in \Omega^+_{\mathrm{H}}$ and $\rho - \omega \in \Omega^+_{\mathrm{H}}$ as $(\mathcal{A}, \Omega)$ is downwards closed.

Let $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ be a non-empty set of coercive and pairwise commuting Stieltjes elements and such that $(Q^{\downarrow})^+_{\mathrm{H}}$ is strongly dense in $\mathcal{A}^+_{\mathrm{H}}$, where Q is the dominant set constructed out of Q' like in Lemma 3.5.17. Consider the abstract O^*-algebra $(Q^{\downarrow}, \Omega)$, construct its GNS-representation $\big(\pi_\rho, \mathcal{D}_\rho\big)$ associated to ρ with $\mathcal{D}_\rho \coloneqq Q^{\downarrow}/\mathcal{G}_\rho$ and $\mathcal{G}_\rho$ the Gelfand ideal, as well as the completion $\mathfrak{H}_\rho$ of $\mathcal{D}_\rho$ to a Hilbert space and the closure $\big(\pi^{\mathrm{cl}}_\rho, \mathcal{D}^{\mathrm{cl}}_\rho\big)$ of the representation $\big(\pi_\rho, \mathcal{D}_\rho\big)$. Then $\pi^{\mathrm{cl}}_\rho(Q)^{\downarrow} \subseteq \mathcal{L}^*\big(\mathcal{D}^{\mathrm{cl}}_\rho\big)$ is a strictly self-adjoint O^*-algebra by Proposition 3.5.18.

Then $(\mathcal{D}_\rho)^2 \ni ([a]_\rho, [b]_\rho) \mapsto s([a]_\rho, [b]_\rho) \coloneqq \langle \omega\,,\, a^*b \rangle \in \mathbb{C}$ is a well-defined and bounded sesquilinear form: This is due to the observation that $0 \leq \langle \omega\,,\, a^*a \rangle \leq \langle \rho\,,\, a^*a \rangle = \big\|[a]_\rho\big\|^2_\rho$ holds for all $a \in \mathcal{A}$, and especially if $a \in \mathcal{G}_\rho$, then $\langle \omega\,,\, a^*b \rangle = \langle \omega\,,\, b^*a \rangle = 0$ for all $b \in \mathcal{A}$ by the Cauchy Schwarz inequality. Moreover,

$$s\Big([a]_\rho, \big(\pi^{\mathrm{cl}}_\rho(q)\big)([b]_\rho)\Big) = \langle \omega\,,\, a^*q\,b \rangle = \langle \omega\,,\, (q^*a)^*b \rangle = s\Big(\big(\pi^{\mathrm{cl}}_\rho(q)\big)([a]_\rho), [b]_\rho\Big)$$

holds for all $[a]_\rho, [b]_\rho \in \mathcal{D}_\rho$ and all $q \in Q$. As $\mathcal{D}_\rho$ is dense in $\mathcal{D}^{\mathrm{cl}}_\rho$ with respect to the graph topology of $\pi^{\mathrm{cl}}_\rho(Q)^{\downarrow}$ and all involved maps are continuous, s extends continuously to a bounded sesquilinear form s^{cl} on $\mathcal{D}^{\mathrm{cl}}_\rho$ fulfilling $s^{\mathrm{cl}}\big(\phi, \big(\pi^{\mathrm{cl}}_\rho(q)\big)(\psi)\big) = s^{\mathrm{cl}}\big(\big(\pi^{\mathrm{cl}}_\rho(q)\big)(\phi), \psi\big)$ for all $\phi, \psi \in \mathcal{D}^{\mathrm{cl}}_\rho$ and all $q \in Q$. By the generalized Lax-Milgram Theorem A.4.58 there exists a bounded $\hat{s} \in \mathcal{L}^*\big(\mathcal{D}^{\mathrm{cl}}_\rho\big)$ fulfilling $s^{\mathrm{cl}}(\phi, \psi) = \langle \hat{s}(\phi) \,|\, \psi \rangle_\rho$ for all $\phi, \psi \in \mathcal{D}^{\mathrm{cl}}_\rho$. Moreover, $\hat{s}$ is continuous in the graph topology and thus $\hat{s} \in \pi^{\mathrm{cl}}_\rho(Q)^{\downarrow}$, and $\hat{s}$ commutes with all $\pi^{\mathrm{cl}}(q)$ for $q \in Q$.

As $\hat{s}$ is also positive due to the positivity of s, Corollary A.4.60 allows to construct a positive $\sqrt{\hat{s}} \in \pi^{\mathrm{cl}}_\rho(Q)^{\downarrow}$ fulfilling $\sqrt{\hat{s}}^{\,2} = \hat{s}$ and still commuting with all $\pi^{\mathrm{cl}}(q)$ with $q \in Q$ as the limit in the operator norm of a sequence of polynomials of $\hat{s}$ in the usual way. This yields a new representation of ω as

$$\langle \omega\,,\, c \rangle = \Big\langle\, [\mathbb{1}]_\rho \,\Big|\, \hat{s}\big(\pi^{\mathrm{cl}}_\rho(c)([\mathbb{1}]_\rho)\big) \,\Big\rangle_\rho = \Big\langle\, \sqrt{\hat{s}}([\mathbb{1}]_\rho) \,\Big|\, \pi^{\mathrm{cl}}_\rho(c)\big(\sqrt{\hat{s}}([\mathbb{1}]_\rho)\big) \,\Big\rangle_\rho = \big\langle\, \xi \,\big|\, \big(\pi^{\mathrm{cl}}_\rho(c)\big)(\xi) \,\big\rangle_\rho$$

for all $c \in Q^{\downarrow}$ with $\xi \coloneqq \sqrt{\hat{s}}([\mathbb{1}]_\rho) \in \mathcal{D}^{\mathrm{cl}}_\rho$. From Lemma 3.5.16 it now follows that $\langle \omega\,,\, c \rangle \geq 0$ for all $c \in (Q^{\downarrow})^+_{\mathrm{H}}$. Moreover, $|\langle \omega\,,\, a \rangle| \leq \langle \omega\,,\, \mathbb{1} \rangle^{1/2} \langle \omega\,,\, a^*a \rangle^{1/2} \leq \langle \omega\,,\, \mathbb{1} \rangle^{1/2} \|a\|_{\rho,\mathrm{st}}$ holds for all $a \in \mathcal{A}$, so ω is strongly continuous. As $(Q^{\downarrow})^+_{\mathrm{H}}$ is strongly dense in $\mathcal{A}^+_{\mathrm{H}}$ by assumption, this implies that $\langle \omega\,,\, c \rangle \geq 0$ for all $c \in \mathcal{A}^+_{\mathrm{H}}$. Finally, note that $\omega' \coloneqq \rho - \omega$ also fulfils the condition that ω' and $\rho - \omega' = \omega$ are algebraically positive, so the above also shows that $\langle \rho - \omega\,,\, c \rangle = \langle \omega'\,,\, c \rangle \geq 0$ for all $c \in \mathcal{A}^+_{\mathrm{H}}$. □

Theorem 3.5.20 *[76, Thm. 5.17] Let* $(\mathcal{A}, \Omega)$ *be a commutative and downwards closed abstract* O^**-algebra and assume that there exists a subset* $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ *of coercive and pairwise commuting Stieltjes elements for which* $(Q^{\downarrow})^+_{\mathrm{H}}$ *is strongly dense in* $\mathcal{A}^+_{\mathrm{H}}$*, where* $Q \subseteq \mathcal{A}_{\mathrm{H}}$ *is constructed out of* Q' *like in Lemma 3.5.17. Then* $\mathcal{S}_{\mathrm{p}}(\mathcal{A}, \Omega) = \mathcal{M}(\mathcal{A}, \Omega)$*.*

PROOF: As $\mathcal{S}_{\mathrm{p}}(\mathcal{A},\Omega) \supseteq \mathcal{M}(\mathcal{A},\Omega)$ by Proposition 3.5.1, it only remains to show that every pure state ω of $(\mathcal{A},\Omega)$ is multiplicative. By assumption, there exists a non-empty set $Q' \subseteq \mathcal{A}^+_{\mathrm{H}}$ of coercive and pairwise commuting elements such that every $\omega \in \mathcal{S}(\mathcal{A},\Omega)$ is a Stieltjes state for every $q' \in Q'$ and such that $(Q^{\downarrow})^+_{\mathrm{H}}$ is strongly dense in $\mathcal{A}^+_{\mathrm{H}}$, where Q is the dominant set constructed out of Q' like in Lemma 3.5.17. Note that this implies that $Q^{\downarrow}$ is strongly dense in $\mathcal{A}$ because $\mathcal{A}$ is the linear span of $\mathcal{A}^+_{\mathrm{H}}$.

Let $\omega \in \mathcal{S}_{\mathrm{p}}(\mathcal{A},\Omega)$ be given, construct the abstract O^*-algebra $(Q^{\downarrow},\Omega)$ and let $\pi^{\mathrm{cl}}_\omega \colon Q^{\downarrow} \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}}_\omega)$ be its closed GNS representation associated to ω. Proposition 3.5.18 shows that Lemma 3.5.14 can be applied to the dominant set $\pi^{\mathrm{cl}}_\omega(Q) \subseteq \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}}_\omega)$, so there exists an inverse $\pi^{\mathrm{cl}}_\omega(q)^{-1} \in \pi^{\mathrm{cl}}_\omega(Q)^{\downarrow} \subseteq \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}}_\omega)$ of $\pi^{\mathrm{cl}}_\omega(q)$ for every $q \in Q$.

Let $a \in Q^{\downarrow}$ be given, then there exists a $q \in Q$ such that $\mathbb{1} + a^*a \lesssim q^2$ holds, so especially $b^*b = b^*q^2b - b^*(q^2 - \mathbb{1})\,b \lesssim b^*q^2b = qb^*bq$ for all $b \in \mathcal{A}$. Define the linear functional $\tilde{\rho}_q \colon Q^{\downarrow} \to \mathbb{C}$,

$$b \mapsto \langle\, \tilde{\rho}_q \,,\, b \,\rangle := \big\langle\, \pi^{\mathrm{cl}}_\omega(q)^{-1}[\mathbb{1}]_\omega \,\big|\, \pi^{\mathrm{cl}}_\omega(b)\big(\pi^{\mathrm{cl}}_\omega(q)^{-1}[\mathbb{1}]_\omega\big) \,\big\rangle_\omega ,$$

then $\langle\, \tilde{\rho}_q \,,\, b \,\rangle \geq 0$ for all $b \in (Q^{\downarrow})^+_{\mathrm{H}}$ and $\langle\, \tilde{\rho}_q \,,\, b^*b \,\rangle \leq \langle\, \tilde{\rho}_q \,,\, qb^*bq \,\rangle = \langle\, \omega \,,\, b^*b \,\rangle$ for all $b \in Q^{\downarrow}$ by Lemma 3.5.16.

Now $(Q^{\downarrow},\Omega)$ is again an abstract O^*-algebra in which all $q \in Q'$ are Stieltjes elements. It is even downwards-closed, as every positive linear functional on $Q^{\downarrow}$ dominated by a functional in Ω^+_{H} is strongly continuous and thus extends to a functional on $\mathcal{A}$, which is in Ω^+_{H} by using that $(Q^{\downarrow})^+_{\mathrm{H}}$ is strongly dense in $\mathcal{A}^+_{\mathrm{H}}$. So the previous Proposition 3.5.19 can be applied to $(Q^{\downarrow},\Omega)$ and shows that $\langle\, \tilde{\rho}_q \,,\, c \,\rangle \geq 0$ and $\langle\, \omega - \tilde{\rho}_q \,,\, c \,\rangle \geq 0$ for all $c \in (Q^{\downarrow})^+_{\mathrm{H}}$. Let ρ_q be the strongly continuous extension of $\tilde{\rho}_q$ to $\mathcal{A}$, then $\rho_q, \omega - \rho_q \in \Omega^+_{\mathrm{H}}$ and Lemma 3.5.3 shows that $\rho_q = \langle\, \rho_q \,,\, \mathbb{1} \,\rangle\, \omega$.

Moreover, $q \triangleright \rho_q \in \Omega^+_{\mathrm{H}}$ is also strongly continuous, and so it follows from $\langle\, q \triangleright \rho_q \,,\, b \,\rangle = \langle\, \omega \,,\, b \,\rangle$ for all $b \in Q^{\downarrow}$ that $q \triangleright \rho_q = \omega$. Consequently,

$$\langle\, \omega - (a \triangleright \rho_q) \,,\, b^*b \,\rangle = \langle\, \omega \,,\, b^*b \,\rangle - \langle\, \rho_q \,,\, b^*a^*a\,b \,\rangle \geq \langle\, \omega \,,\, b^*b \,\rangle - \langle\, \rho_q \,,\, b^*q^2b \,\rangle = \langle\, \omega \,,\, b^*b \,\rangle - \langle\, q \triangleright \rho_q \,,\, b^*b \,\rangle = 0$$

holds for all $b \in Q^{\downarrow}$ and shows that $a \triangleright \rho_q \leq \omega$ because $(Q^{\downarrow},\Omega)$ is hyper-regular by the previous Proposition 3.5.19, hence especially regular. But this yields $a \triangleright \omega = \langle\, \rho_q \,,\, \mathbb{1} \,\rangle^{-1}(a \triangleright \rho_q) \leq \langle\, \rho_q \,,\, \mathbb{1} \,\rangle^{-1}\omega$, where $\langle\, \rho_q \,,\, \mathbb{1} \,\rangle^{-1} > 0$ because $\rho_q \neq 0$ due to $q \triangleright \rho_q = \omega$, and then Proposition 3.5.4 shows that ω is multiplicative on $Q^{\downarrow}$.

Finally, ω is multiplicative on all of $\mathcal{A}$ because $Q^{\downarrow}$ is strongly dense in $\mathcal{A}$ and because $\mathrm{Var}_\omega \colon \mathcal{A} \to \mathbb{C}$ is strongly continuous and vanishes on $Q^{\downarrow}$, hence on whole $\mathcal{A}$. □

Note that the abstract O^*-algebra $(\mathcal{A},\Omega)$ of polynomial functions from $\mathbb{R}$ to $\mathbb{C}$ from Example 3.4.22 fulfils the conditions of the previous Theorem 3.5.20: As $\mathcal{A}$ is generated as a unital *-algebra by $x = \mathsf{id}_{\mathbb{R}}$ and as $4x = (x + \mathbb{1})^2 - (x - \mathbb{1})^2$ with $(x + \mathbb{1})^2$ and $(x - \mathbb{1})^2$ Stieltjes elements, even $\mathcal{A} = Q^{\downarrow}$ is fulfilled. As a consequence, all pure states in this example are characters and thus evaluation functionals on points of $\mathbb{R}$. This of course also follows from the possibility to represent all $\omega \in \Omega^+_{\mathrm{H}}$ by integrals over positive measures. A less trivial application of this theorem will be discussed in Corollary 4.2.30.

3.6 Application of Freudenthal's Spectral Theorem

Freudenthal's spectral theorem (see Theorem A.2.6 in the appendix) is a theorem about very well-behaved ordered vector spaces, namely Dedekind-σ-complete Riesz spaces with weak order unit $\mathbb{1}$, and essentially states that every positive element of such a space R is the supremum of an increasing sequence of linear combinations of components of the unit. Here, a component of the unit is an element $p \in R$ for which the infimum of p and $\mathbb{1} - p$ is 0.

Under certain circumstances, this theorem can be applied to (commutative) abstract O^*-algebras, or, more generally, ordered *-algebras. Of course, only very special ordered *-algebras $\mathcal{A}$ have the property that their Hermitian elements $\mathcal{A}_{\mathrm{H}}$ form a Dedekind-σ-complete Riesz space with weak order unit $\mathbb{1}$ (one class of examples are commutative W^*-algebras, which will be discussed as Example 3.6.3). Finding good sufficient conditions for a commutative abstract O^*-algebra to allow the application of Freudenthal's spectral theorem will be left open for future projects. For now, it shall be enough to discuss what the implications of this theorem are if it can be applied, i.e. what components of the unit are:

Proposition 3.6.1 *Given two projections $p = p^* = p^2$ and $q = q^* = q^2$ in a commutative ordered *-algebra $\mathcal{A}$, then pq is the infimum of p and q in $\mathcal{A}_{\mathrm{H}}$. Moreover, a Hermitian element $p \in \mathcal{A}_{\mathrm{H}}$ is a projector, i.e. $p = p^2$, if and only if 0 is the infimum of p and $\mathbb{1} - p$ in $\mathcal{A}_{\mathrm{H}}$.*

PROOF: Every projector $p \in \mathcal{A}_{\mathrm{H}}$ is positive because $0 \le p^2 = p$. As $\mathbb{1} - p$ is also a projector due to $(\mathbb{1} - p)^2 = \mathbb{1}^2 - 2p + p^2 = \mathbb{1} - p$, also $0 \le \mathbb{1} - p$, so $0 \le p \le \mathbb{1}$.

If $p, q \in \mathcal{A}_{\mathrm{H}}$ are commuting projectors, then $pq = qpq \le q\mathbb{1}q = q$ and $pq = pqp \le p\mathbb{1}p = p$ show that pq is a lower bound of p and q. It is even the greatest lower bound because every other lower bound $\ell \in \mathcal{A}_{\mathrm{H}}$ of p and q fulfils $\ell = \ell p + \ell(\mathbb{1} - p) = p\ell p + (\mathbb{1} - p)\ell(\mathbb{1} - p) \le pqp + (\mathbb{1} - p)p(\mathbb{1} - p) = pq$ by using the commutativity of $\mathcal{A}$.

Now assume that $p \in \mathcal{A}_{\mathrm{H}}$ is a projector, then $\mathbb{1} - p$ is also a projector and by the previous discussion, the infmum of p and $\mathbb{1} - p$ is $p(\mathbb{1} - p) = 0$. Conversely, if $p \in \mathcal{A}_{\mathrm{H}}$ has the property that 0 is the infimum of p and $\mathbb{1} - p$, then $0 \le \mathbb{1} - p$ shows that $p \le \mathbb{1}$, hence $p^3 \le p^2$. Moreover, $0 \le p$ implies $0 \le (\mathbb{1} - p)p(\mathbb{1} - p) = p - 2p^2 + p^3$ and shows that $2p^2 \le p + p^3$, hence $p^2 \le p$ by using $p^3 \le p^2$. Conversely, from $p - p^2 \le p$ and $p - p^2 = (\mathbb{1} - p) - (\mathbb{1} - p)^2 \le \mathbb{1} - p$ it follows that $p - p^2 \le 0$ as 0 was assumed to be the greatest lower bound of p and $\mathbb{1} - p$, so $p \le p^2$. All in all, this shows that $p = p^2$. □

Theorem 3.6.2 *Let $(\mathcal{A}, \Omega)$ be a commutative Hausdorff abstract O^*-algebra and such that $\mathcal{A}_{\mathrm{H}}$ is a Dedekind-σ-complete Riesz space, $\mathbb{1}$ a weak order unit and assume that the supremum $\sup_{n\in\mathbb{N}} a_n$ of every monotonely increasing and bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\mathrm{H}}$ is its weak limit. Write*

$$\mathscr{S}(\mathcal{A}) := \langle\!\langle \{ p \in \mathcal{A}_{\mathrm{H}} \mid p^2 = p \} \rangle\!\rangle_{\mathrm{lin}} \tag{3.6.1}$$

for the set of simple elements *$\mathcal{A}$. Then $\mathscr{S}(\mathcal{A})$ is a weakly dense *-subalgebra of $\mathcal{A}$, and, more precisely, for every $a \in \mathcal{A}_{\mathrm{H}}^+$ there exists an increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathcal{A})$ of simple and positive Hermitian elements, whose supremum and weak limit is a.*

PROOF: It is easy to check that the linear subspace of $\mathcal{A}$ generated by all projectors of $\mathcal{A}$ is closed under the *-involution (as every projector is Hermitian by definition) and closed under multiplication (as the product of two commuting projectors is again a projector). Applying Freudenthal's spectral theorem to $\mathcal{A}_\mathrm{H}$ shows that every $a \in \mathcal{A}_\mathrm{H}^+$ is the supremum of a countable increasing sequence $(s_n)_{n\in\mathbb{N}}$ of linear combinations of components of the unit. By the previous Proposition 3.6.1, these s_n are simple elements of $\mathcal{A}$ and by assumption, their supremum is their weak limit. As $\mathcal{A}_\mathrm{H}^+$ spans whole $\mathcal{A}$, we see that $\mathscr{S}(\mathcal{A})$ is weakly dense in $\mathcal{A}$. $\square$

Example 3.6.3 *A W^*-algebra is defined as a C^*-algebra $\mathcal{M}$, which – as a Banach space – is the dual of another Banach space $\mathcal{M}_*$, i.e. $\mathcal{M}$ is the vector space of all continuous linear functionals on $\mathcal{M}_*$ with norm*

$$\|a\| := \sup_{\omega\in\mathcal{M}_*,\|\omega\|=1} |\langle\, \omega\,,\, a\,\rangle| \tag{3.6.2}$$

for all $a \in \mathcal{M}$, where $\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathcal{M}_* \times \mathcal{M} \to \mathbb{C}$ is the dual pairing, see [73, Def. 1.1.2].

Besides the $\|\cdot\|$-topology, a W^*-algebra $\mathcal{M}$ thus also carries the weak topology induced on it by the predual $\mathcal{M}_*$, and this predual $\mathcal{M}_*$ can be embedded canonically in the dual $\mathcal{M}^*$ of $\mathcal{M}$ and be identified with the space of weakly-continuous linear functionals on $\mathcal{M}$.

Following the notation of [73], let $T \subseteq \mathcal{M}_*$ be the cone of algebraically positive linear functionals in $\mathcal{M}_*$. Then the linear hull of T is $\mathcal{M}_*$ by [73, Thms. 1.13.2 and 1.14.3]. Moreover, T is stable under the $\mathcal{M}$-monoid action $\triangleright$, as left- and right multiplication in $\mathcal{M}$ are weakly continuous by [73, Th. 1.7.8]. So $(\mathcal{M},\mathcal{M}_*)$ is an abstract O^*-algebra with the order on $\mathcal{M}_\mathrm{H}$ being the usual one on C^*-algebras, i.e. $\mathcal{M}_\mathrm{H}^+ = \mathcal{M}_\mathrm{H}^{++}$, which is indeed weakly closed by [73, Lemma 1.7.1].

One can now show that the supremum of every bounded, monotonely increasing directed set in $\mathcal{M}_\mathrm{H}$ exists and is the weak limit of this set [73, Lemma 1.7.4], so especially every bounded, monotonely increasing sequence in $\mathcal{M}_\mathrm{H}$ has a supremum which is the weak limit. Moreover, if $\mathcal{M}$ is commutative, then $\mathcal{M}_\mathrm{H}$ is, as a real algebra, isomorphic to the space of real-valued continuous functions on a compact topological Hausdorff space by [73, Thm. 1.2.1], hence is a Riesz space with weak order unit $\mathbb{1}$, and even Dedekind-σ-complete by the discussion before.

So the previous Theorem 3.6.2 can be applied to all commutative W^*-algebras, and essentially yields a version of the well-known spectral theorem for such algebras.

Of course, it would be interesting to extend this to more general, especially unbounded, commutative abstract O^*-algebras. This is left open for future projects. Note, however, that it is clear that there exist also unbounded examples: Take, e.g., the ordered *-algebra of measurable functions from some measurable space to $\mathbb{C}$ with the pointwise operations and pointwise comparison, or suitable *-subalgebras thereof. They can be turned into abstract O^*-algebras fulfilling the conditions of Theorem 3.6.2, e.g. like in Definition 3.2.1.

3.7 Examples and Counterexamples

The first example is a well-known *-algebra (even Banach-*-algebra), that does not admit any algebraically positive linear functionals. This shows that even the rather strong assumption of Banach-*-algebras does not exclude pathological cases with, e.g., no non-trivial representations.

Example 3.7.1 *Let $\mathbb{S}_1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the circle and $\mathcal{A} := \mathscr{C}(\mathbb{S}_1, \mathbb{C})$ the complex algebra of continuous complex-valued functions on $\mathbb{S}_1$ with the pointwise addition and multiplications. Then $\mathcal{A} \ni f \mapsto f^* \in \mathcal{A}$ with $f^*(z) := \overline{f(-z)}$ for all $z \in \mathbb{S}_1$ is a *-involution on $\mathcal{A}$ and together with this and the usual norm $\|f\|_\infty := \sup_{z \in \mathbb{S}_1} |f(z)|$, the algebra $\mathcal{A}$ becomes a Banach-*-algebra.*

However, there are no algebraically positive linear functionals on $\mathcal{A}$ besides the trivial constant-0-functional. The only abstract O^*-algebra one can construct over $\mathcal{A}$ thus is $(\mathcal{A}, \{0\})$, and the resulting order on $\mathcal{A}_{\mathrm{H}}$ is the one with $\mathcal{A}_{\mathrm{H}}^{+} = \mathcal{A}_{\mathrm{H}}^{++} = \mathcal{A}_{\mathrm{H}}$.

PROOF: It is clear that $\cdot^*$ is an antilinear involution on $\mathcal{A}$ and $(fg)^*(z) = \overline{f(-z)g(-z)} = (f^*g^*)(-z)$ for all $z \in \mathbb{S}_1$ together with the commutativity of $\mathcal{A}$ shows that $\mathcal{A}$ with $\cdot^*$ is a *-algebra. As $\mathbb{S}_1$ is compact, $\mathcal{A}$ is complete with respect to $\|\cdot\|_\infty$ and continuity of all algebraic operations is easy to check.

However, consider the function $\mathbb{S}_1 \ni z \mapsto e(z) := z \in \mathbb{C}$, which is clearly continuous, thus $e \in \mathcal{A}$, but $e^*e = -\mathbb{1}$ because $(e^*e)(z) = \overline{-z}\, z = -1$ for all $z \in \mathbb{S}_1$. As a consequence, if $\omega \in \mathcal{A}_{\mathrm{H}}^{*,++}$ then $0 \le \langle \omega \,,\, \mathbb{1} \rangle = -\langle \omega \,,\, -\mathbb{1} \rangle = -\langle \omega \,,\, e^*e \rangle \le 0$ and thus $\mathcal{A}_{\mathrm{H}}^{*,++} = \{0\}$. So there is no other abstract O^*-algebra over $\mathcal{A}$ than $(\mathcal{A}, \{0\})$, and every $a \in \mathcal{A}_{\mathrm{H}}$ is positive. Moreover, every $a \in \mathcal{A}_{\mathrm{H}}$ is even algebraically positive because

$$a = \left(\frac{\mathbb{1} + a}{2}\right)^2 - \left(\frac{\mathbb{1} - a}{2}\right)^2 = \left(\frac{\mathbb{1} + a}{2}\right)^2 + \left(e\,\frac{\mathbb{1} - a}{2}\right)^* \left(e\,\frac{\mathbb{1} - a}{2}\right). \qquad \square$$

The next example demonstrates that there can be interesting and relevant orders on *-algebras besides the one whose positive elements are the algebraically positive ones.

Example 3.7.2 *Let $\mathcal{A} = \mathbb{C}[x]$ be the *-algebra of polynomials in one variable x with the *-involution $\big(\sum_{n=0}^\infty a_n x^n\big)^* := \sum_{n=0}^\infty \overline{a_n} x^n$. Then $a \in \mathcal{A}_{\mathrm{H}}$ is algebraically positive if and only if it is pointwise positive on $\mathbb{R}$, i.e. if and only if $a(x) \ge 0$ for all $x \in \mathbb{R}$.*

However, due to the Stone-Weierstrass theorem, $\mathcal{A}$ is also dense in the C^*-algebra $\mathscr{C}\big([-1,1], \mathbb{C}\big)$ of continuous complex-valued functions on $[-1,1]$ with the pointwise operations, by assigning to every $a \in \mathcal{A}$ the function $[-1,1] \ni x \mapsto a(x) \in \mathbb{C}$. But with respect to the usual order on $\mathscr{C}\big([-1,1], \mathbb{C}\big)_{\mathrm{H}}$ by pointwise comparison on points of $[-1,1]$, the polynomial $1 - x^2 \in \mathcal{A}$ is positive, even though it is not algebraically positive.

Note that $\mathscr{C}\big([-1,1], \mathbb{C}\big)_{\mathrm{H}}^{+} = \mathscr{C}\big([-1,1], \mathbb{C}\big)_{\mathrm{H}}^{++}$, but that this does no longer hold for its unital *-subalgebra $\mathcal{A}$, i.e. $1 - x^2 \in \mathcal{A}_{\mathrm{H}} \cap \mathscr{C}\big([-1,1], \mathbb{C}\big)_{\mathrm{H}}^{+}$ but $1 - x^2 \notin \mathcal{A}_{\mathrm{H}}^{++}$.

PROOF: If $a \in \mathcal{A}_{\mathrm{H}}$ is pointwise positive on $\mathbb{R}$, then its real roots all have even multiplicity. Its complex roots always come in pairs as $a^* = a$: If $z \in \mathbb{C} \backslash \mathbb{R}$ is a root of a with multiplicity n, then $\overline{z}$ is also a root of a with multiplicity n. The fundamental theorem of algebra thus allows to express a as $a = b^*b$ with

$$b = \left(\prod_{i=1}^k (x - \lambda_i)^{n_i/2}\right)\left(\prod_{j=1}^\ell (x - z_j)^{m_j}\right),$$

where $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ are the real roots of a and $n_1, \dots, n_k \in 2\mathbb{N}$ their multiplicities, and $z_1, \dots, z_\ell \in \{ z \in \mathbb{C} \mid \mathsf{Im}(z) > 0 \}$ its truely complex roots in the upper half plane and $m_1, \dots, m_k \in \mathbb{N}$ their multiplicities. The rest is clear. $\square$

The last example shows that the left multiplication on abstract O^*-algebras is not always strongly continuous:

Example 3.7.3 *Let $\mathcal{D} = \mathscr{C}_0^\infty(\mathbb{R}, \mathbb{C})$ be the pre-Hilbert space of smooth complex-valued functions on $\mathbb{R}$ with compact support and the usual inner product*

$$\langle \phi \,|\, \psi \rangle := \int \overline{\phi(x)}\, \psi(x)\, \mathrm{d}x$$

for all $\phi, \psi \in \mathcal{D}$. Consider the abstract O^*-algebra $\big(\mathcal{L}^*(\mathcal{D}), \mathcal{X}(\mathcal{D})\big)$. Then the multiplication operator $a\colon \mathcal{D} \to \mathcal{D}$, $\phi \mapsto a(\phi)$ with $a(\phi)(x) := x\phi(x)$ for all $x \in \mathbb{R}$ is Hermitian, i.e. $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$, and the translation operators $t_\delta\colon \mathcal{D} \to \mathcal{D}$, $\phi \mapsto t_\delta(\phi)$ with $t_\delta(\phi)(x) := \phi(x - \delta)$ for all $x \in \mathbb{R}$ are all unitary. Nevertheless, $\|a\, t_\delta\|_{\chi_\phi,\mathrm{st}} \xrightarrow{\delta\to\infty} \infty$ for all $\phi \in \mathcal{D}\backslash\{0\}$ even though $\|t_\delta\|_{\omega,\mathrm{st}} = \langle \omega\,,\, \mathbb{1} \rangle^{1/2}$ for all $\delta \in \mathbb{R}$ and all $\omega \in \mathcal{X}(\mathcal{D})^+_{\mathrm{H}}$.

PROOF: It is easy to check that a is Hermitian and that the t_δ for all $\delta \in \mathbb{R}$ are unitary with $t_\delta^* = t_{-\delta}$. Because of this, $\|t_\delta\|_{\omega,\mathrm{st}} = \langle \omega\,,\, t_\delta^*\, t_\delta \rangle^{1/2} = \langle \omega\,,\, \mathbb{1} \rangle^{1/2}$ for all $\omega \in \mathcal{X}(\mathcal{D})^+_{\mathrm{H}}$, but

$$\|a\, t_\delta\|^2_{\chi_\phi,\mathrm{st}} = \|a\big(t_\delta(\phi)\big)\|^2 = \int \big|x\, \phi(x-\delta)\big|^2\, \mathrm{d}x = \int \big|(x+\delta)\, \phi(x)\big|^2\, \mathrm{d}x \geq n^2 \int \big|\phi(x)\big|^2\, \mathrm{d}x = n^2 \|\phi\|^2$$

holds for all $n \in \mathbb{N}$ and all δ large enough so that $x + \delta \geq n$ for all x in the support of ϕ. So $\|a\, t_\delta\|_{\chi_\phi,\mathrm{st}} \xrightarrow{\delta\to\infty} \infty$ if $\phi \neq 0$. □

Chapter 4

Convergent Star Products – Example I

In this chapter we are going to construct a deformation of locally convex *-algebras $\big(\mathcal{S}^\bullet(V), \star\big)$ that generalizes the exponential star product from (2.5.1) on polynomials, and examine its properties as well as the properties of the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\hbar, \mathcal{T}\big)$ for fixed $\hbar \geq 0$. One – at first glance rather innocent looking – feature of this construction is that for $\hbar > 0$, there can be elements $P, Q \in \mathcal{S}^\bullet(V)$ fulfilling the canonical commutation relation

$$[\,P\,,\,Q\,]_{\star_\hbar} = \lambda \mathbb{1} \tag{4.0.1}$$

with some $\lambda \in \mathbb{C}\backslash\{0\}$. So before describing the actual construction, it might be instructive to discuss the consequences of this relation, and to understand which properties we cannot expect the algebras $\big(\mathcal{S}^\bullet(V), \star_\hbar\big)$ with $\hbar > 0$ to fulfil:

Proposition 4.0.1 *Let $\mathcal{A}$ be a *-algebra and $P, Q \in \mathcal{A}$ such that* (4.0.1) *holds with some $\lambda \in \mathbb{C}\backslash\{0\}$. Then:*

- *The only submultiplicative seminorm on $\mathcal{A}$ is the trivial seminorm which is constant* 0.
- *There is no algebraic state on $\mathcal{A}$ which is bounded on P and Q.*
- *There is no algebraic state ω on $\mathcal{A}$ which is an eigenstate of P or Q, i.e. which fulfils* $\mathrm{Var}_\omega(P) = 0$ *or* $\mathrm{Var}_\omega(Q) = 0$.

PROOF: The first point is well-known and the second is a direct consequence thereof: Let $\|\cdot\|$ be a submultiplicative seminorm on $\mathcal{A}$ and write $\mathrm{ad}_P \colon \mathcal{A} \to \mathcal{A}$ for the linear map defined by $\mathrm{ad}_P(a) := [\,P\,,\,a\,]$ for all $a \in \mathcal{A}$. Then $\|\mathrm{ad}_P(a)\| \leq 2\|P\|\|a\|$ due to submultiplicativity, and thus $\|(\mathrm{ad}_P)^n\big(Q^n\big)\| \leq 2^n\|P\|^n\|Q^n\| \leq 2^n\|P\|^n\|Q\|^n$ for all $n \in \mathbb{N}$. However, using $\mathrm{ad}_P(Q) = \lambda\mathbb{1}$ and the Leibniz rule for the commutator one gets $\mathrm{ad}_P(Q^n) = n\lambda Q^{n-1}$ for all $n \in \mathbb{N}$, thus $(\mathrm{ad}_P)^n\big(Q^n\big) = n!\,\lambda^n\mathbb{1}$, which yields the estimate

$$n!\,|\lambda|^n\|\mathbb{1}\| = \big\|n!\,\lambda^n\mathbb{1}\big\| = \big\|(\mathrm{ad}_P)^n\big(Q^n\big)\big\| \leq 2^n\|P\|^n\|Q\|^n$$

for all $n \in \mathbb{N}$, which implies $\|\mathbb{1}\| = 0$. But using submultiplicativity again, this shows $\|a\| \leq \|a\|\|\mathbb{1}\| = 0$ for all $a \in \mathcal{A}$ and proves the first point. Now assume that there was an algebraic state ω on $\mathcal{A}$ which is bounded on P and Q, then $\|\cdot\|_{\omega,\infty}$ would be a C^*-seminorm (especially submultiplicative) on the unital *-subalgebra $\mathcal{B}_\omega(\mathcal{A})$ of $\mathcal{A}$ and $P, Q \in \mathcal{B}_\omega(\mathcal{A})$, which contradicts the first point as $\|\mathbb{1}\|_{\omega,\infty} = 1$.

Finally, if ω was an algebraic state on $\mathcal{A}$ fulfilling $\operatorname{Var}_\omega(P) = 0$ or $\operatorname{Var}_\omega(Q) = 0$, then $\lambda\langle\, \omega\, ,\, \mathbb{1}\, \rangle = \langle\, \omega\, ,\, PQ\, \rangle - \langle\, \omega\, ,\, QP\, \rangle = \langle\, \omega\, ,\, P\, \rangle\langle\, \omega\, ,\, Q\, \rangle - \langle\, \omega\, ,\, Q\, \rangle\langle\, \omega\, ,\, P\, \rangle = 0$ holds due to (1.2.9), which is a contradiction again. □

Because of this, the locally convex *-algebras $\big(\mathcal{S}^\bullet(V), \star_\hbar\big)$ with $\hbar > 0$ cannot be lmc *-algebras, they cannot have a non-trivial representation by bounded operators, and in no representation will P or Q have eigenvectors, as every eigenvector ϕ would yield an eigenstates $\chi_\phi / \|\phi\|^2$.

4.1 Deformation of a Locally Convex *-Algebra of Symmetric Tensors

The rest of this chapter follows closely the publication [77] by Stefan Waldmann and the author. The construction presented here is also similar to the one in [83], but uses Hilbert tensor products instead of projective tensor products. Because of this, we have to restrict our attention to locally convex spaces whose topology is given by Hilbert seminorms.

Let V be a locally convex space. Recall from Section 1.2 that a positive Hermitian form on V is a sesquilinear Hermitian and positive semi-definite form $\langle\, \cdot \mid \cdot\, \rangle_\alpha \colon V \times V \to \mathbb{C}$ and yields a Hilbert seminorm $\|\cdot\|_\alpha$ on V. Then $\mathcal{I}_V$ will denote the set of all continuous positive Hermitian forms on V and we distinguish different positive Hermitian forms by a lowercase greek subscript. Out of $p, q \geq 0$ and $\langle\, \cdot \mid \cdot\, \rangle_\alpha, \langle\, \cdot \mid \cdot\, \rangle_\beta \in \mathcal{I}_V$ we get a new continuous positive Hermitian form

$$\langle\, \cdot \mid \cdot\, \rangle_{p\alpha+q\beta} := p\langle\, \cdot \mid \cdot\, \rangle_\alpha + q\langle\, \cdot \mid \cdot\, \rangle_\beta\,.$$

Similarly, the set of all continuous Hilbert seminorms on V will be denoted by $\mathcal{P}_V$ (note that the correspondence between $\mathcal{I}_V$ and $\mathcal{P}_V$ is one-to-one). With the usual partial ordering of seminorms by pointwise comparison, $\mathcal{P}_V$ is an upwards directed poset.

For the rest of this chapter we will always assume that V is a complex Hausdorff locally convex space whose topology is defined by its continuous Hilbert seminorms (i.e. it is "hilbertisable" in the language of [46]). In other words, we assume that $\mathcal{P}_V$ is cofinal in the upwards directed set of all continuous seminorms on V: for every continuous seminorm $\|\cdot\|$ we assume that there is a continuous Hilbert seminorm $\|\cdot\|_\alpha \in \mathcal{P}_V$ such that $\|\cdot\| \leq \|\cdot\|_\alpha$. Important examples of such spaces are (pre-) Hilbert spaces and nuclear spaces (see [46, Corollary 21.2.2]) and, in general, all projective limits of pre-Hilbert spaces in the category of locally convex spaces. Of course, all $\mathbb{C}^n$ with $n \in \mathbb{N}$ and the standard inner product are very special examples thereof.

4.1.1 Extension of Hilbert Seminorms to the Tensor Algebra

Write $\mathcal{T}^k_{\mathrm{alg}}(V)$ for the space of degree k-tensors, $k \in \mathbb{N}_0$, over V and $\mathcal{T}^\bullet_{\mathrm{alg}} := \bigoplus_{k\in\mathbb{N}_0} \mathcal{T}^k_{\mathrm{alg}}(V)$ for the vector space underlying the tensor algebra. The projections on the tensors of degree k are denoted by $\langle\,\cdot\,\rangle_k \colon \mathcal{T}^\bullet_{\mathrm{alg}}(V) \to \mathcal{T}^k_{\mathrm{alg}}(V)$.

Analogous to [83], we extend all Hilbert seminorms from V to $\mathcal{T}^\bullet_{\mathrm{alg}}(V)$ with the difference that we first extend the $\langle\, \cdot \mid \cdot\, \rangle_\alpha$ and construct the seminorms out of these extensions:

Definition 4.1.1 *[77, Def. 2.1] For every continuous positive Hermitian form $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_V$ we define the sesquilinear extension $\langle \cdot \,|\, \cdot \rangle_\alpha^\bullet \colon \mathcal{T}^\bullet_{\mathrm{alg}}(V) \times \mathcal{T}^\bullet_{\mathrm{alg}}(V) \to \mathbb{C}$*

$$(X,Y) \mapsto \langle X \,|\, Y \rangle_\alpha^\bullet := \sum_{k=0}^\infty \big\langle \langle X \rangle_k \,\big|\, \langle Y \rangle_k \big\rangle_\alpha^\bullet, \tag{4.1.1}$$

where

$$\big\langle x_1 \otimes \dots \otimes x_k \,\big|\, y_1 \otimes \dots \otimes y_k \big\rangle_\alpha^\bullet := k! \prod_{m=1}^k \langle x_m \,|\, y_m \rangle_\alpha \tag{4.1.2}$$

for all $k \in \mathbb{N}_0$ and all $x, y \in V^k$.

It is well-known (e.g. from a good course in linear algebra) that this is a positive Hermitian form on all homogeneous tensor spaces and then it is clear that $\langle \cdot \,|\, \cdot \rangle_\alpha^\bullet$ is a positive Hermitian form on $\mathcal{T}^\bullet_{\mathrm{alg}}(V)$. We write $\|\cdot\|_\alpha^\bullet$ for the resulting seminorm on $\mathcal{T}^\bullet_{\mathrm{alg}}(V)$ and $\mathcal{T}^\bullet(V)$ for the locally convex space of $\mathcal{T}^\bullet_{\mathrm{alg}}(V)$ with the topology defined by the extensions of all $\|\cdot\|_\alpha \in \mathcal{P}_V$, as well as $\mathcal{T}^k(V)$ for the linear subspace $\mathcal{T}^k_{\mathrm{alg}}(V)$ with the subspace topology. Note that $\|\cdot\|_\alpha^\bullet \le \|\cdot\|_\beta^\bullet$ holds if and only if $\|\cdot\|_\alpha \le \|\cdot\|_\beta$ and that, in general, for a fixed tensor degree the resulting topology on $\mathcal{T}^k(V)$ is *not* the projective topology used in [83].

The factor $k!$ in (4.1.2) for the extensions of positive Hermitian forms corresponds roughly to the factor $(n!)^R$ for $R = 1/2$ in [83, Eq. (3.7)] for the extensions of seminorms (where $R = 1/2$ yields the coarsest topology for which the continuity of the star product could be shown in [83]). This special case is the only one that will be interesting here because of the characterization in Section 4.1.5.

The following is an easy consequence of the definition of the topology on $\mathcal{T}^\bullet(V)$:

Proposition 4.1.2 *[77, Prop. 2.2] The locally convex space $\mathcal{T}^\bullet(V)$ is Hausdorff and is metrizable if and only if V is metrizable.*

Proof: The "only if" part is clear because $V \cong \mathcal{T}^1(V)$ can be identitfied with a linear subspace of $\mathcal{T}^\bullet(V)$. Conversely, if V is Hausdorff, then for every finite-dimensional linear subspace U of V there is a $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_V$, whose restriction to U is non-degenerate. From basic linear algebra it follows that $\langle \cdot \,|\, \cdot \rangle_\alpha^\bullet$ is also non-degenerate on the tensor algebra of U. As every $X \in \mathcal{T}^\bullet(V)$ is an element of the tensor algebra of U for some finite-dimensional linear subspace U of V, it thus follows that $\mathcal{T}^\bullet(V)$ is Hausdorff as well. Similarly, if V is metrizable then $\mathcal{T}^\bullet(V)$ is again metrizable because there is an order-preserving one-to-one correspondence between the seminorms in $\mathcal{P}_V$ and their extensions to $\mathcal{T}^\bullet(V)$. □

For working with these extensions of not necessarily positive-definite positive Hermitian forms, the following technical lemma will be helpful:

Lemma 4.1.3 *[77, Lemma 2.3] Let $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_V$, $k \in \mathbb{N}$ and $X \in \mathcal{T}^k(V)$ be given. Then X can be expressed as $X = X_0 + \tilde{X}$ with tensors $X_0, \tilde{X} \in \mathcal{T}^k(V)$ that have the following properties:*

i.) One has $\|X_0\|_\alpha^\bullet = 0$ and there exists a finite (possibly empty) set A and tuples $x_a \in V^k$ for all $a \in A$ that fulfil $\prod_{n=1}^k \|x_{a,n}\|_\alpha = 0$ and $X_0 = \sum_{a \in A} x_{a,1} \otimes \dots \otimes x_{a,k}$.

ii.) *There exist a $d \in \mathbb{N}_0$ and a $\langle \cdot \mid \cdot \rangle_\alpha$-orthonormal tuple $e \in V^d$ as well as complex coefficients $X^{a'}$, such that*

$$\tilde{X} = \sum_{a' \in \{1,\dots,d\}^k} X^{a'} e_{a'_1} \otimes \dots \otimes e_{a'_k} \quad \text{and} \quad \|X\|_\alpha^{\bullet,2} = \|\tilde{X}\|_\alpha^{\bullet,2} = k! \sum_{a' \in \{1,\dots,d\}^k} \big|X^{a'}\big|^2. \tag{4.1.3}$$

PROOF: We can express X as a finite sum of simple tensors, $X = \sum_{b \in B} x_{b,1} \otimes \dots \otimes x_{b,k}$ with a finite set B and vectors $x_{b,i} \in V$. Let

$$V_X := \big\langle\!\!\big\langle \big\{ x_{b,i} \bigm| b \in B, i \in \{1,\dots,k\} \big\} \big\rangle\!\!\big\rangle_{\mathrm{lin}} \text{ and } V_{X_0} := \{ v \in V_X \mid \|v\|_\alpha = 0 \}.$$

Construct a complementary linear subspace $V_{\tilde{X}}$ of V_{X_0} in V_X, then we can also assume without loss of generality that $x_{b,i} \in V_{X_0} \cup V_{\tilde{X}}$ for all $b \in B$ and $i \in \{1,\dots,k\}$. Note that V_X, V_{X_0} and $V_{\tilde{X}}$ are all finite-dimensional. Now define $A := \big\{ a \in B \bigm| \exists_{n \in \{1,\dots,k\}} \colon x_{a,n} \in V_{X_0} \big\}$ and $X_0 := \sum_{a \in A} x_{a,1} \otimes \dots \otimes x_{a,k}$, then $\prod_{n=1}^k \|x_{a,n}\|_\alpha = 0$ by construction and so $\|X_0\|_\alpha^\bullet = 0$ and $\|X - X_0\|_\alpha^\bullet = \|X\|_\alpha^\bullet$. Restricted to $V_{\tilde{X}}$, the positive Hermitian form $\langle \cdot \mid \cdot \rangle_\alpha$ is even positive definite, i.e. an inner product. Let $d := \dim(V_{\tilde{X}})$ and $e \in V^d$ be an $\langle \cdot \mid \cdot \rangle_\alpha$-orthonormal base of $V_{\tilde{X}}$. Define $\tilde{X} := X - X_0$, then $\tilde{X} = \sum_{a' \in \{1,\dots,d\}^k} X^{a'} e_{a'_1} \otimes \dots \otimes e_{a'_k}$ with complex coefficients $X^{a'}$ and

$$\|X\|_\alpha^{\bullet,2} = \|\tilde{X}\|_\alpha^{\bullet,2} = \sum_{a' \in \{1,\dots,d\}^k} \big|X^{a'}\big|^2 \|e_{a'_1} \otimes \dots \otimes e_{a'_k}\|_\alpha^{\bullet,2} = \sum_{a' \in \{1,\dots,d\}^k} \big|X^{a'}\big|^2 k!. \qquad \square$$

On the locally convex space $\mathcal{T}^\bullet(V)$, the tensor product is indeed continuous. This is equivalent to the continuity of the following function:

Definition 4.1.4 *[77, Def. 2.4] Define the linear map $\mu_\otimes \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V)$ by*

$$X \otimes_\pi Y \;\mapsto\; \mu_\otimes(X \otimes_\pi Y) := X \otimes Y. \tag{4.1.4}$$

Algebraically, $\mu_\otimes$ is of course just the product of the tensor algebra. The emphasize here lies on the topologies involved: $\otimes_\pi$ denotes the projective tensor product. This means that the topology on $\mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ is described by the seminorms $\|\cdot\|_{\alpha \otimes_\pi \beta}^\bullet \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to [0, \infty[$

$$Z \;\mapsto\; \|Z\|_{\alpha \otimes_\pi \beta}^\bullet := \inf \sum_{i \in I} \|X_i\|_\alpha^\bullet \|Y_i\|_\beta^\bullet, \tag{4.1.5}$$

where the infimum runs over all possibilities to express Z as a sum $Z = \sum_{i \in I} X_i \otimes_\pi Y_i$ indexed by a finite set I, and $\|\cdot\|_\alpha^\bullet, \|\cdot\|_\beta^\bullet$ run over all extensions of continuous Hilbert seminorms on V. The only property of the projective tensor product relevant for us is the following well-known lemma:

Lemma 4.1.5 *[77, Lemma 2.5] Let W be a locally convex space, $\|\cdot\|$ a continuous seminorm on W and $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_V$. Let $\Phi \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to W$ be a linear map. Then the two statements*

i.) $\big\|\Phi(X \otimes_\pi Y)\big\| \le \|X\|_\alpha^\bullet \|Y\|_\beta^\bullet$ *for all $X, Y \in \mathcal{T}^\bullet(V)$*

ii.) $\big\|\Phi(Z)\big\| \le \|Z\|_{\alpha \otimes_\pi \beta}^\bullet$ *for all $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$*

are equivalent. Continuity of the bilinear map $\mathcal{T}^\bullet(V) \times \mathcal{T}^\bullet(V) \ni (X,Y) \mapsto \Phi(X \otimes_\pi Y) \in W$ is therefore equivalent to continuity of Φ.

PROOF: If i.) holds, let $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ be given. If Z can be expressed as a finite sum $Z = \sum_{i \in I} X_i \otimes_\pi Y_i$ with $X_i, Y_i \in \mathcal{T}^\bullet(V)$, then

$$\big\|\Phi(Z)\big\| \leq \sum_{i\in I} \big\|\Phi(X_i \otimes_\pi Y_i)\big\| \leq \sum_{i \in I} \|X_i\|^\bullet_\alpha \; \|Y_i\|^\bullet_\beta \,.$$

As this holds for all such representations of Z as a finite sum of factorizing tensors, it follows that $\|\Phi(Z)\| \leq \|Z\|^\bullet_{\alpha\otimes_\pi\beta}$. Conversely, if ii.) holds, let $X, Y \in \mathcal{T}^\bullet(V)$ be given. Then

$$\big\|\Phi(X \otimes_\pi Y)\big\| \leq \|X \otimes_\pi Y\|^\bullet_{\alpha\otimes_\pi\beta} \leq \|X\|^\bullet_\alpha \; \|Y\|^\bullet_\beta \,.$$

□

Proposition 4.1.6 *[77, Prop. 2.6] The linear map $\mu_\otimes$ is continuous and the estimate*

$$\|\mu_\otimes(Z)\|^\bullet_\gamma \leq \|Z\|^\bullet_{2\gamma\otimes_\pi 2\gamma} \tag{4.1.6}$$

holds for all $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ and all $\|\cdot\|_\gamma \in \mathcal{P}_V$. Moreover, all $X \in \mathcal{T}^k(V)$ and $Y \in \mathcal{T}^\ell(V)$ with $k, \ell \in \mathbb{N}_0$ fulfil for all $\|\cdot\|_\gamma \in \mathcal{P}_V$ the estimate

$$\big\|\mu_\otimes(X \otimes_\pi Y)\big\|^\bullet_\gamma \leq \binom{k+\ell}{k}^{1/2} \|X\|^\bullet_\gamma \|Y\|^\bullet_\gamma. \tag{4.1.7}$$

PROOF: Let $X \in \mathcal{T}^k(V)$ and $Y \in \mathcal{T}^\ell(V)$ with $k, \ell \in \mathbb{N}_0$ be given. Then

$$\|X \otimes Y\|^\bullet_\gamma = \sqrt{\langle\, X \otimes Y \,|\, X \otimes Y \,\rangle^\bullet_\gamma} = \binom{k+\ell}{k}^{\frac{1}{2}} \|X\|^\bullet_\gamma \|Y\|^\bullet_\gamma$$

holds. It now follows for all $X, Y \in \mathcal{T}^\bullet(V)$ that

$$\begin{aligned}
\|X \otimes Y\|^{\bullet,2}_\gamma &= \sum_{m=0}^\infty \|\langle X \otimes Y\rangle_m\|^{\bullet,2}_\gamma \\
&\leq \sum_{m=0}^\infty \left(\sum_{n=0}^m \|\langle X\rangle_{m-n} \otimes \langle Y\rangle_n\|^\bullet_\gamma\right)^2 \\
&= \sum_{m=0}^\infty \left(\sum_{n=0}^m \binom{m}{n}^{\frac{1}{2}} \|\langle X\rangle_{m-n}\|^\bullet_\gamma \|\langle Y\rangle_n\|^\bullet_\gamma\right)^2 \\
&= \sum_{m=0}^\infty \left(\sum_{n=0}^m \left(\binom{m}{n}\frac{1}{2^m}\right)^{\frac{1}{2}} \|\langle X\rangle_{m-n}\|^\bullet_{2\gamma} \|\langle Y\rangle_n\|^\bullet_{2\gamma}\right)^2 \\
&\overset{\text{CS}}{\leq} \sum_{m=0}^\infty \left(\sum_{n=0}^m \binom{m}{n}\frac{1}{2^m}\right)\left(\sum_{n=0}^m \|\langle X\rangle_{m-n}\|^{\bullet,2}_{2\gamma} \|\langle Y\rangle_n\|^{\bullet,2}_{2\gamma}\right) \\
&= \|X\|^{\bullet,2}_{2\gamma} \|Y\|^{\bullet,2}_{2\gamma},
\end{aligned}$$

by the Cauchy Schwarz inequality. □

4.1.2 Symmetrisation

Let $\mathfrak{S}_k \subseteq \{1, \dots, k\}^k$ be the symmetric group of degree k (in the case $k = 0$ this is $\mathfrak{S}_0 = \{\mathsf{id}_\emptyset\}$), then $\mathfrak{S}_k$ acts linearly on $\mathcal{T}^k_{\mathrm{alg}}(V)$ from the right via $(x_1 \otimes \dots \otimes x_k)^\sigma := x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$. This allows to define the *symmetrisation operators* $\mathscr{S}^k \colon \mathcal{T}^k_{\mathrm{alg}}(V) \to \mathcal{T}^k_{\mathrm{alg}}(V)$ by $X \mapsto \mathscr{S}^k(X) := \frac{1}{k!}\sum_{\sigma \in \mathfrak{S}_k} X^\sigma$ and $\mathscr{S}^\bullet \colon \mathcal{T}^\bullet_{\mathrm{alg}}(V) \to \mathcal{T}^\bullet_{\mathrm{alg}}(V)$ by $X \mapsto \mathscr{S}^\bullet(X) := \sum_{k\in\mathbb{N}_0} \mathscr{S}^k\big(\langle X\rangle_k\big)$. These are projectors on subspaces of $\mathcal{T}^k_{\mathrm{alg}}(V)$ and $\mathcal{T}^\bullet_{\mathrm{alg}}(V)$ which we will denote by $\mathcal{S}^k_{\mathrm{alg}}(V)$ and $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$. Like before, $\mathcal{S}^\bullet(V)$ and $\mathcal{S}^k(V)$ denote the subspaces $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$ and $\mathcal{S}^k_{\mathrm{alg}}(V)$ of $\mathcal{T}^\bullet(V)$ with the subspace topology. The star product will be defined on the symmetric tensor algebra with undeformed product $X \vee Y := \mathscr{S}^\bullet(X \otimes Y)$ for $X, Y \in \mathcal{S}^\bullet(V)$, which is well-known to be associative and indeed is continuous:

Proposition 4.1.7 *[77, Prop. 2.7] The symmetrisation operator is continuous and $\|\mathscr{S}^\bullet(X)\|^\bullet_\gamma \leq \|X\|^\bullet_\gamma$ holds for all $X \in \mathcal{T}^\bullet(V)$ and $\|\cdot\|_\gamma \in \mathcal{P}_V$.*

PROOF: From Definition 4.1.1 it is clear that $\langle\, X^\sigma \,|\, Y^\sigma \,\rangle^\bullet_\gamma = \langle\, X \,|\, Y \,\rangle^\bullet_\gamma$ for all $k \in \mathbb{N}_0$, $X, Y \in \mathcal{T}^k(V)$ and $\sigma \in \mathfrak{S}_k$, because this holds for all simple tensors and because both sides are (anti-)linear in X and Y. Therefore $\|X^\sigma\|^\bullet_\gamma = \|X\|^\bullet_\gamma$ and $\|\mathscr{S}^k(X)\|^\bullet_\gamma \leq \|X\|^\bullet_\gamma$ and we get the desired estimate

$$\|\mathscr{S}^\bullet(X)\|^{\bullet,2}_\gamma = \sum_{k=0}^\infty \big\|\mathscr{S}^k\big(\langle X\rangle_k\big)\big\|^{\bullet,2}_\gamma \leq \sum_{k=0}^\infty \|\langle X\rangle_k\|^{\bullet,2}_\gamma = \|X\|^{\bullet,2}_\gamma$$

on $\mathcal{T}^\bullet(V)$. □

Analogously to $\mu_\otimes$, define the linear map $\mu_\vee := \mathscr{S}^\bullet \circ \mu_\otimes \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V)$. Then the restriction of $\mu_\vee$ to $\mathcal{S}^\bullet(V)$ describes the symmetric tensor product $\vee$ and Propositions 4.1.6 and 4.1.7 yield:

Corollary 4.1.8 *[77, Cor. 2.8] The linear map $\mu_\vee$ is continuous and $\|\mu_\vee(Z)\|^\bullet_\gamma \leq \|Z\|^\bullet_{2\gamma\otimes_\pi 2\gamma}$ holds for all $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ and all $\|\cdot\|_\gamma \in \mathcal{P}_V$.*

4.1.3 The Star Product

The following star product is based on a bilinear form and generalizes the usual exponential star products like the Weyl or Wick star product, which have already been discussed as an example in Section 2.5.1, to arbitrary dimensions:

Definition 4.1.9 *[77, Def. 2.9] For every continuous bilinear form Λ on V, define the product $\mu_{\star_\Lambda} \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V)$,*

$$X \otimes_\pi Y \;\mapsto\; \mu_{\star_\Lambda}(X \otimes_\pi Y) := \sum_{t=0}^\infty \frac{1}{t!}\mu_\vee\Big(\big(\mathrm{P}_\Lambda\big)^t(X \otimes_\pi Y)\Big), \tag{4.1.8}$$

where the linear map $\mathrm{P}_\Lambda \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to \mathcal{T}^{\bullet-1}(V) \otimes_\pi \mathcal{T}^{\bullet-1}(V)$ is given on factorizing tensors of degree $k, \ell \in \mathbb{N}$ by

$$\mathrm{P}_\Lambda\big((x_1 \otimes \dots \otimes x_k) \otimes_\pi (y_1 \otimes \dots \otimes y_\ell)\big) := k\ell\,\Lambda(x_k, y_1)(x_1 \otimes \dots \otimes x_{k-1}) \otimes_\pi (y_2 \otimes \dots \otimes y_\ell) \tag{4.1.9}$$

for all $x \in V^k$ and $y \in V^\ell$. Moreover, the product $\star_\Lambda$ on $\mathcal{S}^\bullet(V)$ is defined as the bilinear map described by the restriction of $\mu_{\star_\Lambda}$ to $\mathcal{S}^\bullet(V)$.

Note that these definitions of P_Λ and $\star_\Lambda$ coincide algebraically on $\mathcal{S}^\bullet(V)$ with the ones in [83, Eq. (2.13) and (2.19)], evaluated at a fixed value for ν in the truely (not graded) symmetric case $V = V_0$. Note that with the convention used here, the deformation parameter $\hbar$ is already part of Λ.

We are now going to prove the continuity of $\star_\Lambda$. Therefore note that continuity of Λ means that there exist $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_V$ such that $|\Lambda(v,w)| \le \|v\|_\alpha \|w\|_\beta$ holds for all $v, w \in V$. So the set

$$\mathcal{P}_{V,\Lambda} := \Big\{ \|\cdot\|_\gamma \in \mathcal{P}_V \;\Big|\; |\Lambda(v,w)| \le \|v\|_\gamma \|w\|_\gamma \text{ for all } v,w \in V \Big\} \tag{4.1.10}$$

contains at least all continuous Hilbert seminorms on V that dominate $\|\cdot\|_{\alpha+\beta}$. Thus this set is cofinal in $\mathcal{P}_V$.

Lemma 4.1.10 *[77, Lemma 2.10] Let Λ be a continuous bilinear form on V, let $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_{V,\Lambda}$ as well as $k, \ell \in \mathbb{N}_0$ and $X \in \mathcal{T}^k(V)$, $Y \in \mathcal{T}^\ell(V)$ be given. Then*

$$\|\mathrm{P}_\Lambda(X \otimes_\pi Y)\|^\bullet_{\alpha\otimes_\pi\beta} \le \sqrt{k\ell}\, \|X\|^\bullet_\alpha \|Y\|^\bullet_\beta. \tag{4.1.11}$$

PROOF: If $k = 0$ or $\ell = 0$ this is clearly true, so assume $k, \ell \in \mathbb{N}$. We use Lemma 4.1.3 to construct $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ and $\tilde{X} = \sum_{a' \in \{1,\dots,c\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ with respect to $\langle \,\cdot\, | \,\cdot\, \rangle_\alpha$ as well as $Y_0 = \sum_{b \in B} y_{b,1} \otimes \cdots \otimes y_{b,\ell}$ and $\tilde{Y} = \sum_{b' \in \{1,\dots,d\}^\ell} Y^{b'} f_{b'_1} \otimes \cdots \otimes f_{b'_\ell}$ with respect to $\langle \,\cdot\, | \,\cdot\, \rangle_\beta$. Then

$$\big\|\mathrm{P}_\Lambda\big((X_0 + \tilde{X}) \otimes_\pi (Y_0 + \tilde{Y})\big)\big\|^\bullet_{\alpha\otimes_\pi\beta} \le \big\|\mathrm{P}_\Lambda(\tilde{X} \otimes_\pi \tilde{Y})\big\|^\bullet_{\alpha\otimes_\pi\beta},$$

because

$$\begin{aligned}
\big\|\mathrm{P}_\Lambda\big((\xi_1 \otimes \cdots \otimes \xi_k) \otimes_\pi (\eta_1 \otimes \cdots \otimes \eta_\ell)\big)\big\|^\bullet_{\alpha\otimes_\pi\beta} &= k\ell\, |\Lambda(\xi_k, \eta_1)|\, \|\xi_1 \otimes \cdots \otimes \xi_{k-1}\|^\bullet_\alpha \|\eta_2 \otimes \cdots \otimes \eta_\ell\|^\bullet_\beta \\
&= 0
\end{aligned}$$

for all $\xi \in V^k$, $\eta \in V^\ell$ for which there is at least one $m \in \{1, \dots, k\}$ with $\|\xi_m\|_\alpha = 0$ or one $n \in \{1, \dots, \ell\}$ with $\|\eta_n\|_\beta = 0$. On the subspaces $V_{\tilde{X}} = \langle\!\langle \{e_1, \dots, e_c\} \rangle\!\rangle_{\mathrm{lin}}$ and $V_{\tilde{Y}} = \langle\!\langle \{f_1, \dots, f_d\} \rangle\!\rangle_{\mathrm{lin}}$ of V, the bilinear form Λ is described by a matrix $\Omega \in \mathbb{C}^{c \times d}$ with entries $\Omega_{gh} = \Lambda(e_g, f_h)$. By using a singular value decomposition, we can even assume without loss of generality that all off-diagonal entries of Ω vanish. We also note that $|\Omega_{gg}| = |\Lambda(e_g, f_g)| \le \|e_g\|_\alpha \|f_g\|_\beta \le 1$. This gives the desired estimate

$$\begin{aligned}
&\|\mathrm{P}_\Lambda(X \otimes_\pi Y)\|^\bullet_{\alpha\otimes_\pi\beta} \\
&\quad \le \big\|\mathrm{P}_\Lambda(\tilde{X} \otimes_\pi \tilde{Y})\big\|^\bullet_{\alpha\otimes_\pi\beta} \\
&\quad = \Big\|\sum\nolimits_{a' \in \{1,\dots,c\}^k} \sum\nolimits_{b' \in \{1,\dots,d\}^\ell} X^{a'} Y^{b'} \mathrm{P}_\Lambda\big((e_{a'_1} \otimes \cdots \otimes e_{a'_k}) \otimes_\pi (f_{b'_1} \otimes \cdots \otimes f_{b'_\ell})\big)\Big\|^\bullet_{\alpha\otimes_\pi\beta} \\
&\quad = k\ell \Bigg\| \sum_{r=1}^{\min\{c,d\}} \sum\nolimits_{\substack{\tilde{a}' \in \{1,\dots,c\}^{k-1} \\ \tilde{b}' \in \{1,\dots,d\}^{\ell-1}}} X^{(\tilde{a}',r)} Y^{(r,\tilde{b}')} \Omega_{rr} (e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-1}}) \otimes_\pi (f_{\tilde{b}'_1} \otimes \cdots \otimes f_{\tilde{b}'_{\ell-1}}) \Bigg\|^\bullet_{\alpha\otimes_\pi\beta}
\end{aligned}$$

$$
\begin{aligned}
&\leq k\ell \sum_{r=1}^{\min\{c,d\}} \Big\| \sum_{\tilde a' \in \{1,\dots,c\}^{k-1}} X^{(\tilde a', r)} e_{\tilde a'_1} \otimes \dots \otimes e_{\tilde a'_{k-1}} \Big\|_\alpha^\bullet \Big\| \sum_{\tilde b' \in \{1,\dots,d\}^{\ell-1}} Y^{(r,\tilde b')} f_{\tilde b'_1} \otimes \dots \otimes f_{\tilde b'_{\ell-1}} \Big\|_\beta^\bullet \\
&\overset{\text{CS}}{\leq} \sqrt{k\ell}\, \|X\|_\alpha^\bullet \|Y\|_\beta^\bullet,
\end{aligned}
$$

by using in the last line after applying the Cauchy Schwarz inequality that

$$
\begin{aligned}
\sum_{r=1}^{\min\{c,d\}} \Big\| \sum_{\tilde a' \in \{1,\dots,c\}^{k-1}} X^{(\tilde a', r)} e_{\tilde a'_1} \otimes \dots \otimes e_{\tilde a'_{k-1}} \Big\|_\alpha^{\bullet,2} &= \sum_{r=1}^{\min\{c,d\}} \sum_{\tilde a' \in \{1,\dots,c\}^{k-1}} \big|X^{(\tilde a', r)}\big|^2 (k-1)! \\
&\leq \frac{1}{k} \|X\|_\alpha^{\bullet,2}
\end{aligned}
$$

and analogously for Y. □

Proposition 4.1.11 *[77, Prop. 2.11] Let Λ be a continuous bilinear form on V, then the function P_Λ is continuous and fulfils the estimate*

$$
\big\| (\mathrm{P}_\Lambda)^t(Z) \big\|_{\alpha\otimes_\pi\beta}^\bullet \leq \frac{c}{c-1} \frac{t!}{c^t} \|Z\|_{2c\alpha\otimes_\pi 2c\beta}^\bullet \tag{4.1.12}
$$

for all $c > 1$, all $t \in \mathbb{N}_0$, all seminorms $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_{V,\Lambda}$, and all $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$.

