Universality and Hypertranscendence of Zeta-Functions

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Würzburg, August 2019 *Athanasios Sourmelidis*

Abstract

The starting point of the thesis is the *universality* property of the Riemann Zeta-function $\zeta(s)$ which was proved by Voronin in 1975:

Given a positive number $\varepsilon > 0$ and an analytic non-vanishing function f defined on a compact subset $\mathcal{K}$ of the strip $\{s \in \mathbb{C} : 1/2 < \Re s < 1\}$ with connected complement, there exists a real number τ such that

$$\max_{s\in\mathcal{K}} |\zeta(s + i\tau) - f(s)| < \varepsilon. \tag{1}$$

In 1980, Reich proved a discrete analogue of Voronin's theorem, also known as *discrete universality theorem* for $\zeta(s)$:

If $\mathcal{K}$, f and ε are as before, then

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \max_{s\in\mathcal{K}} |\zeta(s + i\Delta n) - f(s)| < \varepsilon \right\} > 0, \tag{2}$$

where Δ is an arbitrary but fixed positive number.

We aim at developing a theory which can be applied to prove the majority of all so far existing discrete universality theorems in the case of Dirichlet L-functions $L(s, \chi)$ and Hurwitz zeta-functions $\zeta(s; \alpha)$, where χ is a Dirichlet character and $\alpha \in (0, 1]$, respectively. Both of the aforementioned classes of functions are generalizations of $\zeta(s)$, since $\zeta(s) = L(s, \chi_0) = \zeta(s; 1)$, where χ_0 is the principal Dirichlet character mod 1.

Amongst others, we prove statement (2) where instead of $\zeta(s)$ we have $L(s, \chi)$ for some Dirichlet character χ or $\zeta(s; \alpha)$ for some transcendental or rational number $\alpha \in (0, 1]$, and instead of $(\Delta n)_{n\in\mathbb{N}}$ we can have:

1. *Beatty sequences,*

2. *sequences of ordinates of c-points of zeta-functions from the Selberg class,*

3. *sequences which are generated by polynomials.*

In all the preceding cases, the notion of *uniformly distributed sequences* plays an important role and we draw attention to it wherever we can. Moreover, for the case of polynomials, we employ more advanced techniques from Analytic Number Theory such as bounds of exponential sums and zero-density estimates for Dirichlet L-functions. This will allow us to prove the existence of discrete second moments of $L(s, \chi)$ and $\zeta(s; \alpha)$ on the left of the vertical line $1 + i\mathbb{R}$, with respect to polynomials.

In the case of the Hurwitz Zeta-function $\zeta(s;\alpha)$, where α is transcendental or rational but not equal to $1/2$ or 1, the target function f in (1) or (2), where $\zeta(\cdot)$ is replaced by $\zeta(\cdot;\alpha)$, is also allowed to have zeros. Until recently there was no result regarding the universality of $\zeta(s;\alpha)$ in the literature whenever α is an algebraic irrational. In the second half of the thesis, we prove that a weak version of statement (1) for $\zeta(s;\alpha)$ holds for all but finitely many algebraic irrational α in $[A,1]$, where $A \in (0,1]$ is an arbitrary but fixed real number.

Lastly, we prove that the ordinary Dirichlet series $\zeta(s;f) = \sum_{n\geq 1} f(n)n^{-s}$ and $\zeta_\alpha(s) = \sum_{n\geq 1} \lfloor P(\alpha n + \beta)\rfloor^{-s}$ are hypertranscendental, where $f : \mathbb{N} \to \mathbb{C}$ is a *Besicovitch almost periodic arithmetical function*, $\alpha, \beta > 0$ are such that $\lfloor \alpha + \beta \rfloor > 1$ and $P \in \mathbb{Z}[X]$ is such that $P(\mathbb{N}) \subseteq \mathbb{N}$.

Zusammenfassung

Der Ausgangspunkt dieser Dissertation ist die folgende *Universalitätseigenschaft* der Riemannschen Zetafunktion $\zeta(s)$, die von Voronin 1975 nachgewiesen wurde:

Zu gegebenem $\varepsilon > 0$ und einer analytischen nullstellenfreien Funktion f, die auf einer kompakten Teilmenge $\mathcal{K}$ des Streifens $\{s \in \mathbb{C} : 1/2 < \Re s < 1\}$ mit zusammenhängendem Komplement definiert ist, existiert eine reelle Zahl τ, so dass

$$\max_{s\in\mathcal{K}} |\zeta(s + i\tau) - f(s)| < \varepsilon. \tag{1}$$

Im Jahr 1980 bewies Reich folgendes diskrete Analogon des Voroninschen Satzes, welches auch als *diskretes Universalitätstheorem* für $\zeta(s)$ bekannt ist:

Sind $\mathcal{K}$, f und ε wie oben, so gilt

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N : \max_{s\in\mathcal{K}} |\zeta(s + i\Delta n) - f(s)| < \varepsilon \right\} > 0, \tag{2}$$

wobei Δ eine beliebige, aber fest gewählte positive reelle Zahl bezeichnet.

Unser Ziel ist die Entwicklung einer Theorie, welche die Mehrheit der bislang bewiesenen diskreten Universalitätstheoreme im Fall Dirichletscher L-Funktionen $L(s, \chi)$ und Hurwitzscher Zetafunktionen $\zeta(s; \alpha)$ (wobei χ ein Dirichlet-Charakter ist und $\alpha \in (0, 1]$) umfasst. Beide genannten Funktionenklassen verallgemeinern $\zeta(s)$, denn $\zeta(s) = L(s, \chi_0) = \zeta(s; 1)$, wobei χ_0 der Hauptcharakter modulo 1 ist.

Neben anderen Resultaten beweisen wir Aussage (2) mit $L(s, \chi)$ für einen beliebigen Dirichlet-Charakter χ bzw. $\zeta(s; \alpha)$ für ein transzendentes oder rationales $\alpha \in (0, 1]$ anstelle von $\zeta(s)$ sowie $(\Delta n)_{n\in\mathbb{N}}$ ersetzt durch eine der nachstehenden Folgen:

1. *Beatty-Folgen,*
2. *Folgen von Imaginärteilen der c-Punkte einer beliebigen Zetafunktion der Selbergklasse,*
3. *Folgen, die durch ein Polynom generiert werden.*

In all diesen Fällen spielt der Begriff einer *gleichverteilten Folge* eine wichtige Rolle, und wir schenken diesem Aspekt besondere Beachtung im Folgenden. Speziell für den Fall der Polynome benutzen wir weitere fortgeschrittene Techniken der Analytischen Zahlentheorie, wie besipielsweise Schranken für Exponentialsummen und Nullstellen-Dichtigkeitsabschätzungen für Dirichletsche L-Funktionen. Dies erlaubt uns, die Existenz gewisser diskreter quadratischer Momente für $L(s, \chi)$ und $\zeta(s; \alpha)$ links der vertikalen Geraden $1+i\mathbb{R}$ im Polynom-Fall zu beweisen.

Im Fall der Hurwitzschen Zetafunktion $\zeta(s;\alpha)$, wobei α transzendent oder rational, aber ungleich 1/2 oder 1 ist, kann die zu approximierende Funktion f in (1) oder (2), wobei $\zeta(\cdot)$ durch $\zeta(\cdot;\alpha)$ zu ersetzen ist, sogar Nullstellen besitzen.

Bis vor kurzem waren hinsichtlich der Universalität von $\zeta(s;\alpha)$ in der Literatur für algebraisch-irrationale α keine Ergebnisse erzielt worden. Im zweiten Teil der Dissertation beweisen wir eine schwache Version der Aussage (1) für $\zeta(s;\alpha)$ für alle algebraisch-irrationalen $\alpha \in [A,1]$ bis auf höchstens endlich viele Ausnahmen, wobei $A \in (0,1]$ eine beliebige, aber fest gewählte reelle Zahl ist.

Schließlich weisen wir die Hypertranszendenz der gewöhnlichen Dirichlet-Reihen $\zeta(s;f) = \sum_{n\geq 1} f(n)n^{-s}$ und $\zeta_\alpha(s) = \sum_{n\geq 1} \lfloor P(\alpha n + \beta)\rfloor^{-s}$ nach, wobei $f : \mathbb{N} \to \mathbb{C}$ irgendeine *Besicovitch-fastperiodische zahlentheoretische Funktion* ist, $\alpha, \beta > 0$ der Ungleichung $\lfloor \alpha + \beta \rfloor > 1$ genügt und $P \in \mathbb{Z}[X]$ die Bedingung $P(\mathbb{N}) \subseteq \mathbb{N}$ erfüllt.

Notations

The *Vinogradov symbols* $\ll$ and $\gg$ have their usual meaning, and if the implied constant depends on some parameter ϵ (say), then we write $\ll_\epsilon$. If we say that some variable *be fixed*, then this usually entails that we drop such subscripts even if the implied constant depends on said variable. The same comments apply to the *Landau symbols* $o(\cdot)$ and $O(\cdot)$.

We indicate some of the notations and conventions used in this thesis. Sometimes we explain some of the notations listed below for convenience.

$\mathscr{A}, \mathscr{B}, \ldots$	usually denote finite sets.
$\mathcal{A}, \mathcal{B}, \ldots$	usually denote infinite sets.
$\mathbb{1}_A$	denotes the characteristic function of a set A.
$d \mid x$	means that d divides x.
$\mathrm{e}(x)$	$= \exp(2\pi i x)$.
$f(x) \asymp g(x)$	means $f(x) \ll g(x)$ and $g(x) \ll f(x)$.
$f(x) \sim g(x)$	means $f(x) = g(x)(1 + o(1))$ as $x \to \infty$.
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	are the sets of positive integers, integers, rational numbers, real numbers and complex numbers, respectively.
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$.
$\mathscr{A}_{>0}$	$= \mathscr{A} \cap (0, +\infty)$ for $\mathscr{A} \subseteq \mathbb{R}$.
$\Re z$, $\Im z$	denote the real part and imaginary part of z, respectively.
$s = \sigma + it$	s is a complex number with $\Re s = \sigma$ and $\Im s = t$.
$\mathcal{D}$	$= \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$.
p	usually denotes a prime number unless stated otherwise.
$\phi(n)$	$= \sharp\{1 \leq m \leq n : m \text{ coprime to } n\}$.
$\pi(x)$	$= \sharp\{p \leq x : p \text{ prime}\}$.
$\sharp\mathscr{A}$	denotes the cardinality of a set $\mathscr{A}$.

$\mathrm{m}(\mathcal{A})$	denotes the Lebesgue measure of a set $\mathcal{A}$.
$\|z\|$	is the absolute value of $z \in \mathbb{C}$.
$\lfloor x \rfloor$	is the largest integer less than or equal to $x \in \mathbb{R}$.
$\{x\}$	is $x - \lfloor x \rfloor$ for $x > 0$.
$\\|x\\|$	$= \min_{y \in \mathbb{Z}} \|x - y\|$.
$\overline{z}$	the conjugate of $z \in \mathbb{C}$.

Chapter 1

Introduction

We start with the definition of the Riemann Zeta-function and we refer to some of its most significant properties. We state Voronin's theorem about the universality of $\zeta(s)$ and then give an exposition of results with respect to universality theorems for large classes of zeta- and L- functions. In particular, we focus on Dirichlet L-functions $L(s,\chi)$ and Hurwitz zeta-functions $\zeta(s;\alpha)$ since their value-distribution is the main scope of this thesis.

1.1 Preludé

1.1.1 The Riemann Zeta-Function $\zeta(s)$

Zeta- and L- functions play a central role in analytic number theory. The Riemann Zeta-function, defined for a complex variable $s := \sigma + it$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1, \tag{1.1}$$

may be regarded as the prototype. Both the Dirichlet series and the Euler product which appear in the identity above, are absolutely convergent in the half-plane $\sigma > 1$ and uniformly convergent in each compact subset of this half-plane. The identity between them was discovered by Euler [17] and it may be interpreted as an analytic version of the unique prime factorization of integers. It was this simple observation that allowed Euler to prove the infinitude of primes, a fact already known since Euclid's elementary proof; however, Euler's reasoning was by means of analysis. Assuming that there were only finitely many primes, the product in (1.1) is finite, and therefore convergent for $s = 1$, contradicting the fact that the Dirichlet series in (1.1) reduces to the divergent harmonic series as $s \to 1^+$.

Riemann [73] was the first who treated $\zeta(s)$ as a function of a complex variable. In his only paper on Number Theory he obtained a series of remarkable results regarding $\zeta(s)$. He proved that $\zeta(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$. Moreover, he discovered and proved the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.2}$$

for all complex numbers s, where $\Gamma(s)$ denotes the Gamma-function.

It can be easily seen from (1.1) and (1.2) that the only zeros of $\zeta(s)$ in the half-planes $\sigma < 0$ and $\sigma > 1$ are at the negative even integers $-2n$, $n \in \mathbb{N}$. Indeed, the absolute convergence in the half-plane $\sigma > 1$ of the Euler product in (1.1) yields that $\zeta(s)$ is zero-free in this half-plane. Then the functional equation (1.2) implies that $\zeta(s)$ has to have zeros in exactly those places where $\Gamma(s/2)$ has poles for $\sigma < 0$; namely, the points $-2n$, $n \in \mathbb{N}$, which are also called the *trivial zeros* of $\zeta(s)$. In addition, the reflection principle for $\zeta(s)$,

$$\zeta(\overline{s}) = \overline{\zeta(s)},$$

allows us to study $\zeta(s)$ in the upper half-plane.

In the same memoir, Riemann

1. outlined how a proof of the celebrated prime number theorem would follow by establishing fundamental properties of $\zeta(s)$,

2. conjectured an asymptotic formula for the number of zeros of $\zeta(s)$ in the region $0 \leq \sigma \leq 1$, $0 < t \leq T$ (counted according multiplicities);

3. conjectured that all *non-trivial zeros* of $\zeta(s)$, that is, complex zeros with real part in the interval $[0, 1]$, lie on the vertical line $1/2 + i\mathbb{R}$.

The prime number theorem states that

$$\pi(x) := \sharp\{p \leq x : p \text{ prime}\} \sim \int_2^x \frac{\mathrm{d}u}{\log u}.$$

This formula was first conjectured by Gauss [23] and it was proved by Hadamard [29] and de la Vallée-Poussin [92] (independently) who have built on ideas of Riemann and showed that $\zeta(1 + it) \neq 0$ for all real $t \neq 0$. This can be seen to be equivalent to the prime number theorem. It also follows from the functional equation that $\zeta(it) \neq 0$ for real numbers t. Therefore, the non-trivial zeros of $\zeta(s)$ lie inside the open strip $0 < \sigma < 1$. This strip is also called the *critical strip* of $\zeta(s)$ since it contains vital information with respect to the zero-distribution of $\zeta(s)$.

Riemann's conjecture on the asymptotic formula

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e},$$

where $N(T) := \{\rho = \beta + i\gamma : 0 < \beta < 1,\, 0 < \gamma \leq T,\, \zeta(\rho) = 0\}$, was proved by von Mangoldt [94, 95]. In particular, von Mangoldt proved more precisely that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T), \;\; T \geq 1.$$

Finally, Riemann's conjecture on the non-trivial zeros of $\zeta(s)$ having all real part $\sigma = 1/2$ is known as the *Riemann Hypothesis (RH)* and it remains unsolved so far. There have been various attempts to prove this hypothesis by several mathematicians over the last 160 years since Riemann posed his conjecture and, although they were not successful, they provided new insights on the zero-distribution of $\zeta(s)$. For example, Hardy [30] showed that infinitely many zeros of $\zeta(s)$ lie on the vertical line $1/2 + i\mathbb{R}$, which is aso known as the *critical line*, while Selberg [80] was the first to prove that a positive proportion of all non-trivial zeros of $\zeta(s)$ have real part $\sigma = 1/2$.

In another direction there has been a lot of effort on proving zero-free regions of $\zeta(s)$ in $0 < \sigma < 1$ because they yield an asymptotic formula of $\pi(x)$ with a remainder term. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov [93] and Korobov [47] (independently) who proved that

$$\zeta(s) \neq 0 \text{ for } \sigma \geq 1 - (\log|t|)^{-1/3}(\log\log|t|)^{-2/3}$$

as $|t| \to +\infty$. This non-vanishing implies

$$\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O\left(x \exp\left(-C\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right),$$

as $x \to \infty$, where C is an absolute positive constant.

Von Koch [44] showed for fixed $\theta \in [1/2, 1)$ that

$$\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O_\epsilon(x^{\theta+\epsilon}) \quad \text{as} \quad x \to \infty \quad \Leftrightarrow \quad \zeta(s) \neq 0 \quad \text{for} \quad \sigma > \theta.$$

Here *RH* would imply that one can take $\theta = 1/2$. Moreover, with regard to known zeros of $\zeta(s)$ on the critical line, it is obvious that θ can not be smaller than $1/2$. Thus, *RH* states that the prime numbers are as uniformly distributed as possible.

This is only the "peak of the iceberg" with respect to the value-distribution theory of $\zeta(s)$, especially inside the critical strip, and its applications in problems arising in Number Theory. We refer to to the monographs of Ivić [38] and Titchmarsh [90] for a detailed exposition of results regarding $\zeta(s)$.

1.1.2 Voronin's Universality Theorem

In the 1910s Bohr developed methods to investigate the value-distribution of $\zeta(s)$ inside the critical strip. In particular, his approach, inspired by concepts from probability theory, led to important insights. At first [5] he obtained a theorem for $\zeta(s)$ which nature is very similar to the Big Picard Theorem:

Theorem 1.1. *In any strip* $1 < \sigma < 1+\epsilon$, $\zeta(s)$ *takes any non-zero value infinitely often.*

Later, in joint work with Courant [7], he proved

Theorem 1.2. *Let $\sigma \in (1/2, 1]$ be fixed. Then the set of $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

Of course, this denseness cannot hold in the half-plane of absolute convergence of the $\zeta(s)$ defining Dirichlet series; whether or not this is also true for the critical line $1/2 + i\mathbb{R}$ is still open (cf. [86]).

Bohr's line of investigation appears to have been forsaken for some decades. It was in 1972 when Voronin [98] proved a multi-dimensional version of Bohr's latter result:

Theorem 1.3. *Given a positive number ε, a positive integer N and a vector $(a_0, a_1, \ldots, a_N) \in \mathbb{C}^{N+1}$ satisfying $a_0 \neq 0$, there exists a positive integer n such that*

$$|\zeta^{(j)}(s + in) - a_j| < \varepsilon, \;\; j = 0, 1, \ldots, N. \tag{1.3}$$

Here s has to lie in the right half of the critical strip, i.e., $1/2 < \mathrm{Re}\, s < 1$ as in Bohr's theorem, and $\zeta^{(j)}(s)$ denotes the j-th derivative of $\zeta(s)$ and $\zeta^{(0)}(s) = \zeta(s)$.

In view of this remarkable approximation property of $\zeta(s)$ and its derivatives one might consider an application in order to approximate not a vector of complex numbers but a a complex-valued function. Of course, Taylor expansion is the tool to step from approximating numbers to approximation of a function. Indeed this construction has been realized by Garunkštis *et al.* [21].

It is not clear whether Voronin had this idea in mind but while writing his doctoral thesis in 1975 he succeeded in proving the following infinite-dimensional analogue [99]:

Theorem 1.4. *Given a positive number ε and an analytic non-vanishing function f defined somewhere on a disc with center 0 and radius $r < 1/4$, there exists a real number τ such that*

$$\max_{|s| \leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon;$$

Since the set of target functions is almost unlimited, this phenomenon is called *universality*. Voronin called his theorem the *theorem about little discs*. Reich [69] and Bagchi [2] improved Voronin's result significantly in replacing the disc by an arbitrary compact set in the right half of the critical strip with connected complement, and by giving a proof in the language of probability theory. The strongest version of Voronin's theorem has the form:

Theorem 1.5. *Let $\mathcal{K}$ be a compact set in the right half of the critical strip $\mathcal{D} := \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complement, and let $f(s)$ be a non-vanishing continuous function defined on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then for every $\varepsilon > 0$*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{m} \left\{ \tau \in [0, T] : \max_{s \in \mathcal{K}} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0. \tag{1.4}$$

We may interpret the absolute value of an analytic function as an analytic landscape over the complex plane. Then the universality theorem states that any finite analytic landscape can be found (up to an arbitrarily small error) in the analytic landscape of $\zeta(s)$.