Proof: Let $X, Y \in \mathcal{T}^\bullet(V)$ be given, then the previous Lemma 4.1.10 together with Lemma 4.1.5 yields

$$
\begin{aligned}
\big\| (\mathrm{P}_\Lambda)^t (X \otimes_\pi Y) \big\|_{\alpha\otimes_\pi\beta}^\bullet &\leq \sum_{k,\ell=0}^\infty \big\| (\mathrm{P}_\Lambda)^t \big(\langle X \rangle_{k+t} \otimes_\pi \langle Y \rangle_{\ell+t} \big) \big\|_{\alpha\otimes_\pi\beta}^\bullet \\
&\leq t! \sum_{k,\ell=0}^\infty \binom{k+t}{t}^{\frac12} \binom{\ell+t}{t}^{\frac12} \|\langle X \rangle_{k+t}\|_\alpha^\bullet \|\langle Y \rangle_{\ell+t}\|_\beta^\bullet \\
&\leq t! \sum_{k,\ell=0}^\infty \|\langle X \rangle_{k+t}\|_{2\alpha}^\bullet \|\langle Y \rangle_{\ell+t}\|_{2\beta}^\bullet \\
&= \frac{t!}{c^t} \sum_{k,\ell=0}^\infty \frac{1}{\sqrt{c}^{k+\ell}} \|\langle X \rangle_{k+t}\|_{2c\alpha}^\bullet \|\langle Y \rangle_{\ell+t}\|_{2c\beta}^\bullet \\
&\overset{\text{CS}}{\leq} \frac{t!}{c^t} \bigg(\sum_{k,\ell=0}^\infty \frac{1}{c^{k+\ell}} \bigg)^{\frac12} \bigg(\sum_{k,\ell=0}^\infty \|\langle X \rangle_{k+t}\|_{2c\alpha}^{\bullet,2} \|\langle Y \rangle_{\ell+t}\|_{2c\beta}^{\bullet,2} \bigg)^{\frac12} \\
&\leq \frac{c}{c-1} \frac{t!}{c^t} \|X\|_{2c\alpha}^\bullet \|Y\|_{2c\beta}^\bullet.
\end{aligned}
$$

□

Lemma 4.1.12 *[77, Lemma 2.12] Let Λ be a continuous bilinear form on V, then $\mu_{\star_\Lambda}$ is continuous and, given $R > 1/2$, the estimate*

$$
\|\mu_{\star_{z\Lambda}}(Z)\|_\gamma^\bullet \leq \sum_{t=0}^\infty \frac{1}{t!} \Big\| \mu_\vee \Big(\big(\mathrm{P}_{z\Lambda}\big)^t (Z) \Big) \Big\|_\gamma^\bullet \leq \frac{4R}{2R-1} \|Z\|_{8R\gamma\otimes_\pi 8R\gamma}^\bullet \tag{4.1.13}
$$

holds for all $\|\cdot\|_\gamma \in \mathcal{P}_{V,\Lambda}$, all $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ and all $z \in \mathbb{C}$ with $|z| \leq R$.

PROOF: The first estimate is just the triangle-inequality. Combining Corollary 4.1.8 and Proposition 4.1.11 with $c = 2R$ yields the second estimate

$$\begin{aligned}\sum_{t=0}^{\infty} \frac{1}{t!} \Big\| \mu_\vee \Big(\big(\mathrm{P}_{z\Lambda}\big)^t (Z) \Big) \Big\|_\gamma^\bullet &\le \sum_{t=0}^{\infty} \frac{|z|^t}{t!} \Big\| \big(\mathrm{P}_\Lambda\big)^t (Z) \Big\|_{2\gamma\otimes_\pi 2\gamma}^\bullet \\ &\le \frac{2R}{2R-1} \sum_{t=0}^{\infty} \frac{1}{2^t} \| Z \|_{8R\gamma\otimes_\pi 8R\gamma} \\ &= \frac{4R}{2R-1} \| Z \|_{8R\gamma\otimes_\pi 8R\gamma}. \qquad \square\end{aligned}$$

This estimate immediately leads to:

Theorem 4.1.13 *[77, Thm. 2.13] Let Λ be a continuous bilinear form on V, then the product $\star_\Lambda$ is continuous on $\mathcal{S}^\bullet(V)$. Moreover, for fixed tensors X, Y from the completion $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$, the product $X \star_{z\Lambda} Y$ converges absolutely and locally uniformly in $z \in \mathbb{C}$ and thus depends holomorphically on z.*

Note that the above estimate also shows that $\big(\mathcal{S}^\bullet(V), \star_{z\Lambda}\big)$ describes a holomorphic deformation (as defined in [67]) of the locally convex algebra $\big(\mathcal{S}^\bullet(V), \vee\big)$. However, in order to examine the star product for fixed values of both Λ and z it is advantageous to absorb z in the bilinear form Λ.

4.1.4 The *-Involution

The final ingredient that is still missing in the construction of a deformation of locally convex *-algebras is of course the *-involution: There is clearly one and only one possibility to extend an antilinear involution $\overline{\,\cdot\,}$ on V to a *-involution $^*\colon \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V)$ on the tensor algebra over V, namely by $(x_1 \otimes \cdots \otimes x_k)^* := \overline{x}_k \otimes \cdots \otimes \overline{x}_1$ for all $k \in \mathbb{N}$ and $x \in V^k$ and antilinear extension. Its restriction to $\mathcal{S}^\bullet(V)$ gives a *-involution on $\big(\mathcal{S}^\bullet(V), \vee\big)$.

Proposition 4.1.14 *[77, Prop. 3.1] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V, then the induced *-involution on $\mathcal{T}^\bullet(V)$ is also continuous.*

PROOF: For $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_V$ define the continuous positive Hermitian form $V^2 \ni (v,w) \mapsto \langle v \,|\, w \rangle_{\alpha^*} := \overline{\langle \overline{v} \,|\, \overline{w} \rangle_\alpha}$. Then $\langle X^* \,|\, Y^* \rangle_\alpha^\bullet = \overline{\langle X \,|\, Y \rangle_{\alpha^*}^\bullet}$ and in particular $\|X^*\|_\alpha^\bullet = \|X\|_{\alpha^*}^\bullet$ for all $X, Y \in \mathcal{T}^\bullet(V)$ because this is clearly true for simple tensors and because both sides are (anti-)linear in X and Y. $\square$

For certain bilinear forms Λ on V we can also show that $\cdot^*$ is a *-involution of $\star_\Lambda$, which is of course not a new result and simply extends the observation, that the star product on $\mathbb{R}^n$ in (2.5.1) is compatible with pointwise complex conjugation, to arbitrary dimensions:

Definition 4.1.15 *[77, Def. 3.2] Let $\overline{\,\cdot\,}\colon V \to V$ be a continuous antilinear involution on V. For every continuous bilinear form $\Lambda\colon V \times V \to \mathbb{C}$ we define its conjugate Λ^* by $\Lambda^*(v,w) := \overline{\Lambda(\overline{w}, \overline{v})}$, which is again a continuous bilinear form on V. Again, Λ is called* Hermitian *if $\Lambda = \Lambda^*$ holds.*

Note that the bilinear form $V^2 \ni (v,w) \mapsto \Lambda(v,w)$ is Hermitian if and only if the sesquilinear form $V^2 \ni (v,w) \mapsto \Lambda(\overline{v}, w)$ is Hermitian. The typical example of a complex vector space V with antilinear involution $\overline{\,\cdot\,}$ is that $V = W \otimes \mathbb{C}$ is the complexification of a real vector space W with the canonical involution $\overline{w \otimes \lambda} \coloneqq w \otimes \overline{\lambda}$. In this case, every bilinear form Λ on V is fixed by two bilinear forms $\Lambda_r, \Lambda_i \colon W \times W \to \mathbb{R}$, the restriction of the real- and imaginary part of Λ to the real subspace $W \cong W \otimes 1$ of V, and Λ is Hermitian if and only if Λ_r is symmetric and Λ_i antisymmetric. Similarly to [83, Prop. 3.25] we get:

Proposition 4.1.16 *[77, Prop. 3.3] Let $\overline{\cdot} \colon V \to V$ be a continuous antilinear involution and Λ a continuous bilinear form on V. Then $(X \star_\Lambda Y)^* = Y^* \star_{\Lambda^*} X^*$ holds for all $X, Y \in \mathcal{S}^\bullet(V)$. Consequently, if Λ is Hermitian, then $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$ is a locally convex *-algebra.*

PROOF: The identities $\cdot^* \circ \mathscr{S}^\bullet = \mathscr{S}^\bullet \circ \cdot^*$ and $\cdot^* \circ \mu_\otimes = \mu_\otimes \circ \tau \circ \big(\cdot^* \otimes_\pi \cdot^*\big)$, with $\tau \colon \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ defined as $\tau(X \otimes_\pi Y) \coloneqq Y \otimes_\pi X$, can easily be checked on simple tensors, so $\cdot^* \circ \mu_\vee = \mu_\vee \circ \tau \circ \big(\cdot^* \otimes_\pi \cdot^*\big)$. Combining this with $\tau \circ (\cdot^* \otimes_\pi \cdot^*) \circ \mathrm{P}_\Lambda = \mathrm{P}_{\Lambda^*} \circ \tau \circ (\cdot^* \otimes_\pi \cdot^*)$ on symmetric tensors, which again can easily be checked on simple symmetric tensors, yields the desired result. □

Theorem 4.1.17 *Let $\overline{\cdot} \colon V \to V$ be a continuous antilinear involution and Λ a continuous Hermitian bilinear form on V, then $\big(\mathcal{S}^\bullet(V), \mathbb{R} \ni \hbar \mapsto \star_{\hbar\Lambda}, \cdot^*\big)$ is a deformation of a locally convex *-algebra.*

Moreover, write $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ for the completion of $\mathcal{S}^\bullet(V)$, then $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \hbar \mapsto \star_{\hbar\Lambda}, \cdot^*\big)$ is also a deformation of a locally convex *-algebra, where $\star_{\hbar\Lambda}$ and $\cdot^*$ now are the continuous extensions to the completion.

PROOF: Theorem 4.1.13 and the previous Proposition 4.1.16 already show that $\big(\mathcal{S}^\bullet(V), \star_{\hbar\Lambda}, \cdot^*\big)$ is a locally convex *-algebra and for all $\hbar \in \mathbb{R}$ and that the product depends continuously on $\hbar$. Moreover, on the commutative *-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$, i.e. on the classical limit for $\hbar = 0$, the bracket

$$\{ X , Y \} \coloneqq \lim_{\hbar \to 0} \frac{1}{\mathrm{i}\hbar} [X , Y]_{\star_{\hbar\Lambda}} = \frac{1}{\mathrm{i}\hbar} \mu_\vee\big(\mathrm{P}_{\hbar\Lambda}(X \otimes Y - Y \otimes X)\big) = \mu_\vee\big(\mathrm{P}_\pi(X \otimes Y)\big),$$

with $\pi \colon V \times V \to \mathbb{C}$ defined as $\pi(v, w) \coloneqq \mathrm{i}\Lambda(w, v) - \mathrm{i}\Lambda(v, w)$ for all $v, w \in V$, is a real Poisson bracket: Antisymmetry and the compatibility with the *-involution are clear from the previous results. Jacobi identity and Leibniz rule follow directly from the analogous properties of the $\star_{\hbar\Lambda}$-commutator $[\,\cdot\,,\,\cdot\,]_{\star_{\hbar\Lambda}}$ by using that the involved products of three elements like $\mathbb{C}^2 \ni (\hbar, \hbar') \mapsto [X , Y \star_{\hbar\Lambda} Z]_{\star_{\hbar'\Lambda}} \in \mathcal{S}^\bullet(V)$ depend holomorphically, hence especially continuously, on the deformation parameter.

Finally, the continuous extension to the completion is possible because the product $\star_{\hbar\Lambda}$ and the *-involution are continuous, see also Proposition A.1.11 in the appendix. The results about the classical limit, including the Poisson bracket, also extend because the star product converges locally uniformly in the deformation parameter by Theorem 4.1.13. □

4.1.5 Characterization of the Topology

Finally, the topology on $\mathcal{S}^\bullet(V)$, which was constructed previously in a rather unmotivated way, has a nice characterization as the essentially coarsest one possible:

Lemma 4.1.18 *[77, Lemma 3.4] Let $\overline{\cdot} \colon V \to V$ be a continuous antilinear involution. For every $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha \in \mathcal{I}_V$ we define a continuous bilinear form Λ_α on V by $\Lambda_\alpha(v, w) \coloneqq \langle\,\overline{v}\,|\,w\,\rangle_\alpha$ for all $v, w \in V$,* *then Λ_α is Hermitian and the identities*

$$\sum_{t=0}^{\infty} \frac{1}{t!} \mu_\otimes \Big(\big(\mathrm{P}_{\Lambda_\alpha}\big)^t \big(\langle X^* \rangle_t \otimes_\pi \langle Y \rangle_t \big) \Big) = \langle\, X \,|\, Y \,\rangle_\alpha^\bullet \tag{4.1.14}$$

and

$$\big\langle \mu_{\star_{\Lambda_\alpha}} (X^* \otimes_\pi Y) \big\rangle_0 = \langle\, X \,|\, Y \,\rangle_\alpha^\bullet \tag{4.1.15}$$

hold for all $X, Y \in \mathcal{T}^\bullet(V)$.

P R O O F: Clearly, Λ_α is Hermitian because $\langle\, \cdot \,|\, \cdot \,\rangle_\alpha$ is Hermitian. Then (4.1.15) follows directly from (4.1.14) because of the grading of $\mu_\vee$ and $\mathrm{P}_{\Lambda_\alpha}$. For proving (4.1.14) it is sufficient to check it for factorizing tensors of the same degree, because both sides are (anti-)linear in X and Y and vanish if X and Y are homogeneous of different degree. If X and Y are of degree 0, then (4.1.14) is clearly fulfilled. Otherwise we get

$$\begin{aligned}
&\frac{1}{k!} \mu_\otimes \Big(\big(\mathrm{P}_{\Lambda_\alpha}\big)^k \big((x_1 \otimes \cdots \otimes x_k)^* \otimes_\pi (y_1 \otimes \cdots \otimes y_k) \big) \Big) \\
&\quad = \frac{1}{k!} \mu_\otimes \Big(\big(\mathrm{P}_{\Lambda_\alpha}\big)^k \big((\overline{x}_k \otimes \cdots \otimes \overline{x}_1) \otimes_\pi (y_1 \otimes \cdots \otimes y_k) \big) \Big) \\
&\quad = \frac{1}{k!} \mu_\otimes \Big(\big(1 \otimes_\pi 1\big) (k!)^2 \prod_{m=1}^{k} \Lambda_\alpha(\overline{x}_m, y_m) \Big) \\
&\quad = k! \prod_{m=1}^{k} \Lambda_\alpha(\overline{x}_m, y_m) \\
&\quad = k! \prod_{m=1}^{k} \langle\, x_m \,|\, y_m \,\rangle_\alpha \\
&\quad = \langle\, x_1 \otimes \cdots \otimes x_k \,|\, y_1 \otimes \cdots \otimes y_k \,\rangle_\alpha^\bullet .
\end{aligned}$$

□

Theorem 4.1.19 *[77, Thm. 3.5] The topology on $\mathcal{S}^\bullet(V)$ is the coarsest locally convex one that makes all star products $\star_\Lambda$ for all continuous and Hermitian bilinear forms Λ on V as well as the *-involution and the projection $\langle \cdot \rangle_0$ onto the scalars continuous. In addition we have for all $X, Y \in \mathcal{S}^\bullet(V)$ and all $\langle\, \cdot \,|\, \cdot \,\rangle_\alpha \in \mathcal{I}_V$*

$$\langle X^* \star_{\Lambda_\alpha} Y \rangle_0 = \langle\, X \,|\, Y \,\rangle_\alpha^\bullet, \tag{4.1.16}$$

with Λ_α like in Lemma 4.1.18.

P R O O F: We have already shown the continuity of the star product and of the *-involution, the continuity of $\langle \cdot \rangle_0$ is clear. Conversely, if these three functions are continuous, their compositions yield the extensions of all $\langle\, \cdot \,|\, \cdot \,\rangle_\alpha \in \mathcal{I}_V$ which then have to be continuous and (4.1.15) gives (4.1.16) for symmetric tensors X and Y. □

4.2 Representations and Properties of the Construction

Having constructed a deformation of locally convex *-algebras $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \hbar \mapsto \star_{\hbar\Lambda}, \cdot^*\big)$, where V is a hilbertisable locally convex space endowed with a continuous antilinear involution, and Λ a Hermitian and continuous bilinear form on V, the next step is to understand the properties of these algebras, especially their representations as functions or operators. In order to do this, the theory of abstract O^*-algebras from Chapter 3 will be helpful at some points: The representations of the locally convex *-algebras $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_{\hbar\Lambda}, \cdot^*\big)$ for real $\hbar$ coincide with those of the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_{\hbar\Lambda}, \cdot^*, \mathcal{T}\big)$ by Theorems 3.3.20, 3.3.21 or 3.3.22 (note that the weakly and strongly continuous representations as operators coincide as the product $\star_{\hbar\Lambda}$ is continuous, see also Proposition 3.3.14), so all the results about existence and properties of representations derived there can be applied.

The first step is to examine the classical, commutative algebra for $\hbar = 0$, prove that it has a faithful representation as functions and explicitly determine its Gelfand transformation. Then we examine, under which conditions two products $\star_\Lambda$ and $\star_{\Lambda'}$ are equivalent, which will also be helpful for constructing continuous algebraically positive linear functionals on the non-commutative algebras with $\hbar > 0$, which proves the existence of faithful representations as operators under certain circumstances.

We have already seen in the discussion at the beginning of this chapter that there cannot be representations as bounded operators except in some trivial cases for Λ. However, we will show that there always are many Stieltjes elements, thus many essentially self-adjoint operators in all representations.

4.2.1 Gelfand Transformation

The Gelfand transformation of general abstract O^*-algebra has been defined in Definition 3.2.3. In order to apply this to the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*, \mathcal{T}\big)$, it is necessary to determine all the characters of $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*, \mathcal{T}\big)$, i.e. all the continuous unital *-homomorphisms from $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$ to $\mathbb{C}$. This construction then yields a faithful continuous representation as functions of the commutative locally convex *-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$.

Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V, then again V_{H} is the real linear subspace of V consisting of Hermitian elements. The inner products compatible with the involution are denoted by

$$\mathcal{I}_{V,\mathrm{H}} := \big\{ \langle \,\cdot\,|\,\cdot\, \rangle_\alpha \in \mathcal{I}_V \;\big|\; \overline{\langle\, v \,|\, w \,\rangle_\alpha} = \langle\, \overline{v} \,|\, \overline{w} \,\rangle_\alpha \text{ for all } v, w \in V \big\} . \tag{4.2.1}$$

Moreover, write V' for the topological dual space of V and V'_{H} again for the real linear subspace of V' consisting of Hermitian elements, i.e.

$$V'_{\mathrm{H}} = \big\{ \rho \in V' \;\big|\; \overline{\rho(v)} = \rho(\overline{v}) \text{ for all } v \in V \big\} . \tag{4.2.2}$$

Finally, recall that a subset $B \subseteq V'_{\mathrm{H}}$ is *bounded* (with respect to the equicontinuous bornology) if there exists a $\langle \,\cdot\,|\,\cdot\, \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ such that $|\rho(v)| \le \|v\|_\alpha$ holds for all $v \in V$ and all $\rho \in B$. This also gives a notion of boundedness of functions from or to V'_{H}: A (multi-)linear function is bounded if it maps bounded sets to bounded ones.

Note that one can identify V'_{H} with the topological dual of V_{H} and $\mathcal{I}_{V,\mathrm{H}}$ with the set of continuous positive bilinear forms on V_{H}. Moreover, $\mathcal{I}_{V,\mathrm{H}}$ is cofinal in $\mathcal{I}_V$: every $\langle \,\cdot\,|\,\cdot\, \rangle_\alpha \in \mathcal{I}_V$ is dominated by $V^2 \ni (v, w) \mapsto \langle\, v \,|\, w \,\rangle_\alpha + \overline{\langle\, \overline{v} \,|\, \overline{w} \,\rangle_\alpha} \in \mathbb{C}$.

Definition 4.2.1 *[77, Def. 3.11] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $\rho \in V'_{\mathrm{H}}$, then define the* derivative in direction of ρ *as the linear map $D_\rho \colon \mathcal{T}^\bullet(V) \to \mathcal{T}^{\bullet-1}(V)$,*

$$x_1 \otimes \cdots \otimes x_k \;\mapsto\; D_\rho\big(x_1 \otimes \cdots \otimes x_k\big) := k\rho(x_k) x_1 \otimes \cdots \otimes x_{k-1} \tag{4.2.3}$$

for all $k \in \mathbb{N}$ and all $x \in V^k$. Next, define the translation by ρ *as the linear map*

$$\tau^*_\rho := \sum_{t=0}^{\infty} \frac{1}{t!} \big(D_\rho\big)^t \colon \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V), \tag{4.2.4}$$

and the evaluation at ρ *by*

$$\delta_\rho := \langle\,\cdot\,\rangle_0 \circ \tau^*_\rho \colon \mathcal{T}^\bullet(V) \to \mathbb{C}. \tag{4.2.5}$$

Finally, for $k \in \mathbb{N}$ and $\rho_1, \dots, \rho_k \in V'_{\mathrm{H}}$ we set $D^{(k)}_{\rho_1,\dots,\rho_k} := D_{\rho_1} \cdots D_{\rho_k} \colon \mathcal{T}^\bullet(V) \to \mathcal{T}^{\bullet-k}(V)$.

Note that τ^*_ρ is well-defined because for every $X \in \mathcal{T}^\bullet(V)$ only finitely many terms contribute to the infinite series $\tau^*_\rho(X) = \sum_{t=0}^\infty \frac{1}{t!}\big(D_\rho\big)^t(X)$. Note also that D_ρ and consequently τ^*_ρ can be restricted to endomorphisms of $\mathcal{S}^\bullet(V)$. Moreover, this restriction of D_ρ is a *-derivation of all the *-algebras $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \,\cdot\,^*\big)$ for all continuous Hermitian bilinear forms Λ on V (see [83, Lem. 2.13, *iii*], the compatibility with the *-involution is clear), so that τ^*_ρ turns out to be a unital *-automorphism of these *-algebras.

Lemma 4.2.2 *[77, Lemma 3.12] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $\rho, \sigma \in V'_{\mathrm{H}}$. Then*

$$\big(D_\rho D_\sigma - D_\sigma D_\rho\big)(X) = \big(\tau^*_\rho D_\sigma - D_\sigma \tau^*_\rho\big)(X) = \big(\tau^*_\rho \tau^*_\sigma - \tau^*_\sigma \tau^*_\rho\big)(X) = 0 \tag{4.2.6}$$

holds for all $X \in \mathcal{S}^\bullet(V)$.

Proof: It is sufficient to show that $\big(D_\rho D_\sigma - D_\sigma D_\rho\big)(X) = 0$ for all $X \in \mathcal{S}^\bullet(V)$, which clearly holds if X is a homogeneous factorizing symmetric tensor and so holds for all $X \in \mathcal{S}^\bullet(V)$ by linearity. □

Lemma 4.2.3 *[77, Lemma 3.13] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $\rho \in V'_{\mathrm{H}}$. Then D_ρ, τ^*_ρ and δ_ρ are all continuous. Moreover, if $\|\cdot\|_\alpha \in \mathcal{P}_V$ fulfils $|\rho(v)| \le \|v\|_\alpha$, then the estimates*

$$\big\|(D_\rho)^t X\big\|^\bullet_\alpha \le \sqrt{t!}\, \|X\|^\bullet_{2\alpha} \tag{4.2.7}$$

and

$$\|\tau^*_\rho(X)\|^\bullet_\alpha \le \sum_{t'=0}^{\infty} \frac{1}{t'!} \big\|(D_\rho)^{t'} X\big\|^\bullet_\alpha \le \frac{2}{\sqrt{2}-1} \|X\|^\bullet_{2\alpha} \tag{4.2.8}$$

hold for all $X \in \mathcal{T}^\bullet(V)$ and all $t \in \mathbb{N}_0$.

Proof: Let $\|\cdot\|_\alpha \in \mathcal{P}_V$ be given such that $|\rho(v)| \le \|v\|_\alpha$ holds for all $v \in V$. For all $d \in \mathbb{N}_0$ and all $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha$-orthonormal $e \in V^d$ we then get

$$\sum_{i=1}^{d} |\rho(e_i)|^2 = \rho\bigg(\sum_{i=1}^{d} e_i \overline{\rho(e_i)}\bigg) \le \bigg\|\sum_{i=1}^{d} e_i \overline{\rho(e_i)}\bigg\|_\alpha = \bigg(\sum_{i=1}^{d} |\rho(e_i)|^2\bigg)^{\frac{1}{2}},$$

hence $\sum_{i=1}^{d}|\rho(e_i)|^2 \le 1$. Given $k \in \mathbb{N}$ and $X \in \mathcal{T}^k(V)$, construct $X_0 = \sum_{a\in A} x_{a,1}\otimes\cdots\otimes x_{a,k}$ and $\tilde{X} = \sum_{a'\in\{1,\dots,d\}^k} X^{a'} e_{a'_1}\otimes\cdots\otimes e_{a'_k}$ like in Lemma 4.1.3. Then $\|D_\rho X_0\|^{\bullet}_{\alpha} = 0$ because

$$\|D_\rho(x_{a,1}\otimes\cdots\otimes x_{a,k})\|^{\bullet}_{\alpha} = k|\rho(x_{a,k})|\|x_{a,1}\otimes\cdots\otimes x_{a,k-1}\|^{\bullet}_{\alpha} \le k\sqrt{(k-1)!}\prod_{m=1}^{k}\|x_{a,m}\|_{\alpha} = 0$$

holds for all $a \in A$. Consequently $\|D_\rho X\|^{\bullet}_{\alpha} \le \|D_\rho \tilde{X}\|^{\bullet}_{\alpha}$ and

$$\begin{aligned}
\|D_\rho X\|^{\bullet,2}_{\alpha} \le \|D_\rho \tilde{X}\|^{\bullet,2}_{\alpha} &= \Bigg\|\sum_{a'\in\{1,\dots,d\}^k} X^{a'} D_\rho(e_{a'_1}\otimes\cdots\otimes e_{a'_k})\Bigg\|^{\bullet,2}_{\alpha} \\
&= k^2 \sum_{\tilde{a}'\in\{1,\dots,d\}^{k-1}} \Bigg\|\sum_{g=1}^{d} X^{(\tilde{a}',g)}\rho(e_g) e_{\tilde{a}'_1}\otimes\cdots\otimes e_{\tilde{a}'_{k-1}}\Bigg\|^{\bullet,2}_{\alpha} \\
&\le k^2(k-1)! \sum_{\tilde{a}'\in\{1,\dots,d\}^{k-1}} \Bigg(\sum_{g=1}^{d}\big|X^{(\tilde{a}',g)}\big||\rho(e_g)|\Bigg)^2 \\
&\overset{\text{CS}}{\le} k^2(k-1)! \sum_{\tilde{a}'\in\{1,\dots,d\}^{k-1}} \Bigg(\sum_{g=1}^{d}\big|X^{(\tilde{a}',g)}\big|^2\Bigg)\Bigg(\sum_{g=1}^{d}|\rho(e_g)|^2\Bigg) \\
&\le k^2(k-1)! \sum_{a'\in\{1,\dots,d\}^{k}} \big|X^{a'}\big|^2 \\
&= k\|X\|^{\bullet,2}_{\alpha}.
\end{aligned}$$

Using this one can derive the estimate (4.2.7), which also proves the continuity of D_ρ: If $t = 0$, then this is clearly fulfilled. Otherwise, let $X \in \mathcal{T}^\bullet(V)$ be given, then

$$\big\|(D_\rho)^t X\big\|^{\bullet,2}_{\alpha} = \sum_{k=t}^{\infty}\big\|(D_\rho)^t\langle X\rangle_k\big\|^{\bullet,2}_{\alpha} \le t!\sum_{k=t}^{\infty}\binom{k}{t}\|\langle X\rangle_k\|^{\bullet,2}_{\alpha} \le t!\sum_{k=t}^{\infty}\|\langle X\rangle_k\|^{\bullet,2}_{2\alpha} \le t!\|X\|^{\bullet,2}_{2\alpha}.$$

From this one can now also deduce the estimate (4.2.8), which shows continuity of τ^*_ρ and of $\delta_\rho = \langle\,\cdot\,\rangle_0 \circ \tau^*_\rho$: The first inequality is just the triangle inequality and then one uses that $t! \ge 2^{t-1}$ for all $t \in \mathbb{N}_0$, so

$$\sum_{t=0}^{\infty}\frac{1}{t!}\big\|(D_\rho)^t X\big\|^{\bullet}_{\alpha} \le \sum_{t=0}^{\infty}\frac{1}{\sqrt{t!}}\|X\|^{\bullet}_{2\alpha} \le \sqrt{2}\sum_{t=0}^{\infty}\frac{1}{\sqrt{2}^t}\|X\|^{\bullet}_{2\alpha} \le \frac{2}{\sqrt{2}-1}\|X\|^{\bullet}_{2\alpha}. \qquad \square$$

Proposition 4.2.4 *[77, Prop. 3.14] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V, then the set $\mathcal{M}$ of all continuous unital *-homomorphisms from $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \cdot^*\big)$ to $\mathbb{C}$ is $\{\,\delta_\rho \mid \rho \in V'_{\mathrm{H}}\,\}$ (strictly speaking, the continuous extensions to $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ of the restrictions of δ_ρ to $\mathcal{S}^\bullet(V)$).*

Proof: On the one hand, every such δ_ρ is a continuous unital *-homomorphism, because $\langle\,\cdot\,\rangle_0$ and τ^*_ρ are. On the other hand, if $\phi\colon \big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \cdot^*\big) \to \mathbb{C}$ is a continuous unital *-homomorphism, then $V \ni v \mapsto \rho(v) := \phi(v) \in \mathbb{C}$ is an element of V'_{H} and fulfils $\delta_\rho = \phi$ because the unital *-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$ is generated by V and because $\mathcal{S}^\bullet(V)$ is dense in its completion. $\square$

Note that $\mathcal{M}$ is just short for $\mathcal{M}\big(\mathcal{S}^\bullet(V), \vee, \cdot^*, \mathcal{T}\big)$. The Gelfand transformation π_{Gelfand} from Definition 3.2.3 would represent $\mathcal{S}^\bullet(V)$ as functions on the set $\mathcal{M}$. In this special case here, however, it will be more convenient to identify $\mathcal{M}$ with V'_{H} like in the previous Proposition 4.2.4 and to use the shorthand $\widehat{\cdot}$ for this new representation map:

Definition 4.2.5 *[77, Def. 3.15] Let $\overline{\cdot}$ be a continuous antilinear involution on V and $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$, then define the function $\widehat{X} \colon V'_{\mathrm{H}} \to \mathbb{C}$,*

$$\rho \;\mapsto\; \widehat{X}(\rho) \coloneqq \delta_\rho(X). \tag{4.2.9}$$

In the following we will show that this construction yields an isomorphism between $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \cdot^*\big)$ and a unital *-algebra of certain analytic functions on V'_{H}:

Definition 4.2.6 *[77, Def. 3.16] Let $f \colon V'_{\mathrm{H}} \to \mathbb{C}$ be a function. For $\rho, \sigma \in V'_{\mathrm{H}}$ we denote by*

$$\big(\widehat{D}_\rho f\big)(\sigma) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\sigma + t\rho) \tag{4.2.10}$$

(if it exists) the directional derivative of f at σ in direction ρ. If the directional derivative of f in direction ρ exists at all $\sigma \in V'_{\mathrm{H}}$, then we denote by $\widehat{D}_\rho f \colon V'_{\mathrm{H}} \to \mathbb{C}$ the function $\sigma \mapsto \big(\widehat{D}_\rho f\big)(\sigma)$. In this case we can also examine directional derivatives of $\widehat{D}_\rho f$ and define the iterated directional derivative

$$\widehat{D}^{(k)}_{\rho} f \coloneqq \widehat{D}_{\rho_1} \cdots \widehat{D}_{\rho_k} f \tag{4.2.11}$$

(if it exists) for $k \in \mathbb{N}$ and $\rho \in (V'_{\mathrm{H}})^k$. For $k = 0$ we define $\widehat{D}^{(0)} f \coloneqq f$. Moreover, we say that f is smooth if all iterated directional derivatives $\widehat{D}^{(k)}_\rho f$ exist for all $k \in \mathbb{N}_0$ and all $\rho \in (V'_{\mathrm{H}})^k$ and describe a bounded symmetric multilinear form $(V'_{\mathrm{H}})^k \ni \rho \mapsto \big(\widehat{D}^{(k)}_\rho f\big)(\sigma) \in \mathbb{C}$ for all $\sigma \in V'_{\mathrm{H}}$. Finally, we write $\mathscr{C}^\infty(V'_{\mathrm{H}})$ for the unital *-algebra of all smooth functions on V'_{H}.

Note that this notion of smoothness is rather weak, we do not even demand that a smooth function is continuous (we did not even endow V'_{H} with a topology). For example, every bounded linear functional on V'_{H} is smooth.

Proposition 4.2.7 *[77, Prop. 3.17] Let $\overline{\cdot}$ be a continuous antilinear involution on V and $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$. Then $\widehat{X} \colon V'_{\mathrm{H}} \to \mathbb{C}$ is smooth and*

$$\widehat{D}^{(k)}_\rho \widehat{X} = \widehat{D^{(k)}_\rho X} \tag{4.2.12}$$

holds for all $k \in \mathbb{N}_0$ and all $\rho \in (V'_{\mathrm{H}})^k$.

Proof: Let $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ be given. As the exponential series $\tau^*_{t\rho}(X)$ with $t \in \mathbb{R}$ is absolutely and (in t) locally uniformly convergent by Lemma 4.2.3, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \tau^*_{t\rho}(X) = D_\rho(X)$ for all $\rho \in V'_{\mathrm{H}}$ and so we conclude that

$$\big(\widehat{D}_\rho \widehat{X}\big)(\sigma) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \delta_{\sigma + t\rho}(X) = \left\langle \tau^*_\sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \tau^*_{t\rho}(X) \right) \right\rangle_0 = \big\langle \tau^*_\sigma\big(D_\rho(X)\big) \big\rangle_0 = \widehat{D_\rho(X)}(\sigma)$$

holds for all $\rho, \sigma \in V_{\mathrm{H}}'$, which proves (4.2.12) in the case $k = 1$. Thus $\widehat{D}_\rho$, for all $\rho \in V_{\mathrm{H}}'$, is an endomorphism of the vector space $\{ \widehat{X} \mid X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}} \}$, so all iterated directional derivatives of such an $\widehat{X}$ exist. By induction it is now easy to see that (4.2.12) holds for arbitrary $k \in \mathbb{N}_0$. Moreover, $D_\rho D_{\rho'} X = D_{\rho'} D_\rho X$ holds for all $\rho, \rho' \in V_{\mathrm{H}}'$ and all $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ by Lemmas 4.2.2 and 4.2.3. Together with (4.2.12) this shows that directional derivatives on $\widehat{X}$ commute. Finally, the multilinear form $(V_{\mathrm{H}}')^k \ni \rho \mapsto \big(\widehat{D}_\rho^{(k)} \widehat{X}\big)(\sigma) \in \mathbb{C}$ is bounded for all $\sigma \in V_{\mathrm{H}}'$: It is sufficient to show this for $\sigma = 0$, because τ_σ^* is a continuous automorphism of $\mathcal{S}^\bullet(V)$ and commutes with $D_\rho^{(k)}$. If $\rho \in (V_{\mathrm{H}}')^k$ fulfils $|\rho_i(v)| \leq \|v\|_\alpha$ for all $i \in \{1, \dots, k\}$, all $v \in V$ and one $\|\cdot\|_\alpha \in \mathcal{P}_V$, then $\|D_{\rho_1} \cdots D_{\rho_k} X\|_\alpha^\bullet \leq \|X\|_{2^k\alpha}$ holds due to Lemma 4.2.3, and gives an upper bound of $\big(\widehat{D}_\rho^{(k)} \widehat{X}\big)(0)$. □

Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and let $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ be given, then the Hermitian degeneracy space of the inner product $\langle \cdot \mid \cdot \rangle_\alpha$ is

$$\mathrm{kern}_{\mathrm{H}} \|\cdot\|_\alpha \coloneqq \big\{ v \in V_{\mathrm{H}} \bigm| \|v\|_\alpha = 0 \big\}. \tag{4.2.13}$$

This yields a well-defined non-degenerate positive bilinear form on the real vector space $V_{\mathrm{H}} \big/ \mathrm{kern}_{\mathrm{H}} \|\cdot\|_\alpha$. Write $V_{\mathrm{H},\alpha}^{\mathrm{cpl}}$ for the completion of this space to a real Hilbert space with inner product $\langle \cdot \mid \cdot \rangle_\alpha$ and define the linear map $\cdot^{\flat_\alpha}$ from $V_{\mathrm{H},\alpha}^{\mathrm{cpl}}$ to V_{H}' as

$$v^{\flat_\alpha}(w) \coloneqq \langle v \mid w \rangle_\alpha \tag{4.2.14}$$

for all $v \in V_{\mathrm{H},\alpha}^{\mathrm{cpl}}$ and all $w \in V$. Note that $\cdot^{\flat_\alpha} \colon V_{\mathrm{H},\alpha}^{\mathrm{cpl}} \to V_{\mathrm{H}}'$ is a bounded linear map due to the Cauchy Schwarz inequality. Analogously, define

$$\mathrm{kern}\, \|\cdot\|_\alpha^\bullet \coloneqq \big\{ X \in \mathcal{T}^\bullet(V) \bigm| \|X\|_\alpha^\bullet = 0 \big\}, \tag{4.2.15}$$

and denote by $\mathcal{T}^\bullet(V)_\alpha^{\mathrm{cpl}}$ the completion of the complex vector space $\mathcal{T}_{\mathrm{alg}}^\bullet(V) \big/ \mathrm{kern}\, \|\cdot\|_\alpha^\bullet$ to a complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle_\alpha^\bullet$. Then $\mathcal{S}^\bullet(V)_\alpha^{\mathrm{cpl}}$ becomes the linear subspace of (equivalence classes of) symmetric tensors, which is closed because $\mathscr{S}^\bullet$ extends to a continuous endomorphism of $\mathcal{T}^\bullet(V)_\alpha^{\mathrm{cpl}}$ by Proposition 4.1.7.

Moreover, for $\langle \cdot \mid \cdot \rangle_\alpha, \langle \cdot \mid \cdot \rangle_\beta \in \mathcal{I}_{V,\mathrm{H}}$ with $\langle \cdot \mid \cdot \rangle_\beta \leq \langle \cdot \mid \cdot \rangle_\alpha$, the linear map $\mathsf{id}_{\mathcal{T}^\bullet(V)} \colon \mathcal{T}^\bullet(V) \to \mathcal{T}^\bullet(V)$ extends to continuous linear maps $\iota_{\infty\alpha} \colon \mathcal{T}^\bullet(V)^{\mathrm{cpl}} \to \mathcal{T}^\bullet(V)_\alpha^{\mathrm{cpl}}$ and $\iota_{\alpha\beta} \colon \mathcal{T}^\bullet(V)_\alpha^{\mathrm{cpl}} \to \mathcal{T}^\bullet(V)_\beta^{\mathrm{cpl}}$, such that $\iota_{\alpha\beta} \circ \iota_{\infty\alpha} = \iota_{\infty\beta}$ and $\iota_{\beta\gamma} \circ \iota_{\alpha\beta} = \iota_{\alpha\gamma}$ hold for all $\langle \cdot \mid \cdot \rangle_\alpha, \langle \cdot \mid \cdot \rangle_\beta, \langle \cdot \mid \cdot \rangle_\gamma \in \mathcal{I}_{V,\mathrm{H}}$ with $\langle \cdot \mid \cdot \rangle_\gamma \leq \langle \cdot \mid \cdot \rangle_\beta \leq \langle \cdot \mid \cdot \rangle_\alpha$. This way, $\mathcal{T}^\bullet(V)^{\mathrm{cpl}}$ is realized as the projective limit of the Hilbert spaces $\mathcal{T}^\bullet(V)_\alpha^{\mathrm{cpl}}$ and similarly, $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ as the projective limit of its closed linear subspaces $\mathcal{S}^\bullet(V)_\alpha^{\mathrm{cpl}}$.

Lemma 4.2.8 *[77, Lemma 3.18] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $f \in \mathscr{C}^\infty(V_{\mathrm{H}}')$. Given $\rho \in V_{\mathrm{H}}'$ and $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ such that $|\rho(v)| \leq \|v\|_\alpha$ holds for all $v \in V$, then*

$$\widehat{D}_\rho f = \sum_{i \in I} \rho(e_i) \widehat{D}_{e_i^{\flat_\alpha}} f \tag{4.2.16}$$

holds for every Hilbert basis $e \in (V_{\mathrm{H},\alpha}^{\mathrm{cpl}})^I$ of $V_{\mathrm{H},\alpha}^{\mathrm{cpl}}$ indexed by a set I.

PROOF: As f is smooth, the function $V'_{\mathrm{H}} \ni \sigma \mapsto \widehat{D}_\sigma f \in \mathbb{C}$ is bounded, which implies that its restriction to the dual space of $V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$ is continuous with respect to the Hilbert space topology on (the dual of) $V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$. As $\rho = \sum_{i\in I} e_i^{\flat_\alpha} \rho(e_i)$ with respect to this topology, it follows that $\widehat{D}_\rho f = \sum_{i\in I} \rho(e_i) \widehat{D}_{e_i^{\flat_\alpha}} f$.□

Definition 4.2.9 *[77, Def. 3.19] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V. We say that a function $f\colon V'_{\mathrm{H}} \to \mathbb{C}$ is* analytic of Hilbert-Schmidt type*, if it is smooth and additionally fulfils the condition that for all $\sigma, \sigma' \in V'_{\mathrm{H}}$ and all $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ there exists a $C_{\sigma,\sigma',\alpha} \in \mathbb{R}$ such that*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} \Big| \big(\widehat{D}^{(k)}_{(e_{i_1}^{\flat_\alpha},\dots,e_{i_k}^{\flat_\alpha})} f\big)(\xi)\Big|^2 \le C_{\sigma,\sigma',\alpha} \tag{4.2.17}$$

holds for one Hilbert base $e \in (V^{\mathrm{cpl}}_{\mathrm{H},\alpha})^I$ of $V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$ indexed by a set I and every ξ from the line-segment between σ and σ', i.e. every $\xi = \lambda\sigma + (1-\lambda)\sigma'$ with $\lambda \in [0,1]$. We write $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ for the set of all complex functions on V'_{H} that are analytic of Hilbert-Schmidt type.

Here and elsewhere a sum over an uncountable Hilbert basis is understood in the usual sense: only countably many terms in the sum are non-zero. The above definition is independent of the choice of the Hilbert basis due to Lemma 4.2.8 and $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ is a complex vector space. It is not too hard to check that $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ is even a unital *-subalgebra of $\mathscr{C}^{\infty}(V'_{\mathrm{H}})$. However, we will indirectly prove this later on. Calling the functions in $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ *analytic* is justified thanks to the following statement:

Proposition 4.2.10 *[77, Prop. 3.20] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $f\colon V'_{\mathrm{H}} \to \mathbb{C}$ analytic of Hilbert-Schmidt type with $\big(\widehat{D}^{(k)}_\rho f\big)(0) = 0$ for all $k \in \mathbb{N}_0$ and all $\rho \in (V'_{\mathrm{H}})^k$. Then $f = 0$.*

PROOF: Given $\sigma \in V'_{\mathrm{H}}$, construct a smooth function $g\colon \mathbb{R} \to \mathbb{C}$ by $t \mapsto g(t) \coloneqq f(t\sigma)$ and write $g^{(k)}(t)$ for the k-th derivative of g at t. Then there exists a $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ that fulfils $|\sigma(v)| \le \|v\|_\alpha$ for all $v \in V$, so $\sigma = \nu e^{\flat_\alpha}$ with a normalized $e \in V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$ and $\nu \in [0,1]$ by the Fréchet-Riesz theorem. Then

$$\bigg(\sum_{k=0}^{\infty} \frac{1}{k!} \big|g^{(k)}(t)\big|\bigg)^2 \overset{\mathrm{CS}}{\le} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \big|g^{(\ell)}(t)\big|^2 \le \mathrm{e} \sum_{\ell=0}^{\infty} \frac{\nu^{2\ell}}{\ell!} \Big| \big(\widehat{D}^{(\ell)}_{(e^{\flat_\alpha},\dots,e^{\flat_\alpha})} f\big)(t\sigma)\Big|^2 \le \mathrm{e}\, C_{-2\sigma,2\sigma,\alpha}$$

holds for all $t \in [-2,2]$ with a constant $C_{-2\sigma,2\sigma,\alpha} \in \mathbb{R}$, which shows that g is an analytic function on $]-2,2[$. As $g^{(k)}(0) = 0$ for all $k \in \mathbb{N}_0$ this implies $f(\sigma) = g(1) = 0$. □

Note that one can derive even better estimates for the derivatives of g. This shows that condition (4.2.17) is even stronger than just analyticity. As an example, consider $V = \mathbb{C}$, $V'_{\mathrm{H}} = \mathbb{R}$, then the function $\mathbb{R} \ni x \mapsto \exp(x^2) \in \mathbb{C}$ is not analytic of Hilbert-Schmidt type.

Definition 4.2.11 *[77, Def. 3.21] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $f, g\colon V'_{\mathrm{H}} \to \mathbb{C}$ analytic of Hilbert-Schmidt type as well as $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$. Because of estimate* (4.2.17) *we can define a function $\langle\!\langle f \,|\, g \rangle\!\rangle^{\bullet}_\alpha\colon V'_{\mathrm{H}} \to \mathbb{C}$,*

$$\rho \;\mapsto\; \langle\!\langle f \,|\, g \rangle\!\rangle^{\bullet}_\alpha(\rho) \coloneqq \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} \overline{\Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f\Big)(\rho)} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} g\Big)(\rho), \tag{4.2.18}$$

where $e \in (V^{\mathrm{cpl}}_{\mathrm{H},\alpha})^I$ is an arbitrary Hilbert base of $V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$ indexed by a set I.

Note that $\langle\!\langle f \,|\, g \rangle\!\rangle^{\bullet}_{\alpha}$ does not depend on the choice of this Hilbert base due to Lemma 4.2.8. Essentially, $\langle\!\langle f \,|\, g \rangle\!\rangle^{\bullet}_{\alpha}(\rho)$ is a weighted ℓ^2-inner product (yet not necessarily positive-definite) of all partial derivatives of f and g at ρ in directions described by (the dual of) a $\langle \,\cdot\,|\,\cdot\, \rangle_{\alpha}$-Hilbert base. Note that the analyticity condition (4.2.17) for a function f is equivalent to demanding that $\langle\!\langle f \,|\, f \rangle\!\rangle^{\bullet}_{\alpha}(\xi)$ exists for all $\xi \in V'_{\mathrm{H}}$ and all $\langle \,\cdot\,|\,\cdot\, \rangle_{\alpha} \in \mathcal{I}_{V,\mathrm{H}}$ and is uniformly bounded on line segments in V'_{H}.

Lemma 4.2.12 *[77, Lemma 3.22] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V. Let $k \in \mathbb{N}$ and $x \in (V_{\mathrm{H}})^k$ as well as $\langle \,\cdot\,|\,\cdot\, \rangle_{\alpha} \in \mathcal{I}_{V,\mathrm{H}}$ be given. Then*

$$\big(\widehat{D}^{(k)}_{x^{\flat_\alpha}} \widehat{Y}\big)(0) = \big\langle D^{(k)}_{x^{\flat_\alpha}} Y \big\rangle_0 = \langle\, x_1 \otimes \cdots \otimes x_k \,|\, Y \,\rangle^{\bullet}_{\alpha} \tag{4.2.19}$$

holds for all $Y \in \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}}$.

P ROOF: The first identity is just Proposition 4.2.7, and for the second one it is sufficient to show that $\big\langle D^{(k)}_{x^{\flat_\alpha}} Y \big\rangle_0 = \langle\, x_1 \otimes \cdots \otimes x_k \,|\, Y \,\rangle^{\bullet}_{\alpha}$ holds for all factorizing tensors Y of degree k, because both sides of this equation vanish on homogeneous tensors of different degree and are linear and continuous in Y by Lemma 4.2.3. However, it is an immediate consequence of the definitions of D, $\cdot^{\flat_\alpha}$, and $\langle \,\cdot\,|\,\cdot\, \rangle^{\bullet}_{\alpha}$ that

$$\Big\langle D^{(k)}_{(x_1^{\flat_\alpha},\dots,x_k^{\flat_\alpha})} y_1 \otimes \cdots \otimes y_k \Big\rangle_0 = k! \prod_{m=1}^{k} \langle\, x_m \,|\, y_m \,\rangle_{\alpha} = \langle\, x_1 \otimes \cdots \otimes x_k \,|\, y_1 \otimes \cdots \otimes y_k \,\rangle^{\bullet}_{\alpha}$$

holds for all $y_1, \dots, y_k \in V$. □

Proposition 4.2.13 *[77, Prop. 3.23] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V, then*

$$\langle\!\langle \widehat{X} \,|\, \widehat{Y} \rangle\!\rangle^{\bullet}_{\alpha}(\rho) = \langle\, \tau^*_\rho X \,|\, \tau^*_\rho Y \,\rangle^{\bullet}_{\alpha} = \widehat{X^* \star_{\Lambda_\alpha} Y}(\rho) \tag{4.2.20}$$

holds for all $X, Y \in \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}}$, all $\rho \in V'_{\mathrm{H}}$, and all $\langle \,\cdot\,|\,\cdot\, \rangle_{\alpha} \in \mathcal{I}_{V,\mathrm{H}}$, where $\Lambda_\alpha \colon V \times V \to \mathbb{C}$ is the continuous bilinear form defined by $\Lambda_\alpha(v,w) := \langle\, \overline{v} \,|\, w \,\rangle_{\alpha}$.

P ROOF: Let $X, Y \in \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}}$, $\rho \in V'_{\mathrm{H}}$ and $\langle \,\cdot\,|\,\cdot\, \rangle_{\alpha} \in \mathcal{I}_{V,\mathrm{H}}$ be given. Let $e \in (V^{\mathrm{cpl}}_{\mathrm{H},\alpha})^I$ be a Hilbert base of $V^{\mathrm{cpl}}_{\mathrm{H},\alpha}$ indexed by a set I. Then

$$\begin{aligned}
\langle\!\langle \widehat{X} \,|\, \widehat{Y} \rangle\!\rangle^{\bullet}_{\alpha}(\rho) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{\Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} \widehat{X}\Big)(\rho)} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} \widehat{Y}\Big)(\rho) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{\Big\langle D^{(k)}_{e_i^{\flat_\alpha}} \tau^*_\rho X \Big\rangle_0} \Big\langle D^{(k)}_{e_i^{\flat_\alpha}} \tau^*_\rho Y \Big\rangle_0 \\
&= \sum_{k=0}^{\infty} \sum_{i \in I^k} \frac{1}{k!} \big\langle\, \tau^*_\rho X \,\big|\, e_{i_1} \otimes \cdots \otimes e_{i_k} \,\big\rangle^{\bullet}_{\alpha} \big\langle\, e_{i_1} \otimes \cdots \otimes e_{i_k} \,\big|\, \tau^*_\rho Y \,\big\rangle^{\bullet}_{\alpha} \\
&= \big\langle\, \tau^*_\rho X \,\big|\, \tau^*_\rho Y \,\big\rangle^{\bullet}_{\alpha}
\end{aligned}$$

holds by Proposition 4.2.7 and Lemma 4.2.2 as well as the previous Lemma 4.2.12 and the fact that the tensors $(k!)^{1/2} e_{i_1} \otimes \cdots \otimes e_{i_k}$ for all $k \in \mathbb{N}_0$ and $i \in I^k$ form a Hilbert base of $\mathcal{T}^{\bullet}(V)^{\mathrm{cpl}}_{\alpha}$. The second identity is a direct consequence of Theorem 4.1.19 because τ_ρ^* is a unital *-automorphism of $\star_{\Lambda_\alpha}$. Indeed, we have

$$\langle\, \tau_\rho^* X \,|\, \tau_\rho^* Y \,\rangle_\alpha^\bullet = \big\langle (\tau_\rho^* X)^* \star_{\Lambda_\alpha} (\tau_\rho^* Y) \big\rangle_0 = \big\langle \tau_\rho^* \big(X^* \star_{\Lambda_\alpha} Y \big) \big\rangle_0 = \widehat{X^* \star_{\Lambda_\alpha} Y}(\rho). \qquad \square$$

Corollary 4.2.14 *[77, Cor. 3.24] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$, then $\widehat{X} \in \mathscr{C}^{\omega_{HS}}(V_{\mathrm{H}}')$.*

PROOF: The function $\widehat{X}$ is smooth by Proposition 4.2.7. By the previous Proposition 4.2.13, we have

$$\sum_{k=0}^\infty \frac{1}{k!} \sum_{i \in I^k} \Big| \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} \widehat{X} \Big)(\xi) \Big|^2 = \langle\!\langle \widehat{X} \,|\, \widehat{X} \rangle\!\rangle_\alpha^\bullet(\xi) = \widehat{X^* \star_{\Lambda_\alpha} X}(\xi)$$

for all $\langle \,\cdot \,|\, \cdot\, \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$, which is finite and depends smoothly on $\xi \in V_{\mathrm{H}}'$ by Proposition 4.2.7 again. Therefore it is uniformly bounded on line segments. $\square$

Lemma 4.2.15 *[77, Lemma 3.25] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V and $\langle \,\cdot \,|\, \cdot\, \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$. For every $f \in \mathscr{C}^{\omega_{HS}}(V_{\mathrm{H}}')$ there exists an $X_f \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ that fulfils $\langle\!\langle f \,|\, f \rangle\!\rangle_\alpha^\bullet(0) = \langle\!\langle \widehat{X}_f \,|\, \widehat{X}_f \rangle\!\rangle_\alpha^\bullet(0)$ and $\langle\!\langle f \,|\, \widehat{Y} \rangle\!\rangle_\alpha^\bullet(0) = \langle\!\langle \widehat{X}_f \,|\, \widehat{Y} \rangle\!\rangle_\alpha^\bullet(0)$ for all $Y \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ and all $\langle \,\cdot \,|\, \cdot\, \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$.*

PROOF: For every $\alpha \in \mathcal{I}_{V,\mathrm{H}}$ construct $X_{f,\alpha} \in \mathcal{S}^\bullet(V)_\alpha^{\mathrm{cpl}}$ as

$$X_{f,\alpha} \coloneqq \sum_{k=0}^\infty \frac{1}{k!} \sum_{i \in I^k} e_{i_1} \otimes \cdots \otimes e_{i_k} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f \Big)(0) \in \mathcal{S}^\bullet(V)_\alpha^{\mathrm{cpl}},$$

where $e \in (V_{\mathrm{H},\alpha}^{\mathrm{cpl}})^I$ is a Hilbert base of $V_{\mathrm{H},\alpha}^{\mathrm{cpl}}$ indexed by a set I. This infinite sum $X_{f,\alpha}$ indeed lies in $\mathcal{S}^\bullet(V)_\alpha^{\mathrm{cpl}}$ and fulfils $\langle\, X_{f,\alpha} \,|\, X_{f,\alpha} \,\rangle_\alpha^\bullet = \langle\!\langle f \,|\, f \rangle\!\rangle_\alpha^\bullet(0)$, because $\big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f\big)(0)$ is invariant under permutations of the $e_{i_1}, \dots, e_{i_k}$ due to the smoothness of f and because

$$\begin{aligned}
&\sum_{k,\ell=0}^\infty \sum_{i \in I^k, i' \in I^\ell} \frac{1}{k!\ell!} \Big\langle\, e_{i_1} \otimes \cdots \otimes e_{i_k} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f \Big)(0) \,\Big|\, e_{i_1'} \otimes \cdots \otimes e_{i_\ell'} \Big(\widehat{D}^{(\ell)}_{e_i^{\flat_\alpha}} f \Big)(0) \,\Big\rangle_\alpha^\bullet \\
&\quad = \sum_{k=0}^\infty \sum_{i \in I^k} \frac{1}{k!} \Big| \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f \Big)(0) \Big|^2 \\
&\quad = \langle\!\langle f \,|\, f \rangle\!\rangle_\alpha^\bullet(0).
\end{aligned}$$

Moreover, for all $Y \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$, the identity

$$\begin{aligned}
\langle\!\langle f \,|\, \widehat{Y} \rangle\!\rangle_\alpha^\bullet(0) &= \sum_{k=0}^\infty \frac{1}{k!} \sum_{i \in I^k} \overline{\Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f \Big)(0)} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} \widehat{Y} \Big)(0) \\
&= \sum_{k=0}^\infty \frac{1}{k!} \sum_{i \in I^k} \langle\, X_{f,\alpha} \,|\, e_{i_1} \otimes \cdots \otimes e_{i_k} \,\rangle_\alpha^\bullet \langle\, e_{i_1} \otimes \cdots \otimes e_{i_k} \,|\, Y \,\rangle_\alpha^\bullet \\
&= \langle\, X_{f,\alpha} \,|\, Y \,\rangle_\alpha^\bullet
\end{aligned}$$

holds due to the construction of $X_{f,\alpha}$ and Lemma 4.2.12 and because the tensors $(k!)^{1/2} e_{i_1} \otimes \cdots \otimes e_{i_k}$ for all $k \in \mathbb{N}_0$ and all $i \in I^k$ are a Hilbert base of $\mathcal{T}^\bullet(V)^{\mathrm{cpl}}_\alpha$.

Next, let $\langle \cdot \,|\, \cdot \rangle_\beta \in \mathcal{I}_{V,\mathrm{H}}$ with $\langle \cdot \,|\, \cdot \rangle_\beta \le \langle \cdot \,|\, \cdot \rangle_\alpha$ and a Hilbert basis $d \in (V^{\mathrm{cpl}}_{h,\beta})^J$ of $V^{\mathrm{cpl}}_{h,\beta}$ indexed by a set J be given. Using the explicit formulas and the identity

$$\Big(\widehat{D}^{(k)}_{d_j^{\flat_\beta}} f\Big)(0) = \frac{1}{k!} \sum_{i \in I^k} \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f\Big)(0) \langle\, d_{j_1} \otimes \cdots \otimes d_{j_k} \,|\, \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \,\rangle^\bullet_\beta$$

from Lemma 4.2.8 one can now calculate that

$$\begin{aligned}
\iota_{\alpha\beta}(X_{f,\alpha}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f\Big)(0) \\
&= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \sum_{i \in I^k} \sum_{j \in J^k} d_{j_1} \otimes \cdots \otimes d_{j_k} \langle\, d_{j_1} \otimes \cdots \otimes d_{j_k} \,|\, \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \,\rangle^\bullet_\beta \Big(\widehat{D}^{(k)}_{e_i^{\flat_\alpha}} f\Big)(0) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j \in J^k} d_{j_1} \otimes \cdots \otimes d_{j_k} \Big(\widehat{D}^{(k)}_{d_j^{\flat_\beta}} f\Big)(0) \\
&= X_{f,\beta}\,.
\end{aligned}$$

As $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ is the projective limit of the Hilbert spaces $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}_\alpha$, this implies that there exists a unique $X_f \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ that fulfils $\iota_{\infty\alpha}(X_f) = X_{f,\alpha}$ for all $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$. Consequently and with the help of Proposition 4.2.13,

$$\big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{Y} \big\rangle\!\big\rangle^\bullet_\alpha(0) = \langle\, X_f \,|\, Y \,\rangle^\bullet_\alpha = \langle\, \iota_{\infty\alpha}(X_f) \,|\, Y \,\rangle^\bullet_\alpha = \langle\, X_{f,\alpha} \,|\, Y \,\rangle^\bullet_\alpha = \big\langle\!\big\langle f \,\big|\, \widehat{Y} \big\rangle\!\big\rangle^\bullet_\alpha(0)$$

holds for all $Y \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ and all $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$, and similarly,

$$\big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) = \langle\, X_f \,|\, X_f \,\rangle^\bullet_\alpha = \langle\, \iota_{\infty\alpha}(X_f) \,|\, \iota_{\infty\alpha}(X_f) \,\rangle^\bullet_\alpha = \langle\, X_{f,\alpha} \,|\, X_{f,\alpha} \,\rangle^\bullet_\alpha = \big\langle\!\big\langle f \,\big|\, f \big\rangle\!\big\rangle^\bullet_\alpha(0). \qquad \square$$

After this preparation we are now able to identify the image of the Gelfand transformation explicitly:

Theorem 4.2.16 *[77, Thm. 3.26] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V, then the Gelfand transformation $\widehat{\,\cdot\,} \colon \big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \cdot^*\big) \to \mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ is an isomorphism of unital *-algebras, and $\big(V'_{\mathrm{H}}, \widehat{\,\cdot\,}\big)$ a faithful continuous representation as functions of the locally convex *-algebra $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \cdot^*\big)$ like in Definition 3.3.11.*

PROOF: Let $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ be given, then $\widehat{X} \in \mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ by Corollary 4.2.14. The Gelfand transformation is a unital *-homomorphism onto its image by construction and injective because $\widehat{X} = 0$ implies $\langle\, X \,|\, X \,\rangle^\bullet_\alpha = \big\langle\!\big\langle \widehat{X} \,\big|\, \widehat{X} \big\rangle\!\big\rangle^\bullet_\alpha(0) = 0$ for all $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ by Proposition 4.2.13, hence $X = 0$. It only remains to show that $\widehat{\,\cdot\,}$ is surjective, so let $f \in \mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ be given. Construct $X_f \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ like in the previous Lemma 4.2.15, then

$$\begin{aligned}
\big\langle\!\big\langle f - \widehat{X}_f \,\big|\, f - \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) &= \big\langle\!\big\langle f \,\big|\, f \big\rangle\!\big\rangle^\bullet_\alpha(0) - \big\langle\!\big\langle f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) - \big\langle\!\big\langle \widehat{X}_f \,\big|\, f \big\rangle\!\big\rangle^\bullet_\alpha(0) + \big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) \\
&= \big\langle\!\big\langle f \,\big|\, f \big\rangle\!\big\rangle^\bullet_\alpha(0) - \big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) - \big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) + \big\langle\!\big\langle \widehat{X}_f \,\big|\, \widehat{X}_f \big\rangle\!\big\rangle^\bullet_\alpha(0) \\
&= 0
\end{aligned}$$

holds for all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$, hence $f = \widehat{X}_f$ due to Proposition 4.2.10. It is now clear that $\big(V'_{\mathrm{H}}, \widehat{\cdot}\big)$ is a faithful continuous representation as functions. $\square$

Let $\overline{\cdot}$ be a continuous antilinear involution on V. For a continuous bilinear form Λ on V the identity

$$\mathrm{P}_\Lambda(X \otimes_\pi Y) = \sum_{i,i' \in I} \Lambda(e_i, e_{i'}) \big(D_{e_i^{\flat_\alpha}} X \otimes_\pi D_{e_{i'}^{\flat_\alpha}} Y\big) \tag{4.2.21}$$

holds for all $X, Y \in \mathcal{S}^\bullet(V)$ and every $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ for which $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$ and for every Hilbert base $e \in (V^{\mathrm{cpl}}_{\mathrm{H},\alpha})^I$ indexed by a set I. Thus

$$\widehat{X} \widehat{\star_\Lambda} \widehat{Y} \coloneqq \widehat{X \star_\Lambda Y} = \mu\bigg(\sum_{t=0}^\infty \frac{1}{t!} \bigg(\sum_{i,i' \in I} \Lambda(e_i, e_{i'}) \big(\widehat{D}_{e_i^{\flat_\alpha}} \otimes \widehat{D}_{e_{i'}^{\flat_\alpha}}\big)\bigg)^t \big(\widehat{X} \otimes \widehat{Y}\big)\bigg)$$

with $\mu\colon \mathscr{C}^\infty(V'_{\mathrm{H}}) \otimes \mathscr{C}^\infty(V'_{\mathrm{H}}) \to \mathscr{C}^\infty(V'_{\mathrm{H}})$ the pointwise product is the usual exponential star product on $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$. Moreover, if $\mathcal{A} \subseteq \mathscr{C}^\infty(V'_{\mathrm{H}})$ is any unital *-subalgebra on which all such products $\widehat{\star_\Lambda}$ for all continuous Hermitian bilinear forms Λ on V converge, then $\mathcal{A} \subseteq \mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$, because analogous to Proposition 4.2.13, every $f \in \mathcal{A}$ fulfils $\langle\!\langle f \mid f \rangle\!\rangle^\bullet_\alpha = f^* \widehat{\star_{\Lambda_\alpha}} f \in \mathcal{A} \subseteq \mathscr{C}^\infty(V'_{\mathrm{H}})$ for all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ with corresponding continuous Hermitian bilinear form $V^2 \ni (v,w) \mapsto \Lambda_\alpha(v,w) \coloneqq \langle \overline{v} \mid w \rangle_\alpha \in \mathbb{C}$. This is of course just Theorem 4.1.19 again.

Note also that the isomorphism of *-algebras constructed in Theorem 4.2.16 becomes an isomorphism of locally convex *-algebras if one simply transfers the topology from $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ to $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$. From Proposition 4.2.13 it then follows that the resulting topology on $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ can be defined intrinsically as the hilbertisable one coming from the positive Hermitian forms $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})^2 \ni (f,g) \mapsto \langle\!\langle f \mid g \rangle\!\rangle(\rho) \in \mathbb{C}$ for $\rho = 0$, or for any other choice of $\rho \in V'_{\mathrm{H}}$ as the translations τ^*_ρ are homeomorphisms of $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$.

4.2.2 Equivalence of Star Products

Before discussing the properties of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$ as an abstract O^*-algebra e.g. whether it is Hausdorff or not, it will be helpful to understand which star products are equivalent, i.e. isomorphic via a continuous unital *-isomorphism. In the formal case, equivalence transformations between star products of exponential type can be constructed using exponentials of a Laplace operator (see [83] for the algebraic background). So the question arises under which conditions this exponential converges.

Definition 4.2.17 *[77, Def. 3.6] Let $b\colon V \times V \to \mathbb{C}$ be a symmetric bilinear form on V, i.e. $b(v,w) = b(w,v)$ for all $v, w \in V$. Then define the* Laplace operator $\Delta_b\colon \mathcal{T}^\bullet(V) \to \mathcal{T}^{\bullet-2}(V)$ *as the linear map given on simple tensors of degree $k \in \mathbb{N}\backslash\{1\}$ by*

$$\Delta_b(x_1 \otimes \cdots \otimes x_k) \coloneqq \frac{k(k-1)}{2} b(x_1, x_2) x_3 \otimes \cdots \otimes x_k. \tag{4.2.22}$$

Note that Δ_b can be restricted to symmetric tensors on which it coincides with the Laplace operator from [83, Eq. (2.31)]. However, there is no need for Δ_b to be continuous even if b is continuous, because the Hilbert tensor product in general does not allow the extension of all continuous multilinear forms.

Note that this is very different from the approach taken in [83] where the projective tensor product was used: this guaranteed the continuity of the Laplace operator directly for all continuous bilinear forms.

For the restriction of Δ_b to $\mathcal{S}^2(V)$, continuity is equivalent to the existence of a $\|\cdot\|_\alpha \in \mathcal{P}_V$ that fulfils $|\Delta_b X| \leq \|X\|_\alpha^\bullet$ for all $X \in \mathcal{S}^2(V)$. This motivates the following:

Definition 4.2.18 *[77, Def. 3.7] A* bilinear form of Hilbert-Schmidt type *on V is a bilinear form $b\colon V \times V \to \mathbb{C}$, for which there is a seminorm $\|\cdot\|_\alpha \in \mathcal{P}_V$ such that the following is fulfilled:*

i.) If $\|v\|_\alpha = 0$ or $\|w\|_\alpha = 0$ for vectors $v, w \in V$, then $b(v,w) = 0$.

ii.) For every tuple of $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha$-orthonormal vectors $e \in V^d$, $d \in \mathbb{N}$, the estimate

$$\sum_{i,j=1}^{d} |b(e_i, e_j)|^2 \leq 1 \tag{4.2.23}$$

holds.

For such a bilinear form of Hilbert-Schmidt type b, define $\mathcal{P}_{V,b,HS}$ as the set of all $\|\cdot\|_\alpha \in \mathcal{P}_V$ that fulfil these two conditions.

Proposition 4.2.19 *[77, Prop. 3.8] Let b be a symmetric bilinear form on V and $\|\cdot\|_\alpha \in \mathcal{P}_V$, then the following two statements are equivalent:*

i.) The bilinear form b is of Hilbert-Schmidt type and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.

ii.) The estimate $|\Delta_b X| \leq 2^{-1/2}\|X\|_\alpha^\bullet$ holds for all $X \in \mathcal{S}^2(V)$.

Moreover, if this holds then $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$ and b is continuous.