1.2 Generalizations

1.2.1 Dirichlet L-Functions and Hurwitz Zeta-Functions

The Dirichlet L-functions were introduced by Dirichlet [14] in order to tackle the problem of the distribution of primes in arithmetic progressions $an + b$, $n \in \mathbb{N}$, for coprime positive integers a and b.

If q is a positive integer, then a Dirichlet character mod q is a non-vanishning group homomorphism from the group $(\mathbb{Z}/q\mathbb{Z})^*$ of prime residue classes modulo q to $\mathbb{C}$. Therefore, the number of Dirichlet characters mod q is $\phi(q)$, where ϕ is Euler's totient function. The character which is identically one, is called the principal character and is denoted usually by χ_0. By setting $\chi(a) = 0$ on the non-prime residue classes, such a character extends via $\chi(n) = \chi(a)$ for $n \equiv a$ mod q, to a completely multiplicative arithmetic function, that is

$$\chi(mn) = \chi(m)\chi(n),$$

for any positive integers m and n. Then the Dirichlet L-function $L(s, \chi)$ attached to a character χ mod q is given by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \ \ \sigma > 1.$$

This class of functions contains $\zeta(s)$, since $\zeta(s) = L(s, \chi_0)$, where χ_0 is the principal Dirichlet character mod 1. From here on $L(s, \chi)$ will always denote a Dirichlet L-function associated with a Dirichlet character χ mod q.

It may happen that for values of n coprime with q, the character $\chi(n)$ may have a period less than q. If that is the case, we say that χ is imprimitive, and otherwise primitive; the princicipal character is not considered to be primitive. It can be seen that any imprimitive character is induced by a primitive character. Moreover, two characters are said to be non-equivalent if they are not induced by the same primitive character. The characters to a common modulus are pairwise non-equivalent.

Dirichlet L-functions share many properties with $\zeta(s)$. They also have a meromorphic continuation to the whole complex plane, with the only difference that $L(s, \chi)$ is regular at $s = 1$ for any non-principal Dirichlet character χ. One can obtain a Riemann-von Mangoldt formula and similar zero-free regions as for $\zeta(s)$. Furthermore, Dirichlet L-functions to primitive characters satisfy a Riemann-type functional equation similar to (1.2). Using analogous techniques as for $\zeta(s)$, one can prove that $L(s, \chi) \neq 0$ for $\sigma \geq 1$ and, therefore, obtain the following prime number theorem for arithmetic progressions:

$$\pi(x; a \mod q) := \sharp \{p \leq x : p \equiv a \mod q, \, p \text{ prime}\} \sim \frac{\pi(x)}{\phi(q)}.$$

Lastly, the analogue of *RH* exists also for Dirichlet L-functions; the so-called *Generalized Riemann Hypothesis (GRH)* states that

$$L(s,\chi) \neq 0 \quad \text{for all} \quad \sigma > \frac{1}{2} \quad \text{and} \quad \chi.$$

Yet another generalization of $\zeta(s)$ is the Hurwitz Zeta-function which is defined by

$$\zeta(s;\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \sigma > 1.$$

Here α is a real parameter from the interval $(0,1]$ and for $\alpha = 1$ we obtain $\zeta(s)$. Hurwitz [37] himself treated only Hurwitz zeta-functions with a rational parameter. In his investigations on Dirichlet's analytic class number formula, he studied Dirichlet series of the form

$$\sum_{n \equiv a \bmod m} \frac{1}{n^s}$$

which can be rewritten as $m^{-s}\zeta(s;a/m)$.

It can be seen that $\zeta(s;\alpha)$ can be analytically continued to the whole complex plane except for a simple pole at $s=1$ and it satisfies a Riemann-type functional equation; however, on the other side of such equation does not appear $\zeta(1-s;\alpha)$ but $L(1-s;\alpha,1)$, where $L(s;\lambda,\alpha)$ is the Lerch Zeta-function defined by

$$L(s;\lambda,\alpha) := \sum_{n=0}^{\infty} \frac{\mathrm{e}(\lambda \mathrm{n})}{(n+\alpha)^s}, \quad \sigma > 1,$$

with parameters $\lambda \in \mathbb{R}$ and $\alpha \in (0,1]$. For a survey on the Lerch Zeta-function we refer to [52].

There exists a very convenient connection between $L(s,\chi)$ and $\zeta(s;\alpha)$ with rational parameter α, which can be described by the following two identities:

$$L(s,\chi) = \frac{1}{q^s} \sum_{r=1}^{q-1} \chi(r) \zeta\left(s; \frac{r}{q}\right) \tag{1.5}$$

and if q and r are positive integers such that $0 < r < q$ and $(r,q) = 1$, then

$$\zeta\left(s; \frac{r}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi(r)} L(s,\chi), \tag{1.6}$$

both valid for all $s \in \mathbb{C}$, where the summation runs over all Dirichlet characters $\chi \bmod q$.

It appears that switching from a rational to an irrational parameter does not affect analytic continuation, functional identities and the order of growth, however, the zero-distribution definitely depends and the general distribution of values might depend on the diophantine nature of the parameter.

A major difference between $\zeta(s;\alpha)$ for $\alpha \neq 1/2, 1$ and $\zeta(s)$ or $L(s,\chi)$ is the absence of an Euler product representation in the half-plane $\sigma > 1$. This already differentiates the zero-distribution of $\zeta(s;\alpha)$ from $L(s,\chi)$. It has been proved by Davenport and Heilbronn [13] and Cassels [12] that $\zeta(s;\alpha)$ has zeros for $\alpha \neq 1/2, 1$ in the half-plane $\sigma > 1$.

In the succeeding chapters we will need the following formula for $\zeta(s;\alpha)$ which holds for all $0<\alpha\le 1$, $0<\sigma_0\le\sigma\le 2$ and $\pi x\ge t\ge t_0>0$:

$$\zeta(s;\alpha)=\sum_{0\le n\le x}\frac{1}{(n+\alpha)^s}+\frac{x^{1-s}}{1-s}+O_{\sigma_0}\left(x^{\sigma}\right). \tag{1.7}$$

For a proof we refer to [43, Chapter III,§2,Theorem 1]. Such a formula is usually called an *approximate functional equation* and has been proven firstly for $\zeta(s)$.

1.2.2 Universality Theorems and Applications of Them

The first who proved universality theorems for Dirichlet L-functions were Voronin [97], Gonek [24] and Bagchi [2] (independently):

Theorem 1.6. *Let $\chi_1,\dots,\chi_J$ be pairwise non-equivalent Dirichlet characters, $\mathcal{K}_1,\dots,\mathcal{K}_J$ be disjoint compact sets of the strip $\mathcal{D}$ with connected complements. Further, for each $1\le k\le J$, let $f_k(s)$ be a continuous non-vanishing function on $\mathcal{K}_k$ which is analytic in the interior of $\mathcal{K}_k$. Then, for any $\varepsilon>0$*

$$\liminf_{T\to\infty}\mathrm{m}\left\{\tau\in[0,T]:\max_{1\le k\le J}\max_{s\in\mathcal{K}_k}|L(s+i\tau,\chi_k)-f_k(s)|<\varepsilon\right\}>0.$$

Observe that the latter universality theorem is proved for a family of L-functions and not just a single L-function. For that reason it is also called *joint universality theorem* and in order to be proved, some sort of "independence" has to be posed upon the L-functions which appear in the theorem. Here the orthogonality relation of Dirichlet characters plays that role:

$$\sum_{a \bmod q}\chi(a)=\begin{cases}\phi(q), & \text{if } \chi=\chi_0,\\ 0, & \text{otherwise.}\end{cases}$$

Gonek [24] and Bagchi [2] in their respective theses proved, additionally to the aforementioned theorem, that $\zeta(s;\alpha)$ is universal whenever α is not algebraic irrational. In fact, with the additional assumption of $\alpha\ne 1/2,1$ they showed:

Theorem 1.7. *Let $\alpha\in(0,1]$ be a transcendental number or a rational number different from $1/2$ and 1. Let also $\mathcal{K}$ be a compact set of the strip $\mathcal{D}$ with connected complement, and let $f(s)$ be a continuous function defined on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\varepsilon>0$*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{m}\left\{\tau\in[0,T]:\max_{s\in\mathcal{K}}|\zeta(s+i\tau;\alpha)-f(s)|<\varepsilon\right\}>0.$$

For the proof of the aforementioned theorem when α is transcendental, it is crucial that the numbers

$$\log\alpha,\ \log(1+\alpha),\ \log(2+\alpha),\dots$$

are linearly independent over $\mathbb{Q}$. On the other hand, when α is rational but not $1/2$ or 1, then the theorem follows almost directly from the joint universality theorem of Dirichlet L-functions and relation (1.6). Unfortunately, when α is algebraic irrational, both of the previous methods fail. We will return to this discussion in Chapter 4.

It is quite remarkable that the target function f in the preceding theorem is allowed to have zeros. When such a universality statement holds without the assumption of f being non-vanishing, we call it *strong universality theorem*. Strong universality has an important application to the distribution of zeros. Let $a < \sigma_1 < \sigma_2 < b$ and let

$$N(t;\sigma_1,\sigma_2;L) := \sharp\left\{\rho \in \mathbb{C} : \sigma_1 \leq \Re\rho \leq \sigma_2,\ 0 \leq \Im\rho \leq t,\ L(\rho) = 0\right\}$$

for some function L. If a strong universality theorem can be proved for L in the strip $a < \sigma < b$, then

$$N(T;\sigma_1,\sigma_2;L) \gg T, \quad T \geq 1.$$

The proof is straightforward and we mention it here briefly. The strong universality theorem implies that there is $\tau > 0$ such that $L(s+i\tau)$ is ε-close to a function f having a zero in a disc inside the strip $a < \sigma < b$, where ε is equal to the minimum of f on the disc. Then Rouché's theorem yields that also $L(s+i\tau)$ has a zero in that disc and from the positive lower density of the set of τ given in the strong universality theorem, the claim follows. This also explains why the strong universality theorem can not be true for $\zeta(s;\alpha)$ when $\alpha = 1/2$ or 1. For, otherwise, $\zeta(s;1/2) = (2^s-1)\zeta(s)$ or $\zeta(s;1) = \zeta(s)$ and, thus, the strong universality theorem would be true for $\zeta(s)$. But from the above discussion, it would follow for some fixed $\epsilon > 0$ that

$$N\left(T;\frac{1}{2}+\epsilon,1;\zeta\right) \gg T,$$

which contradicts known zero-density estimates for $\zeta(s)$ in the strip $\mathcal{D}$.

We conclude this section by discussing one more type of universality. The first *discrete universality theorem* is due to Reich [71] for Dedekind zeta-functions. Let $\mathbb{K}$ be an algebraic number field (i.e., a finite extension of $\mathbb{Q}$) of degree $d_{\mathbb{K}} = [\mathbb{K}:\mathbb{Q}]$. Then the associated Dedekind zeta-function is for $\sigma > 1$ defined by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \quad \sigma > 1;$$

here the sum is taken over all non-zero integral ideals, the product is taken over all prime ideals of the ring of integers of $\mathbb{K}$ and $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$. Reich proved the following:

Theorem 1.8. *Let $\mathcal{K}$ be a compact set of $\left\{s \in \mathbb{C} : 1 - \max\left\{2, d_{\mathbb{K}}\right\}^{-1} < \sigma < 1\right\}$ with connected complement, and f be a non-vanishing continuous function on $\mathcal{K}$ that is analytic in its interior. Then, for any $h, \varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N : \max_{s\in\mathcal{K}} |\zeta_{\mathbb{K}}(s+ihn) - f(s)| < \varepsilon\right\} > 0. \tag{1.8}$$

Observe that the case of $\zeta(s)$ is also included in the latter theorem since $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ and $d_{\mathbb{Q}} = 1$. Also, if the degree $d_{\mathbb{K}}$ is large, then the vertical strip where we have universality gets thinner and its width depends on $d_{\mathbb{K}}$. However, in case of abelian fields $\mathbb{K}$, the strip of universality can be extended to the whole strip $\mathcal{D}$ independently of the degree $d_{\mathbb{K}}$, as Gonek [24] showed.

Bagchi [2] obtained the joint discrete universality of Dirichlet L-functions. Based on his work, Sander and Steuding [76] obtained a discrete joint universality theorem for Hurwitz zeta-functions. There exists a rich literature on discrete universality results for zeta-functions and there are at least twice as many as the current ones for the *continuous universality.* The reason is that in such theorems we can also ask what other sequences could replace the arithmetic progressions $(hn)_{n\in\mathbb{N}}$ in (1.8). It is apparent that discrete universality provides more information regarding the value-distribution of the zeta- or L-function we wish to study, since we can examine this function's behaviour over a prescribed (discrete) set.

For more results on (discrete) universality theorems and more references we suggest to the interested reader the book of Laurinčikas [50], the survey of Matsumoto [56] and the monograph of Steuding [86].

The most spectacular application of universality is Riemann's hypothesis. It has been discovered by Bagchi [3] that *RH* is equivalent to the property that $\zeta(s)$ can approximate itself in the sense of Voronin's theorem (1.4). Indeed, if there are zeros to the right of the critical line, then the approximation property in combination with Rouché's theorem would imply the existence of infinitely many complex zeros in a domain where there should not be any zero at all; a classical zero-density estimate for the number of hypothetical zeros off the critical line and the positive lower density for the approximating shifts conclude the reasoning in the same way we described above.

Another consequence of universality for a zeta- or an L- function is the functional independence of the corresponding function. We say that the functions $f_1(s), \ldots, f_m(s)$ are *functionally independent* if for any continuous functions $F_0, F_1, \ldots, F_N : \mathbb{C}^m \to \mathbb{C}$, not all identically vanishing, the function

$$\sum_{k=0}^{N} s^k F_k(f_1(s), \ldots, f_m(s))$$

is non-zero for some values of s. It was Voronin [96] who first proved that Dirichlet L-functions are functionally independent by applying his universality theorem (in fact, it may have been his original aim until he discovered the universality property of $\zeta(s)$). We give here a general statement of Voronin's theorem and a proof can be found in [86, Theorem 10.3].

Theorem 1.9. *Assume that L is a function satisfying the universality property* (1.4) *in place of $\zeta(s)$, $\mathbf{z} = (z_0, z_1, \ldots, z_{m-1}) \in \mathbb{C}^m$ and suppose that $F_0(\mathbf{z})$, $F_1(\mathbf{z}), \ldots, F_n(\mathbf{z})$ are given continuous functions, not all identically zero. Then there is $s \in \mathbb{C}$ such that*

$$\sum_{k=0}^{N} s^k F_k \left(L(s), L'(s), \ldots, L^{(m-1)}(s) \right) \neq 0$$

1.3 Outline of the Thesis

We aim at a deeper understanding of the value-distribution of Dirichlet L-functions and Hurwitz zeta-functions inside the strip $\mathcal{D}$.

In Chapter 2 we obtain discrete second moments for the aforementioned zeta- and L-functions with respect to polynomials. We prove by elementary methods that if $\sigma > 1$ and $d \in \mathbb{N}$, then for almost all vectors $(a_1, \ldots, a_d) \in [0, +\infty)^d$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\zeta\left(\sigma + i\left(a_1 n + \cdots + a_d n^d\right); \alpha\right)\right|^2 = \zeta(2\sigma; \alpha)$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|L\left(\sigma + i\left(a_1 n + \cdots + a_d n^d\right), \chi\right)\right|^2 = L(2\sigma, \chi_0).$$

Moreover, we obtain the same result on the left of the vertical line $1 + i\mathbb{R}$ by using more advanced tools from Analytic Number Theory, such as estimates for Weyl sums and order estimates of $L(s, \chi)$ and $\zeta(s; \alpha)$ near the line $1 + i\mathbb{R}$.

In Chapter 3 we prove a series of discrete universality criteria for $L(s; \chi)$ and $\zeta(s; \alpha)$ with respect to uniformly distributed sequences $(x_n)_{n\in\mathbb{N}}$ without posing growth conditions on x_n as it is usually done in such theorems. These criteria were motivated by our previous work [53, 83, 84]. To justify our results we give three examples of such sequences that will satisfy our universality criteria. They all have less *structure* than an arithmetic progression $(hn)_{n\in\mathbb{N}}$ which is the most attractive (but easy) candidate for discrete universality theorems. The example on the c-points that will be mentioned further ahead is the outcome of joint work with Jörn Steuding and Teerapat Srichan. Building on results from Chapter 2 and using basic zero-density estimates for $L(s; \chi)$, we prove in the final part of this chapter a discrete universality theorem by using the logarithms of $L(s, \chi)$.

In Chapter 4 we tackle the open question of whether $\zeta(s; \alpha)$ with algebraic irrational has the universality property or not. This is joint work with Jörn Steuding and we give partially an affirmative answer. We also obtain effective universality results.

In Chapter 5, we prove by elementary means, a weak version of the functional independence property for Dirichlet series for which is not known yet whether they are universal or not. The statements can also be found in [82]. This series are generated by almost periodic arithmetical functions and Beatty sets. Beyond the scope of this thesis, we have also proved analytic continuation [85] of such series.

Chapter 2

Discrete Moments with respect to Polynomials

In this chapter we obtain discrete second moments for $\zeta(\sigma+it;\alpha)$ and $L(\sigma+it,\chi)$ in the half-plane $\sigma > 1/2$ with respect to shifts of the imaginary argument t. In particular, we generalize some results regarding the Riemann Zeta-function and we study the case when these *vertical* shifts are generated by a polynomial of arbitrary degree. In the last section we recall the Lindelöf Hypothesis and state a conditional result.

2.1 In the Half-Plane $\sigma > 1$

From here on $\underline{a}, \underline{b}, \dots$ will always denote a vector of $[0,+\infty)^d$ while $P_{\underline{a}}(x) := a_1 x + \cdots + a_d x^d$ define a real-valued polynomial. Since $\zeta(s;\alpha)$ and $L(s,\chi)$ are absolutely convergent in the half-plane $\sigma > 1$, computing their discrete second moments with respect to $P_{\underline{a}}(x)$ is quite easy if one is willing to omit a negligible set of $[0,+\infty)^d$. This set can be written explicitly:

$$\mathcal{L}(d,\alpha) := \bigcap_{\underline{r}\in\mathbb{Q}^d} \bigcap_{\substack{k,\ell=0 \\ k\neq\ell}}^{\infty} \left\{ \underline{a} \in [0,+\infty)^d : a_i \log \frac{k+\alpha}{\ell+\alpha} \neq 2\pi r_i,\ 1 \le i \le d \right\}.$$

We can prove now the following theorem:

Theorem 2.1. *Let $d \geq 1$ be an integer, $\alpha \in (0,1]$ and $\sigma > 1$. Then, for any $\underline{a} \in \mathcal{L}(d,\alpha)$ and any $\underline{b} \in \mathcal{L}(d,1)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\zeta(\sigma + iP_{\underline{a}}(n);\alpha)|^2 = \zeta(2\sigma;\alpha)$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |L(\sigma + iP_{\underline{b}}(n),\chi)|^2 = L(2\sigma,\chi_0).$$

Proof. We have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\zeta(\sigma+iP_{\underline{a}}(n);\alpha)|^2 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{k,\ell=0}^{\infty}\frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}\\ &= \sum_{k,\ell=0}^{\infty}\frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}\\ &=: A,\end{aligned}$$

where interchanging summation and the limit operator is valid by the absolute convergence of the double series $\sum_{k,\ell}((k+\alpha)(\ell+\alpha))^{-\sigma}$. If we split now the latter double sum in sums of diagonal and non-diagonal terms, we get

$$A=\zeta(2\sigma;\alpha)+\sum_{\substack{k,\ell=0\\k\neq\ell}}^{\infty}\frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}\left(\frac{P_{\underline{a}}(n)}{2\pi}\log\frac{k+\alpha}{\ell+\alpha}\right).$$

Observe that for any pair (k,ℓ) with $k\neq\ell$, $\log((k+\alpha)/(\ell+\alpha))\,P_{\underline{a}}(x)/2\pi$ is a polynomial with at least one irrational coefficient as can be seen from our choice of the vector $\underline{a}$. Therefore, Theorems A.3 and A.4 yield

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}\left(\frac{P_{\underline{a}}(n)}{2\pi}\log\frac{k+\alpha}{\ell+\alpha}\right)=0$$

for any pair (k,ℓ) with $k\neq\ell$, and the first part of the theorem follows.

The case of $L(s,\chi)$ is similar. Indeed, since $\underline{b}\in\mathcal{L}(d,1)$, we obtain that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|L(\sigma+iP_{\underline{b}}(n),\chi)|^2 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{k,\ell=1}^{\infty}\frac{\chi(k)\overline{\chi(\ell)}}{k^{\sigma}\ell^{\sigma}}\left(\frac{k}{\ell}\right)^{iP_{\underline{b}}(n)}\\ &= \sum_{k=1}^{\infty}\frac{|\chi(k)|^2}{k^{2\sigma}}+\\ &\quad+\sum_{\substack{k,\ell=1\\k\neq\ell}}^{\infty}\frac{\chi(k)\overline{\chi(\ell)}}{k^{\sigma}\ell^{\sigma}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}\left(\frac{P_{\underline{b}}(n)}{2\pi}\log\frac{k}{\ell}\right)\\ &= L(2\sigma,\chi_0).\end{aligned}$$

□

We do not discuss the case of $\underline{a}\in[0,+\infty)^d\setminus\mathcal{L}(d,\alpha)$, mainly because the primary results of this chapter, which are given in the next section, are also of metric nature. We state here, however, the case of $d=1$ for $L(s,\chi)$, which was proved by Reich [70]. Before that we need to introduce the finite Euler products

$$L_{\mathscr{M}}(s,\chi):=\prod_{p\in\mathscr{M}}\left(1-\frac{\chi(p)}{p^s}\right)^{-1},\ \ \sigma>0,$$

where $\mathscr{M}$ is a finite set of primes and χ a Dirichlet character mod q.

Theorem 2.2. *Let* $\mathscr{A} = \left\{2\pi m \left(\log(k/\ell)\right)^{-1} : k, \ell, m \in \mathbb{N} \ \textit{ with } \ k \neq \ell\right\}$.

i. *For* $0 < a_1 \notin \mathscr{A}$ *and* $\sigma > 1/2$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |L(\sigma + ia_1 n, \chi)|^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T |L(\sigma + ia_1 t, \chi)|^2 \mathrm{d}t.$$

ii. *For* $0 < a_1 \in \mathscr{A}$ *and* $\sigma > 1/2$, *there exists a finite set of prime numbers* $\mathscr{M}$, *independently from* L *and* σ, *such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |L(\sigma + ia_1 n, \chi)|^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |L_{\mathscr{M}}(\sigma + ia_1 n, \chi)|^2 \times$$
$$\times \lim_{T\to\infty}\frac{1}{T}\int_0^T \left|\frac{L}{L_{\mathscr{M}}}(\sigma + ia_1 t, \chi)\right|^2 \mathrm{d}t.$$

We expect similar statements to hold also for $d \geq 2$, as well as for $\zeta(s;\alpha)$, although there is no Euler product for $\alpha \neq 1/2, 1$.

2.2 Inside the Strip $1/2 < \sigma \leq 1$

Our aim is to obtain similar discrete second moments like in the previous section inside $\mathcal{D} = \{s \in \mathbb{C} : 1/2 < \sigma \leq 1\}$. As we mentioned earlier, the case of linear polynomials $P_{a_1}(x)$ has been resolved by Reich [70] for $L(s,\chi)$. In subsection 2.2.1 we sketch a proof also in the case of $\zeta(s;\alpha)$.

The situation of higher degree polynomials is much more difficult. At first, we have to use an approximate functional equation for $\zeta(s;\alpha)$ or $L(s,\chi)$. Of course, the length of the Dirichlet polynomial in any such equation is depending on t. Therefore, when the degree of $P_{\underline{a}}(x)$ is large, the known approximate functional equations contain rather long Dirichlet polynomials. In subsection 2.2.2 we provide a method to control the length of those Dirichlet polynomials, by taking its summands running up to a small power μ of t. The repercussion of this approach is that we are forced to narrow the strip where we can prove an asymptotic formula for the discrete moments. The left abscissa of this strip can be computed explicitly, depends on the degree of $P_{\underline{a}}(x)$ and is less than one.

The second and major difficulty is that one is led to estimate finite exponential sums $\sum_n \mathrm{e}\left(P_{\underline{a}}(n)\right)$ which are known as Weyl sums. In the case of linear polynomials the latter sum is a geometric series and can be estimated rather efficiently. It is worth mentioning the ingenious work of Good [25], who proved an asymptotic formula for the discrete fourth moments of $\zeta(s)$ in $\mathcal{D}$ by estimating such sums. His approach can easily be adapted for the discrete second moments of $\zeta(s;\alpha)$ and $L(s,\chi)$, when someone wishes to have an explicit error term as N tends to infinity. On the other hand, for higher degree polynomials the known estimates for such sums rely heavily on the rational approximations of the coefficients of $P_{\underline{a}}(x)$ and the corresponding length of the sum. At this point lies the main difference with the case of the continuous moments, where exponential integrals $\int_a^b \mathrm{e}\left(P_{\underline{a}}(x)\right) \mathrm{d}x$, have to be estimated and they are much easier to handle than the Weyl sums (see for example [38, Chapter 2]). At the expense of a negligible set of vectors $\underline{a}$, we will be able to overcome such difficulties in subsection 2.2.3 by making use of known mean value estimates for Weyl sums.

2.2.1 The Case of Linear Polynomials

In order to obtain discrete second moments for $\zeta(s;\alpha)$ with respect to linear polynomials, we will employ basic properties of Besicovitch almost-periodic functions, as well as the so-called Gallagher's lemma. We refer to [78, Chapter VI] for a treatment of almost-periodic arithmetical functions or Section A.5. We state here Gallagher's lemma, which is proven to be extremely useful in analytic number theory.

Lemma 2.1 (Gallagher's Lemma). *Let $T_0 > 0$, $T \geq \delta > 0$ and $\mathscr{B}$ be a finite subset of $\subseteq [T_0 + \delta/2, T + T_0 - \delta/2]$. Define also $N_\delta(x) := \sum_{t\in\mathscr{B}, |t-x|<\delta} 1$ and assume that $f(x)$ is a complex-valued continuous function on $[T_0, T + T_0]$ continuously differentiable on $(T_0, T + T_0)$. Then*

$$\sum_{t\in\mathscr{B}} N_\delta^{-1}(t)|f(t)|^2 \leq \frac{1}{\delta}\int_{T_0}^{T+T_0} |f(x)|^2 \mathrm{d}x + \left(\int_{T_0}^{T+T_0} |f(x)|^2 \mathrm{d}x \int_{T_0}^{T+T_0} |f'(x)|^2 \mathrm{d}x\right)^{1/2}.$$

Proof. For a proof see [59, Lemma 1.4]. □

Before stating the next theorem, we define the following Dirichlet polynomials

$$(s,\alpha) \longmapsto \zeta_Q(s,\alpha) := \sum_{n=0}^{Q-1} \frac{1}{(n+\alpha)^s}, \tag{2.1}$$

for every $(s,\alpha) \in \mathbb{C} \times (0,1]$.

Theorem 2.3. *Let $\alpha \in (0,1]$, $a_1 \in \mathcal{L}(1,\alpha)$ and $\sigma > 1/2$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\zeta(\sigma + ia_1 n;\alpha)|^2 = \zeta(2\sigma;\alpha).$$

Proof. If we set $f_Q(t) := \zeta(\sigma+it;\alpha) - \zeta_Q(\sigma+it;\alpha)$, for any $t > 0$ and $Q \in \mathbb{N}$, then Lemma 2.1 implies that, for any integer $N \geq \max\{Q, Q/a_1\}$, $T_0 = a_1(N-1/2)$, $T = a_1 N$, $\delta = a_1$ and $\mathscr{B} = \{a_1 n : N \leq n \leq 2N\}$,

$$\begin{aligned}&\sum_{n=N}^{2N} |\zeta(\sigma + ia_1 n;\alpha) - \zeta_Q(\sigma + ia_1 n;\alpha)|^2 \\ &\leq \frac{1}{a_1}\int_{a_1(N-1/2)}^{2a_1N} |f_Q(t)|^2 \mathrm{d}t + \left(\int_{a_1(N-1/2)}^{2a_1N} |f_Q(t)|^2 \mathrm{d}t \int_{a_1(N-1/2)}^{2a_1N} |f_Q'(t)|^2 \mathrm{d}t\right)^{1/2}.\end{aligned} \tag{2.2}$$

The definition (2.1) of $\zeta_Q(s;\alpha)$ and the approximate functional equation (1.7) for $\zeta(s;\alpha)$ imply that

$$f_Q(t) = \sum_{Q\le n\le t} \frac{1}{(n+\alpha)^{\sigma+it}} + O(t^{-\sigma})$$

for any $0 < a_1 N \le t \le 2a_1 N$. Therefore,

$$\int_{a_1(N-1/2)}^{2a_1N} |f_Q(t)|^2 \, \mathrm{d}t \ll \int_{a_1(N-1/2)}^{2a_1N} \left| \sum_{Q\le n\le t} \frac{1}{(n+\alpha)^{\sigma+it}} \right|^2 \mathrm{d}t + \int_{a_1(N-1/2)}^{2a_1N} t^{-2\sigma} \mathrm{d}t. \tag{2.3}$$

The second term in the right-hand side of (2.3) is $O_{a_1}\left(N^{1-2\sigma}\right)$. We also have

$$\begin{aligned}
&\int_{a_1(N-1/2)}^{2a_1N} \left| \sum_{Q\le n\le t} \frac{1}{(n+\alpha)^{\sigma+it}} \right|^2 \mathrm{d}t \\
&\quad \ll_{a_1} N \sum_{Q\le k\le 2a_1N} \frac{1}{(k+\alpha)^{2\sigma}} + \\
&\quad + \sum_{Q\le k\ne \ell\le 2a_1N} \frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}} \int_{\max\{k,\ell,a_1(N-1/2)\}}^{2a_1N} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{it} \mathrm{d}t \qquad (2.4) \\
&\quad \ll_{a_1} NQ^{1-2\sigma} + \sum_{0\le \ell< k\le 2a_1N} \frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}} \left(\log \frac{k+\alpha}{\ell+\alpha}\right)^{-1} \\
&\quad \ll_{a_1,\alpha} NQ^{1-2\sigma} + N^{2-2\sigma}(\log N)^2,
\end{aligned}$$

where the square of the logarithm appears for the case of $\sigma = 1$. Hence, relations (2.3) and (2.4) yield that

$$\int_{a_1(N-1/2)}^{2a_1N} |f_Q(t)|^2 \, \mathrm{d}t \ll_{a_1,\alpha} NQ^{1-2\sigma} + N^{2-2\sigma}(\log N)^2. \tag{2.5}$$

We can prove in the same manner that

$$\int_{a_1(N-1/2)}^{2a_1N} \left|f'_Q(t)\right|^2 \mathrm{d}t \ll_{a_1,\alpha} NQ^{1-2\sigma} + N^{2-2\sigma}(\log N)^3, \tag{2.6}$$

where

$$f'_Q(t) = \sum_{Q\le n\le t} \frac{-\log(n+\alpha)}{(n+\alpha)^{\sigma+it}} + O(t^{-\sigma})$$

for any $0 < a_1 N \le t \le 2a_1 N$. We deduce now from relations (2.2), (2.5) and (2.6) that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\zeta\left(\sigma + ia_1 n;\alpha\right) - \zeta_Q\left(\sigma + ia_1 n;\alpha\right)\right|^2 \ll_{a_1,\alpha} Q^{1-2\sigma}.$$

Since the functions

$$\mathbb{R} \ni t \longmapsto \zeta_Q(\sigma + ia_1 t; \alpha),$$

$Q \in \mathbb{N}$, are elements of the set $\mathcal{A} := \mathrm{span}_{\mathbb{C}} \{t \longmapsto \mathrm{e}(\beta t) : \beta \in [0,1)\}$, the latter relation implies that the function

$$\mathbb{N} \ni n \longmapsto \zeta(\sigma + ia_1 n; \alpha)$$

is a B^2-almost periodic arithmetical function (see Section A.5), its mean value exists and, from Theorem A.24 it follows that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\zeta(\sigma + ia_1 n; \alpha)| \\
&\qquad = \lim_{Q\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\zeta_Q(\sigma + ia_1 n; \alpha)|^2 \\
&\qquad = \lim_{Q\to\infty} \left[\sum_{k=0}^{Q-1} \frac{1}{(k+\alpha)^{2\sigma}} + \right. \\
&\qquad\qquad \left. + \sum_{0\le k\neq \ell \le Q-1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}} \mathrm{e}\left(\frac{a_1 n}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right) \right] \\
&\qquad = \zeta(2\sigma; \alpha).
\end{aligned}$$

□

2.2.2 An Approximate Functional Equation

We start with another approximate functional equation for $\zeta(s;\alpha)$ and $L(s,\chi)$, respectively. All constants appearing in this subsection, implicit or not, are effectively computable unless stated otherwise.

Theorem 2.4. *For every $\mu > 0$, there exists a positive number $\nu = \nu(\mu, \sigma_0)$, such that*

$$\zeta(s;\alpha) = \sum_{0 \le n \le t^{\mu}} \frac{1}{(n+\alpha)^s} + O_{\mu,\sigma_0}\left(t^{-\nu}\right), \quad t \ge t_1 > 1,$$

and

$$L(s,\chi) = \sum_{n \le q(t^{\mu}+1)} \frac{\chi(n)}{n^s} + O_{\mu,\sigma_0}\left(t^{-\nu}\right), \quad t \ge t_1 > 1,$$

uniformly in $\mathbf{A}(\mu) < \sigma_0 \le \sigma \le 1$ and $0 < \alpha \le 1$, where

$$\mathbf{A}(\mu) := \begin{cases} 1 - \mu^{-1}, & \text{if } \mu \ge 1, \\ \min\left\{ \dfrac{1}{2\mu}, 1 - \theta\mu^2 \right\}, & \text{if } 0 < \mu < 1, \end{cases} \tag{2.7}$$

$\theta = 4/(27\eta^2)$ and $\eta = 4.45$. Moreover, if $\mu \ge 1$ or $\mathbf{A}(\mu) = 1 - \theta\mu^2$ for $0 < \mu < 1$, the approximate functional equations hold uniformly in $\mathbf{A}(\mu) < \sigma_0 \le \sigma \le 2$ and $0 < \alpha \le 1$.

We will prove the theorem only in the case of $\zeta(s;\alpha)$, since we can obtain the above approximate functional for $L(s,\chi)$ directly from its expression (1.5) as a linear combination of finitely many Hurwitz zeta-functions. The proof of the theorem is based on some well-known order estimates and approximate functional equations for $\zeta(s;\alpha)$, as well as Perron's formula.

Lemma 2.2. *If* $\epsilon \in (0,1)$, *then*

$$\zeta_1(s;\alpha) := \zeta(s;\alpha) - \alpha^{-s} \ll_\epsilon |t|^\epsilon, \quad |t| \geq t_0 > 0,$$

uniformly in $1-\epsilon \leq \sigma \leq 2$ *and* $0 < \alpha \leq 1$.

Proof. For a proof see [1, Theorem 12.23]. □

The next lemma has its origins in the work of Vinogradov and Korobov regarding zero-free regions of the Riemann Zeta-function. It has undergone through the decades many generalizations and improvements. We present here the latest version due to Ford [18, Theorem 1]:

Lemma 2.3. *The following bound*

$$\zeta_1(s;\alpha) \ll |t|^{\eta(1-\sigma)^{3/2}} \log^{2/3}|t|, \quad |t| \geq t_1 > 1, \tag{2.8}$$

holds uniformly in $1/2 \leq \sigma \leq 1$ *and* $0 < \alpha \leq 1$.

Proof of Theorem 2.4. The case of $\mu \geq 1$ follows immediately from the approximate functional equation (1.7) for $\zeta(s;\alpha)$:

$$\zeta(s;\alpha) = \sum_{0\leq n\leq x} \frac{1}{(n+\alpha)^s} + \frac{x^{1-s}}{1-s} + O_{\sigma_0}(x^{-\sigma}), \;\; t \geq t_0 > 0,$$

where $0 < \sigma_0 \leq \sigma \leq 2$ and $\pi x \geq t$. We only need to set $x = t^\mu$.

If now $0 < \mu < 1$, then we see that

$$\mathbf{A}(\mu) := \begin{cases} \dfrac{1}{2\mu}, & \text{if} \quad \mu_0 < \mu < 1, \\[2ex] 1 - \theta\mu^2, & \text{if} \;\; 0 < \mu \leq \mu_0, \end{cases}$$

where $\mu_0 \in [1/2, 3/4]$ is the unique real root of the polynomial $Q(x) = 2\theta x^3 - 2\theta x + 1$ in the interval $[0,1]$.

Let $\mu_0 < \mu < 1$. We consider the approximate functional equation for $\zeta(s;\alpha)$ due to Miyagawa [58, Theorem 2], which is a generalization of the approximate functional equation for $\zeta(s)$ due to Hardy and Littlewood [31, Theorem 1]:

$$\begin{aligned} \zeta(s;\alpha) = &\sum_{0\leq n\leq x} \frac{1}{(n+\alpha)^s} + O\left(x^{-\sigma} + |t|^{1/2-\sigma} y^{\sigma-1}\right) + \\ &+ \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left[\mathrm{e}\left(\frac{1-s}{4}\right) \sum_{n\leq y} \frac{\mathrm{e}\left((1-\alpha)n\right)}{n^{1-s}} + \mathrm{e}\left(\frac{s-1}{4}\right) \sum_{n\leq y} \frac{\mathrm{e}\left(\alpha n\right)}{n^{1-s}} \right], \end{aligned}$$

where $0 \le \sigma \le 1$ and $x \ge 1$, $y \ge 1$ are such that $2\pi xy = |t|$. If we set $x = t^\mu$, then Stirling's formula (see Theorem A.8) yields for $t \ge t_0 > 0$ and $0 \le \sigma \le 1$ that

$$\begin{aligned}\zeta(s;\alpha) - \sum_{0\le n\le t^\mu} \frac{1}{(n+\alpha)^s} &\ll_\mu t^{1/2-\sigma} \sum_{n\le t^{1-\mu}/(2\pi)} \frac{1}{n^{1-\sigma}} + t^{-\sigma\mu} + t^{1/2-\sigma+(\sigma-1)(1-\mu)} \\ &\ll_\mu t^{1/2-\sigma+(1-\mu)\sigma} + t^{-\sigma\mu} + t^{-\sigma\mu+\mu-1/2} \\ &\ll_\mu t^{1/2-\sigma\mu} + t^{-1/2-\sigma\mu+\mu},\end{aligned}$$

where the exponent of t in the latter relation is negative for

$$\frac{1}{2\mu} < \sigma \le 1.$$

Lastly, let $0 < \mu \le \mu_0$. If we set $c = 1 + b$, where $b = b(\mu) \in (0, 1]$ will be determined later on, and $x = m + 1/2$, $m \in \mathbb{N}$, then the absolute convergence of $\zeta_1(s;\alpha)$ in the half-plane $\sigma > 1$ and Perron's formula (see Theorem A.9) imply that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta_1(s+z;\alpha)\frac{(x+\alpha)^z}{z}\mathrm{d}z = \sum_{n=1}^{m} \frac{1}{(n+\alpha)^s} + \\ + O_{\sigma_0}\left(\frac{1}{T}\sum_{n=1}^{\infty}\left(\frac{x+\alpha}{n+\alpha}\right)^c \left|\log\frac{x+\alpha}{n+\alpha}\right|^{-1}\right),\end{aligned} \tag{2.9}$$

uniformly in $\sigma \ge \sigma_0 > 0$ and $0 < \alpha \le 1$. We estimate the sum in the error term. Observe that

$$\begin{aligned}\left\{\sum_{n<\frac{x}{2}} + \sum_{n>2x}\right\}\left(\frac{x+\alpha}{n+\alpha}\right)^c \left|\log\frac{x+\alpha}{n+\alpha}\right|^{-1} &\ll x^c \left\{\sum_{n<\frac{x}{2}} + \sum_{n>2x}\right\} \frac{\max\{x,n\}+\alpha}{n^c|x-n|} \\ &\ll x^c \sum_{n=1}^{\infty}\frac{1}{n^c} \\ &\ll \frac{x^c}{b},\end{aligned} \tag{2.10}$$

while if we set $q = m - n$ for $x/2 \le n < x$ and $r = n - m$ for $x < n \le 2x$, we have

$$\begin{aligned}\sum_{\frac{x}{2}\le n\le 2x}\left(\frac{x+\alpha}{n+\alpha}\right)^c \left|\log\frac{x+\alpha}{n+\alpha}\right|^{-1} &\ll \sum_{\frac{x}{2}\le n\le 2x} \frac{x^c}{n^c}\frac{\max\{x,n\}}{|x-n|} \\ &\ll x\left[\sum_{0\le q\le\frac{x-1}{2}}\frac{1}{q+\frac12} + \sum_{r\le\frac{2x+1}{2}}\frac{1}{r-\frac12}\right] \\ &\ll x\log x\end{aligned} \tag{2.11}$$

Hence, we deduce from (2.9)-(2.11) that

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\zeta_1(s+z;\alpha)\frac{(x+\alpha)^z}{z}\mathrm{d}z=\sum_{n=1}^{m}\frac{1}{(n+\alpha)^s}+O_{\sigma_0}\left(\frac{x^c}{bT}+\frac{x\log x}{T}\right), \quad (2.12)$$

uniformly in $\sigma\geq\sigma_0>0$ and $0<\alpha\leq 1$.