Proof: If the first point holds, let $X \in \mathcal{T}^2(V)$ be given. Construct $X_0 = \sum_{a\in A} x_{a,1} \otimes x_{a,2}$ and $\tilde{X} = \sum_{a_1',a_2'=1}^{d} X^{a_1',a_2'} e_{a_1'} \otimes e_{a_2'} \in \mathcal{T}^2(V)$ like in Lemma 4.1.3. Then $b(x_{a,1}, x_{a,2}) = 0$ for all $a \in A$ because $\|x_{a,1}\|_\alpha = 0$ or $\|x_{a,2}\|_\alpha = 0$. Moreover,

$$\begin{aligned}
|\Delta_b X| &\leq \left| \sum\nolimits_{a_1',a_2'=1}^{d} X^{a_1',a_2'} b(e_{a_1'}, e_{a_2'}) \right| \\
&\overset{\text{CS}}{\leq} \left(\sum\nolimits_{a_1',a_2'=1}^{d} \left|X^{a_1',a_2'}\right|^2 \right)^{\frac{1}{2}} \left(\sum\nolimits_{a_1',a_2'=1}^{d} \left|b(e_{a_1'}, e_{a_2'})\right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}} \|X\|_\alpha^\bullet
\end{aligned}$$

shows that the second point holds. Conversely, the second point yields $|b(v,w)| = |\Delta_b(v \vee w)| \leq 2^{-1/2}\|v \vee w\|_\alpha^\bullet \leq \|v\|_\alpha \|w\|_\alpha$ for all $v, w \in V$. Hence $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$, the bilinear form b is continuous, and $b(v,w) = 0$ if one of v or w is in the kernel of $\|\cdot\|_\alpha$. Moreover, given an $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha$-orthonormal set of vectors $e \in V^d$, $d \in \mathbb{N}$, construct $X := \sum_{i,j=1}^{d} \overline{b(e_i, e_j)} e_i \otimes e_j \in \mathcal{S}^2(V)$, then

$$0 \leq \sum_{i,j=1}^{d} |b(e_i, e_j)|^2 = |\Delta_b X| \leq \frac{1}{\sqrt{2}} \|X\|_\alpha^\bullet = \left(\sum\nolimits_{i,j=1}^{d} |b(e_i, e_j)|^2 \right)^{\frac{1}{2}},$$

which implies $\sum_{i,j=1}^{d} |b(e_i, e_j)|^2 \leq 1$. $\square$

Note that this also implies that for a bilinear form of Hilbert-Schmidt type b, the set $\mathcal{P}_{V,b,HS}$ is cofinal in $\mathcal{P}_V$, because if $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$, $\|\cdot\|_\beta \in \mathcal{P}_V$ and $\|\cdot\|_\beta \geq \|\cdot\|_\alpha$, then $|\Delta_b X| \leq 2^{-1/2}\|X\|^\bullet_\alpha \leq 2^{-1/2}\|X\|^\bullet_\beta$ and so $\|\cdot\|_\beta \in \mathcal{P}_{V,b,HS}$.

As a consequence of the above characterization we see that a symmetric bilinear form b on V has to be of Hilbert-Schmidt type if we want Δ_b to be continuous. We are going to show now that this is also sufficient:

Proposition 4.2.20 *[77, Prop. 3.9] Let b be a symmetric bilinear form of Hilbert-Schmidt type on V, then the Laplace operator Δ_b is continuous and fulfils the estimate*

$$\left\|(\Delta_b)^t X\right\|^\bullet_\alpha \leq \frac{\sqrt{(2t)!}}{(2r)^t}\|X\|^\bullet_{2r\alpha} \tag{4.2.24}$$

for all $X \in \mathcal{T}^\bullet(V)$, $t \in \mathbb{N}_0$, $r \geq 1$, and all $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.

Proof: First, let $X \in \mathcal{T}^k(V)$, $k \geq 2$, and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$ be given. Construct $X_0 = \sum_{a\in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ and $\tilde{X} = \sum_{a'\in\{1,\dots,d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ like in Lemma 4.1.3. Then again

$$\|\Delta_b X_0\|^\bullet_\alpha \leq \frac{k(k-1)\sqrt{(k-2)!}}{2}\sum_{a\in A}|b(x_{a_1},x_{a_2})|\prod_{m=3}^k \|x_{a_m}\|_\alpha = 0$$

shows that $\|\Delta_b X\|^\bullet_\alpha \leq \|\Delta_b \tilde{X}\|^\bullet_\alpha$. For $\tilde{X}$ we get:

$$\begin{aligned}
\|\Delta_b\tilde{X}\|^{\bullet,2}_\alpha &= \left\|\frac{k(k-1)}{2}\sum_{a'\in\{1,\dots,d\}^k} X^{a'} b\big(e_{a'_1},e_{a'_2}\big)e_{a'_3}\otimes\cdots\otimes e_{a'_k}\right\|^{\bullet,2}_\alpha \\
&= \frac{k^2(k-1)^2}{4}\sum_{\tilde{a}'\in\{1,\dots,d\}^{k-2}}\left\|\sum_{g,h=1}^d X^{(g,h,\tilde{a}')}b(e_g,e_h)e_{\tilde{a}'_1}\otimes\cdots\otimes e_{\tilde{a}'_{k-2}}\right\|^{\bullet,2}_\alpha \\
&= \frac{k^2(k-1)^2}{4}\sum_{\tilde{a}'\in\{1,\dots,d\}^{k-2}}\left|\sum_{g,h=1}^d X^{(g,h,\tilde{a}')}b(e_g,e_h)\right|^2(k-2)! \\
&\leq \frac{k(k-1)k!}{4}\sum_{\tilde{a}'\in\{1,\dots,d\}^{k-2}}\left(\sum_{g,h=1}^d \big|X^{(g,h,\tilde{a}')}\big||b(e_g,e_h)|\right)^2 \\
&\overset{\text{CS}}{\leq} \frac{k(k-1)k!}{4}\sum_{\tilde{a}'\in\{1,\dots,d\}^{k-2}}\left(\sum_{g,h=1}^d \big|X^{(g,h,\tilde{a}')}\big|^2\right)\left(\sum_{g,h=1}^d |b(e_g,e_h)|^2\right) \\
&\leq \frac{k(k-1)k!}{4}\sum_{a'\in\{1,\dots,d\}^k}|X^{a'}|^2 \\
&= \frac{k(k-1)}{4}\|X\|^{\bullet,2}_\alpha.
\end{aligned}$$

Using this we get

$$\left\|(\Delta_b)^t X\right\|^{\bullet,2}_\alpha = \sum_{k=2t}^\infty \left\|(\Delta_b)^t\langle X\rangle_k\right\|^{\bullet,2}_\alpha$$

$$
\begin{aligned}
&\leq \sum_{k=2t}^{\infty} \binom{k}{2t} \frac{(2t)!}{4^t} \|\langle X\rangle_k\|_{\alpha}^{\bullet,2} \\
&\leq \frac{(2t)!}{4^t} \sum_{k=2t}^{\infty} \frac{1}{r^k} \|\langle X\rangle_k\|_{2r\alpha}^{\bullet,2} \\
&\leq \frac{(2t)!}{(2r)^{2t}} \|X\|_{2r\alpha}^{\bullet,2}
\end{aligned}
$$

for arbitrary $X \in \mathcal{T}^\bullet(V)$ and $t \in \mathbb{N}$. Finally, the estimate (4.2.24) also holds in the case $t = 0$. □

Theorem 4.2.21 *[77, Thm. 3.10] Let b be a symmetric bilinear form on V, then the linear operator $\mathrm{e}^{\Delta_b} = \sum_{t=0}^{\infty} \frac{1}{t!}(\Delta_b)^t$ as well as its restriction to $\mathcal{S}^\bullet(V)$ are continuous if and only if b is of Hilbert-Schmidt type. In this case*

$$
\mathrm{e}^{\Delta_b}\big(X \star_\Lambda Y\big) = \big(\mathrm{e}^{\Delta_b} X\big) \star_{\Lambda+b} \big(\mathrm{e}^{\Delta_b} Y\big) \tag{4.2.25}
$$

holds for all $X, Y \in \mathcal{S}^\bullet(V)$ and all continuous bilinear forms Λ on V. Hence e^{Δ_b} describes an isomorphism of the locally convex algebras $\big(\mathcal{S}^\bullet(V), \star_\Lambda\big)$ and $\big(\mathcal{S}^\bullet(V), \star_{\Lambda+b}\big)$. Moreover, for fixed $X \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}}$, the series $\mathrm{e}^{z\Delta_b} X$ converges absolutely and locally uniformly in $z \in \mathbb{C}$ and thus depends holomorphically on z.

Proof: As $|\Delta_b X| \leq \|\mathrm{e}^{\Delta_b} X\|_\alpha^\bullet$ holds for all $\|\cdot\|_\alpha \in \mathcal{P}_V$ and all $X \in \mathcal{S}^2(V)$, it follows from Proposition 4.2.19 that continuity of the restriction of e^{Δ_b} to $\mathcal{S}^\bullet(V)$ implies that b is of Hilbert-Schmidt type. Conversely, for all $X \in \mathcal{T}^\bullet(V)$, all $\alpha \in \mathcal{P}_{V,b,HS}$, and $r > 1$, the estimate

$$
\big\|\mathrm{e}^{z\Delta_b} X\big\|_\alpha \leq \sum_{t=0}^{\infty} \frac{1}{t!} \big\|\big(z\Delta_b\big)^t(X)\big\|_\alpha \leq \sum_{t=0}^{\infty} \frac{|z|^t}{(4r)^t} \binom{2t}{t}^{\frac{1}{2}} \|X\|_{4r\alpha}^\bullet \leq \sum_{t=0}^{\infty} \frac{1}{2^t} \|X\|_{4r\alpha}^\bullet = 2\|X\|_{4r\alpha}^\bullet
$$

holds for all $z \in \mathbb{C}$ with $|z| \leq r$ due to the previous Proposition 4.2.20 if b is of Hilbert-Schmidt type, which proves the continuity of $\mathrm{e}^{z\Delta_b}$ for all $z \in \mathbb{C}$ as well as the absolute and locally uniform convergence of the series $\mathrm{e}^{z\Delta_b} X$. The algebraic relation (4.2.25) is well-known, see e.g. [83, Prop. 2.18]. Finally, as e^{Δ_b} is invertible with inverse $\mathrm{e}^{-\Delta_b}$, and because Δ_b and thus e^{Δ_b} map symmetric tensors to symmetric ones, we conclude that the restriction of e^{Δ_b} to $\mathcal{S}^\bullet(V)$ is an isomorphism of the locally convex algebras $\big(\mathcal{S}^\bullet(V), \star_\Lambda\big)$ and $\big(\mathcal{S}^\bullet(V), \star_{\Lambda+b}\big)$. □

Note that, if V is endowed with a continuous antilinear involution $\overline{\,\cdot\,}$ and b is a symmetric Hermitian bilinear form of Hilbert-Schmidt type on V, then one immediately sees that Δ_b, hence also e^{Δ_b}, commutes with $\cdot^*$.

4.2.3 Existence of Continuous Algebraically Positive Linear Functionals

In order to prove that the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$ is Hausdorff under a certain condition on Λ, and thus has a faithful representation as operators, we have to show that there exist continuous algebraically positive linear functionals on $\mathcal{S}^\bullet(V)$. In order to do so, one can apply an argument similar to the one used in [20] in the formal case, and construct algebraically positive linear functionals on the quantum *-algebra with $\hbar > 0$ out of those on the classical one with $\hbar = 0$.

Lemma 4.2.22 *[77, Lemma 3.28] Let $\overline{\cdot}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear form on V such that $\Lambda(\overline{v}, v) \geq 0$ holds for all $v \in V$. Then for all $X \in \mathcal{S}^\bullet(V)$ and all $t \in \mathbb{N}_0$ there exist $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \mathcal{S}^\bullet(V)$ such that*

$$\big(\mathrm{P}_\Lambda\big)^t(X^* \otimes_\pi X) = \sum_{i=1}^n X_i^* \otimes_\pi X_i. \tag{4.2.26}$$

PROOF: This is trivial for $t = 0$ and for the remaining cases it is sufficient to consider $t = 1$, the others then follow by induction. So let $X \in \mathcal{S}^\bullet(V)$ be given and assume that $\langle X \rangle_0 = 0$. This is not a restriction because $\mathrm{P}_\Lambda(X^* \otimes_\pi X)$ is independent of the scalar component of X. Expand X as $X = \sum_{j=1}^m x_{j,1} \vee \dots \vee x_{j,k_j}$ with $m \in \mathbb{N}$ and vectors $x_{1,1}, \dots, x_{m,k_m} \in V$. Then

$$\mathrm{P}_\Lambda(X^* \otimes_\pi X) = \sum_{j',j=1}^m \sum_{\ell',\ell=1}^{k_{j'},k_j} \Lambda(\overline{x_{j',\ell'}}, x_{j,\ell})(x_{j',1} \vee \cdots \widehat{x_{j',\ell'}} \cdots \vee x_{j',k_j})^* \otimes_\pi (x_{j,1} \vee \cdots \widehat{x_{j,\ell}} \cdots \vee x_{j,k_j}),$$

where $\widehat{\cdot}$ denotes omission of a vector in the product. The complex $s \times s$ -matrix, $s = \sum_{j=1}^m k_j$, with entries $\Lambda(\overline{x_{j',\ell'}}, x_{j,\ell})$ is positive semi-definite due to the positivity condition on Λ, which implies that it has a Hermitian square root $R \in \mathbb{C}^{s \times s}$ that fulfils $\Lambda(\overline{x_{j',\ell'}}, x_{j,\ell}) = \sum_{p=1}^m \sum_{q=1}^{k_p} \overline{R_{(p,q),(j',\ell')}} R_{(p,q),(j,\ell)}$ for all $j, j' \in \{1, \dots, m\}$ and $\ell \in \{1, \dots, k_j\}$, $\ell' \in \{1, \dots, k_{j'}\}$. Consequently,

$$\begin{aligned} \mathrm{P}_\Lambda(X^* \otimes_\pi X) &= \sum_{p=1}^m \sum_{q=1}^{k_p} X_{(p,q)}^* \otimes_\pi X_{(p,q)} \\ \text{with} \qquad X_{(p,q)} &\coloneqq \sum_{j=1}^m \sum_{\ell=1}^{k_j} R_{(p,q),(j,\ell)}(x_{j,1} \vee \cdots \widehat{x_{j,\ell}} \cdots \vee x_{j,k}) \end{aligned}$$

holds, which proves the lemma. □

Proposition 4.2.23 *[77, Prop. 3.29] Let $\overline{\cdot}$ be a continuous antilinear involution of V and Λ, Λ' as well as b three continuous Hermitian bilinear forms on V such that b is symmetric and of Hilbert-Schmidt type and such that $\Lambda'(\overline{v}, v) + b(\overline{v}, v) \geq 0$ holds for all $v \in V$. Given a continuous linear functional ω on $\mathcal{S}^\bullet(V)$ that is algebraically positive for $\star_\Lambda$, define $\omega_{zb} \colon \mathcal{S}^\bullet(V) \to \mathbb{C}$ as*

$$X \;\mapsto\; \omega_{zb}(X) \coloneqq \omega\big(\mathrm{e}^{z\Delta_b} X\big) \tag{4.2.27}$$

for all $z \in \mathbb{R}$. Then ω_{zb} is a continuous linear functional and algebraically positive for $\star_{\Lambda + z\Lambda'}$.

PROOF: It follows from Theorem 4.2.21 that ω_{zb} is continuous, and given $X \in \mathcal{S}^\bullet(V)$, then

$$\begin{aligned} \omega\big(\mathrm{e}^{z\Delta_b}(X^* \star_{\Lambda+z\Lambda'} X)\big) &= \omega\big((\mathrm{e}^{z\Delta_b} X)^* \star_{\Lambda+z(\Lambda'+b)} (\mathrm{e}^{z\Delta_b} X)\big) \\ &= \sum_{r=0}^\infty \frac{1}{r!} \omega\Big(\mu_\vee\Big(\big(\mathrm{P}_\Lambda + \mathrm{P}_{z(\Lambda'+b)}\big)^r \big((\mathrm{e}^{z\Delta_b} X)^* \otimes_\pi (\mathrm{e}^{z\Delta_b} X)\big)\Big)\Big) \\ &= \sum_{s,t=0}^\infty \frac{1}{s!t!} \omega\Big(\mu_\vee\Big(\big(\mathrm{P}_\Lambda\big)^s \big(\mathrm{P}_{z(\Lambda'+b)}\big)^t \big((\mathrm{e}^{z\Delta_b} X)^* \otimes_\pi (\mathrm{e}^{z\Delta_b} X)\big)\Big)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty} \frac{1}{t!} \omega\Big(\mu_{\star_\Lambda}\Big(\big(\mathrm{P}_{z(\Lambda'+b)}\big)^t\big((\mathrm{e}^{z\Delta_b}X)^* \otimes_\pi (\mathrm{e}^{z\Delta_b}X)\big)\Big)\Big) \\
&\geq 0
\end{aligned}
$$

holds because P_Λ and $\mathrm{P}_{z(\Lambda'+b')}$ commute on symmetric tensors and because of Lemma 4.2.22. □

Note that Theorem 4.2.21 also shows that ω_{zb} depends holomorphically on $z \in \mathbb{C}$ in so far as $\mathbb{C} \ni z \mapsto \omega_{zb}(X) \in \mathbb{C}$ is holomorphic for all $X \in \mathcal{S}^\bullet(V)$. This is the analog of statements in [49, 53] in the C^*-algebra setting.

Proposition 4.2.24 *[77, Prop. 3.30] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear forms on V. If there exists a continuous linear functional ω on $\mathcal{S}^\bullet(V)$ that is algebraically positive for $\star_\Lambda$ and fulfils $\omega(1) = 1$, then the bilinear form $V^2 \ni (v, w) \mapsto b_\omega(v, w) := \omega(v \vee w) \in \mathbb{C}$ is symmetric, Hermitian, of Hilbert-Schmidt type and fulfils $\Lambda(\overline{v}, v) + b_\omega(\overline{v}, v) \geq 0$ for all $v \in V$.*

PROOF: It follows immediately from the construction of b_ω that this bilinear form is symmetric and it is Hermitian because $\overline{b_\omega(v, w)} = \overline{\omega(v \vee w)} = \omega(\overline{w} \vee \overline{v}) = b_\omega(\overline{w}, \overline{v})$ holds for all $v, w \in V$. Continuity of ω especially implies that there exists a $\langle \,\cdot\, | \,\cdot\, \rangle_\alpha \in \mathcal{I}_V$ such that $|\omega(X)| \leq 2^{-1/2}\|X\|_\alpha^\bullet$ holds for all $X \in \mathcal{S}^2(V)$, hence b_ω is of Hilbert-Schmidt type by Proposition 4.2.19 and because $\Delta_{b_\omega} X = \omega(X)$ for $X \in \mathcal{S}^2(V)$. Finally, $0 \leq \omega(v^* \star_\Lambda v) = \Lambda(\overline{v}, v) + b_\omega(\overline{v}, v)$ holds due to the positivity of ω. □

Theorem 4.2.25 *[77, Thm. 3.31] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear form on V. Assume $V \neq \{0\}$. There exists a non-zero continuous algebraically positive linear functional on $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \,\cdot\,^*\big)$ if and only if there exists a symmetric and Hermitian bilinear form of Hilbert-Schmidt type b on V such that $\Lambda(\overline{v}, v) + b(\overline{v}, v) \geq 0$ holds for all $v \in V$. In this case, the continuous algebraically positive linear functionals on the completion $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_\Lambda, \,\cdot\,^*\big)$ are point-separating, i.e. their common kernel is $\{0\}$, the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_\Lambda, \,\cdot\,^*, \mathcal{T}\big)$ is Hausdorff and $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_\Lambda, \,\cdot\,^*\big)$ has a faithful continuous representation as operators.*

PROOF: If a non-zero continuous algebraically positive linear functional ω exists on $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \,\cdot\,^*\big)$, then $\omega(1) \neq 0$ due to the Cauchy Schwarz inequality and we can rescale ω such that $\omega(1) = 1$. Then the previous Proposition 4.2.24 shows the existence of such a bilinear form b. Conversely, if such a bilinear form b exists, then Proposition 4.2.23 shows that all continuous linear functionals on $\mathcal{S}^\bullet(V)$ that are algebraically positive for $\vee$ can be deformed to continuous linear functionals that are algebraically positive for $\star_\Lambda$ by taking the pull-back with e^{Δ_b}. As e^{Δ_b} is invertible, it only remains to show that the continuous algebraically positive linear functionals on $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \vee, \,\cdot\,^*\big)$ are point-separating. This is an immediate consequence of Theorem 4.2.16, which especially shows that the evaluation functionals δ_ρ with $\rho \in V'_{\mathrm{H}}$ are point-separating. By definition, $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \,\cdot\,^*, \mathcal{T}\big)$ is Hausdorff and so the locally convex *-algebra $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_\Lambda, \,\cdot\,^*\big)$ has a faithful continuous representation as operators by Theorems 3.2.11 and 3.3.21 or 3.3.22. □

4.2.4 Stieltjes Elements

We have already seen (e.g. in Section 3.5), that the concept of Stieltjes elements of abstract O^*-algebras is very helpful, especially in cases of algebras allowing canonical commutation relations, which prohibit some elements from being bounded or even uniformly bounded. However, in order to prove the existence of many Stieltjes elements by applying Proposition 3.4.20, an estimate for the growth of continuous seminorms on powers of algebra elements is necessary. Note that the n-th power of $X \in \mathcal{S}^\bullet(V)$ with respect to a product $\diamond$ will be denoted by $X^{\diamond n}$ and the exponential by $\sum_{n=0}^{\infty} X^{\diamond n}/n! =: \exp_\diamond(X)$.

Definition 4.2.26 *[77, Def. 3.32] Define, for all $k \in \mathbb{N}_0$, the linear subspace*

$$\mathcal{S}^{(k)}(V) := \bigoplus_{\ell=0}^{k} \mathcal{S}^{\ell}(V), \tag{4.2.28}$$

of $\mathcal{S}^\bullet(V)$ as well as its closure $\mathcal{S}^{(k)}(V)^{\mathrm{cpl}}$ in $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$.

Lemma 4.2.27 *[77, Lemma 3.33] One has*

$$\binom{m}{\ell}\binom{m-\ell+t}{t} \leq \binom{\ell+t}{t}\binom{k(n+1)}{k} \tag{4.2.29}$$

for all $k, n \in \mathbb{N}_0$, $m \in \{0, \ldots, kn\}$, $t \in \{0, \ldots, k\}$, and all $\ell \in \big\{0, \ldots, \min\{m, k-t\}\big\}$.

PROOF: This inequality is equivalent to

$$\frac{m!}{(m-\ell)!}\,\frac{(m-\ell+t)!}{(m-\ell)!}\,\frac{k!}{(\ell+t)!} \leq \frac{(kn+k)!}{(kn)!}\,,$$

which is true because

$$\begin{aligned}
\frac{m!}{(m-\ell)!} &= \prod_{i=1}^{\ell}(m-\ell+i) &&\leq \prod_{i=1}^{\ell}(kn+i) \\
\frac{(m-\ell+t)!}{(m-\ell)!} &= \prod_{i=\ell+1}^{\ell+t}(m-2\ell+i) &&\leq \prod_{i=\ell+1}^{\ell+t}(kn+i) \\
\text{and} \qquad \frac{k!}{(\ell+t)!} &= \prod_{i=\ell+t+1}^{k} i &&\leq \prod_{i=\ell+t+1}^{k}(kn+i)\,.
\end{aligned}$$

□

Lemma 4.2.28 *[77, Lemma 3.34] Let Λ be a continuous bilinear form on V. Let $k, n \in \mathbb{N}_0$ and $X_1, \ldots, X_n \in \mathcal{S}^{(k)}(V)^{\mathrm{cpl}}$ be given. Then the estimates*

$$\big\|\big\langle X_1 \star_\Lambda \cdots \star_\Lambda X_n\big\rangle_m\big\|^\bullet_\alpha \leq \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} \big(2\mathrm{e}^2\big)^{kn}\|X_1\|^\bullet_\alpha \cdots \|X_n\|^\bullet_\alpha \tag{4.2.30}$$

and

$$\big\|X_1 \star_\Lambda \cdots \star_\Lambda X_n\big\|^\bullet_\alpha \leq \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} \big(2\mathrm{e}^3\big)^{kn}\|X_1\|^\bullet_\alpha \cdots \|X_n\|^\bullet_\alpha \tag{4.2.31}$$

hold for all $m \in \{0, \ldots, kn\}$ and all $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$.

P r o o f: The first estimate implies the second, because $\|X_1 \star_\Lambda \cdots \star_\Lambda X_n\|_\alpha^\bullet$ has at most $(1+kn)$ non-vanishing homogeneous components, namely those of degree $m \in \{0,\dots,kn\}$, and $(1+kn) \le \mathrm{e}^{kn}$. The first estimate can be proven by induction over n: If $n=0$ or $n=1$, then the estimate is clearly fulfilled for all possible k and m, and if it holds for one $n \in \mathbb{N}$, then

$$
\begin{aligned}
&\big\|\big\langle X_1 \star_\Lambda \cdots \star_\Lambda X_{n+1}\big\rangle_m\big\|_\alpha^\bullet \\
&\le \sum_{t=0}^{k} \frac{1}{t!}\Big\|\Big\langle \mu_\vee\Big(\big(\mathrm{P}_\Lambda\big)^t\big((X_1 \star_\Lambda \cdots \star_\Lambda X_n)\otimes_\pi X_{n+1}\big)\Big)\Big\rangle_m\Big\|_\alpha^\bullet \\
&\le \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \frac{1}{t!}\Big\|\mu_\vee\Big(\big(\mathrm{P}_\Lambda\big)^t\big(\langle X_1 \star_\Lambda \cdots \star_\Lambda X_n\rangle_{m-\ell+t}\otimes_\pi \langle X_{n+1}\rangle_{\ell+t}\big)\Big)\Big\|_\alpha^\bullet \\
&\le \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \frac{1}{t!}\binom{m}{\ell}^{\frac{1}{2}}\Big\|\big(\mathrm{P}_\Lambda\big)^t\big(\langle X_1 \star_\Lambda \cdots \star_\Lambda X_n\rangle_{m-\ell+t}\otimes_\pi \langle X_{n+1}\rangle_{\ell+t}\big)\Big\|_{\alpha\otimes_\pi\alpha}^\bullet \\
&\le \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \binom{m}{\ell}^{\frac{1}{2}}\binom{m-\ell+t}{t}^{\frac{1}{2}}\binom{\ell+t}{t}^{\frac{1}{2}}\|\langle X_1 \star_\Lambda \cdots \star_\Lambda X_n\rangle_{m-\ell+t}\|_\alpha^\bullet\|\langle X_{n+1}\rangle_{\ell+t}\|_\alpha^\bullet \\
&\le \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \binom{\ell+t}{t}\binom{k(n+1)}{k}^{\frac{1}{2}}\|\langle X_1 \star_\Lambda \cdots \star_\Lambda X_n\rangle_{m-\ell+t}\|_\alpha^\bullet\|\langle X_{n+1}\rangle_{\ell+t}\|_\alpha^\bullet \\
&\le \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \binom{\ell+t}{t}\binom{k(n+1)}{k}^{\frac{1}{2}}\bigg(\frac{(kn)!}{(k!)^n}\bigg)^{\frac{1}{2}}\big(2\mathrm{e}^2\big)^{kn}\|X_1\|_\alpha^\bullet\cdots\|X_n\|_\alpha^\bullet\|X_{n+1}\|_\alpha^\bullet \\
&= \sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}} \binom{\ell+t}{t}\bigg(\frac{(k(n+1))!}{(k!)^{n+1}}\bigg)^{\frac{1}{2}}\big(2\mathrm{e}^2\big)^{kn}\|X_1\|_\alpha^\bullet\cdots\|X_{n+1}\|_\alpha^\bullet \\
&\le \bigg(\frac{(k(n+1))!}{(k!)^{n+1}}\bigg)^{\frac{1}{2}}\big(2\mathrm{e}^2\big)^{k(n+1)}\|X_1\|_\alpha^\bullet\cdots\|X_{n+1}\|_\alpha^\bullet
\end{aligned}
$$

holds due to the grading of $\mu_\vee$ and P_Λ, the estimates from Propositions 4.1.6 as well as 4.1.7 and Lemma 4.1.10 for $\mu_\vee$ and P_Λ, and the previous Lemma 4.2.27. The very last step is a rather trivial estimate: The binomial coefficient of $\ell+t$ over t is at most 2^k, and the two remaining sums have at most $(1+k)^2$ terms, so

$$
\sum_{t=0}^{k}\sum_{\ell=0}^{\min\{m,k-t\}}\binom{\ell+t}{t} \le (1+k)^2 2^k \le (\mathrm{e}^k)^2 2^k = \big(2\mathrm{e}^2\big)^k . \qquad \square
$$

This estimate now allows to prove the following slight generalization of [77, Thm. 3.40]:

Theorem 4.2.29 *Let $\overline{\,\cdot\,}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear forms on V. Then all $X \in \mathcal{S}^{(4)}(V)$ which are positive in the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$, i.e. which fulfil $\langle\, \omega\,,\, X\,\rangle \ge 0$ for all continuous linear functionals ω on $\mathcal{S}^\bullet(V)$ which are algebraically positive for $\star_\Lambda$, are Stieltjes elements of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$. Moreover, in every continuous representation as operators of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$, all Hermitian elements in $\mathcal{S}^{(2)}(V)$ as well as and all semibounded Hermitian elements in $\mathcal{S}^{(4)}(V)$ will be essentially self-adjoint.*

PROOF: The previous Lemma 4.2.28 with $k = 4$ shows that

$$\big\| X^{\star_\Lambda n} \big\|^\bullet_\alpha \le \left(\frac{(4n)!}{(4!)^n}\right)^{\frac{1}{2}} \big(2\mathrm{e}^3\big)^{4n} \big(\|X\|^\bullet_\alpha\big)^n = (2n)! \binom{4n}{2n}^{\frac{1}{2}} \left(\frac{2\mathrm{e}^{12}\|X\|^\bullet_\alpha}{3}\right)^n \le (2n)^{2n} \left(\frac{8\mathrm{e}^{12}\|X\|^\bullet_\alpha}{3}\right)^n$$

holds for all $X \in \mathcal{S}^{(4)}(V)$ and $\|\cdot\|_\alpha \in \mathcal{P}_V$, so $\big(\big\|X^{\star_\Lambda n}\big\|^\bullet_\alpha\big)^{1/(2n)} \le 2n\sqrt{8\mathrm{e}^{12}\|X\|^\bullet_\alpha/3}$ for all $n \in \mathbb{N}$, which implies that either $\big\|X^{\star_\Lambda n}\big\|^\bullet_\alpha = 0$ for one $n \in \mathbb{N}$ or $\sum_{n=1}^\infty \|X^{\star_\Lambda n}\|_p^{-1/(2n)} = \infty$.

By Proposition 3.4.20, this implies that all $X \in \mathcal{S}^{(4)}(V)$ which are positive in the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$ are Stieltjes elements of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$. Then Corollary 3.4.19 together with Theorem 3.3.21 or 3.3.22 shows that in every continuous representation as operators of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$, all Hermitian elements $X \in \mathcal{S}^{(2)}(V)$ and all semibounded Hermitian elements in $\mathcal{S}^{(4)}(V)$ will be essentially self-adjoint. $\square$

This also allows to characterize the pure states in the commutative case as characters:

Corollary 4.2.30 *Let $\overline{\cdot}$ be a continuous antilinear involution of V, then the set of pure states of the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$ coincides with the set of characters, i.e. with the set $\{\, \delta_\rho \mid \rho \in V'_{\mathrm{H}} \,\}$.*

PROOF: This is just an application of Theorem 3.5.20 and the previous Theorem 4.2.29 to the commutative abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*, \mathcal{T}\big)$, which is downwards closed by Proposition 3.3.19, and generated as a unital *-algebra by the subset of coercive and pairwise commuting Stieltjes elements $Q' := \{\, \mathbb{1} + X^* \vee X \mid X \in \mathcal{S}^1(V) \,\}$, because $4Y = (\mathbb{1}+Y)^{\vee 2} - (\mathbb{1}-Y)^{\vee 2} = (\mathbb{1} + X_+^* \vee X_+) - (\mathbb{1} + X_-^* \vee X_-)$ with $X_+ = \mathbb{1} + Y$ and $X_- = \mathbb{1} - Y$ holds for all Hermitian $Y \in \mathcal{S}^1(V)$, which generate $\big(\mathcal{S}^\bullet(V), \vee, \cdot^*\big)$ as a unital *-algebra. The set of characters has explicitly been determined in Proposition 4.2.4 as $\{\, \delta_\rho \mid \rho \in V'_{\mathrm{H}} \,\}$. $\square$

It is interesting to note that the estimate provided by Lemma 4.2.28 is actually too good, or at least significantly better than needed in order to apply Theorem 3.5.20: Not only are coercive elements in $\mathcal{S}^{(2)}(V)$ Stieltjes elements (which already generate the whole algebra), but also coercive elements in $\mathcal{S}^{(4)}(V)$. As an example from physics, if $P, Q \in \mathcal{S}^1(V)$ are two Hermitian elements of first order, then not only the Hamiltonian of the harmonic oscillator $P^2 + Q^2$, but also of the perturbed oscillator $P^2 + Q^2 + \lambda Q^4$ with $\lambda \ge 0$ would be a Stieltjes element, hence essentially self-adjoint in every representation. Moreover, this estimate also allows to directly construct exponentials of first order elements in the algebra:

Proposition 4.2.31 *[77, Prop. 3.35] Let Λ be a continuous bilinear form on V, then $\exp_{\star_\Lambda}(v)$ is absolutely convergent and*

$$\exp_{\star_\Lambda}(v) = \mathrm{e}^{\frac{1}{2}\Lambda(v,v)} \exp_\vee(v) \tag{4.2.32}$$

holds for all $v \in V$. Moreover,

$$\exp_\vee(v) \star_\Lambda \exp_\vee(w) = \mathrm{e}^{\Lambda(v,w)} \exp_\vee(v+w) \tag{4.2.33}$$

and

$$\big\langle \exp_\vee(v) \,\big|\, \exp_\vee(w) \big\rangle^\bullet_\alpha = \mathrm{e}^{\langle\, v \,|\, w \,\rangle_\alpha} \tag{4.2.34}$$

hold for all $v, w \in V$ and all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$. Finally, $\exp_\vee(v)^* = \exp_\vee(\overline{v})$ for all $v \in V$ if V is equipped with a continuous antilinear involution $\overline{\cdot}$.

PROOF: The existence and absolute convergence of $\star_\Lambda$-exponentials of vectors follows directly from Lemma 4.2.28 with $k = 1$ and $X_1 = \dots = X_n = v$:

$$\sum_{n=0}^{\infty} \frac{\|v^{\star_\Lambda n}\|_\alpha^\bullet}{n!} \le \sum_{n=0}^{\infty} \frac{\big(4\mathrm{e}^3\|v\|_\alpha\big)^n}{\sqrt{n!}} \frac{1}{2^n} \overset{\mathrm{CS}}{\le} \bigg(\sum_{n=0}^{\infty} \frac{\big(4\mathrm{e}^3\|v\|_\alpha\big)^{2n}}{n!}\bigg)^{\frac{1}{2}} \bigg(\sum_{n=0}^{\infty} \frac{1}{4^n}\bigg)^{\frac{1}{2}} < \infty$$

The explicit formula can then be derived like in [83, Lem. 5.5]. For (4.2.33) just note that

$$\begin{aligned} \mathrm{P}_\Lambda\big(\exp_\vee(v) \otimes_\pi \exp_\vee(w)\big) &= \sum_{k,\ell=0}^{\infty} \mathrm{P}_\Lambda\bigg(\frac{v^{\vee k}}{k!} \otimes_\pi \frac{w^{\vee \ell}}{\ell!}\bigg) \\ &= \Lambda(v,w) \sum_{k,\ell=1}^{\infty} \frac{k v^{\vee(k-1)}}{k!} \otimes_\pi \frac{\ell w^{\vee(\ell-1)}}{\ell!} \\ &= \Lambda(v,w) \exp_\vee(v) \otimes_\pi \exp_\vee(w), \end{aligned}$$

and so

$$\exp_\vee(v) \star_\Lambda \exp_\vee(w) = \sum_{t=0}^{\infty} \frac{1}{t!} \mu_\vee\Big((\mathrm{P}_\Lambda)^t\big(\exp_\vee(v) \otimes_\pi \exp_\vee(w)\big)\Big) = \mathrm{e}^{\Lambda(v,w)} \exp_\vee(v) \vee \exp_\vee(w).$$

The remaining two identities are the results of straightforward calculations. □

As a consequence there exists a dense *-subalgebra consisting of uniformly bounded elements:

Definition 4.2.32 *[77, Def. 3.36] Let $\overline{\cdot}$ be a continuous antilinear involution on V, then define the linear subspace*

$$\mathcal{S}^\bullet_{\mathrm{per}}(V) := \big\langle\!\big\langle \{ \exp_\vee(\mathrm{i}v) \in \mathcal{S}^\bullet(V)^{\mathrm{cpl}} \mid v \in V_{\mathrm{H}} \} \big\rangle\!\big\rangle_{\mathrm{lin}} \tag{4.2.35}$$

of $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$.

Proposition 4.2.33 *[77, Prop. 3.37] Let $\overline{\cdot}$ be a continuous antilinear involution on V. Then $\mathcal{S}^\bullet_{\mathrm{per}}(V)$ is a dense *-subalgebra of $\big(\mathcal{S}^\bullet(V)^{\mathrm{cpl}}, \star_\Lambda, \cdot^*\big)$ with respect to all products $\star_\Lambda$ for all continuous bilinear Hermitian forms Λ on V. Moreover, for a fixed such Λ,*

$$\|X\|_\infty = \sup \sqrt{\omega(X^* \star_\Lambda X)} < \infty \tag{4.2.36}$$

holds for all $X \in \mathcal{S}^\bullet_{\mathrm{per}}(V)$, where the supremum runs over all continuous algebraic states ω of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$, i.e. $\|\cdot\|_\infty$ is just the seminorm from Definition 3.4.3 for the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$.

PROOF: Proposition 4.2.31 shows that $\mathcal{S}^\bullet_{\mathrm{per}}(V)$ is a *-subalgebra of $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ with respect to all products $\star_\Lambda$ for all continuous bilinear Hermitian forms Λ on V. As $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\big|_{z=0} \exp_\vee(\mathrm{i}zv) = v$ for all $v \in V$ with $v = \overline{v}$, we see that the closure of the subalgebra $\mathcal{S}^\bullet_{\mathrm{per}}(V)$ contains V, hence $\mathcal{S}^\bullet(V)$ which is (as a unital algebra) generated by V, and so the closure of $\mathcal{S}^\bullet_{\mathrm{per}}(V)$ coincides with $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$.

As $\mathcal{S}^\bullet_{\mathrm{per}}(V)$ is spanned by exponentials and $\omega\big(\exp_\vee(\mathrm{i}v)^* \star_\Lambda \exp_\vee(\mathrm{i}v)\big) = \mathrm{e}^{\Lambda(v,v)}\omega\big(\exp_\vee(0)\big) = \mathrm{e}^{\Lambda(v,v)}$ holds for all algebraic states ω on $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \,\cdot\,^*\big)$ by Proposition 4.2.31, it follows that $\|X\|_{\infty,\Lambda} < \infty$ for all $X \in \mathcal{S}^\bullet_{\mathrm{per}}(V)$. $\square$

In contrast to the existence of exponentials of vectors, there are strict constraints on the existence of exponentials of quadratic elements:

Proposition 4.2.34 *[77, Prop. 3.38] Let $\overline{\,\cdot\,}$ be a continuous antilinear involution on V. Then there is no locally convex topology τ on $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$ with the property that any (undeformed) exponential $\exp_\vee(X) = \sum_{n=0}^\infty \frac{X^{\vee n}}{n!}$ of any $X \in \mathcal{S}^2(V)\backslash\{0\}$ exists in the completion of $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$ under τ and such that all the products $\star_\Lambda$ for all continuous Hermitian bilinear forms Λ on V as well as the *-involution and the projection $\langle\,\cdot\,\rangle_0$ on the scalars are continuous.*

PROOF: Analogously to the proof of Theorem 4.1.19 we see that, if all the products $\star_\Lambda$ for all continuous Hermitian bilinear forms Λ on V as well as the *-involution and the projection $\langle\,\cdot\,\rangle_0$ on the scalars are continuous, then all the extended positive Hermitian forms $\langle\,\cdot\,|\,\cdot\,\rangle^\bullet_\alpha$ for all $\langle\,\cdot\,|\,\cdot\,\rangle_\alpha \in \mathcal{I}_V$ would have to be continuous and thus extend to the completion of $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$.

Now let $X \in \mathcal{S}^2(V) \setminus \{0\}$ be given. There exist $k \in \mathbb{N}$ and $x \in V^k$ such that $x_1, \dots, x_k$ are linearly independent and $X = \sum_{i=1}^k \sum_{j=i}^k \tilde{X}^{ij} x_i \vee x_j$ with complex coefficients $\tilde{X}^{ij}$. If there exists an $i \in \{1, \dots, k\}$ such that $\tilde{X}^{ii} \neq 0$, then we can assume without loss of generality that $i = 1$ and $\tilde{X}^{11} = 1$ and define a continuous positive Hermitian form on V by $\langle\, v \,|\, w \,\rangle_\omega := \overline{\omega(v)}\omega(w)$, where $\omega\colon V \to \mathbb{C}$ is a continuous linear form on V that satisfies $\omega(x_1) = 1$ and $\omega(x_i) = 0$ for $i \in \{2, \dots, k\}$. Otherwise we can assume without loss of generality that $\tilde{X}^{11} = \tilde{X}^{22} = 0$ and $\tilde{X}^{12} = 1$ and define a continuous positive Hermitian form on V by $\langle\, v \,|\, w \,\rangle_\omega := \overline{\omega(v)}^T \omega(w)$, where $\omega\colon V \to \mathbb{C}^2$ is a continuous linear map that satisfies $\omega(x_1) = \binom{1}{0}$, $\omega(x_2) = \binom{0}{1}$ and $\omega(x_i) = 0$ for $i \in \{3, \dots, k\}$.

In the first case, this results in $\langle\, X^{\vee n} \,|\, X^{\vee n} \,\rangle^\bullet_\omega = (2n)!$ and in the second, $\langle\, X^{\vee n} \,|\, X^{\vee n} \,\rangle^\bullet_\omega = (n!)^2$. So $\sum_{n=0}^\infty \frac{X^{\vee n}}{n!}$ cannot converge in the completion of $\mathcal{S}^\bullet_{\mathrm{alg}}(V)$ because

$$\left\langle \sum_{n=0}^N \frac{X^{\vee n}}{n!} \,\middle|\, \sum_{n=0}^N \frac{X^{\vee n}}{n!} \right\rangle^\bullet_\omega \geq \sum_{n=0}^N 1 \xrightarrow{N\to\infty} \infty. \qquad \square$$

A similar result has already been obtained by Omori, Maeda, Miyazaki and Yoshioka in the special 2-dimensional case in [63], where they show that associativity of the Weyl star product breaks down on exponentials of quadratic functions. Note that the above proposition does not exclude the possibility that exponentials of *some* quadratic functions exist if one only demands that *some* special deformations are continuous.

Furthermore, as all Hermitian quadratic elements will be essentially self-adjoint in all continuous representations by Theorem 4.2.29, which actually exist under certain circumstances by Theorem 4.2.25, it is of course possible to construct e.g. exponentials of such elements in representations using the functional calculus for self-adjoint operators. However, this leads to non-trivial domain issues, and [63] shows, that this means that the algebraic structure can not always be extended to these exponentials in a sensible way.

4.3 Special Cases and Examples

Finally, before closing this chapter, it makes sense to discuss two special cases that have appeared in the literature before, namely that V is a nuclear space (in which case it is well-known that its topology can be described by continuous Hilbert seminorms, i.e. is hilbertisable, see e.g. [46, Cor. 2]), or a Hilbert space.

4.3.1 Deformation Quantization of Nuclear Spaces

The case of a nuclear space V is especially interesting in light of the enormous importance of bilinear forms of Hilbert-Schmidt type for the equivalence of star products and the existence of continuous algebraically positive linear functionals, as all continuous bilinear forms on a nuclear space V are automatically of Hilbert-Schmidt type (see [46, Chap. 21.3, Thm. 5]). So in this case we get:

Theorem 4.3.1 *[77, Thm. 4.7] Let V be a Hausdorff nuclear space and $\overline{\cdot}$ a continuous antilinear involution of V as well as Λ a continuous Hermitian bilinear form on V, then there exist point-separating many continuous algebraically positive linear functionals of $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$, so the abstract O^*-algebra $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*, \mathcal{T}\big)$ is Hausdorff and $\big(\mathcal{S}^\bullet(V), \star_\Lambda, \cdot^*\big)$ has a faithful continuous representation as operators.*

PROOF: Choose some $\langle \cdot \,|\, \cdot \rangle_\alpha \in \mathcal{I}_{V,\mathrm{H}}$ such that $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$ and define a bilinear form b on V by $b(v,w) := \langle \overline{v} \,|\, w \rangle_\alpha$ for all $v, w \in V$. Then b is continuous and Hermitian by construction and symmetric due to the compatibility of $\langle \cdot \,|\, \cdot \rangle_\alpha$ with $\overline{\cdot}$. Moreover, $\Lambda(\overline{v}, v) \leq \|\overline{v}\|_\alpha \|v\|_\alpha = \|v\|_\alpha^2 = \langle v \,|\, v \rangle_\alpha = b(\overline{v}, v)$ holds for all $v \in V$ and b is of Hilbert-Schmidt type because every continuous bilinear form on a nuclear space is of Hilbert-Schmidt type (see [46, Chap. 21.3, Thm. 5]). Because of this, Theorem 4.2.25 applies. □

As Theorem 4.3.1 shows the existence of many continuous positive linear functionals in the nuclear case, this might be the best candidate for applications, because it allows to combine most of our results: The space $\mathcal{S}^\bullet(V)^{\mathrm{cpl}}$ has a clear interpretation as a space of certain analytic functions (Theorem 4.2.16) and its topology is essentially the coarsest possible one (Theorem 4.1.19). The usual equivalences of star products that are generated by continuous bilinear forms that differ only in the symmetric part still holds due to Theorem 4.2.21 and because all continuous bilinear forms on a nuclear space are of Hilbert-Schmidt type. Finally, the existence of many continuous positive linear functionals assures that there exist non-trivial representations of the deformed algebras, in which all Hermitian elements of up to degree 2, or 4 if semibounded – including many important elements from the point of view of physics – are represented by essentially self-adjoint operators (Theorem 4.2.29). Note that these results are very similar to the well-known properties that make C^*-algebras interesting for applications in physics, even though the topology on the algebra that we have considered here is far from C^*, indeed not even submultiplicative.

Moreover, it is also worthwhile to mention that for a nuclear space V, the construction presented here coincides with the one in [83]: The topology of the Hilbert tensor product on $\mathcal{S}^k(V)$ coincides with the topology of the projective tensor product which was examined in [83]. However, for the comparison of the topologies on $\mathcal{S}^\bullet(V)$, one has to be more careful: Let $\|\cdot\|_\alpha \in \mathcal{P}_V$ be given. Define the seminorm $\|\cdot\|^\bullet_{\alpha,\mathrm{pr}}$ as

$$\|X\|^\bullet_{\alpha,\mathrm{pr}} := \big|\langle X\rangle_0\big| + \sum_{k=1}^{\infty} \sqrt{k!}\inf \sum_{i\in I}\prod_{m=1}^{k}\|x_{i,m}\|_\alpha \tag{4.3.1}$$

for all $X \in \mathcal{T}^\bullet_{\mathrm{alg}}(V)$, where the infimum runs over all possibilities to express $\langle X\rangle_k$ as a finite sum of factorizing tensors, i.e. as $\langle X\rangle_k = \sum_{i\in I} x_{i,1}\otimes\cdots\otimes x_{i,k}$ with $x_i \in V^k$.

Lemma 4.3.2 *[77, Lemma 4.3] The estimate*

$$\|X\|^\bullet_\alpha \le \|X\|^\bullet_{\alpha,\mathrm{pr}} \tag{4.3.2}$$

holds for all $X \in \mathcal{T}^\bullet_{\mathrm{alg}}(V)$. Moreover, if there is a $\|\cdot\|_\beta \in \mathcal{P}_V$, $\|\cdot\|_\beta \ge \|\cdot\|_\alpha$, such that for every $\langle\,\cdot\,|\,\cdot\,\rangle_\beta$-orthonormal $e \in V^d$ and all $d \in \mathbb{N}$ the estimate $\sum_{i=1}^{d}\|e_i\|^2_\alpha \le 1$ holds, then

$$\|X\|^\bullet_{\alpha,\mathrm{pr}} \le \|X\|^\bullet_\beta \tag{4.3.3}$$

for all $X \in \mathcal{T}^\bullet_{\mathrm{alg}}(V)$.

PROOF: Let $X \in \mathcal{T}^\bullet_{\mathrm{alg}}(V)$ be given, then $\|X\|^\bullet_\alpha \le \sum_{k=0}^{\infty}\|\langle X\rangle_k\|^\bullet_\alpha$ and $\|X\|^\bullet_{\alpha,\mathrm{pr}} = \sum_{k=0}^{\infty}\|\langle X\rangle_k\|^\bullet_{\alpha,\mathrm{pr}}$. Thus it is sufficient for the first estimate to show that $\|\langle X\rangle_k\|^\bullet_\alpha \le \|\langle X\rangle_k\|^\bullet_{\alpha,\mathrm{pr}}$ for all $k \in \mathbb{N}_0$. Fix $k \in \mathbb{N}_0$ and assume that $\langle X\rangle_k = \sum_{i\in I} x_{i,1}\otimes\cdots\otimes x_{i,k}$ with $x_i \in V^k$. Then

$$\|\langle X\rangle_k\|^\bullet_\alpha \le \sum\nolimits_{i\in I}\|x_{i,1}\otimes\cdots\otimes x_{i,k}\|^\bullet_\alpha = \sqrt{k!}\sum\nolimits_{i\in I}\prod\nolimits_{m=1}^{k}\|x_{i,m}\|_\alpha$$

shows that $\|\langle X\rangle_k\|^\bullet_\alpha \le \|\langle X\rangle_k\|^\bullet_{\alpha,\mathrm{pr}}$, hence $\|X\|^\bullet_\alpha \le \|X\|^\bullet_{\alpha,\mathrm{pr}}$. For the second estimate, let $\|\cdot\|_\beta$ with the stated properties and $X \in \mathcal{T}^k_{\mathrm{alg}}(V)$ be given. Use Lemma 4.1.3 to construct $X_0 = \sum_{a\in A} x_{a,1}\otimes\cdots\otimes x_{a,k}$ and $\tilde{X} = \sum_{a'\in\{1,\dots,d\}^k} X^{a'} e_{a'_1}\otimes\cdots\otimes e_{a'_k}$ with $e \in V^k$ orthonormal with respect to $\langle\,\cdot\,|\,\cdot\,\rangle_\beta$. Clearly $\|X_0\|^\bullet_{\alpha,\mathrm{pr}} = 0$ and so

$$\begin{aligned}
\|X\|^\bullet_{\alpha,\mathrm{pr}} &\le \|\tilde{X}\|^\bullet_{\alpha,\mathrm{pr}} \\
&\le \sqrt{k!}\sum_{a'\in\{1,\dots,d\}^k}\big|X^{a'}\big|\prod_{m=1}^{k}\|e_{a'_m}\|_\alpha \\
&\overset{\mathrm{cs}}{\le} \Bigg(k!\Big(\sum_{a'\in\{1,\dots,d\}^k}\big|X^{a'}\big|^2\Big)\Big(\sum_{a'\in\{1,\dots,d\}^k}\prod_{m=1}^{k}\|e_{a'_m}\|^2_\alpha\Big)\Bigg)^{\frac{1}{2}} \\
&\le \Bigg(k!\Big(\sum_{a'\in\{1,\dots,d\}^k}\big|X^{a'}\big|^2\Big)\Big(\sum_{i=1}^{d}\|e_i\|^2_\alpha\Big)^k\Bigg)^{\frac{1}{2}} \\
&\le \|X\|^\bullet_\beta.
\end{aligned}$$

□

Proposition 4.3.3 *[77, Prop. 4.4] Let V be a nuclear space, then the topology on $\mathcal{S}^\bullet(V)$ coincides with the one constructed in [83] for $R = \frac{1}{2}$.*

Proof: This is a direct consequence of the preceding lemma because the locally convex topology constructed in [83] for $R = \frac{1}{2}$ is the one defined by the seminorms $\|\cdot\|^{\bullet}_{\alpha,\mathrm{pr}}$ for all $\|\cdot\|_{\alpha} \in \mathcal{P}_V$ and because in a nuclear space, such seminorms $\|\cdot\|_{\beta}$ as required in the lemma exist for all $\|\cdot\|_{\alpha} \in \mathcal{P}_V$, see [46, Chap. 21.2, Thm. 1]. □

From [83, Thm. 4.10] we get:

Corollary 4.3.4 *[77, Cor. 4.5] Let V be a nuclear space, then $\mathcal{S}^{\bullet}(V)$ is nuclear.*

And conversely Theorem 4.1.19 here implies:

Corollary 4.3.5 *[77, Cor. 4.6] Let V be a nuclear space, then the $R = \frac{1}{2}$ topology constructed in [83] is the coarsest one possible under the conditions of Theorem 4.1.19 in the truely (not graded) symmetric case.*

4.3.2 Deformation Quantization of Hilbert Spaces

Finally, assume that V is a (complex) Hilbert space with inner product $\langle\,\cdot\,|\,\cdot\,\rangle_1$. In this case $\mathcal{S}^{\bullet}(V)$ is not a pre-Hilbert space but only a countable projective limit of pre-Hilbert spaces, because the extensions $\langle\,\cdot\,|\,\cdot\,\rangle^{\bullet}_{\alpha}$ of the (equivalent) inner products $\langle\,\cdot\,|\,\cdot\,\rangle_{\alpha} := \alpha\langle\,\cdot\,|\,\cdot\,\rangle_1$ for $\alpha \in\,]0,\infty[$ are not equivalent. If V is a Hilbert space, then its topological dual and, more generally, all spaces of bounded multilinear functionals on V are Banach spaces. This allows a more detailed analysis of the continuity of functions in $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ and of the dependence of the product $\star_{\Lambda}$ on $\Lambda \in \mathfrak{Bil}(V)$. These are again results from [77], where one can also find the proofs:

Theorem 4.3.6 *[77, Thm. 4.1] Let V be a (complex) Hilbert space with inner product $\langle\,\cdot\,|\,\cdot\,\rangle_1$ and unit ball $U \subseteq V$ and let $\mathfrak{Bil}(V)$ be the Banach space of all continuous bilinear forms on V with norm $\|\Lambda\| := \sup_{v,w\in U}|\Lambda(v,w)|$. Then the map $\mathfrak{Bil}(V) \times \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}} \times \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}} \to \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}}$*

$$(\Lambda, X, Y) \;\mapsto\; X \star_{\Lambda} Y \tag{4.3.4}$$

is continuous.

Theorem 4.3.7 *[77, Thm. 4.2] Let V be a (complex) Hilbert space with inner product $\langle\,\cdot\,|\,\cdot\,\rangle_1$ and a continuous antilinear involution $\overline{\cdot}$ that fulfils $\overline{\langle\, v\,|\,w\,\rangle_1} = \langle\,\overline{v}\,|\,\overline{w}\,\rangle_1$ for all $v, w \in V$, then $\widehat{X}\colon V'_{\mathrm{H}} \to \mathbb{C}$ is smooth in the Fréchet sense for all $X \in \mathcal{S}^{\bullet}(V)^{\mathrm{cpl}}$.*

The formal deformation quantization of a Hilbert space in a very similar setting has already been examined in [30] by Dito. There the formal deformations of exponential type of a certain algebra $\mathcal{F}_{HS}$ of smooth functions on a Hilbert space $\mathfrak{H}$ was constructed. More precisely, $\mathcal{F}_{HS}$ consists of all smooth (in the Fréchet sense) functions f whose derivatives fulfil the additional condition that for all $\sigma \in \mathfrak{H}$

$$k!\langle\!\langle f\,|\,f\rangle\!\rangle^k(\sigma) := \sum_{i\in I^k}\bigl|\bigl(\widehat{D}^{(k)}_{(e_{i_1},\dots,e_{i_k})}f\bigr)(\sigma)\bigr|^2 < \infty \tag{4.3.5}$$

holds and depends continuously on σ for one (hence all) Hilbert base $e \in \mathfrak{H}^I$ of $\mathfrak{H}$ indexed by a set I. In this case $\langle\!\langle f\,|\,f\rangle\!\rangle^k \in \mathcal{F}_{HS}$ holds.

The convergent deformations discussed here and the formal deformations discussed by Dito in [30] are very much analogous: In both cases it is necessary to restrict the construction to a subalgebra of all smooth functions, $\mathcal{F}_{HS}$ or $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$, where the additional requirement is that all the derivatives of fixed order (in the formal case) or of all orders (in the convergent case) at every point σ obey a Hilbert-Schmidt condition and that the square of the corresponding Hilbert-Schmidt norms, $\langle\!\langle f \,|\, f \rangle\!\rangle^k(\sigma)$ or $\langle\!\langle f \,|\, f \rangle\!\rangle^\bullet(\sigma)$, respectively, depend in a sufficiently nice way on σ such that one can prove that $\langle\!\langle f \,|\, f \rangle\!\rangle^k$ and $\langle\!\langle f \,|\, f \rangle\!\rangle^\bullet$ are again elements of $\mathcal{F}_{HS}$ or $\mathscr{C}^{\omega_{HS}}(V'_{\mathrm{H}})$ (see the proof of Proposition 3.4 in [30] and Proposition 4.2.13 here). Moreover, the results concerning equivalence of the deformations are similar: In [30, Thm. 2] it is shown that two (formal) deformations are equivalent if and only if they differ by bilinear forms of Hilbert-Schmidt type, while Theorem 4.2.21 here shows that the corresponding equivalence transformations are continuous if and only if they are generated by bilinear forms of Hilbert-Schmidt type.

Chapter 5

Convergent Star Products – Example II

Having constructed a whole class of examples of convergent star products in the previous chapter, symmetry reduction provides a method to produce new examples by essentially dividing out the action of a symmetry group. In a rather general formulation, starting with a Poisson *-algebra $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}\big)$ and a *-ideal $\mathcal{I} \subseteq \mathcal{A}$, one constructs the *normalizer*

$$\mathcal{N}(\mathcal{I}) := \big\{\, a \in \mathcal{A} \;\big|\; \forall_{i\in\mathcal{I}} : \{\, a\,,\, i\,\} \in \mathcal{I} \,\big\}$$

of $\mathcal{I}$ in $\mathcal{A}$, which is a unital *-subalgebra of $\mathcal{A}$ because $\{\,\cdot\,,\,\cdot\,\}$ is real and fulfils a Leibniz rule in both arguments, and in which $\mathcal{I}$ is a Poisson *-ideal by construction, such that on $\mathcal{A}_{\mathrm{red}} := \mathcal{N}(\mathcal{I})/\mathcal{I}$, the Poisson bracket remains well defined and turns $\mathcal{A}_{\mathrm{red}}$ into a new Poisson *-algebra, see e.g. [17].

In the context of the example from the previous chapter, this raises two questions: Can such a construction be carried out for all $\hbar$ and in such a way that the result is again a deformation of locally convex *-algebras? And what about the continuous algebraically positive linear functionals, will the reduction preserve some, or all of them?

Our previous discussion in Chapter 2 suggests that one should rather formulate reduction for physical systems, i.e. for triples $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}, \Omega^+_{\mathrm{H}}\big)$ of a Poisson-*-algebra $\big(\mathcal{A}, \{\,\cdot\,,\,\cdot\,\}\big)$ describing the observables and a convex cone Ω^+_{H} of algebraically positive linear functionals determining the states. However, we will to some extend ignore this guideline here and construct a reduced deformation of a locally convex *-algebra first and then examine its continuous algebraically positive linear functionals like in the previous chapter. As a side effect, this will also demonstrate an unexpected effect in Theorem 4.2.16: On the reduced algebra there can exist continuous algebraically positive linear functionals that have not yet been present on the original one!

This chapter closely follows the preprint [52] by Kraus, Roth, Waldmann and the author: In the first part, a deformation of a locally convex *-algebra of analytic functions on the complex disc (and its higher-dimensional analoga) will be constructed. This makes it necessary to discuss some more involved geometric problems than in the flat case of the previous Chapter 4. Besides being able to construct another example of a deformation of a locally convex *-algebra, the major result here is a characterization of the classical limit, i.e. the $\hbar = 0$-case, as a Fréchet *-algebra of all analytic functions which have an extension to holomorphic functions on a certain open complex submanifold of $\mathbb{CP}^n \times \mathbb{CP}^n$ (Theorem 5.1.30). The second part is then again devoted to the study of representations.

5.1 Construction of the Reduced Algebra

In the first half of this section, the essential steps of the construction of the star product on the Poincaré disc D_n, $n \in \mathbb{N}$, will be presented along with some notation and some additional structures that will be helpful later on. Essentially everything there is either standard concerning the Marsden-Weinstein reduction or can be found in the previous works [6, 7, 15, 16] concerning the construction of the star product, see also [23–26] for another approach to this star product by means of Berezin's deformation procedure, as well as [10, 11] for related C^*-algebraic deformation quantizations.

In the following we are going to introduce various manifolds and maps between them that are important for the reduction procedure. Then the classical Poisson *-algebra on the space $\mathscr{C}^\infty(D_n)$ of all complex-valued smooth functions on D_n, its subalgebras $\mathcal{A}(D_n)$ and $\mathcal{P}(D_n)$ of analytic and polynomial functions, as well as the classical reduction map Ψ_0 from functions on $\mathbb{C}^{1+n}$ to such on D_n will be constructed. The deformed quantum algebra on D_n can then be obtained by a similar reduction procedure starting from the space of polynomials on $\mathbb{C}^{1+n}$ with a Wick-type star product.

After that, starting in Section 5.1.4, the topology used in the first example in Chapter 4 will be transfered to the star product on the Poincaré disc and studied in detail. In Section 5.1.5 it will be shown that by completing the deformed algebra of polynomials on D_n, one obtains an algebra of real-analytic functions with a very clear extension property. Finally, Section 5.1.6 deals with the classical limit of this completion. Like in the previous example of Chapter 4, the deformation depends holomorphically on the deformation parameter $\hbar$, but only as long as $\hbar \in H$ with H a true subset of $\mathbb{C}$. As it will turn out that $\hbar = 0$ is only a boundary point of H, this discussion of the classical limit is now much more involved as before.

5.1.1 The Poincaré Disc D_n

In the rather well-behaved case that is considered here and for $\hbar = 0$, the general reduction of Poisson *-algebras outlined at the beginning of this chapter boils down to what is known as Marsden-Weinstein reduction: Given a symplectic manifold (M, ω) and an Hamiltonian action of a connected Lie Group G with Lie algebra $\mathfrak{g}$ and ad^*-equivariant moment map $J \colon M \to \mathfrak{g}^*$ (see Definition 2.2.2), then one considers the level set $J^{-1}(\{\mu\})$ for some $\mu \in \mathfrak{g}^*$ and, provided that $J^{-1}(\{\mu\})$ is a submanifold of M on which G acts free and proper, the quotient $M_{\mathrm{red}} := J^{-1}(\{\mu\})/G$. One can now show (see e.g. [82, Satz 3.3.55]) that on M_{red} there exists a unique symplectic form ω_{red} for which $\iota^*(\omega) = \mathrm{pr}^*(\omega_{\mathrm{red}})$ with $\iota \colon J^{-1}(\{\mu\}) \to M$ the canonical embedding and $\mathrm{pr} : J^{-1}(\{\mu\}) \to M_{\mathrm{red}}$ the canonical projection. In our case here, M will be $\mathbb{C}^{1+n}$ and $G = \mathrm{U}(1)$, so $\mathfrak{g} \cong \mathbb{R}$. The level set will be denoted by Z and the reduced manifold will be the Poincaré disc D_n. Moreover, it will be helpful to also have a realization of D_n as an open submanifold of the complex projective space $\mathbb{CP}^n = \big(\mathbb{C}^{1+n} \backslash \{0\}\big)/\mathbb{C}_*$, where $\mathbb{C}_* := \mathbb{C}\backslash\{0\}$ is the multiplicative group.

For a more detailed study of the function spaces involved in the construction, it will be necessary to also introduce extensions of all these manifolds to complex manifolds of twice the (real) dimension.

The whole geometric construction can be summarized by the following commutative diagram (the detailed description follows). The upper horizontal row consists of complex manifolds, each of which is equipped with an anti-holomorphic involution τ and a smooth action of the Lie group $\mathrm{U}(1, n) =$

$\mathrm{U}(1) \times \mathrm{SU}(1,n)$ by holomorphic automorphisms that commute with τ. The arrows between them are holomorphic maps and equivariant with respect to the $\mathrm{U}(1,n)$-action and the involution τ. All other objects are (at least) smooth manifolds equipped with a smooth action of $\mathrm{U}(1,n)$ and all other arrows are $\mathrm{U}(1,n)$-equivariant smooth maps:

$$\begin{tikzcd}
\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} & \hat{Z} \arrow[l, "\hat{\iota}"'] \arrow[r, "\hat{\mathrm{pr}}"] & \hat{D}_n \arrow[rr, "\iota_{\mathbb{P}\times\mathbb{P}}"] & & \mathbb{CP}^n \times \mathbb{CP}^n \\
& & & D_{n,\mathrm{ext}} \arrow[ul, "\Delta_{\mathrm{ext}}"'] \arrow[dr, "\iota_{\mathrm{x},\mathbb{P}}"] & \\
\mathbb{C}^{1+n} \arrow[uu, "\Delta"'] & Z \arrow[l, "\iota"'] \arrow[r, "\mathrm{pr}"] \arrow[uu, "\Delta_Z"'] & D_n \arrow[rr, "\iota_{\mathbb{P}}"] \arrow[uu, "\Delta_D"'] \arrow[ur, "\iota_{\mathrm{ext}}"] & & \mathbb{CP}^n \arrow[uu, "\Delta_{\mathbb{P}}"']
\end{tikzcd}$$

Bottom row:

On the smooth manifold $\mathbb{C}^{1+n}$ with standard (complex) coordinates $z^0, \dots, z^n \colon \mathbb{C}^{1+n} \to \mathbb{C}$, the Lie group $\mathrm{U}(1,n)$ acts from the left via $U \triangleright r := Ur$ for $U \in \mathrm{U}(1,n)$ and $r \in \mathbb{C}^{1+n}$. Define

$$g := h_{\mu\nu} z^\mu \overline{z}^\nu \in \mathscr{C}^\infty(\mathbb{C}^{1+n}) , \tag{5.1.1}$$

where $h_{00} := -1$, $h_{ii} := 1$ for $i \in \{1, \dots, n\}$ and $h_{\mu\nu} := 0$ otherwise, then

$$Z := g^{-1}\big(\{-1\}\big) = \Big\{ r \in \mathbb{C}^{1+n} \;\Big|\; |z^0(r)|^2 = 1 + \sum\nolimits_{i=1}^n |z^i(r)|^2 \Big\} \tag{5.1.2}$$

is the orbit of $(1, 0, \dots, 0)^T \in \mathbb{C}^{1+n}$ under the $\mathrm{U}(1,n)$-action and a submanifold of $\mathbb{C}^{1+n}$. Consequently, the action of $\mathrm{U}(1,n)$ can be restricted to Z and the canonical inclusion ι of the subset Z in $\mathbb{C}^{1+n}$ is of course $\mathrm{U}(1,n)$-equivariant.

Next consider the (compact) Lie subgroup $\mathrm{U}(1) \cong \{\, \mathrm{e}^{\mathrm{i}\phi} \mathbb{1}_{1+n} \mid \phi \in \mathbb{R} \,\} \subseteq \mathrm{U}(1,n)$ and construct the quotient manifold

$$D_n := Z \big/ \mathrm{U}(1) . \tag{5.1.3}$$

As the $\mathrm{U}(1)$-subgroup lies in the center of $\mathrm{U}(1,n)$, the action of $\mathrm{U}(1,n)$ remains well-defined on D_n (of course, only the complementary $\mathrm{SU}(1,n)$-subgroup acts non-trivially) and is still transitive. The canonical projection $\mathrm{pr} \colon Z \to D_n$ then is an $\mathrm{U}(1,n)$-equivariant smooth map.