Let $1-\kappa\leq\sigma\leq 2$ be arbitrary, where $\kappa=\kappa(\mu)\in[0,1/2]$ will be determined later on. Let also $T\geq t$ and consider the rectangle $\mathcal{R}$ with vertices $1-3\kappa-\sigma\pm iT$, $c\pm iT$. By the calculus of residues we get

$$\frac{1}{2\pi i}\int_{\mathcal{R}}\zeta_1(s+z;\alpha)\frac{(x+\alpha)^z}{z}\mathrm{d}z=\zeta_1(s)+\frac{(x+\alpha)^{1-s}}{1-s}=\zeta_1(s;\alpha)+O\left(x^{1-\sigma}t^{-1}\right). \quad (2.13)$$

Observe that Lemma 2.2 implies that

$$\left\{\int\limits_{1-3\kappa-\sigma-iT}^{c-iT}+\int\limits_{c+iT}^{1-3\kappa-\sigma+iT}\right\}\zeta_1(s+z;\alpha)\frac{(x+\alpha)^z}{z}\mathrm{d}z\ll_\kappa\frac{x^cT^{3\kappa}}{T}, \quad (2.14)$$

while Lemma 2.3 yields

$$\begin{aligned}\int\limits_{1-3\kappa-\sigma+iT}^{1-3\kappa-\sigma-iT}\zeta_1(s+z;\alpha)\frac{(x+\alpha)^z}{z}\mathrm{d}z&\ll x^{1-3\kappa-\sigma}\int\limits_{-T}^{T}\frac{|\zeta_1\left(1-3\kappa+i(t+u)\right)|}{|1-3\kappa+iu|}\mathrm{d}u\\&\ll_\kappa x^{-2\kappa}T^{(3\kappa)^{3/2}\eta}\left(\log T\right)^2.\end{aligned} \quad (2.15)$$

From relations (2.12)-(2.15) we deduce

$$\begin{aligned}\zeta_1(s;\alpha)=&\sum_{n=1}^{m}\frac{1}{(n+\alpha)^s}+O_{\sigma_0}\left(\frac{x^c}{bT}+\frac{x\log x}{T}\right)+O\left(x^{1-\sigma}t^{-1}\right)+\\&+O_\kappa\left(x^cT^{-1+3\kappa}+x^{-2\kappa}T^{(3\kappa)^{3/2}\eta}\left(\log T\right)^2\right).\end{aligned}$$

If we set $m=\lfloor t^\mu\rfloor$, then the last three terms in the latter relation are bounded above by

$$C(\sigma_0,\kappa,b)\left(t^{(1+b)\mu-1}b^{-1}+t^{\mu(1-\sigma)-1}+t^{(1+b)\mu+3\kappa-1}+t^{\kappa\left(-2\mu+3^{3/2}\kappa^{1/2}\eta\right)}\left(\log t\right)^2\right),$$

where $C(\sigma_0,\kappa,b)>0$ is a constant. It is clear now that for $\kappa=4\mu^2/(27\eta^2)$ and $b\ll_\mu 1$ sufficiently small, the theorem follows also for $0<\mu\leq\mu_0<3/4$. □

2.2.3 Estimates of Exponential Sums

Bounds for exponential sums, especially in the case of Weyl sums, lie at the heart of analytic number theory. There is a vast literature regarding methods to estimate them as well as their numerous applications. We refer to [38], [39] and [77] for an exposition of such results.

In our case we focus on mean value estimates for Weyl sums, that is, on upper bounds for the quantity

$$J_{h,d}(N) := \int_{[0,1]^d} \left| \sum_{n=1}^{N} \mathrm{e}\left(a_1 n + \cdots + a_d n^d\right) \right|^{2h} \mathrm{d}a_1 \ldots \mathrm{d}a_d,$$

where h, d and N are positive integers. Observe that $J_{h,d}(N)$ denotes the number of integral solutions of the system

$$\begin{array}{ccccccccccc} X_1 & + & \ldots & + & X_h & = & X_{h+1} & + & \ldots & + & X_{2h} \\ X_1^2 & + & \ldots & + & X_h^2 & = & X_{h+1}^2 & + & \ldots & + & X_{2h}^2 \\ & & \vdots & & & \vdots & & & \vdots & & \\ X_1^d & + & \ldots & + & X_h^d & = & X_{h+1}^d & + & \ldots & + & X_{2h}^d \end{array}$$

with $1 \leq X_1, \ldots, X_{2h} \leq N$. Recently, Bourgain, Demeter and Guth [9] proved the so-called main conjecture in Vinogradov's Mean Value Theorem:

Theorem 2.5. *For any integers $h \geq 1$ and $d, N \geq 2$,*

$$J_{h,d}(N) \ll_{h,d,\epsilon} N^{h+\epsilon} + N^{2h-d(d+1)/2+\epsilon}$$

It should be mentioned here that the case of $d = 2$ follows from elementary estimates for the divisor function, while the case of $d = 3$ was first solved by Wooley [101]. We use the latter theorem to obtain a rather useful metric result.

Lemma 2.4. *Let $d \geq 2$ be an integer, $\alpha \in (0, 1]$ and $\epsilon, \mu > 0$. Then, there exists a set $\mathcal{F}(d, \alpha, \mu, \epsilon) \subseteq [0, +\infty)^d$ of full Lebesgue measure with elements satisfying the following property:*

If $\underline{a} \in \mathcal{F}(d, \alpha, \mu, \epsilon)$ is a vector of real numbers which coefficients are bounded by an $M_{\underline{a}} \in \mathbb{N}$, then there exists $K_{\underline{a}} \in \mathbb{N}$ such that

$$\left| \sum_{n=1}^{N} \left(\frac{k+\alpha}{\ell+\alpha} \right)^{iP_{\underline{a}}(n)} \right|^{d(d+1)} \ll_{d,\epsilon} \left(\frac{\left\lfloor \frac{M_{\underline{a}}}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right\rfloor + 1}{\log \frac{k+\alpha}{\ell+\alpha}} \right)^d N^{d(d+1)/2+1+2\mu d+3\epsilon} \quad (2.16)$$

for every integer $N \geq K_{\underline{a}}$ and any integers $0 \leq \ell < k \leq (dM_{\underline{a}}N^d)^\mu$.

Proof. For any positive integers M, N, k, any integer $0 \leq \ell < k$ and for $h := d(d+1)/2$, Theorem 2.5 yields that

$$\begin{aligned} \int_{[0,M]^d} \left| \sum_{n=1}^{N} \left(\frac{k+\alpha}{\ell+\alpha} \right)^{iP_{\underline{a}}(n)} \right|^{2h} \mathrm{d}\underline{a} &= \left(\frac{2\pi}{\log \frac{k+\alpha}{\ell+\alpha}} \right)^d \int_{[0, \frac{M}{2\pi} \log \frac{k+\alpha}{\ell+\alpha}]^d} \left| \sum_{n=1}^{N} \mathrm{e}\left(P_{\underline{a}}(n)\right) \right|^{2h} \mathrm{d}\underline{a} \\ &\leq \left(\frac{2\pi \left(\left\lfloor \frac{M}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right\rfloor + 1 \right)}{\log \frac{k+\alpha}{\ell+\alpha}} \right)^d J_{h,d}(N) \\ &\leq C(d,\epsilon) \left(\frac{\left\lfloor \frac{M}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right\rfloor + 1}{\log \frac{k+\alpha}{\ell+\alpha}} \right)^d N^{h+\epsilon}, \end{aligned}$$

where $C(d,\epsilon) > 0$ is a constant. Therefore, the set $\mathcal{E}(d,\alpha,\mu,\epsilon,M,N,k,\ell)$ of those $\underline{a} \in [0,M]^d$ satisfying

$$\left|\sum_{n=1}^{N}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}\right|^{d(d+1)} \geq C(d,\epsilon)\left(\frac{\left\lfloor \frac{M}{2\pi}\log\frac{k+\alpha}{\ell+\alpha}\right\rfloor+1}{\log\frac{k+\alpha}{\ell+\alpha}}\right)^{d} N^{d(d+1)/2+1+2\mu d+3\epsilon},$$

has Lebesgue measure $\mathrm{m}\left(\mathcal{E}(d,\alpha,\mu,\epsilon,M,N,k,\ell)\right) \leq N^{-(1+2\mu d+\epsilon)}$. Hence, the set

$$\mathcal{G}(d,\alpha,\mu,\epsilon,M,K) := \bigcup_{N=K}^{\infty}\ \bigcup_{1\leq\ell<k\leq\left(dMN^d\right)^{\mu}} \mathcal{E}(d,\alpha,\mu,\epsilon,M,N,k,\ell)$$

has Lebesgue measure $\mathrm{m}(\mathcal{G}(d,\alpha,\mu,\epsilon,M,K)) \ll_{d,\mu,\epsilon,M} K^{-\epsilon}$, for every positive integers M and K, and, thus, the set

$$\mathcal{F}(d,\alpha,\mu,\epsilon) := [0,+\infty)^d \setminus \left(\bigcup_{M=1}^{\infty}\bigcap_{K=1}^{\infty}\mathcal{G}(d,\alpha,\mu,\epsilon,M,K)\right), \tag{2.17}$$

is of full Lebesgue measure. □

2.2.4 A Metric Result

Here we prove the main theorems of this chapter, that is, discrete second moments for $\zeta(s;\alpha)$ and $L(s,\chi)$ on the left of the vertical line $1+i\mathbb{R}$. But before that we need to introduce the quantities

$$\mathbf{B}(d,\mu) := \frac{1}{2\mu d}\left(\frac{1}{2}-\frac{2\mu}{d+1}-\frac{1}{d(d+1)}\right) \tag{2.18}$$

and

$$\mathbf{S}(d) := \min_{0<\mu<\frac{d^2+d-2}{4d}} \max\left\{\mathbf{A}(\mu), 1-\mathbf{B}(d,\mu)\right\} \in (0,1), \tag{2.19}$$

where $d \geq 2$ is integer, $\mu > 0$ and $\mathbf{A}(\mu)$ is as in (2.7). It will become apparent from the subsequent proofs how we came up with these numbers.

Theorem 2.6. *For every integer $d \geq 2$, any $\alpha \in (0,1]$ and any $\sigma_0 \in (\mathbf{S}(d),1]$, there are effectively computable positive numbers $\mu = \mu(d)$, $\epsilon = \epsilon(d,\sigma_0)$ and $\nu = \nu(d,\sigma_0)$ such that, for any $\underline{a} \in \mathcal{F}(d,\alpha,\mu,\epsilon)$*

$$\frac{1}{N}\sum_{n=1}^{N}\left|\zeta\left(\sigma+iP_{\underline{a}}(n);\alpha\right)\right|^2 = \zeta(2\sigma;\alpha) + O_{d,\sigma_0,\underline{a},\alpha}\left(N^{-\nu}\right), \quad N \geq 1,$$

uniformly in $\sigma_0 \leq \sigma \leq 1$. Moreover, for $\alpha = 1$ and any $\underline{a} \in \mathcal{F}(d,1,\mu,\epsilon)$

$$\frac{1}{N}\sum_{n=1}^{N}\left|L\left(\sigma+iP_{\underline{a}}(n),\chi\right)\right|^2 = L(2\sigma,\chi_0) + O_{d,\sigma_0,\underline{a}}\left(N^{-\nu}\right), \quad N \geq 1,$$

uniformly in $\sigma_0 \leq \sigma \leq 1$.

Proof of Theorem 2.6. Let $\underline{a} = (a_1, \ldots, a_d) \in \mathcal{F}(d, \alpha, \mu, \epsilon) \setminus \{\underline{0}\}$ be such that $|a_i| \leq M_{\underline{a}}$ for some $M_{\underline{a}} \in \mathbb{N}$ and every $i = 1, \ldots, d$. The numbers μ and ϵ will be suitably chosen later on.

Let $N_{\underline{a}} > K_{\underline{a}}$ be such that $P_{\underline{a}}(N_{\underline{a}}) \geq 2$. Notice here that all polynomials we consider are stricLty increasing functions in $[0, +\infty)$. Using the approximate functional equation for $\zeta(s;\alpha)$ from Theorem 2.4 and applying the Cauchy-Schwarz inequality we obtain for every $N \geq N_{\underline{a}}$ that

$$\begin{aligned}
&\sum_{n=1}^{N} |\zeta(\sigma + iP_{\underline{a}}(n);\alpha)|^2 \\
&\quad = O_{\underline{a}}(1) + \sum_{n=N_{\underline{a}}}^{N} \left| \sum_{0 \leq k \leq P_{\underline{a}}^{\mu}(n)} \frac{1}{(k+\alpha)^{\sigma + iP_{\underline{a}}(n)}} + O_{\mu,\sigma_0}\left(P_{\underline{a}}^{-\nu}(n)\right) \right|^2 \\
&\quad = S_N + O_{\underline{a}}\left(1 + T_N + (S_N T_N)^{1/2}\right),
\end{aligned} \tag{2.20}$$

where $\nu = \nu(\mu, \sigma_0) > 0$ is as in Theorem 2.4, and

$$T_N := \sum_{n=N_{\underline{a}}}^{N} \left| O_{\mu,\sigma_0}\left(P_{\underline{a}}^{-\nu}(n)\right)\right|^2 \ll_{\mu,\sigma_0,\underline{a}} \sum_{n=N_{\underline{a}}}^{N} n^{-2d\nu} \ll_{\mu,\sigma_0,\underline{a}} 1 + N^{1-2d\nu} \tag{2.21}$$

and

$$S_N := \sum_{n=N_{\underline{a}}}^{N} \sum_{0 \leq \ell, k \leq P_{\underline{a}}^{\mu}(n)} \frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}.$$

Splitting S_N into sum of diagonal and non-diagonal terms yields

$$\begin{aligned}
S_N &= \sum_{n=N_{\underline{a}}}^{N} \left[\zeta(2\sigma;\alpha) + O\left((P_{\underline{a}}^{\mu}(n))^{1-2\sigma}\right)\right] + \\
&\quad + \sum_{n=N_{\underline{a}}}^{N} \sum_{0 \leq \ell \neq k \leq P_{\underline{a}}^{\mu}(n)} \frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)} \\
&= (N - N_{\underline{a}})\zeta(2\sigma;\alpha) + O_{\mu,\sigma_0,\underline{a}}\left(\sum_{n=N_{\underline{a}}}^{N} n^{d\mu(1-2\sigma)}\right) + R_N \\
&= N\zeta(2\sigma;\alpha) + O_{\mu,\sigma_0,\underline{a}}\left(N^{1+d\mu(1-2\sigma)}\right) + R_N,
\end{aligned} \tag{2.22}$$

with

$$R_N := \sum_{0 \leq \ell \neq k \leq P_{\underline{a}}^{\mu}(N)} \frac{1}{(k+\alpha)^{\sigma}(\ell+\alpha)^{\sigma}} \sum_{n \in \mathscr{A}(N,\underline{a},k,\ell,\alpha)} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}$$

and

$$\begin{aligned}
\mathscr{A}(N, \underline{a}, k, \ell, \alpha) &:= \left\{N_{\underline{a}} \leq n \leq N : P_{\underline{a}}(n) \geq \max\left\{(k+\alpha)^{1/\mu}, (\ell+\alpha)^{1/\mu}\right\}\right\} \\
&= \{N_1, \ldots, N\}.
\end{aligned}$$

Observe that

$$R_N \ll \sum_{0\le \ell<k\le P_{\underline{a}}^{\mu}(N)} \frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}} \left[\left| \sum_{n=1}^{N} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)} \right| + \left| \sum_{n=1}^{N_1-1} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)} \right| \right]. \tag{2.23}$$

By our choice of the vector $\underline{a}$ and Lemma 2.4, it follows that

$$\left| \sum_{n=1}^{N} \left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)} \right| \ll_{\epsilon} \left(\frac{\left\lfloor \frac{M_{\underline{a}}}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right\rfloor + 1}{\log \frac{k+\alpha}{\ell+\alpha}} \right)^{1/(d+1)} N^{1/2+2\mu/(d+1)+1/(d(d+1))+\epsilon}$$

for every $N \ge K_{\underline{a}}$ and any $0 \le \ell < k \le \left(dM_{\underline{a}}N^d\right)^{\mu}$. Implementing the latter bound to (2.23), we obtain

$$R_N \ll_{\epsilon} N^{1-2\mu d\mathbf{B}(d,\mu)+\epsilon} \tilde{R}_N, \tag{2.24}$$

where

$$\mathbf{B}(d,\mu) = \frac{1}{2\mu d}\left(\frac{1}{2} - \frac{2\mu}{d+1} - \frac{1}{d(d+1)}\right) \tag{2.25}$$

and

$$\begin{aligned}
\tilde{R}_N &:= M_{\underline{a}}^{1/(d+1)} \sum_{0\le \ell<k\le P_{\underline{a}}^{\mu}(N)} \frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}} \left(\frac{\left\lfloor \frac{M_{\underline{a}}}{2\pi} \log \frac{k+\alpha}{\ell+\alpha} \right\rfloor + 1}{\frac{M_{\underline{a}}}{2\pi} \log \frac{k+\alpha}{\ell+\alpha}} \right)^{1/(d+1)} \\
&\ll_{d,\underline{a}} \sum_{\substack{0\le \ell<k\le P_{\underline{a}}^{\mu}(N) \\ \log \frac{k+\alpha}{\ell+\alpha} > \frac{2\pi}{M_{\underline{a}}}}} \frac{(k+\alpha)^{-\sigma}}{(\ell+\alpha)^{\sigma}} + \sum_{\substack{0\le \ell<k\le P_{\underline{a}}^{\mu}(N) \\ \log \frac{k+\alpha}{\ell+\alpha} \le \frac{2\pi}{M_{\underline{a}}}}} \frac{(k+\alpha)^{\sigma}}{(\ell+\alpha)^{\sigma}} \left(\log \frac{k+\alpha}{\ell+\alpha}\right)^{-1} \\
&\ll_{d,\underline{a},\alpha} \left(P_{\underline{a}}^{\mu}(N)\right)^{(2-2\sigma)} (\log N)^2 \\
&\ll_{d,\mu,\sigma_0,\underline{a},\alpha} N^{(2-2\sigma)\mu d} (\log N)^2
\end{aligned} \tag{2.26}$$

for every $N \ge K_{\underline{a}}$. The square for the logarithm results with respect to the case of $\sigma = 1$.

Gathering up the terms and estimates from (2.20)-(2.26), we deduce that

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} \left|\zeta\left(\sigma + iP_{\underline{a}}(n); \alpha\right)\right|^2 - \zeta(2\sigma;\alpha) &\ll_{d,\mu,\sigma_0,\underline{a},\alpha,\epsilon} N^{d\mu(1-2\sigma)/2} + N^{-1/2} + N^{-2d\nu} + \\
&\quad + N^{-\mu d(\mathbf{B}(d,\mu)+\sigma-1)+\epsilon}
\end{aligned} \tag{2.27}$$

for every $N \ge N_{\underline{a}}$, where $\sigma \le 1$ and $\mu > 0$ are such that $\mathbf{B}(d,\mu) + \sigma - 1$ is positive. Notice that this is possible only when $\mathbf{B}(d,\mu) > 0$, or equivalently from (2.25), when

$$0 < \mu < \frac{d^2+d-2}{4d}.$$

It remains to choose $\mu = \mu(d)$ in such a way that we can obtain the wider strip possible containing its right boundary $1 + i\mathbb{R}$, bound to the method we have used. In view of (2.7), (2.25) and the preceding discussion, the abscissa of the left boundary of the wider strip is given by

$$\mathbf{S}(d) = \min_{0<\mu<\frac{d^2+d-2}{4d}} \max\left\{\mathbf{A}(\mu), 1 - \mathbf{B}(d,\mu)\right\}$$

and the desired μ is the one for which the aforementioned well-defined minimum is attained. Now the first part of the theorem follows from (2.27) and for an arbitrary $0 < \epsilon < \mu d(\sigma_0 - \mathbf{S}(d))/2$.

The proof for the discrete moments of $L(s,\chi)$ follows the same reasoning as above with slight changes. This time we use the second approximate functional equation from Lemma 2.4, the sum of the diagonal terms of the corresponding S_N gives the main term $L(2\sigma, \chi_0)$, while the sum of the non-diagonal terms is treated in the same way as in (2.23)-(2.26) by applying once more Lemma 2.5 for $\alpha = 1$. □

2.2.5 Discrete Moments with respect to Monomials

Of particular interest, especially regarding the next chapter, is the special case of $P_a(x) = ax^d$. It is clear from the previous section that as soon as we have estimates for

$$M_{h,d}(N) := \int_0^1 \left|\sum_{n=1}^{N} \mathrm{e}\left(an^d\right)\right|^{2h} \mathrm{d}a$$

analogous to the one of Theorem 2.5, we can obtain similar results as before. Observe that $M_{d,h}(N)$ denotes the number of $(2h)$-tuples $(X_1, \ldots, X_{2h})$ of positive integers not exceeding N for which

$$X_1^d + \cdots + X_h^d = X_{h+1}^d + \cdots + X_{2h}^d.$$

In that direction, Salberger and Wooley [75] proved the following theorem:

Theorem 2.7. *Suppose that d and h are positive integers with $d \geq 2h - 1 \geq 3$. Let also $T_h(N)$ denote the number of $(2h)$-tuples $(X_1, \ldots, X_{2h})$ of positive integers not exceeding N for which the h-tuple $(X_1, \ldots, X_h)$ is a permutation of $(X_{h+1}, \ldots, X_{2h})$. Then*

$$M_{h,d}(N) - T_h(N) \ll_{h,\epsilon} N^{h+\lambda(d,h)+\epsilon},$$

where

$$\lambda(d,h) := \begin{cases} -2 + 2/\sqrt{3} + \kappa(d, h-1, 2h-2), & \text{if } d \geq 2h-1, \\ -1 + \kappa(d, h-1, 2h-2), & \text{if } d \geq (2h-1)^2, \\ -\dfrac{1}{2}, & \text{if } d \geq (2h)^{4h}, \end{cases} \tag{2.28}$$

and

$$\kappa(d,k,m) := \sum_{r=k+1}^{m} (r+1)/\sqrt[r]{d},$$

for any positive integers d, k and m with $k < m$.

By definition $T_h(N) \sim h!N^h$. Therefore,

$$M_{h,d}(N) \ll_{h,\epsilon} N^h + N^{h+\lambda(d,h)+\epsilon},$$

and we can prove a lemma similar to Lemma 2.4. We omit its proof.

Lemma 2.5. *Suppose that d and h are positive integers with $d \geq 2h-1 \geq 3$, $\alpha \in (0,1]$ and $\epsilon, \mu > 0$. Then, there exists a set $\mathcal{F}_{mo}(d,\alpha,\mu,\epsilon) \subseteq [0,+\infty)$ of full Lebesgue measure with elements satisfying the following property:*

If $a \in \mathcal{F}_{mo}(d,\alpha,\mu,\epsilon)$ is a real number bounded by an $M_a \in \mathbb{N}$, then there exists $K_a \in \mathbb{N}$ such that

$$\left|\sum_{n=1}^{N}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{ian^d}\right|^{2h} \ll_{h,\epsilon} \frac{\left\lfloor \frac{M_a}{2\pi}\log\frac{k+\alpha}{\ell+\alpha}\right\rfloor+1}{\log\frac{k+\alpha}{\ell+\alpha}} N^{h+\lambda(d,h)+1+2\mu d+3\epsilon}$$

for every integer $N \geq K_a$ and any integers $0 \leq \ell < k \leq (dM_aN^d)^{\mu}$.

Now in order to obtain a theorem for the monomials $P_a(x)$, similar to Theorem 2.6, we need to choose $2 \leq h \leq (d+1)/2$ and $\mu > 0$ in such way that the number

$$\mathbf{B}_{mo}(d,\mu,h) := \frac{1}{2\mu d}\left(\frac{1}{2} - \frac{\lambda(d,h)+1+2\mu d}{2h}\right)$$

is positive. The argumentation of how we come up with the quantity $\mathbf{B}_{mo}(d,\mu,h)$ is the same as in the proof of Theorem 2.6. From definition (2.28) of $\lambda(d,h)$ we know that for any $d \geq 3$ and $h = 2$

$$\frac{1}{2} - \frac{\lambda(d,2)+1}{4} > 0.$$

Hence, there is $2 \leq h_{mo} \leq (d+1)/2$ such that

$$e_{mo} := \frac{1}{2} - \frac{\lambda(d,h_{mo})+1}{2h_{mo}} = \max_{2\leq h\leq \frac{d+1}{2}} \frac{1}{2} - \frac{\lambda(d,h)+1}{2h} > 0.$$

If we set now

$$\mathbf{S}_{mo}(d) := \min_{0<\mu<e_{mo}/(2d)} \max\left\{\mathbf{A}(\mu), 1-\mathbf{B}_{mo}(d,\mu,h_{mo})\right\}, \tag{2.29}$$

then we can prove the following theorem in the same way as Theorem 2.6.

Theorem 2.8. *For every integer $d \geq 2$, any $\alpha \in (0,1]$ and any $\sigma_0 \in (\mathbf{S}_{mo}(d), 1]$, there are effectively computable positive numbers $\mu = \mu(d)$, $\epsilon = \epsilon(d, \sigma_0)$ and $\nu = \nu(d, \sigma_0)$ such that, for any $a \in \mathcal{F}_{mo}(d, \alpha, \mu, \epsilon)$*

$$\frac{1}{N}\sum_{n=1}^{N}\left|\zeta\left(\sigma + ian^d; \alpha\right)\right|^2 = \zeta(2\sigma;\alpha) + O_{d,\sigma_0,a,\alpha}\left(N^{-\nu}\right), \quad N \geq 1,$$

uniformly in $\sigma_0 \leq \sigma \leq 1$. Moreover, for $\alpha = 1$ and any $a \in \mathcal{F}_{mo}(d, 1, \mu, \epsilon)$

$$\frac{1}{N}\sum_{n=1}^{N}\left|L\left(\sigma + ian^d, \chi\right)\right|^2 = \sum_{n=1}^{\infty}\frac{|\chi(n)|^2}{n^{2\sigma}} + O_{d,\sigma_0,a}\left(N^{-\nu}\right), \quad N \geq 1,$$

uniformly in $\sigma_0 \leq \sigma \leq 1$.

Notice that in the latter theorem we consider also $d = 2$. In order to treat this case, our starting point is a classical estimate due to Hua [36]:

$$\int_0^1 \left|\sum_{n=1}^{N} \mathrm{e}\left(an^2\right)\right|^4 \mathrm{d}a \ll_\epsilon N^{2+\epsilon},$$

We can then argue as before to compute the corresponding quantity $\mathbf{S}_{mo}(2)$.

2.3 The Lindelöf Hypothesis

As we have already mentioned in the beginning of the previous section, in our attempt to control the length of the Dirichlet polynomials appearing in the approximate functional equations of Lemma 2.4, we are forced to narrow the strip inside $\mathcal{D}$ for which we can prove discrete second moments. Indeed, one could argue that there is some σ_1 with $1/2 < \sigma_1 < 1$ such that $\mathbf{S}(d_k) \leq \sigma_1$ for infinitely many integers $d_k \geq 2$, $k \in \mathbb{N}$. But that would require at first

$$\max\left\{\mathbf{A}(\mu_k), 1 - \mathbf{B}(d_k, \mu_k)\right\} \leq \sigma_1$$

for all $k \in \mathbb{N}$ and some $0 < \mu_k < (d_k^2 + d_k - 2)/(4d_k)$, or, equivalently,

$$\mathbf{A}(\mu_k) \leq \sigma_1 \quad \text{and} \quad \mu_k \leq \frac{d_k^2 + d_k - 2}{4d_k^2\left((1-\sigma_1)(d_k+1)+1\right)}$$

for all $k \in \mathbb{N}$. Then we would have $\lim\limits_{k\to\infty} \mu_k = 0$ and, by the definition (2.7) of $\mathbf{A}(\mu)$, that $1 = \lim\limits_{k\to\infty} \mathbf{A}(\mu_k) \leq \sigma_1$, which contradicts our assumption about σ_1. Hence, $\mathbf{S}(d)$ tends to 1 from the left as d tends to infinity. The same holds true for $\mathbf{S}_{mo}(d)$.

Of course, it would be desirable to prove $\mathbf{S}(d) = \mathbf{S}_{mo}(d) = 1/2$, for any $d \geq 2$. The definitions (2.19) and (2.22) already give us a hint of how that could be achieved. It would suffice if we were able to take $\mathbf{A}(\mu) = 1/2$ for any $0 < \mu < 1$, since $\lim_{\mu\to 0^+} \mathbf{B}(d,\mu) = \lim_{\mu\to 0^+} \mathbf{B}_{mo}(d,\mu,h_{mo}) = +\infty$. This is where the *Lindelöf hypothesis (LH)* for $\zeta(s;\alpha)$ and $L(s,\chi)$ can be used:

$$\zeta\left(\frac{1}{2}+it;\alpha\right) \ll_\epsilon |t|^\epsilon \quad \text{and} \quad L\left(\frac{1}{2}+it,\chi\right) \ll_\epsilon |t|^\epsilon, \ \ |t| \geq t_0 > 0.$$

In fact, the truth of the Lindelöf hypothesis for all $L(s,\chi)$ is also known as the *Generalized Lindelöf Hypothesis (GLH)*. For a discussion on the *LH* for the Riemann zeta-function we refer to [43] and [90]. All results presented there can be seen to hold true for $L(s,\chi)$ as well, the most notable being that the *GRH* implies the *GLH*. Naturally, such an implication has no meaning in the case of $\zeta(s;\alpha)$ when $\alpha \neq 1/2, 1$. Garunkštis and Steuding [22] study the *LH* for the Lerch zeta-function which is a generalization of $\zeta(s;\alpha)$.

In our case a classical result for $\zeta(s)$ (see for example Theorem 13.3 in [90]) was the inspiration for our Lemma 2.4, where the number $\mathbf{A}(\mu)$ first appears:

If the LH for $\zeta(s)$ is true, then for any $\sigma > 1/2$ and any $0<\mu<1$, there is an $\nu = \nu(\sigma,\mu)$ such that

$$\zeta(s) = \sum_{n\leq t^\mu} \frac{1}{n^s} + O(t^{-\nu}), \ \ t \geq t_0 > 0.$$

It is then quite straightforward to prove the following conditional results:

Lemma 2.6. *Under LH for $\zeta(s;\alpha)$ and GLH, the approximate functional equations of Lemma 2.4 hold true with $\mathbf{A}(\mu) = 1/2$ for any $0<\mu<1$.*

Theorem 2.9. *Under LH for $\zeta(s;\alpha)$ and GLH, Theorem 2.6 and Theorem 2.8 hold true with $\mathbf{S}(d) = \mathbf{S}_{mo}(d) = 1/2$, for any integer $d \geq 2$.*

Chapter 3

Discrete Universality Theorems

We continue with the study of the value-distribution of $L(s, \chi)$ and $\zeta(s; \alpha)$. In particular, we begin with developing two sufficient criteria to prove discrete universality theorems. We then provide examples of sequences satisfying these criteria, such as Beatty sequences, ordinates of c-points of zeta-functions, and sequences generated by polynomials.

3.1 Criteria for Discrete Universality

From the first proof of the continuous universality for $\zeta(s)$ due to Voronin until today, the tools used to prove such theorems have been refined by numerous mathematicians. The setting, however, is essentially the one initiated by Voronin. One may say that there is indeed a "silver bullet" to prove universality theorems for most of the zeta- and L- functions known so far. The probabilistic approach introduced by Bagchi is based, in fact, on the same three components. The first one consists of a denseness theorem regarding twisted Euler products and Dirichlet polynomials in some suitable Hilbert space of analytic functions. The second amounts to none other than the theory of uniformly distributed sequences, while the last one involves the approximation in the discrete mean square of our selected zeta- or L- function by a sequence of Dirichlet polynomials. In the following subsections we try to make perceivable how these notions "manufacture the silver bullet" that leads to discrete universality for $L(s, \chi)$ and $\zeta(s; \alpha)$. We favor Voronin's approach over Bagchi's, since it seems to us more natural.

A second way to prove universality theorems is straightforward and rather efficient. One needs only to find a representation of the zeta- or L- function as a linear combination or a product representation of jointly universal functions. This argument is the only way so far to prove that $\zeta(s; \alpha)$ with $\alpha \neq 1/2, 1$ rational is universal and it was shown by Gonek [24] and Bagchi [2] (independently). This is also why we discuss discrete joint universality of Dirichlet L-functions.

Lastly, there is a third way to prove universality due to Good [26]. As a matter of fact Good's method yields in some cases quantitative results for the value-distibution of $\zeta(s)$ inside $\mathcal{D}$. However, it is quite technical and it has been adapted by very few researchers, such as for example Garunkštis [21] and Gonek [24]. We will return to this discussion in Chapter 4.

3.1.1 The case of Dirichlet L-Functions

If χ is a Dirichlet character mod q and $\mathscr{M}$ is a finite set of primes, we define the truncated and twisted Euler product

$$(s,\underline{\omega}) \longmapsto L_{\mathscr{M}}(s,\chi,\underline{\omega}) := \prod_{p\in\mathscr{M}} \left(1 - \frac{\chi(p)\mathrm{e}(-\omega_p)}{p^s}\right)^{-1},$$

for every $(s,\underline{\omega}) \in \{s\in\mathbb{C} : \sigma > 0\} \times \mathbb{R}^{\mathbb{P}}$. We also set $L_{\mathscr{M}}(s,\chi) := L(s,\chi,\underline{0})$ for brevity. Lastly, we say that an open set $\mathcal{R} \subseteq \mathbb{C}$ is *admissible* if for every positive number ε the set

$$\mathcal{R}_\varepsilon := \{w \in \mathbb{C} : \text{ there is } s \in \mathcal{R} \text{ such that } |s-w| < \varepsilon\}$$

has connected complement.

The first lemma establishes a denseness theorem for the aforementioned Euler products in the space of analytic functions which are defined in the strip $\mathcal{D}$.

Lemma 3.1. *Let $\mathcal{R}$ be a bounded admissible set together with its closure $\overline{\mathcal{R}}$ contained in the vertical strip $1/2 < \sigma < 1$. Let $\chi_1,\ldots,\chi_J$ be pairwise non-equivalent Dirichlet characters and let $f_1,\ldots,f_J$ be arbitrary functions which are non-vanishing continuous on $\mathcal{R}$ and analytic in the interior. Then there exists a sequence of real numbers $\underline{\omega}_0 = (\omega_{0p})_{p\in\mathbb{P}}$ depending only on $\chi_1,\ldots,\chi_J$, such that for every $\varepsilon > 0$ and arbitrary $0 < z < y$, there exists a finite set of primes $\mathscr{M}$ containing all primes $z < p < y$ but no primes $p \leq z$ with*

$$\max_{1\leq j\leq J} \max_{s\in\mathcal{R}} |L_{\mathscr{M}}(s,\chi_j,\underline{\omega}_0) - f(s)| < \varepsilon.$$

Proof. For a proof see [41, Lemma 7]. □

The next lemma points out where the theory of uniformly distributed sequences is applied in order to prove discrete universality theorems.

Lemma 3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the sequence of vectors*

$$\left(\left(x_n \frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}\right), \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 for every finite set of primes $\mathscr{M}$. Then for any pairwise non-equivalent Dirichlet characters $\chi_1,\ldots,\chi_J$, any $z > 0$, any real numbers ξ_p, $p \leq z$, any compact set $\mathcal{K} \subseteq \mathcal{D}$ with connected complement, any functions $f_1,\ldots,f_J$ continuous non-vanishing on $\mathcal{K}$ and analytic in its interior, and any $\varepsilon > 0$, there exist positive numbers c and v such that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \begin{array}{c} \max\limits_{1\leq j\leq J} \max\limits_{s\in\mathcal{K}} \left| L_{\{p\in\mathbb{P}\,:\, z<p\leq u\}}(s+ix_n,\chi_j) - f_j(s)\right| < \varepsilon \\ \max\limits_{p\leq z} \left\| x_n \frac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\} > c$$

for every $u \geq v$.

Proof. By Mergelyan's theorem (see Theorem A.21), we may assume without loss of generality that $f_1, \dots, f_J$ are polynomials which are non-vanishing in $\mathcal{K}$. Therefore, we can find a bounded admissable set $\mathcal{R}$ satisfying $\mathcal{K} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{D}$ where $f_1, \dots, f_J$ are non-vanishing.