Finally, D_n can be embedded injectively in $\mathbb{CP}^n$ via $\iota_{\mathbb{P}} \colon D_n \to \mathbb{CP}^n$,

$$[r]_{\mathrm{U}(1)} \mapsto \iota_{\mathbb{P}}\big([r]_{\mathrm{U}(1)}\big) := [r]_{\mathbb{C}_*} , \tag{5.1.4}$$

where we denote the equivalence classes with respect to the different group actions by $[\,\cdot\,]_{\mathrm{U}(1)}$ and $[\,\cdot\,]_{\mathbb{C}_*}$, respectively, to indicate the different equivalence relations. This embedding is also $\mathrm{U}(1,n)$-equivariant with respect to the $\mathrm{U}(1,n)$-action on $\mathbb{CP}^n$ inherited from $\mathbb{C}^{1+n} \setminus \{0\}$, i.e. $U \triangleright [r] := [Ur]$ for $U \in \mathrm{U}(1,n)$ and $[r] \in \mathbb{CP}^n$. Note that $\mathbb{CP}^n$ is the disjoint union

$$\mathbb{CP}^n = \big\{\, [r] \in \mathbb{CP}^n \;\big|\; g(r) < 0 \,\big\} \cup \big\{\, [r] \in \mathbb{CP}^n \;\big|\; g(r) = 0 \,\big\} \cup \big\{\, [r] \in \mathbb{CP}^n \;\big|\; g(r) > 0 \,\big\} , \tag{5.1.5}$$

and that the image of $\iota_{\mathbb{P}}$ is $\{ [r] \in \mathbb{CP}^n \mid g(r) < 0 \}$. From $\mathbb{CP}^n$, the disc D_n inherits the structure of a complex n-dimensional manifold, and can even be covered by a single holomorphic chart $\varphi^{\mathrm{std}} = (w^1, \dots, w^n)^T : D_n \to \mathbb{D}_n \subseteq \mathbb{C}^n$, where $\mathbb{D}_n$ is the n-ball $\mathbb{D}_n = \big\{ r \in \mathbb{C}^n \bigm| \sum_{i=1}^n |z^i(r)|^2 < 1 \big\}$ and

$$w^i([r]) \coloneqq \frac{z^i(r)}{z^0(r)} \tag{5.1.6}$$

for all $i \in \{1, \dots, n\}$ and $[r] \in D_n$. This chart actually is a biholomorphic mapping from D_n to $\mathbb{D}_n$ and the action of $\mathrm{U}(1,n)$ on D_n in this chart is described by Möbius transformations

$$\varphi^{\mathrm{std}}\big(U \triangleright [r]\big) = \varphi^{\mathrm{std}}([Ur]) = \frac{c + A\varphi^{\mathrm{std}}([r])}{\alpha + b \cdot \varphi^{\mathrm{std}}([r])} \quad \text{for} \quad [r] \in D_n \text{ and } U = \begin{pmatrix} \alpha & b^T \\ c & A \end{pmatrix} \in \mathrm{SU}(1,n)\,, \tag{5.1.7}$$

with $\alpha \in \mathbb{C}$, $b, c \in \mathbb{C}^n$, and $A \in \mathbb{C}^{n \times n}$ such that $U \in \mathrm{SU}(1,n)$. Note that here and in the following, $x \cdot y \coloneqq \sum_{i=1}^n x_i y_i$ if $x, y \in \mathbb{C}^n$.

Top row:

As we are interested in a non-formal star product on D_n, which cannot be constructed on all smooth functions but only on certain analytic functions on D_n, we have to extend the above construction in a certain way that allows to describe these analytic functions appropriately, i.e. as pullback with a smooth map $\Delta_D : D_n \to \hat{D}_n$ of holomorphic functions on some complex manifold $\hat{D}_n$. This leads to the top row of the diagram above.

On the complex manifold $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ with standard holomorphic coordinate functions $x^0, \dots, x^n$, $y^0, \dots, y^n$, the group $\mathrm{U}(1,n)$ acts from the left via $U \triangleright (p,q) \coloneqq (Up, \overline{U}q)$ for all $U \in \mathrm{U}(1,n)$ and $p, q \in \mathbb{C}^{1+n}$. Define the $\mathrm{U}(1,n)$-equivariant anti-holomorphic involution $\tau(p,q) \coloneqq (\overline{q}, \overline{p})$ on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ as well as the $\mathrm{U}(1,n)$-equivariant diagonal inclusion $\Delta : \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$

$$r \mapsto \Delta(r) \coloneqq (r, \overline{r})\,, \tag{5.1.8}$$

then Δ describes a diffeomorphism from $\mathbb{C}^{1+n}$ to $\{ (p,q) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \mid \tau(p,q) = (p,q) \}$. Moreover, define

$$\hat{g} \coloneqq h_{\mu\nu} x^\mu y^\nu \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})\,, \tag{5.1.9}$$

then $\hat{g} \circ \Delta = g$ holds. Let

$$\hat{Z} \coloneqq \hat{g}^{-1}\big(\{-1\}\big)\,, \tag{5.1.10}$$

then $\hat{Z}$ is a holomorphic submanifold of $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, the $\mathrm{U}(1,n)$-action as well as the anti-holomorphic involution τ can be restricted to $\hat{Z}$ and so the canonical inclusion $\hat{\iota} : \hat{Z} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is $\mathrm{U}(1,n)$- and τ-equivariant. Define $\Delta_Z : Z \to \hat{Z}$ as the restriction of Δ, then Δ_Z is still $\mathrm{U}(1,n)$-equivariant, the left rectangle of the above diagram commutes and Δ_Z describes a diffeomorphism from Z to $\{ (p,q) \in \hat{Z} \mid \tau(p,q) = (p,q) \}$.

The action of the Lie subgroup $\mathrm{U}(1) \cong \{ \mathrm{e}^{\mathrm{i}\phi} \mathbb{1}_{1+n} \mid \phi \in \mathbb{R} \} \subseteq \mathrm{U}(1,n)$ on $\hat{Z}$ can be extended to a holomorphic action of $\mathbb{C}_*$ via $z \triangleright (p,q) \coloneqq (zp, q/z)$ for all $(p,q) \in \hat{Z}$ and $z \in \mathbb{C}_*$. As this action is free and proper, one can construct the holomorphic quotient manifold

$$\hat{D}_n \coloneqq \hat{Z}/\mathbb{C}_* \,. \tag{5.1.11}$$

The $\mathbb{C}_*$-action on $\hat{Z}$ commutes with the $\mathrm{U}(1,n)$-action, so the $\mathrm{U}(1,n)$-action remains well-defined on $\hat{D}_n$ (again, only the $\mathrm{SU}(1,n)$-subgroup acts non-trivially) and the canonical projection $\hat{\mathrm{pr}}\colon \hat{Z} \to \hat{D}_n$ is $\mathrm{U}(1,n)$-equivariant. Moreover, there is a unique anti-holomorphic involution τ on $\hat{D}_n$ such that $\hat{\mathrm{pr}}$ becomes also τ-equivariant, namely $\tau\big([p,q]\big) \coloneqq [(\overline{q},\overline{p})]$, which is indeed well-defined and commutes with the action of $\mathrm{U}(1,n)$. The map Δ_Z remains well-defined on the quotients, so $\Delta_D\colon D_n \to \hat{D}_n$,

$$[r] \mapsto \Delta_D([r]) \coloneqq \Delta_Z(r) \tag{5.1.12}$$

is well-defined and a smooth and $\mathrm{U}(1,n)$-equivariant map. It is now easy to see that the central rectangle in the above diagram commutes as well.

Finally, we can again embed $\hat{D}_n$ injectively in $\mathbb{CP}^n \times \mathbb{CP}^n$ via $\iota_{\mathbb{P}\times\mathbb{P}}\colon \hat{D}_n \to \mathbb{CP}^n \times \mathbb{CP}^n$

$$[p,q]_{\mathbb{C}_*} \mapsto \iota_{\mathbb{P}\times\mathbb{P}}\big([p,q]_{\mathbb{C}_*}\big) \coloneqq \big([p]_{\mathbb{C}_*},[q]_{\mathbb{C}_*}\big)\,, \tag{5.1.13}$$

which is equivariant with respect to the $\mathrm{U}(1,n)$-action on $\mathbb{CP}^n \times \mathbb{CP}^n$ defined as $U \triangleright \big([p],[q]\big) \coloneqq \big([Up],[\overline{U}q]\big)$ for all $[p],[q] \in \mathbb{CP}^n$ and $U \in \mathrm{U}(1,n)$. It is also τ-equivariant if one defines the anti-holomorphic involution τ on $\mathbb{CP}^n \times \mathbb{CP}^n$ as $\tau([p],[q]) \coloneqq ([\overline{q}],[\overline{p}])$. The image of $\iota_{\mathbb{P}\times\mathbb{P}}$ in $\mathbb{CP}^n \times \mathbb{CP}^n$ is then $\big\{\, \big([p],[q]\big) \in \mathbb{CP}^n \times \mathbb{CP}^n \;\big|\; \hat{g}(p,q) \neq 0 \,\big\}$ and pulling back the usual charts from $\mathbb{CP}^n \times \mathbb{CP}^n$ to $\hat{D}_n$ yields suitable holomorphic charts on $\hat{D}_n$. In particular, one can define the standard chart $\hat{\varphi}^{\mathrm{std}} = (u^1,\dots,u^n,v^1,\dots,v^n)^T\colon \hat{D}_n^{\mathrm{std}} \to C^{\mathrm{std}} \subseteq \mathbb{C}^n \times \mathbb{C}^n$ by

$$u^i([p,q]) \coloneqq \frac{x^i(p,q)}{x^0(p,q)} \quad \text{and} \quad v^i([p,q]) \coloneqq \frac{y^i(p,q)}{y^0(p,q)} \tag{5.1.14}$$

for all $i \in \{1,\dots,n\}$ and $[p,q] \in \hat{D}_n$, with domain

$$\hat{D}_n^{\mathrm{std}} \coloneqq \big\{\, [p,q] \in \hat{D}_n \;\big|\; x^0(p,q) \neq 0 \text{ and } y^0(p,q) \neq 0 \,\big\} \tag{5.1.15}$$

and image $C^{\mathrm{std}} \coloneqq \big\{\, (p,q) \in \mathbb{C}^n \times \mathbb{C}^n \;\big|\; p \cdot q \neq 1 \,\big\}$. Moreover, let $\Delta_{\mathbb{P}}\colon \mathbb{CP}^n \to \mathbb{CP}^n \times \mathbb{CP}^n$ be the $\mathrm{U}(1,n)$-equivariant diagonal inclusion

$$[r] \mapsto \Delta_{\mathbb{P}}\big([r]\big) \coloneqq \big([r],[\overline{r}]\big), \tag{5.1.16}$$

then the right rectangle in the above diagram commutes.

The extended disc $D_{n,\mathrm{ext}}$:

Comparing the images of the embeddings of D_n and $\hat{D}_n$ in $\mathbb{CP}^n$ and $\mathbb{CP}^n \times \mathbb{CP}^n$, respectively, shows that the image of D_n under Δ_D in $\hat{D}_n$ is only contained in, but not the whole set,

$$D_{n,\mathrm{ext}} \coloneqq \big\{\, [p,q] \in \hat{D}_n \;\big|\; \tau\big([p,q]\big) = [p,q] \,\big\}\,, \tag{5.1.17}$$

which is a smooth submanifold of $\hat{D}_n$ and stable under the $\mathrm{U}(1,n)$-action. Let $\Delta_{\mathrm{ext}}\colon D_{n,\mathrm{ext}} \to \hat{D}_n$ be the canonical embedding, which is of course $\mathrm{U}(1,n)$-equivariant. Then the diagonal inclusion Δ_D of D_n in $\hat{D}_n$ factors through Δ_{ext}, such that $\Delta_D = \Delta_{\mathrm{ext}} \circ \iota_{\mathrm{ext}}$ with a unique $\mathrm{U}(1,n)$-equivariant smooth map $\iota_{\mathrm{ext}}\colon D_n \to D_{n,\mathrm{ext}}$. At last, the above commutative diagram is completed by the smooth $\mathrm{U}(1,n)$-equivariant map $\iota_{\mathrm{ext},\mathbb{P}}\colon D_{n,\mathrm{ext}} \to \mathbb{CP}^n$,

$$[p,q]_{\mathbb{C}_*} \mapsto \iota_{\mathrm{ext},\mathbb{P}}\big([p,q]_{\mathbb{C}_*}\big) \coloneqq [p]_{\mathbb{C}_*}, \tag{5.1.18}$$

which is an injective embedding of $D_{n,\mathrm{ext}}$ in $\mathbb{CP}^n$ with image $\{\,[p] \in \mathbb{CP}^n \mid g(p) \neq 0\,\}$: This is a consequence of the observation that, given $p \in \mathbb{C}^{1+n}$ with $g(p) \neq 0$, there is a unique $q \in \mathbb{C}^{1+n}$ such that $[p,q] \in D_{n,\mathrm{ext}}$, namely $q = -\overline{p}/g(p)$, and $\iota_{\mathrm{ext},\mathbb{P}}\big([p,q]\big) = [p]$.

5.1.2 The Classical Poisson Algebra

The Poisson structure on D_n comes from a Kähler structure which can be obtained from a variant of Marsden-Weinstein reduction from $\mathbb{C}^{1+n}$: consider the complex 2-form $h = h_{\mu\nu}\,\mathrm{d}\overline{z}^\mu \otimes \mathrm{d}z^\nu$, then h endows $\mathbb{C}^{1+n}$ with the structure of a pseudo Kähler manifold with pseudo Riemannian metric $g \coloneqq \mathsf{Re}(h) = h_{\mu\nu}\,\mathrm{d}\overline{z}^\mu \vee \mathrm{d}z^\nu$ and Kähler symplectic form $\omega \coloneqq \mathsf{Im}(h) = -\mathrm{i}h_{\mu\nu}\,\mathrm{d}\overline{z}^\mu \wedge \mathrm{d}z^\nu$. Note that, in order to keep notation consistent throughout the thesis, the symmetric and antisymmetric tensor products here are defined as $\mathrm{d}\overline{z}^\mu \vee \mathrm{d}z^\nu = \big(\mathrm{d}\overline{z}^\mu \otimes \mathrm{d}z^\nu + \mathrm{d}z^\nu \otimes \mathrm{d}\overline{z}^\mu\big)/2$ and $\mathrm{d}\overline{z}^\mu \wedge \mathrm{d}z^\nu = \big(\mathrm{d}\overline{z}^\mu \otimes \mathrm{d}z^\nu - \mathrm{d}z^\nu \otimes \mathrm{d}\overline{z}^\mu\big)/2$. The inverse tensors are then $g^{-1} = 4h^{\mu\nu}\partial_{z^\mu} \vee \partial_{\overline{z}^\nu}$ and $\pi \coloneqq \omega^{-1} = -4\mathrm{i}h^{\mu\nu}\partial_{z^\mu} \wedge \partial_{\overline{z}^\nu}$ with $h^{\mu\nu} = h_{\mu\nu}$ for all $\mu,\nu \in \{0,\dots,n\}$, so that the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathbb{C}^{1+n}$ is given by

$$\{\,a\,,\,b\,\} \coloneqq \pi\big(\mathrm{d}a \otimes \mathrm{d}b\big) = -2\mathrm{i}h^{\mu\nu}\bigg(\frac{\partial a}{\partial z^\mu}\frac{\partial b}{\partial \overline{z}^\nu} - \frac{\partial b}{\partial z^\mu}\frac{\partial a}{\partial \overline{z}^\nu}\bigg) \tag{5.1.19}$$

for all $a,b \in \mathscr{C}^\infty(\mathbb{C}^{1+n})$. It fulfils $\{\,a\,,\,b\,\}^* = \{\,a^*\,,\,b^*\,\}$ because the Poisson tensor π is real. Note that h, hence also g and ω, are $\mathrm{U}(1,n)$-invariant, so the Poisson bracket is $\mathrm{U}(1,n)$-equivariant, i.e. $\{\,a \triangleleft U\,,\,b \triangleleft U\,\} = \{\,a\,,\,b\,\} \triangleleft U$ for all $a,b \in \mathscr{C}^\infty(\mathbb{C}^{1+n})$ and $U \in \mathrm{U}(1,n)$, where $\triangleleft$ describes the induced right action of $\mathrm{U}(1,n)$ on functions on $\mathbb{C}^{1+n}$ by pullback.

The action of $\mathrm{U}(1,n)$ is not only Kähler, i.e. preserves ω and g, but also Hamiltonian with an equivariant moment map. Explicitly, the infinitesimal action of $\mathfrak{u}(1,n)$ on spaces of tensor fields on $\mathbb{C}^{1+n}$ is given by $X \triangleleft u = \mathscr{L}_{\xi_u+\overline{\xi_u}}X$, where $\mathfrak{u}(1,n) \ni u \mapsto \xi_u = u^\mu_\nu z^\nu \partial_{z^\mu} \in \Gamma(\mathbb{T}\mathbb{C}^{1+n})$ is an anti-morphism of Lie algebras. An equivariant moment map for this action is then given by $\mathscr{J}(u)\colon \mathfrak{u}(1,n) \to \mathscr{C}^\infty(\mathbb{C}^{1+n})$,

$$\mathscr{J}(u) \coloneqq \frac{\mathrm{i}}{2}h_{\mu\nu}u^\mu_\rho z^\rho \overline{z}^\nu, \tag{5.1.20}$$

because a straightforward calculation shows that $\{\,f\,,\,\mathscr{J}(u)\,\} = \xi_u(f) + \overline{\xi_u}(f) = f \triangleleft u$ as well as $\{\,\mathscr{J}(u)\,,\,\mathscr{J}(v)\,\} = \mathscr{J}\big([\,u\,,\,v\,]\big)$ holds for all $u,v \in \mathfrak{u}(1,n)$ and $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})$.

Note that $g = \mathscr{J}(-2\mathrm{i}\mathbb{1}_{1+n})$ with $-2\mathrm{i}\mathbb{1}_{1+n} \in \mathfrak{u}(1,n)$ generating $\{\,\mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_{1+n} \mid \phi \in \mathbb{R}\,\}$, the $\mathrm{U}(1)$-subgroup of $\mathrm{U}(1,n)$, so Z is a $\mathfrak{u}(1)$-level set and $D_n = Z/\mathrm{U}(1)$ the resulting Marsden-Weinstein quotient. The Kähler structure on D_n in the standard coordinates $\varphi^{\mathrm{std}} = (w^1,\dots,w^n)^T$ is then given by the

2-form

$$h_d = (1 - \overline{w} \cdot w)^{-2} \big(\delta_{ij}(1 - \overline{w} \cdot w) + \overline{w}^k w^\ell \delta_{ki} \delta_{\ell j} \big) \, \mathrm{d}\overline{w}^i \otimes \mathrm{d}w^j \,, \tag{5.1.21}$$

the usual Fubini-Study form, and thus the Kähler metric becomes $g_d = \mathsf{Re}(h_d)$ while the Kähler symplectic form is $\omega_d = \mathsf{Im}(h_d)$. Finally, the Poisson tensor is

$$\pi_d = \omega_d^{-1} = -4\mathrm{i}(1 - \overline{w} \cdot w)(\delta^{ij} - \overline{w}^i w^j) \, \partial_{w^i} \wedge \partial_{\overline{w}^j} \,. \tag{5.1.22}$$

From the point of view of deformation quantization, the description of this geometric reduction in terms of function algebras becomes more important: Let $\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z$ be the *-ideal in $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ consisting of all functions in $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ that vanish on Z, then $\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z$ is the ideal generated by $g+1$ and $\mathscr{C}^\infty(Z) \cong \mathscr{C}^\infty(\mathbb{C}^{1+n})/\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z$ as *-algebras. Moreover, restricted to $\mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, the Poisson *-subalgebra of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ consisting of all $\mathrm{U}(1)$-invariant functions, the *-ideal $\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z$ is even a Poisson *-ideal, and consequently $\mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)} / \big(\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z \cap \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}\big)$ is even a Poisson *-algebra and isomorphic to the reduced Poisson *-algebra $\mathscr{C}^\infty(D_n)$. This isomorphism is described by the *classical reduction map*:

Definition 5.1.1 *[52, Def. 2.1] Define the map $\Psi_0 \colon \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \to \mathscr{C}^\infty(D_n)$,*

$$a \mapsto \Psi_0(a) \quad \textit{with} \quad \Psi_0(a)([r]) \coloneqq a(r) \quad \textit{for all} \quad [r] \in D_n \,. \tag{5.1.23}$$

One can check that Ψ_0 is indeed a well-defined unital and $\mathrm{U}(1,n)$-equivariant Poisson *-homomorphism. Moreover, its kernel is clearly $\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z \cap \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, and as every smooth $\mathrm{U}(1)$-invariant function on Z can be extended to a smooth $\mathrm{U}(1)$-invariant function on $\mathbb{C}^{1+n}$ (just take an arbitrary extension and make it $\mathrm{U}(1)$-invariant by averaging over the action of $\mathrm{U}(1)$), Ψ_0 is also surjective and thus descends to an isomorphism from the quotient $\mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)} / \big(\mathscr{C}^\infty(\mathbb{C}^{1+n})_Z \cap \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}\big)$ to $\mathscr{C}^\infty(D_n)$.

However, as we are interested in non-formal deformation quantizations of these Poisson *-algebras, which cannot be constructed on the whole spaces $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ and $\mathscr{C}^\infty(D_n)$, we have to restrict our attention to suitable subalgebras:

Definition 5.1.2 *[52, Def. 2.2] Define*

$$\mathcal{A}(\mathbb{C}^{1+n}) \coloneqq \{ \hat{a} \circ \Delta \mid \hat{a} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \} \quad \textit{and} \quad \mathcal{A}(D_n) \coloneqq \{ \hat{a} \circ \Delta_D \mid \hat{a} \in \mathcal{O}(\hat{D}_n) \} \,. \tag{5.1.24}$$

Note that these are unital Poisson *-subalgebras of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ and $\mathscr{C}^\infty(D_n)$, respectively, because the coefficients of the Poisson tensors π and π_d with respect to the standard charts are polynomials in the coordinate functions and because $(\hat{a} \circ \Delta)^* = \overline{\,\cdot\,} \circ \hat{a} \circ \tau \circ \Delta$ as well as $(\hat{b} \circ \Delta_D)^* = \overline{\,\cdot\,} \circ \hat{b} \circ \tau \circ \Delta_D$ holds for all $\hat{a} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ and $\hat{b} \in \mathcal{O}(\hat{D}_n)$, where $\overline{\,\cdot\,} \circ \hat{a} \circ \tau$ and $\overline{\,\cdot\,} \circ \hat{b} \circ \tau$ are compositions of two anti-holomorphic and one holomorphic function, hence are holomorphic. Moreover, the pullback $\Delta^* \colon \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \to \mathcal{A}(\mathbb{C}^{1+n})$ is an isomorphism of vector spaces, because $\frac{\partial \hat{a}}{\partial x^\mu} \circ \Delta = \frac{\partial}{\partial z^\mu}(\hat{a} \circ \Delta)$ and $\frac{\partial \hat{a}}{\partial y^\mu} \circ \Delta = \frac{\partial}{\partial \overline{z}^\mu}(\hat{a} \circ \Delta)$ holds for all $\mu \in \{0, \dots, n\}$ and $\hat{a} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$, so $\hat{a} \circ \Delta = 0$ implies that all derivatives of $\hat{a}$ vanish at all points in the image of Δ, and thus $\hat{a} = 0$. Similarly, the pullback $\Delta_D^* \colon \mathcal{O}(\hat{D}_n) \to \mathcal{A}(D_n)$ is an isomorphism as well. The inverse of these isomorphisms will simply be denoted by $\hat{\cdot}$, i.e. given $a \in \mathcal{A}(\mathbb{C}^{1+n})$, then $\hat{a}$ is the unique element in $\mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ that fulfils $a = \hat{a} \circ \Delta$, similarly for $\mathcal{A}(D_n)$. Note that this notation has already been used for $g = \hat{g} \circ \Delta \in \mathcal{A}(\mathbb{C}^{1+n})$ in (5.1.9).

As $\mathcal{A}(\mathbb{C}^{1+n})$ and $\mathcal{A}(D_n)$ are isomorphic to spaces of holomorphic functions, they are Fréchet spaces with respect to the topology of locally uniform convergence of the holomorphic extensions. We will only need the topology of $\mathcal{A}(D_n)$:

Definition 5.1.3 *[52, Def. 2.3] Let $K \subseteq \hat{D}_n$ be compact, then define the seminorm $\|\cdot\|_{D_n,K}$ on $\mathcal{A}(D_n)$ as*

$$\|a\|_{D_n,K} \coloneqq \sup_{[p,q]\in K} \big|\hat{a}\big([p,q]\big)\big| \tag{5.1.25}$$

for all $a \in \mathcal{A}(D_n)$.

It will be helpful to describe $\mathcal{A}(\mathbb{C}^{1+n})$ and $\mathcal{A}(D_n)$ as completions of algebras of polynomials:

Definition 5.1.4 *[52, Def. 2.4] For all multiindices $P, Q \in \mathbb{N}_0^{1+n}$, define the monomial*

$$\mathrm{d}_{P,Q} \coloneqq z^P \overline{z}^Q \coloneqq (z^0)^{P_0} \cdots (z^n)^{P_n} (\overline{z}^0)^{Q_0} \cdots (\overline{z}^n)^{Q_n} \in \mathcal{A}(\mathbb{C}^{1+n}) \tag{5.1.26}$$

and write $\mathcal{P}(\mathbb{C}^{1+n})$ for their span, i.e. for the space of polynomial functions.

It is then clear that the polynomial functions form a dense unital Poisson *-subalgebra of $\mathcal{A}(\mathbb{C}^{1+n})$.

Similarly like before, denote the (closed) subspaces of U(1)-invariant analytic functions and polynomials on $\mathbb{C}^{1+n}$ by $\mathcal{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ and $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. Then $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ is spanned by the U(1)-invariant monomials $\mathrm{d}_{P,Q}$ with $P, Q \in \mathbb{N}_0^{1+n}$ and $|P| = |Q|$ (using the usual multiindex notation), and is dense in $\mathcal{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. This both is a consequence of the observation that the Cauchy formula for reconstructing the Taylor coefficient of $\mathrm{d}_{P,Q}$ by means of circular integrals around the origin of $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is U(1)-invariant only if $|P| = |Q|$. Of course, $\mathcal{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ and $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ are again unital Poisson *-subalgebras of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$. Using the reduction map we can now construct polynomials on D_n:

Definition 5.1.5 *[52, Def. 2.5] Define $\mathrm{f}_{P,Q} \coloneqq \Psi_0(\mathrm{d}_{P,Q})$ for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and write $\mathcal{P}(D_n)$ for the image of $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ under Ψ_0, i.e. for the span of the functions $\mathrm{f}_{P,Q}$.*

Note that we will show in Theorem 5.1.29 that $\mathcal{P}(D_n)$ is dense in $\mathcal{A}(D_n)$ with respect to its Fréchet topology. As Ψ_0 is not injective, we cannot expect the monomials $\mathrm{f}_{P,Q}$ on D_n to be a basis of $\mathcal{P}(D_n)$. A suitable choice for a basis is the following (see [7, Lemma 4.20]):

Definition 5.1.6 *[52, Def. 2.6] For all $P, Q \in \mathbb{N}_0^n$, define the* fundamental monomial

$$\mathrm{f}_{\mathrm{red};P,Q} \coloneqq \begin{cases} \mathrm{f}_{(|Q|-|P|,P_1,\dots,P_n),(0,Q_1,\dots,Q_n)} & \textit{if } |Q| \geq |P| \\ \mathrm{f}_{(0,P_1,\dots,P_n),(|P|-|Q|,Q_1,\dots,Q_n)} & \textit{if } |Q| \leq |P|. \end{cases} \tag{5.1.27}$$

Note that, with respect to the coordinates of the standard chart $\varphi^{\mathrm{std}} = (w^1, \dots, w^n)^T$, the monomials on D_n are represented as

$$\mathrm{f}_{P,Q} = \frac{(w^1)^{P_1} \cdots (w^n)^{P_n} (\overline{w}^1)^{Q_1} \cdots (\overline{w}^n)^{Q_n}}{(1 - w \cdot \overline{w})^{|P|}} \tag{5.1.28}$$

for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$. In particular,

$$\mathrm{f}_{\mathrm{red};P,Q} = \frac{(w^1)^{P_1} \cdots (w^n)^{P_n} (\overline{w}^1)^{Q_1} \cdots (\overline{w}^n)^{Q_n}}{(1 - w \cdot \overline{w})^{\max\{|P|,|Q|\}}} = \frac{w^P \overline{w}^Q}{(1 - w \cdot \overline{w})^{\max\{|P|,|Q|\}}} \tag{5.1.29}$$

for all $P, Q \in \mathbb{N}_0^n$. Using this one can already show that the $\mathrm{f}_{\mathrm{red};P,Q}$ are linearly independent, and they span $\mathcal{P}(D_n)$ because every $\mathrm{f}_{P,Q}$ with $P, Q \in \mathbb{N}_0^{1+n}$ and $|P| = |Q|$ can be rewritten as

$$\mathrm{f}_{P,Q} = \sum_{\substack{T \in \mathbb{N}_0^n \\ |T| \leq \min\{P_0, Q_0\}}} \binom{\min\{P_0, Q_0\}}{|T|} \frac{|T|!}{T!} \mathrm{f}_{\mathrm{red};P'+T,Q'+T}\,, \tag{5.1.30}$$

where $P' = (P_1, \ldots, P_n) \in \mathbb{N}_0^n$ and analogously for Q. So we get ([7, Lemma 4.20]):

Proposition 5.1.7 *[52, Prop. 2.7] The fundamental monomials on D_n form a basis of $\mathcal{P}(D_n)$.*

Note that the product of two fundamental monomials on D_n is more complicated than the product of monomials on $\mathbb{R}^n$: Given $\mathrm{f}_{\mathrm{red};P,Q}, \mathrm{f}_{\mathrm{red};R,S}$ with $P, Q, R, S \in \mathbb{N}_0^n$, then $\mathrm{f}_{\mathrm{red};P,Q}\,\mathrm{f}_{\mathrm{red};R,S} = \mathrm{f}_{\mathrm{red};P+R,Q+S}$ holds only in the cases that $|P| \geq |Q|$ and $|R| \geq |S|$ or that $|P| \leq |Q|$ and $|R| \leq |S|$. If $|P| \geq |Q|$ and $|R| \leq |S|$ then

$$\mathrm{f}_{\mathrm{red};P,Q}\,\mathrm{f}_{\mathrm{red};R,S} = \sum_{\substack{T \in \mathbb{N}_0^n \\ |T| \leq \min\{|S|-|R|,|P|-|Q|\}}} \binom{\min\{|S| - |R|, |P| - |Q|\}}{|T|} \frac{|T|!}{T!} \mathrm{f}_{\mathrm{red};P+R+T,Q+S+T}\,, \tag{5.1.31}$$

and similarly if $|P| \leq |Q|$ and $|R| \geq |S|$. This especially shows that identifying $\mathrm{f}_{\mathrm{red};P,Q}$ with a monomial $z^P \overline{z}^Q$ on $\mathbb{C}^n$ does not extend to an isomorphism of algebras.

The unital Poisson-*-algebras $\mathcal{P}(\mathbb{C}^{1+n})$ and $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ are of course graded by the degree of polynomials. However, the induced filtration will be even more important in the following, because it remains well-defined after reduction to $\mathcal{P}(D_n)$ and will also be respected by the deformed product:

Definition 5.1.8 *[52, Def. 2.8] For all $m \in \mathbb{N}_0$, define $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)}$ as the space of $\mathrm{U}(1)$-invariant polynomials of up to degree $2m$, i.e. as the span of $\mathrm{d}_{P,Q}$ for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q| \leq m$. Similarly, $\mathcal{P}(D_n)^{(m)}$ is defined as the image of $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)}$ under Ψ_0, i.e. as the span of $\mathrm{f}_{\mathrm{red};P,Q}$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| \leq m$, $|Q| \leq m$.*

Similarly to [7, Lemma 4.18], we get:

Proposition 5.1.9 *[52, Prop. 2.9] For all $m \in \mathbb{N}_0$ the following holds:*

i.) $\dim \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)} = \sum_{k=0}^m \binom{n+k}{k}^2$.

ii.) $\dim \mathcal{P}(D_n)^{(m)} = \binom{n+m}{m}^2$.

Moreover, the kernel of the restriction of Ψ_0 to $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ is the ideal in $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ generated by $g + 1$, i.e.

$$\ker \Psi_0 \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} = \big\{ (g + 1)a \;\big|\; a \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \big\}. \tag{5.1.32}$$

PROOF: Given $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ then the set $\{ P \in \mathbb{N}_0^\ell \mid |P| = k \}$ has $\binom{\ell-1+k}{k}$ elements. From this one can easily deduce the first dimension formula and also the second because

$$\dim \mathcal{P}(D_n)^{(m)} = \Bigg(\sum_{k=0}^{m} \binom{n-1+k}{k} \Bigg)^2 = \binom{n+m}{m}^2 .$$

Moreover, it is easy to see that the *-ideal in $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ that is generated by $g+1$ is in the kernel of Ψ_0 and in order to show that it is the whole of $\ker \Psi_0 \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ it is sufficient to show for all $m \in \mathbb{N}_0$ that $\ker \Psi_0 \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)} \subseteq \{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \} \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)}$, or

$$\dim \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)} - \dim \mathcal{P}(D_n)^{(m)} \leq \dim \Big(\{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \} \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)} \Big) .$$

Due to the above dimension formulas, the left hand side of this reduces to $\sum_{k=0}^{m-1} \binom{n+k}{k}^2$. In the case that $m = 0$, this inequality is certainly true. But if it holds for one $m \in \mathbb{N}_0$ then also for $m+1$ because the span of all $(g+1)\mathrm{d}_{P,Q}$ with $P, Q \in \mathbb{N}_0$ and $|P| = |Q| = m$ has dimension $\binom{n+m}{m}^2$ and is a subspace of $\{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \} \cap \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m+1)}$ that has trivial intersection with $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(m)}$. □

So the algebraic description of the reduction stays the same for the polynomials, i.e. $\mathcal{P}(D_n)$ is isomorphic as a Poisson-*-algebra to the quotient of $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ over the ideal generated by $g+1$.

5.1.3 The Deformed Quantum Algebra

Analogously to the Marsden-Weinstein reduction in the classical case, the star product on the Poincaré disc can be constructed by quantum reduction: In order to obtain a formal deformation quantization of D_n, one can start with the Wick star product on $\mathbb{C}^{1+n}$ given by

$$a \mathbin{\tilde{\star}} b := \sum_{t=0}^{\infty} \frac{(2\hbar)^t}{t!} \sum_{i_1,\dots,i_t,j_1,\dots,j_t=0}^{n} h^{i_1 j_1} \cdots h^{i_t j_t} \frac{\partial^t a}{\partial z^{i_1} \cdots \partial z^{i_t}} \frac{\partial^t b}{\partial \overline{z}^{j_1} \cdots \partial \overline{z}^{j_t}} \tag{5.1.33}$$

for all $a, b \in \mathscr{C}^\infty(\mathbb{C}^{1+n})[\![\hbar]\!]$, which is just a special case of the one in (2.5.1) that has subsequently been examined in Chapter 4. It is also compatible with the *-involution of pointwise complex conjugation, i.e. $(a \mathbin{\tilde{\star}} b)^* = b^* \mathbin{\tilde{\star}} a^*$ holds for all $a, b \in \mathscr{C}^\infty(\mathbb{C}^{1+n})[\![\hbar]\!]$. Its commutator is the classical Poisson bracket up to terms of higher order, $\frac{1}{\mathrm{i}\hbar}[\, a \,,\, b \,]_{\tilde{\star}} = \{\, a \,,\, b \,\} + \hbar \cdots$. Note that $\mathcal{J}$ is not only a classical, but also a quantum moment map: $\frac{1}{\mathrm{i}\hbar}[\, a \,,\, \mathcal{J}(u) \,]_{\tilde{\star}} = \{\, a \,,\, \mathcal{J}(u) \,\} = a \triangleleft u$ for all $u \in \mathfrak{u}(1,n)$, because in commutators with $\mathcal{J}(\,\cdot\,)$, only the first order in $\hbar$ contributes due to the linearity of the moment map in the z- and $\overline{z}$-coordinates. Analogously to the classical Poisson bracket, the Wick star product is also $\mathrm{U}(1,n)$-equivariant, i.e. $(a \triangleleft U) \mathbin{\tilde{\star}} (b \triangleleft U) = (a \mathbin{\tilde{\star}} b) \triangleleft U$ holds for all $a, b \in \mathscr{C}^\infty(\mathbb{C}^{1+n})[\![\hbar]\!]$ and $U \in \mathrm{U}(1,n)$. The reduced star product algebra on D_n can then be obtained from the one on $\mathbb{C}^{1+n}$ by restriction to the subalgebra of $\mathrm{U}(1)$-invariant elements in $\mathscr{C}^\infty(\mathbb{C}^{1+n})[\![\hbar]\!]$ and dividing out the ideal generated by $g+1$.

On $\mathcal{P}(\mathbb{C}^{1+n})$, the Wick star product converges trivially for every $\hbar \in \mathbb{C}$, which yields an associative product $\tilde{\star}_\hbar$ that fulfils $(a \mathbin{\tilde{\star}_\hbar} b)^* = b^* \mathbin{\tilde{\star}_{\overline{\hbar}}} a^*$, hence $\big(\mathcal{P}(\mathbb{C}^{1+n}), \tilde{\star}_\hbar, \,\cdot^*\big)$ is a *-algebra for all $\hbar \in \mathbb{R}$. On the basis of $\mathcal{P}(\mathbb{C}^{1+n})$ given by the monomials $\mathrm{d}_{P,Q}$, the Wick star product can be expressed as

$$\mathrm{d}_{P,Q}\,\tilde{\star}_\hbar\,\mathrm{d}_{R,S} = \sum_{T=0}^{\min\{P,S\}} (-1)^{T_0}(2\hbar)^{|T|}T!\binom{P}{T}\binom{S}{T}\mathrm{d}_{P+R-T,Q+S-T} \tag{5.1.34}$$

for all $P,Q,R,S \in \mathbb{N}_0^{1+n}$. As the classical *-ideal generated by $g+1$ in $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, i.e. the kernel of Ψ_0, is no longer an ideal with respect to the Wick star product, one has to perform an equivalence transformation first that assures that the star product with g is the classical product [7, 15], and can then restrict to functions on D_n. This procedure results in the following *deformed reduction map*:

Definition 5.1.10 *[52, Def. 2.10] Let $H \coloneqq \mathbb{C}_*\backslash\{\,-1/(2m) \mid m \in \mathbb{N}\,\}$ and define for all $\hbar \in H$ the deformed reduction map $\Psi_\hbar\colon \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \to \mathcal{P}(D_n)$ by linear extension of*

$$\Psi_\hbar(\mathrm{d}_{P,Q}) \coloneqq (2\hbar)^{|P|}\left(\frac{1}{2\hbar}\right)_{|P|}\Psi_0(\mathrm{d}_{P,Q}) = (2\hbar)^{|P|}\left(\frac{1}{2\hbar}\right)_{|P|}\mathrm{f}_{P,Q} \tag{5.1.35}$$

for all $P,Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, where $(z)_m$ denotes the Pochhammer symbol, or rising factorial,

$$(z)_m \coloneqq \prod_{k=0}^{m-1}(z+k) \tag{5.1.36}$$

for all $z \in \mathbb{C}$ and $m \in \mathbb{N}_0$.

Proposition 5.1.11 *[52, Prop. 2.11] For all $\hbar \in H$ the kernel of the deformed reduction map $\Psi_\hbar$ is the *-ideal generated by $g+1$ with respect to the Wick star product $\tilde{\star}_\hbar$ on $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$.*

PROOF: Using the explicit formula (5.1.34) one can check that indeed

$$\mathrm{d}_{P,Q}\,\tilde{\star}_\hbar(g+1) = (g+1)\,\tilde{\star}_\hbar\,\mathrm{d}_{P,Q} = (g+1+2\hbar|P|)\mathrm{d}_{P,Q}$$

is in the kernel of $\Psi_\hbar$ for all $P,Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, because

$$\Psi_\hbar\big((g+1+2\hbar|P|)\mathrm{d}_{P,Q}\big) = (2\hbar)^{|P|+1}\left(\frac{1}{2\hbar}\right)_{|P|+1}\Psi_0\big((g+1)\mathrm{d}_{P,Q}\big) = 0\,.$$

So the *-ideal generated by $g+1$ with respect to the Wick product on $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ is in the kernel of $\Psi_\hbar$. Conversely, in order to show that this is indeed the whole kernel of $\Psi_\hbar$ one can use the same argument as in the proof of Proposition 5.1.9 and count dimensions. □

As a consequence, the following *reduced product on $\mathcal{P}(D_n)$* is indeed well-defined and associative:

Definition 5.1.12 *[52, Def. 2.12] For all $\hbar \in H$, define the product $\star_\hbar\colon \mathcal{P}(D_n) \times \mathcal{P}(D_n) \to \mathcal{P}(D_n)$*

$$a \star_\hbar b \coloneqq \Psi_\hbar\big(a'\,\tilde{\star}_\hbar\,b'\big)\,, \tag{5.1.37}$$

where $a', b' \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ are arbitrary preimages of a and b under $\Psi_\hbar$.

Note also that $(a \star_\hbar b)^* = b^* \star_{\overline{\hbar}} a^*$ holds for all $\hbar \in H$ and $a, b \in \mathcal{P}(D_n)$, so pointwise complex conjugation is a *-involution for $\star_\hbar$ if $\hbar \in H \cap \mathbb{R}$. An explicit formula for $\star_\hbar$ on the monomials on D_n is

$$\mathrm{f}_{P,Q} \star_\hbar \mathrm{f}_{R,S} = \sum_{T=0}^{\min\{P,S\}} (-1)^{T_0} \frac{(\frac{1}{2\hbar})_{|P+S-T|} T!}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} \binom{P}{T} \binom{S}{T} \mathrm{f}_{P+R-T,Q+S-T} \tag{5.1.38}$$

for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$. Moreover, as the Wick star product on $\mathbb{C}^{1+n}$ and the deformed reduction map are $\mathrm{U}(1,n)$-equivariant, the reduced star product $\star_\hbar$ is also $\mathrm{U}(1,n)$-equivariant.

5.1.4 The Topology

By constructing a locally convex topology on $\mathcal{P}(D_n)$ under which $\star_\hbar$ is continuous, we can extend the star product to the completion of $\mathcal{P}(D_n)$. A well-behaved topology on $\mathcal{P}(\mathbb{C}^{1+n})$ has already been examined in Chapter 4. Transferring these results to the Poincaré disc is straightforward. Note that in [7, Thm. 4.21, (viii)] another topology was constructed for which the star product is also continuous. However, the topology discussed here is slightly coarser which is ultimately the reason for being able to determine the completion explicitly in geometric terms.

Definition 5.1.13 *[52, Def. 3.1] For all $\rho > 0$ define the norm*

$$\Big\| \sum_{P,Q \in \mathbb{N}_0^{1+n}} a_{P,Q} \mathrm{d}_{P,Q} \Big\|_{\mathbb{C}^{1+n},\rho} := \sum_{P,Q \in \mathbb{N}_0^{1+n}} |a_{P,Q}| \rho^{|P+Q|} \sqrt{|P+Q|!} \tag{5.1.39}$$

on $\mathcal{P}(\mathbb{C}^{1+n})$, where $a_{P,Q} \in \mathbb{C}$ for all $P, Q \in \mathbb{N}_0^{1+n}$.

The locally convex topology defined by all these norms on $\mathcal{P}(\mathbb{C}^{1+n})$ is essentially the one constructed in the previous example of Chapter 4:

Proposition 5.1.14 *The usual isomorphism between the *-algebras $\mathcal{P}(\mathbb{C}^{1+n})$ and $\mathcal{S}^\bullet(\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n})$, i.e. $\Xi\colon \mathcal{P}(\mathbb{C}^{1+n}) \to \mathcal{S}^\bullet(\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n})$,*

$$\mathrm{d}_{P,Q} \mapsto \Xi(\mathrm{d}_{P,Q}) := (b_0)^{P_0} \vee \cdots \vee (b_n)^{P_n} \vee (c_0)^{Q_0} \vee \cdots \vee (c_n)^{Q_n} \qquad \textit{for all } P, Q \in \mathbb{N}_0^{1+m}, \tag{5.1.40}$$

with $b_0, \dots, b_n, c_0, \dots, c_n$ the standard basis of $\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n}$, is a homeomorphism with respect to the locally convex topology of the norms $\|\cdot\|_{\mathbb{C}^{1+n},\rho}$ on $\mathcal{P}(\mathbb{C}^{1+n})$ and the locally convex topology on $\mathcal{S}^\bullet(\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n})$ used in Chapter 4.

PROOF: Note that the $\Xi(\mathrm{d}_{P,Q})$ are an orthogonal basis of $\mathcal{S}^\bullet(\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n})$ with respect to the extensions $\langle\, \cdot \mid \cdot \,\rangle_\rho^\bullet$ of all multiples $\langle\, \cdot \mid \cdot \,\rangle_\rho := \rho \langle\, \cdot \mid \cdot \,\rangle$ with $\rho > 0$ of the standard inner product on $\mathbb{C}^{1+n} \oplus \mathbb{C}^{1+n}$. Moreover, one can check that $\big\|\Xi(\mathrm{d}_{P,Q})\big\|_\rho^2 = \rho^{|P+Q|} P!\, Q!$ holds for all $P, Q \in \mathbb{N}_0^{1+n}$, which yields the estimates

$$\Big\| \Xi\Big(\sum_{P,Q \in \mathbb{N}_0^{1+n}} a_{P,Q} \mathrm{d}_{P,Q} \Big) \Big\|_\rho^\bullet = \Big(\sum_{P,Q \in \mathbb{N}_0^{1+n}} |a_{P,Q}|^2 \rho^{|P+Q|} P!\, Q! \Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le \sum_{P,Q\in\mathbb{N}_0^{1+n}} |a_{P,Q}| \sqrt{\rho}^{\,|P+Q|} \sqrt{|P+Q|!} \\
&= \Big\| \sum_{P,Q\in\mathbb{N}_0^{1+n}} a_{P,Q} \mathrm{d}_{P,Q} \Big\|_{\mathbb{C}^{1+n},\sqrt{\rho}}
\end{aligned}
$$

and also

$$
\begin{aligned}
&\Big\| \sum_{P,Q\in\mathbb{N}_0^{1+n}} a_{P,Q} \mathrm{d}_{P,Q} \Big\|_{\mathbb{C}^{1+n},\rho} = \\
&\qquad = \sum_{P,Q\in\mathbb{N}_0^{1+n}} |a_{P,Q}| \rho^{|P+Q|} \sqrt{|P+Q|!} \\
&\qquad = \sum_{P,Q\in\mathbb{N}_0^{1+n}} \left(\frac{|P+Q|!}{P!\,Q!}\right)^{\frac{1}{2}} \sqrt{4(1+n)}^{\,-|P+Q|} |a_{P,Q}| \big(\sqrt{4(1+n)}\,\rho\big)^{|P+Q|} \sqrt{P!\,Q!} \\
&\qquad \stackrel{\mathrm{CS}}{\le} \left(\sum_{P,Q\in\mathbb{N}_0^{1+n}} \frac{|P+Q|!}{P!\,Q!} \big(4(1+n)\big)^{-|P+Q|} \right)^{\frac{1}{2}} \left(\sum_{P,Q\in\mathbb{N}_0^{1+n}} |a_{P,Q}|^2 \big(4(1+n)\rho^2\big)^{|P+Q|} P!\,Q! \right)^{\frac{1}{2}} \\
&\qquad = 2 \Big\| \Xi\Big(\sum_{P,Q\in\mathbb{N}_0^{1+n}} a_{P,Q} \mathrm{d}_{P,Q} \Big) \Big\|^{\bullet}_{4(1+n)\rho^2}
\end{aligned}
$$

using the multinomial formula in the last step. □

Because of this, we can immediately transfer the previous results about the continuity and absolute convergence of the star product from Lemma 4.1.12, as well as about the growth of powers from Lemma 4.2.28:

Lemma 5.1.15 *[52, Lemma 3.2] For every compact $K \subseteq \mathbb{C}$ and every $\rho > 0$ there exist $C, \rho' > 0$ such that the estimate*

$$
\|a \mathbin{\tilde{\star}_\hbar} b\|_{\mathbb{C}^{1+n},\rho} \le \sum_{P,Q,R,S\in\mathbb{N}_0^{1+n}} |a_{P,Q}||b_{R,S}| \, \|\mathrm{d}_{P,Q} \mathbin{\tilde{\star}_\hbar} \mathrm{d}_{R,S}\|_{\mathbb{C}^{1+n},\rho} \le C \|a\|_{\mathbb{C}^{1+n},\rho'} \|b\|_{\mathbb{C}^{1+n},\rho'} \tag{5.1.41}
$$

holds for all $\hbar \in K$ and all $a = \sum_{P,Q\in\mathbb{N}_0^{1+n}} a_{P,Q}\mathrm{d}_{P,Q}, b = \sum_{R,S\in\mathbb{N}_0^{1+n}} b_{R,S}\mathrm{d}_{R,S} \in \mathcal{P}(\mathbb{C}^{1+n})$.

PROOF: The first estimate is just the triangle inequality, and the second follows from the previous Proposition 5.1.14 together with Lemma 4.1.12: Given K and ρ there exist $C, \rho' > 0$ such that $\|\mathrm{d}_{P,Q} \mathbin{\tilde{\star}_\hbar} \mathrm{d}_{R,S}\|_{\mathbb{C}^{1+n},\rho} \le C \|\mathrm{d}_{P,Q}\|_{\mathbb{C}^{1+n},\rho'} \|\mathrm{d}_{R,S}\|_{\mathbb{C}^{1+n},\rho'}$ holds for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ and $\hbar \in K$. □

Lemma 5.1.16 *[52, Lemma 3.3] Let $\hbar \in \mathbb{R}$ as well as a linear functional $\omega\colon \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \to \mathbb{C}$ be given, such that ω is continuous with respect to the locally convex topology on $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ defined by the norms $\|\cdot\|_{\mathbb{C}^{1+n},\rho}$ for all $\rho > 0$ and such that ω is algebraically positive with respect $\tilde{\star}_\hbar$. Then for all $k \in \mathbb{N}_0$ and all $a \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1),(k)}$ there exist $C, D > 0$ with the property that*

$$
\omega\big((a^*)^{\tilde{\star}_\hbar m} \mathbin{\tilde{\star}_\hbar} a^{\tilde{\star}_\hbar m}\big)^{\frac{1}{2}} \le C D^m (km)! \tag{5.1.42}
$$

holds for all $m \in \mathbb{N}_0$, where $a^{\tilde{\star}_\hbar m}$ denotes the m-th power of a with respect to the product $\tilde{\star}_\hbar$.

PROOF: Let such $\hbar, \omega, k$ and a be given, then it follows from the previous Lemma 5.1.15 and the continuity of ω that there exist $C', \rho > 0$ with the property that $\omega(b^* \tilde{\star}_\hbar b)^{1/2} \le C' \|b\|_{\mathbb{C}^{1+n},\rho}$ holds for all $b \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, and especially

$$\omega\big((a^*)^{\tilde{\star}_\hbar m} \,\tilde{\star}_\hbar\, a^{\tilde{\star}_\hbar m}\big)^{\frac{1}{2}} \le C' \|a^{\tilde{\star}_\hbar m}\|_{\mathbb{C}^{1+n},\rho}$$

for all $m \in \mathbb{N}_0$. Then Proposition 5.1.14 and Lemma 4.2.28 show that there exist $C'', D' > 0$ such that $\|a^{\tilde{\star}_\hbar m}\|_{\mathbb{C}^{1+n},\rho} \le C'' \sqrt{(2km)!} D'^m$ holds for all $m \in \mathbb{N}_0$. As $\sqrt{(2km)!} \le 2^{km}(km)!$, this proves the claim with $C = C'C''$ and $D = 2^k D'$. □

Using the explicit basis for $\mathcal{P}(D_n)$ one can define norms in the same spirit as before. Note that with the normalization conventions for the basis functions $\mathrm{f}_{\mathrm{red};P,Q}$ used here, the following weighted ℓ^1-like norms yield a slightly coarser topology than the original one in [7]:

Definition 5.1.17 *[52, Def 3.4] For all $\rho > 0$, define the norm*

$$\Big\| \sum_{P,Q \in \mathbb{N}_0^n} a_{P,Q} \mathrm{f}_{\mathrm{red};P,Q} \Big\|_{D_n,\rho} := \sum_{P,Q \in \mathbb{N}_0^n} |a_{P,Q}| \rho^{|P+Q|} \tag{5.1.43}$$

on $\mathcal{P}(D_n)$, where $a_{P,Q} \in \mathbb{C}$ for all $P, Q \in \mathbb{N}_0^n$.

We will soon identify the resulting topology as the quotient topology with respect to the reduction map $\Psi_\hbar$ for $\hbar \neq 0$, but this requires the following well-known estimate for the Pochhammer symbols:

Lemma 5.1.18 *[52, Lemma 3.5] For every compact subset $K \subseteq \mathbb{C} \backslash (-\mathbb{N}_0)$ there exist two constants $\alpha, \omega > 0$ such that*

$$\alpha^m m! \le |(z)_m| \le \omega^m m! \tag{5.1.44}$$

holds for all $z \in K$ and all $m \in \mathbb{N}_0$.

PROOF: By dividing (5.1.44) by $m!$ it becomes clear that one can choose

$$\alpha := \min_{z \in K} \inf_{\ell \in \mathbb{N}} \left| \frac{z-1}{\ell} + 1 \right| \qquad \text{and} \qquad \omega := \max_{z \in K} \sup_{\ell \in \mathbb{N}} \left| \frac{z-1}{\ell} + 1 \right|,$$

both of which exist because $\mathbb{C} \ni z \mapsto \inf_{\ell \in \mathbb{N}} |(z-1)/\ell + 1| \in \mathbb{R}$ and $\mathbb{C} \ni z \mapsto \sup_{\ell \in \mathbb{N}} |(z-1)/\ell + 1| \in \mathbb{R}$, as pointwise infima and suprema of an equicontinuous set of functions, are continuous. Moreover, $\alpha > 0$ holds because $\lim_{\ell \to \infty} |(z-1)/\ell + 1| = 1$ and $|(z-1)/\ell + 1| > 0$ for all $z \in K$ and all $\ell \in \mathbb{N}$ shows that $\inf_{\ell \in \mathbb{N}} |(z-1)/\ell + 1| > 0$ for all $z \in K$. Finally, $\omega \ge \alpha$ is clear. □

The next lemma allows to relate the topologies before and after the (quantum) reduction procedure:

Lemma 5.1.19 *[52, Lemma 3.6] Let $K \subseteq H$ be a compact subset and let $\rho > 0$. Then there exists a $\rho' > 0$ such that*

$$\|\Psi_\hbar(a)\|_{D_n,\rho} \le \|a\|_{\mathbb{C}^{1+n},\rho'} \tag{5.1.45}$$

holds for all $\hbar \in K$ and all $a \in \mathcal{P}(\mathbb{C}^{1+n})$. Conversely, there exists a $\rho'' > 0$ such that

$$\|\Phi_\hbar(a)\|_{\mathbb{C}^{1+n},\rho} \le \|a\|_{D_n,\rho''} \tag{5.1.46}$$

holds for all $\hbar \in K$ and all $a \in \mathcal{P}(D_n)$, where $\Phi_\hbar \colon \mathcal{P}(D_n) \to \mathcal{P}(\mathbb{C}^{1+n})$ is the (non-multiplicative) right inverse of $\Psi_\hbar$ that is defined as the linear extension of

$$\Phi_\hbar(\mathrm{f}_{\mathrm{red};P,Q}) \coloneqq \left((2\hbar)^{\max\{|P|,|Q|\}}\left(\frac{1}{2\hbar}\right)_{\max\{|P|,|Q|\}}\right)^{-1} \mathrm{d}_{\tilde{P},\tilde{Q}} \tag{5.1.47}$$

for $\tilde{P} \coloneqq \big(\max\{|Q|-|P|,0\}, P_1,\dots,P_n\big) \in \mathbb{N}_0^{1+n}$ and $\tilde{Q} \coloneqq \big(\max\{|P|-|Q|,0\}, Q_1,\dots,Q_n\big) \in \mathbb{N}_0^{1+n}$.

PROOF: Let $K \subseteq H$ and, without loss of generality, $\rho \geq 1$ be given. Then the previous Lemma 5.1.18 shows that there exist $\alpha, \omega > 0$ such that $\alpha^m m! \leq |(1/(2\hbar))_m| \leq \omega^m m!$ holds for all $m \in \mathbb{N}_0$ and all $\hbar \in K$. Define $r_{\max} \coloneqq \max_{\hbar\in K}|2\hbar|$ and $r_{\min} \coloneqq \min_{\hbar\in K}|2\hbar| > 0$.

For all $\hbar \in K$ and $a = \sum_{P,Q\in\mathbb{N}_0^{1+n},|P|=|Q|} a_{P,Q}\mathrm{d}_{P,Q} \in \mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ one derives with the help of identity (5.1.30) and the prime-notation for omission of the 0-component in tuples used there:

$$\begin{aligned}
&\|\Psi_\hbar(a)\|_{D_n,\rho}\\
&\quad= \Bigg\| \sum_{\substack{P,Q\in\mathbb{N}_0^{1+n}\\|P|=|Q|}} a_{P,Q}(2\hbar)^{|P|}\left(\frac{1}{2\hbar}\right)_{|P|}\mathrm{f}_{P,Q}\Bigg\|_{D_n,\rho}\\
&\quad\leq \sum_{\substack{P,Q\in\mathbb{N}_0^{1+n}\\|P|=|Q|}} |a_{P,Q}||2\hbar|^{|P|}\left|\left(\frac{1}{2\hbar}\right)_{|P|}\right| \sum_{\substack{T\in\mathbb{N}_0^n\\|T|\leq\min\{P_0,Q_0\}}} \binom{\min\{P_0,Q_0\}}{|T|}\frac{|T|!}{T!}\rho^{|P'+Q'+2T|}\\
&\quad\leq \sum_{\substack{P,Q\in\mathbb{N}_0^{1+n}\\|P|=|Q|}} |a_{P,Q}|\Big(\rho\sqrt{\omega r_{\max}}\Big)^{|P+Q|}|P|! \sum_{\substack{T\in\mathbb{N}_0^n\\|T|\leq\min\{P_0,Q_0\}}} \binom{\min\{P_0,Q_0\}}{|T|}\frac{|T|!}{T!}\\
&\quad= \sum_{\substack{P,Q\in\mathbb{N}_0^{1+n}\\|P|=|Q|}} |a_{P,Q}|\Big(\rho\sqrt{\omega r_{\max}}\Big)^{|P+Q|}(1+n)^{\min\{P_0,Q_0\}}|P|!\\
&\quad\leq \sum_{\substack{P,Q\in\mathbb{N}_0^{1+n}\\|P|=|Q|}} |a_{P,Q}|\Big(\rho\sqrt{\omega r_{\max}(1+n)}\Big)^{|P+Q|}\sqrt{|P+Q|!}\\
&\quad= \|a\|_{\mathbb{C}^{1+n},\rho\sqrt{\omega r_{\max}(1+n)}}.
\end{aligned}$$

This shows the first estimate with $\rho' = \rho\sqrt{\omega r_{\max}(1+n)}$. Conversely, for all $\hbar \in K$ and all $b = \sum_{P,Q\in\mathbb{N}_0^n} b_{P,Q}\mathrm{f}_{\mathrm{red};P,Q} \in \mathcal{P}(D_n)$ we get

$$\begin{aligned}
&\|\Phi_\hbar(b)\|_{\mathbb{C}^{1+n},\rho}\\
&\quad= \Bigg\|\sum_{P,Q\in\mathbb{N}_0^n} b_{P,Q}\left((2\hbar)^{\max\{|P|,|Q|\}}\left(\frac{1}{2\hbar}\right)_{\max\{|P|,|Q|\}}\right)^{-1}\mathrm{d}_{\tilde{P},\tilde{Q}}\Bigg\|_{\mathbb{C}^{1+n},\rho}\\
&\quad\leq \sum_{P,Q\in\mathbb{N}_0^n} |b_{P,Q}|\Big((r_{\min}\alpha)^{\max\{|P|,|Q|\}}\big(\max\{|P|,|Q|\}\big)!\Big)^{-1}\rho^{2\max\{|P|,|Q|\}}\sqrt{\big(2\max\{|P|,|Q|\}\big)!}\\
&\quad= \sum_{P,Q\in\mathbb{N}_0^n} |b_{P,Q}|\left(\frac{\rho^2}{r_{\min}\alpha}\right)^{\max\{|P|,|Q|\}}\binom{2\max\{|P|,|Q|\}}{\max\{|P|,|Q|\}}^{\frac{1}{2}}
\end{aligned}$$

$$\leq \sum_{P,Q\in\mathbb{N}_0^n} |b_{P,Q}| \left(\frac{2\rho^2}{r_{\min}\alpha}\right)^{\max\{|P|,|Q|\}}$$
$$\leq \|b\|_{D_n,\rho''}.$$

With $\rho'' = \max\{2\rho^2/(r_{\min}\alpha), 1\}$ as $\max\{|P|,|Q|\} \leq |P+Q|$. □

The previous Lemma 5.1.19 shows that for all $\hbar \in H$ the norms $\|\cdot\|_{D_n,\rho}$ with $\rho > 0$ induce the quotient topology of $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}/\ker\Psi_\hbar$ with the locally convex topology of the norms $\|\cdot\|_{\mathbb{C}^{1+n},\rho}$ with $\rho > 0$. Together with the continuity of $\tilde{\star}_\hbar$ from Lemma 5.1.15 this yields:

Theorem 5.1.20 *[52, Thm. 3.7] For every $\hbar \in H$ the star product $\star_\hbar$ on $\mathcal{P}(D_n)$ is continuous in the locally convex topology of the norms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$.*

So $\big(\mathcal{P}(D_n), \star_\hbar, \cdot^*\big)$, with $\cdot^*$ the pointwise complex conjugation, is a locally convex *-algebra with continuous product for all $\hbar \in H \cap \mathbb{R}$ as continuity of $\cdot^*$ is clear and compatibility between product and *-involution is an immediate consequence of the construction of $\star_\hbar$.

5.1.5 Characterization of the Completion

Having constructed a suitable locally convex topology on $\mathcal{P}(D_n)$, the next step is to characterize the topology as well as the completion of the space $\mathcal{P}(D_n)$ under this topology. Understanding various charts on $\hat{D}_n$ will be especially helpful. Recall that we have already defined the standard chart $\hat{\varphi}^{\mathrm{std}} \colon \hat{D}_n^{\mathrm{std}} \to C^{\mathrm{std}}$ in (5.1.14) such that

$$\hat{\varphi}^{\mathrm{std}} \circ \hat{\mathrm{pr}} := \left(\frac{x^1}{x^0}, \dots, \frac{x^n}{x^0}, \frac{y^1}{y^0}, \dots, \frac{y^n}{y^0}\right)\bigg|_{\hat{Z}}, \tag{5.1.48}$$

where $\hat{D}_n^{\mathrm{std}} = \big\{\, [p,q] \in \hat{D}_n \;\big|\; x^0(p,q) \neq 0 \text{ and } y^0(p,q) \neq 0 \,\big\}$ and $C^{\mathrm{std}} := \big\{\, (p,q) \in \mathbb{C}^n \times \mathbb{C}^n \;\big|\; p\cdot q \neq 1 \,\big\}$. We will also need the following two charts:

Definition 5.1.21 *[52, Def. 3.8] Let $\hat{D}_n^P := \big\{\, [p,q] \in \hat{D}_n \;\big|\; y^0(p,q) \neq 0 \,\big\}$ and $\hat{D}_n^Q := \big\{\, [p,q] \in \hat{D}_n \;\big|\; x^0(p,q) \neq 0 \,\big\}$ and define the P-*chart *$\hat{\varphi}^P \colon \hat{D}_n^P \to \mathbb{C}^n \times \mathbb{C}^n$ as well as the Q-*chart *$\hat{\varphi}^Q \colon \hat{D}_n^Q \to \mathbb{C}^n \times \mathbb{C}^n$ by*

$$\hat{\varphi}^P \circ \hat{\mathrm{pr}} := \left(x^1y^0, \dots, x^ny^0, \frac{y^1}{y^0}, \dots, \frac{y^n}{y^0}\right)\bigg|_{\hat{Z}} \tag{5.1.49}$$

and

$$\hat{\varphi}^Q \circ \hat{\mathrm{pr}} := \left(\frac{x^1}{x^0}, \dots, \frac{x^n}{x^0}, x^0y^1, \dots, x^0y^n\right)\bigg|_{\hat{Z}}, \tag{5.1.50}$$

respectively.

Note that $\hat{\varphi}^{\mathrm{std}}$, $\hat{\varphi}^P$ and $\hat{\varphi}^Q$ are all well-defined biholomorphic maps. With respect to these charts, the monomials $\hat{\mathrm{f}}_{P,Q}$ with $P, Q \in \mathbb{N}_0^{1+n}$, $|P| = |Q|$, are represented as

$$\hat{\mathrm{f}}_{P,Q} \circ (\hat{\varphi}^{\mathrm{std}})^{-1} = \frac{x^{P'}y^{Q'}}{(1 - x\cdot y)^{|P|}}\bigg|_{C^{\mathrm{std}}}, \tag{5.1.51}$$

$$\hat{\mathrm{f}}_{P,Q} \circ (\hat{\varphi}^P)^{-1} = (1 + x \cdot y)^{P_0} x^{P'} y^{Q'}\,, \tag{5.1.52}$$

and

$$\hat{\mathrm{f}}_{P,Q} \circ (\hat{\varphi}^Q)^{-1} = (1 + x \cdot y)^{Q_0} x^{P'} y^{Q'}\,, \tag{5.1.53}$$

where $P' = (P_1, \dots, P_n) \in \mathbb{N}_0^n$ and $Q' = (Q_1, \dots, Q_n) \in \mathbb{N}_0^n$. In particular, this implies that $\hat{\mathrm{f}}_{\mathrm{red},P,Q} \circ (\hat{\varphi}^P)^{-1} = x^P y^Q$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| \geq |Q|$ and $\hat{\mathrm{f}}_{\mathrm{red},P,Q} \circ (\hat{\varphi}^Q)^{-1} = x^P y^Q$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| \leq |Q|$, which motivates the following definition:

Definition 5.1.22 *[52, Def. 3.9] For all $P, Q \in \mathbb{N}_0^n$, define the linear functional $\mathrm{f}'_{\mathrm{red};P,Q} \colon \mathcal{A}(D_n) \to \mathbb{C}$ as the Cauchy integral*

$$\mathrm{f}'_{\mathrm{red};P,Q}(a) \coloneqq \frac{1}{(-4\pi^2)^n} \oint_C \cdots \oint_C \frac{\hat{a} \circ (\hat{\varphi}^P)^{-1}}{x^{P+1} y^{Q+1}} \,\mathrm{d}^n x \wedge \mathrm{d}^n y \qquad \text{if } |P| \geq |Q| \tag{5.1.54}$$

and

$$\mathrm{f}'_{\mathrm{red};P,Q}(a) \coloneqq \frac{1}{(-4\pi^2)^n} \oint_C \cdots \oint_C \frac{\hat{a} \circ (\hat{\varphi}^Q)^{-1}}{x^{P+1} y^{Q+1}} \,\mathrm{d}^n x \wedge \mathrm{d}^n y \qquad \text{if } |P| < |Q| \tag{5.1.55}$$

for all $a \in \mathcal{A}(D_n)$, where $C \subseteq \mathbb{C}$ is a circle around 0 with arbitrary positive radius and where $P + 1 \coloneqq (P_1 + 1, \dots, P_n + 1)$, analogous for Q.

Using the explicit formulas (5.1.52) and (5.1.53) we immediately get:

Proposition 5.1.23 *[52, Prop. 3.10] For all $P, Q, R, S \in \mathbb{N}_0^n$, the identity*

$$\mathrm{f}'_{\mathrm{red};R,S}\big(\mathrm{f}_{\mathrm{red};P,Q}\big) = \delta_{P,R}\, \delta_{Q,S} \tag{5.1.56}$$

holds.

Proposition 5.1.24 *[52, Prop. 3.11] The two formulas for $\mathrm{f}'_{\mathrm{red};P,Q}$ in the P- and Q-chart can be combined into one single formula in the standard-chart, namely*

$$\mathrm{f}'_{\mathrm{red};P,Q}(a) = \frac{1}{(-4\pi^2)^n} \oint_C \cdots \oint_C \frac{\hat{a} \circ (\hat{\varphi}^{\mathrm{std}})^{-1}}{x^{P+1} y^{Q+1}} (1 - x \cdot y)^{\max\{|P|,|Q|\}-1} \,\mathrm{d}^n x \wedge \mathrm{d}^n y \tag{5.1.57}$$

for all $a \in \mathcal{A}(D_n)$, where $C \subseteq \mathbb{C}$ is a circle around 0 with radius in $]0, 1/\sqrt{n}[$ and where again $P + 1 \coloneqq (P_1 + 1, \dots, P_n + 1)$, analogous for Q.

Proof: The change of coordinates from the standard- to the P-chart is given by

$$\begin{aligned} \Psi^P = \hat{\varphi}^P \circ (\hat{\varphi}^{\mathrm{std}})^{-1} \colon C^{\mathrm{std}} &\to \{\, (\xi, \eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid \xi \cdot \eta = -1 \,\} \\ (\xi, \eta) &\mapsto \Psi^P(\xi, \eta) = \left(\frac{\xi}{1 - \xi \cdot \eta}, \eta \right). \end{aligned}$$

Then

$$\mathrm{f}'_{\mathrm{red};P,Q}(a) \coloneqq \frac{1}{(-4\pi^2)^n} \int_{\Psi^P(C^{2n})} \frac{\hat{a} \circ (\hat{\varphi}^P)^{-1}}{x^{P+1} y^{Q+1}} \,\mathrm{d}^n x \wedge \mathrm{d}^n y$$

$$
\begin{aligned}
&= \frac{1}{(-4\pi^2)^n} \int_{C^{2n}} \frac{\hat{a} \circ (\hat{\varphi}^P)^{-1} \circ \Psi^P}{(x^{P+1} y^{Q+1}) \circ \Psi^P} \, \mathrm{d}^n(x \circ \Psi^P) \wedge \mathrm{d}^n(y \circ \Psi^P) \\
&= \frac{1}{(-4\pi^2)^n} \int_{C^{2n}} \frac{\hat{a} \circ (\hat{\varphi}^{\mathrm{std}})^{-1}}{x^{P+1} y^{Q+1}} (1 - x \cdot y)^{|P+1|} \frac{\mathrm{d}^n x \wedge \mathrm{d}^n y}{(1 - x \cdot y)^{1+n}},
\end{aligned}
$$

which yields (5.1.57) if $|P| \geq |Q|$. Note that the calculation of $\mathrm{d}^n(x \circ \Psi^P)|_{\xi,\eta}$ is easy for $\eta = (1, 0 \dots, 0)^T \in \mathbb{C}^n$, which is already sufficient by symmetry. If $|P| < |Q|$, the argument is analogous using the Q-chart. □

Recall that $\mathcal{A}(D_n)$ is endowed with a Fréchet topology given by the seminorms $\|\cdot\|_{D_n,K}$ defined for all compact $K \subseteq \hat{D}_n$ in Definition 5.1.3. We would of course like to understand the relation between this topology and the topology defined by the norms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$.