Lemma 3.1 yields the existence of a sequence of real numbers $\underline{\omega}_0 = (\omega_{0p})_{p\in\mathbb{P}}$ such that, for every y with $0 < z < y$, there is a finite set of primes $\mathscr{M}_y$ containing all primes $z < p \leq y$ but no primes $p \leq z$ with

$$\max_{1\leq j\leq J} \max_{s\in\mathcal{R}} \left| L_{\mathscr{M}_y}(s, \chi_j, \underline{\omega}_0) - f_j(s) \right| < \frac{\varepsilon}{4}. \tag{3.1}$$

Let $u > 0$ be such that $\mathscr{N}_u := \{p \in \mathbb{P} : z < p \leq u\} \supseteq \mathscr{M}_y$. By continuity it follows that there is a $\delta > 0$ such that, for every $\underline{\phi} := (\phi_p)_{p\leq u}$ belonging to the closed and Jordan measurable set

$$\mathcal{L} := \left\{ (\phi_p)_{p\leq u} \in [0,1]^{\pi(u)} : \begin{array}{c} \max_{p\in\mathscr{M}_y} \|\phi_p - \omega_{0p}\| \leq \delta \\ \max_{p\leq z} \|\phi_p - \xi_p\| < \varepsilon \end{array} \right\},$$

we have

$$\max_{1\leq j\leq J} \max_{s\in\mathcal{R}} \left| L_{\mathscr{M}_y}\left(s, \chi_j, (\phi_p)_{p\in\mathscr{M}_y}\right) - L_{\mathscr{M}_y}\left(s, \chi_j, (\omega_{0p})_{p\in\mathscr{M}_y}\right) \right| < \frac{\varepsilon}{4}.$$

It then follows from inequality (3.1) that

$$\max_{1\leq j\leq J} \max_{s\in\mathcal{R}} \left| L_{\mathscr{M}_y}\left(s, \chi_j, (\phi_p)_{p\in\mathscr{M}_y}\right) - f_j(s) \right| < \frac{\varepsilon}{2}. \tag{3.2}$$

Now by assumption we know that the sequence

$$\underline{x}_n := \left(x_n \frac{\log p}{2\pi} \right)_{p\leq u}, \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 in $\mathbb{R}^{\pi(u)}$. Thus, by definition

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} 1 = \lim_{N\to\infty} \frac{1}{N} \sharp \{1 \leq n \leq N : \{\underline{x}_n\} \in \mathcal{L}\} = \mathrm{m}(\mathcal{L}), \tag{3.3}$$

which in this case coincides with the Jordan measure on $\mathbb{R}^{\pi(u)}$. In addition, if we define for every $j = 1, \dots, J$ and $s \in \overline{\mathcal{R}}$ the Riemann integrable and 1-periodic function

$$F_{j,s}\left(\underline{\phi}\right) := \left| L_{\mathscr{M}_y}\left(s, \chi_j, (\phi_p)_{p\in\mathscr{M}_y}\right) \left(L_{\mathscr{N}_u\setminus\mathscr{M}_y}\left(s, \chi_j, (\phi_p)_{p\in\mathscr{N}_u\setminus\mathscr{M}_y}\right) - 1 \right) \right|^2,$$

whenever $\left\{\underline{\phi}\right\} \in \mathcal{L}$ and 0 otherwise, then

$$\sum_{\substack{n=1 \\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} \left| L_{\mathscr{N}_u}(s + ix_n, \chi_j, \underline{0}) - L_{\mathscr{M}_y}(s + ix_n, \chi_j, \underline{0}) \right|^2 = \sum_{n=1}^{N} F_{j,s}\left(\underline{x}_n\right).$$

In view of Theorem A.7 and relation (3.2) we have that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F_{j,s}\left(\underline{x}_n\right) &= \int_{\mathcal{L}} F_{j,s}(\underline{x})\mathrm{d}\underline{x}\\ &\le \left(\max_{s\in\overline{\mathcal{R}}}|f_j(s)|+\frac{\varepsilon}{2}\right)^2 \mathrm{m}(\mathcal{L})\times\\ &\quad\times\int_0^1\cdots\int_0^1\left|L_{\mathscr{N}_u\setminus\mathscr{M}_y}\left(s,\chi_j,(\phi_p)_{p\in\mathscr{N}_u\setminus\mathscr{M}_y}\right)-1\right|^2\prod_{p\in\mathscr{N}_u\setminus\mathscr{M}_y}\mathrm{d}\phi_p,\end{aligned}\tag{3.4}$$

uniformly for $j=1,\ldots,J$ and $s\in\overline{\mathcal{R}}$, since the family of functions

$$\left\{F_{j,s}:\mathcal{L}\to\mathbb{C}:1\le j\le J,\ s\in\overline{\mathcal{R}}\right\}$$

is uniformly bounded and equicontinuous.

The set $\mathscr{N}_u\setminus\mathscr{M}_y$ contains only primes greater than y. Therefore, in view of (3.4) we can choose $y>0$ sufficiently large such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\ \{\underline{x}_n\}\in\mathcal{L}}}^{N}\iint_{\mathcal{R}}\left|\left(L_{\mathscr{N}_u}-L_{\mathscr{M}_y}\right)(s+ix_n,\chi_j)\right|^2\mathrm{d}\sigma\mathrm{d}t<\frac{\pi\left(d(\partial\mathcal{R},K)\right)^2\mathrm{m}(\mathcal{L})\varepsilon^2}{8J}.$$

Then it follows from Theorem A.16 that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\ \{\underline{x}_n\}\in\mathcal{L}}}^{N}\left(\max_{s\in\mathcal{K}}\left|\left(L_{\mathscr{N}_u}-L_{\mathscr{M}_y}\right)(s+ix_n,\chi_j)\right|\right)^2<\frac{\mathrm{m}(\mathcal{L})\varepsilon^2}{8J}$$

for every $j=1,\ldots,J$. Taking also into account relation (3.3), we deduce that

$$\liminf_{N\to\infty}\frac{1}{N}\sharp\left\{1\le n\le N:\ \begin{array}{c}\max\limits_{1\le j\le J}\max\limits_{s\in\mathcal{K}}\left|\left(L_{\mathscr{N}_u}-L_{\mathscr{M}_y}\right)(s+ix_n,\chi_j)\right|<\dfrac{\varepsilon}{2}\\ \{\underline{x}_n\}\in\mathcal{L}\end{array}\right\}>\frac{\mathrm{m}(\mathcal{L})}{2}.\tag{3.5}$$

Now the lemma follows from (3.2) and (3.5) if we set $v=\max\{p:p\in\mathscr{M}_y\}$ and $c=\mathrm{m}(\mathcal{L})/2$. As a last remark we notice that for fixed v the number c is the same for all $u\ge v$. □

We conclude with the first criterion to obtain discrete joint universality for Dirichlet L-functions with respect to a given sequence of real numbers $(x_n)_{n\in\mathbb{N}}$. In fact, our theorem is the discrete analogue of [41, Theorem 3].

Theorem 3.1 (Discrete Joint Universality Criterion for $\mathbf{L}(\mathbf{s},\boldsymbol{\chi})$). *Let $\chi_1,\ldots,\chi_J$ be pairwise non-equivalent Dirichlet characters. Let also $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the sequence of vectors*

$$\left(\left(x_n\frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}\right),\quad n\in\mathbb{N},$$

is uniformly distributed mod 1 for every finite set of primes $\mathscr{M}$, AND

$$\lim_{w\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{n=N}^{2N}\left|L(s+ix_n,\chi_j)-L_{\{p\in\mathbb{P}\,:\,p\le w\}}(s+ix_n,\chi_j)\right|^2=0,$$

$j=1,\dots,J$, uniformly in compact subsets of $\mathcal{D}_{\sigma_m}:=\{s\in\mathbb{C}:\sigma_m<\sigma<1\}$, where $\sigma_m\in[1/2,1)$. Then for any compact set $\mathcal{K}\subseteq\mathcal{D}_{\sigma_m}$ with connected complement, any $z>0$, any real numbers ξ_p, $p\le z$, any functions $f_1,\dots,f_J$ continuous non-vanishing on $\mathcal{K}$ and analytic in its interior, and any $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N}\sharp\left\{1\le n\le N:\begin{array}{c}\max\limits_{1\le j\le J}\max\limits_{s\in\mathcal{K}}\left|L\left(s+ix_n,\chi_j\right)-f_j(s)\right|<\varepsilon\\ \max\limits_{p\le z}\left\|x_n\dfrac{\log p}{2\pi}-\xi_p\right\|<\varepsilon\end{array}\right\}>0.$$

Proof. Let $\tilde{\varepsilon}$ be a suitable positive number which will be determined later on. Define the functions

$$\tilde{f}_j(s):=f_j(s)\prod_{p\le z}\left(1-\frac{\chi_j(p)\mathrm{e}(-\xi_p)}{p^s}\right),$$

for every $s\in\mathcal{K}$ and $j=1,\dots,J$. By the assumption on the uniform distribution of $(x_n)_{n\in\mathbb{N}}$ and Lemma 3.2 there exist positive numbers c and v, both depending on $\tilde{\varepsilon}$, such that

$$\liminf_{N\to\infty}\frac{1}{N}\sharp\left\{1\le n\le N:\begin{array}{c}\max\limits_{1\le j\le J}\max\limits_{s\in\mathcal{K}}\left|L_{\{p\in\mathbb{P}\,:\,z<p\le u\}}(s+ix_n,\chi_j)-\tilde{f}_j(s)\right|<\tilde{\varepsilon}\\ \max\limits_{p\le z}\left\|x_n\dfrac{\log p}{2\pi}-\xi_p\right\|<\tilde{\varepsilon}\end{array}\right\}>c \tag{3.6}$$

for every $u\ge v$. If $1\le n\le N$ is an element of the preceding set, then, for every $s\in\mathcal{K}$ and $j=1,\dots,J$,

$$\begin{aligned}&\left|L_{\{p\in\mathbb{P}\,:\,p\le u\}}\left(s+ix_n,\chi_j\right)-f_j(s)\right|\\&=\prod_{p\le z}\left|1-\frac{\chi_j(p)}{p^{s+ix_n}}\right|^{-1}\left|L_{\{p\in\mathbb{P}\,:\,z<p\le u\}}\left(s+ix_n,\chi_j\right)-f_j(s)\prod_{p\le z}\left(1-\frac{\chi_j(p)}{p^{s+ix_n}}\right)\right|\\&\le\max_{s\in\mathcal{K}}\prod_{p\le z}\left(1-\frac{1}{p^\sigma}\right)^{-1}\left(\tilde{\varepsilon}+\left|\tilde{f}_j(s)-f_j(s)\prod_{p\le z}\left(1-\frac{\chi_j(p)}{p^{s+ix_n}}\right)\right|\right)\\&\le 2^{\pi(z)/2}\left[\tilde{\varepsilon}+\max_{s\in\mathcal{K}}|f_j(s)|\left|\prod_{p\le z}\left(1-\frac{\chi_j(p)\mathrm{e}(-\xi_p)}{p^s}\right)-\prod_{p\le z}\left(1-\frac{\chi_j(p)}{p^{s+ix_n}}\right)\right|\right].\end{aligned} \tag{3.7}$$

Observe that the functions

$$\left(s,(\omega_p)_{p\le z}\right)\longmapsto\prod_{p\le z}\left(1-\frac{\chi_j(p)\mathrm{e}(\omega_p)}{p^s}\right),\quad j=1,\dots,J,$$

defined for fixed $z > 0$ and any $\left(s, (\omega_p)_{p \le z}\right) \in \mathcal{K} \times \mathbb{R}^{\pi(z)}$, are uniformly continuous. Hence, if we take $\tilde{\varepsilon} = \tilde{\varepsilon}(z, \mathcal{K}, f_1, \ldots, f_J, \varepsilon)$ to be a sufficiently small positive number, then for the positive integers $n \le N$ it follows from relations (3.7) and

$$\max_{p \le z} \left\| x_n \frac{\log p}{2\pi} - \xi_p \right\| < \tilde{\varepsilon} \le \varepsilon$$

that

$$\max_{1 \le j \le J} \max_{s \in \mathcal{K}} \left| L_{\{p \in \mathbb{P} : p \le u\}} (s + i x_n, \chi_j) - f_j(s) \right| < \frac{\varepsilon}{2}.$$

Hence, the latter inequlaties hold true for any positive integer $n \le N$ from (3.6). Therefore, if

$$\mathscr{A}(u, N) := \left\{ 1 \le n \le N : \begin{array}{c} \max\limits_{1 \le j \le J} \max\limits_{s \in \mathcal{K}} \left| L_{\{p \in \mathbb{P} : p \le u\}} (s + i x_n, \chi_j) - f_j(s) \right| < \dfrac{\varepsilon}{2} \\ \max\limits_{p \le z} \left\| x_n \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\},$$

then

$$\liminf_{N \to \infty} \frac{\sharp \mathscr{A}(u, N)}{N} > c$$

for every $u \ge v$.

Now let $\mathcal{R}$ be a bounded admissable set such that $K \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{D}_{\sigma_m}$. Then the second assertion of the theorem implies that for every $\delta > 0$ there is $w = w(\delta) > 0$ such that

$$\limsup_{N \to \infty} \frac{1}{N+1} \sum_{n=N}^{2N} \iint_{\mathcal{R}} \left| \left(L - L_{\{p \in \mathbb{P} : p \le u\}} \right) (s + i x_n, \chi_j) \right|^2 \mathrm{d}\sigma \mathrm{d}t < \delta^2$$

for every $u \ge w$ and $j = 1, \ldots, J$. In view of Theorem A.16 and since any interval (and, therefore, any sum), can be divided in a dyadic manner

$$[1, N] \subseteq \bigcup_{j=0}^{J} \left[\frac{N}{2^{j+1}}, \frac{N}{2^j} \right], \quad J = \left\lfloor \frac{\log N}{\log 2} \right\rfloor,$$

we obtain

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left(\max_{s \in \mathcal{K}} \left| \left(L - L_{\{p \in \mathbb{P} : p \le u\}} \right) (s + i x_n, \chi_j) \right| \right)^2 \mathrm{d}\sigma \mathrm{d}t \ll_{\mathcal{R}, \mathcal{K}} \delta^2$$

for every $u \ge w$ and $j = 1, \ldots, J$. The dependence of the implicit constant on $\mathcal{R}$ is in fact on σ_m. It is clear from the latter relation that for any sufficiently small positive number $\delta = \delta(\sigma_m, \mathcal{K}, J)$, if

$$\mathscr{B}(u, \delta, N) := \left\{ 1 \le n \le N : \max_{1 \le j \le J} \max_{s \in \mathcal{K}} \left| \left(L - L_{\{p \in \mathbb{P} : p \le u\}} \right) (s + i x_n, \chi_j) \right| < \delta \right\},$$

then

$$\liminf_{N\to\infty} \frac{\sharp\mathscr{B}(u,\delta,N)}{N} > 1-\delta$$

for every $u \geq w$.

If we take δ sufficiently small such that $\delta < \max\{c, \varepsilon/2\}$ and $u = \max\{v, w\}$, then

$$\begin{aligned}\liminf_{N\to\infty} \frac{\sharp\left(\mathscr{A}(u,N)\cap\mathscr{B}(u,\delta,N)\right)}{N} &\geq \liminf_{N\to\infty} \frac{\sharp\mathscr{A}(u,N)}{N} + \liminf_{N\to\infty} \frac{\sharp\mathscr{B}(u,\delta,N)}{N} - 1\\ &> c+1-\delta-1\\ &> 0.\end{aligned} \tag{3.8}$$

The theorem follows now from (3.8) and an application of the triangle inequality. □

As it is stated, the theorem supersedes a previous result due to Macaitienė [55], where she proved a similar criterion for $\zeta(s)$. The significance of our theorem does not lie only on the fact that it is a generalization of the aforementioned result to the more general setting of Dirichlet L-functions, and even more a joint universality theorem, but also to the fact that it is stronger in the sense of relaxing the conditions which were set in [55] for the sequence $(x_n)_{n\in\mathbb{N}}$ to satisfy. In particular, the sequence $(\alpha x_n)_{n\in\mathbb{N}}$ should be uniformly distributed mod 1 for every irrational α, $x_n \ll n$ as $n \to +\infty$ and, for any integers $1 \leq n \neq m \leq N$, $|x_n - x_m| \geq 1/y_N$, where $(y_n)_{n\in\mathbb{N}}$ is a sequence of real numbers satisfying $x_n y_n \ll n$ as $n \to +\infty$. It can be seen that the last two conditions are sufficient to prove the second assertion in our theorem with $\sigma_m = 1/2$, as it would follow by an application of Gallagher's Lemma (Lemma 2.1) and a classical result due to Bohr [6] on the approximation of $\zeta(s)$ by truncated Euler products $\zeta_{\mathscr{M}}(s)$ in the mean-square. However, these conditions are not necessary as we will show in the next chapter when we consider sequences of polynomials $P_{\underline{a}}(x) = a_n x^n + \cdots + a_1 x + a_0$. In this case it is obvious that $P_{\underline{a}}(n) = O(n)$ does not hold for polynomials of degree greater than one.

A last remark about the number σ_m, which appears only in the second condition for the discrete mean-square. We do not expect that σ_m can be smaller than 1/2, because it is the least possible number provided by Lemma 3.1.

3.1.2 The case of Hurwitz Zeta-Functions

The succeeding results are similar to the ones of the previous subsection, but in the case of $\zeta(s;\alpha)$ the target function f does not need to be non-vanishing. Actually, this is the first time in the literature where applying Voronin's approach is used to prove discrete universality for $\zeta(s;\alpha)$ instead of applying Bagchi's probabilistic treatment. There is an extensive literature regarding the probabilistic model and we encourage the interested reader to study the bibliographies in [56] and [86].

The following lemma is a special case of Lemma 2.2 in Gonek's thesis [24] for $\Lambda := \{\log(n+\alpha) : n \in \mathbb{N}_0\}$ and $\alpha \in (0,1]$. It should be mentioned that its proof is, in principle, different from the one of Lemma 3.1, which depends on a theorem of Pechersky [63]. Basically, Pechersky's theorem is an extension of Riemann's theorem on rearrangements of conditionally convergent series of real numbers to Hilbert space. Gonek derives his lemma by incorporating techniques introduced by Good [26].

Lemma 3.3. *Let $\mathcal{K}$ be a compact set with connected complement in a strip $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$ and f be continuous on $\mathcal{K}$ and analytic in its interior. Then, for any $\alpha \in (0,1]$ and any $R \in \mathbb{N}$, there exists an ineffectively computable positive number $Q_0 = Q_0(\sigma_1, \sigma_2, \mathcal{K}, \alpha, f, R)$ such that for any integer $Q \geq Q_0$, there are real numbers θ_k, $k = R, \ldots, Q-1$, for which*

$$\max_{s\in\mathcal{K}} \left| f(s) - \sum_{k=R}^{Q-1} \frac{\mathrm{e}(\theta_k)}{(k+\alpha)^s} \right| \ll_{\sigma_1,\sigma_2,\mathcal{K},\alpha} R^{-1/2}.$$

Note that the condition of f being non-vanishing has already been dropped.

Using the latter lemma we prove

Lemma 3.4. *Let $\mathcal{K}$ be a compact set with connected complement in a strip $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$ and f be continuous on $\mathcal{K}$ and analytic in its interior. Then, for any $A \in (0,1]$ and any $R \in \mathbb{N}$, there exists an ineffectively computable positive number $Q_0 = Q_0(\sigma_1, \sigma_2, K, A, f, R)$, such that for any integer $Q \geq Q_0$ and any real number $\alpha \in [A,1]$, there are real numbers θ_k, $k = R, \ldots, Q-1$, for which*

$$\max_{s\in\mathcal{K}} \left| f(s) - \sum_{k=0}^{R-1} \frac{1}{(k+\alpha)^s} - \sum_{k=R}^{Q-1} \frac{\mathrm{e}(\theta_k)}{(k+\alpha)^s} \right| \ll_{\sigma_1,\sigma_2,\mathcal{K},A} R^{-1/2}.$$

Proof. Let $A \in (0,1]$ and $R \in \mathbb{N}$. If

$$0 < \rho \ll \frac{R^{-1/2}}{\zeta(1+\sigma_1; A)} \tag{3.9}$$

is sufficiently small, we divide the interval $[A,1]$ into finitely many disjoint subintervals of length at most ρ, say L_m, $m = 1, \ldots, M$, and we choose one number from each such interval $\alpha_m \in L_m$. Then, for every $m = 1, \ldots, M$, Lemma 3.3 yields the existence of a positive number $Q_m = Q_m(\sigma_1, \sigma_2, \mathcal{K}, \alpha_m, f, R)$ such that for any integer $Q \geq Q_m$ there are real numbers θ_{mk} with

$$\max_{s\in\mathcal{K}} \left| f(s) - \sum_{k=0}^{R-1} \frac{1}{(k+\alpha_m)^s} - \sum_{k=R}^{Q-1} \frac{\mathrm{e}(\theta_{mk})}{(k+\alpha_m)^s} \right| \ll R^{-1/2}, \tag{3.10}$$

where the implicit constant depends on $\sigma_1, \sigma_2, \mathcal{K}$ and α_m. We set

$$Q_0 := \max\{Q_m : m = 1, \ldots, M\}$$

and assume that $Q \geq Q_0$ is an integer and $\alpha \in [A, 1]$. Then $\alpha \in L_m$ for some $m = 1, \ldots, M$ and, if we set $\theta_k := \theta_{mk}$, for all $k = R, \ldots, Q-1$, we deduce from (3.10) and the triangle inequality that

$$\begin{aligned}
&\max_{s\in\mathcal{K}} \left| f(s) - \sum_{k=0}^{R-1} \frac{1}{(k+\alpha)^s} - \sum_{k=R}^{Q-1} \frac{\mathrm{e}(\theta_k)}{(k+\alpha)^s} \right| \\
&\ll_{\sigma_1,\sigma_2,\mathcal{K}} \max_{s\in\mathcal{K}} \left| \sum_{k=0}^{R-1} \left(\frac{1}{(k+\alpha)^s} - \frac{1}{(k+\alpha_m)^s} \right) + \sum_{k=R}^{Q-1} \left(\frac{\mathrm{e}(\theta_k)}{(k+\alpha)^s} - \frac{\mathrm{e}(\theta_{mk})}{(k+\alpha_m)^s} \right) \right| \\
&\qquad + R^{-1/2} \\
&\ll_{\sigma_1,\sigma_2,\mathcal{K}} \max_{s\in\mathcal{K}} |s| \sum_{k=0}^{Q-1} \left| \int_{\alpha}^{\alpha_m} \frac{1}{(k+u)^{s+1}} \right| + R^{-1/2} \\
&\ll_{\sigma_1,\sigma_2,\mathcal{K}} |\alpha - \alpha_m| \sum_{k=0}^{Q-1} \frac{1}{(k+A)^{1+\sigma_1}} + R^{-1/2} \\
&\ll_{\sigma_1,\sigma_2,\mathcal{K},\alpha_m} R^{-1/2}.
\end{aligned}$$

Since the choice of α_m depends only on A, the lemma follows. □

The proof of the succeeding lemma is almost the same as the one for Lemma 3.2. In fact, in some points it is even simpler. Nevertheless, we give a detailed proof. Before proceeding we furthermore need to define the following Dirichlet polynomials

$$(s, \underline{\theta}, \alpha) \longmapsto \zeta_Q(s, \underline{\theta}, \alpha) := \sum_{k=0}^{Q-1} \frac{\mathrm{e}(\theta_k)}{(k+\alpha)^s}, \tag{3.11}$$

for every $(s, \underline{\theta}, \alpha) \in \mathbb{C} \times \mathbb{R}^{\mathbb{N}} \times (0, 1]$. We also set $\zeta_Q(s; \alpha) := \zeta(s, \underline{0}, \alpha)$ for brevity.

Lemma 3.5. *Assume that $\alpha \in (0, 1]$ is a transcendental number. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the sequence of vectors*

$$\left(\left(x_n \frac{\log(k+\alpha)}{2\pi} \right)_{0\leq k\leq Q-1} \right), \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 for any positive integer Q. Then, for any compact set $\mathcal{K} \subseteq \mathcal{D}$ with connected complement, any function f continuous on $\mathcal{K}$ and analytic in its interior, and any $\varepsilon > 0$, there exist positive numbers c and Q_0 such that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \max_{s\in\mathcal{K}} |\zeta_Q(s + ix_n; \alpha) - f(s)| < \varepsilon \right\} > 0$$

for every integer $Q > Q_0$.

Proof. Let $\mathcal{R}$ be a bounded admissable set satisfying the relation $K \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{D}$. If R is a positive integer such that $R \gg (\varepsilon/2)^{-1/2}$, then Lemma 3.4 yields the existence of a positive integer Q_0 and real numbers θ_k, $k = 0, \ldots, Q_0 - 1$, with

$$\max_{s\in\mathcal{R}} \left|\zeta_{Q_0}\left(s, (\theta_k)_{0\le n\le Q_0-1}, \alpha\right) - f(s)\right| < \frac{\varepsilon}{4}. \tag{3.12}$$

There is $\delta > 0$ such that, for any integer $Q > Q_0$ and any

$$\underline{\phi} \in \mathcal{L} := \left\{(\phi_k)_{0\le k\le Q-1} \in [0,1]^Q : \max_{0\le k\le Q_0-1} \|\phi_k - \theta_k\| \le \delta\right\},$$

we have

$$\max_{s\in\mathcal{R}} \left|\zeta_{Q_0}\left(s, (\phi_k)_{0\le k\le Q_0-1}, \alpha\right) - \zeta_{Q_0}\left(s, (\theta_k)_{0\le k\le Q_0-1}, \alpha\right)\right| < \frac{\varepsilon}{4}.$$

It then follows from relation (3.12) that

$$\max_{s\in\mathcal{R}} \left|\zeta_{Q_0}\left(s, (\phi_k)_{0\le k\le Q_0-1}, \alpha\right) - f(s)\right| < \frac{\varepsilon}{2}. \tag{3.13}$$

Since the sequence

$$\underline{x}_n := \left(x_n \frac{\log(k+\alpha)}{2\pi}\right)_{0\le k\le Q-1}, \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 in $\mathbb{R}^Q$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} 1 = \lim_{N\to\infty} \frac{1}{N} \sharp \left\{1 \le n \le N : \{\underline{x}_n\} \in \mathcal{L}\right\} = \mathrm{m}(\mathcal{L}). \tag{3.14}$$

If we define for every $s \in \overline{\mathcal{R}}$ the Riemann integrable and 1-periodic function

$$F_s\left(\underline{\phi}\right) := \left|\zeta_Q\left(s, -(\phi_k)_{0\le k\le Q-1}, \alpha\right) - \zeta_{Q_0}\left(s, -(\phi_k)_{0\le n\le Q_0-1}, \alpha\right)\right|^2,$$

whenever $\left\{\underline{\phi}\right\} \in \mathcal{L}$ and 0 otherwise, then Theorem A.7 yields

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} \left|\zeta_Q\left(s + ix_n, \underline{0}, \alpha\right) - \zeta_{Q_0}\left(s + ix_n, \underline{0}, \alpha\right)\right|^2 \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_s(\{\underline{x}_n\}) \\
&= \int_{\mathcal{L}} F_s(\underline{x}) \mathrm{d}\underline{x} \qquad (3.15) \\
&= \mathrm{m}(\mathcal{L}) \int_0^1 \cdots \int_0^1 \left|\sum_{k=Q_0}^{Q-1} \frac{\mathrm{e}(-\phi_k)}{(k+\alpha)^s}\right|^2 \prod_{k=Q_0}^{Q-1} \mathrm{d}\phi_k \\
&= \mathrm{m}(\mathcal{L}) \sum_{k=Q_0}^{Q-1} \frac{1}{(k+\alpha)^{2\sigma}},
\end{aligned}$$

uniformly for $s \in \overline{\mathcal{R}}$, since the family of functions

$$\left\{F_s : \mathcal{L} \to \mathbb{C} : s \in \overline{\mathcal{R}}\right\}$$

is uniformly bounded and equicontinuous.

In view of (3.15) we can choose a sufficiently large positive integer Q_0 such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} \iint_{\mathcal{R}} \left|(\zeta_Q - \zeta_{Q_0})(s + ix_n; \alpha)\right|^2 \mathrm{d}\sigma \mathrm{d}t < \frac{\pi\left(d(\partial\mathcal{R}, K)\right)^2 \mathrm{m}(\mathcal{L})\varepsilon^2}{8}.$$

It then follows from Theorem A.16 that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ \{\underline{x}_n\}\in\mathcal{L}}}^{N} \left(\max_{s\in K} \left|(\zeta_Q - \zeta_{Q_0})(s + ix_n; \alpha)\right|\right)^2 < \frac{\mathrm{m}(\mathcal{L})\varepsilon^2}{8}.$$

Taking also into account relation (3.14) we deduce that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ n \in \mathbb{N} : \begin{array}{c} \max_{s\in K} \left|(\zeta_Q - \zeta_{Q_0})(s + ix_n; \alpha)\right| < \frac{\varepsilon}{2} \\ \{\underline{x}_n\} \in \mathcal{L} \end{array} \right\} > \frac{\mathrm{m}(\mathcal{L})}{2}. \tag{3.16}$$

Now the lemma follows from (3.13) and (3.16) if we set $c = \mathrm{m}(\mathcal{L})/2$, which is the same for fixed Q_0 and any integer $Q > Q_0$. □

One could consider α in Lemma 3.5 to be a rational or an algebraic irrational number in $(0, 1]$, without any changes in the proof. Let us assume for the time being that α is a rational or an algebraic irrational in $(0, 1]$. According to the lemma, we would have to prove that the sequence

$$\left(\left(x_n \frac{\log(k + \alpha)}{2\pi}\right)_{0\leq k\leq Q-1}\right), \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 for any positive integer Q, or equivalently, that for any positive integer Q and any lattice point $\underline{h} = (h_k)_{0\leq k\leq Q-1} \in \mathbb{Z}^Q$, $\underline{h} \neq \underline{0}$, the sequence

$$\left(\frac{x_n}{2\pi} \sum_{k=0}^{Q-1} h_k \log(k + \alpha)\right)_{n\in\mathbb{N}} \tag{3.17}$$

is uniformly distributed mod 1. However, the numbers $\log(k+\alpha)$, $0 \leq k \leq Q-1$, could be linearly dependent over $\mathbb{Q}$ for some Q and, thus, for a suitable lattice point $\underline{h} \neq \underline{0}$, the latter sum could be zero. But then the sequence in (3.17) could never be uniformly distributed mod 1 for any sequence $(x_n)_{n\in\mathbb{N}}$ and the lemma would have no meaning to begin with. That is why we assume that α is a transcendental number, in which case the numbers $\log(k + \alpha)$, $k \in \mathbb{N}_0$, are linearly independent over $\mathbb{Q}$.

Now arguing similarly as in the proof of Theorem 3.1, we obtain

Theorem 3.2 (**Discrete Universality Criterion for $\zeta(s;\alpha)$**). *Assume that $\alpha \in (0,1]$ is a transcendental number. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the sequence of vectors*

$$\left(\left(x_n \frac{\log(k+\alpha)}{2\pi}\right)_{0\le k\le Q-1}\right), \quad n \in \mathbb{N},$$

is uniformly distributed mod 1 for any positive integer Q, AND

$$\lim_{Q\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{n=N}^{2N} |\zeta(s+ix_n;\alpha) - \zeta_Q(s+ix_n;\alpha)|^2 = 0,$$

uniformly in compact subsets of $\mathcal{D}_{\sigma_m}$, where $\sigma_m \in [1/2, 1)$. Then, for any compact set $\mathcal{K} \subseteq \mathcal{D}_{\sigma_m}$ with connected complement, any function f continuous on $\mathcal{K}$ and analytic in its interior, and any $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N : \max_{s\in\mathcal{K}} |\zeta(s+ix_n;\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Proof. From the assumption on the uniform distribution of $(x_n)_{n\in\mathbb{N}}$ and Lemma 3.5 we know that there are positive numbers c and Q_0, both depending on ε, such that, if

$$\mathscr{A}(Q,N) := \left\{ 1 \le n \le N : \max_{s\in\mathcal{K}} |\zeta_Q(s+ix_n;\alpha) - f(s)| < \varepsilon \right\},$$

then

$$\liminf_{N\to\infty} \frac{\sharp\mathscr{A}(Q,N)}{N} > c \tag{3.18}$$

for every $Q > Q_0$.

If $\mathcal{R}$ is a bounded admissable set such that $K \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{D}_{\sigma_m}$, then the second assertion of the theorem implies that, for every $\delta > 0$, there is a positive integer $Q_1 = Q_1(\delta)$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{n=N}^{2N} \iint_{\mathcal{R}} |(\zeta - \zeta_Q)(s+ix_n;\alpha)|^2 \, d\sigma dt < \delta^2$$

or

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\max_{s\in\mathcal{K}} |(\zeta - \zeta_Q)(s+ix_n;\alpha)| \right)^2 d\sigma dt \ll_{\sigma_m,\mathcal{K}} \delta^2$$

for every $Q \ge Q_0$. Hence, for any sufficiently small positive number $\delta = \delta(\sigma_m, \mathcal{K})$, if

$$\mathscr{B}(Q,\delta,N) := \left\{ 1 \le n \le N : \max_{s\in\mathcal{K}} |(\zeta - \zeta_Q)(s+ix_n;\alpha)| < \delta \right\},$$

then

$$\liminf_{N\to\infty} \frac{\sharp\mathscr{B}(Q,\delta,N)}{N} > 1-\delta$$

for every $Q \geq Q_1$.

If now δ is sufficiently small such that $\delta < \max\{c, \varepsilon/2\}$ and $Q = \max\{Q_0, Q_1\}$, then

$$\liminf_{N\to\infty} \frac{\sharp\left(\mathscr{A}(Q,N)\cap\mathscr{B}(Q,\delta,N)\right)}{N} > c-\delta > 0$$

and the theorem follows. □

Although we can not obtain directly by the same methods discrete universality theorems for $\zeta(s;\alpha)$ when α is a rational number in $(0,1]$, the expression of such Hurwitz Zeta-function as a linear combination of Dirichlet L-functions (1.6) and Theorem 3.1 yield the following theorem. The proof has no significant differences with the one given in [24, Theorem 4.1], but we will provide one for the sake of completeness regarding our discrete universality criteria.

Theorem 3.3 (**Discrete Universality Criterion for $\boldsymbol{\zeta(s;r/q)}$**)**.** *Let $r, q \geq 1$ be integers such that $(r,q)=1$. Let also $(x_n)_{n\in\mathbb{N}}$ satisfy the conditions of Theorem 3.1 for all Dirichlet characters mod q. Then, for any compact set $\mathcal{K} \subseteq \mathcal{D}_{\sigma_m}$ with connected complement, any function f continuous on $\mathcal{K}$, analytic in its interior and non-vanishing when $q=1$ or 2, and any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\sharp\left\{1\leq n\leq N : \max_{s\in\mathcal{K}}\left|\zeta\left(s+ix_n;\frac{r}{q}\right)-f(s)\right| < \varepsilon\right\} > 0.$$

Proof. If $q=1$, then $\zeta(s;1)=\zeta(s)=L(s,\chi_0)$, χ_0 being the unique Dirichlet character mod 1, and the Theorem 3.1 yields the the result.

If $q=2$, then $\zeta(s;1/2)=2^sL(s,\chi_0)$, χ_0 being the unique Dirichlet character mod 2. We apply Theorem 3.1 for $z=2$, $\xi_2=0$, $\tilde{f}(s):=2^{-s}f(s)$, $s\in\mathcal{K}$, and a sufficiently small $\tilde{\varepsilon}$. Then the set of integers $n>0$ such that

$$\max_{s\in\mathcal{K}}\left|L\left(s+ix_n,\chi_0\right)-\tilde{f}(s)\right| < \tilde{\varepsilon}$$

and

$$\left\|x_n\frac{\log 2}{2\pi}\right\| < \tilde{\varepsilon}, \tag{3.19}$$

has positive lower density. But for those integers n we also have that, for every $s\in\mathcal{K}$

$$\begin{aligned}\left|\zeta\left(s+ix_n;\frac{1}{2}\right)-f(s)\right| &= \left|2^{s+ix_n}\right|\left|L\left(s+ix_n,\chi_0\right)-2^{-s-ix_n}f(s)\right| \\ &\leq 2\left(\tilde{\varepsilon}+\left|\tilde{f}(s)-2^{-s-ix_n}f(s)\right|\right) \\ &\leq 2\left(\tilde{\varepsilon}+\max_{s\in\mathcal{K}}\left|2^{-s}f(s)\right|\left|1-2^{-ix_n}\right|\right),\end{aligned}$$

which can be made arbitrarily small by choosing sufficiently small $\tilde{\varepsilon}$, as follows by the uniform continuity of $\theta \longmapsto 2^{i\theta}$, $\theta \in \mathbb{R}$, and relation (3.19). Therefore, the same set of integers, which has positive lower density, can be used for $\zeta(s;1/2)$.

Lastly, for $q \geq 3$, let $\chi_1, \ldots, \chi_{\varphi(q)}$, be the pairwise non-equivalent Dirichlet characters mod q and f a function which can also have zeros in $\mathcal{K}$. If we fix a positive number c such that

$$c > \frac{\max_{s\in\mathcal{K}} |f(s)q^{-s}|}{\varphi(q)-1},$$

then the functions $s \longmapsto f(s)q^{-s} + c$ and $s \longmapsto f(s)q^{-s} - c(\varphi(q)-1)$, $s \in \mathcal{K}$, do not vanish in $\mathcal{K}$. Therefore, the functions

$$s \longmapsto f_j(s) := \begin{cases} \chi_1(r)\left(f(s)q^{-s} - c(\varphi(q)-1)\right), & j = 1, \\ \chi_j\left(f(s)q^{-s} + c\right), & j = 2, \ldots, \varphi(q), \end{cases}$$

are continuous non-vanishing on $\mathcal{K}$, analytic in its interior and

$$f(s) = \frac{q^s}{\varphi(q)} \sum_{j=1}^{\varphi(q)} \overline{\chi_j(r)} f_j(s)$$

for every $s \in \mathcal{K}$. We apply Theorem 3.1 with $z = q$, $\xi_p = 0$, $p \leq z$, and sufficiently small $\tilde{\varepsilon}$. Then the set of positive integers n such that

$$\max_{1\leq j\leq\varphi(q)} \max_{s\in\mathcal{K}} |L(s+ix_n,\chi_j) - f_j(s)| < \tilde{\varepsilon}$$

and

$$\left\| x_n \frac{\log q}{2\pi} \right\| \ll \max_{p|q} \left\| x_n \frac{\log p}{2\pi} \right\| \ll \max_{p\leq q} \left\| x_n \frac{\log p}{2\pi} \right\| \ll \tilde{\varepsilon}, \tag{3.20}$$

has positive lower density. But for those integers n we also have that, for every $s \in \mathcal{K}$,

$$\begin{aligned}
\left|\zeta\left(s+ix_n;\frac{r}{q}\right) - f(s)\right| &= \left|\frac{q^s}{\varphi(q)}\right| \left|\sum_{j=1}^{\varphi(q)} \overline{\chi_j(r)} (q^{ix_n} L(s+ix_n,\chi_j) - f_j(s))\right| \\
&\leq \frac{q}{\varphi(q)} \sum_{j=1}^{\varphi(q)} \left|q^{ix_n} L(s+ix_n,\chi_j) - f_j(s)\right| \\
&\leq \frac{q}{\varphi(q)} \sum_{j=1}^{\varphi(q)} \left[\tilde{\varepsilon} + \max_{s\in\mathcal{K}} |f_j(s)| \left|q^{ix_n} - 1\right|\right],
\end{aligned}$$

which can be made arbitrarily small by choosing sufficiently small $\tilde{\varepsilon}$, as follows from the uniform continuity of $\theta \longmapsto q^{i\theta}$, $\theta \in \mathbb{R}$, and relation (3.20). Therefore, the same set of integers, which has positive lower density, can be used for $\zeta(s;r/q)$. □

We do not treat the case of algebraic irrational $\alpha \in (0,1]$ here. In fact, it is still an open problem whether $\zeta(s;\alpha)$ for such an α is universal or not! We will return to this topic in the next chapter which is devoted completely to this case and where we give a partial answer to this problem.

3.2 Sequences leading to Universality

This section is dedicated, as already implied by the title, to sequences which satisfy the criteria we proved in the previous section. The usual candidates are arithmetic progressions $(an)_{n\in\mathbb{N}}$, for fixed $a > 0$, since they behave quite well and the proofs for discrete universality are much alike the ones for continuous universality. An interesting and open question is whether continuous universality implies the discrete one or vice versa. If that is the case, then one has to consider with respect to which sequences the discrete universality is equivalent to the continuous one. Is it with respect to any sequence or just, for example, arithmetic progressions?

It is apparent that discrete universality provides more information regarding the value-distribution of the zeta- or L-function we wish to study, since we can examine this function's behaviour over a prescribed (discrete) set. We give three examples of such sets which are generated by sequences with less structure than the one of arithmetic progressions. We start with Beatty sequences $(\lfloor an\rfloor)_{n\in\mathbb{N}}$, which behave almost like arithmetic progressions, then we present a discrete universality result with respect to ordinates of c-points of a suitable zeta-function. Lastly, we treat the most difficult case of polynomial sequences $(P_{\underline{a}}(n))_{n\in\mathbb{N}}$.