Proposition 5.1.25 *[52, Prop. 3.12] For all $P, Q \in \mathbb{N}_0^n$ the linear functional $\mathrm{f}'_{\mathrm{red};P,Q} \colon \mathcal{A}(D_n) \to \mathbb{C}$ is continuous. Moreover, for every $\rho > 0$ there exists a compact $K \subseteq \hat{D}_n$ such that the estimate*

$$
\sum_{P,Q \in \mathbb{N}_0^n} |\mathrm{f}'_{\mathrm{red};P,Q}(a)| \rho^{|P+Q|} \leq 2^{2n} \|a\|_{D_n,K} \tag{5.1.58}
$$

holds for all $a \in \mathcal{A}(D_n)$.

PROOF: It is sufficient to show that the estimate (5.1.58) holds, which is much stronger than mere continuity of $\mathrm{f}'_{\mathrm{red};P,Q}$. So let $\rho > 0$ be given and define K^P and K^Q as the images of the polydiscs with radius 2ρ in $\mathbb{C}^n \times \mathbb{C}^n$ under the holomorphic maps $(\hat{\varphi}^P)^{-1}$ and $(\hat{\varphi}^Q)^{-1}$, respectively, and $K \coloneqq K^P \cup K^Q \subseteq \hat{D}_n$. Then K is compact and from the usual estimate for the Cauchy integral over the boundary of a polydisc with radius 2ρ it follows for all $a \in \mathcal{P}(D_n)$ that

$$
\sum_{P,Q \in \mathbb{N}_0^n} |\mathrm{f}'_{\mathrm{red};P,Q}(a)| \rho^{|P+Q|} \leq \sum_{P,Q \in \mathbb{N}_0^n} \|a\|_{D_n,K} \frac{\rho^{|P+Q|}}{(2\rho)^{|P+Q|}} = 2^{2n} \|a\|_{D_n,K} \,. \qquad \square
$$

Lemma 5.1.26 *[52, Lemma 3.13] For all compact $K \subseteq \hat{D}_n$ there exists a $\rho > 0$ such that the estimate $\|\mathrm{f}_{\mathrm{red};P,Q}\|_{D_n,K} \leq \rho^{|P+Q|}$ holds for all $P, Q \in \mathbb{N}_0^n$.*

PROOF: Given such a $K \subseteq \hat{D}_n$, then one possible choice for ρ is the maximum of $\|\mathrm{f}_{E_\mu,E_\nu}\|^2_{D_n,K}$ over all $\mu, \nu \in \{0, \dots, n\}$, where $E_\mu, E_\nu \in \mathbb{N}_0^{1+n}$ are the standard unit vectors. Submultiplicativity of $\|\cdot\|_{D_n,K}$ with respect to the pointwise product yields $\|\mathrm{f}_{P,Q}\|_{D_n,K} \leq \sqrt{\rho}^{\,|P+Q|}$ for all $P, Q \in \mathbb{N}_0^{1+n}$ and thus $\|\mathrm{f}_{\mathrm{red};P,Q}\|_{D_n,K} \leq \sqrt{\rho}^{\,2|P+Q|} = \rho^{|P+Q|}$ for all $P, Q \in \mathbb{N}_0^n$. □

Proposition 5.1.27 *[52, Prop. 3.14] On $\mathcal{P}(D_n)$ the locally convex topology defined by the seminorms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$ coincides with the subspace topology inherited from $\mathcal{A}(D_n)$.*

PROOF: Let a compact $K \subseteq \hat{D}_n$ be given, then by the previous Lemma 5.1.26 there exists a $\rho > 0$ such that $\|\mathrm{f}_{\mathrm{red};P,Q}\|_{D_n,K} \leq \rho^{|P+Q|}$ holds for all $P, Q \in \mathbb{N}_0^n$, so

$$
\|a\|_{D_n,K} \leq \sum_{P,Q \in \mathbb{N}_0^n} |a_{P,Q}| \|\mathrm{f}_{\mathrm{red};P,Q}\|_{D_n,K} \leq \|a\|_{D_n,\rho}
$$

holds for all $a = \sum_{P,Q\in\mathbb{N}_0^n} a_{P,Q}\mathrm{f}_{\mathrm{red};P,Q} \in \mathcal{P}(D_n)$ with complex coefficients $a_{P,Q}$. The converse estimate follows directly from Proposition 5.1.25, which shows that for every $\rho > 0$ there exists a compact $K \subseteq \hat{D}_n$ such that

$$\|a\|_{D_n,\rho} = \sum_{P,Q\in\mathbb{N}_0^n} |\mathrm{f}'_{\mathrm{red};P,Q}(a)|\, \rho^{|P+Q|} \leq 2^{2n+2}\|a\|_{D_n,K}$$

holds for all $a \in \mathcal{P}(D_n)$. □

Lemma 5.1.28 *[52, Lemma 3.15] If $a \in \mathcal{A}(D_n)$ fulfils $\mathrm{f}'_{\mathrm{red};P,Q}(a) = 0$ for all $P, Q \in \mathbb{N}_0^n$, then $a = 0$.*

PROOF: Given $a \in \mathcal{A}(D_n)$, then $\hat{a} \circ (\hat{\varphi}^P)^{-1} \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n)$, so there exist unique complex coefficients $\tilde{a}_{P,Q}$ such that $\hat{a} \circ (\hat{\varphi}^P)^{-1} = \sum_{P,Q\in\mathbb{N}_0^n} \tilde{a}_{P,Q} x^P y^Q$ (and the series converges absolutely and locally uniformly). It is sufficient to show that all these coefficients vanish, because the domain of the P-chart is dense in $\hat{D}_n$. From the definition of $\mathrm{f}'_{\mathrm{red};P,Q}$ it is immediately clear that $\tilde{a}_{P,Q} = 0$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| \geq |Q|$. Now assume that there is a non-vanishing coefficient $\tilde{a}_{P,Q}$, then there is a minimal $N \in \mathbb{N}_0$ such that $\tilde{a}_{P,Q} \neq 0$ for some $P, Q \in \mathbb{N}_0^n$ with $|P| < |Q|$ and $|P + Q| = N$, so

$$\hat{a} \circ (\hat{\varphi}^P)^{-1} = \sum_{\substack{P,Q\in\mathbb{N}_0^n \\ |P|<|Q| \text{ and } |P+Q|\geq N}} \tilde{a}_{P,Q} x^P y^Q.$$

Consider $\Psi := \hat{\varphi}^P \circ (\hat{\varphi}^Q)^{-1}|_{C^\Psi} \colon C^\Psi \to C^\Psi$ with $C^\Psi := \{\, (\xi, \eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid \xi \cdot \eta \neq 1 \,\}$, which is explicitly given by $\Psi(\xi, \eta) = \big(\xi(1 - \xi \cdot \eta), \frac{\eta}{1-\xi\cdot\eta}\big)$ and describes the change of coordinates between P- and Q-chart. Then $\hat{a} \circ (\hat{\varphi}^P)^{-1} \circ \Psi = \hat{a} \circ (\hat{\varphi}^Q)^{-1}|_{C^\Psi}$ can be represented as the absolutely and locally uniformly convergent series

$$\hat{a} \circ (\hat{\varphi}^Q)^{-1}|_{C^\Psi} = \sum_{\substack{P,Q\in\mathbb{N}_0^n \\ |P|<|Q| \text{ and } |P+Q|\geq N}} \tilde{a}_{P,Q} \frac{x^P y^Q}{(1 - x \cdot y)^{|P|-|Q|}}.$$

It follows that $\tilde{a}_{P,Q} = \mathrm{f}'_{\mathrm{red};P,Q}(a)$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| < |Q|$ and $|P + Q| = N$ by evaluating the Cauchy-integral for $\mathrm{f}'_{\mathrm{red};P,Q}(a)$ on sufficiently small circles in the Q-chart. So $\tilde{a}_{P,Q} = 0$ and we have a contradiction. □

Theorem 5.1.29 *[52, Thm. 3.16] The Fréchet *-algebra $\mathcal{A}(D_n)$ with the pointwise operations is the completion of the *-algebra $\mathcal{P}(D_n)$ with the pointwise operations and the locally convex topology defined by the seminorms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$. Moreover, the functions $\mathrm{f}_{\mathrm{red};P,Q}$ with $P, Q \in \mathbb{N}_0^n$ form an absolute Schauder basis of $\mathcal{A}(D_n)$ and the coefficients of the expansion in this basis can be calculated explicitly by means of the integral formulas for $\mathrm{f}'_{\mathrm{red};P,Q}$ from Definition 5.1.22 or Proposition 5.1.24:*

$$a = \sum_{P,Q\in\mathbb{N}_0^n} \mathrm{f}'_{\mathrm{red};P,Q}(a)\, \mathrm{f}_{\mathrm{red};P,Q} = \sum_{P,Q\in\mathbb{N}_0^n} \frac{\mathrm{f}_{\mathrm{red};P,Q}}{(-4\pi^2)^n} \oint_C \cdots \oint_C \hat{a}\, \frac{(1 - u \cdot v)^{\max\{|P|,|Q|\}-1}}{u^{P+1} v^{Q+1}}\, \mathrm{d}^n u \wedge \mathrm{d}^n v, \tag{5.1.59}$$

for all $a \in \mathcal{A}(D_n)$, where $u^1, \dots, u^n, v^1, \dots, v^n$ are the coordinate functions of the standard chart (5.1.14).

PROOF: Proposition 5.1.27 shows that the $\|\cdot\|_{D_n,\rho}$-topology on $\mathcal{P}(D_n)$ coincides with the relative topology inherited from $\mathcal{A}(D_n)$. Moreover, given $a \in \mathcal{A}(D_n)$, then $\tilde{a} := \sum_{P,Q\in\mathbb{N}_0^n} \mathrm{f}'_{\mathrm{red};P,Q}(a)\,\mathrm{f}_{\mathrm{red};P,Q}$ converges absolutely in $\mathcal{A}(D_n)$ due to the estimates in Proposition 5.1.25 and Lemma 5.1.26. As $\mathrm{f}'_{\mathrm{red};R,S}(a) = \mathrm{f}'_{\mathrm{red};R,S}(\tilde{a})$ for all $R, S \in \mathbb{N}_0^n$ due to the continuity of $\mathrm{f}'_{\mathrm{red};R,S}$ shown in Proposition 5.1.25 and due to the identity from Proposition 5.1.23, it follows from Lemma 5.1.28 that $a = \tilde{a}$. So a is an element of the closure of $\mathcal{P}(D_n)$ in $\mathcal{A}(D_n)$. As the functions $\mathrm{f}_{\mathrm{red};P,Q}$ with $P, Q \in \mathbb{N}_0^n$ are linearly independent, this also shows that they form an absolute Schauder basis of $\mathcal{A}(D_n)$ and that the coefficients of a with respect to this basis are the $\mathrm{f}'_{\mathrm{red};P,Q}(a)$. □

Note that this also showns that $\mathcal{O}(\hat{D}_n)$ is isomorphic to $\mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n)$ as a Fréchet *space* via the isomorphism

$$\mathcal{O}(\hat{D}_n) \ni \sum_{P,Q\in\mathbb{N}_0^n} a_{P,Q}\hat{\mathrm{f}}_{\mathrm{red},P,Q} \mapsto \sum_{P,Q\in\mathbb{N}_0^n} a_{P,Q}x^P y^Q \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n). \tag{5.1.60}$$

However, this is not an isomorphism of Fréchet *algebras* due to the more complicated formula (5.1.31) for the (commutative) product on $\mathcal{P}(D_n)$.

Theorem 5.1.30 *[52, Thm. 3.17] For all $\hbar \in H$ the product $\star_\hbar$ on $\mathcal{P}(D_n)$ extends continuously to the completion $\mathcal{A}(D_n)$, such that $\mathcal{A}(D_n)$ with the product $\star_\hbar$ becomes a Fréchet algebra. The product can explicitly be calculated as the series*

$$a \star_\hbar b = \sum_{P,Q,R,S\in\mathbb{N}_0^n} a_{P,Q}b_{R,S}\,\mathrm{f}_{\mathrm{red};P,Q} \star_\hbar \mathrm{f}_{\mathrm{red};R,S}\,, \tag{5.1.61}$$

which converges absolutely and locally uniformly in $\hbar \in H$ for all $a, b \in \mathcal{A}(D_n)$ with coefficients $a_{P,Q} := \mathrm{f}'_{\mathrm{red};P,Q}(a)$ and $b_{R,S} := \mathrm{f}'_{\mathrm{red};R,S}(b)$. If $\hbar$ is even real, then this Fréchet algebra is a Fréchet *-algebra with pointwise complex conjugation as *-involution.

PROOF: Continuity of $\star_\hbar$ on $\mathcal{P}(D_n)$ has already been shown in Theorem 5.1.20, so $\star_\hbar$ extends continuously to the completion of $\mathcal{P}(D_n)$, which is $\mathcal{A}(D_n)$ by the previous Theorem 5.1.29.

From the construction of $\star_\hbar$ out of the star product $\tilde{\star}_\hbar$ on $\mathbb{C}^{1+n}$ in Definition 5.1.12, the locally uniform estimate for $\tilde{\star}_\hbar$ in Lemma 5.1.15 and the locally uniform estimates for the reduction map $\Psi_\hbar$ in Lemma 5.1.19 it follows that the explicit formula for $\star_\hbar$ converges absolutely and locally uniformly in $\hbar \in H$.

Finally, if $\hbar \in \mathbb{R}$, then pointwise complex conjugation is a *-involution for this product by construction, and this also extends to the completion by continuity. □

5.1.6 Holomorphic Dependence on $\hbar$ and Classical Limit

In order to finally show that $\mathcal{A}(D_n)$ together with the star product $\star$ describes a deformation of a locally convex *-algebra, it only remains to show that the classical limit is well-behaved.

Theorem 5.1.31 *[52, Thm. 4.1] For all $a, b \in \mathcal{A}(D_n)$ the function*

$$H \ni \hbar \mapsto a \star_\hbar b \in \mathcal{A}(D_n) \tag{5.1.62}$$

is holomorphic. The singularities at $\hbar = -1/(2m)$ with $m \in \mathbb{N}$ are at most poles of order 1.

PROOF: If $a, b \in \mathcal{P}(D_n)$, then this all is clear because the explicit formula (5.1.38) for $\star_\hbar$ shows that $a \star_\hbar b$ is even a rational function of $\hbar$ with finitely many poles of at most order 1 only at the points $-1/(2m)$ with $m \in \mathbb{N}$. From the explicit formula for $\star_\hbar$ in Theorem 5.1.30 and its absolute and locally uniform convergence it follows that this result extends to the completion. $\square$

Note that the above theorem does not give any information about the classical limit $\hbar \to 0$. In fact, this limit is (contrary to the case of the ordinary Wick star product on $\mathbb{C}^{1+n}$) non-trivial because the following example shows that there can indeed occur a pole at every $\hbar = -1/(2m)$ with $m \in \mathbb{N}$:

Example 5.1.32 *[52, Expl. 4.2] Let $j, k \in \mathbb{N}$ be given and write $E_1 := (0, 1, 0, \dots, 0) \in \mathbb{N}_0^{1+n}$, then*

$$\mathrm{f}_{jE_1,jE_1} \star_\hbar \mathrm{f}_{kE_1,kE_1} = \sum_{t=0}^{\min\{j,k\}} \frac{(\frac{1}{2\hbar})_{j+k-t}\, t!}{(\frac{1}{2\hbar})_j (\frac{1}{2\hbar})_k} \binom{j}{t}\binom{k}{t} \mathrm{f}_{(j+k-t)E_1,(j+k-t)E_1}.$$

Moreover,

$$\frac{(z)_{j+k-t}}{(z)_j (z)_k} = \frac{\prod_{i=\max\{j,k\}}^{j+k-t-1}(z+i)}{\prod_{i=0}^{\min\{j,k\}-1}(z+i)}$$

has first order poles at all $z = -m$ with $m \in \big\{0, \dots, \min\{j,k\} - 1\big\}$ and residue

$$(-1)^m \frac{(j+k-t-m-1)!}{m!(j-m-1)!(k-m-1)!},$$

whose sign only depends on m, but not on j, k or t. This implies that if $a, b \in \mathcal{A}(D_n)$ are of the form

$$a = \sum_{j=0}^{\infty} a_j \mathrm{f}_{jE_1,jE_1} \qquad \text{and} \qquad b = \sum_{k=0}^{\infty} b_k \mathrm{f}_{kE_1,kE_1},$$

with positive coefficients $a_j, b_k \in \,]0, \infty[$, e.g. $a_j = b_j = 1/j!$, then

$$a \star_\hbar b = \sum_{j,k=0}^{\infty} a_j b_k \big(\mathrm{f}_{jE_1,jE_1} \star_\hbar \mathrm{f}_{kE_1,kE_1}\big)$$

has simple poles at each of the points $\hbar = -1/(2m)$, $m \in \mathbb{N}$.

Lemma 5.1.33 *[52, Lemma 4.3] For all $p, s \in \mathbb{N}_0$ and $x \in [0, 1]$, the estimate*

$$1 \le \frac{\prod_{i=0}^{p+s-1}(1+xi)}{\big(\prod_{j=0}^{p-1}(1+xj)\big)\big(\prod_{k=0}^{s-1}(1+xk)\big)} \le 1 + x2^{p+s} \tag{5.1.63}$$

holds.

PROOF: Without loss of generality we can assume that $p \ge s$. Note that

$$\frac{\prod_{i=0}^{p+s-1}(1+xi)}{\big(\prod_{j=0}^{p-1}(1+xj)\big)\big(\prod_{k=0}^{s-1}(1+xk)\big)} = \prod_{k=0}^{s-1} \frac{1+x(p+k)}{1+xk}$$

holds. So the first estimate $1 \leq \dots$ is trivial, for the second one we will show by induction over s that $\prod_{k=0}^{s-1} \frac{1+x(p+k)}{1+xk} \leq 1 + x\binom{p+s}{s} - x$ holds. If $s = 0$ or $s = 1$, then this is certainly true, and if it holds for one $s \in \mathbb{N}$ with $s < p$, then also for $s+1$, because then

$$
\begin{aligned}
\prod_{k=0}^{s} \frac{1+x(p+k)}{1+xk} &\leq \left(1 + x\binom{p+s}{s} - x\right)\frac{1+x(p+s)}{1+xs} \\
&= 1 + x\binom{p+s}{s}\frac{1+xp+xs}{1+xs} - x + \frac{(1-x)xp}{1+xs} \\
&= 1 + x\binom{p+s}{s}\frac{1+p+s}{1+s} - x\binom{p+s}{s}\frac{(1-x)p}{(1+xs)(1+s)} - x + \frac{(1-x)xp}{1+xs} \\
&\leq 1 + x\binom{p+s+1}{s+1} - x. \qquad \square
\end{aligned}
$$

Lemma 5.1.34 *[52, Lemma 4.4] For all $t, k_0 \in \mathbb{N}_0$ and all $x \in [0,1]$, the estimate*

$$
\frac{x^t t!}{\prod_{k=k_0}^{k_0+t-1}(1+xk)} \leq x^m 2^t m!
$$

holds for all $m \in \{0, \dots, t\}$.

PROOF: As $\frac{1}{1+xk} \leq \frac{1}{1+x(k-k_0)}$ it is sufficient to prove the estimate for the special case $k_0 = 0$. If $t = m = 0$ then this is certainly true, and otherwise $t \geq 1$ and thus:

$$
\frac{x^t t!}{\prod_{k=0}^{t-1}(1+xk)} = x^m\left(\prod_{k=0}^{m-1}\frac{1}{1+xk}\right)\left(\prod_{k=m}^{t-1}\frac{xk}{1+xk}\right)t(m-1)! \leq x^m t(m-1)! \leq x^m 2^t m! \qquad \square
$$

Theorem 5.1.35 *[52, Thm. 4.5] For all $a, b \in \mathcal{A}(D_n)$ the functions*

$$
]0,\infty[\, \ni \hbar \mapsto a \star_\hbar b \in \mathcal{A}(D_n) \quad \textit{and} \quad]0,\infty[\, \ni \hbar \mapsto \frac{1}{\mathrm{i}\hbar}[\,a\,,\,b\,]_{\star_\hbar} \in \mathcal{A}(D_n) \tag{5.1.64}
$$

are continuous and can be extended continuously to $[0,\infty[$ by

$$
\lim_{\hbar \to 0^+} a \star_\hbar b = ab \quad \textit{and} \quad \lim_{\hbar \to 0^+} \frac{1}{\mathrm{i}\hbar}[\,a\,,\,b\,]_{\star_\hbar} = \{\,a\,,\,b\,\}\,. \tag{5.1.65}
$$

PROOF: The continuity of these functions on $]0,\infty[$ is a direct consequence of the holomorphic dependence of $\star_\hbar$ on $\hbar$ from Theorem 5.1.31. For the limit $\hbar \to 0^+$ we first consider only products of $\mathrm{f}_{P,Q}$ and $\mathrm{f}_{R,S}$ with $P, Q, R, S \in \mathbb{N}_0^{1+n}$ as well as $|P| = |Q|$ and $|R| = |S|$. It will be helpful to use both the fundamental system of continuous seminorms $\|\cdot\|_{D_n,K}$ of $\mathcal{A}(D_n)$ with K running over all compact subsets of $\hat{D}_n$ and the fundamental system of continuous seminorms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$, extended continuously from $\mathcal{P}(D_n)$ to $\mathcal{A}(D_n)$. Recall that these two systems are equivalent by Theorem 5.1.29. Let $\hbar \in\,]0, 1/2]$ and a compact $K \subseteq \hat{D}_n$ be given, then the estimate

$$
\left\|\mathrm{f}_{P,Q} \star_\hbar \mathrm{f}_{R,S} - \mathrm{f}_{P,Q}\mathrm{f}_{R,S}\right\|_{D_n,K} \leq
$$

$$\leq \left|\frac{\left(\frac{1}{2\hbar}\right)_{|P+S|}}{\left(\frac{1}{2\hbar}\right)_{|P|}\left(\frac{1}{2\hbar}\right)_{|S|}} - 1\right| \big\|\mathrm{f}_{P+R,Q+S}\big\|_{D_n,K} + \sum_{\substack{T\in\mathbb{N}_0^{1+n},\\ |T|>0 \text{ and}\\ T\leq\min\{P,S\}}} \frac{\left(\frac{1}{2\hbar}\right)_{|P+S-T|}T!}{\left(\frac{1}{2\hbar}\right)_{|P|}\left(\frac{1}{2\hbar}\right)_{|S|}}\binom{P}{T}\binom{S}{T}\big\|\mathrm{f}_{P+R-T,Q+S-T}\big\|_{D_n,K}$$

holds by the formula (5.1.38) for $\star_\hbar$. Using the results of the previous two lemmas, we get

$$\left|\frac{\left(\frac{1}{2\hbar}\right)_{|P+S|}}{\left(\frac{1}{2\hbar}\right)_{|P|}\left(\frac{1}{2\hbar}\right)_{|S|}} - 1\right| = \left|\frac{\prod_{i=0}^{|P+S|-1}(1+2\hbar i)}{\left(\prod_{j=0}^{|P|-1}(1+2\hbar j)\right)\left(\prod_{k=0}^{|S|-1}(1+2\hbar k)\right)} - 1\right| \leq 2\hbar\, 2^{|P+S|}$$

by Lemma 5.1.33 and

$$\begin{aligned}\frac{\left(\frac{1}{2\hbar}\right)_{|P+S-T|}T!}{\left(\frac{1}{2\hbar}\right)_{|P|}\left(\frac{1}{2\hbar}\right)_{|S|}} &= \frac{\left(\prod_{i=0}^{|P+S-T|-1}(1+2\hbar i)\right)(2\hbar)^{|T|}T!}{\left(\prod_{j=0}^{|P|-1}(1+2\hbar j)\right)\left(\prod_{k=0}^{|S-T|-1}(1+2\hbar k)\right)\left(\prod_{\ell=|S-T|}^{|S|-1}(1+2\hbar \ell)\right)}\\ &\leq \binom{P+S-T}{P} 2\hbar\, 2^{|T|}\\ &\leq 2\hbar\, 2^{|P+S|}\end{aligned}$$

by Lemma 5.1.34 with $m=1$ and by using that $\frac{1+2\hbar i}{1+2\hbar k} \leq \frac{1+i}{1+k}$ as long as $i \geq k$. Now define

$$\rho := 2 + \max_{\mu,\nu\in\{1,\dots,n\}} \|\mathrm{f}_{E_\mu,E_\nu}\|_{D_n,K},$$

then $\|\mathrm{f}_{P+R-T,Q+S-T}\|_{D_n,K} \leq \rho^{|P+R+Q+S-2T|/2} = \rho^{|P+S-T|}$ by submultiplicativity, and this yields

$$\begin{aligned}\big\|\mathrm{f}_{P,Q}\star_\hbar \mathrm{f}_{R,S} - \mathrm{f}_{P,Q}\mathrm{f}_{R,S}\big\|_{D_n,K} &\leq 2\hbar\, 2^{|P+S|}\sum_{T=0}^{\min\{P,S\}}\binom{P}{T}\binom{S}{T}\rho^{|P+S-T|}\\ &\leq 2\hbar\,(4\rho)^{|P+S|}\sum_{T=0}^{\min\{P,S\}}\rho^{-|T|}\\ &\leq 2\hbar\, 2^{1+n}\,(4\rho)^{|P+S|}\end{aligned}$$

as $\rho \geq 2$. From the definition of the fundamental monomials it now follows that especially

$$\big\|\mathrm{f}_{\mathrm{red};P,Q}\star_\hbar \mathrm{f}_{\mathrm{red};R,S} - \mathrm{f}_{\mathrm{red};P,Q}\mathrm{f}_{\mathrm{red};R,S}\big\|_{D_n,K} \leq 2\hbar\, 2^{1+n}\,(4\rho)^{\max\{|P|,|Q|\}+\max\{|R|,|S|\}}$$

holds for all $P,Q,R,S\in\mathbb{N}_0^n$. Thus, for all $a = \sum_{P,Q\in\mathbb{N}_0^n} a_{P,Q}\mathrm{f}_{\mathrm{red};P,Q}$ and $b = \sum_{R,S\in\mathbb{N}_0^n} b_{R,S}\mathrm{f}_{\mathrm{red};R,S}$ with complex coefficients $a_{P,Q}$ and $b_{R,S}$ we get

$$\begin{aligned}\|a\star_\hbar b - ab\|_{D_n,K} &\leq \sum_{P,Q,R,S\in\mathbb{N}_0^n}|a_{P,Q}b_{R,S}|\big\|\mathrm{f}_{\mathrm{red};P,Q}\star_\hbar \mathrm{f}_{\mathrm{red};R,S} - \mathrm{f}_{\mathrm{red};P,Q}\mathrm{f}_{\mathrm{red};R,S}\big\|_{D_n,K}\\ &\leq 2\hbar\, 2^{1+n}\sum_{P,Q,R,S\in\mathbb{N}_0^n}|a_{P,Q}b_{R,S}|(4\rho)^{\max\{|P|,|Q|\}+\max\{|R|,|S|\}}\\ &\leq 2\hbar\, 2^{1+n}\sum_{P,Q,R,S\in\mathbb{N}_0^n}|a_{P,Q}b_{R,S}|(4\rho)^{|P+Q+R+S|}\end{aligned}$$

$$= 2\hbar \, 2^{1+n} \, \|a\|_{D_n,4\rho} \|b\|_{D_n,4\rho} \, ,$$

which proves that $\lim_{\hbar\to 0^+} a \star_\hbar b = ab$. In order to prove the result for the limit of the commutator, we proceed analogously and start with commutators of $\mathrm{f}_{P,Q}$ and $\mathrm{f}_{R,S}$ with $P, Q, R, S \in \mathbb{N}_0^{1+n}$. Let $\hbar \in \,]0, 1/2]$ and a compact $K \subseteq \hat{D}_n$ be given, then the estimate

$$\begin{aligned}
&\left\| \frac{1}{\mathrm{i}\hbar} [\, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \,]_{\star_\hbar} - \{ \, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \, \} \right\|_{D_n,K} \leq \\
&\leq \frac{1}{\hbar} \sum_{m=0}^{n} \left| \frac{(\frac{1}{2\hbar})_{|P+S|-1}}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} - 2\hbar \right| \big(P_m S_m + Q_m R_m\big) \big\| \mathrm{f}_{P+R-E_m, Q+S-E_m} \big\|_{D_n,K} \\
&\quad + \frac{1}{\hbar} \sum_{\substack{T \in \mathbb{N}_0^{1+n}, \\ |T|>1 \text{ and } T \leq \min\{P,S\}}} \frac{(\frac{1}{2\hbar})_{|P+S-T|} T!}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} \left(\binom{P}{T} \binom{S}{T} + \binom{Q}{T} \binom{R}{T} \right) \big\| \mathrm{f}_{P+R-T, Q+S-T} \big\|_{D_n,K}
\end{aligned}$$

holds by the formula (5.1.38) for $\star_\hbar$ and because

$$\{ \, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \, \} = \lim_{\hbar \to 0^+} \frac{1}{\mathrm{i}\hbar} [\, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \,]_{\star_\hbar} = 2\mathrm{i} \big(P_m S_m - Q_m R_m\big) \mathrm{f}_{P+R-E_m, Q+S-E_m}$$

by construction of $\star_\hbar$. For the first term we can use

$$\begin{aligned}
\left| \frac{(\frac{1}{2\hbar})_{|P+S|-1}}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} - 2\hbar \right| &= 2\hbar \left| \frac{(\frac{1}{2\hbar})_{|P+S|}}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} \frac{1}{1 + 2\hbar(|P+S|-1)} - 1 \right| \\
&\leq 2\hbar \left| \frac{(\frac{1}{2\hbar})_{|P+S|}}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} - 1 \right| \frac{1}{1 + 2\hbar(|P+S|-1)} + \frac{(2\hbar)^2 (|P+S|-1)}{1 + 2\hbar(|P+S|-1)} \\
&\leq (2\hbar)^2 \left(2^{|P+S|} + |P+S| \right) \\
&\leq 2 \, (2\hbar)^2 \, 2^{|P+S|}
\end{aligned}$$

by Lemma 5.1.33 as long as $|P+S| \geq 1$, which is of course the only case of interest. For the second term, an analogous argument as before using Lemma 5.1.34 with $m = 2$ yields

$$\frac{(\frac{1}{2\hbar})_{|P+S-T|} T!}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} \leq 2 \, (2\hbar)^2 \, 2^{|P+S|}$$

and by putting all of this together we see that

$$\begin{aligned}
&\left\| \frac{1}{\mathrm{i}\hbar} [\, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \,]_{\star_\hbar} - \{ \, \mathrm{f}_{P,Q} \, , \, \mathrm{f}_{R,S} \, \} \right\|_{D_n,K} \leq \\
&\qquad \leq 8\hbar \, 2^{|P+S|} \sum_{\substack{T \in \mathbb{N}_0^{1+n}, \\ |T|>0 \text{ and } T \leq \min\{P,S\}}} \left(\binom{P}{T} \binom{S}{T} + \binom{Q}{T} \binom{R}{T} \right) \rho^{|P+S-T|} \\
&\qquad \leq 16\hbar \, (4\rho)^{|P+S|} \sum_{\substack{T \in \mathbb{N}_0^{1+n}, \\ |T|>0 \text{ and } T \leq \min\{P,S\}}} \rho^{-|T|} \\
&\qquad \leq 16\hbar \, 2^{1+n} \, (4\rho)^{|P+S|} \, ,
\end{aligned}$$

which leads to $\lim_{\hbar\to 0^+} \frac{1}{\mathrm{i}\hbar}[\, a\, ,\, b\,]_{\star_\hbar} = \{\, a\, ,\, b\,\}$ like before. $\square$

Corollary 5.1.36 $\big(\mathcal{A}(D_n),\,]0,\infty[\, \ni \hbar \mapsto \star_\hbar,\, \cdot^*\big)$ *is a deformation of a locally convex* **-algebra. Its classical limit is the restriction to* $\mathcal{A}(D_n)$ *of the Poisson* **-algebra* $\mathscr{C}^\infty(D_n,\mathbb{C})$ *with the Poisson bracket determined by the Kähler structure of* D_n *and with the canonical Fréchet topology of the space of* $\mathcal{A}(D_n)$.

Note that the classical limit $\mathcal{A}(D_n)$ of the reduced star product $\star$ on D_n is *not* the reduction of the classical limit of the star product $\tilde{\star}$ on $\mathbb{C}^{1+n}$: the topology on $\mathcal{A}(D_n)$, or rather $\mathcal{P}(D_n)$, is not the quotient topology of (the completion of) $\mathcal{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ as one can check that Lemma 5.1.19 does not hold for $\hbar = 0$. It is only this – at first sight rather unpleasent – observation that allows the characterization of the completion of $\mathcal{P}(D_n)$ as the space of *all* real-analytic functions $\mathcal{A}(D_n)$ which extend to holomorphic functions on $\hat{D}_n$, a result that could not be achieved in the flat case on $\mathbb{C}^{1+n}$.

5.2 Properties of the Construction

In a next step we investigate the classically positive linear functionals on $\mathcal{A}(D_n)$ and show that the continuous algebraic characters of $\mathcal{A}(D_n)$ are given by $D_{n,\mathrm{ext}}$. This means that the most natural representation of $\mathcal{A}(D_n)$ as functions is not a representation on D_n, but on the larger $D_{n,\mathrm{ext}}$. After that the deformation $\star_\hbar$ for $\hbar > 0$ is shown to have a faithful continuous representation as operators and we discuss the question whether and how the infinitesimal action of $\mathfrak{su}(1,n)$ exponentiates to the global symmetry under the group $\mathrm{SU}(1,n)$. Note, however, that continuity of algebraic characters, of algebraically positive linear functionals in general, and thus of all representations, is automatically guaranteed because $\mathcal{A}(D_n)$ is a Fréchet *-algebra, see the discussion in Proposition 3.3.19.

5.2.1 Gelfand Transformation and Classically Positive Linear Functionals

From the construction of the commutative *-algebra $\mathcal{A}(D_n)$ with the pointwise product it is clear that $\mathcal{A}(D_n)$ has a faithful (continuous) representation as functions on D_n. Nevertheless, this is to some extend artificial. The most natural representation as functions of $\mathcal{A}(D_n)$ is on the space of its (continuous) algebraic characters via Gelfand transformation.

Proposition 5.2.1 *[52, Prop. 4.6] The map* $M\colon \hat{D}_n \to \mathbb{C}^{(1+n)\times(1+n)}$ *with components* $M^{\mu\nu} := \hat{\mathrm{f}}_{E_\mu,E_\nu}$ *is a holomorphic embedding that realizes* $\hat{D}_n$ *as the submanifold*

$$\mathcal{S} := \big\{\, A \in \mathbb{C}^{(1+n)\times(1+n)} \;\big|\; h_{\mu\nu}A^{\mu\nu} = -1 \textit{ and } A^{\mu\nu}A^{\rho\sigma} = A^{\mu\sigma}A^{\rho\nu} \textit{ for } \mu,\nu,\rho,\sigma \in \{0,\dots,n\} \,\big\}\,. \quad (5.2.1)$$

P R O O F : First of all note that $h_{\mu\nu}M^{\mu\nu} = -\hat{\mathrm{f}}_{E_0,E_0} + \sum_{i=1}^n \hat{\mathrm{f}}_{E_i,E_i} = -1$ by (5.1.30), where E_i is the unit vector with 1 at the i-th position and 0 elsewhere. Note also that $M^{\mu\nu}\circ\hat{\mathrm{pr}} = (x^\mu y^\nu)\circ\hat{\imath}$ by construction of the $\hat{\mathrm{f}}_{E_\mu,E_\nu}$, so

$$\big(M^{\mu\nu}M^{\rho\sigma}\big)([p,q]) = x^\mu(p,q)y^\nu(p,q)x^\rho(p,q)y^\sigma(p,q) = \big(M^{\mu\sigma}M^{\rho\nu}\big)([p,q])$$

for all $[p,q] \in \hat{D}_n$. As these polynomials are holomorphic, M is a holomorphic mapping to $\mathcal{S}$.

Given $A \in \mathcal{S}$, then $h_{\mu\nu}A^{\mu\nu} = -1$ implies that there exists a $\rho \in \{0, \dots, n\}$ such that $A^{\rho\rho} \neq 0$. If there exists $[p, q] \in \hat{D}_n$ with $M([p, q]) = A$, then $A^{\mu\nu} = p^\mu q^\nu$ for all $\mu, \nu \in \{0, \dots, n\}$. This determines a unique class $[p, q] \in \hat{D}_n$: As $p^\rho \neq 0$ and $q^\rho \neq 0$ it follows that $p^\mu = A^{\mu\rho}/q^\rho$ for all $\mu \in \{0, \dots, n\}$ and $q^\nu = A^{\rho\nu}/p^\rho = q^\rho A^{\rho\nu}/A^{\rho\rho}$ for all $\nu \in \{0, \dots, n\}$. So if a preimage $[p, q] \in \hat{D}_n$ of A under M exists, then it is unique, and choosing $p^\mu = A^{\mu\rho}$ and $q^\nu = A^{\rho\nu}/A^{\rho\rho}$ yields such a preimage as $A \in \mathcal{S}$ fulfils $A^{\mu\nu} = A^{\mu\rho}A^{\rho\nu}/A^{\rho\rho}$ for all $\mu, \nu \in \{0, \dots, n\}$. So M is a holomorphic bijection from $\hat{D}_n$ to $\mathcal{S}$, hence a biholomorphic map by the open mapping theorem. □

Note that this especially implies that the holomorphic functions on $\hat{D}_n$ separate points, because the holomorphic functions on $\mathbb{C}^{(1+n)\times(1+n)}$ do.

Definition 5.2.2 *[52, Def. 4.7] For $[p, q] \in \hat{D}_n$ define the evaluation functional $\delta_{[p,q]} : \mathcal{A}(D_n) \to \mathbb{C}$, $a \mapsto \langle\, \delta_{[p,q]}\,,\, a \,\rangle := \hat{a}([p, q])$.*

Proposition 5.2.3 *[52, Prop. 4.8] Given $[p, q] \in \hat{D}_n$, then $\delta_{[p,q]}$ is a continuous unital homomorphism from $\mathcal{A}(D_n)$ with the pointwise product to $\mathbb{C}$ and it is a continuous algebraic character of $\mathcal{A}(D_n)$ if and only if $[p, q] \in D_{n,ext}$.*

PROOF: It is clear from its definition that $\delta_{[p,q]}$ is a continuous unital homomorphism. Moreover, $\langle\, \delta_{[p,q]}\,,\, a^* \,\rangle = \overline{(\hat{a} \circ \tau)([p, q])}$ and $\overline{\langle\, \delta_{[p,q]}\,,\, a \,\rangle} = \overline{\hat{a}([p, q])}$, so $\delta_{[p,q]}$ is a character if and only if $\big(\hat{a} \circ \tau\big)([p, q]) = \hat{a}([p, q])$ holds for all $a \in \mathcal{A}(D_n)$. As the holomorphic functions on $\hat{D}_n$ separate points by Proposition 5.2.1, this is equivalent to $\tau([p, q]) = [p, q]$, i.e. to $[p, q] \in D_{n,\mathrm{ext}}$. □

Theorem 5.2.4 (Gelfand transformation) *[52, Thm. 4.9] Let $\mathrm{Spec}\big(\mathcal{A}(D_n)\big)$ be the set of continuous unital homomorphisms from $\mathcal{A}(D_n)$ to $\mathbb{C}$ with the weak-*-topology. Then $\delta : \hat{D}_n \to \mathrm{Spec}\big(\mathcal{A}(D_n)\big)$, $[p, q] \mapsto \delta_{[p,q]}$ is a well-defined homeomorphism. Moreover, let $\mathcal{M}\big(\mathcal{A}(D_n), \mathcal{T}\big) \subseteq \mathrm{Spec}\big(\mathcal{A}(D_n)\big)$ be the set of continuous algebraic characters of $\mathcal{A}(D_n)$ again with the weak-*-topology, then δ restricts to a homeomorphism from $D_{n,ext}$ to $\mathcal{M}\big(\mathcal{A}(D_n), \mathcal{T}\big)$.*

PROOF: Proposition 5.2.3 already shows that δ maps to the continuous unital homomorphisms, and δ is injective because the holomorphic functions on $\hat{D}_n$ separate points due to Proposition 5.2.1.

Now let a continuous unital homomorphism $\omega : \mathcal{A}(D_n) \to \mathbb{C}$ be given. Let $A^{\mu\nu} := \langle\, \omega\,,\, \mathrm{f}_{E_\mu,E_\nu} \,\rangle$ for all $\mu, \nu \in \{0, \dots, n\}$, then $h_{\mu\nu}A^{\mu\nu} = -\big\langle\, \omega\,,\, \mathrm{f}_{E_0,E_0} - \sum_{i=1}^n \mathrm{f}_{E_i,E_i} \,\big\rangle = -\langle\, \omega\,,\, \mathrm{f}_{0,0} \,\rangle = -1$ by (5.1.30) and $A^{\mu\nu}A^{\rho\sigma} = \big\langle\, \omega\,,\, \mathrm{f}_{E_\mu,E_\nu}\mathrm{f}_{E_\rho,E_\sigma} \,\big\rangle = \big\langle\, \omega\,,\, \mathrm{f}_{E_\mu,E_\sigma}\mathrm{f}_{E_\rho,E_\nu} \,\big\rangle = A^{\mu\sigma}A^{\rho\nu}$, so A is in the image of the holomorphic embedding M from Proposition 5.2.1 and there exists a unique $[p, q] \in \hat{D}_n$ with $\langle\, \omega\,,\, \mathrm{f}_{E_\mu,E_\nu} \,\rangle = A^{\mu\nu} = M^{\mu\nu}([p, q]) = \langle\, \delta_{[p,q]}\,,\, \mathrm{f}_{E_\mu,E_\nu} \,\rangle$ for all $\mu, \nu \in \{0, \dots, n\}$. As these monomials generate $\mathcal{P}(D_n)$ as a unital algebra, $\delta_{[p,q]}$ and ω coincide on $\mathcal{P}(D_n)$, and as $\mathcal{P}(D_n)$ is dense in $\mathcal{A}(D_n)$ we can conclude that $\delta_{[p,q]} = \omega$.

By now we have seen that δ is a bijection, and it is even a homeomorphism, because the embedding M of $\hat{D}_n$ in $\mathbb{C}^{(1+n)\times(1+n)}$ shows that $\hat{D}_n$ carries the weak topology of its holomorphic functions (because $\mathbb{C}^{(1+n)\times(1+n)}$ does), which under δ corresponds to the weak-*-topology.

The analogous statements about the space of characters of $\mathcal{A}(D_n)$ are now an immediate consequence of the above and of Proposition 5.2.3. □

Note that this result is to some extend unfortunate, as it shows that the interpretation of $\mathcal{A}(D_n)$ as a *-algebra of functions on D_n is not really natural. At the center of the problem lies the fact that the function $\frac{1}{1-w\cdot\overline{w}} = \mathrm{f}_{E_0,E_0} \in \mathcal{P}(D_n)$ is not algebraically positive even though $\mathrm{f}_{E_0,E_0} = \Psi_0\big((\mathrm{d}_{E_0,0})^*\mathrm{d}_{E_0,0}\big)$. As the algebra $\mathcal{P}(D_n)$ arises from $\mathcal{P}(\mathbb{C}^{1+n})$ by a reduction procedure (in the classical case as well as in the quantum case), one might consider only those positive linear functionals on $\mathcal{P}(D_n)$ to be "relevant" that come from a U(1)-invariant functional on $\mathcal{P}(\mathbb{C}^{1+n})$. This would especially eliminate all characters $\phi \in \mathcal{M}\big(\mathcal{A}(D_n)\big)$ for which $\langle\, \phi\,,\, \mathrm{f}_{E_0,E_0}\,\rangle < 0$ and leave only the evaluation functionals at points in D_n.

Corollary 5.2.5 *[52, Cor. 4.10] Let $\phi\colon \mathcal{A}(D_n) \to \mathbb{C}$ be a continuous algebraically positive linear functional (with respect to the pointwise product), then there exists a compact $K \subseteq D_{n,ext}$ and a Radon measure μ on K such that*

$$\langle\, \phi\,,\, a\,\rangle = \int_K \hat{a}\,\mathrm{d}\mu$$

holds for all $a \in \mathcal{A}(D_n)$.

PROOF: It is sufficient to treat the case that ϕ is normalized to $\langle\, \phi\,,\, \mathbb{1}\,\rangle = 1$, as $\langle\, \phi\,,\, \mathbb{1}\,\rangle = 0$ implies $\phi = 0$ by the Cauchy Schwarz inequality, and to show that there exists a compact $K \subseteq D_{n,\mathrm{ext}}$ such that $|\phi(a)| \leq \|a\|_{D_n,K}$ holds for all $a \in \mathcal{A}(D_n)$, in which case ϕ extends continuously to $\mathscr{C}(K)$, the completion of $\mathcal{A}(D_n)$ under $\|\cdot\|_{D_n,K}$ by the Stone-Weierstraß theorem, and can be represented by integration over a Radon measure μ on K by the Riesz-Markov theorem.

As $\mathcal{A}(D_n)$ is an lmc *-algebra (its locally convex topology is defined by the submultiplicative *-seminorms $\|\cdot\|_{D_n,K}$ with $K \subseteq \hat{D}_n$ compact and stable under τ), the abstract O^*-algebra $(\mathcal{A}(D_n), \mathcal{T})$ is bounded by Proposition 3.4.14 and so $\|\cdot\|_{\phi,\infty}$ like in Definition 3.4.9 is a continuous C^*-seminorm on $\mathcal{A}(D_n)$ for which $|\langle\, \phi\,,\, a\,\rangle| \leq \|a\|_{\phi,\infty}$ holds for all $a \in \mathcal{A}(D_n)$.

By dividing out the zeros of $\|\cdot\|_{\phi,\infty}$ and completing with respect to $\|\cdot\|_{\phi,\infty}$, one constructs a commutative C^*-algebra $\mathcal{B}$ and the continuous $\iota\colon \mathcal{A}(D_n) \to \mathcal{B}$ as the composition of the projection on the quotient and the inclusion in the completion. Let $\mathcal{M}(\mathcal{B}, \mathcal{T})$ be the (compact) set of characters of $\mathcal{B}$, then the C^*-norm $\|\cdot\|_{\phi,\infty}$ on $\mathcal{B}$ is the uniform norm on the Gelfand transformation of $\mathcal{B}$, hence especially $\|a\|_{\phi,\infty} = \sup_{\psi \in \mathcal{M}(\mathcal{B},\mathcal{T})} |\langle\, \psi\,,\, \iota(a)\,\rangle|$ for all $a \in \mathcal{A}(D_n)$. The pullback $\iota^*\colon \mathcal{M}(\mathcal{B}, \mathcal{T}) \to \mathcal{M}(\mathcal{A}(D_n), \mathcal{T})$ is weak-*-continuous by construction of ι and by the previous Theorem 5.2.4, the compact $\iota^*\big(\mathcal{M}(\mathcal{B}, \mathcal{T})\big) \subseteq \mathcal{M}(\mathcal{A}(D_n), \mathcal{T})$ is the image of a compact $K \subseteq D_{n,\mathrm{ext}}$ under δ, so

$$|\langle\, \phi\,,\, a\,\rangle| \leq \|a\|_{\phi,\infty} = \sup_{\psi \in \mathcal{M}(\mathcal{B},\mathcal{T})} \big|\langle\, \psi\,,\, \iota(a)\,\rangle\big| = \sup_{[p,q] \in K} \big|\langle\, \delta_{[p,q]}\,,\, a\,\rangle\big| = \sup_{[p,q] \in K} |\hat{a}([p,q])| = \|a\|_{D_n,K}$$

for all $a \in \mathcal{A}(D_n)$. □

5.2.2 Positive Linear Functionals and Representations of the Deformed Algebra

From the point of view of physics, the most important problem after having constructed a *-algebra of observables is, whether there exist many positive linear functionals and thus faithful representations.

Proposition 5.2.6 *[52, Prop. 4.11] For every $[r] \in D_n$, the evaluation functional $\delta_{[r]} := \delta_{\Delta_D([r])}$, i.e. $\delta_{[r]}\colon \mathcal{A}(D_n) \to \mathbb{C}$, $a \mapsto \langle\, \delta_{[r]}\,,\, a\,\rangle = a([r])$ is continuous and algebraically positive with respect to every product $\star_\hbar$ for all $\hbar \geq 0$.*

PROOF: Like in Proposition 5.2.3, continuity of all evaluation functionals $\delta_{[r]}$ with $[r] \in D_n$ is clear. It is thus also sufficient to prove positivity of $\delta_{[r]}$ only on the dense unital *-subalgebra $\mathcal{P}(D_n)$ of $\mathcal{A}(D_n)$, which has already been done in [7, Lemma 5.21]. For sake of completeness, the proof is repeated here:

Due to the $\mathrm{SU}(1,n)$-invariance of the star product it is sufficient to check positivity of δ_0, the evaluation functional at the point of D_n which is mapped to 0 in the standard chart, because for all $[r] \in D_n$ there exists a $U \in \mathrm{SU}(1,n)$ such that $U \triangleright 0 = [r]$, and then $\langle\, \delta_{[r]}\,,\, a^* \star_\hbar a \,\rangle = \langle\, \delta_0\,,\, (a^* \star_\hbar a) \triangleleft U \,\rangle = \langle\, \delta_0\,,\, (a \triangleleft U)^* \star_\hbar (a \triangleleft U) \,\rangle$ for all $a \in \mathcal{P}(D_n)$. For this case, however, it is not hard to check by an explicit calculation that $\langle\, \delta_0\,,\, a^* \star_\hbar a \,\rangle \geq 0$: From the explicit formula (5.1.38) it follows that

$$\langle\, \delta_0\,,\, \mathrm{f}_{P,Q} \star_\hbar \mathrm{f}_{R,S} \,\rangle = \sum_{T=0}^{\min\{P,S\}} (-1)^{T_0} \frac{(\frac{1}{2\hbar})_{|P+S-T|} T!}{(\frac{1}{2\hbar})_{|P|} (\frac{1}{2\hbar})_{|S|}} \binom{P}{T} \binom{S}{T} \langle\, \delta_0\,,\, \mathrm{f}_{P+R-T,Q+S-T} \,\rangle$$

for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ and then (5.1.30) together with the observation that $\langle\, \delta_0\,,\, \mathrm{f}_{\mathrm{red};P',Q'} \,\rangle \neq 0$ only for $P' = Q' = 0$ implies that the above gives a non-zero result only if $P_i + R_i = Q_i + S_i = T_i$ for all $i \in \{1, \dots, n\}$. As $T \leq \min\{P,S\}$ this is the case only if $P_i = S_i = T_i$ and $Q_i = R_i = 0$ for all $i \in \{1, \dots, n\}$. Especially for the fundamental monomials this implies that $\langle\, \delta_0\,,\, \mathrm{f}_{\mathrm{red};P,Q} \,\tilde{\star}_\hbar\, \mathrm{f}_{\mathrm{red};R,S} \,\rangle \neq 0$ with $P, Q, R, S \in \mathbb{N}_0^n$ only if $P = S$ and $Q = R = 0$, in which case

$$\langle\, \delta_0\,,\, \mathrm{f}_{\mathrm{red};S,0} \,\tilde{\star}_\hbar\, \mathrm{f}_{\mathrm{red};0,S} \,\rangle = \langle\, \delta_0\,,\, \mathrm{f}_{(0,S),(|S|,0)} \,\tilde{\star}_\hbar\, \mathrm{f}_{(|S|,0),(0,S)} \,\rangle = \frac{(\frac{1}{2\hbar})_{|S|} S!}{(\frac{1}{2\hbar})_{|S|} (\frac{1}{2\hbar})_{|S|}} \langle\, \delta_0\,,\, \mathrm{f}_{(|S|,0),(|S|,0)} \,\rangle,$$

and thus

$$\langle\, \delta_0\,,\, a^* \star_\hbar a \,\rangle = \sum_{S \in \mathbb{N}_0^n} |a_{0,S}|^2 \frac{S!}{(\frac{1}{2\hbar})_{|S|}} \geq 0$$

for all $a = \sum_{R,S \in \mathbb{N}_0^n} a_{R,S} \mathrm{f}_{\mathrm{red};R,S}$. □

Corollary 5.2.7 *[52, Cor. 4.12] Let $\hbar \geq 0$. Then every algebraically positive linear functional on $\mathscr{C}^\infty(D_n, \mathbb{C})$ restricts to a continuous algebraically positive linear functional on $\mathcal{A}(D_n)$ with respect to $\star_\hbar$.*

PROOF: Indeed, every such positive linear functional is an integration with respect to a Radon measure on a compact subset $K \subseteq D_n$ due to an argument similar to that of Corollary 5.2.5 using that $\mathscr{C}^\infty(D_n, \mathbb{C})$ is an lmc Fréchet *-algebra, and using that the only algebraic characters of $\mathscr{C}^\infty(D_n, \mathbb{C})$ are the evaluation functionals in D_n: Every such character is also an algebraic character of $\mathcal{A}(D_n)$ and yields a non-negative result on $\frac{1}{1 - w \cdot \overline{w}} = \mathrm{f}_{E_0,E_0} \in \mathcal{P}(D_n)$, because this function has a smooth square root on D_n.

Since all evaluation functionals at points of D_n are positive with respect to $\star_\hbar$, this also holds for the convex combinations needed for general Radon measures. □

As these evaluation functionals are clearly point-separating, we conclude by using Theorems 3.2.11 and 3.3.21 or 3.3.22:

Theorem 5.2.8 *[52, Thm. 4.14] Let $\hbar \geq 0$ be given, then the abstract O^*-algebra $(\mathcal{A}(D_n), \star_\hbar, \cdot^*, \mathcal{T})$ is Hausdorff and there exists a faithful continuous representation as operators of the Fréchet *-algebra $\big(\mathcal{A}(D_n), \star_\hbar, \cdot^*\big)$.*

Having established the existence of interesting representations by (unbounded) operators, the question arises which algebra elements are actually essentially self-adjoint in representations. Therefore, recall Definition 5.1.8 of the filtration of $\mathcal{P}(D_n)$ by degree and Lemma 5.1.16 for the estimate on the growth of $\star_\hbar$-powers. Note that the formula (5.1.38) also shows immediately that the deformed products $\star_\hbar$ are also filtered with respect to the above filtration, i.e. $a \star_\hbar b \in \mathcal{P}(D_n)^{(k+\ell)}$ holds for all $a \in \mathcal{P}(D_n)^{(k)}, b \in \mathcal{P}(D_n)^{(\ell)}$ and all $\hbar \in H$.

Theorem 5.2.9 *[52, Thm. 4.15] Fix $\hbar \geq 0$ and let $(\mathcal{D}, \pi)$ be a continuous *-representation of $\big(\mathcal{A}(D_n), \star_\hbar, \cdot^*\big)$. Then $\pi(a)$ is essentially self-adjoint for every Hermitian $a \in \mathcal{P}(D_n)^{(1)}$ and for every Hermitian $a \in \mathcal{P}(D_n)^{(2)}$ that is semi-bounded, i.e. for which the set of all $\langle\, \phi\,,\, a\,\rangle$, with ϕ running over all continuous algebraic states of $\big(\mathcal{A}(D_n), \star_\hbar, \cdot^*\big)$, is bounded from above or below.*

PROOF: Lemma 5.1.16 shows that every continuous algebraic state of $\big(\mathcal{A}(D_n), \star_\hbar, \cdot^*\big)$ is a Stieltjes state for all Hermitian and algebraically positive $a \in \mathcal{P}(D_n)^{(2)}$, so this all is a direct consequence of Corollary 3.4.19. □

5.2.3 Exponentiation of the $\mathfrak{su}(1, n)$-Action

By reduction to D_n, the $\mathrm{U}(1, n)$-symmetry of $\mathbb{C}^{1+n}$ is reduced to a $\mathrm{SU}(1, n)$-symmetry. Recall that $\mathfrak{u}(1, n) \ni u \mapsto \mathcal{J}(u) := \frac{\mathrm{i}}{2} h_{\mu\nu} u^\mu_\rho z^\rho \overline{z}^\nu \in \mathcal{P}(\mathbb{C}^{1+n}) \subseteq \mathscr{C}^\infty(\mathbb{C}^{1+n})$ is a (classical) equivariant moment map for this action. As $\mathcal{J}(u)$ is linear in the z and $\overline{z}$-coordinates, only terms up to first order in $\hbar$ will contribute to the Wick star product with $\mathcal{J}(u)$, so $f \triangleleft u = \{\, f\,,\, \mathcal{J}(u)\,\} = \frac{1}{\mathrm{i}\hbar}[\, f\,,\, \frac{1}{\mathrm{i}\hbar}\mathcal{J}(u)\,]$ and $\frac{1}{\mathrm{i}\hbar}\mathcal{J}([\, u\,,\, v\,]) = \frac{1}{\mathrm{i}\hbar}\{\, \mathcal{J}(u)\,,\, \mathcal{J}(v)\,\} = [\, \frac{1}{\mathrm{i}\hbar}\mathcal{J}(u)\,,\, \frac{1}{\mathrm{i}\hbar}\mathcal{J}(v)\,]$ hold for all $u, v \in \mathfrak{u}(1, n)$ and $f \in \mathcal{P}(\mathbb{C}^{1+n})$, i.e. $\frac{1}{\mathrm{i}\hbar}\mathcal{J}$ is an equivariant quantum moment map. Reduction to D_n then yields the following well-known result, see e.g. [16, Lemma 5] for the case of reduction to $\mathbb{CP}^n$:

Proposition 5.2.10 *[52, Prop. 4.16] The map $\mathcal{J}_{D_n} \colon \mathfrak{su}(1, n) \to \mathcal{P}(D_n)$,*

$$u \mapsto \mathcal{J}_{D_n}(u) := \big(\Psi_0 \circ \mathcal{J}\big)(u) = \frac{\mathrm{i}}{2} h_{\mu\nu} u^\mu_\rho \mathrm{f}_{E_\rho, E_\nu} \tag{5.2.2}$$

is a classical equivariant moment map (with respect to the Poisson tensor π_d on D_n) and $\Psi_\hbar \circ \frac{1}{\mathrm{i}\hbar}\mathcal{J} = \frac{1}{\mathrm{i}\hbar}\mathcal{J}_{D_n}$ an equivariant quantum moment map (with respect to $\star_\hbar$) for all $\hbar \in H$.

PROOF: This follows directly from Ψ_0 and $\Psi_\hbar$ being $\mathrm{U}(1, n)$-equivariant and the algebraic version of the reduction procedure, i.e. that Ψ_0 is a morphism of Poisson-*-algebras, or the construction of $\star_\hbar$ such that $\Psi_\hbar$ becomes a morphism of *-algebras, respectively. □

It would of course be a nice property of the deformed algebra if we could exponentiate the inner action of the $\mathfrak{su}(1, n)$-algebra to an inner action of the $\mathrm{SU}(1, n)$-group. So note that the image of $\mathcal{J}_{D_n}$ is in $\mathcal{P}(D_n)^{(1)}$. However, from Lemma 5.1.16 one cannot deduce that the $\star_\hbar$-exponential series of all elements $a \in \mathcal{P}(D_n)^{(1)}$ converges. In fact:

Example 5.2.11 *[52, Expl. 4.17] Let $n = 1$, $\hbar = 1/2$ and $a = \mathrm{f}_{E_0, E_1} = \mathrm{f}_{\mathrm{red};0,1}$, then the m-th $\star_\hbar$-power of a is*

$$a^{\star_\hbar m} = m!\, \mathrm{f}_{mE_0, mE_1} = m!\, \mathrm{f}_{\mathrm{red};0,m}\,, \tag{5.2.3}$$

and so $\lim_{M\to\infty}\sum_{m=0}^{M} a^{\star_\hbar m}/m! = \lim_{M\to\infty}\sum_{m=0}^{M} \mathrm{f}_{\mathrm{red};0,m}$ *does not converge in any topology on* $\mathcal{P}(D_n)$ *that makes the evaluation functionals* $\delta_{[r]}$ *at all* $[r] \in D_n$ *continuous, as this series does not converge in the point* $[r] \in D_n$ *with* $\varphi^{\mathrm{std}}([r]) = w^1([r]) = 1/\sqrt{2}$*, where* $\mathrm{f}_{\mathrm{red};0,m}([r]) = 2^{m/2}$.

Note that this also rules out the existence of any locally *multiplicatively* convex topology on $\mathcal{P}(D_n)$ that makes all these evaluation functionals continuous: the example shows that there is no entire calculus which for a locally multiplicatively convex algebra would exist.

Nevertheless, by Theorem 5.2.9, all elements in the image of $\mathscr{J}_{D_n}$ are essentially self-adjoint in every continuous *-representation for all $\hbar > 0$. Moreover, Nelson's theorem even allows to exponentiate this inner Lie algebra action to an inner Lie group action in such representations:

Theorem 5.2.12 *[52, Thm. 4.18] Fix* $\hbar > 0$ *and let* $(\mathcal{D}, \pi)$ *be a continuous* **-representation of* $\big(\mathcal{A}(D_n), \star_\hbar, \cdot^*\big)$ *and* $\mathfrak{H}$ *the completion of* $\mathcal{D}$*. Then there exists a unique unitary representation* $\boldsymbol{U}\colon \mathrm{SU}(1,n) \to \mathfrak{U}(\mathfrak{H})$ *such that* $\pi(\mathscr{J}_{D_n}(u))^{\mathrm{cl}} = \mathrm{d}\,\boldsymbol{U}(u)^{\mathrm{cl}}$ *holds for all* $u \in \mathfrak{su}(1,n)$*, where* $\cdot^{\mathrm{cl}}$ *denotes the closure of an operator on* $\mathfrak{H}$ *and* $\mathrm{d}\,\boldsymbol{U}(u)$ *the derivation of the representation* $\boldsymbol{U}$ *at the neutral element in direction* u*, i.e.* $\mathrm{d}\,\boldsymbol{U}(u)$ *is the operator in* $\mathfrak{H}$ *whose domain is* $\mathscr{C}^\infty(\boldsymbol{U})$*, the set of all vectors* $\phi \in \mathfrak{H}$ *for which the map* $\mathrm{SU}(1,n) \ni g \mapsto \langle\, \psi \mid \boldsymbol{U}(g)(\phi)\,\rangle_{\mathfrak{H}} \in \mathbb{C}$ *is smooth for all* $\psi \in \mathfrak{H}$*, and is defined as*

$$\mathrm{d}\,\boldsymbol{U}(u)(\phi) \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{U}\big(\exp(tu)\big)(\phi) \tag{5.2.4}$$

for all $\phi \in \mathscr{C}^\infty(\boldsymbol{U})$.

PROOF: By [57, Thm. 5] (see also [74, Thm. 10.5.6]) we only have to show that the Nelson Laplacian $\Delta \coloneqq \sum \frac{1}{\hbar^2}\mathscr{J}_{D_n}(u_i)^2$, with u_i running over a basis of $\mathfrak{su}(1,n)$, is represented by an essentially self-adjoint operator. As the image of $\mathscr{J}$ is in $\mathcal{P}(D_n)^{(1)}$ and Hermitian, it follows that $\Delta \in \mathcal{P}(D_n)^{(2)}$, and as Δ is clearly bounded from below, one can apply Theorem 5.2.9. $\square$

Chapter 6

Conclusion and Outlook

We have discussed how a physical system from classical or quantum mechanics can usually be desribed by a Poisson *-algebra describing its observabless and a cone of algebraically positive linear functionals describing its states, and how the classical and quantum system are related. However, the *-algebras occuring in many models are not very well-behaved: Proposition 4.0.1 shows that it is not unusual that one has to deal with not locally multiplicatively convex *-algebras, which are not yet very well understood, e.g. from the point of view of spectral theory.

This thesis tries to present a way how such algebras can be treated, the essential observation is that it is still possible for such a non-lmc *-algebra to be generated by Hermitian elements which are essentially self-adjoint in all representations, which can be guaranteed by proving an estimate on the growth of powers of algebra elements like in Lemma 4.2.28.

With respect to the example in Chapter 4 it has already been known that one can construct a reasonable (but non-lmc) locally convex topology that makes the multiplication continuous, and in which some elements can be exponentiated, see [83]. However, this result was not yet good enough for many applications, as e.g. the exponential of the Hamiltonian of the harmonic oscillator would not converge. This thesis improves on this result especially by showing that significantly more elements (including e.g. the Hamiltonian of the harmonic oscillator) are essentially self-adjoint in all continuous representations, and thus can at least be exponentiated in all such representations.

While this first example is not restricted to systems with finitely many degrees of freedom, it still has a rather trivial geometry. However, the second example in Chapter 5 shows that this approach is also viable for less trivial geometries. Again, in this example, a reasonable topology making the product continuous has already been known from [7], but the crucial estimate on the growth of powers of algebra elements like in Lemma 5.1.16 has been missing. The characterization of the classical limit on the Poincaré disc D_n as a space of real-analytic functions that have an extension to holomorphic functions on a larger complex manifold $\hat{D}_n$ from 5.1.29 could be rather a mathematical curiosity than a fundamental property of e.g. convergent star products on Kähler manifolds, but it might be very useful for the comparison of D_n with $\mathbb{CP}^n$ by Wick transformation in a future project: one simply restricts the extended holomorphic functions to a (smooth) embedding of $\mathbb{CP}^n$ in $\hat{D}_n$.

The discussion of characters and pure states of *-algebras in Section 3.5 demonstrates that it is also possible to prove non-trivial theorems about certain types of *-algebras not by assuming a well-behaved topology on them, but by assuming that they are endowed with a cone of (for whatever reason) "interesting" algebraically positive linear functionals fulfilling certain growth conditions on powers of algebra elements like in Definition 3.4.17. As was demonstrated by the two examples in Chapters 4 and 5, these two approaches to understanding *-algebras fit together quite nicely: A locally convex topology can be used to single out the well-behaved algebraically positive linear functionals.

For future projects, the results of Chapters 4 and 5 give some motivation to examine the concept of abstract O^*-algebras that where described in Chapter 3 in more detail, as this seems to be a sufficiently general setting for dealing with all the *-algebras arising in models of classical and quantum mechanics, but still allows to derive powerful results. The most important open question from the point of view of physics is, of course, whether a spectral theorem for abstract O^*-algebras can be formulated. Theorem 3.6.2 already gives a strong hint, but it is not yet clear how exactly this can be applied to e.g. the examples in Chapters 4 and 5. Moreover, it might be interesting to study reduction in terms of abstract O^*-algebras in order to prevent effects like the one observed in Section 5.2.1, that the reduced observable algebra admits states that where not yet present on the original one.

With respect of concrete applications of the ideas developed here, one might consider revisiting the topology on the universal enveloping algebra of a Lie algebra that was constructed in [34]: The question whether this topology fulfils estimates similar to those in lemma 4.2.28 seems to be still open. If this holds it would guarantee that Nelsons theorem can be applied in all its continuous representation.

Another class of *-algebras one should look at are those arising from deformation quantization of cotangent bundles: This is certainly one of the most important examples from the point of view of physics, and in the formal sense they are already rather well-understood. One should now continue to examine whether the same techniques used in Chapters 4 and 5 also apply there.

Appendix A

Appendix

The following appendix gives an overview over most of the mathematical results the thesis builds upon: The theory of locally convex spaces is indispensable for the treatment of real or complex vector spaces of infinite dimension. A short overview over ordered (real) vector spaces will then, on the one hand, just be necessary for understanding the order structure on *-algebras. On the other, Freudenthal's spectral theorem for a very special class of ordered vector spaces also provides the basis for the spectral theorems e.g. for operators on Hilbert spaces. The next section deals with the concepts of hull operators and Galois connections, which occur in seemingly all branches of mathematics. With respect to this thesis, they will be especially helpful for understanding how an order on a vector space determines an order on its dual and vice versa. The largest section of the appendix deals with unbounded operators on a Hilbert space and tries to give a clear and – for the needs of this thesis – complete introduction into the subject and its relation to O^*-algebras. Finally, the thesis closes with the very basic definitions of category theory.

A.1 Locally Convex Spaces

When doing analysis on real or complex vector spaces of infinite dimension, the theory of locally convex spaces is usually a sufficiently general setting. Locally convex spaces are vector spaces which are also toplogical spaces, and whose topology has some compatibility with the algebraic structure: Besides the rather obvious requirement that the operations of addition and scalar multiplication should be continuous, one also requires that the topology can be described by a system of seminorms. This section will provide the basic notions and results needed in the rest of the thesis. A more detailed introduction can be found in the standard text books on the subject, e.g. the classic [72].

A.1.1 Topological Spaces

Some basics about set theoretic topology will be necessary in the following. A more detailed introduction can be found e.g. in [81]. Recall that a topological space is a set X endowed with a *topology* $\tau \subseteq \mathscr{P}(X)$, with $\mathscr{P}(X)$ denoting the set of all subsets of X, whose elements are called the *open sets* and which has to fulfil the conditions that X itself and $\emptyset$ are open, and that finite intersections as well as arbitrary unions of open sets are again open. The *closed sets* are the complements in X of the open sets. Topological spaces yield a general approach towards the concepts of continuous maps and limits of nets (e.g. sequences):

A map $\Phi\colon X \to Y$ between two topological spaces is called *continuous* if $\Phi^{-1}(V)$ is open in X for every open set V in Y (or equivalently, if $\Phi^{-1}(V)$ is closed in X for every closed set V in Y). Moreover, if X is a topological space and $x \in X$, then a subset $U \subseteq X$ is called a *neighbourhood* of x if there exists an open subset $U' \subseteq X$ such that $x \in U'$ and $U' \subseteq U$ hold. The knowledge of all neighbourhoods of all points of a topological space is equivalent to the knowledge of its topology, because a subset U of a topological space X is open if and only if it is a neighbourhood of all its points $x \in U$. This also allows to talk about continuity of maps at a single point: If $\Phi\colon X \to Y$ is a map between two topological spaces and $x \in X$, then Φ is called *continuous at* x if $\Phi^{-1}(V)$ is a neighbourhood of x for every neighbourhood V of $\Phi(x)$. It is then straightforward to show that a function between topological spaces is continuous if and only if it is continuous at every point.

In order to define limits of nets, recall that an *upwards directed set* is a non-empty set I with a partial order $\preccurlyeq$ (i.e. reflexive, transitive and antisymmetric relation) and the property that for all $i, i' \in I$ there exists a $j \in I$ such that $i \preccurlyeq j$ and $i' \preccurlyeq j$ hold. A *net* in a set X is then a map $x\colon I \to X$, usually denoted as $x \mapsto x_i$ or $(x_i)_{i\in I}$, from an upwards directed set I to X. So sequences are just nets over the upwards directed set $\mathbb{N}$ or $\mathbb{N}_0$. If X is a topological space, then a net $(x_i)_{i\in I}$ is said to *converge* against a limit $\hat{x} \in X$, if for every neighbourhood U of $\hat{x}$ there exists an $i \in I$ such that $x_j \in U$ for all $j \in I$ with $i \preccurlyeq j$. Note that it can of course happen that the limit $\hat{x}$ does not exist, but also that it need not be unique if it exists (for instance, cosider a topological space X whose open sets are only $\emptyset$ and X itself, then every net in X converges against every point in X). Because of this, one defines a topological space X to be *Hausdorff* if for all distinct two points $x_1, x_2 \in X$ there exist two non-intersecting neighbourhoods U_1, U_2 of x_1 and x_2, respectively. One can check that this is precisely equivalent to demanding that all limits of nets in X are unique (if they exist). This (as many other arguments concerning nets in topological spaces) makes use of the fact that the set of all neighbourhoods of a given point in a topological space is an upwards directed set with respect to the opposite order of inclusion, i.e. $U \preccurlyeq U'$ if and only if $U \supseteq U'$ for two neighbourhoods U and U' of the same point $\hat{x}$. So assigning to every neighbourhood U of $\hat{x}$ a point $x \in U$ yields a net that converges against $\hat{x}$. In the case of Hausdorff topological spaces one writes $\lim_{i\to\infty} x_i \coloneqq \hat{x}$ for the limit $\hat{x}$ of a net $(x_i)_{i\in I}$ over an upwards directed set I (if the limit exists).

Convergent nets give a very helpful characterization of closed sets and of continuous functions: One can check that a subset $U \subseteq X$ of a toplogical space X is closed if and only if for every every net $(x_i)_{i\in I}$ in U which converges against some $\hat{x} \in X$ it follows that $\hat{x} \in U$. Similarly, a map $\Phi\colon X \to Y$ between two topological spaces is continuous at a point $\hat{x} \in X$ if and only if for every net $(x_i)_{i\in I}$ in X that converges against $\hat{x}$, the net $\big(\Phi(x_i)\big)_{i\in I}$ in Y converges against $\Phi(\hat{x})$. Now assume that a topological space X has the property that for every $x \in X$ there exists a sequence $\big(U_{x,n}\big)_{n\in\mathbb{N}}$ of neighbourhoods of x, such that $U_{x,n+1} \subseteq U_{x,n}$ for all $n \in N$ and such that for every neighbourhood U' of x there exists an $n \in \mathbb{N}$ with $U_{x,n} \subseteq U'$. Such a topological space is called *first countable*. In this case, it is sufficient to consider only convergent sequences when testing closedness of a subset or continuity of a function from X into another (not necessarily second countable) topological space, by using that, for every fixed $\hat{x} \in X$, assigning to every $n \in \mathbb{N}$ an element $x_n \in U_{n,\hat{x}}$ yields a sequence that converges against $\hat{x}$.

A.1.2 Basic Results on Locally Convex Spaces

Definition A.1.1 *Let V be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $S \subseteq V$.*

- *S is called* absorbing *if for all $v \in V$ there exists a $\lambda \in \left]0, \infty\right[$ such that $\lambda v \in S$.*
- *S is called* balanced *if $\lambda s \in S$ for all $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ and all $s \in S$.*
- *S is called* convex *if $\lambda s + (1 - \lambda)t \in S$ for all $s, t \in S$ and all $\lambda \in [0, 1]$.*

Proposition A.1.2 *Let V be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $\mathcal{N} \subseteq \mathscr{P}(V)$ a non-empty set of subsets of X with the following properties:*

- *$0 \in U$ for all $U \in \mathcal{N}$.*
- *$\mathcal{N}$ is a* filter, *i.e. if $U, U' \in \mathcal{N}$ then every $V \subseteq X$ fulfilling $U \cap U' \subseteq V$ is an element of $\mathcal{N}$.*
- *$\mathcal{N}$ has a filter-base consisting of absorbing, balanced and convex sets, i.e. for every $U \in \mathcal{N}$ there exists an absorbing, balanced and convex set $C \in \mathcal{N}$ with $C \subseteq U$.*
- *If $U \in \mathcal{N}$, then $\lambda U := \{ \lambda u \mid u \in U \} \in \mathcal{N}$ for all $\lambda \in \left]0, \infty\right[$.*

Then there exists a unique topology on V whose systems of neighbourhoods are determined by $\mathcal{N}$ as follows: If $v \in V$ and if $U \subseteq V$ with $v \in U$, then U is a neighbourhood of v if and only if $U - v := \{ u - v \mid u \in U \} \in \mathcal{N}$. The addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ are continuous in this topology. Moreover, let $\mathcal{P}$ be the set of all continuous seminorms on V, then $\mathcal{P}$ is closed under pointwise addition of seminorms and their multiplication with non-negative scalars. In addition, for every $v \in V$ and every neighbourhood $U \subseteq V$ of v there exists a $\|\cdot\| \in \mathcal{P}$ such that $\big\{ v \in V \bigm| \|v - u\| \leq 1 \big\} \subseteq U$, i.e. the set of continuous seminorms determines the system of neighbourhoods, hence the topology, of V.

PROOF: A detailed proof can be found in [72, Thms. 1.35 – 1.37]. Roughly speaking, one constructs a topology on V whose open sets are those $U \subseteq V$ for which $U - u \in \mathcal{N}$ for all $u \in U$ and checks that this is indeed a topology on V and that it has the correct neighbourhoods and makes addition and scalar multiplication continuous. It thus has to be uniquely determined because every other topology would have different neighbourhoods. As addition $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication with a fixed scalar $\mathbb{F} \to \mathbb{F}$ are continuous, the set $\mathcal{P}$ of continuous seminorms on V is closed under pointwise addition and scalar multiplication with non-negative scalars. Then one constructs for every absorbing, balanced and convex $C \in \mathcal{N}$ a function $\|\cdot\|_C \colon V \to \left[0, \infty\right[$,

$$v \mapsto \|v\|_C := \inf\big\{ r \in \left]0, \infty\right[\bigm| v/r \in C \big\}$$

(the Minkowski functional of C) and checks that this is a continuous seminorm on V. Given $v \in V$ and a neighbourhood $U \subseteq V$ of v, then there exists an absorbing, balanced and convex $C \in \mathcal{N}$ with $v + C \subseteq U$ and then $\big\{ v \in V \bigm| 2\|v - u\|_C \leq 1 \big\} \subseteq U$. □

Note that the filter $\mathcal{N}$ above is the filter of neighbourhoods of 0.

Definition A.1.3 *A* locally convex topology *is a toplogy on a real or complex vector space like in the previous Proposition A.1.2, and a* locally convex space *is a real or complex vector space with a locally convex topology.*

Note that the set of continuous seminorms $\mathcal{P}$ on such a locally convex space is upwards directed as $\|\cdot\|_\alpha + \|\cdot\|_\beta \geq \|\cdot\|_\alpha$ and $\|\cdot\|_\alpha + \|\cdot\|_\beta \geq \|\cdot\|_\beta$ for all $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}$. Moreover, a seminorm $\|\cdot\|$ on a locally convex space V is continuous if and only if its 1-ball $\{\, v \in V \mid \|v\| \leq 1 \,\}$ is a neighbourhood of 0.

If V is a real or complex vector space and $\mathcal{P}'$ a non-empty set of seminorms on it, then one can construct a filter of 0-neighbourhoods $\mathcal{N}$ fulfilling the conditions of Proposition A.1.2 as $U \in \mathcal{N}$ if and only if U is a subset of V for which there exist $N \in \mathbb{N}$ and $\|\cdot\|_1, \dots, \|\cdot\|_N \in \mathcal{P}'$ as well as $\lambda_1, \dots, \lambda_N \in [0, \infty[$ with the property that $\big\{\, v \in V \;\big|\; \sum_{n=1}^N \lambda_n \|v\|_n \leq 1 \,\big\} \subseteq U$. The resulting locally convex topology τ then is the weakest one (i.e. the smallest one with respect to the order by inclusion) that makes all the seminorms in $\mathcal{P}'$ continuous and a seminorm $\|\cdot\|'$ on V is continuous with respect to τ if and only if there exist $N \in \mathbb{N}$ and $\|\cdot\|_1, \dots, \|\cdot\|_N \in \mathcal{P}'$ as well as $\lambda_1, \dots, \lambda_N \in [0, \infty[$ such that $\|\cdot\|' \leq \sum_{n=1}^N \lambda_n \|\cdot\|_n$ holds.

Proposition A.1.4 *Let V be a locally convex space, $\hat{v} \in V$ and $(v_i)_{i\in I}$ a net in V over an upwards-directed set I. Then $(v_i)_{i\in I}$ converges against $\hat{v}$ if and only if for every continuous seminorm $\|\cdot\|$ on V there exists an $i \in I$ such that $\|\hat{v} - v_j\| \leq 1$ for all $j \in I$ with $i \preccurlyeq j$.*

PROOF: This is an immediate consequence of the description of neighbourhoods of locally convex spaces by continuous seminorms. □

Proposition A.1.5 *Let $N \in \mathbb{N}$ and locally convex spaces $V_1, \dots, V_N, W$ (over $\mathbb{R}$ or $\mathbb{C}$) as well as a multi-$\mathbb{R}$-linear map $\Phi\colon V_1 \times \dots \times V_N \to W$ be given. Then Φ is continuous if and only if for every continuous seminorm $\|\cdot\|'$ on W there exist continuous seminorms $\|\cdot\|_1, \dots, \|\cdot\|_N$ on $V_1, \dots, V_N$ such that*

$$\|\Phi(v_1, \dots, v_N)\|' \leq \prod_{n=1}^N \|v_n\|_n \tag{A.1.1}$$

holds for all $v_1 \in V_1, \dots, v_N \in V_N$.

PROOF: This uses that a multilinear map is continuous if and only if it is continuous at 0 and the description of neighbourhoods of locally convex spaces by continuous seminorms. □

Note that this covers $\mathbb{C}$-linear, $\mathbb{C}$-antilinear and mixed $\mathbb{C}$-linear, $\mathbb{C}$-antilinear maps over complex spaces, which are of course all $\mathbb{R}$-linear! With respect to the Hausdorff property one has:

Proposition A.1.6 *Let V be a locally convex space, then V is Hausdorff if and only if for every $v \in V\backslash\{0\}$ there exists a continuous seminorm $\|\cdot\|$ on V with $\|v\| \neq 0$.*

PROOF: This can easily be checked using that the continuous seminorms determine the neighbourhoods of a locally convex space. □

A.1.3 Completion of Locally Convex Spaces

Topological spaces in general are not suitable for defining Cauchy sequences or Cauchy nets, because, heuristically, one would need a notion of distance between two arbitrary points, while the system of neighbourhoods of a given point only gives a notion of distance from this one point. So usually, one needs *uniform spaces* to be able to construct completions. However, on vector spaces one can define a "distance" of two arbitrary vectors v and w as the "distance" of $v-w$ from 0, which is given by the filter of 0-neighbourhoods, or equivalently for locally convex spaces, the continuous seminorms. Because of this, the algebraic structure of a vector space allows to turn locally convex spaces into uniform spaces in a natural way.

While the theory of uniform spaces has been treated in some textbooks, e.g. [81], the important result about completion of uniform spaces and the possibility to continuously extend certain functions to the completion is oftentimes formulated in a too restrictive way, i.e. it is usually shown that uniformly continuous functions extend to uniformly continuous functions on the completion. A more general treatment is given in [78] using Cauchy continuous functions (also called C-regular functions). See Remark 10 there for a discussion why this is necessary for the extension of multilinear maps between locally convex spaces.

Definition A.1.7 *Let V be a locally convex space and $(v_i)_{i\in I}$ a net in V over an upwards directed set I. Then $(v_i)_{i\in I}$ is called a* Cauchy net *if for every continuous seminorm $\|\cdot\|$ on V there exists an $i \in I$ such that $\|v_j - v_{j'}\| \leq 1$ holds for all $j, j' \in I$ with $i \preccurlyeq j$ and $i \preccurlyeq j'$. Similarly, a* Cauchy sequence *is a Cauchy net over the upwards directed set $\mathbb{N}$.*

As usual it is easy to see that every convergent net is a Cauchy net, conversely, one defines:

Definition A.1.8 *Let V be a locally convex space, then V is said to be* complete *if for every Cauchy net $(v_i)_{i\in I}$ there exists a $\hat{v} \in V$ such that $(v_i)_{i\in I}$ converges against $\hat{v}$.*

For every locally convex space one can construct a completion that allows the continuous extension of many functions:

Definition A.1.9 *Let $N \in \mathbb{N}$ and $V_1, \dots, V_N, W$ be locally convex spaces. Then $\Phi\colon V_1 \times \dots \times V_N \to W$ is called* Cauchy continuous, *if for all Cauchy nets $(v_{1,i_1})_{i_1\in I_1}, \dots, (v_{N,i_N})_{i_N\in I_N}$ over upwards directed nets $I_1, \dots, I_N$ the net $\big(\Phi(v_{1,i_1}, \dots, v_{N,i_N})\big)_{(i_1,\dots,i_N)\in I_1\times\dots\times I_N}$ in W over the upwards directed set $I_1 \times \dots \times I_N$ with the elementwise comparison is again a Cauchy net.*

Note that Cauchy continuous functions are automatically continuous and conversely, that every continuous function from a complete locally convex space V into a locally convex space W is Cauchy continuous because every Cauchy net in V is convergent, thus mapped to a convergent net in W, which is a Cauchy net.

Proposition A.1.10 *Let V be a locally convex space, then there exists a tuple $(V^{\mathrm{cpl}}, \iota)$, called a* completion *of V, consisting of a complete Hausdorff locally convex space V^{cpl} and a continuous and Cauchy continuous linear map $\iota\colon V \to V^{\mathrm{cpl}}$, which have the universal property that for every complete Hausdorff locally convex space W and every Cauchy continuous linear function $\Phi\colon V \to W$ there exists a unique continuous linear function $\Phi^{\mathrm{cpl}}\colon V^{\mathrm{cpl}} \to W$ such that $\Phi^{\mathrm{cpl}} \circ \iota = \Phi$.*

Furthermore, this tuple $(V^{\mathrm{cpl}}, \iota)$ *is unique in so far as for every other tuple* $(V'^{,\mathrm{cpl}}, \iota')$ *that fulfils this universal property there exists a unique pair of mutually inverse continuous isomorphisms* $\Psi\colon V^{\mathrm{cpl}} \to V'^{,\mathrm{cpl}}$ *and* $\Psi'\colon V'^{,\mathrm{cpl}} \to V^{\mathrm{cpl}}$ *fulfilling* $\Psi \circ \iota = \iota'$ *and* $\Psi' \circ \iota' = \iota$.

Finally, if $N \in \mathbb{N}$ *and if* $V_1, \dots, V_N$ *are locally convex spaces with completions* $(V_1, \iota_1), \dots, (V_N, \iota_N)$, *and* W *is a complete Hausdorff locally convex space, and if* $\Phi\colon V_1, \dots, V_N \to W$ *is Cauchy continuous, then there exists a unique continuous map* $\Phi^{\mathrm{cpl}}\colon V_1^{\mathrm{cpl}} \times \dots \times V_N^{\mathrm{cpl}} \to W$ *fulfilling* $\Phi^{\mathrm{cpl}} \circ (\iota_1 \times \dots \times \iota_N) = \Phi$.

PROOF: A construction of the completion of general uniform spaces is given in [81, Chap. 12]. The equivalence up to unique isomorphism of such constructions of a completion is a standard argument and the extension of Cauchy continuous functions to the completion is shown in [78]. □

Note that it is not required that the locally convex space V is Hausdorff, but its completion will be. This means that the map $\iota\colon V \to V^{\mathrm{cpl}}$ is not injective for non-Hausdorff V but has the closure of $\{0\}$ as its kernel. With respect to multilinear functions one can show (see essentially [78, Thm. 9]):

Proposition A.1.11 *Let* $N \in \mathbb{N}$ *and* $V_1, \dots, V_N, W$ *be locally convex spaces. Then an* $\mathbb{R}$*-multilinear and continuous map* $\Phi\colon V_1 \times \dots \times V_N \to W$ *is also Cauchy continuous, and thus extends to the completions of* $V_1, \dots, V_N$ *if* W *is complete.*

This is especially helpful for locally convex *-algebras: If the antilinear involution and the multiplication are continuous, then they extend to the completion.

A.1.4 Duality of Locally Convex Spaces

Definition A.1.12 *Let* V *and* W *be two vector spaces over the same field* $\mathbb{F}$ *of real or complex numbers, and let* $\langle \,\cdot\,, \,\cdot\, \rangle\colon V \times W \to \mathbb{F}$ *be a bilinear map. Then the* weak topology *defined by* V *on* W *is the locally convex topology on* W *that is defined by the seminorms* $\|\cdot\|_{v,\mathrm{wk}}\colon W \to [0, \infty[$,

$$w \mapsto \|w\|_{v,\mathrm{wk}} \coloneqq |\langle v, w \rangle| , \tag{A.1.2}$$

for all $v \in V$. *Conversely, the* weak topology *defined by* W *on* V *is the locally convex topology on* V *that is defined by the seminorms* $\|\cdot\|_{w,\mathrm{wk}}\colon V \to [0, \infty[$,

$$v \mapsto \|v\|_{w,\mathrm{wk}} \coloneqq |\langle v, w \rangle| , \tag{A.1.3}$$

for all $w \in W$.

One special example is of course given by the vector space V and its algebraic dual V^*, the vector space of all linear functions from V to the scalars, together with the dual pairing $\langle \,\cdot\,, \,\cdot\, \rangle\colon V^* \times V \to \mathbb{F}$ that evaluates a linear functional from V^* on a vector from V. The weak topology defined by V on V^* is called the *weak-*-topology.*

Note that the above definition does allow $\langle \,\cdot\,, \,\cdot\, \rangle$ to be degenerate, i.e. it may happen that $\langle v, w \rangle = 0$ holds for one $v \in V \backslash \{0\}$ and all $w \in W$ or vice versa. In this case, the weak topology defined on V by W is not Hausdorff.

Such two vector spaces V, W with a bilinear map $\langle \cdot\,, \cdot \rangle$ to the scalars give rise to some geometric questions that will be relevant later on: Under which conditions can two sets in V be separated by a linear functional of the form $\langle \cdot\,, w \rangle$ with $w \in W$? It will be sufficient to restrict to $\mathbb{R}$-bilinear maps $\langle \cdot\,, \cdot \rangle$ to the real numbers. The following well-known separation lemma can be seen as a consequence of the Hahn-Banach theorem or be proven directly (as it only refers to a weakly closed subset in this version):

Lemma A.1.13 *Let V and W be two real vector spaces as well as $S \subseteq V$ convex and closed in the weak topology defined by W on V. Moreover, let $\hat{v} \in V$ be a vector outside of S, then there exists a $\hat{w} \in W$ such that $\langle s\,, \hat{w} \rangle \geq \langle \hat{v}\,, \hat{w} \rangle + 1$ holds for all $s \in S$.*

PROOF: As $\hat{v} \in V \backslash S$ and as S is weakly closed, there exist $N \in \mathbb{N}$ as well as $w_1, \dots, w_N \in W$ such that $\big\{ v \in V \bigm| \sum_{n=1}^N |\langle v - \hat{v}\,, w_n \rangle| \leq 1 \big\} \cap S = \emptyset$. Then $\Phi \colon V \to \mathbb{R}^N$, $v \mapsto \Phi(v) \coloneqq \big(\langle v\,, w_1 \rangle, \dots, \langle v\,, w_N \rangle\big)$ is a linear map, $\Phi(S) \subseteq \mathbb{R}^N$ convex and $\Phi(\hat{v})$ is not in the closure of $\Phi(S)$. As $\mathbb{R}^N$ is locally compact, there exists a $\Lambda = (\lambda_1, \dots, \lambda_N)$ in the closure of $\Phi(S)$ with minimal Euclidean distance to $\Phi(\hat{v})$. Let $d^2 \coloneqq \sum_{n=1}^N \big(\lambda_n - \langle \hat{v}\,, w_n \rangle\big)^2 > 0$ be the square of this distance, then it is not hard to check that $\sum_{n=1}^N \langle s - \hat{v}\,, w_n \rangle \big(\lambda_n - \langle \hat{v}\,, w_n \rangle\big) \geq d^2$ for all $s \in S$. So $\hat{w} \coloneqq \sum_{n=1}^N w_n \big(\lambda_n - \langle \hat{v}\,, w_n \rangle\big)/d^2$ has the desired properties. □

A.1.5 Hahn-Banach Theorem

One important property of locally convex spaces, besides the convenience of being able to describe the topology by means of systems of seminorms, is that one can prove the existence of many continuous linear functionals using the Hahn-Banach theorem, see e.g. [72, Thm. 3.3]:

Definition A.1.14 *Let V be a locally convex space, then denote by V' the linear subspace of V^* consisting of all continuous linear functionals on V.*

Theorem A.1.15 (Hahn-Banach) *Let V be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, $\|\cdot\|$ a seminorm on V, $U \subseteq V$ a linear subspace and $\omega|_U \colon U \to \mathbb{F}$ a linear functional such that $|\langle \omega\,, u \rangle| \leq \|u\|$ holds for all $u \in U$. Then there exists a linear functional $\omega \colon V \to \mathbb{F}$ such that $\langle \omega\,, u \rangle = \langle \omega|_U\,, u \rangle$ for all $u \in U$ and $|\langle \omega\,, v \rangle| \leq \|v\|$ for all $v \in V$.*

PROOF: Using Zorn's lemma one shows that under all extensions of $\omega|_U$ that remain dominated by $\|\cdot\|$ there exist maximal (with respect to their domain) elements, and proves that these maximal elements are defined on whole V. □

Especially if V is a locally convex space and $\|\cdot\|$ a continuous seminorm, then such an extended linear functional ω is of course continuous. This shows that every continuous seminorm $\|\cdot\|$ on a locally convex space V over a field $\mathbb{F}$ of real or complex numbers can be expressed as

$$\|v\| = \sup_{\omega \in T} |\langle \omega\,, v \rangle| \tag{A.1.4}$$

$$\text{with} \qquad T = \big\{ \omega \in V' \bigm| |\langle \omega\,, v \rangle| \leq \|v\| \text{ for all } v \in V \big\} \tag{A.1.5}$$

by extending for all $v \in V$ with $\|v\| = 1$ the linear map $\{ \lambda v \mid \lambda \in \mathbb{F} \} \ni \lambda v \mapsto \langle \omega_v , \lambda v \rangle := \lambda$. For Hausdorff locally convex spaces this also implies:

Corollary A.1.16 *Let V be a Hausdorff locally convex space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $\langle \cdot , \cdot \rangle \colon V' \times V \to \mathbb{F}$ the dual pairing, then the weak topology defined by V' on V is Hausdorff.*

Another consequence of the Hahn-Banach theorem is [72, Thm. 3.12]:

Proposition A.1.17 *Let V be a locally convex space and $C \subseteq V$ convex and closed. Then C is even closed in the weak topology defined by V' on V.*

This result is especially helpful with respect to Lemma A.1.13.

A.1.6 Special Locally Convex Spaces

There are some special types of locally convex spaces that deserve more attention:

Metrizable Locally Convex Spaces and Fréchet Spaces

Proposition A.1.18 *Let V be a locally convex space, then the topology on V can be described by a metric $d \colon V \times V \to [0, \infty[$ if and only if V is Hausdorff and first countable. In this case, the metric can even be chosen to be invariant, i.e. $d(u+v, u+w) = d(v, w)$ for all $u, v, w \in V$, to have absorbing, balanced and convex open balls around 0 and to describe not only the same topology, but also the same uniform structure, i.e. to lead to the same Cauchy sequences (and thus same completion) as the locally convex topology on V.*

Proof: As every metric space is Hausdorff and first countable, these conditions are clearly necessary. They are also sufficient as is shown in [72, Thm. 1.24] by explicit construction of a metric with all the mentioned properties. □

This leads to the usual definition of Fréchet spaces:

Definition A.1.19 *A* Fréchet space *is a complete Hausdorff first countable locally convex space.*

As Fréchet spaces are also complete metric spaces, they have some especially nice properties, many of which are a consequence of Baire's theorem. To mention just one, one can show [72, Thm. 2.17]:

Proposition A.1.20 *Let V and W be two Fréchet spaces and X a locally convex space, then a bilinear map $b \colon V \times W \to X$ is continuous if and only if it is separately continuous, i.e. if and only if the linear maps $b_v \colon W \to X$, $w \mapsto b_v(w) := b(v, w)$ and $b^w \colon V \to X$, $v \mapsto b^w(v) := b(v, w)$ are continuous for all $v \in V$, $w \in W$.*

Banach Spaces

Banach spaces do not play an important role in this thesis, their definition is just given for sake of completeness:

Definition A.1.21 *A* Banach space *is a real or complex vector space V together with a norm $\|\cdot\|$ on V such that V with the locally convex topology defined by $\|\cdot\|$ is complete.*

One nice property of Banach spaces is the well-known fact that the continuous linear maps between two fixed Banach spaces are again a Banach space:

Proposition A.1.22 *Let V and W be two Banach spaces and $\Phi\colon V \to W$ a linear map. Then Φ is continuous if and only if*

$$\|\Phi\| := \sup_{v\in V, \|v\|=1} \|\Phi(v)\| < \infty\,, \tag{A.1.6}$$

and this defines a norm on the vector space of all continuous linear maps from V to W under which this space is complete, hence a Banach space.

Hilbertisable and Hilbert Spaces

Hilbertisable locally convex spaces prove to be useful in Chapter 4, not only for technical reasons. This definition is less standard but can be found e.g. in [46].

Definition A.1.23 *A locally convex space V is called* hilbertisable *if for every continuous seminorm $\|\cdot\|$ on V there exists a continuous positive Hermitian form (see Section 1.2) $\langle\,\cdot\,|\,\cdot\,\rangle'$ on V such that $\|\cdot\| \le \|\cdot\|'$, where $\|\cdot\|'$ is the (continuous) Hilbert seminorm that is constructed out of $\langle\,\cdot\,|\,\cdot\,\rangle'$ in the usual way as $\|v\|' := \big(\langle\, v\,|\,v\,\rangle'\big)^{1/2}$ for all $v \in V$.*

As an example, if V and W are two vector spaces over the field $\mathbb{F}$ of real or complex numbers, and if $\langle\,\cdot\,,\,\cdot\,\rangle\colon V\times W \to \mathbb{F}$ is a bilinear form, then the weak topologies on V and W like in Definition A.1.12 are hilbertisable: Every weakly continuous seminorm on, say, V is dominated by a continuous seminorm of the form $V \ni v \mapsto \sum_{n=1}^N \|v\|_{w_n,\mathrm{wk}}$ with $N\in\mathbb{N}$ and $w_1,\dots,w_N \in W$, which itself is dominated by the seminorm induced by the continuous inner product

$$V^2 \ni (v,v') \mapsto \langle\, v\,|\,v'\,\rangle := N\sum_{n=1}^N \overline{\langle\, v\,,\,w_n\,\rangle}\langle\, v'\,,\,w_n\,\rangle\,, \tag{A.1.7}$$

because the Cauchy Schwarz inequality yields

$$\sum_{n=1}^N \|v\|_{w_n,\mathrm{wk}} \overset{\mathrm{CS}}{\le} \bigg(\sum_{n=1}^N 1\bigg)^{\frac12}\bigg(\sum_{n=1}^N \|v\|^2_{w_n,\mathrm{wk}}\bigg)^{\frac12} = \bigg(N\sum_{n=1}^N \overline{\langle\, v\,,\,w_n\,\rangle}\langle\, v\,,\,w_n\,\rangle\bigg)^{\frac12}\,.$$

This has already been used in the proof of Lemma A.1.13. However, the most well-known examples are of course the (pre-)Hilbert spaces, see also Section 1.2:

Definition A.1.24 *A (complex)* pre-Hilbert space *is a vector space $\mathcal{D}$ over the field $\mathbb{F}=\mathbb{C}$ endowed with an inner product $\langle\,\cdot\,|\,\cdot\,\rangle\colon \mathcal{D}\times\mathcal{D}\to\mathbb{C}$, i.e. a positive Hermitian form that is non-degenerate. A* Hilbert space *is a pre-Hilbert space $\mathfrak{H}$ such that $\mathfrak{H}$ is complete in the locally convex topology defined by the norm $\|\cdot\|$ that is constructed out of the inner product $\langle\,\cdot\,|\,\cdot\,\rangle$ like in the previous Definition A.1.23.*

There is also an analogous definition of Hilbert spaces over the field $\mathbb{F}=\mathbb{R}$. However, in this thesis only the complex ones are relevevant.

In a pre-Hilbert space $\mathcal{D}$ it is clear without the use of the Hahn-Banach theorem that the space $\mathcal{D}'$ of continuous linear functionals is not trivial, as every $\mathcal{D} \ni \phi \mapsto \langle \psi \,|\, \phi \rangle \in \mathbb{C}$ for all $\psi \in \mathcal{D}$ describes a non-trivial continuous linear functional. However, in general there exist more continuous linear functionals than just these. Nevertheless, for Hilbert spaces one can show [72, Thms. 12.4 and 12.5]:

Proposition A.1.25 *Let $\mathfrak{H}$ be a Hilbert space and $U \subseteq \mathfrak{H}$ a closed linear subspace, then $\mathfrak{H}$ decomposes as the direct sum $\mathfrak{H} = U \oplus V$ with $V = \{\, \psi \in \mathfrak{H} \mid \forall_{\phi \in U} \colon \langle \phi \,|\, \psi \rangle = 0 \,\}$. Especially if $U = \operatorname{kern} \omega$ for a continuous linear functional $\omega \in \mathfrak{H}'$, then $\mathfrak{H} = \big(\operatorname{kern} \omega\big) \oplus \{\, \lambda \omega^\sharp \mid \lambda \in \mathbb{C} \,\}$ with a unique $\omega^\sharp \in \mathfrak{H}$ fulfilling $\langle \omega^\sharp \,|\, \phi \rangle = \langle \omega \,,\, \phi \rangle$ for all $\phi \in \mathfrak{H}$.*

The resulting map $\cdot^\sharp \colon \mathfrak{H}' \to \mathfrak{H}$ then is an (anti-linear) isomorphism between $\mathfrak{H}$ and $\mathfrak{H}'$. This is the Fréchet-Riesz representation theorem.

A.2 Ordered Vector Spaces

Ordered and quasi-ordered vector spaces are very important throughout this thesis. Freudenthal's spectral theorem even shows that the key to a spectral theorem for abstract O^*-algebras is to understand, under which conditions their order on the Hermitian elements is well-behaved. The definition of ordered vector spaces is standard, but sometimes a generalization to quasi-ordered vector spaces is necessary:

Definition A.2.1 *A* quasi-ordered vector space *is a real vector space V together with a quasi-order (i.e. reflexive and transitive relation) $\lesssim$ fulfilling*

$$u + v \lesssim u + w \qquad \text{and} \qquad \lambda v \lesssim \lambda w \tag{A.2.1}$$

for all $u, v, w \in V$ with $v \lesssim w$ and all $\lambda \in [0, \infty[$. If V is a quasi-ordered vector space, then the set of its positive *elements is*

$$V^+ := \{\, v \in V \mid 0 \lesssim v \,\}. \tag{A.2.2}$$

An ordered vector space *is a quasi-ordered vector space whose order $\lesssim$ is even a partial order, i.e. is also antisymmetric. In this case we also write $\leq$ for $\lesssim$.*

The ordering on a quasi-ordered vector space is determined by its cone of positive elements:

Proposition A.2.2 *Let V be a quasi-ordered vector space, then V^+ is a convex cone in V, i.e. $0 \in V^+$ and $\lambda v + \mu w \in V^+$ for all $v, w \in V^+$ and all $\lambda, \mu \in [0, \infty[$. Conversely, if V is a real vector space and $C \subseteq V$ a convex cone, then the relation*

$$v \lesssim w \qquad :\Longleftrightarrow \qquad w - v \in C$$

is a quasi-order on V that turns V into a quasi-ordered vector space with cone of positive elements $V^+ = C$. Moreover, a quasi-ordered vector space is even an ordered vector space if and only if the only linear subspace of V contained in V^+ is $\{0\}$.

PROOF: This is all straightforward to show. $\square$

The usual notion of functions compatible with the structure of quasi-ordered vector spaces are the positive linear functions:

Definition A.2.3 *Let V and W be two quasi-ordered vector spaces and $L\colon V \to W$ a linear function, then L is called* positive *if $L(v) \in W^+$ for all $v \in V^+$. If L additionally fulfils $L(v) \notin W^+$ for all $v \in V \backslash V^+$, the L is called an* order embedding.

For a general quasi-ordered vector space V, the cone of positive elements V^+ could be as small as $\{0\}$, or as large as whole V. In the first case, $v \lesssim w$ holds only if $v = w$, in the second, $v \lesssim w$ is always true. Because of this, the most interesting examples are usually ordered vector spaces V which are generated by V^+.

On ordered vector spaces one can ask whether the supremum or infimum of a subset exists. This leads to the idea of Riesz spaces, see e.g. the textbook [54] for the following definitions and properties and a more in-depth treatment of the subject:

Definition A.2.4 *A* Riesz space *is an ordered vector space V in which the supremum* $\sup\{v, w\}$ *exist for all $v, w \in V$.*

Note that one can show by multiplication with -1 that the existence of all (finite, countable, all) suprema in an ordered vector space also implies the existence of all (finite, countable, all) infima, and that every Riesz space V is the linear hull of its cone V^+ of positive elements as $v = \sup\{v, 0\} + \inf\{v, 0\}$ for all $v \in V$.

A typical example of Riesz spaces is the vector space $\mathscr{C}(X, \mathbb{R})$ of all continuous real-valued functions on a topological space X, ordered by pointwise comparison, i.e. $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. In this case, the supremum of two $f, g \in \mathscr{C}(X, \mathbb{R})$ is the pointwise maximum $\big(\sup\{f, g\}\big)(x) = \max\{f(x), g(x)\}$ and the infimum the pointwise minimum $\big(\inf\{f, g\}\big)(x) = \min\{f(x), g(x)\}$.

In a especially well-behaved class of Riesz spaces, one can proof Freudenthal's spectral theorem:

Definition A.2.5 *Let V be a Riesz space. Then an element $\mathbb{1} \in V^+$ is called a* weak order unit *if $v = \sup_{n\in\mathbb{N}} \inf\{v, n\mathbb{1}\}$ holds for all $v \in V^+$. In a Riesz space V with weak order unit $\mathbb{1}$ one says that a $p \in V$ is a* component of the unit *if $\inf\{p, \mathbb{1} - p\} = 0$ holds, and defines the linear subspace of* simple elements *of V as the linear span of the components of the unit.*

In the above example of the Riesz space $\mathscr{C}(X, \mathbb{R})$ with the constant 1-function as weak order unit, one immediately sees that the components of the unit are precisely those $p \in \mathscr{C}(X, \mathbb{R})$, whose image is (at most) $\{0, 1\}$. In this case, there typically do not exist many such functions, for example, the only components of the unit in $\mathscr{C}(\mathbb{R}, \mathbb{R})$ are $\mathbb{1}$ and 0 as $\mathbb{R}$ is connected. However, for certain types of Riesz spaces one can prove the existence of many components of the unit. This is Freudenthal's spectral theorem [54, Thm. 40.3]:

Theorem A.2.6 *Let V be a Riesz space with weak order unit $\mathbb{1} \in V$. If V is Dedekind-σ-complete, i.e. if the supremum of every monotonely increasing and bounded sequence exists in V, then for every $v \in V^+$ there exists a monotonely increasing sequence $(s_n)_{n\in\mathbb{N}}$ of simple elements in V, whose supremum is v.*

A.3 Hull Operators and Galois Connections

Hull operators and Galois connections are concepts that occur in virtually every branch of mathematics, but are oftentimes not emphasized very much. While it might be easier to simply discuss those examples of Hull operators and Galois connections needed in the rest of this work without any surrounding theory, it will, in the long run, be advantageous to say some words about the general case. For more details and a more general formulation in terms of ordered sets rather than the concrete case of a power set $\mathscr{P}(X)$ of a set X, ordered by inclusion, see e.g. [12, Chap. 1.4 and 1.6]. The examples considered are standard.

A.3.1 Hull Operators and Hull Systems

Definition A.3.1 *Let X be a set, then a* hull operator on X *is a function* $\langle\!\langle \cdot \rangle\!\rangle \colon \mathscr{P}(X) \to \mathscr{P}(X)$ *on the set $\mathscr{P}(X)$ of all subsets of X, which fulfils for all $S \subseteq T \subseteq X$ the properties*

$$
\begin{aligned}
\langle\!\langle S \rangle\!\rangle &\subseteq \langle\!\langle T \rangle\!\rangle && \text{i.e. } \langle\!\langle \cdot \rangle\!\rangle \text{ is monotone,} \\
\langle\!\langle S \rangle\!\rangle &\supseteq S && \text{i.e. } \langle\!\langle \cdot \rangle\!\rangle \text{ is extensive,} \\
\langle\!\langle \langle\!\langle S \rangle\!\rangle \rangle\!\rangle &= \langle\!\langle S \rangle\!\rangle && \text{i.e. } \langle\!\langle \cdot \rangle\!\rangle \text{ is idempotent.}
\end{aligned}
$$

Recall that $\mathscr{P}(X)$ is, for every set X, a complete lattice with respect to $\subseteq$, i.e. the inclusion $\subseteq$ is a partial order on $\mathscr{P}(X)$ with respect to which all suprema and all infima in $\mathscr{P}(X)$ exist: The supremum of some $\mathscr{S} \subseteq \mathscr{P}(X)$ is simply the union $\bigcup \mathscr{S}$ and the infimum is the intersection $\bigcap \mathscr{S}$ (where the union and intersection over the empty set of sets are understood to be $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = X$).

It turns out that the image of a hull operator $\langle\!\langle \cdot \rangle\!\rangle$ is again a complete lattice, even though it is not necessarily a sublattice of $\mathscr{P}(X)$, because the suprema in the image of $\langle\!\langle \cdot \rangle\!\rangle$ and in $\mathscr{P}(X)$ need not coincide:

Proposition A.3.2 *Let X be a set and $\langle\!\langle \cdot \rangle\!\rangle$ a hull operator on X, then* $\operatorname{image}\langle\!\langle \cdot \rangle\!\rangle \subseteq \mathscr{P}(X)$*, the image of $\langle\!\langle \cdot \rangle\!\rangle$, together with the partial order $\subseteq$ on it, is a complete lattice. The suprema and infima are given by*

$$
\sup \mathscr{S} = \langle\!\langle \textstyle\bigcup \mathscr{S} \rangle\!\rangle \qquad \textit{and} \qquad \inf \mathscr{S} = \textstyle\bigcap \mathscr{S} \tag{A.3.1}
$$

for all $\mathscr{S} \subseteq \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$, and this especially means that $\bigcap \mathscr{S} \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$.

PROOF: The partial order $\subseteq$ on $\mathscr{P}(X)$ can always be restricted to an arbitrary subset of $\mathscr{P}(X)$, especially to $\operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$. Now let $\mathscr{S} \subseteq \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$ be given, then $\langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle \supseteq \bigcup \mathscr{S}$ is certainly an upper bound in $\operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$ of all sets $S \in \mathscr{S}$, and it is the least upper bound in $\operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$, because given another upper bound $\langle\!\langle T \rangle\!\rangle \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$ with $T \subseteq X$ of all $S \in \mathscr{S}$, then $\langle\!\langle T \rangle\!\rangle \supseteq \bigcup \mathscr{S}$ and thus $\langle\!\langle T \rangle\!\rangle = \langle\!\langle \langle\!\langle T \rangle\!\rangle \rangle\!\rangle \supseteq \langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle$. Moreover, $\bigcap \mathscr{S} = \langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$: The inclusion $\subseteq$ follows immediately from the definition of hull operators and for the converse inclusion $\supseteq$, note that $S \supseteq \bigcap \mathscr{S}$, hence $S = \langle\!\langle S \rangle\!\rangle \supseteq \langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle$, holds for all $S \in \mathscr{S}$ by using the properties of hull operators and that $S \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$. Consequently $\bigcap \mathscr{S} \supseteq \langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle$. This shows that $\bigcap \mathscr{S} \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$ and it is then clear that $\bigcap \mathscr{S}$ is not only the infimum of $\mathscr{S}$ in $\mathscr{P}(X)$, but also in $\operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$. □

Hull operators are compatible with arbitrary suprema when taken in the correct complete lattices:

Proposition A.3.3 *Let X be a set, $\langle\!\langle \cdot \rangle\!\rangle$ a hull operator on X and $\mathscr{S} \subseteq \mathscr{P}(X)$, then*

$$\langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle \subseteq \bigcap_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle = \inf_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle \tag{A.3.2}$$

$$\text{and} \qquad \langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle = \left\langle\!\!\left\langle \bigcup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle \right\rangle\!\!\right\rangle = \sup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle \tag{A.3.3}$$

holds, where inf *and* sup *denote the supremum and infimum in the complete lattice* image$\langle\!\langle \cdot \rangle\!\rangle$ *like in the previous Proposition A.3.2.*

PROOF: The expressions for inf and sup in image$\langle\!\langle \cdot \rangle\!\rangle$ have already been derived in the previous Proposition A.3.2. Moreover, $\langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle \subseteq \bigcap_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle$ is true because $\bigcap \mathscr{S} \subseteq S$, hence $\langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle$, holds for every $S \in \mathscr{S}$. Finally, $S \subseteq \langle\!\langle S \rangle\!\rangle$ for all $S \in \mathscr{S}$ shows that $\bigcup \mathscr{S} \subseteq \bigcup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle$, hence $\langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle \subseteq \langle\!\langle \bigcup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle \rangle\!\rangle$, and conversely, $\bigcup \mathscr{S} \supseteq S$, hence $\langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle \supseteq \langle\!\langle S \rangle\!\rangle$, for all $S \in \mathscr{S}$ shows that $\langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle \supseteq \bigcup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle$ and thus $\langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle = \langle\!\langle \langle\!\langle \bigcup \mathscr{S} \rangle\!\rangle \rangle\!\rangle \supseteq \langle\!\langle \bigcup_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle \rangle\!\rangle$. □

In A.3.2 we will see an example that shows that in general indeed $\langle\!\langle \bigcap \mathscr{S} \rangle\!\rangle \not\supseteq \bigcap_{S \in \mathscr{S}} \langle\!\langle S \rangle\!\rangle$. A hull operator is characterized completely by the properties of its image:

Definition A.3.4 *Let X be a set, then a* hull system on X *is a non-empty set of subsets $\mathscr{S} \subseteq \mathscr{P}(X)$, which is closed under arbitrary intersections, i.e.*

$$\bigcap \mathscr{T} \in \mathscr{S} \tag{A.3.4}$$

holds for all $\mathscr{T} \subseteq \mathscr{S}$.

Note that this expecially means that $X = \bigcap \emptyset \in \mathscr{S}$ for every hull system $\mathscr{S}$ on a set X.

Theorem A.3.5 *Let X be a set. If $\langle\!\langle \cdot \rangle\!\rangle$ is a hull operator on X, then its image* image$\langle\!\langle \cdot \rangle\!\rangle \subseteq \mathscr{P}(X)$ *is a hull system on X and*

$$\langle\!\langle T \rangle\!\rangle = \bigcap \{\, S \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle \mid S \supseteq T \,\} \tag{A.3.5}$$

holds for all $T \subseteq X$. Conversely, if $\mathscr{S} \subseteq \mathscr{P}(X)$ is a hull system on X, then the map $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}} \colon \mathscr{P}(X) \to \mathscr{P}(X)$, defined by

$$\langle\!\langle T \rangle\!\rangle_{\mathscr{S}} \coloneqq \bigcap \{\, S \in \mathscr{S} \mid S \supseteq T \,\} \tag{A.3.6}$$

is a hull operator on X with image$\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}} = \mathscr{S}$.

PROOF: Proposition A.3.2 already shows that the image of every hull operator is a hull system. In order to prove (A.3.5), let $T \subseteq X$ be given, then $\langle\!\langle T \rangle\!\rangle \supseteq T$ already shows that the inclusion $\langle\!\langle T \rangle\!\rangle \supseteq \bigcap \{\, S \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle \mid S \supseteq T \,\}$ holds. The converse follows from $\langle\!\langle T \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle = S$ for all $S \in \operatorname{image}\langle\!\langle \cdot \rangle\!\rangle$ which fulfil $S \supseteq T$.

Now let $\mathscr{S} \subseteq \mathscr{P}(X)$ be a hull system on X. It is clear from (A.3.6) that $\langle\!\langle T \rangle\!\rangle_{\mathscr{S}} \supseteq T$ for all $T \subseteq X$, i.e. $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}}$ is an extensive map, but before checking the other properties of a hull operator, we show that image$\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}} = \mathscr{S}$. The inclusion $\subseteq$ is clear because $\mathscr{S}$ is closed under arbitrary intersections. Conversely, given $S \in \mathscr{S}$ then $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}} = S$: On the one hand, $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}} \supseteq S$ because

$\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}}$ is extensive, and $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}} \subseteq S$ because $S \in \mathscr{S}$ and $S \supseteq S$, so S is one of the sets on the right-hand side of (A.3.6). Note that image $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}} = \mathscr{S}$ and $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}} = S$ for all $S \in \mathscr{S}$ together also imply that $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}}$ is idempotent. Finally, $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}}$ is monotone, because given $T_1, T_2 \subseteq X$ with $T_1 \subseteq T_2$, then $\{ S \in \mathscr{S} \mid S \supseteq T_1 \} \supseteq \{ S \in \mathscr{S} \mid S \supseteq T_2 \}$, thus $\langle\!\langle T_1 \rangle\!\rangle_{\mathscr{S}} \subseteq \langle\!\langle T_2 \rangle\!\rangle_{\mathscr{S}}$. $\square$

The above Theorem A.3.5 shows that hull operators and hull systems are equivalent concepts, and that the connection between them is that the hull operator $\langle\!\langle \cdot \rangle\!\rangle$ on a set X corresponding to a hull system $\mathscr{S} \subseteq \mathscr{P}(X)$ assigns to every $T \subseteq X$ the "smallest $S \in \mathscr{S}$ that contains T", i.e. the minimum of all $S \in \mathscr{S}$ with $S \supseteq T$.

In many cases, it is easy to check that certain sets of subsets $\mathscr{S}$ are hull systems, thus yield a hull operator $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}}$, but one would like to also determine an alternative description of this hull operator that, unlike (A.3.6), constructs $\langle\!\langle T \rangle\!\rangle_{\mathscr{S}}$ out of the elements of T and not indirectly using the other sets in $\mathscr{S}$.

Moreover, if $\mathscr{S}_1$ and $\mathscr{S}_2$ are two hull systems on a set X, then it is easy to check that $\mathscr{S}_1 \cap \mathscr{S}_2$ again is a hull system. With respect to the corresponding hull operators we can say the following:

Proposition A.3.6 *Let X be a set and $\mathscr{S}_1$ as well as $\mathscr{S}_2$ two hull systems on X, then*

$$\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} = \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2} \tag{A.3.7}$$

holds for all $T \in X$ if and only if $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}_2} \in \mathscr{S}_1$ for every $S \in \mathscr{S}_1$.

PROOF: Assume that there is an $S \in \mathscr{S}_1$ such that $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}_2} \notin \mathscr{S}_1$, then $\langle\!\langle \langle\!\langle S \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} = \langle\!\langle S \rangle\!\rangle_{\mathscr{S}_2} \notin \mathscr{S}_1$, while $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2} \in \mathscr{S}_1 \cap \mathscr{S}_2 \subseteq \mathscr{S}_1$. So $\langle\!\langle \langle\!\langle S \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \neq \langle\!\langle S \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2}$.

Conversely, assume that $\langle\!\langle S \rangle\!\rangle_{\mathscr{S}_2} \in \mathscr{S}_1$ for every $S \in \mathscr{S}_1$ and let $T \subseteq X$ be given. Then $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \in \mathscr{S}_1$ due to $\langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \in$ image $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}_1} = \mathscr{S}_1$ and $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \in \mathscr{S}_2$ due to image $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}_2} = \mathscr{S}_2$, so $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \in \mathscr{S}_1 \cap \mathscr{S}_2$. Moreover, $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \supseteq \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \supseteq T$, and thus $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \supseteq \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2}$ due to (A.3.6). Conversely, $\{ S \in \mathscr{S}_1 \mid S \supseteq T \} \supseteq \{ S \in \mathscr{S}_1 \cap \mathscr{S}_2 \mid S \supseteq T \}$, hence $\langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \subseteq \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2}$. Applying $\langle\!\langle \cdot \rangle\!\rangle_{\mathscr{S}_2}$ on both sides and using that $\langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2} \in \mathscr{S}_2$ then yields $\langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1} \rangle\!\rangle_{\mathscr{S}_2} \subseteq \langle\!\langle \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2} \rangle\!\rangle_{\mathscr{S}_2} = \langle\!\langle T \rangle\!\rangle_{\mathscr{S}_1 \cap \mathscr{S}_2}$. $\square$

A.3.2 Examples of Hull Operators

A vast class of hull systems are given by the various subsets of algebraic objects that are compatible with the algebraic structure:

Definition A.3.7 *Let V be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $S \subseteq V$.*

- *S is called a* convex cone *if $0 \in S$ and $\lambda s + \mu t \in S$ for all $s, t \in S$ and all $\lambda, \mu \in [0, \infty[$.*
- *S is called a* linear subspace *if $0 \in S$ and $\lambda s + \mu t \in S$ for all $s, t \in S$ and all $\lambda, \mu \in \mathbb{F}$.*

Note that the definition of a convex cone varies in the literature. Recall also the definition of convex subsets, e.g. from Definition A.1.1. It is easy to check that the sets of all convex subsets (or convex cones, or linear subspaces) of a real or complex vector space form hull systems, and it is not much harder to show that the corresponding hull operators $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{conv}}$, $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cone}}$ and $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{lin}}$ can, in addition to (A.3.6), also be described as

$$
\begin{aligned}
\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{conv}} &= \Big\{ \sum\nolimits_{n=1}^N \lambda_n s_n \;\Big|\; N \in \mathbb{N};\; s_1, \dots, s_N \in S;\; \lambda_1, \dots, \lambda_N \in [0,1] \text{ with } \sum\nolimits_{n=1}^N \lambda_n = 1 \Big\} \\
\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{cone}} &= \Big\{ \sum\nolimits_{n=1}^N \lambda_n s_n \;\Big|\; N \in \mathbb{N}_0;\; s_1, \dots, s_N \in S;\; \lambda_1, \dots, \lambda_N \in [0,\infty[\Big\} \\
\text{and} \qquad \langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{lin}} &= \Big\{ \sum\nolimits_{n=1}^N \lambda_n s_n \;\Big|\; N \in \mathbb{N}_0;\; s_1, \dots, s_N \in S;\; \lambda_1, \dots, \lambda_N \in \mathbb{R} \Big\}
\end{aligned}
$$

for every subset $S \subseteq V$ of a real or complex vector space V, where $\sum_{n=1}^0 \dots \coloneqq 0$ is used.

We can now show that the missing inclusion in Proposition A.3.3 actually is not fulfilled in general. Take, as an example, the convex hull of subsets of $\mathbb{R}$, then $\langle\!\langle\, \{0,2\} \,\rangle\!\rangle_{\mathrm{conv}} = [0,2]$ and $\langle\!\langle\, \{1,3\} \,\rangle\!\rangle_{\mathrm{conv}} = [1,3]$, but $\langle\!\langle\, \{0,2\} \cap \{1,3\} \,\rangle\!\rangle_{\mathrm{conv}} = \langle\!\langle\, \emptyset \,\rangle\!\rangle_{\mathrm{conv}} = \emptyset \neq [1,2] = [0,2] \cap [1,3]$.

Another important hull system is the set of all closed subsets of a topological space X, whose corresponding hull operator is of course the closure operator $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}}$. Again, the closure of a set $S \subseteq X$ has also an alternative description, namely as the set of all limit points of all nets in S that converge in X. It is worthwhile to mention that in this case,

$$
\langle\!\langle\, S \cup T \,\rangle\!\rangle_{\mathrm{cl}} = \langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{cl}} \cup \langle\!\langle\, T \,\rangle\!\rangle_{\mathrm{cl}} \qquad \text{for all } S, T \in X\,, \tag{A.3.8}
$$

and that, conversely, every hull operator $\langle\!\langle \,\cdot\, \rangle\!\rangle$ on a set X fulfilling (A.3.8) defines a unique topology on X for which $\langle\!\langle \,\cdot\, \rangle\!\rangle$ is the closure operator. Thus it makes sense to call a hull operator fulfilling (A.3.8) a *closure operator*. However, there are different definitions in use in the literature: Sometimes "hull operator" and "closure operator" are replaced by "closure operator" and "topological closure operator", so caution is advised.

With respect to the hull systems of closed convex sets, closed convex cones and closed linear subspaces of a real or complex locally convex vector space we just note that one can check that the closure of a convex set remains convex, the closure of a convex cone remains a convex cone and that the closure of a linear subspace remains a linear subspace. Because of this, Proposition A.3.6 applies and the corresponding hull operators are just the compositions $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}} \circ \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{conv}}$, $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}} \circ \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cone}}$ and $\langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}} \circ \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{lin}}$.

A.3.3 Galois Connections

Another source of hull operators are Galois connections:

Definition A.3.8 *Let X and Y be two sets and $\ltimes$ a relation between them. Then define two maps $\cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(Y)$ and $\cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ by*

$$
S^{\ltimes} \coloneqq \big\{\, y \in Y \;\big|\; \forall_{s \in S} : s \ltimes y \,\big\} \qquad \textit{and} \qquad T^{\ltimes} \coloneqq \big\{\, x \in X \;\big|\; \forall_{t \in T} : x \ltimes t \,\big\} \tag{A.3.9}
$$

for all $S \subseteq X$ and $T \subseteq Y$. These maps are called the Galois connection associated to $\ltimes$. *Moreover, write* $\mathrm{image}_X \ltimes$ *for the image in $\mathscr{P}(X)$ of $\cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ and* $\mathrm{image}_Y \ltimes$ *for the image in $\mathscr{P}(Y)$ of $\cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(Y)$.*

Using the same symbol for both maps usually creates no problems as they have different domains of definition as long as $X \neq Y$. In the case that $X = Y$ there is, again, no problem if $\ltimes$ is a symmetric relation, because then both maps coincide. As all examples here will be of one of these types, this slightly sloppy notation will be good enough.

Lemma A.3.9 *Let X and Y be two sets and $\ltimes$ a relation between them, then $\cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(Y)$ and $\cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ are antitone, i.e.*

$$S_1^{\ltimes} \supseteq S_2^{\ltimes} \qquad \text{and} \qquad T_1^{\ltimes} \supseteq T_2^{\ltimes} \tag{A.3.10}$$

hold for all $S_1 \subseteq S_2 \subseteq X$ and $T_1 \subseteq T_2 \subseteq Y$, respectively. Furthermore, for all $S \subseteq X$ and $T \subseteq Y$,

$$S^{\ltimes\ltimes} \supseteq S \qquad \text{and} \qquad T^{\ltimes\ltimes} \supseteq T \tag{A.3.11}$$

$$\text{as well as} \qquad S^{\ltimes\ltimes\ltimes} = S^{\ltimes} \qquad \text{and} \qquad T^{\ltimes\ltimes\ltimes} = T^{\ltimes}\,. \tag{A.3.12}$$

PROOF: Given $S_1 \subseteq S_2 \subseteq X$ and $y \in S_2^{\ltimes}$, then $s \ltimes y$ holds for all $s \in S_2$, therefore especially for all $s \in S_1$, i.e. $y \in S_1^{\ltimes}$. So $S_1^{\ltimes} \supseteq S_2^{\ltimes}$, and analogously one shows that $T_1^{\ltimes} \supseteq T_2^{\ltimes}$ if $T_1 \subseteq T_2 \subseteq Y$.

Now if $S \subseteq X$ and $s \in S$, then $s \in S^{\ltimes\ltimes}$, because given any $t \in S^{\ltimes}$ then $s \ltimes t$ holds by definition of $S^{\ltimes}$. So $S^{\ltimes\ltimes} \supseteq S$ and analogously one shows that $T^{\ltimes\ltimes} \supseteq T$ for all $T \subseteq Y$.

As S and T can be chosen as an arbitrary subset of X and Y, respectively, the above also shows that $S^{\ltimes\ltimes\ltimes} \supseteq S^{\ltimes}$ and $T^{\ltimes\ltimes\ltimes} \supseteq T^{\ltimes}$ for all $S \subseteq X$ and $T \subseteq Y$, but conversely, applying the antitone $\cdot^{\ltimes}$ to $S^{\ltimes\ltimes} \supseteq S$ yields $S^{\ltimes\ltimes\ltimes} \subseteq S^{\ltimes}$, so $S^{\ltimes\ltimes\ltimes} = S^{\ltimes}$, analogously for T. □

Theorem A.3.10 *Let X and Y be two sets and $\ltimes$ a relation between them, then* $\mathrm{image}_X \ltimes \subseteq \mathscr{P}(X)$ *and* $\mathrm{image}_Y \ltimes \subseteq \mathscr{P}(Y)$ *are hull systems, and the corresponding hull operators are*

$$\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(X) \qquad \text{as well as} \qquad \cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(Y)\,. \tag{A.3.13}$$

Furthermore, the antitone maps $\cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(Y)$ and $\cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ turn arbitrary unions into intersections, i.e.

$$\Big(\bigcup \mathscr{S}\Big)^{\ltimes} = \bigcap_{S \in \mathscr{S}} S^{\ltimes} \qquad \text{and} \qquad \Big(\bigcup \mathscr{T}\Big)^{\ltimes} = \bigcap_{T \in \mathscr{T}} T^{\ltimes} \tag{A.3.14}$$

hold for all $\mathscr{S} \subseteq \mathscr{P}(X)$ and $\mathscr{T} \subseteq \mathscr{P}(Y)$, respectively.

PROOF: As $\cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(Y)$ and $\cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ are antitone by the previous Lemma A.3.9, the maps $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(X)$ and $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(Y)$ are monotone. Lemma A.3.9 also shows that they are extensive and idempotent as $\cdot^{\ltimes} \circ \cdot^{\ltimes} \circ \cdot^{\ltimes} \circ \cdot^{\ltimes} = (\cdot^{\ltimes} \circ \cdot^{\ltimes} \circ \cdot^{\ltimes}) \circ \cdot^{\ltimes} = \cdot^{\ltimes} \circ \cdot^{\ltimes}$ in both versions. Moreover, the image of the hull operator $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(X)$ is certainly a subset of $\mathrm{image}_X \ltimes \subseteq \mathscr{P}(X)$, and even coincides with $\mathrm{image}_X \ltimes \subseteq \mathscr{P}(X)$ because of (A.3.12) of Lemma A.3.9. Analogously one sees that the image of $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(Y)$ is $\mathrm{image}_Y \ltimes \subseteq \mathscr{P}(Y)$. It is now an immediate consequence of Theorem A.3.5 that image_X and image_Y are hull systems.

Finally, let $\mathscr{S} \subseteq \mathscr{P}(X)$ be given, then $\bigcup \mathscr{S} \supseteq S$ for all $S \in \mathscr{S}$ implies $(\bigcup \mathscr{S})^{\ltimes} \subseteq S^{\ltimes}$ for all $S \in \mathscr{S}$, hence $\big(\bigcup \mathscr{S}\big)^{\ltimes} \subseteq \bigcap_{S \in \mathscr{S}} S^{\ltimes}$. Conversely, given $y \in \bigcap_{S \in \mathscr{S}} S^{\ltimes}$, then $y \in S^{\ltimes}$ for all $S \in \mathscr{S}$, hence $s \ltimes y$ for all $s \in S$ and all $S \in \mathscr{S}$, i.e. for all $s \in \bigcup \mathscr{S}$, which shows $y \in (\bigcup \mathscr{S})^{\ltimes}$. Analogously for $\mathscr{T} \subseteq \mathscr{P}(Y)$. $\square$

Corollary A.3.11 *Let X and Y be two sets and $\ltimes$ a relation between them, then the Galois connection associated to $\ltimes$ describes an anti-isomorphism of the complete lattices* $\operatorname{image}_X \ltimes$ *and* $\operatorname{image}_Y \ltimes$*, i.e. the restrictions of the $\cdot^{\ltimes}$ to maps from* $\operatorname{image}_X \ltimes$ *to* $\operatorname{image}_Y \ltimes$ *and back are mutually inverse and compatible with all suprema and infima:*

$$(\sup \mathscr{S})^{\ltimes} = \inf_{S \in \mathscr{S}} S^{\ltimes} \qquad \text{and} \qquad (\inf \mathscr{S})^{\ltimes} = \sup_{S \in \mathscr{S}} S^{\ltimes} \tag{A.3.15}$$

$$\text{as well as} \qquad (\sup \mathscr{T})^{\ltimes} = \inf_{T \in \mathscr{T}} T^{\ltimes} \qquad \text{and} \qquad (\inf \mathscr{T})^{\ltimes} = \sup_{T \in \mathscr{T}} T^{\ltimes} \tag{A.3.16}$$

hold for all $\mathscr{S} \subseteq$ $\operatorname{image}_X \ltimes$ *and all $\mathscr{T} \subseteq$* $\operatorname{image}_Y \ltimes$ *with supremum and infimum understood in* $\operatorname{image}_X \ltimes$ *and* $\operatorname{image}_Y \ltimes$*, i.e. like in Proposition A.3.2.*

PROOF: The restrictions of the $\cdot^{\ltimes}$ to maps from $\operatorname{image}_X \ltimes$ to $\operatorname{image}_Y \ltimes$ are mutually inverse because $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(X) \to \mathscr{P}(X)$ and $\cdot^{\ltimes} \circ \cdot^{\ltimes} \colon \mathscr{P}(Y) \to \mathscr{P}(Y)$ are idempotent with images $\operatorname{image}_X \ltimes$ and $\operatorname{image}_Y \ltimes$. The compatibility with suprema and infima follows from

$$(\sup \mathscr{S})^{\ltimes} = \big(\bigcup \mathscr{S}\big)^{\ltimes\ltimes\ltimes} = \big(\bigcup \mathscr{S}\big)^{\ltimes} = \bigcap_{S \in \mathscr{S}} S^{\ltimes} = \inf_{S \in \mathscr{S}} S^{\ltimes}$$

$$\text{and} \qquad (\inf \mathscr{S})^{\ltimes} = \big(\bigcap \mathscr{S}\big)^{\ltimes} = \Big(\bigcap_{S \in \mathscr{S}} S^{\ltimes\ltimes}\Big)^{\ltimes} = \Big(\bigcup_{S \in \mathscr{S}} S^{\ltimes}\Big)^{\ltimes\ltimes} = \sup_{S \in \mathscr{S}} S^{\ltimes}$$

for all $\mathscr{S} \subseteq \operatorname{image}_X \ltimes$ by using (A.3.14). This also implies the analogous statement for $\mathscr{T} \subseteq \operatorname{image}_Y \ltimes$ as the $\cdot^{\ltimes}$ are mutually inverse. $\square$

A.3.4 Examples of Galois Connections

As Galois connections always yield two hull operators, a typical problem is to find an easier, more direct description of these operators than (A.3.13). In many cases, this leads to highly non-trivial theorems.

For example, let $\mathbb{F}$ be a field, then consider the relation $\ltimes$ between $\mathfrak{Aut}(\mathbb{F})$, the group of field automorphisms of $\mathbb{F}$, and $\mathbb{F}$, defined as $g \ltimes f :\Longleftrightarrow g(f) = f$ for all $g \in \mathfrak{Aut}(\mathbb{F})$ and $f \in \mathbb{F}$. Given $S \subseteq \mathfrak{Aut}(f)$, then it is straightforward to check that $S^{\ltimes} \subseteq \mathbb{F}$ is a subfield of $\mathbb{F}$, and conversely, given $T \subseteq \mathbb{F}$, then it is equally easy to check that $T^{\ltimes}$ is a subgroup of $\mathfrak{Aut}(\mathbb{F})$. So the Galois connection associated to $\ltimes$ describes an anti-isomorphism between $\operatorname{image}_{\mathfrak{Aut}(\mathbb{F})} \ltimes$, a complete lattice of subgroups of $\mathfrak{Aut}(\mathbb{F})$, and $\operatorname{image}_{\mathbb{F}} \ltimes$, a complete lattice of subfields of $\mathbb{F}$ – but it is not easy to say which subgroups or subfields are in $\operatorname{image}_{\mathfrak{Aut}(\mathbb{F})} \ltimes$ or $\operatorname{image}_{\mathbb{F}} \ltimes$. The answer to a very similar question is given by the fundamental theorem of Galois theory, which explains the term "Galois connection". However, the Galois connections relevant in this thesis are others:

Definition A.3.12 *Let $\mathcal{D}$ be a pre-Hilbert space, then consider the symmetric relation $\perp$ on $\mathcal{D}$ that is defined by*

$$\phi \perp \psi \qquad :\Longleftrightarrow \qquad \langle \phi \,|\, \psi \rangle = 0\,. \tag{A.3.17}$$

The associated Galois connection will then be denoted by $\cdot^{\perp}\colon \mathscr{P}(\mathcal{D}) \to \mathscr{P}(\mathcal{D})$ *and is called the* orthogonal complement.

In this example, the two versions of $\cdot^{\perp}$ coincide because $\perp$ is a symmetric relation. It is easy to check that $S^{\perp}$ is always a closed linear subspace of $\mathcal{D}$, and for Hilbert spaces one gets:

Proposition A.3.13 *Let* $\mathfrak{H}$ *be a Hilbert space, then* $\operatorname{image}_{\mathfrak{H}} \perp$ *is the set of all closed linear subspaces of* $\mathfrak{H}$. *The corresponding hull operator thus is*

$$\cdot^{\perp} \circ \cdot^{\perp} = \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}} \circ \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{lin}}\,. \tag{A.3.18}$$

PROOF: It only remains to show that every closed linear subspace S of $\mathfrak{H}$ is in $\operatorname{image}_{\mathfrak{H}} \perp$, so let such $S \subseteq \mathfrak{H}$ be given as well as $\phi \in \mathfrak{H}$, then $\phi = \phi_S + \phi_{\perp}$ with unique vectors $\phi_S \in S$ and $\phi_{\perp} \in S^{\perp}$ by A.1.25. If $\phi \notin S$, i.e. $\phi_{\perp} \neq 0$, then $\langle \phi \,|\, \phi_{\perp} \rangle = \|\phi_{\perp}\|^2 > 0$ and thus $\phi \notin S^{\perp\perp}$. This shows $S^{\perp\perp} \subseteq S$, and $S^{\perp\perp} \supseteq S$ is clear because $\cdot^{\perp} \circ \cdot^{\perp}$ is a hull operator. □

Definition A.3.14 *Let* V *and* W *be two real vector spaces and* $\langle \,\cdot\,,\,\cdot\, \rangle\colon V \times W \to \mathbb{R}$ *a bilinear form. Define the relation* $\uparrow$ *between* V *and* W *as*

$$v \uparrow w \qquad :\Longleftrightarrow \qquad \langle v\,,\, w \rangle \geq 0\,. \tag{A.3.19}$$

The associated Galois connection will then be denoted by $\cdot^{\uparrow}\colon \mathscr{P}(V) \to \mathscr{P}(W)$ *and* $\cdot^{\uparrow}\colon \mathscr{P}(W) \to \mathscr{P}(V)$.

This time one easily checks that $S^{\uparrow} \subseteq W$ and $T^{\uparrow} \subseteq V$ for all $S \subseteq V$ and $T \subseteq W$ are convex cones and closed with respect to the weak topology on V and W via $\langle \,\cdot\,,\,\cdot\, \rangle$. This actually characterizes $\operatorname{image}_V \uparrow$ and $\operatorname{image}_W \uparrow$:

Proposition A.3.15 *Let* V *and* W *be two real vector spaces and* $\langle \,\cdot\,,\,\cdot\, \rangle\colon V \times W \to \mathbb{R}$ *a bilinear form, then* $\operatorname{image}_V \uparrow$ *and* $\operatorname{image}_W \uparrow$ *are the sets of weakly closed (with respect to* $\langle \,\cdot\,,\,\cdot\, \rangle$*) convex cones in* V *and* W*, respectively. The corresponding hull operators thus are both*

$$\cdot^{\uparrow} \circ \cdot^{\uparrow} = \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cl}} \circ \langle\!\langle \,\cdot\, \rangle\!\rangle_{\mathrm{cone}}\,. \tag{A.3.20}$$

PROOF: As the problem is symmetric in V and W, it is sufficient to shows that every weakly closed convex cone $S \subseteq V$ is in $\operatorname{image}_V \uparrow$, so let such an $S \subseteq V$ as well as $v \in V \backslash S$ be given. As S is weakly closed and convex, Lemma A.1.13 shows that there exists a $w \in W$ with the property that $\langle s\,,\, w \rangle \geq \langle v\,,\, w \rangle + 1$ holds for all $s \in S$.

Especially for $s = 0$ this yields $-1 \geq \langle v\,,\, w \rangle$, from which one can deduce that $\langle s\,,\, w \rangle \geq 0$ for all $s \in S$: If $\langle s\,,\, w \rangle < 0$ for one $s \in S$ then $\langle \lambda s\,,\, w \rangle = \langle v\,,\, w \rangle$ for $\lambda = \langle v\,,\, w \rangle / \langle s\,,\, w \rangle \in [0, \infty[$, which is a contradiction because $\lambda s \in S$.

So $w \in S^{\uparrow}$, which, together with $\langle v\,,\, w \rangle \leq -1$, shows that $v \notin S^{\uparrow\uparrow}$ and we conclude that $S^{\uparrow\uparrow} \subseteq S$. The converse inclusion $S^{\uparrow\uparrow} \supseteq S$ is clear because $\cdot^{\uparrow} \circ \cdot^{\uparrow}$ is a hull operator. □

A.4 Operators on a Hilbert Space

When modelling the observables of a quantum system by unbounded operators, one would like the observables to have a good algebraic structure, e.g. be a *-algebra, and be sufficiently well-behaved such that, for example, they admit a meaningful spectral theory. The first property can easily be guaranteed by considering *-algebras of adjointable endomorphisms on a pre-Hilbert space $\mathcal{D}$, or unital *-subalgebras thereof, i.e. O^*-algebras. However, the spectral theory is best understood for (possibly unbounded) operators on a Hilbert space $\mathfrak{H}$, which are usually defined as tuples $(\mathcal{D}, A)$ of a dense linear subspace $\mathcal{D} \subseteq \mathfrak{H}$ and a linear map $A\colon \mathcal{D} \to \mathfrak{H}$.

The set of all operators on a fixed Hilbert space $\mathfrak{H}$ quite obviously does not permit the usual algebraic operations: The sum of two operators can only be defined on their common domain, which might be $\{0\}$. The product, i.e. composition, of two operators is only defined as long as the image of the first one is a subset of the domain of the second one, which need not be true. And finally, it is not even obvious how to define the adjoint of an operator. Nevertheless, this notion of operators on a Hilbert space is the key towards a good spectral theory and the spectral theorem for unbounded operators, see e.g. [75, Part II].

Of course, every adjointable endomorphism a of a pre-Hilbert space $\mathcal{D}$ can be seen as an operator $(\mathcal{D}, a)$ on the completion $\mathfrak{H}$ of $\mathcal{D}$. This way, the algebraic properties of O^*-algebras can be combined with the results about operators on Hilbert spaces. In the following, some of these results about operators and their application to adjointable endomorphisms will be discussed. All results are standard, for a more detailed treatment of the subject, see e.g. [75]. Note that every Hilbert space $\mathfrak{H}$ and every dense linear subspace $\mathcal{D}$ of $\mathfrak{H}$ will – if not explicitly stated differently – always be endowed with the norm topology defined by the usual Hilbert norm $\|\phi\| := \langle\, \phi \,|\, \phi \,\rangle^{1/2}$. We will also treat every pre-Hilbert space $\mathcal{D}$ as a dense linear subspace of its completion $\mathfrak{H}$, which is only a slight abuse of notation in light of Section A.1.3 as every Hausdorff locally convex space can be embedded densely and injectively in its completion.

A.4.1 The Operator Theoretic Adjoint

The most important definition regarding operators on a Hilbert space is the one of the adjoint operator, which is different from the definition of the adjoint of an endomorphism on a pre-Hilbert space. This is easiest to understand by looking at the graph of an operator. See [75, Chap. 1] for more details, but note that some definitions for ill-behaved cases differ (e.g. operators with non-dense domain).

Graphs

Recall that a function from a set X to a set Y is typically defined as a subset $G \subseteq X \times Y$, the graph of the function, with the property that for every $x \in X$ there exists one and only one $y \in Y$ such that $(x, y) \in G$. In this case, the function described by G maps x to y and we write $G(x) = y$, so that $G = \big\{ \big(x, G(x)\big) \;\big|\; x \in X \big\}$.

Now let V and W be two vector spaces over a field $\mathbb{F}$, then $V \times W$ together with the componentwise addition and scalar multiplication, $\lambda(v, w) + \lambda'(v', w') := (\lambda v + \lambda' v', \lambda' w + \lambda w')$ for all $\lambda, \lambda' \in \mathbb{F}$, $(v, w), (v', w') \in V \times W$, is a new vector space. Even more:

Proposition A.4.1 *Let V and W be two vector spaces over a field $\mathbb{F}$ and $G \subseteq V \times W$ the graph of a function. Then G is the graph of a linear function if and only if G is a linear subspace of $V \times W$.*

PROOF: First assume that G is a linear subspace and let $\lambda, \lambda' \in \mathbb{F}$ as well as $v, v' \in V$ be given. Then it follows from $\big(v, G(v)\big), \big(v', G(v')\big) \in G$ and because G is a linear subspace that $\big(\lambda v + \lambda' v', \lambda G(v) + \lambda' G(v')\big) = \lambda\big(v, G(v)\big) + \lambda'\big(v', G(v')\big) \in G$, so $G(\lambda v + \lambda' v') = \lambda G(v) + \lambda' G(v')$.

Conversely, if the function described by G is linear, then $(0,0) \in G$ because $G(0) = 0$, and given $\lambda, \lambda' \in \mathbb{F}$ as well as $\big(v, G(v)\big), \big(v', G(v')\big) \in G$, then linearity of G implies $G(\lambda v + \lambda' v') = \lambda G(v) + \lambda' G(v')$, hence $\lambda\big(v, G(v)\big) + \lambda'\big(v', G(v')\big) = \big(\lambda v + \lambda' v', \lambda G(v) + \lambda' G(v')\big) \in G$. □

If $V = W = \mathfrak{H}$ is a Hilbert space, then the cartesian product $\mathfrak{H} \times \mathfrak{H}$ is again a Hilbert space with the componentwise inner product $\langle\, (\phi, \psi) \,|\, (\phi', \psi') \,\rangle := \langle\, \phi \,|\, \phi' \,\rangle + \langle\, \psi \,|\, \psi' \,\rangle$ for all $(\phi, \psi), (\phi', \psi') \in \mathfrak{H} \times \mathfrak{H}$: Checking that $\langle\, \cdot \,|\, \cdot \,\rangle$ on $\mathfrak{H} \times \mathfrak{H}$ is again sesquilinear, positive and non-degenerate is straightforward, and $\mathfrak{H} \times \mathfrak{H}$ is even complete with respect to the induced norm because for every Cauchy sequence $(\phi_n, \psi_n)_{n \in \mathbb{N}}$ in $\mathfrak{H} \times \mathfrak{H}$ the components $(\phi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ are Cauchy sequences in $\mathfrak{H}$, which converge against $\hat{\phi}, \hat{\psi} \in \mathfrak{H}$, and then $(\hat{\phi}, \hat{\psi})$ is the limit of $(\phi_n, \psi_n)_{n \in \mathbb{N}}$ in $\mathfrak{H} \times \mathfrak{H}$.

So we see that linear functions from $\mathfrak{H}$ to $\mathfrak{H}$ are described by special linear subspaces of the Hilbert space $\mathfrak{H} \times \mathfrak{H}$. As operators on $\mathfrak{H}$ generalize the concept of a linear function from $\mathfrak{H}$ to $\mathfrak{H}$, they can be understood as more general linear subspaces of $\mathfrak{H} \times \mathfrak{H}$:

Definition A.4.2 *Let $\mathfrak{H}$ be a Hilbert space and $G \subseteq \mathfrak{H} \times \mathfrak{H}$ a linear subspace.*

i.) The multiplicity space *of G is defined as* $\mathrm{mul}(G) := \{\, \psi \in \mathfrak{H} \mid (0, \psi) \in G \,\}$ *and G is called a* graph *if* $\mathrm{mul}(G) = \{0\}$.

ii.) The domain *of G is defined as* $\mathrm{dom}(G) := \{\, \phi \in \mathfrak{H} \mid \exists_{\psi \in \mathfrak{H}} : (\phi, \psi) \in G \,\}$ *and G is called* everywhere defined *if* $\mathrm{dom}(G) = \mathfrak{H}$*, and* densely defined *if* $\mathrm{dom}(G)$ *is dense in $\mathfrak{H}$.*

iii.) The kernel *of G is defined as* $\ker(G) := \{\, \phi \in \mathfrak{H} \mid (\phi, 0) \in G \,\}$ *and G is called* injective *if* $\ker(G) = \{0\}$.

iv.) The image *of G is defined as* $\mathrm{img}(G) := \{\, \psi \in \mathfrak{H} \mid \exists_{\phi \in \mathfrak{H}} : (\phi, \psi) \in G \,\}$ *and G is called* surjective *if* $\mathrm{img}(G) = \mathfrak{H}$*, and* quasi surjective *if* $\mathrm{img}(G)$ *is dense in $\mathfrak{H}$.*

Note that $\mathrm{mul}(G)$, $\mathrm{dom}(G)$, $\ker(G)$ and $\mathrm{img}(G)$ are linear subspaces of $\mathfrak{H}$, as they are the kernel and image of G under the canonical projections from $\mathfrak{H} \times \mathfrak{H}$ on the first and second component. It is not so hard to see that the operators on $\mathfrak{H}$ are in one to one correspondence to the densely defined graphs $G \subseteq \mathfrak{H} \times \mathfrak{H}$:

Proposition A.4.3 *Let $(\mathcal{D}, A)$ be an operator on a Hilbert space $\mathfrak{H}$, then the graph G_A of A, seen as a subset of $\mathfrak{H} \times \mathfrak{H}$ (and not only of $\mathcal{D} \times \mathfrak{H}$), is a densely defined graph in the sense of the previous Proposition A.4.2 with* $\mathrm{dom}(G_A) = \mathcal{D}$.

Conversely, if $G \subseteq \mathfrak{H} \times \mathfrak{H}$ is a densely defined graph, then $G \subseteq \mathrm{dom}(G) \times \mathfrak{H}$ is the graph of a linear function $A_G \colon \mathrm{dom}(G) \to \mathfrak{H}$ and $\big(\mathrm{dom}(G), A_G\big)$ is an operator on $\mathfrak{H}$.

PROOF: If $(\mathcal{D}, A)$ is an operator on $\mathfrak{H}$, then $\operatorname{mul}(G_A) = \{0\}$ because only $\psi = A(0) = 0$ fulfils $(0, \psi) \in G_A$, and it is easy to check that $\operatorname{dom}(G_A) = \mathcal{D}$ is dense in $\mathfrak{H}$.

Conversely, if $G \subseteq \mathfrak{H} \times \mathfrak{H}$ is a densely defined graph, then $G \subseteq \operatorname{dom}(G) \times \mathfrak{H}$ holds by the definition of $\operatorname{dom}(G)$, and this is the graph of a function $A_G \colon \operatorname{dom}(G) \to \mathfrak{H}$ because, again by the definition of $\operatorname{dom}(G)$, for every $\phi \in \operatorname{dom}(G)$ there exists a $\psi \in \mathfrak{H}$ with $(\phi, \psi) \in G$, and this vector ψ is unique: If $\psi, \psi' \in \mathfrak{H}$ both fulfil $(\phi, \psi), (\phi, \psi') \in G$, then $(0, \psi - \psi') = (\phi, \psi) - (\phi, \psi') \in G$ as G is a linear subspace of $\mathfrak{H} \times \mathfrak{H}$, and $\psi = \psi'$ because $\operatorname{mul}(G) = \{0\}$. The function A_G is linear by Proposition A.4.1. $\square$

Of course, the linear functions from $\mathfrak{H}$ to $\mathfrak{H}$ correspond to the operators on $\mathfrak{H}$ with domain $\mathfrak{H}$, hence to the everywhere defined graphs $G \subseteq \mathfrak{H} \times \mathfrak{H}$. Moreover, if $G \subseteq \mathfrak{H} \times \mathfrak{H}$ is a densely defined graph, hence describes an operator on $\mathfrak{H}$, then $\ker(G)$ and $\operatorname{img}(G)$ are indeed just the kernel and image of the corresponding operator, so the notions of injectivity and surjectivity are the usual ones in this case. The beauty of Definition A.4.2 is that these notions make sense completely independently of whether or not $G \subseteq \mathfrak{H} \times \mathfrak{H}$ describes some kind of function.

The Inverse

It is obvious that there is a relation between the definitions of $\operatorname{mul}(G)$ and $\ker(G)$ as well as between $\operatorname{dom}(G)$ and image (G).

Definition A.4.4 *Let $\mathfrak{H}$ be a Hilbert space, then define the linear map $\tau \colon \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H} \times \mathfrak{H}$,*

$$(\phi, \psi) \mapsto \tau(\phi, \psi) := (\psi, \phi)\,. \tag{A.4.1}$$

Moreover, for every linear subspace $G \subseteq \mathfrak{H} \times \mathfrak{H}$, define the inverse of G *as*

$$G^{-1} := \tau(G) = \big\{\, (\psi, \phi) \;\big|\; (\phi, \psi) \in G \,\big\}\,. \tag{A.4.2}$$

It is clear that τ is a continuous linear and involutive automorphism of $\mathfrak{H} \times \mathfrak{H}$, and consequently $\,\cdot^{-1}$ is an involution of the set (even hull system) of linear subspaces of $\mathfrak{H} \times \mathfrak{H}$. Recall that G^{-1} is indeed the graph of the inverse function if G is the graph of an injective and surjective linear function. However, we can be more precise here:

Proposition A.4.5 *Let $\mathfrak{H}$ be a Hilbert space and $G \subseteq \mathfrak{H} \times \mathfrak{H}$ a linear subspace, then the following holds:*

i.) $\operatorname{mul}(G^{-1}) = \ker(G)$ *and G^{-1} is a graph if and only if G is injective.*

ii.) $\operatorname{dom}(G^{-1}) = \operatorname{img}(G)$ *and G^{-1} is everywhere defined if and only if G is surjective, and densely defined if and only if G is quasi surjective.*

PROOF: $\phi \in \operatorname{mul}(G^{-1})$ is equivalent to $(0, \phi) \in G^{-1}$ and thus to $(\phi, 0) \in G$, which is equivalent to $\phi \in \ker(G)$, so $\operatorname{mul}(G^{-1}) = \ker(G)$. Similarly, $\psi \in \operatorname{dom}(G^{-1})$ is equivalent to the existence of a $\phi \in \mathfrak{H}$ with $(\psi, \phi) \in G^{-1}$, i.e. $(\phi, \psi) \in G$, which is equivalent to $\psi \in \operatorname{img}(G)$, so $\operatorname{dom}(G^{-1}) = \operatorname{img}(G)$. The rest is just the application of Definition A.4.2. $\square$

Using that $\cdot^{-1}$ is an involution of the set of linear subspaces of $\mathfrak{H} \times \mathfrak{H}$ yields:

Corollary A.4.6 *Let $\mathfrak{H}$ be a Hilbert space and $G \subseteq \mathfrak{H} \times \mathfrak{H}$ a linear subspace, then the following holds:*

i.) $\ker(G^{-1}) = \mathrm{mul}(G)$ *and G^{-1} is injective if and only if G is a graph.*

ii.) $\mathrm{img}(G^{-1}) = \mathrm{dom}(G)$ *and G^{-1} is surjective if and only if G is everywhere defined, and quasi surjective if and only if G is densely defined.*

Note that, as operators are in one-to-one correspondence to densely defined graphs, this shows that an operator is invertible (i.e. the inverse of its graph is again the graph of an operator) if and only if it is injective and quasi surjective.

The Adjoint

Let $a \in \mathcal{L}^*(\mathcal{D})$ be an adjointable endomorphism of a pre-Hilbert space $\mathcal{D}$, then its adjoint endomorphism a^* is fixed by the condition $\langle \phi \,|\, a(\phi') \rangle = \langle a^*(\phi) \,|\, \phi' \rangle$ for all $\phi, \phi' \in \mathcal{D}$. This motivates:

Definition A.4.7 *Let $\mathfrak{H}$ be a Hilbert space, then define the symmetric relation $\dagger$ on $\mathfrak{H} \times \mathfrak{H}$ as*

$$(\phi, \psi) \dagger (\phi', \psi') \qquad :\Longleftrightarrow \qquad \langle \phi \,|\, \psi' \rangle = \langle \psi \,|\, \phi' \rangle \tag{A.4.3}$$

for all $(\phi, \psi), (\phi', \psi') \in \mathfrak{H} \times \mathfrak{H}$. The corresponding Galois connection is denoted by $\cdot^\dagger \colon \mathscr{P}(\mathfrak{H} \times \mathfrak{H}) \to \mathscr{P}(\mathfrak{H} \times \mathfrak{H})$.

Note that $\dagger$ is indeed a symmetric relation because $\langle \phi \,|\, \psi' \rangle = \langle \psi \,|\, \phi' \rangle$ is equivalent (by complex conjugation and exchange of the left and right side) to $\langle \phi' \,|\, \psi \rangle = \langle \psi' \,|\, \phi \rangle$, so the two versions of $\cdot^\dagger \colon \mathscr{P}(\mathfrak{H} \times \mathfrak{H}) \to \mathscr{P}(\mathfrak{H} \times \mathfrak{H})$ coincide. The relation to the adjoint endomorphism is:

Proposition A.4.8 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $a, b \colon \mathcal{D} \to \mathcal{D}$ two linear maps. Then a is an adjointable endomorphism of $\mathcal{D}$ with adjoint $a^* = b$ if and only if*

$$(G_a)^\dagger \supseteq G_b \,, \tag{A.4.4}$$

where $G_a, G_b \subseteq \mathfrak{H} \times \mathfrak{H}$ are the graphs of a and b, respectively.

Proof: Given $\big(\phi, a(\phi)\big) \in G_a$ and $\big(\psi, b(\psi)\big) \in G_b$, then $\langle \phi \,|\, a(\psi) \rangle = \langle b(\phi) \,|\, \psi \rangle$ is equivalent to $\big(\phi, b(\phi)\big) \dagger \big(\psi, a(\psi)\big)$. So $\langle \phi \,|\, a(\psi) \rangle = \langle b(\phi) \,|\, \psi \rangle$ holds for all $\phi, \psi \in \mathcal{D}$ if and only if $\big(\phi, b(\phi)\big) \in (G_a)^\dagger$ for all $\phi \in \mathcal{D}$. $\square$

So especially $(G_a)^\dagger \supseteq G_{a^*}$ holds for all adjointable endomorphisms. Of course, it would be desireable to have equality here, but this is typically too restrictive: From [74, Prop. 2.1.10] it follows that in this case $\mathcal{D} = \mathrm{dom}(G_{a^*}) = \mathfrak{H}$, so a would have to be a bounded operator on $\mathfrak{H}$ by the Hellinger-Toeplitz theorem.

From the properties of Galois connections it is immediately clear that $G^{\dagger\dagger\dagger} = G^\dagger$ for every subset $G \subseteq \mathfrak{H} \times \mathfrak{H}$. As the map to the adjoint should be an involution, the question arises under which conditions $G^{\dagger\dagger} = G$ holds, i.e. what $\mathrm{image}_{\mathfrak{H} \times \mathfrak{H}} \dagger$ is.

Definition A.4.9 *Let $\mathfrak{H}$ be a Hilbert space, then define the linear map $I\colon \mathfrak{H}\times\mathfrak{H}\to\mathfrak{H}\times\mathfrak{H}$,*

$$(\phi,\psi)\mapsto I(\phi,\psi) := (\psi,-\phi)\,. \tag{A.4.5}$$

One easily checks that I is a continuous linear automorphism of $\mathfrak{H}$ with $I^{-1}=-I$ and that I commutes with τ. This map is interesting because

$$\begin{aligned}(\phi,\psi)\dagger(\phi',\psi') \quad&\Longleftrightarrow\quad \langle\,\phi\,|\,\psi'\,\rangle = \langle\,\psi\,|\,\phi'\,\rangle\\ &\Longleftrightarrow\quad \langle\,(\phi,\psi)\,|\,I(\phi',\psi')\,\rangle = 0\\ &\Longleftrightarrow\quad (\phi,\psi)\perp I(\phi',\psi')\end{aligned}$$

holds for all $(\phi,\psi),(\phi',\psi')\in\mathfrak{H}\times\mathfrak{H}$. This observation allows to easily characterize $\mathrm{image}_{\mathfrak{H}\times\mathfrak{H}}\,\dagger$:

Lemma A.4.10 *Let $\mathfrak{H}$ be a Hilbert space and $G\subseteq\mathfrak{H}\times\mathfrak{H}$, then*

$$G^\dagger = I(G)^\perp = -I\big(G^\perp\big) = I\big(G^\perp\big)\,. \tag{A.4.6}$$

PROOF: Given $(\phi,\psi)\in\mathfrak{H}\times\mathfrak{H}$, then $(\phi,\psi)\in G^\dagger$ if and only if $(\phi,\psi)\perp I(\phi',\psi')$ holds for all $(\phi',\psi')\in G$, i.e. if and only if $(\phi,\psi)\in I(G)^\perp$. Moreover, as $\dagger$ is symmetric, $(\phi,\psi)\in G^\dagger$ if and only if $I(\phi,\psi)\perp(\phi',\psi')$, i.e. if and only if $I(\phi,\psi)\in G^\perp$, which is equivalent to $(\phi,\psi)\in I^{-1}(G^\perp)$, hence to $(\phi,\psi)\in -I(G^\perp)$ because $I^{-1}=-I$. As $G^\perp$ is a linear subspace of $\mathfrak{H}\times\mathfrak{H}$, it follows that $-I\big(G^\perp\big)=I\big(G^\perp\big)$. □

Proposition A.4.11 *Let $\mathfrak{H}$ be a Hilbert space, then the hull system* $\mathrm{image}_{\mathfrak{H}\times\mathfrak{H}}\,\dagger$ *on $\mathfrak{H}\times\mathfrak{H}$ coincides with* $\mathrm{image}_{\mathfrak{H}\times\mathfrak{H}}\perp$*, i.e. it consists of all closed linear subspaces of $\mathfrak{H}\times\mathfrak{H}$, hence*

$$\,\cdot\,^\dagger\circ\,\cdot\,^\dagger = \,\cdot\,^\perp\circ\,\cdot\,^\perp = \langle\!\langle\,\cdot\,\rangle\!\rangle_{\mathrm{cl}}\circ\langle\!\langle\,\cdot\,\rangle\!\rangle_{\mathrm{lin}}\,. \tag{A.4.7}$$

PROOF: The previous Lemma A.4.10 implies that $G^{\dagger\dagger}=I\big(G^\perp\big)^\dagger = \big(-I\big(I\big(G^\perp\big)\big)\big)^\perp = G^{\perp\perp}$ holds for all $G\subseteq\mathfrak{H}\times\mathfrak{H}$, and $\,\cdot\,^\perp\circ\,\cdot\,^\perp = \langle\!\langle\,\cdot\,\rangle\!\rangle_{\mathrm{cl}}\circ\langle\!\langle\,\cdot\,\rangle\!\rangle_{\mathrm{lin}}$ has been shown in Proposition A.3.13. □

So we have seen that $G^\dagger$ is, for all $G\subseteq\mathfrak{H}\times\mathfrak{H}$, a closed linear subspace of $\mathfrak{H}\times\mathfrak{H}$ and that $G^{\dagger\dagger}=G$ holds if and only if G was already a closed linear subspace of $\mathfrak{H}\times\mathfrak{H}$.

This Galois connection $\,\cdot\,^\dagger$ is extremely helpful, e.g. because it provides a relation between $\mathrm{img}(G)$ and $\ker(G^\dagger)$ of e.g. the graph G of an operator, and thus gives a relation between surjectivity and injectivity:

Lemma A.4.12 *Let $\mathfrak{H}$ be a Hilbert space and $G\subseteq\mathfrak{H}\times\mathfrak{H}$ a linear subspace, then $(G^\dagger)^{-1}=(G^{-1})^\dagger$.*

PROOF: The identity $(G^\dagger)^{-1}=\{\,(\phi,\psi)\in\mathfrak{H}\times\mathfrak{H}\mid\forall_{(\phi',\psi')\in G}:(\psi,\phi)\dagger(\phi',\psi')\,\}$ follows immediately from the definitions of $\,\cdot\,^\dagger$ and $\,\cdot\,^{-1}$, similarly $(G^{-1})^\dagger=\{\,(\phi,\psi)\in\mathfrak{H}\times\mathfrak{H}\mid\forall_{(\phi',\psi')\in G}:(\phi,\psi)\dagger(\psi',\phi')\,\}$. However, the conditions $(\psi,\phi)\dagger(\phi',\psi')$ and $(\phi,\psi)\dagger(\psi',\phi')$ are both, for all $(\phi,\psi),(\phi',\psi')\in\mathfrak{H}\times\mathfrak{H}$, equivalent to $\langle\,\psi\,|\,\psi'\,\rangle=\langle\,\phi\,|\,\phi'\,\rangle$. □

Proposition A.4.13 *Let $\mathfrak{H}$ be a Hilbert space and $G\subseteq\mathfrak{H}\times\mathfrak{H}$ a linear subspace, then the following holds:*

i.) $\mathrm{mul}(G^\dagger) = \mathrm{dom}(G)^\perp$ *and so* $G^\dagger$ *is a graph if and only if* G *is densely defined.*

ii.) $\ker(G^\dagger) = \mathrm{img}(G)^\perp$ *and so* $G^\dagger$ *is injective if and only if* G *is quasi surjective.*

PROOF: The first identity is due to

$$\mathrm{mul}(G^\dagger) = \big\{\, \psi \in \mathfrak{H} \,\big|\, \forall_{(\phi',\psi')\in G} : (0,\psi) \dagger (\phi',\psi') \,\big\} = \big\{\, \psi \in \mathfrak{H} \,\big|\, \forall_{\phi'\in \mathrm{dom}(G)} : 0 = \langle\, \psi \,|\, \phi' \,\rangle \,\big\} = \mathrm{dom}(G)^\perp$$

and so $G^\dagger$ is a graph if and only if $\mathrm{dom}(G)^\perp = \{0\}$, which, by Proposition A.3.13 and the general properties of Galois connections, is the case if and only if $\mathrm{dom}(G)$ is dense in $\mathfrak{H}$. The second one could be derived in a similar way, or follows from the first one by combining Corollary A.4.6, the previous Lemma A.4.12 and Proposition A.4.5:

$$\ker(G^\dagger) = \mathrm{mul}\big((G^\dagger)^{-1}\big) = \mathrm{mul}\big((G^{-1})^\dagger\big) = \mathrm{dom}(G^{-1})^\perp = \mathrm{img}(G)^\perp\,. \qquad \square$$

Corollary A.4.14 *Let* $\mathfrak{H}$ *be a Hilbert space and* $G \subseteq \mathfrak{H}\times\mathfrak{H}$ *a linear subspace, then the following holds:*

i.) $\mathrm{dom}(G^\dagger)^\perp = \mathrm{mul}\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big)$*, so* $G^\dagger$ *is densely defined if and only if* $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ *is a graph.*

ii.) $\mathrm{img}(G^\dagger)^\perp = \ker\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big)$*, so* $G^\dagger$ *is quasi surjective if and only if* $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ *is injective.*

PROOF: Proposition A.4.13 together with Proposition A.4.11 shows that $\mathrm{dom}(G^\dagger)^\perp = \mathrm{mul}(G^{\dagger\dagger}) = \mathrm{mul}\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big)$ and that $\mathrm{img}(G^\dagger)^\perp = \ker(G^{\dagger\dagger}) = \ker\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big)$. Then Proposition A.3.13 shows that $\mathrm{dom}(G^\dagger)$ is dense in $\mathfrak{H}$ if and only if $\mathrm{mul}\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big) = \{0\}$, and analogously, $\mathrm{img}(G^\dagger)$ is dense in $\mathfrak{H}$ if and only if $\ker\big(\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}\big) = \{0\}$. $\square$

Another immediate consequence is:

Corollary A.4.15 *Let* $\mathfrak{H}$ *be a Hilbert space and* $G \subseteq \mathfrak{H}\times\mathfrak{H}$ *a closed linear subspace, then* $\mathrm{mul}(G)$ *and* $\ker(G)$ *are closed linear subspaces of* $\mathfrak{H}$*.*

Closable Operators

From the definition of the Galois connection $\,\cdot^\dagger$ it is not clear whether $G^\dagger$ for a linear subspace and densely defined graph $G \subseteq \mathfrak{H}\times\mathfrak{H}$ is again a densely defined graph, and in general it is not true that $\,\cdot^\dagger$, applied to the graph of an operator on a Hilbert space, gives again the graph of another operator on a Hilbert space (which then could be called the adjoint operator): Proposition A.4.13 shows that $G^\dagger$ is automatically a graph if G is densely defined, but Corollary A.4.14 shows that $G^\dagger$ is densely defined if and only if $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ is a graph. This is not necessarily the case even if G is a densely defined graph, see e.g. [75, Expl. 1.1]. However, this shows:

Proposition A.4.16 *Let* $\mathfrak{H}$ *be a Hilbert space and* $G \subseteq \mathfrak{H}\times\mathfrak{H}$ *a linear subspace and a densely defined graph, then the following is equivalent:*

- $G^\dagger$ *is densely defined.*
- $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ *is a graph.*

- $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ *is a densely defined graph.*
- $G^\dagger$ *is a densely defined graph.*

PROOF: It has already been discussed that the first two points are equivalent by Corollary A.4.14. As G is densely defined by assumption, it follows that $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$ is trivially densely defined and that $G^\dagger$ is always a graph by Proposition A.4.13, which yields the remaining equivalences. $\square$

Because of this it makes sense to define:

Definition A.4.17 *Let $(\mathcal{D}, A)$ be an operator on a Hilbert space $\mathfrak{H}$ and $G_A \subseteq \mathcal{D} \times \mathfrak{H} \subseteq \mathfrak{H} \times \mathfrak{H}$ its graph, then $(\mathcal{D}, A)$ is said to be* closable *if G_A fulfils one (hence all) of the equivalent conditions of the previous Proposition A.4.16.*

For such a closable operator $(\mathcal{D}, A)$ on $\mathfrak{H}$ with graph G one can now define the adjoint and the closure as the operators on $\mathfrak{H}$ whose graphs are $G^\dagger$ and $\langle\!\langle\, G \,\rangle\!\rangle_{\mathrm{cl}}$, respectively. However, the special case of adjointable endomorphisms on a pre-Hilbert space is all that is relevant here. Because of this, the following treats specifically such adjointable endomorphisms, even though many results can be shown to hold for general closable operators.

Proposition A.4.18 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $a \in \mathcal{L}^*(\mathcal{D})$ an adjointable endomorphism of $\mathcal{D}$ with graph $G_a \subseteq \mathcal{D} \times \mathcal{D} \subseteq \mathfrak{H} \times \mathfrak{H}$. Then G_a fulfils the four equivalent conditions of Proposition A.4.16.*

PROOF: Proposition A.4.8 shows that $(G_a)^\dagger \supseteq G_{a^*}$, where G_{a^*} is the graph of the adjoint $a^* \in \mathcal{L}^*(\mathcal{D})$ of a, whose domain $\mathcal{D}$ is already dense, so $(G_a)^\dagger$ is densely defined. $\square$

This is an important result: For all adjointable endomorphisms a of a pre-Hilbert space $\mathcal{D}$ we can construct a closure and an operator theoretic adjoint, which are operators on the completion $\mathfrak{H}$ of $\mathcal{D}$, whose domain is typically larger than $\mathcal{D}$ (assuming that $\mathcal{D}$ is embedded in $\mathfrak{H}$ as a subspace):

Definition A.4.19 *Let $\mathfrak{H}$ be a Hilbert space and $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace. For every adjointable endomorphism $a \in \mathcal{L}^*(\mathcal{D})$ we define the* closure *$(\mathcal{D}_{a^{\mathrm{cl}}}, a^{\mathrm{cl}})$ and the* operator theoretic adjoint *$(\mathcal{D}_{a^\dagger}, a^\dagger)$ as follows:*

Write $G_a \subseteq \mathcal{D} \times \mathcal{D} \subseteq \mathfrak{H} \times \mathfrak{H}$ for the graph of a, then $\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}$ and $(G_a)^\dagger$ are densely defined graphs by the previous Proposition A.4.18 and Proposition A.4.16. So define $\mathcal{D}_{a^{\mathrm{cl}}} := \operatorname{dom}\big(\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}\big)$ and $\mathcal{D}_{a^\dagger} := \operatorname{dom}\big((G_a)^\dagger\big)$ and let $a^{\mathrm{cl}} \colon \mathcal{D}_{a^{\mathrm{cl}}} \to \mathfrak{H}$ and $a^\dagger \colon \mathcal{D}_{a^\dagger} \to \mathfrak{H}$ be the linear maps whose graphs are $\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}$ and $(G_a)^\dagger$, respectively.

Proposition A.4.20 *Let $\mathfrak{H}$ be a Hilbert space and $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace, then for all $a \in \mathcal{L}^*(\mathcal{D})$ the inclusion $\mathcal{D}_{a^\dagger} \supseteq \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ holds, and $a^\dagger$, restricted to $\mathcal{D}_{(a^*)^{\mathrm{cl}}}$, coincides with $(a^*)^{\mathrm{cl}}$.*

PROOF: Let G_a and G_{a^*} be the graphs of a and a^*, then $(G_a)^\dagger \supseteq G_{a^*}$ by Proposition A.4.8. As $(G_a)^\dagger$ is closed, even $(G_a)^\dagger \supseteq \langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}}$ holds. So $\mathcal{D}_{a^\dagger} = \operatorname{dom}\big((G_a)^\dagger\big) \supseteq \operatorname{dom}\big(\langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}}\big) = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ and the restriction of $a^\dagger$ to $\mathcal{D}_{(a^*)^{\mathrm{cl}}}$ coincides with a^{cl}. $\square$

As the notions of closure and adjoint of an adjointable endomorphism have been developed by using the geometry of graphs, we yet have to find more direct expressions for these concepts:

Proposition A.4.21 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $a \in \mathcal{L}^*(\mathcal{D})$, then*

$$\mathcal{D}_{a^\dagger} = \left\{ \psi \in \mathfrak{H} \;\middle|\; \mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, a(\phi) \,\rangle \in \mathbb{C} \text{ is continuous } \right\}, \tag{A.4.8}$$

and $a^\dagger \colon \mathcal{D}_{a^\dagger} \to \mathfrak{H}$ is fixed by the requirement that

$$\left\langle\, a^\dagger(\psi) \,\middle|\, \phi \,\right\rangle = \langle\, \psi \,|\, a(\phi) \,\rangle \tag{A.4.9}$$

holds for all $\psi \in \mathcal{D}_{a^\dagger}$ and all $\phi \in \mathcal{D}$.

Proof: Let $G_a \coloneqq \left\{ \big(\phi, a(\phi)\big) \;\middle|\; \phi \in \mathcal{D} \right\} \subseteq \mathfrak{H} \times \mathfrak{H}$ be the graph of a and $\psi \in \mathfrak{H}$. Then $\psi \in \operatorname{dom}\big((G_a)^\dagger\big)$ if and only if there exists a vector $\xi \in \mathfrak{H}$ such that $(\psi, \xi) \in (G_a)^\dagger$, i.e. such that $\langle\, \psi \,|\, a(\phi) \,\rangle = \langle\, \xi \,|\, \phi \,\rangle$ for all $\phi \in \mathcal{D}$. By the Fréchet-Riesz theorem this is the case if and only if the linear functional $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, a(\phi) \,\rangle \in \mathbb{C}$ is continuous (in which case it extends to a continuous linear functional on whole $\mathfrak{H}$). This proves the description of $\mathcal{D}_{a^\dagger}$, and (A.4.9) is then an immediate consequence of the definition of $a^\dagger$, as $a^\dagger(\psi) = \xi$ for all $\psi \in \mathcal{D}_{a^\dagger}$ with ξ as above. □

Proposition A.4.22 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $a \in \mathcal{L}^*(\mathcal{D})$. Then $\langle \,\cdot\, | \,\cdot\, \rangle_{\mathbb{1}+a^*a} \colon \mathcal{D}_{a^{\mathrm{cl}}} \times \mathcal{D}_{a^{\mathrm{cl}}} \to \mathbb{C}$,*

$$(\phi, \psi) \mapsto \left\langle\, \phi \,\middle|\, \psi \,\right\rangle_{\mathbb{1}+a^*a} \coloneqq \left\langle\, \phi \,\middle|\, \psi \,\right\rangle + \left\langle\, a^{\mathrm{cl}}(\phi) \,\middle|\, a^{\mathrm{cl}}(\psi) \,\right\rangle \tag{A.4.10}$$

is an inner product on $\mathcal{D}_{a^{\mathrm{cl}}}$ and $\langle\, \phi \,|\, \psi \,\rangle_{\mathbb{1}+a^*a} = \left\langle\, \phi \,\middle|\, \big(\mathbb{1} + a^*a\big)(\psi) \,\right\rangle$ for all $\phi, \psi \in \mathcal{D}$. Write $\|\cdot\|_{\mathbb{1}+a^*a}$ for the induced norm on $\mathcal{D}_{a^{\mathrm{cl}}}$, then the following is equivalent for every sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_{a^{\mathrm{cl}}}$:

i.) $(\phi_n)_{n\in\mathbb{N}}$ *and* $\big(a^{\mathrm{cl}}(\phi_n)\big)_{n\in\mathbb{N}}$ *are both Cauchy sequences with respect to the norm* $\|\cdot\|$ *of* $\mathfrak{H}$.

ii.) $\big(\phi_n, a^{\mathrm{cl}}(\phi_n)\big)$ *is a Cauchy sequence with respect to the norm* $\|\cdot\|$ *of the cartesian product* $\mathfrak{H} \times \mathfrak{H}$.

iii.) $(\phi_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence with respect to the norm* $\|\cdot\|_{\mathbb{1}+a^*a}$ *of* $\mathcal{D}_{a^{\mathrm{cl}}}$.

The linear subspace $\mathcal{D}_{a^{\mathrm{cl}}}$ can then be described explicitly as the set of the $\|\cdot\|$-limits of all those sequences in $\mathcal{D}$ which fulfil one (hence all) of the equivalent conditions above.

Moreover, $\mathcal{D}$ is dense in $\mathcal{D}_{a^{\mathrm{cl}}}$ with respect to the $\|\cdot\|_{\mathbb{1}+a^*a}$-topology and $\mathcal{D}_{a^{\mathrm{cl}}}$ is complete in the $\|\cdot\|_{\mathbb{1}+a^*a}$-topology. So $\mathcal{D}_{a^{\mathrm{cl}}}$ with inner product $\langle \,\cdot\, | \,\cdot\, \rangle_{\mathbb{1}+a^*a}$, is the completion of the pre-Hilbert space $\mathcal{D}$ with inner product $\langle \,\cdot\, | \,\cdot\, \rangle_{\mathbb{1}+a^*a}$ to a Hilbert space.

Finally, $a^{\mathrm{cl}} \colon \mathcal{D}_{a^{\mathrm{cl}}} \to \mathfrak{H}$ is continuous with respect to the $\|\cdot\|_{\mathbb{1}+a^*a}$-topology on $\mathcal{D}_{a^{\mathrm{cl}}}$ and the $\|\cdot\|$-topology on $\mathfrak{H}$, and thus is the continuous extension of $a \colon \mathcal{D} \to \mathcal{D} \subseteq \mathfrak{H}$ to the completion $\mathcal{D}_{a^{\mathrm{cl}}}$.

Proof: As $\left\langle\, \big(\phi, a^{\mathrm{cl}}(\phi)\big) \,\middle|\, \big(\phi, a^{\mathrm{cl}}(\phi)\big) \,\right\rangle = \langle\, \phi \,|\, \phi \,\rangle + \left\langle\, a^{\mathrm{cl}}(\phi) \,\middle|\, a^{\mathrm{cl}}(\phi) \,\right\rangle = \langle\, \phi \,|\, \phi \,\rangle_{\mathbb{1}+a^*a}$ holds for all $\phi \in \mathcal{D}_{a^{\mathrm{cl}}}$, ii.) and iii.) are equivalent. It is also clear that i.) and ii.) are equivalent.

Write $G_a \coloneqq \left\{ \big(\phi, a(\phi)\big) \;\middle|\; \phi \in \mathcal{D} \right\}$ for the graph of a. Then $\left\langle\!\left\langle\, G_a \,\right\rangle\!\right\rangle_{\mathrm{cl}}$ is the set of all limits of sequences $\big(\phi_n, a(\phi_n)\big)_{n\in\mathbb{N}}$ in $\mathfrak{H} \times \mathfrak{H}$, where $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$. As in this case $a(\phi_n) = a^{\mathrm{cl}}(\phi_n)$

holds for all $n \in \mathbb{N}$, it follows that $\operatorname{dom}\big(\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}\big)$ is the set of all $\|\cdot\|$-limits in $\mathfrak{H}$ of sequences $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ for which ii.) (or any of the other equivalent conditions) holds. This proves the explicit description of $\mathcal{D}_{a^{\mathrm{cl}}}$.

Now it follows immediately from iii.) that $\mathcal{D}$ is $\|\cdot\|_{\mathbb{1}+a^*a}$-dense in $\mathcal{D}_{a^{\mathrm{cl}}}$. Moreover, $\mathcal{D}_{a^{\mathrm{cl}}}$ is $\|\cdot\|_{\mathbb{1}+a^*a}$-complete because every Cauchy sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_{a^{\mathrm{cl}}}$ with respect to $\|\cdot\|_{\mathbb{1}+a^*a}$ yields a Cauchy sequence $\big(\phi_n, a^{\mathrm{cl}}(\phi_n)\big)$ in $\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}} \subseteq \mathfrak{H}\times\mathfrak{H}$ with respect to the Hilbert norm of the cartesian product $\mathfrak{H}\times\mathfrak{H}$, which thus converges against some $\big(\hat{\phi}, a^{\mathrm{cl}}(\hat{\phi})\big) \in \langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}$, so that $\hat{\phi} \in \operatorname{dom}\big(\langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}\big) = \mathcal{D}_{a^{\mathrm{cl}}}$ is the $\|\cdot\|_{\mathbb{1}+a^*a}$-limit of $(\phi_n)_{n\in\mathbb{N}}$.

Finally, continuity of $a^{\mathrm{cl}}\colon \mathcal{D}_{a^{\mathrm{cl}}} \to \mathfrak{H}$ with respect to the $\|\cdot\|_{\mathbb{1}+a^*a}$-topology on $\mathcal{D}_{a^{\mathrm{cl}}}$ and the $\|\cdot\|$-topology on $\mathfrak{H}$ is clear, and as the restriction of a^{cl} to $\mathcal{D}$ coincides with a, it is the continuous extension to the completion. □

Corollary A.4.23 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $a, b \in \mathcal{L}^*(\mathcal{D})$. If there exists a $\lambda \in [0,\infty[$ such that $\lambda\|\phi\|_{\mathbb{1}+a^*a} \geq \|\phi\|_{\mathbb{1}+b^*b}$ holds for all $\phi \in \mathcal{D}$, then $\mathcal{D}_{a^{\mathrm{cl}}} \subseteq \mathcal{D}_{b^{\mathrm{cl}}}$.*

PROOF: If such a $\lambda \in [0,\infty[$ exists, then every sequence in $\mathcal{D}$ which is a Cauchy sequence with respect to $\|\cdot\|_{\mathbb{1}+a^*a}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathbb{1}+b^*b}$, so $\mathcal{D}_{a^{\mathrm{cl}}} \subseteq \mathcal{D}_{b^{\mathrm{cl}}}$. □

A.4.2 Criteria for Essential Self-Adjointness

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$. In order to be able to fully make use of the properties of the operator theoretic adjoint, one would like the domain of $a^\dagger$ not only to extend the domain of $(a^*)^{\mathrm{cl}}$ like in Proposition A.4.20, but to be identical to it. Note that this also implies $(a^*)^{\mathrm{cl}} = a^\dagger$, hence is equivalent to $\langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}} = (G_a)^\dagger$ with $G_a := \big\{\, \big(\phi, a(\phi)\big) \;\big|\; \phi \in \mathcal{D} \,\big\}$ and $G_{a^*} := \big\{\, \big(\phi, a^*(\phi)\big) \;\big|\; \phi \in \mathcal{D} \,\big\}$ the graphs of a and a^* as before. This fact will be used a lot in the following, but without mentioning it explicitly.

In the general case, this property $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ does not seem to have been given a specific name in the literature, even though it comes up several times, e.g. in [74, Lemma 7.1.2 and Prop. 7.1.3], where it is proven that, if $a\,a^* = a^*a$ holds for an adjointable endomorphism on a pre-Hilbert space, then a^{cl} is normal in an operator theoretic sense if and only if $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$. If a is Hermitian, however, then this property is just essential self-adjointness of a:

Definition A.4.24 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace, and $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ a Hermitian endomorphism. Then a is called* essentially self-adjoint *if $\mathcal{D}_{a^\dagger} = \mathcal{D}_{a^{\mathrm{cl}}}$.*

However, it remains to find good sufficient conditions for an Hermitian endomorphism to be essentially self-adjoint.

Some General Results

Recall that a linear map $\Phi\colon V \to W$ between normed vector spaces is called *bounded* if

$$\|\Phi\| := \sup_{v\in V, \|v\|=1} \|\Phi(v)\| < \infty \tag{A.4.11}$$

holds, which is easily seen to be equivalent to continuity of Φ. Most of the following results are standard, see e.g. [75, Chap. 3].

Proposition A.4.25 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$ bounded, then also a^* is bounded with $\|a^*\| = \|a\|$, and $\mathcal{D}_{a^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger} = \mathfrak{H}$ hold. If a is in addition Hermitian, then a is essentially self-adjoint.*

PROOF: If a is bounded and $\psi \in \mathcal{D}$, let $\phi := a^*(\psi)/\|a^*(\psi)\|$, then

$$\big\|a^*(\psi)\big\| = \frac{\langle\, a^*(\psi) \,|\, a^*(\psi) \,\rangle}{\|a^*(\psi)\|} = \langle\, \phi \,|\, a^*(\psi) \,\rangle = \langle\, a(\phi) \,|\, \psi \,\rangle \leq \|a\|\|\phi\|\|\psi\| = \|a\|\|\psi\|$$

holds. So a^* is also bounded with $\|a^*\| \leq \|a\|$, hence $\|a^*\| = \|a\|$ as $\cdot^*$ is an involution.

Moreover, given $\hat{\phi} \in \mathfrak{H}$ then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging against $\hat{\phi}$, and the sequence $\big(a(\phi_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{H}$, hence converges, because a is continuous. This shows that $\hat{\phi} \in \mathcal{D}_{a^{\mathrm{cl}}}$ and we conclude that $\mathcal{D}_{a^{\mathrm{cl}}} = \mathfrak{H}$.

As a^* is also bounded, it follows that $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathfrak{H}$ as well, and as $\mathcal{D}_{a^\dagger} \supseteq \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ is always true due to Proposition A.4.20, $\mathcal{D}_{a^\dagger} = \mathfrak{H}$. □

Proposition A.4.26 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$ such that $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$. Then:*

- $\mathcal{D}_{(a^*)^\dagger} = \mathcal{D}_{a^{\mathrm{cl}}}$.
- $\mathcal{D}_{((\lambda a)^*)^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger} = \mathcal{D}_{(\lambda a)^\dagger}$ *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.
- $\mathcal{D}_{((a+b)^*)^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger} = \mathcal{D}_{(a+b)^\dagger}$ *for all bounded* $b \in \mathcal{L}^*(\mathcal{D})$.

PROOF: Write $G_a := \big\{\, \big(\phi, a(\phi)\big) \;\big|\; \phi \in \mathcal{D} \,\big\}$ and $G_{a^*} := \big\{\, \big(\phi, a^*(\phi)\big) \;\big|\; \phi \in \mathcal{D} \,\big\}$ for the graphs of a and a^*, respectively. Then $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$ means that $\langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}} = (G_a)^\dagger$, thus $(G_{a^*})^\dagger = \langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}}^\dagger = (G_a)^{\dagger\dagger} = \langle\!\langle\, G_a \,\rangle\!\rangle_{\mathrm{cl}}$ by the properties of the Galois connection $\cdot^\dagger$, i.e. $(a^*)^\dagger = a^{\mathrm{cl}}$.

Given $\lambda \in \mathbb{C}\backslash\{0\}$, as well as a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ that converges in $\mathfrak{H}$, then $\big(a^*(\phi_n)\big)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$ if and only if $\big(\overline{\lambda} a^*(\phi_n)\big)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$, which shows that $\mathcal{D}_{((\lambda a)^*)^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ due to Proposition A.4.22. Similarly, $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(\lambda a)^\dagger}$ by Proposition A.4.21, because given $\psi \in \mathfrak{H}$, then $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, a(\phi) \,\rangle \in \mathbb{C}$ is continuous if and only if $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, \lambda a(\phi) \,\rangle \in \mathbb{C}$ is continuous. As $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$ by assumption, all these domains coincide.

Finally, let a bounded $b \in \mathcal{L}^*(\mathcal{D})$ be given, then again $\mathcal{D}_{((a+b)^*)^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$, because a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ that converges in $\mathfrak{H}$, has the property that $\big(a^*(\phi_n)\big)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$ if and only if $\big((a+b)^*(\phi_n)\big)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$, because $\big(b^*(\phi_n)\big)_{n\in\mathbb{N}}$ is always convergent as b^* is bounded by the previous Proposition A.4.25 (hence is continuous). Similarly, $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a+b)^\dagger}$, because given $\psi \in \mathfrak{H}$, then $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, a(\phi) \,\rangle \in \mathbb{C}$ is continuous if and only if $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, (a+b)(\phi) \,\rangle \in \mathbb{C}$ is continuous due to the observation that $\mathcal{D} \ni \phi \mapsto \langle\, \psi \,|\, b(\phi) \,\rangle \in \mathbb{C}$ is always continuous. □

We have already seen that a bounded adjointable endomorphism $a \in \mathcal{L}^*(\mathcal{D})$ always fulfils $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$. A very similar result is:

Proposition A.4.27 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$. Assume that there exists an $\epsilon > 0$ such that $\|a(\phi)\| \geq \epsilon\|\phi\|$ and $\|a^*(\phi)\| \geq \epsilon\|\phi\|$ hold for all $\phi \in \mathcal{D}$, then $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$ if and only if the images of a and of a^* in $\mathcal{D}$ are dense. In this case $(a^*)^{\mathrm{cl}}$ is a linear isomorphism from $\mathcal{D}_{(a^*)^{\mathrm{cl}}}$ to $\mathfrak{H}$.*

PROOF: The assumption $\|a(\phi)\| \geq \epsilon\|\phi\|$ for all $\phi \in \mathcal{D}$ implies that, whenever $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$ for which $\big(a(\phi_n)\big)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$ (hence is a Cauchy sequence), then $(\phi_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathfrak{H}$ and thus converges, and $\lim_{n\to\infty}\|\phi_n\| \leq \lim_{n\to\infty}\|a(\phi_n)\|/\epsilon$ (analogously for a^*). As a special case, this shows that a^{cl} and $(a^*)^{\mathrm{cl}}$ are injective: Given any $\hat\phi \in \mathcal{D}_{a^{\mathrm{cl}}}$ with $a^{\mathrm{cl}}(\hat\phi) = 0$, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging against $\hat\phi$ and for which $\big(a(\phi_n)\big)_{n\in\mathbb{N}}$ converges against 0, hence $(\phi_n)_{n\in\mathbb{N}}$ converges against 0 as well, i.e. $\hat\phi = 0$. Analogously for a^*.

Now write $G_a := \big\{ \big(\phi, a(\phi)\big) \;\big|\; \phi \in \mathcal{D} \big\}$ and $G_{a^*} := \big\{ \big(\phi, a^*(\phi)\big) \;\big|\; \phi \in \mathcal{D} \big\}$ for the graphs of a and a^*, respectively. If $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$, then $\langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}} = (G_a)^\dagger$, and from Proposition A.4.13 it follows that $\mathrm{img}(G_a)^\perp = \ker\big((G_a)^\dagger\big) = \ker\big(\langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}}\big) = \{0\}$, hence $\langle\!\langle\, \mathrm{img}(G_a) \,\rangle\!\rangle_{\mathrm{cl}} = \mathfrak{H}$ and a has dense image. By the previous Proposition A.4.26, $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$ also implies $\mathcal{D}_{a^{\mathrm{cl}}} = \mathcal{D}_{(a^*)^\dagger}$, hence the same argument applies to a^* as well and thus a^* has dense image.

Conversely, assume that a^* has dense image. Given $\xi \in \mathfrak{H}$, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $a^*(\psi_n)$ converges against ξ. Then also $(\psi_n)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$ against some $\hat\psi \in \mathfrak{H}$, and even $\hat\psi \in \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ and $(a^*)^{\mathrm{cl}}(\hat\psi) = \xi$ by construction. So $(a^*)^{\mathrm{cl}}$ is surjective and as $a^\dagger$ extends $(a^*)^{\mathrm{cl}}$, this implies that $a^\dagger$ is surjective as well. Moreover, if a has dense image as well, then $a^\dagger$ is injective by Proposition A.4.13, so $a^\dagger$ is a linear isomorphism from $\mathcal{D}_{a^\dagger}$ to $\mathfrak{H}$. But this also shows that $\mathcal{D}_{a^\dagger} \subseteq \mathcal{D}_{(a^*)^{\mathrm{cl}}}$, because $(a^*)^{\mathrm{cl}}$ was already surjective. Thus $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$. □

Corollary A.4.28 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$ coercive, i.e. there exists an $\epsilon > 0$ such that $\langle\, \phi \,|\, a(\phi) \,\rangle \geq \epsilon\langle\, \phi \,|\, \phi \,\rangle$ holds for all $\phi \in \mathcal{D}$. Then a is essentially self-adjoint if and only if the image of a in $\mathcal{D}$ is dense. In this case, a^{cl} is a linear isomorphism from $\mathcal{D}_{a^{\mathrm{cl}}}$ to $\mathfrak{H}$.*

PROOF: In order to apply the previous Proposition A.4.27 it is sufficient to show that $\|a(\phi)\| \geq \epsilon\|\phi\|$ holds for all $\phi \in \mathcal{D}$, and this is true because

$$\|a(\phi)\|^2 = \langle\, \phi \,|\, a^2(\phi) \,\rangle = \langle\, \phi \,|\, \big(a - \epsilon\mathbb{1}\big)^2(\phi) \,\rangle + 2\epsilon\langle\, \phi \,|\, a(\phi) \,\rangle - \epsilon^2\langle\, \phi \,|\, \phi \,\rangle \geq \|a(\phi) - \epsilon\phi\|^2 + \epsilon^2\langle\, \phi \,|\, \phi \,\rangle \geq \epsilon^2\|\phi\|^2 \,.$$

□

Corollary A.4.29 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$, then a is essentially self-adjoint if and only if $a + \mathrm{i}\mathbb{1}$ and $a - \mathrm{i}\mathbb{1}$ have dense image.*

PROOF: By definitition, a is essentially self-adjoint if and only if $\mathcal{D}_{a^\dagger} = \mathcal{D}_{a^{\mathrm{cl}}}$. Proposition A.4.26 shows that this is equivalent to $\mathcal{D}_{(a+\mathrm{i}\mathbb{1})^\dagger} = \mathcal{D}_{((a+\mathrm{i}\mathbb{1})^*)^{\mathrm{cl}}}$ and $\mathcal{D}_{(a-\mathrm{i}\mathbb{1})^\dagger} = \mathcal{D}_{((a-\mathrm{i}\mathbb{1})^*)^{\mathrm{cl}}}$ as $\pm\mathrm{i}\mathbb{1}$ is bounded. Moreover, as

$$\|(a \pm \mathrm{i}\mathbb{1})(\phi)\|^2 = \langle\, a(\phi) \,|\, a(\phi) \,\rangle + \langle\, \phi \,|\, \phi \,\rangle \geq \|\phi\|^2$$

for all $\phi \in \mathcal{D}$, Proposition A.4.27 applies and shows that $\mathcal{D}_{(a+\mathrm{i}\mathbb{1})^\dagger} = \mathcal{D}_{((a+\mathrm{i}\mathbb{1})^*)^{\mathrm{cl}}}$ and $\mathcal{D}_{(a-\mathrm{i}\mathbb{1})^\dagger} = \mathcal{D}_{((a-\mathrm{i}\mathbb{1})^*)^{\mathrm{cl}}}$ hold if and only if $a + \mathrm{i}\mathbb{1}$ and $a - \mathrm{i}\mathbb{1}$ have dense image. □

The last result so far is due to [58, Lemma 2.3], see also [74, Lemma 7.1.2].

Proposition A.4.30 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$. If $\mathbb{1} + aa^*$ is essentially self-adjoint, then $\mathcal{D}_{(a^*)^{\mathrm{cl}}} = \mathcal{D}_{a^\dagger}$.*

PROOF: If $\mathbb{1} + aa^*$ is essentially self-adjoint, then it has dense image by Corollary A.4.28. It is to show that this implies that $\mathcal{D}_{(a^*)^{\mathrm{cl}}} \supseteq \mathcal{D}_{a^\dagger}$.

Let $\phi \in \mathcal{D}_{a^\dagger}$ be given and write $G_a := \{ (\phi, a(\phi) \mid \phi \in \mathcal{D} \}$ as well as $G_{a^*} := \{ (\phi, a^*(\phi) \mid \phi \in \mathcal{D} \}$ for the graphs of a and a^*, respectively, then $\big(\phi, a^\dagger(\phi)\big) \in \mathfrak{H} \times \mathfrak{H}$ can be decomposed into $\big(\phi, a^\dagger(\phi)\big) = (\phi_\perp, \psi_\perp) + (\phi_\parallel, \psi_\parallel)$ with unique $(\phi_\perp, \psi_\perp) \in (G_{a^*})^\perp$ and $(\phi_\parallel, \psi_\parallel) \in \langle\!\langle\, G_{a^*} \,\rangle\!\rangle_{\mathrm{cl}}$ by Proposition A.1.25. So $\phi_\parallel \in \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ and $\psi_\parallel = (a^*)^{\mathrm{cl}}(\phi_\parallel)$, and especially $\phi_\parallel \in \mathcal{D}_{a^\dagger}$ and $\psi_\parallel = a^\dagger(\phi_\parallel)$, as $a^\dagger$ extends $(a^*)^{\mathrm{cl}}$. Consequently also $\phi_\perp = \phi - \phi_\parallel \in \mathcal{D}_{a^\dagger}$ and $\psi_\perp = a^\dagger(\phi) - a^\dagger(\phi_\parallel) = a^\dagger(\phi_\perp)$. However, as $\big(\phi_\perp, a^\dagger(\phi_\perp)\big) \in (G_{a^*})^\perp$ by construction, it follows that $0 = \langle\, \phi_\perp \,|\, \xi \,\rangle + \langle\, a^\dagger(\phi_\perp) \,|\, a^*(\xi) \,\rangle = \big\langle\, \phi_\perp \,\big|\, \big(\mathbb{1} + aa^*\big)(\xi) \,\big\rangle$ holds for all $\xi \in \mathcal{D}$, thus $\phi_\perp = 0$ and $\phi = \phi_\parallel \in \mathcal{D}_{(a^*)^{\mathrm{cl}}}$. □

Jacobi Matrices

The aim of this section is to prove a sufficient criterium for essential self-adjointness of special coercive Hermitian endomorphism $J_{\alpha,\beta}$ of

$$\mathbb{C}^{(\mathbb{N}_0)} := \big\{ x \in \mathbb{C}^{\mathbb{N}_0} \;\big|\; x_n = 0 \text{ for all but finitely many } n \in \mathbb{N}_0 \big\}, \tag{A.4.12}$$

with the standard inner product

$$\langle\, x \,|\, y \,\rangle := \sum_{n=0}^{\infty} \overline{x_n} y_n \tag{A.4.13}$$

for all $x, y \in \mathbb{C}^{(\mathbb{N}_0)}$. This can then also be applied to more general cases. The results presented here are classic, but usually not part of standard textbooks on unbounded operators. See e.g. [2] for more details.

Definition A.4.31 *A* real Jacobi matrix *is a linear endomorphism $J_{\alpha,\beta} \colon \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ described for all $x \in \mathbb{C}^{\mathbb{N}_0}$ by*

$$\big(J_{\alpha,\beta}(x)\big)_0 = \alpha_0 x_0 + \beta_0 x_1 \tag{A.4.14}$$

$$\text{and} \qquad \big(J_{\alpha,\beta}(x)\big)_n = \beta_{n-1} x_{n-1} + \alpha_n x_n + \beta_n x_{n+1} \qquad \text{for all } n \in \mathbb{N}, \tag{A.4.15}$$

where $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ are two sequences of real numbers.

So, heuristically, a real Jacobi matrix is the linear map described by the infinite matrix

$$J_{\alpha,\beta} = \begin{pmatrix} \alpha_0 & \beta_0 & & \\ \beta_0 & \alpha_1 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Of course, every such real Jacobi matrix restricts to a Hermitian endomorphism of the pre-Hilbert space $\mathbb{C}^{(\mathbb{N}_0)}$, and one can try to understand under which condition this Hermitian endomorphism is essentially self-adjoint. In the following, only very special real Jacobi matrices will be of interest:

Lemma A.4.32 *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ be two sequences, then the restriction of $J_{\alpha,\beta}$ to the pre-Hilbert space $\mathbb{C}^{(\mathbb{N}_0)}$ is a positive Hermitian endomorphism if and only if*

$$\sum_{n=0}^{N} \big(\alpha_n\, x_n^2 + 2\beta_n x_n x_{n+1}\big) \geq 0 \tag{A.4.16}$$

holds for all $N \in \mathbb{N}_0$ and all $x_0, \dots, x_N \in \mathbb{R}$.

PROOF: For all $x \in \mathbb{C}^{(\mathbb{N}_0)}$, the identity

$$\begin{aligned}
\langle\, x \,|\, J_{\alpha,\beta}(x)\,\rangle &= \alpha_0 |x_0|^2 + \beta_0 \overline{x_0} x_1 + \sum_{n=1}^{\infty} \big(\beta_{n-1}\overline{x_n} x_{n-1} + \alpha_n |x_n|^2 + \beta_n \overline{x_n} x_{n+1}\big) \\
&= \sum_{n=0}^{\infty} \big(\alpha_n |x_n|^2 + \beta_n(\overline{x_n} x_{n+1} + \overline{x_{n+1}} x_n)\big) \\
&= \sum_{n=0}^{\infty} \big(\alpha_n \operatorname{Re}(x_n)^2 + 2\beta_n \operatorname{Re}(x_n)\operatorname{Re}(x_{n+1})\big) + \sum_{n=0}^{\infty} \big(\alpha_n \operatorname{Im}(x_n)^2 + 2\beta_n \operatorname{Im}(x_n)\operatorname{Im}(x_{n+1})\big)
\end{aligned}$$

holds. So (A.4.16) implies that $\langle\, x \,|\, J_{\alpha,\beta}(x)\,\rangle \geq 0$ for all $x \in \mathbb{C}^{(\mathbb{N}_0)}$, and convesely, if $\langle\, x \,|\, J_{\alpha,\beta}(x)\,\rangle \geq 0$ for all $x \in \mathbb{C}^{(\mathbb{N}_0)}$, then (A.4.16) holds by extending $x_0, \dots, x_N \in \mathbb{R}$ to a sequence $x \in \mathbb{C}^{(\mathbb{N}_0)}$ with $x_n = 0$ for $n \in \mathbb{N}_0$, $n > N$. □

For simplicity, a real Jacobi matrix $J_{\alpha,\beta}$ will be called positive if its restriction to the pre-Hilbert space $\mathbb{C}^{(\mathbb{N}_0)}$ is positive, and coercive if this restriction is coercive (defined like in Corollary A.4.28).

By Corollary A.4.28, the restriction of a coercive real Jacobi matrix to the pre-Hilbert space $\mathbb{C}^{(\mathbb{N}_0)}$ is essentially self-adjoint if and only if its image is dense in $\mathbb{C}^{(\mathbb{N}_0)}$, or equivalently, dense in the completion of $\mathbb{C}^{(\mathbb{N}_0)}$ to a Hilbert space, which is of course

$$\ell^2(\mathbb{N}_0) = \Big\{\, x \in \mathbb{C}^{\mathbb{N}_0} \;\Big|\; \textstyle\sum_{n=0}^{\infty} |x_n|^2 < \infty \,\Big\} \subseteq \mathbb{C}^{\mathbb{N}_0}\,. \tag{A.4.17}$$

As a linear subspace of $\ell^2(\mathbb{N}_0)$ is dense if and only if the orthogonal complement of its image is $\{0\}$ (this is due to Proposition A.3.13), we are led to:

Proposition A.4.33 *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ be two sequences, then the image of $\mathbb{C}^{(\mathbb{N}_0)}$ under $J_{\alpha,\beta}$ is dense in $\ell^2(\mathbb{N}_0)$ if and only if the only $x \in \ell^2(\mathbb{N}_0)$ with $J_{\alpha,\beta}(x) = 0$ is $x = 0$, i.e. if and only if the restriction of $J_{\alpha,\beta}$ to a map from $\ell^2(\mathbb{N}_0)$ to $\mathbb{C}^{\mathbb{N}_0}$ is injective.*

PROOF: As mentioned above, the image of $\mathbb{C}^{(\mathbb{N}_0)}$ under $J_{\alpha,\beta}$ is dense in $\ell^2(\mathbb{N}_0)$ if and only if the only $x \in \ell^2(\mathbb{N}_0)$ fulfilling $\langle\, x \,|\, J_{\alpha,\beta}(y)\,\rangle = 0$ for all $y \in \mathbb{C}^{(\mathbb{N}_0)}$ is $x = 0$. Now observe that

$$\langle\, x \,|\, J_{\alpha,\beta}(y)\,\rangle = \alpha_0 \overline{x_0} y_0 + \beta_0 \overline{x_0} y_1 + \sum_{n=1}^{\infty} \big(\beta_{n-1}\overline{x_n} y_{n-1} + \alpha_n \overline{x_n} y_n + \beta_n \overline{x_n} y_{n+1}\big)$$

$$
\begin{aligned}
&= \alpha_0 \overline{x_0} y_0 + \beta_0 \overline{x_1} y_0 + \sum_{n=1}^{\infty} \big(\beta_n \overline{x_{n+1}} y_n + \alpha_n \overline{x_n} y_n + \beta_{n-1} \overline{x_{n-1}} y_n \big) \\
&= \sum_{n=0}^{\infty} \overline{\big(J_{\alpha,\beta}(x)\big)_n}\, y_n
\end{aligned}
$$

for all $x \in \ell^2(\mathbb{N}_0)$ and all $y \in \mathbb{C}^{(\mathbb{N}_0)}$, so $\langle\, x \,|\, J_{\alpha,\beta}(y) \,\rangle = 0$ for one fixed $x \in \ell^2(\mathbb{N}_0)$ and all $y \in \mathbb{C}^{(\mathbb{N}_0)}$ if and only if $J_{\alpha,\beta}(x) = 0$. □

While this result is very similar to the second part of Proposition A.4.13 by using that $J_{\alpha,\beta}$ is Hermitian, it does not make use of any operator theoretic adjoint, but only of the observation that $J_{\alpha,\beta}$ also describes a linear function from $\ell^2(\mathbb{N}_0)$ to $\mathbb{C}^{\mathbb{N}_0}$!

In many cases, the kernel of a real Jacobi matrix $J_{\alpha,\beta}$ (as endomorphism of $\mathbb{C}^{\mathbb{N}_0}$) can be described explicitly and is usually larger than $\{0\}$. The more difficult question is which of these elements are actually in $\ell^2(\mathbb{N}_0)$.

Lemma A.4.34 *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ be two sequences with $\beta_n \neq 0$ for all $n \in \mathbb{N}_0$ and $D \in \mathbb{R}^{\mathbb{N}_0}$ a sequence fulfilling*

$$
D_{n+1} = \alpha_n D_n - \beta_{n-1}^2 D_{n-1}\,, \tag{A.4.18}
$$

for all $n \in \mathbb{N}$, then the sequence $x \in \mathbb{R}^{\mathbb{N}_0}$, defined as

$$
x_n \coloneqq (-1)^n D_n \prod_{m=0}^{n-1} \frac{1}{\beta_m} \tag{A.4.19}
$$

for all $n \in \mathbb{N}_0$ fulfils

$$
x_{n+1} = -\big(\beta_{n-1} x_{n-1} + \alpha_n x_n\big)/\beta_n\,, \tag{A.4.20}
$$

i.e. $\big(J_{\alpha,\beta}(x)\big)_n = 0$, for all $n \in \mathbb{N}$.

PROOF: Combining (A.4.18) and (A.4.19) yields

$$
x_{n+1} = (-1)^{n+1} \big(\alpha_n D_n - \beta_{n-1}^2 D_{n-1}\big) \prod_{m=0}^{n} \frac{1}{\beta_m} = -\big(\alpha_n x_n + \beta_{n-1} x_{n-1}\big) \frac{1}{\beta_n}
$$

for all $n \in \mathbb{N}$, and comparison with the definition of $J_{\alpha,\beta}$ shows immediately that this is equivalent to $\big(J_{\alpha,\beta}(x)\big)_n = 0$ for all $n \in \mathbb{N}$. □

Note that this does not imply that $J_{\alpha,\beta}(x) = 0$ holds for all sequences $x \in \mathbb{C}^{\mathbb{N}_0}$ constructed like in the above Lemma A.4.34, as $\big(J_{\alpha,\beta}(x)\big)_0 = 0$ need not be fulfilled.

Proposition A.4.35 *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ be two sequences with $\beta_n \neq 0$ for all $n \in \mathbb{N}_0$. Then the kernel of $J_{\alpha,\beta}$, i.e. $\big\{\, x \in \mathbb{C}^{\mathbb{N}_0} \,\big|\, J_{\alpha,\beta}(x) = 0 \,\big\}$, is a one-dimensional linear subspace of $\mathbb{C}^{\mathbb{N}_0}$, which is generated by the sequence $\hat{x} \in \mathbb{C}^{\mathbb{N}_0}$ defined as*

$$
\hat{x}_n \coloneqq (-1)^n \det(J_{\alpha,\beta;n}) \prod_{m=0}^{n-1} \frac{1}{\beta_m} \tag{A.4.21}
$$

for all $n \in \mathbb{N}_0$, where $J_{\alpha,\beta;n} \in \mathbb{C}^{n\times n}$ is the matrix of the first n rows and columns of $J_{\alpha,\beta}$, i.e.

$$J_{\alpha,\beta;n} = \begin{pmatrix} \alpha_0 & \beta_0 & & \\ \beta_0 & \alpha_1 & \ddots & \\ & \ddots & \ddots & \beta_{n-2} \\ & & \beta_{n-2} & \alpha_{n-1} \end{pmatrix}, \tag{A.4.22}$$

and for $n = 0$ it is understood that the determinant of the empty 0×0-matrix is $\det(J_{\alpha,\beta,0}) = 1$.

PROOF: From Definition A.4.31 it follows immediately that a sequence $x \in \mathbb{C}^{\mathbb{N}_0}$ fulfils $J_{\alpha,\beta}(x) = 0$ if and only if

$$\begin{aligned} & x_1 = -\alpha_0 x_0/\beta_0 \\ \text{and} \qquad & x_{n+1} = -\big(\beta_{n-1}x_{n-1} + \alpha_n x_n\big)/\beta_n \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

hold, so the solutions of $J_{\alpha,\beta}(x) = 0$ are the one-dimensional linear subspace of $\mathbb{C}^{\mathbb{N}_0}$ consisting of all those sequences $x \in \mathbb{C}^{\mathbb{N}_0}$ fulfilling this linear recursion formula for an arbitrary initial value of x_0, and this solution space is spanned by the sequence $\hat{x} \in \mathbb{C}^{\mathbb{N}_0}$ that fulfils this with initial value $\hat{x}_0 = 1$.

It only remains to show that the formula (A.4.21) is correct: For $n = 0$ one sees that this formula indeed yields $\hat{x}_0 = 1$, and for $n = 1$ it again gives the correct result $\hat{x}_1 := -\alpha_0/\beta_0$. For larger n it follows from Laplace's formula that $\det(J_{\alpha,\beta;n+1}) = \alpha_n \det(J_{\alpha,\beta;n}) - \beta_{n-1}^2 \det(J_{\alpha,\beta;n-1})$ holds for all $n \in \mathbb{N}$ so that the previous Lemma A.4.34 can be applied. □

Proposition A.4.36 *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ be two sequences with $\beta_n \neq 0$ for all $n \in \mathbb{N}_0$. If $x, y \in \mathbb{C}^{\mathbb{N}_0}$ fulfil*

$$\big(J_{\alpha,\beta}(x)\big)_n = 0 = \big(J_{\alpha,\beta}(y)\big)_n \tag{A.4.23}$$

for all $n \in \mathbb{N}$ (but not necessarily for $n = 0$), then the complex number

$$W(x, y) := \beta_{n-1}(x_n\, y_{n-1} - x_{n-1}\, y_n) \tag{A.4.24}$$

is independent of $n \in \mathbb{N}$ (this number is called the Wronskian of the sequences x and y).

PROOF: This is an immediate consequence of the definition of $J_{\alpha,\beta}$: Given $n \in \mathbb{N}\backslash\{1\}$, then

$$\begin{aligned} \beta_{n-1}(x_n y_{n-1} - x_{n-1} y_n) &= -(x_{n-2}\beta_{n-2} + x_{n-1}\alpha_{n-1})y_{n-1} + x_{n-1}(y_{n-2}\beta_{n-2} + y_{n-1}\alpha_{n-1}) \\ &= \beta_{n-2}(-x_{n-2}y_{n-1} + x_{n-1}y_{n-2}) . \end{aligned}$$

By induction it follows that $\beta_{n-1}(x_n y_{n-1} - x_{n-1} y_n) = \beta_0(x_1 y_0 - x_0 y_1)$ for all $n \in \mathbb{N}\backslash\{0\}$. □

The following lemma and theorem are due to [85], see also [2, Chap. 1, Prob. 4].

Lemma A.4.37 *Let $J_{\alpha,\beta}$ be a real Jacobi matrix with $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ and assume that $\beta_n > 0$ for all $n \in \mathbb{N}_0$ and that $J_{\alpha',\beta}$ is positive, where the sequence $\alpha' \in \mathbb{R}^{\mathbb{N}_0}$ is defined as $\alpha'_n := \alpha_n - 1$ for all*

$n \in \mathbb{N}_0$. *Then the sequence* $\hat{x} \in \mathbb{C}^{\mathbb{N}_0}$ *from Proposition A.4.35 that spans the kernel of* $J_{\alpha,\beta}$ *fulfils the estimate*

$$\sum_{n=0}^{N-1} \frac{1}{\sqrt{\beta_n}} \leq \left(\sum_{n=0}^{N} |\hat{x}_n|^2 \right)^{\frac{1}{2}} \tag{A.4.25}$$

for all $N \in \mathbb{N}$.

PROOF: As $J_{\alpha',\beta}$ is positive, it follows that $\langle x \,|\, J_{\alpha,\beta}(x) \rangle = \langle x \,|\, J_{\alpha',\beta}(x) \rangle + \langle x \,|\, x \rangle \geq 0$ for all $x \in \mathbb{C}^{(\mathbb{N}_0)}$, i.e. $J_{\alpha,\beta}$ is positive as well and the matrices $J_{\alpha,\beta,n}$ are positive-definite matrices. From the definition of the sequence $\hat{x}$ in (A.4.21) it thus follows that $\hat{x}_{2n} > 0$ and $\hat{x}_{2n+1} < 0$ hold for all $n \in \mathbb{N}_0$. Now define another sequence $\hat{y} \in \mathbb{R}^{\mathbb{N}_0}$ by

$$\hat{y}_n \coloneqq (-1)^n \det(K_{\alpha,\beta;n}) \prod_{m=0}^{n-1} \frac{1}{\beta_m}$$

for all $n \in \mathbb{N}_0$, where $K_{\alpha,\beta;0} \coloneqq 1 \in \mathbb{R}^{1\times 1}$ and

$$K_{\alpha,\beta;n} \coloneqq \left(\begin{array}{c|cccc} 1 & 1 & 0 & \dots & 0 \\ \hline 1 & & & & \\ 0 & & & & \\ \vdots & & J_{\alpha,\beta;n} & & \\ 0 & & & & \end{array} \right)$$

for $n \in \mathbb{N}$. Then Laplace's formula for $\det(K_{\alpha,\beta;n})$ shows that $\det(K_{\alpha,\beta;n+1}) = \alpha_n \det(K_{\alpha,\beta;n}) - \beta_{n-1}^2 \det(K_{\alpha,\beta;n-1})$ holds for all $n \in \mathbb{N}$, and so it follows from Lemma A.4.34 that $\big(J_{\alpha,\beta}(\hat{y})\big)_n = 0$ for all $n \in \mathbb{N}$. As $\hat{x}$ also fulfils $\big(J_{\alpha,\beta}(\hat{x})\big)_n = 0$ for all $n \in \mathbb{N}$, the previous Proposition A.4.36 can be applied and shows that

$$\beta_{n-1}(\hat{x}_n\hat{y}_{n-1} - \hat{x}_{n-1}\hat{y}_n) = \beta_0(\hat{x}_1\hat{y}_0 - \hat{x}_0\hat{y}_1) = \beta_0(\hat{x}_1 - \hat{y}_1) = -1$$

for all $n \in \mathbb{N}$.

Moreover, the matrix $K_{\alpha,\beta;n}$ is positive semi-definite for all $n \in \mathbb{N}_0$, because this is clear for $n = 0$ and because

$$\langle z \,|\, K_{\alpha,\beta;n}(z) \rangle = z_0^2 + 2z_0z_1 + \big\langle \tilde{z} \,\big|\, J_{\alpha,\beta,n-1}(\tilde{z}) \big\rangle = (z_0 + z_1)^2 + \sum_{m=2}^{n} z_m^2 + \big\langle \tilde{z} \,\big|\, J_{\alpha',\beta,n-1}(\tilde{z}) \big\rangle \geq 0$$

holds for all $n \in \mathbb{N}$ and all $z_0, \dots, z_n \in \mathbb{R}$ with $\tilde{z} = (z_1, \dots, z_n)^T \in \mathbb{R}^n$. So $\det(K_{\alpha,\beta;n}) \geq 0$ and consequently $\hat{y}_{2n} \geq 0$ and $\hat{y}_{2n+1} \leq 0$ for all $n \in \mathbb{N}_0$.

With the help of the sequence $\hat{y}$ it is now easy to prove the estimate of the claim: First of all one notes that $|\hat{x}_n\hat{x}_{n-1}| = -\hat{x}_n\hat{x}_{n-1}$ for all $n \in \mathbb{N}$ due to the alternating sign of the $\hat{x}_n$ and derives the identity

$$\sum_{n=1}^{N} \frac{1}{\beta_{n-1}|\hat{x}_n \hat{x}_{n-1}|} = \sum_{n=1}^{N} \frac{-1}{\beta_{n-1}\hat{x}_n \hat{x}_{n-1}} = \sum_{n=1}^{N} \left(\frac{\hat{y}_{n-1}}{\hat{x}_{n-1}} - \frac{\hat{y}_n}{\hat{x}_n} \right) = \frac{\hat{y}_0}{\hat{x}_0} - \frac{\hat{y}_N}{\hat{x}_N} = 1 - \frac{\hat{y}_N}{\hat{x}_N}$$

for all $N \in \mathbb{N}$. As $\hat{x}_N$ and $\hat{y}_N$ always have the same sign, this yields the estimate

$$0 \leq \sum_{n=1}^{N} \frac{1}{\beta_{n-1}|\hat{x}_n \hat{x}_{n-1}|} = 1 - \frac{\hat{y}_N}{\hat{x}_N} \leq 1$$

for all $N \in \mathbb{N}$. Finally, this leads to

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{\sqrt{\beta_{n-1}}} &= \sum_{n=1}^{N} \frac{1}{\sqrt{\beta_{n-1}|\hat{x}_n \hat{x}_{n-1}|}} \sqrt{|\hat{x}_n \hat{x}_{n-1}|} \\
&\overset{\text{CS}}{\leq} \left(\sum_{n=1}^{N} \frac{1}{\beta_{n-1}\,|\hat{x}_n \hat{x}_{n-1}|} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |\hat{x}_n \hat{x}_{n-1}| \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=1}^{N} |\hat{x}_n \hat{x}_{n-1}| \right)^{\frac{1}{2}} \\
&\overset{\text{CS}}{\leq} \left(\sum_{n=1}^{N} |\hat{x}_n|^2 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{N} |\hat{x}_{n-1}|^2 \right)^{\frac{1}{4}} \\
&\leq \left(\sum_{n=0}^{N} |\hat{x}_n|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

for all $N \in \mathbb{N}$. □

An immediate consequence of this is a sufficient criterium for some real Jacobi matrices to restrict to essentially self-adjoint endomorphism of $\mathbb{C}^{(\mathbb{N}_0)}$:

Theorem A.4.38 *Let $J_{\alpha,\beta}$ be a real Jacobi matrix with $\alpha, \beta \in \mathbb{R}^{\mathbb{N}_0}$ and assume that $\beta_n > 0$ for all $n \in \mathbb{N}_0$ and that $J_{\alpha',\beta}$ is positive, where the sequence $\alpha' \in \mathbb{R}^{\mathbb{N}_0}$ is defined as $\alpha'_n := \alpha_n - 1$ for all $n \in \mathbb{N}_0$. Then the restriction of $J_{\alpha,\beta}$ to a Hermitian endomorphism of the pre-Hilbert space $\mathbb{C}^{(\mathbb{N}_0)}$ is positive and even coercive. If $\sum_{n=0}^{\infty} (\beta_n)^{-1/2} = \infty$, then this Hermitian endomorphism is even essentially self-adjoint and the image of $\mathbb{C}^{(\mathbb{N}_0)}$ under $J_{\alpha,\beta}$ is dense in $\ell^2(\mathbb{N}_0)$.*

PROOF: By construction, $J_{\alpha,\beta}$ restricts to a Hermitian endomorphism of $\mathbb{C}^{(\mathbb{N}_0)}$ fulfilling the identity $\langle\, x \,|\, J_{\alpha,\beta}(x) \,\rangle = \langle\, x \,|\, J_{\alpha',\beta}(x) \,\rangle + \|x\|^2$ for all $x \in \mathbb{C}^{(\mathbb{N}_0)}$, hence is positive and even coercive.

If $\sum_{n=0}^{\infty} (\beta_n)^{-1/2} = \infty$, then the previous Lemma A.4.37 implies that $\hat{x} \in \mathbb{C}^{\mathbb{N}_0}$, the element that generates the kernel of $J_{\alpha,\beta}$ by Proposition A.4.35, is not an element of $\ell^2(\mathbb{N}_0)$. For this case, Proposition A.4.33 shows that the image of $\mathbb{C}^{(\mathbb{N}_0)}$ under $J_{\alpha,\beta}$ is dense in $\ell^2(\mathbb{N}_0)$, hence the restriction of $J_{\alpha,\beta}$ to $\mathbb{C}^{(\mathbb{N}_0)}$ is essentially self-adjoint by Corollary A.4.28. □

Stieltjes Vectors

It should not be surprising that the previous Theorem A.4.38 leads to a sufficient criterium for essential self-adjointness of arbitrary coercive Hermitian endomorphisms of pre-Hilbert spaces by constructing a suitable basis. This will lead to the notion of Stieltjes vectors, which are vectors that fulfil a certain growth condition:

Lemma A.4.39 *Let $\mathcal{D}$ be a pre-Hilbert space, $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ and $\phi \in \mathcal{D}$, then $\|a^n(\phi)\| = 0$ for one $n \in \mathbb{N}$ implies $\|a^n(\phi)\| = 0$ for all $n \in \mathbb{N}$. Otherwise, $\|a^n(\phi)\| > 0$ and*

$$\frac{\|a^n(\phi)\|}{\|a^{n-1}(\phi)\|} \le \frac{\|a^{n+1}(\phi)\|}{\|a^n(\phi)\|}$$
$$\text{as well as} \qquad \|a^n(\phi)\|^{\frac{1}{n}} \le \|a^{n+1}(\phi)\|^{\frac{1}{n+1}}$$

hold for all $n \in \mathbb{N}$.

PROOF: This is just an application of Lemma 3.1.17 on the growth of powers of positive elements of a quasi-ordered *-algebra on positive linear functionals: Consider the positive element $b := a^2 = a^*a$ in the ordered *-algebra $\mathcal{L}^*(\mathcal{D})$ as well as the vector functional $\chi_\phi \colon \mathcal{L}^*(\mathcal{D}) \to \mathbb{C}$, $\chi_\phi(a) = \langle \phi \,|\, a(\phi) \rangle$, which is a positive linear functional on $\mathcal{L}^*(\mathcal{D})$ by definition of the order on $\mathcal{L}^*(\mathcal{D})$. Then $\langle \chi_\phi \,,\, b^n \rangle = \langle a^n(\phi) \,|\, a^n(\phi) \rangle = \|a^n(\phi)\|^2$, and all the claims here follow from Lemma 3.1.17 and from the monotony of the square root function. □

Definition A.4.40 *Let $\mathcal{D}$ be a pre-Hilbert space and $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$, then a vector $\phi \in \mathcal{D}$ is said to be a* Stieltjes vector *of a if*

$$\|a(\phi)\| = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \|a^n(\phi)\|^{-\frac{1}{2n}} = \infty \tag{A.4.26}$$

holds.

Note that Lemma A.4.39 guarantees that if $\|a(\phi)\| > 0$, then $\|a^n(\phi)\| > 0$ for all $n \in \mathbb{N}$.

Lemma A.4.41 *Let $\mathcal{D}$ be a pre-Hilbert space and $a \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$, then $\langle \phi \,|\, a^n(\phi) \rangle \ge 0$ for all $\phi \in \mathcal{D}$ and all $n \in \mathbb{N}_0$, and a vector $\phi \in \mathcal{D}$ is a Stieltjes vector of a if and only if*

$$\langle \phi \,|\, a(\phi) \rangle = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \langle \phi \,|\, a^n(\phi) \rangle^{-\frac{1}{2n}} = \infty \tag{A.4.27}$$

holds.

PROOF: Again, this is an application of Lemma 3.1.17 to the positive algebra element $a \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$ and the positive linear functional $\chi_\phi \colon \mathcal{L}^*(\mathcal{D}) \to \mathbb{C}$, $\chi_\phi(a) = \langle \phi \,|\, a(\phi) \rangle$:

The Lemma shows that $\langle \phi \,|\, a^n(\phi) \rangle = \langle \chi_\phi \,,\, a^n \rangle \ge 0$ for all $n \in \mathbb{N}_0$ as $a^n \ge 0$, and if $\langle \phi \,|\, a^n(\phi) \rangle = \langle \chi_\phi \,,\, a^n \rangle = 0$ for one $n \in \mathbb{N}$ then for all $n \in \mathbb{N}$. So especially $\langle \phi \,|\, a(\phi) \rangle = 0$ is equivalent to $\langle \phi \,|\, a^2(\phi) \rangle = 0$, hence to $\|a(\phi)\| = 0$, and that otherwise $\langle \phi \,|\, a^n(\phi) \rangle > 0$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
\sum_{n=1}^{\infty} \|a^n(\phi)\|^{-\frac{1}{2n}} &= \sum_{n=1}^{\infty} \langle \phi \,|\, a^{2n}(\phi) \rangle^{-\frac{1}{4n}} \\
&\le \sum_{n=1}^{\infty} \langle \phi \,|\, a^n(\phi) \rangle^{-\frac{1}{2n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \langle\, \phi \,|\, a(\phi)\,\rangle^{\frac{1}{2}} + 2\sum_{n=1}^{\infty} \langle\, \phi \,|\, a^{2n}(\phi)\,\rangle^{-\frac{1}{4n}} \\
&= \langle\, \phi \,|\, a(\phi)\,\rangle^{\frac{1}{2}} + 2\sum_{n=1}^{\infty} \|a^{n}(\phi)\|^{-\frac{1}{2n}}
\end{aligned}
$$

because $\langle\, \phi \,|\, a^{(2n+1)}(\phi)\,\rangle^{1/(2n+1)} \geq \langle\, \phi \,|\, a^{(2n)}(\phi)\,\rangle^{1/(2n)}$. As a consequence, $\sum_{n=1}^{\infty}\langle\, \phi \,|\, a^{n}(\phi)\,\rangle^{-1/(2n)}$ is divergent if and only if $\sum_{n=1}^{\infty}\|a^{n}(\phi)\|^{-1/(2n)}$ is divergent. $\square$

In order to apply Theorem A.4.38 in the general case, some inequalities are needed. The first one is a very easy version of Stirling's approximation for factorials:

Lemma A.4.42 *The estimate $\big((n+1)/\mathrm{e}\big)^{n} \leq n!$ holds for all $n \in \mathbb{N}$.*

PROOF: This is equivalent to $(n+1)^n/n! \leq \mathrm{e}^n$ for all $n \in \mathbb{N}$, which is true because

$$
\frac{(n+1)^n}{n!} = \sum_{k=0}^{n} \binom{n}{k} \frac{n^k}{n!} \leq \sum_{k=0}^{n} \frac{n^k}{k!} \leq \mathrm{e}^n\,. \qquad \square
$$

The next one is the well-known inequality between geometric and arithmetic mean:

Lemma A.4.43 *The estimate $\prod_{m=1}^{n} \sqrt[n]{\alpha_m} \leq \sum_{m=1}^{n} \frac{\alpha_m}{n}$ holds for all $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in [0,\infty[$.*

PROOF: If at least one of the $\alpha_1, \dots, \alpha_n$ is 0, then the estimate is trivially fulfilled. Otherwise this is a consequence of the concavity of the logarithm and the monotony of the exponential function:

$$
\prod_{m=1}^{n} \sqrt[n]{\alpha_m} = \exp\left(\sum_{m=1}^{n} \frac{\ln(\alpha_m)}{n}\right) \leq \exp\left(\ln\left(\sum_{m=1}^{n} \frac{\alpha_m}{n}\right)\right) = \sum_{m=1}^{n} \frac{\alpha_m}{n} \qquad \square
$$

By combining the last two estimates one derives Carleman's inequality (see [31] for an overview over some proofs that have been found for this estimate):

Lemma A.4.44 *Let $N \in \mathbb{N}$ and $a_1, \dots, a_N \in [0,\infty[$ be given, then*

$$
\sum_{n=1}^{N} \left(\prod_{m=1}^{n} a_m\right)^{\frac{1}{n}} \leq \mathrm{e} \sum_{n=1}^{N} a_n\,. \tag{A.4.28}
$$

PROOF: With the help of the previous two Lemmas A.4.42 and A.4.43 one gets:

$$
\begin{aligned}
\sum_{n=1}^{N} \left(\prod_{m=1}^{n} a_m\right)^{\frac{1}{n}} &= \sum_{n=1}^{N} \frac{1}{(n!)^{1/n}} \left(\prod_{m=1}^{n} a_m m\right)^{\frac{1}{n}} \\
&\leq \sum_{n=1}^{N} \frac{\mathrm{e}}{n+1} \prod_{m=1}^{n} \sqrt[n]{a_m m} \\
&\leq \mathrm{e} \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{a_m m}{(n+1)n} \\
&= \mathrm{e} \sum_{m=1}^{N} \sum_{n=m}^{N} \frac{a_m m}{(n+1)n}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e} \sum_{m=1}^{N} a_m m \sum_{n=m}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \mathrm{e} \sum_{m=1}^{N} a_m m \left(\frac{1}{m} - \frac{1}{N+1} \right) \\
&\leq \mathrm{e} \sum_{m=1}^{N} a_m
\end{aligned}
\qquad \square
$$

This inequality is the key to applying Lemma A.4.37 to more general cases (see [55] and also [62]):

Theorem A.4.45 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$ coercive, i.e. there exists an $\epsilon > 0$ such that $\langle \phi \,|\, a(\phi) \rangle \geq \epsilon \|\phi\|^2$ for all $\phi \in \mathcal{D}$. If all $\phi \in \mathcal{D}$ are Stieltjes vectors for a, then the image of $\mathcal{D}$ under a is dense in $\mathfrak{H}$ and a is essentially self-adjoint.*

PROOF: By rescaling a with a $\lambda \in]0, \infty[$ one can assume that $\langle \phi \,|\, a(\phi) \rangle \geq \|\phi\|^2$ for all $\phi \in \mathcal{D}$, hence $a - \mathbb{1} \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$, because Stieltjes vectors of a are also Stieltjes vectors of λa and because λa has dense image and is essentially self-adjoint if and only if this holds for a.

In order to show that the image of $\mathcal{D}$ under a is dense in $\mathfrak{H}$ (which then also proves essential self-adjointness by Corollary A.4.28), it is sufficient to show that its orthogonal complement (in $\mathfrak{H}$) is $\{0\}$, i.e. that $\langle \phi \,|\, \psi \rangle = 0$ for every $\phi \in \mathcal{D}$ and every $\psi \in \mathfrak{H}$ which is orthogonal to the image of $\mathcal{D}$ under a. Assume that this is not true, i.e. that there is a vector $\psi \in \mathfrak{H}$ such that $\langle a(\xi) \,|\, \psi \rangle = 0$ for all $\xi \in \mathcal{D}$ but $\psi \neq 0$. Then $\langle \phi \,|\, \psi \rangle \neq 0$ for some $\phi \in \mathcal{D}$ with $\|\phi\| = 1$, and by rescaling ψ we can even assume $\langle \phi \,|\, \psi \rangle = 1$.

Define the finite dimensional linear subspaces $U_{-1} \coloneqq \{0\}$ and $U_n \coloneqq \langle\!\langle \{a^0(\phi), \dots, a^n(\phi)\} \rangle\!\rangle_{\mathrm{lin}}$ of $\mathcal{D}$ for all $n \in \mathbb{N}_0$ and note that $a(U_{n-1}) \subseteq U_n$ for all $n \in \mathbb{N}_0$. If there exists an $n \in \mathbb{N}_0$ such that $a^n(\phi) \in U_{n-1}$, then a restricts to a coercive Hermitian endomorphism of U_{n-1}. As this restriction is especially injective, it is also surjective by elementary linear algebra. So there exists a $\xi \in U_{n-1}$ fulfilling $\phi = a(\xi)$ and $\langle \phi \,|\, \psi \rangle = \langle a(\xi) \,|\, \psi \rangle = 0$, a contradiction.

If $a^n(\phi) \notin U_{n-1}$ for all $n \in \mathbb{N}_0$, then $\dim U_n = n + 1$ for all $n \in \mathbb{N}_0$ and one can construct orthonormal vectors $u_n \in \mathcal{D}$ with the help of the Gram-Schmidt method as

$$
\begin{aligned}
& u'_0 \coloneqq \phi \\
& u'_n \coloneqq a^n(\phi) - \sum_{m=0}^{n-1} u'_m \frac{\langle u'_m \,|\, a^n(\phi) \rangle}{\langle u'_m \,|\, u'_m \rangle} \qquad \text{for all } n \in \mathbb{N} \\
\text{and} \qquad & u_n \coloneqq \frac{u'_n}{\|u'_n\|} \qquad \text{for all } n \in \mathbb{N}_0 \,.
\end{aligned}
$$

Then $u_0 = \phi, \dots, u_n$ are an orthonormal basis of U_n, more precisely $u_n \in U_n \cap (U_{n-1})^\perp$ for all $n \in \mathbb{N}_0$, and as $a(u_n) \in U_{n+1}$ it follows that $\langle u_m \,|\, a(u_n) \rangle = 0$ for all $m, n \in \mathbb{N}_0$ with $m > n + 1$. As a is Hermitian, it also follows that $\langle u_m \,|\, a(u_n) \rangle = \langle a(u_m) \,|\, u_n \rangle = 0$ for all $m, n \in \mathbb{N}_0$ with $n > m + 1$, so $\langle u_m \,|\, a(u_n) \rangle = 0$ for all $m, n \in \mathbb{N}_0$ with $|m - n| > 1$. Moreover, for all $n \in \mathbb{N}_0$ the identity $a^n(\phi) = u'_n + \Delta_n$ holds with some $\Delta_n \in U_{n-1}$, so $a(u'_n) = a^{n+1}(\phi) - a(\Delta_n) = u'_{n+1} + \Delta_{n+1} - a(\Delta_n)$ and thus $a(u'_n) - u'_{n+1} \in U_n$, which shows $\|u'_{n+1}\|^2 = \langle u'_{n+1} \,|\, u'_{n+1} \rangle = \langle u'_{n+1} \,|\, a(u'_n) \rangle$ for all $n \in \mathbb{N}_0$ and also $\|u'_{n+1}\|^2 = \langle u'_n \,|\, a(u'_{n+1}) \rangle$ by complex conjugation.

Define $\alpha_n \coloneqq \langle\, u_n \,|\, a(u_n) \,\rangle \geq \langle\, u_n \,|\, u_n \,\rangle = 1$ and $\beta_n \coloneqq \langle\, u_{n+1} \,|\, a(u_n) \,\rangle = \|u'_{n+1}\|/\|u'_n\|$ for all $n \in \mathbb{N}_0$, which are the coefficients of the Jacobi matrix $J_{\alpha,\beta}$ that represents (the restriction of) a in the orthonormal base $u_0, \dots, u_n, \dots$ of $\bigcup_{n=0}^{\infty} U_n$. Then $\beta_n > 0$ and $J_{\alpha,\beta}$ also fulfils the criterium of Lemma A.4.37 that $J_{\alpha',\beta}$ is positive. Moreover,

$$\|u'_n\| = \|u'_0\| \prod_{m=1}^{n} \beta_{m-1} = \|\phi\| \prod_{m=1}^{n} \beta_{m-1}$$

for all $n \in \mathbb{N}_0$, together with the estimate $\|a^n(\phi)\| = \|u'_n + \Delta_n\| \geq \|u'_n\|$ (using that u'_n and Δ_n are orthogonal) and Carleman's inequality from Lemma A.4.44 yield

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{\|a^n(\phi)\|^{1/(2n)}} &\leq \sum_{n=1}^{N} \frac{1}{\|u'_n\|^{1/(2n)}} \\
&= \sum_{n=1}^{N} \frac{1}{\|\phi\|^{1/(2n)}} \Big(\prod_{m=1}^{n} \frac{1}{\sqrt{\beta_{m-1}}} \Big)^{\frac{1}{n}} \\
&\leq \Big(\frac{1}{\sqrt{\|\phi\|}} + 1 \Big) \sum_{n=1}^{N} \Big(\prod_{m=1}^{n} \frac{1}{\sqrt{\beta_{m-1}}} \Big)^{\frac{1}{n}} \\
&\leq \mathrm{e} \Big(\frac{1}{\sqrt{\|\phi\|}} + 1 \Big) \sum_{n=1}^{N} \frac{1}{\sqrt{\beta_{n-1}}}
\end{aligned}$$

for all $N \in \mathbb{N}$, so $\sum_{n=0}^{N-1} (\beta_n)^{-1/2} \xrightarrow{N\to\infty} \infty$ because ϕ is a Stieltjes vector of A.

Now define $x_n \coloneqq \langle\, u_n \,|\, \psi \,\rangle$ for all $n \in \mathbb{N}_0$, which are the components of ψ parallel to u_n. As ψ is orthogonal to the image of $\mathcal{D}$ under a, the sequence $(x_n)_{n\in\mathbb{N}}$ fulfils the system of linear equations

$$\begin{aligned}
0 &= \langle\, a(u_n) \,|\, \psi \,\rangle \\
&= \langle\, a(u_n) \,|\, u_{n-1} \,\rangle \langle\, u_{n-1} \,|\, \psi \,\rangle + \langle\, a(u_n) \,|\, u_n \,\rangle \langle\, u_n \,|\, \psi \,\rangle + \langle\, a(u_n) \,|\, u_{n+1} \,\rangle \langle\, u_{n+1} \,|\, \psi \,\rangle \\
&= \beta_{n-1}\, x_{n-1} + \alpha_n\, x_n + \beta_n\, x_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$ as well as

$$\begin{aligned}
0 &= \langle\, a(u_0) \,|\, \psi \,\rangle \\
&= \langle\, a(u_0) \,|\, u_0 \,\rangle \langle\, u_0 \,|\, \psi \,\rangle + \langle\, a(u_0) \,|\, u_1 \,\rangle \langle\, u_1 \,|\, \psi \,\rangle \\
&= \alpha_0\, x_0 + \beta_0\, x_1 \,.
\end{aligned}$$

and $x_0 = \langle\, u_0 \,|\, \psi \,\rangle = \langle\, \phi \,|\, \psi \,\rangle = 1$. So Lemma A.4.37 applies and shows that

$$\sum_{n=0}^{N-1} \frac{1}{\sqrt{\beta_n}} \leq \Big(\sum_{n=0}^{N} |x_n|^2 \Big)^{\frac{1}{2}} \leq \|\psi\|$$

holds for all $N \in \mathbb{N}$, which is a contradiction as $\sum_{n=0}^{N-1} (\beta_n)^{-1/2} \xrightarrow{N\to\infty} \infty$. □

This criterium of essential self-adjointness of a coercive Hermitian endomorphism of a pre-Hilbert space now also yields similar results by applying our previous, more general considerations:

Corollary A.4.46 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and and $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ semi-bounded, i.e. there exists a $\lambda \in \mathbb{R}$ such that $a+\lambda\mathbb{1}$ or $-a+\lambda\mathbb{1}$ are positive. If every $\phi \in \mathcal{D}$ is a Stieltjes vector for a, then a is essentially self-adjoint.*

PROOF: Let $\mu := \lambda+1$, then $a+\mu\mathbb{1}$ or $-a+\mu\mathbb{1}$ fulfil the conditions of the previous Theorem A.4.45, because every Stieltjes vector $\phi \in \mathcal{D}$ for a is also a Stieltjes vector for $\pm a + \mu\mathbb{1}$: In order to see this, first note that

$$\begin{aligned}\big\|\big(\pm a+\mu\mathbb{1}\big)^n(\phi)\big\| &\leq \sum_{m=0}^{n}\binom{n}{m}|\mu|^{n-m}\|a^m(\phi)\| \\ &\leq \sum_{m=0}^{n}\binom{n}{m}|\mu|^{n-m}\|a^n(\phi)\|^{\frac{m}{n}} \\ &= \big(|\mu| + \|a^n(\phi)\|^{1/n}\big)^n \\ &\leq \big(\sqrt{|\mu|} + \|a^n(\phi)\|^{1/(2n)}\big)^{2n}\end{aligned}$$

holds for all $\phi \in \mathcal{D}$ by Lemma A.4.39.

If the sequence $\big(\|a^n(\phi)\|^{1/(2n)}\big)_{n\in\mathbb{N}}$, which is monotonely increasing by Lemma A.4.39, is bounded from above by some $B \in [0,\infty[$, then $\sum_{n=1}^{\infty}\big(\sqrt{|\mu|} + \|a^n(\phi)\|^{1/(2n)}\big)^{-1} \geq \sum_{n=1}^{\infty}\big(\sqrt{|\mu|}+B\big)^{-1} = \infty$. If this sequence is unbounded, then there especially exists an $N \in \mathbb{N}$ such that $\|a^N(\phi)\|^{1/(2N)} \geq |\mu|$, and then even $\|a^n(\phi)\|^{1/(2n)} \geq |\mu|$ for all $n \in \mathbb{N}$ with $n \geq N$ due to Lemma A.4.39 again. In this case $\sum_{n=1}^{\infty}\big(\sqrt{|\mu|} + \|a^n(\phi)\|^{1/(2n)}\big)^{-1} \geq \sum_{n=N}^{\infty}\big(2\|a^n(\phi)\|^{1/(2n)}\big)^{-1} = \infty$ by using that ϕ is a Stieltjes vector for a. So both cases yield

$$\sum_{n=0}^{\infty}\big\|\big(\pm a+\mu\mathbb{1}\big)^n(\phi)\big\|^{-\frac{1}{2n}} \geq \sum_{n=0}^{\infty}\frac{1}{\sqrt{|\mu|}+\|a^n(\phi)\|^{1/(2n)}} = \infty\,,$$

hence the coercive one of $a \pm \mu\mathbb{1}$ is essentially self-adjoint, and thus a is also essentially self-adjoint by Proposition A.4.26. □

Corollary A.4.47 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$. If every $\phi \in \mathcal{D}$ is a Stieltjes vector for a^*a, then $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$.*

PROOF: By the previous Corollary A.4.46, a^*a is essentially self-adjoint, thus also $\mathbb{1}+a^*a$ by Proposition A.4.26, which implies that $\mathcal{D}_{a^\dagger} = \mathcal{D}_{(a^*)^{\mathrm{cl}}}$ by Proposition A.4.30. □

Corollary A.4.48 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $a \in \mathcal{L}^*(\mathcal{D})$. If every $\phi \in \mathcal{D}$ is a* quasi-analytic vector *of a, i.e. if*

$$\|a(\phi)\| = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty}\|a^n(\phi)\|^{-\frac{1}{n}} = \infty \tag{A.4.29}$$

holds for all $\phi \in \mathcal{D}$, then a is essentially self-adjoint.

PROOF: This is the previous Corollary A.4.47 again, using $\|a^n(\phi)\|^{1/n} = \langle \phi \,|\, a^{2n}(\phi) \rangle^{1/(2n)}$ and Lemma A.4.41 to show that a vector $\phi \in \mathcal{D}$ is a quasi-analytic for a if and only if it is a Stieltjes vector for a^2. $\square$

Corollary A.4.49 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$, $N \in \mathbb{N}$ and $a_1, \dots a_N \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{H}}$ coercive and pairwise commuting. If every $\phi \in \mathcal{D}$ is a Stieltjes vector of all $a_1, \dots, a_N$, then $\prod_{n=1}^N a_n^2$ is essentially self-adjoint.*

PROOF: For all $n \in \{1, \dots, N\}$ there exists an $\epsilon_n > 0$ such that $a_n - \epsilon_n \geq 0$, thus $a_n^2 - \epsilon_n^2 = (a_n - \epsilon_n)^2 + 2\epsilon_n(a_n - \epsilon_n) \geq 0$ for all $n \in \{1, \dots, N\}$ shows that the a_n^2 are again coercive. By Lemma 3.5.11, the product $\prod_{n=1}^N a_n^2$ is also coercive, so it is sufficient to prove that the image of $\mathcal{D}$ under $\prod_{n=1}^N a_n^2$ is dense in $\mathfrak{H}$.

This however, is an immediate consequence of Theorem A.4.45. In fact, the image of $\mathcal{D}$ under every product $\prod_{m=1}^M b_m$ with $M \in \mathbb{N}$ and $b_1, \dots, b_M \in \{a_1, \dots, a_N\}$ is dense in $\mathfrak{H}$, as can be seen by induction over M: For $M = 1$ this is just Theorem A.4.45, and if it holds for one $M \in \mathbb{N}$, then also for $M+1$: Let $\mathcal{D}' \subseteq \mathcal{D}$ be the image of $\mathcal{D}$ under $\prod_{m=1}^M b_m$, which is dense in $\mathfrak{H}$ by assumption. Then b_{M+1} restricts to a coercive endomorphism of $\mathcal{D}'$ as $b_{M+1}\big(\big(\prod_{m=1}^M b_m\big)(\phi)\big) = \big(\prod_{m=1}^M b_m\big)(b_{M+1}(\phi)) \in \mathcal{D}'$ for all $\phi \in \mathcal{D}$. So Theorem A.4.45 can be applied to this restriction, showing that the image of $\mathcal{D}'$ under b_{M+1}, i.e. the image of $\mathcal{D}$ under $\big(\prod_{m=1}^{M+1} b_m\big)$, is dense in $\mathfrak{H}$. $\square$

Finally, it only remains to mention that Theorem A.4.45 and its corollaries cover the cases of similar, and more well-known, theorems using analytic and semi-analytic vectors:

If $\mathcal{D}$ is a pre-Hilbert space, $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ a Hermitian endomorphism on it and $\phi \in \mathcal{D}$ an *analytic vector* of a, i.e. if there exist $C, D \in [0, \infty[$ such that

$$\|a^n(\phi)\| \leq C D^n n! \tag{A.4.30}$$

for all $n \in \mathbb{N}$, then the estimate $\big(CD^n n!\big)^{1/n} \leq C^{1/n} D n \leq \max\{C, 1\} D n$ for all $n \in \mathbb{N}$ shows that $\sum_{n=1}^\infty \|a^n(\phi)\|^{-1/n} \geq \sum_{n=1}^\infty 1/\big(\max\{C, 1\} D n\big) = \infty$, i.e. ϕ is a quasi-analytic vector of a.

Similarly, if $\phi \in \mathcal{D}$ is a *semi-analytic vector* of a, i.e. if there exist $C, D \in [0, \infty[$ such that

$$\|a^n(\phi)\| \leq C D^{2n} (2n)! \tag{A.4.31}$$

for all $n \in \mathbb{N}$, then the estimate $\big(CD^{2n}(2n)!\big)^{1/(2n)} \leq C^{1/(2n)} D n \leq \max\{C, 1\} D n$ for all $n \in \mathbb{N}$ shows that $\sum_{n=1}^\infty \|a^n(\phi)\|^{-1/(2n)} \geq \sum_{n=1}^\infty 1/\big(\max\{C, 1\} D n\big) = \infty$, i.e. a is a Stieltjes vector of a.

So in Theorem A.4.45 and its corollaries one can replace "Stieltjes vector" by "semi-analytic vector" and "quasi-analytic vector" by "analytic vector".

A.4.3 Application to O^*-Algebras

The definitions and results of this section are from [74]. Recall that an O^*-algebra on a dense linear subspace $\mathcal{D}$ of a Hilbert space $\mathfrak{H}$ is a unital *-subalgebra $\mathcal{A}$ of the ordered *-algebra $\mathcal{L}^*(\mathcal{D})$, hence a *-algebra of closable operators by Proposition A.4.16. Such an O^*-algebra $\mathcal{A}$ is especially well-behaved if its domain $\mathcal{D}$ is somehow related to the domains of its elements:

Definition A.4.50 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ an O^*-algebra. Then $\mathcal{A}$ is said to be* closed *if*

$$\bigcap_{a\in\mathcal{A}} \mathcal{D}_{a^{\mathrm{cl}}} = \mathcal{D}\,. \tag{A.4.32}$$

It was already shown that every adjointable endomorphism is closable. Similarly, every O^*-algebra is closable as well:

Definition A.4.51 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ an O^*-algebra. Then define*

$$\mathcal{D}^{\mathrm{cl}} := \bigcap_{a\in\mathcal{A}} \mathcal{D}_{a^{\mathrm{cl}}} \tag{A.4.33}$$

and endow $\mathcal{D}^{\mathrm{cl}}$ with the locally convex topology of the (restrictions of the) seminorms $\|\cdot\|_{\mathbb{1}+a^*a}$ to $\mathcal{D}^{\mathrm{cl}}$ for all $a \in \mathcal{A}$. This topology on $\mathcal{D}^{\mathrm{cl}}$ is called the graph topology *of $\mathcal{A}$.*

Note that $\mathcal{D}^{\mathrm{cl}}$ depends on the O^*-algebra $\mathcal{A}$, not only on $\mathcal{D}$!

Proposition A.4.52 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ an O^*-algebra, then $\mathcal{D}^{\mathrm{cl}}$ is complete under the graph topology of $\mathcal{A}$ and $\mathcal{D}$ is a dense linear subspace of $\mathcal{D}^{\mathrm{cl}}$ (again with respect to the graph topology). Moreover, all $a \in \mathcal{A}$ are continuous with respect to the (restriction of the) graph topology on $\mathcal{D}$.*

PROOF: By Proposition A.4.22, every $\mathcal{D}_{a^{\mathrm{cl}}}$ with $a \in \mathcal{A}$ is complete with respect to the topology defined by the norm $\|\cdot\|_{\mathbb{1}+a^*a}$ on it, and $\mathcal{D}$ is dense in $\mathcal{D}_{a^{\mathrm{cl}}}$ with respect to this topology. It is then straightforward to show that $\mathcal{D}^{\mathrm{cl}} = \bigcap_{a\in\mathcal{A}} \mathcal{D}_{a^{\mathrm{cl}}}$ is complete under the locally convex topology of all the seminorms $\|\cdot\|_{\mathbb{1}+a^*a}$ for all $a \in \mathcal{A}$ and $\mathcal{D}$ dense in $\mathcal{D}^{\mathrm{cl}}$ with respect to this topology. Moreover, all $a \in \mathcal{A}$ are continuous with respect to the graph topology, because $b^*(\mathbb{1}+a^*a)b = b^*b+b^*a^*a\,b \leq \mathbb{1}+c^*c$ with $c := \mathbb{1} + b^*b + b^*a^*a\,b$, so $\|b(\phi)\|_{\mathbb{1}+a^*a} \leq \|\phi\|_{\mathbb{1}+c^*c}$ for all $\phi \in \mathcal{D}$. □

Because of this, every element a of an O^*-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{D}$ has a unique continuous (with respect to the graph topology) extension a^{cl} to $\mathcal{D}^{\mathrm{cl}}$, which should not be confused with the extension of a to the larger $\mathcal{D}_{a^{\mathrm{cl}}}$. Of course, all the algebraic operations and relations (addition, both multiplications, *-involution, order, associativity, distributivity...) are preserved by this extension $\cdot^{\mathrm{cl}} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}})$ because of the continuity of the involved maps and the density of $\mathcal{D}$ in $\mathcal{D}^{\mathrm{cl}}$. So one can define:

Definition A.4.53 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ an O^*-algebra, then define the* closure *of $\mathcal{A}$ as the positive injective unital *-homomorphism $\cdot^{\mathrm{cl}} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D}^{\mathrm{cl}})$ that assigns to every $a \in \mathcal{A}$ its continuous (with respect to the graph topology) extension $a^{\mathrm{cl}} \colon \mathcal{D}^{\mathrm{cl}} \to \mathcal{D}^{\mathrm{cl}}$.*

Note that it will be clear from the context whether $\cdot^{\mathrm{cl}}$ denotes the closure of only one adjointable endomorphism like in Definition A.4.19 or the closure of a whole O^*-algebra like in Definition A.4.53. It is not hard to check that $a^{\mathrm{cl}} \colon \mathcal{D}_{a^{\mathrm{cl}}} \to \mathfrak{H}$ is an extension of $a^{\mathrm{cl}} \colon \mathcal{D}^{\mathrm{cl}} \to \mathcal{D}^{\mathrm{cl}}$.

It is often more transparent to use a slightly extended system of seminorms to describe the graph topology:

Proposition A.4.54 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ an O^*-algebra, then define for all $a \in \mathcal{A}_{\mathrm{H}}^+$ the positive Hermitian form $\langle \cdot \mid \cdot \rangle_a$ on $\mathcal{D}^{\mathrm{cl}}$ as*

$$\langle \phi \mid \psi \rangle_a := \langle \phi \mid a^{\mathrm{cl}}(\psi) \rangle \qquad \textit{for all } \phi, \psi \in \mathcal{D}^{\mathrm{cl}}. \tag{A.4.34}$$

Then this is consistent with the definition of the inner product $\langle \cdot \mid \cdot \rangle_{\mathbb{1}+a^*a}$ on $\mathcal{D}_{a^{\mathrm{cl}}}$ from Proposition A.4.22 in so far as

$$\langle \phi \mid \psi \rangle + \langle a^{\mathrm{cl}}(\phi) \mid a^{\mathrm{cl}}(\psi) \rangle = \langle \phi \mid \big(\mathbb{1} + a^*a\big)^{\mathrm{cl}}(\psi) \rangle \tag{A.4.35}$$

holds for all $\phi, \psi \in \mathcal{D}^{\mathrm{cl}}$, and the locally convex topology on $\mathcal{D}^{\mathrm{cl}}$ defined by all the seminorms $\|\cdot\|_a$ with $a \in \mathcal{A}_{\mathrm{H}}^+$ coincides with the graph topology.

PROOF: On $\mathcal{D}$ it is clear that (A.4.35) holds, but then it even holds on whole $\mathcal{D}^{\mathrm{cl}}$ by continuity with respect to the graph topology on $\mathcal{D}^{\mathrm{cl}}$. In order to show that the locally convex topology defined by all the seminorms $\|\cdot\|_a$ with $a \in \mathcal{A}_{\mathrm{H}}^+$ on $\mathcal{D}^{\mathrm{cl}}$ coincides with the graph topology, it is sufficient to show that for every $a \in \mathcal{A}_{\mathrm{H}}^+$ there exists a $b \in \mathcal{A}$ such that $\|\cdot\|_a \leq \|\cdot\|_{\mathbb{1}+b^*b}$ holds pointwise on $\mathcal{D}^{\mathrm{cl}}$ as the converse estimate is trivial. Given $a \in \mathcal{A}_{\mathrm{H}}^+$, choose $b := \mathbb{1} + a$. □

Closed O^*-algebras whose domain is not just the intersection of the domains of all closures of its elements, but even of the domains of all (operator theoretic) adjoints of its elements, are especially interesting. An even more well-behaved type of such algebras is the following:

Definition A.4.55 *[74, Def. 7.3.5] A* strictly self-adjoint *O^*-algebra on a dense linear subspace $\mathcal{D}$ of a Hilbert space $\mathfrak{H}$ is a closed O^*-algebra $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ in which there exists a subset $Q \subseteq \mathcal{A}$ with the following properties:*

- *i.) $q\,q^* = q^*q$ for all $q \in Q$.*
- *ii.) q^*q is essentially self-adjoint for all $q \in Q$.*
- *iii.) The set $\{\, q^*q \mid q \in Q \,\}$ is upwards directed and the graph topology defined by $\mathcal{A}$ on $\mathcal{D}$ is the locally convex one defined by all the seminorms $\|\cdot\|_{q^*q}$ with $q \in Q$.*

Proposition A.4.56 *Let $\mathfrak{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathfrak{H}$ a dense linear subspace and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ a strictly self-adjoint O^*-algebra with $Q \subseteq \mathcal{A}$ like in the previous Definition A.4.55. Then*

$$\bigcap_{q\in Q} \mathcal{D}_{q^\dagger} = \bigcap_{q\in Q} \mathcal{D}_{(q^*)^{\mathrm{cl}}} = \mathcal{D} = \bigcap_{q\in Q} \mathcal{D}_{q^{\mathrm{cl}}} = \bigcap_{q\in Q} \mathcal{D}_{(q^*)^\dagger}\,. \tag{A.4.36}$$

PROOF: First of all, $\mathcal{D}_{q^\dagger} = \mathcal{D}_{(q^*)^{\mathrm{cl}}}$ and $\mathcal{D}_{(q^*)^\dagger} = \mathcal{D}_{q^{\mathrm{cl}}}$ hold for all $q \in Q$ due to Proposition A.4.30, using that $\mathbb{1} + q^*q = \mathbb{1} + qq^*$ are both essentially self-adjoint by Proposition A.4.26 and because $\mathcal{A}$ is strictly self-adjoint. So the first and last identity hold.

It is also clear that $\mathcal{D} \subseteq \bigcap_{q\in Q} \mathcal{D}_{q^{\mathrm{cl}}}$ and $\mathcal{D} \subseteq \bigcap_{q\in Q} \mathcal{D}_{(q^*)^{\mathrm{cl}}}$. Conversely, given $a \in \mathcal{A}$, then there exist $q \in Q$ and $\lambda \in [0,\infty[$ with $\lambda\|\cdot\|_{q^*q} \geq \|\cdot\|_{\mathbb{1}+a^*a}$ and thus also $\lambda\|\cdot\|_{\mathbb{1}+q^*q} \geq \|\cdot\|_{\mathbb{1}+a^*a}$. From Corollary A.4.23 it then follows that $\mathcal{D}_{q^{\mathrm{cl}}} \subseteq \mathcal{D}_{a^{\mathrm{cl}}}$ and also that $\mathcal{D}_{(q^*)^{\mathrm{cl}}} \subseteq \mathcal{D}_{a^{\mathrm{cl}}}$ as $\mathbb{1}+q^*q = \mathbb{1}+qq^*$. This yields $\mathcal{D} = \mathcal{D}^{\mathrm{cl}} = \bigcap_{a\in\mathcal{A}} \mathcal{D}_{a^{\mathrm{cl}}} \subseteq \bigcap_{q\in Q} \mathcal{D}_{q^{\mathrm{cl}}}$ and $\mathcal{D} = \mathcal{D}^{\mathrm{cl}} = \bigcap_{a\in\mathcal{A}} \mathcal{D}_{a^{\mathrm{cl}}} \subseteq \bigcap_{q\in Q} \mathcal{D}_{(q^*)^{\mathrm{cl}}}$ as $\mathcal{A}$ is a closed O^*-algebra. □

This description of the domain of strictly self-adjoint O^*-algebras allows to formulate e.g. generalizations of the Fréchet-Riesz theorem and the Lax-Milgram theorem.

Theorem A.4.57 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ a strictly self-adjoint O^*-algebra as well as $\omega\colon \mathcal{D} \to \mathbb{C}$ a linear functional. Then the following is equivalent:*

i.) There exists a $\phi \in \mathcal{D}$ such that $\langle\, \omega\,,\, \psi \,\rangle = \langle\, \phi \,|\, \psi \,\rangle$ for all $\psi \in \mathcal{D}$.

ii.) $\omega \circ a\colon \mathcal{D} \to \mathbb{C}$ is continuous (with respect to the usual $\|\cdot\|$-topology) for every $a \in \mathcal{A}$.

iii.) $\omega \circ q\colon \mathcal{D} \to \mathbb{C}$ is continuous (with respect to the usual $\|\cdot\|$-topology) for every $q \in Q \cup \{\mathbb{1}\}$ and $Q \subseteq \mathcal{A}$ like in Definition A.4.55.

Proof: The first statement implies the second, because if there exists $\phi \in \mathcal{D}$ such that $\langle\, \omega\,,\, \psi \,\rangle = \langle\, \phi \,|\, \psi \,\rangle$ holds for all $\psi \in \mathcal{D}$, then $\mathcal{D} \ni \psi \mapsto \langle\, \omega\,,\, a(\psi) \,\rangle = \langle\, \phi \,|\, a(\psi) \,\rangle = \langle\, a^*(\phi) \,|\, \psi \,\rangle \in \mathbb{C}$ is $\|\cdot\|$-continuous.

The second statement trivially implies the third, which itself implies the first: If $\omega\colon \mathcal{D} \to \mathbb{C}$ is continuous, then there exists a $\phi \in \mathfrak{H}$ with $\langle\, \omega\,,\, \psi \,\rangle = \langle\, \phi \,|\, \psi \,\rangle$ for all $\psi \in \mathcal{D}$ by the Fréchet-Riesz theorem. But the continuity of $\omega \circ q$ with $q \in Q$ shows that $\mathcal{D} \ni \psi \mapsto \langle\, \phi \,|\, q(\psi) \,\rangle \in \mathbb{C}$ is continuous, thus $\phi \in \mathcal{D}_{q^\dagger}$ by Proposition A.4.21. As $\mathcal{D} = \bigcap_{q\in Q} \mathcal{D}_{q^\dagger}$ by the previous Proposition A.4.56 it follows that $\phi \in \mathcal{D}$. □

This version of the Lax-Milgram theorem uses techniques from [74, Prop. 7.2.5, Thm. 7.3.6]:

Theorem A.4.58 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ a strictly self-adjoint O^*-algebra as well as $s\colon \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ a sesquilinear functional with the following properties:*

i.) s is separately $\|\cdot\|$-continuous, i.e. for every $\phi \in \mathcal{D}$ there exists $\lambda_\phi \in [0, \infty[$ fulfilling $|s(\phi,\psi)| \leq \lambda_\phi \|\psi\|$ for all $\psi \in \mathcal{D}$ and for every $\psi \in \mathcal{D}$ there exists $\mu_\psi \in [0,\infty[$ fulfilling $|s(\phi,\psi)| \leq \mu_\psi \|\phi\|$ for all $\phi \in \mathcal{D}$.

ii.) $s\big(\phi, q(\psi)\big) = s\big(q^*(\phi), \psi\big)$ and $s\big(\phi, q^*(\psi)\big) = s\big(q(\phi), \psi\big)$ hold for all $\phi, \psi \in \mathcal{D}$ and all $q \in Q$ with $Q \subseteq \mathcal{A}$ like in Definition A.4.55.

Then there exists a unique $\hat{s} \in \mathcal{L}^*(\mathcal{D})$ fulfilling $\langle\, \hat{s}(\phi) \,|\, \psi \,\rangle = s(\phi, \psi)$ for all $\phi, \psi \in \mathcal{D}$ and $\hat{s}\, q = q\, \hat{s}$ as well as $\hat{s}\, q^* = q^* \hat{s}$ hold for all $q \in Q$.

If in addition s is continuous with respect to the graph topology on $\mathcal{D}$, then $\hat{s}$ is also continuous with respect to the graph topology on $\mathcal{D}$.

Proof: Construct $\hat{s}\colon \mathcal{D} \to \mathcal{D}$ as follows: For every $\phi \in \mathcal{D}$ there exists a (necessarily unique) $\hat{s}(\phi) \in \mathcal{D}$ fulfilling $\langle\, \hat{s}(\phi) \,|\, \psi \,\rangle = s(\phi, \psi)$ for all $\psi \in \mathcal{D}$ by the previous Theorem A.4.57 because $\mathcal{D} \ni \psi \mapsto s\big(\phi, q(\psi)\big) = s\big(q^*(\phi), \psi\big) \in \mathbb{C}$ is $\|\cdot\|$-continuous for all $q \in Q \cup \{\mathbb{1}\}$. Similarly, one can construct

$\hat{s}^* \colon \mathcal{D} \to \mathcal{D}$ fulfilling $\langle\, \hat{s}^*(\psi) \,|\, \phi \,\rangle = \overline{s(\phi, \psi)}$ for all $\phi \in \mathcal{D}$. It is now clear that $\hat{s}$ and $\hat{s}^*$ are mutually adjoint endomorphisms of $\mathcal{D}$. Moreover,

$$\big\langle\, \big(\hat{s}q\big)(\phi) \,\big|\, \psi \,\big\rangle = s\big(q(\phi), \psi\big) = s\big(\phi, q^*(\psi)\big) = \big\langle\, \hat{s}(\phi) \,\big|\, q^*(\psi) \,\big\rangle = \big\langle\, \big(q\hat{s}\big)(\phi) \,\big|\, \psi \,\big\rangle$$

for all $\phi, \psi \in \mathcal{D}$ implies $\hat{s}\, q = q\, \hat{s}$, analogously also $\hat{s}\, q^* = q^* \hat{s}$.

Now assume that s is also continuous with respect to the graph topology on $\mathcal{D}$, then there exist $r \in Q$ and $C \in [0, \infty[$ such that $s(\phi, \psi) \leq C\, \|\phi\|_{r^*r} \|\psi\|_{r^*r}$ holds for all $\phi, \psi \in \mathcal{D}$, so

$$\big\|\hat{s}(\phi)\big\|_{r^*r}^2 = \big\langle\, \hat{s}(\phi) \,\big|\, \big(r^*r\, \hat{s}\big)(\phi) \,\big\rangle = \big\langle\, \big(\hat{s}\, r^*r\big)(\phi) \,\big|\, \hat{s}(\phi) \,\big\rangle = s\big(\big(r^*r\big)(\phi), \hat{s}(\phi)\big) \leq C\, \big\|\big(r^*r\big)(\phi)\big\|_{r^*r} \big\|\hat{s}(\phi)\big\|_{r^*r}$$

and thus $\big\|\hat{s}(\phi)\big\|_{r^*r} \leq C\, \big\|\big(r^*r\big)(\phi)\big\|_{r^*r} = C\, \|\phi\|_{(r^*r)^3}$ hold. Due to the directedness of the set of seminorms $\{\, \|\cdot\|_{q^*q} \mid q \in Q \,\}$ one can assume that there exists $D \in [0, \infty[$ such that $D\, \|\cdot\|_{r^*r} \geq \|\cdot\|$, so for all $q \in Q$ the estimate

$$\|\hat{s}(\phi)\|_{q^*q} = \big\|\big(q\hat{s}\big)(\phi)\big\| = \big\|\big(\hat{s}q\big)(\phi)\big\| \leq D\, \big\|\big(\hat{s}q\big)(\phi)\big\|_{r^*r} \leq CD\, \big\|q(\phi)\big\|_{(r^*r)^3} = CD\, \big\|\phi\big\|_{q^*(r^*r)^3 q}$$

holds for all $\phi \in \mathcal{D}$, proving continuity of $\hat{s}$. □

Note: Especially for $\mathcal{D} = \mathfrak{H}$ and $\mathcal{A} = \mathcal{B}(\mathfrak{H})$, the strictly self-adjoint O^*-algebra of all bounded (and thus automatically adjointable) endomorphisms of $\mathfrak{H}$ with $Q = \{\mathbb{1}\}$, this yields the usual Lax-Milgram theorem: Every separately continuous sesquilinear form on $\mathfrak{H}$ can be represented by an inner product with an adjointable endomorphism $\hat{s}$ on $\mathfrak{H}$, which therefore is also bounded due to the Hellinger-Toeplitz theorem.

For strictly self-adjoint O^*-algebras $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ there is thus an easy sufficient condition for bounded operators on the whole Hilbert space to restrict to elements of $\mathcal{L}^*(\mathcal{D})$:

Corollary A.4.59 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ a strictly self-adjoint O^*-algebra as well as $B \in \mathcal{L}^*(\mathfrak{H})$ a bounded linear operator. If $\langle\, B(\phi) \,|\, q(\psi) \,\rangle = \big\langle\, B\big(q^*(\phi)\big) \,\big|\, \psi \,\big\rangle$ and $\langle\, B(\phi) \,|\, q^*(\psi) \,\rangle = \big\langle\, B\big(q(\phi)\big) \,\big|\, \psi \,\big\rangle$ hold for all $\phi, \psi \in \mathcal{D}$ and all $q \in Q$ like in Definition A.4.55, then B can be restricted to some $b \in \mathcal{L}^*(\mathcal{D})$ which is bounded, commutes with all q and q^* with $q \in Q$ and is continuous with respect to the graph topology on $\mathcal{D}$.*

P ROOF: This follows immediately from the previous Theorem A.4.58 by considering the sesquilinear functional $\mathcal{D}^2 \ni (\phi, \psi) \mapsto \langle\, B(\phi) \,|\, \psi \,\rangle \in \mathbb{C}$, which shows that there exists a (necessarily unique) $b \in \mathcal{L}^*(\mathcal{D})$ fulfilling $\langle\, b(\phi) \,|\, \psi \,\rangle = \langle\, B(\phi) \,|\, \psi \,\rangle$ for all $\phi, \psi \in \mathcal{D}$ and which commutes with q and q^* for all $q \in Q$. As $\mathcal{D}^2 \ni (\phi, \psi) \mapsto \langle\, B(\phi) \,|\, \psi \,\rangle \in \mathbb{C}$ is continuous in the $\|\cdot\|$-topology, it is also continuous with respect to the graph topology so that b is also continuous. But b is also bounded as B was bounded.□

As an application, the continuous calculus for bounded operators defined on the whole Hilbert space can, under some conditions, be restricted to bounded operators on pre-Hilbert spaces:

Corollary A.4.60 *Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathfrak{H}$ and $\mathcal{A} \subseteq \mathcal{L}^*(\mathcal{D})$ a strictly self-adjoint O^*-algebra. If $b \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ commutes with q and q^* for all $q \in Q$ like in Definition A.4.55 and $\big(p_n\big)_{n\in\mathbb{N}}$ is a sequence of polynomials in $\mathbb{R}[x]$ such that $\lim_{n\to\infty} p_n(b) = \hat{b} \colon \mathcal{D} \to \mathfrak{H}$ converges in* *the operator norm, then even $\hat{b} \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$. Moreover, $\hat{b}$ is bounded, continuous with respect to the graph topology and commutes with all q and q^* for all $q \in Q$.*

PROOF: As $p_n(b)$ converges in the operator norm against $\hat{b}$, $\hat{b}$ is still bounded and fulfils

$$\langle \hat{b}(\phi) \mid q(\psi) \rangle = \lim_{n\to\infty} \langle \big(p_n(b)\big)(\phi) \mid q(\psi) \rangle = \lim_{n\to\infty} \langle \big(p_n(b)q^*\big)(\phi) \mid (\psi) \rangle = \langle \big(\hat{b}q^*\big)(\phi) \mid \psi \rangle$$

for all $\phi, \psi \in \mathcal{D}$, analogously also $\langle \hat{b}(\phi) \mid q^*(\psi) \rangle = \langle \big(\hat{b}q\big)(\phi) \mid \psi \rangle$. So the previous Corollary A.4.59 can be applied to $\hat{b}$ (or rather to its continuous extension to a bounded Hermitian operator on whole $\mathfrak{H}$).□

In contrast to the analogous results for bounded operators on the whole Hilbert space, these results for bounded operators on a pre-Hilbert space always require some additional algebraic assumptions: The relevant operators (or sesquilinear forms) have to commute with all $q \in Q$, where Q is the set of operators that describe the domain. While this is obviously no problem for commutative algebras, it might lead to problems when dealing with non-commutative ones.

A.5 Category Theory

In this thesis, category theory is only referred to at some points in order to describe some more general, and rather trivial, observations. Because of this, it should be sufficient to give the most basic definitions here, the ones of categories and functors. For a true introduction into this subject and more details, see one of the standard textbooks, e.g. [1]:

Definition A.5.1 *[1, Def. 3.1] A* category *is a quadruple $(O, \mathrm{Hom}, \mathsf{id}, \circ)$ consisting of*

i.) a class O of objects*,*

ii.) for each pair of objects $A, B \in O$ a set $\mathrm{Hom}(A, B)$ of morphisms from A to B*,*

iii.) for each object $A \in O$ a morphism $\mathsf{id}_A \in \mathrm{Hom}(A, A)$, the identity on A*,*

iv.) for each triple of objects $A, B, C \in O$ a composition $\circ\colon \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$,

such that the following conditions are fulfilled:

i.) The composition is associative (when defined).

ii.) The identities are left and right neutral (where the composition is defined).

iii.) The sets of morphisms between different pairs of objects are disjoint.

In this thesis, all relevant categories have sets with some additional structure as objects and special functions between them as morphisms. In this case it is understood that the composition is the usual composition of functions and that the identities are the $\mathsf{id}_X\colon X \to X$, $x \mapsto \mathsf{id}_X(x) := x$ for sets X.

Definition A.5.2 *[1, Def. 3.17] Let $\mathscr{C} = (O, \mathrm{Hom}, \mathsf{id}, \circ)$ and $\mathscr{C}' = (O', \mathrm{Hom}', \mathsf{id}', \circ')$ be two categories, then a* covariant functor *F from $\mathscr{C}$ to $\mathscr{C}'$ is a function that assigns to each object $A \in O$ an object $F(A) \in O'$ and to each morphism $f \in \mathrm{Hom}(A, B)$ with $A, B \in O$ a morphism $F(f) \in \mathrm{Hom}\big(F(A), F(B)\big)$, such that the following holds:*

i.) $F(f \circ g) = F(f) \circ F(g)$ *for all composable morphisms* f *and* g *of* $\mathscr{C}$*.*

ii.) $F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$ *for all* $A \in O$*.*

A contravariant functor *is defined in the same way, with the only changes being that a morphism* $f \in \operatorname{Hom}(A, B)$ *with* $A, B \in O$ *is mapped to a morphism* $F(f) \in \operatorname{Hom}\big(F(B), F(A)\big)$ *and that, instead of i.),* $F(f \circ g) = F(g) \circ F(f)$ *holds for all composable morphisms* f *and* g *of* $\mathscr{C}$*.*

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