For the sake of simplicity we say that a sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathfrak{U}(\zeta,\alpha,\sigma_m)$-*universal*, if it satisfies the conditions of Theorems 3.2 and 3.3 inside a strip $1/2 \leq \sigma_m < \sigma < 1$. Respectively, a sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathfrak{U}(L,\sigma_m)$-*universal*, if it satisfies the conditions of Theorem 3.1 for any finite collection $L(s,\chi_j)$, $j = 1,\dots,J$, inside a strip $1/2 \leq \sigma_m < \sigma < 1$, where $\chi_1,\dots,\chi_J$, are assumed to be pairwise non-equivalent Dirchlet characters. In addition, since a sequence is $\mathfrak{U}(\zeta,\alpha,\sigma_m)$-*universal* for rational $\alpha \in (0,1]$ if it is $\mathfrak{U}(L,\sigma_m)$-universal with respect to all Dirichlet characters mod q by Theorem 3.3, we will not mention this case furthermore.

3.2.1 Beatty Sequences

If a is a positive real number, then $(\lfloor an\rfloor)_{n\in\mathbb{N}}$ is called a Beatty sequence. We present here only our principal result regarding such sequences, since we mainly wish to highlight the discrete universality criteria of the previous chapter. Moreover, we will give a more detailed introduction and description of Beatty sequences in the last chapter. Lastly, we only consider the case when $a > 1$, for, otherwise, the Beatty sequence contains the sequence of all positive integers and this case has been studied already in the literature with respect to discrete universality theorems.

Theorem 3.4. *Let $\alpha \in (0,1]$ be a transcendental number. Then for almost all $a > 1$ the sequence $(\lfloor an\rfloor)_{n\in\mathbb{N}}$ is $\mathfrak{U}(\zeta,\alpha,1/2)$- and $\mathfrak{U}(L,1/2)$-universal.*

Proof. We start by defining the sets

$$\mathcal{L}_1(\alpha) := \bigcap_{Q=2}^{\infty} \bigcap_{(m_1,\dots,m_Q)\in\mathbb{Z}^Q\setminus\{\underline{0}\}} \left\{ a > 1 : \frac{m_1}{a} + m_2 + \sum_{k=0}^{Q-2} \frac{m_{k+2}\log(k+\alpha)}{2\pi} \notin \mathbb{Z} \right\}$$

and

$$\mathcal{L}_2 := \bigcap_{r\in\mathbb{Q}_+\setminus\{1\}} \bigcap_{(m_1,m_2,m_3)\in\mathbb{Z}^3\setminus\{\underline{0}\}} \left\{ a>1 : \frac{m_1}{a} + m_2 + m_3\frac{\log r}{2\pi} \notin \mathbb{Z} \right\}.$$

Observe that both sets $\mathcal{L}_1(\alpha)$ and $\mathcal{L}_2$ are of full Lebesgue measure. In addition, if a is an element of $\mathcal{L}_1(\alpha)$ or $\mathcal{L}_2$, then the numbers

$$1, a, a\frac{\log\alpha}{2\pi}, \dots, a\frac{\log(Q-1+\alpha)}{2\pi}$$

are linearly independent over $\mathbb{Q}$ for any positive integer Q, or the numbers

$$1, a, \left(a\frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}$$

are linearly independent over $\mathbb{Q}$ for any finite set of primes $\mathscr{M}$, respectively. In view of Corollary A.2 and Theorem A.4, this implies that, if a is an element of $\mathcal{L}_1(\alpha)$ or $\mathcal{L}_2$, then the sequence

$$\left(\lfloor an\rfloor\frac{\log(k+\alpha)}{2\pi}\right)_{0\le k\le Q-1}, \quad n\in\mathbb{N},$$

is uniformly distributed mod 1 for any positive integer Q, or the sequence

$$\left(\lfloor an\rfloor\frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}, \quad n\in\mathbb{N},$$

is uniformly distributed mod 1 for any finite set of primes $\mathscr{M}$, respectively. Therefore, the first condition of Theorems 3.1 and 3.2 on uniformly distributed sequences is satisfied.

It is left to show that the second condition of Theorems 3.1 and 3.2 regarding the discrete mean-squares also holds. To this end, let $a\in\mathcal{L}_1(\alpha)$, $H>0$, $1/2<\sigma_2<\sigma_1<1$ and set

$$f_Q(\sigma,h,t) := \zeta(\sigma+ih+it;\alpha) - \zeta_Q(\sigma+ih+it;\alpha),$$

for any $\sigma_2\le\sigma\le\sigma_1$, any $|h|\le H$, any $t>0$, and any $Q\in\mathbb{N}$. Then Gallagher's lemma (Lemma 2.1) implies that, for any $N\gg_{Q,H}1$ sufficiently large integer, $T_0=aN-(a+1)/2$, $T=a(N+1)$, $\delta=a-1$ and $\mathscr{B}=\{\lfloor an\rfloor : N\le n\le 2N\}$,

$$\begin{aligned}\sum_{n=N}^{2N} &\left|\zeta\left(\sigma+ih+i\lfloor an\rfloor;\alpha\right) - \zeta_Q\left(\sigma+ih+i\lfloor an\rfloor;\alpha\right)\right|^2 \\ \le&\frac{1}{a-1}\int\limits_{aN-(a+1)/2-H}^{2aN+(a-1)/2+H} \left|f_Q(\sigma,0,t)\right|^2\mathrm{d}t+ \\ &+\left[\int\limits_{aN-(a+1)/2-H}^{2aN+(a-1)/2+H}|f_Q(\sigma,0,t)|^2\mathrm{d}t \int\limits_{aN-(a+1)/2-H}^{2aN+(a-1)/2+H}|f_Q'(\sigma,0,t)|^2\mathrm{d}t\right]^{1/2}\end{aligned} \tag{3.21}$$

whenever $\sigma_2 \leq \sigma \leq \sigma_1$, $|h| \leq H$ and $Q \in \mathbb{N}$. Since the interval

$$[aN - (a+1)/2 - H, 2aN + (a-1)/2 + H]$$

is a subset of

$$\left[a\left(\frac{N}{2} - \frac{1}{2}\right), aN\right] \cup \left[a\left(N - \frac{1}{2}\right), 2aN\right] \cup \left[a\left(2N - \frac{1}{2}\right), 4aN\right],$$

for sufficienlty large integer $N \gg_{Q,H} 1$, we deduce from relations (2.5) and (2.6) of the previous chapter, and relation (3.21), that

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{n=N}^{2N} |\zeta(\sigma + ih + i\lfloor an\rfloor; \alpha) - \zeta_Q(\sigma + ih + i\lfloor an\rfloor; \alpha)|^2 \ll_{a,\alpha} Q^{1-2\sigma_1}.$$

Hence,

$$\lim_{Q\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{n=N}^{2N} |\zeta(\sigma + ih + i\lfloor an\rfloor; \alpha) - \zeta_Q(\sigma + ih + i\lfloor an\rfloor; \alpha)|^2 = 0,$$

uniformly in $\sigma_2 \leq \sigma \leq \sigma_1$ and $|h| \leq H$. Thus, the second condition of Theorem 3.2 is also satisfied and we deduce that the sequence $(\lfloor an\rfloor)_{n\in\mathbb{N}}$ is $\mathfrak{U}(\zeta, \alpha, 1/2)$-universal.

To prove now that $(\lfloor an\rfloor)_{n\in\mathbb{N}}$ is $\mathfrak{U}(L, 1/2)$- universal for any $a \in \mathcal{L}_2$, we only need to show that for any Dirichlet character χ

$$\lim_{w\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{n=N}^{2N} \left|\left(L - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}\right)(\sigma + ih + i\lfloor an\rfloor, \chi)\right|^2 = 0,$$

uniformly in $\sigma_2 \leq \sigma \leq \sigma_1$ and $|h| \leq H$. This will follow, in the same manner as in the case of $\zeta(s; \alpha)$, by applying Gallagher's lemma to

$$f_Q(\sigma, h, t) := L(\sigma + ih + it, \chi) - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}(\sigma + ih + it, \chi)$$

and using the identities

$$\lim_{w\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\left(L - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}\right)(s + it, \chi)\right|^2 \mathrm{d}t = 0$$

and

$$\lim_{w\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\left(L - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}\right)'(s + it, \chi)\right|^2 \mathrm{d}t = 0,$$

which hold uniformly in compact sets of the critical strip $\mathcal{D}$. The first identity was proved by Bohr [6, Hilfssatz 2] for $\zeta(s)$ and it is straightforward to prove it also for $L(s, \chi)$. The second identity follows from the first one and an application of Cauchy's integral formula, that is,

$$\left(L - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}\right)'(s + it, \chi) = \frac{1}{2\pi i} \int_{|s-z|=\epsilon} \frac{\left(L - L_{\{p\in\mathbb{P}\,:\,p\leq w\}}\right)(z + it, \chi)}{(z - s)^2} \mathrm{d}z,$$

where ϵ is a sufficienlty small and fixed positive number. This concludes the proof of the theorem. □

3.2.2 c-points of Zeta-Functions from the Selberg Class

Given a complex number c and a suitable zeta-function $\mathfrak{L}(s)$, the roots of the equation

$$\mathfrak{L}(s) = c$$

are called c-points of $\mathfrak{L}(s)$ and will be denoted by $\rho_c = \beta_c + i\gamma_c$. Here and in the sequel the c-points are counted according to multiplicities. Moreover, we restrict ourselves to the upper half-plane of $\mathbb{C}$ and we assume that a sequence of c-points $(\rho_{c,n})_{n\in\mathbb{N}}$, is taken in such a way that, if $n \leq m$, then $0 \leq \gamma_{c,n} \leq \gamma_{c,m}$.

Garunkštis, Laurinčikas and Macaitienė [20] have proved that the sequence $(\gamma_{0,n})_{n\in\mathbb{N}}$, which corresponds to the non-trivial zeros of $\zeta(s)$, is $\mathfrak{U}(\zeta, 1, 1/2)$-universal under assumption of what they call a *weak Montgomery conjecture* on the spacing of the imaginary parts of the non-trivial zeros (as it would follow from the pair correlation conjecture of Montgomery [60]). Later, Garunkštis and Laurinčikas [19] obtained the same universality theorem assuming *RH*.

In this subsection we give an unconditional proof of the aforementioned result in the more general setting of this thesis, as well as in the more general context of c-points of zeta-functions $\mathfrak{L}(s)$ from the Selberg class. This class, which is usually denoted by $\mathcal{S}$, was introduced by Selberg [81] and consists of Dirichlet series

$$\mathfrak{L}(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

satisfying the following hypotheses:

i. *Ramanujan hypothesis.* $a(n) \ll_\epsilon n^\epsilon$

ii. *Analytic continuation.* There exists a non-negative integer k such that $(s-1)^k\mathfrak{L}(s)$ is an entire function of finite order, that is,

$$(s-1)^k\mathfrak{L}(s) \ll \exp\left(r^A\right), \quad |s| = r \to \infty$$

for some positive number A.

iii. *Functional equation.* $\mathfrak{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathfrak{L}}(s) = \omega\overline{\Lambda_{\mathfrak{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathfrak{L}}(s) := \mathfrak{L}(s)Q^s \prod_{j=1}^{J} \Gamma\left(\lambda_j s + \mu_j\right)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\Re\mu_j \geq 0$ and $|\omega| = 1$.

iv. *Euler product.* $\mathfrak{L}(s)$ has a product representation

$$\mathfrak{L}(s) = \prod_p \mathfrak{L}_p(s),$$

where

$$\mathfrak{L}_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$.

Axioms (i) and (ii) imply that a function $\mathfrak{L}(s)$ from $\mathcal{S}$ is a Dirichlet series which is absolutely convergent for $\sigma > 1$ and it has an analytic continuation to the whole complex plane except for a possible pole at $s = 1$. From axiom (iii) one can obtain the quantity

$$d_{\mathfrak{L}} := 2\sum_{j=1}^{J} \lambda_j,$$

which is called the degree of $\mathfrak{L}(s)$ and, although the data from the functional equation is not unique, $d_{\mathfrak{L}}$ is well-defined. Axiom (iv) implies that if $\mathfrak{L} \in \mathcal{S}$ and it is not the zero function, then $a(1) = 1$. Moreover, one can prove that if $d_{\mathfrak{L}} \in [0, 1)$, then $d_{\mathfrak{L}} = 0$ and $\mathfrak{L} \equiv 1$, which is the only constant function in $\mathcal{S}$ (see [86, Theorem 6.1]).

For a survey on the Selberg class and the value-distribution of its functions we refer to Kaczorowski and Perelli [42], Perelli [64] and Steuding [86]. The simplest examples of functions belonging to $\mathcal{S}$ are, to name a few, shifts $L(s + i\theta), \chi)$ of Dirichlet L-functions attached to primitive characters χ with $\theta \in \mathbb{R}$, and Dedekind zeta-functions to number fields $\mathbb{K}$. The first family is of degree one while the second one has degree equal to the degree of the field extension $\mathbb{K}/\mathbb{Q}$.

Steuding [87] proved that if c is a complex number and $(\rho_{c,n})_{n\in\mathbb{N}}$, is the sequence of *non-trivial* c-points of $\zeta(s)$, then the sequence $(a\gamma_{c,n})_{n\in\mathbb{N}}$, is uniformly distributed mod 1 for every real number $a \neq 0$. The adjective *non-trivial* means that the c-points in question are not located in the neighbourhood of a trivial zero of $\zeta(s)$. Jakhlouti, Mazhouda and Steuding [40] proved a similar result for c-points of zeta-functions from $\mathcal{S}$ by assuming the truth of the Lindelöf hypothesis. Such sequences are seemingly good canditates for our universality theorems since they already behave, or it is assumed that they should behave, as uniformly distributed sequences. However, the second condition on discrete mean-squares is still unknown unconditionally even for the simplest examples of zeta-functions. This explains why in [19] and [20], assumptions which have not yet proven to be true are considered.

Our approach is rather elementary and arises from the fact that, in general, there are about $T \log T$ c-points of $\mathfrak{L}(s)$ with $T < \gamma_c \leq 2T$ (see [86, Corollary 7.4]). So we just need to find a subsequence of the original sequence $(\gamma_{c,n})_{n\in\mathbb{N}}$, say $(\gamma_{c,n_k})_{k\in\mathbb{N}}$, such that the sequence $(a\gamma_{c,n_k})_{k\in\mathbb{N}}$, is uniformly distributed mod 1 for every real number $a \neq 0$, but it also satisfies the second condition for the discrete mean-squares in Theorems 3.1 and 3.2. This becomes possible by making use of a classical order result for functions from $\mathcal{S}$ and by generalizing a result due to Littlewood [54] regarding $\zeta(s)$.

Lemma 3.6. *Let $\mathfrak{L} \in \mathcal{S}$. Then*

$$\mathfrak{L}(\sigma + it) \asymp |t|^{(1/2-\sigma)d_{\mathfrak{L}}} |\mathfrak{L}(1 - \sigma + it), \ \ |t| \geq t_0 > 0,$$

uniformly in any strip $\sigma_1 \leq \sigma \leq \sigma_2$.

Proof. For a proof see [86, Theorem 6.8]. □

Lemma 3.7. *Let $\mathfrak{L} \in \mathcal{S} \setminus \{1\}$ and c be a complex number. Then there exists a positive number $A = A(c)$ such that for every $T \geq \exp(3)$ one can find a c-point $\beta_c + i\gamma_c$ of $\mathfrak{L}(s)$ satisfying*

$$|\gamma_c - T| < \frac{A}{\log \log \log T}.$$

Proof. Let $n_0 > 1$ be the first positive integer n such that the n-th coefficient of the Dirichlet series represantation of $\mathfrak{L}(s)$, $a(n)$, is non-zero. We know that such a number exists, because $\mathfrak{L}(s)$ differs from the only constant function in $\mathcal{S}$. This also implies that $d_{\mathfrak{L}} \geq 1$. In the sequel we make use of the notation $a_n := a(n)$, for all $n \in \mathbb{N}$.

Now we define the function

$$g(s) := \begin{cases} \dfrac{|a_{n_0}| n_0^s}{a_{n_0}} (\mathfrak{L}(s) - 1), & \text{if } c = 1, \\ \mathfrak{L}(s) - c, & \text{if } \Re c < 1, \\ c - \mathfrak{L}(s), & \text{if } \Re c > 1, \\ -i\left(\mathfrak{L}(s) - c\right), & \text{if } \Re c = 1 \text{ and } \Im c > 0, \\ i\left(\mathfrak{L}(s) - c\right), & \text{if } \Re c = 1 \text{ and } \Im c < 0, \end{cases}$$

for every $s \in \mathbb{C} \setminus \{1\}$. Then there exists $\sigma_0 = \sigma_0(c) > 1$ such that

$$g(s) \asymp \Re(g(s)) \asymp 1, \tag{3.22}$$

uniformly in the half-plane $\sigma \geq \sigma_0 - 1$. Indeed, the absolute convergence of $\mathfrak{L}(s)$ in the half-plane $\sigma > 1$ implies that

$$\Re(g(s)) \leq |g(s)| \ll 1,$$

uniformly in any half-plane $\sigma \geq \sigma_1 > 1$. On the other hand, for $\sigma > 1$,

$$\Re(g(s)) = \begin{cases} |a_{n_0}| + \sum\limits_{n > n_0} (n_0/n)^{\sigma} E(n,t), & c = 1, \\ \Re c - 1 + \sum\limits_{n \geq n_0} n^{-\sigma} F(n,t), & \Re c < 1, \\ 1 - \Re c - \sum\limits_{n \geq n_0} n^{-\sigma} F(n,t), & \Re c > 1, \\ \Im c + \sum\limits_{n \geq n_0} n^{-\sigma} G(n,t), & \Re c = 1 \text{ and } \Im c > 0, \\ -\Im c - \sum\limits_{n \geq n_0} n^{-\sigma} G(n,t), & \Re c = 1 \text{ and } \Im c < 0, \end{cases}$$

where

$$E(n,t) := |a_{n_0}|^{-1}\left[\Re\left(\overline{a_{n_0}}a_n\right)\cos\left(t\log\tfrac{n_0}{n}\right) - \Im\left(\overline{a_{n_0}}a_n\right)\sin\left(t\log\frac{n_0}{n}\right)\right],$$
$$F(n,t) := \Re a_n\cos\left(t\log n\right) + \Im a_n\sin\left(t\log n\right),$$
$$G(n,t) := \Im a_n\cos\left(t\log n\right) - \Re a_n\sin\left(t\log n\right).$$

Thus, for sufficiently large σ_0 we have

$$|g(s)| \geq \Re(g(z)) \gg 1,$$

uniformly in $\sigma \geq \sigma_0 - 1$. Moreover, by arguing similarly, we can take σ_0 large enough such that also

$$|\mathfrak{L}(s)| \gg 1, \tag{3.23}$$

uniformly in $\sigma \geq \sigma_0 - 1$.

Lastly, we define for every $s \in \mathbb{C} \setminus \{1\}$ the function $f(s) := \log g(s)$, where the logarithm takes its principal value for $\sigma \geq \sigma_0 - 1$, and for other points s we define $f(s)$ to be the value obtained from $f(\sigma_0)$ by continuous variation along the line segments $[\sigma_0, \sigma_0 + it]$ and $[\sigma_0 + it, \sigma + it]$, provided that the path does not cross a zero or a pole of $g(s)$; if it does, then we take $f(s) = f(s+0)$. From the latter definition and (3.22) it follows that

$$f(s) \ll 1, \tag{3.24}$$

uniformly in $\sigma \geq \sigma_0 - 1$.

For the second part of the proof we employ a technique used by Titchmarsh (see [90, Theorem 9.12]). In the sequel $A_1, \ldots, A_5$ denote positive constants, which may depend on $d_{\mathfrak{L}}$, c and some suitable positive numbers σ_0 and ϵ. Since these quantities are considered to be fixed here, we do not indicate this dependence in any of the implicit constants in the sequel of the proof.

It suffices to show that for every sufficiently large positive number T, if there is no c-point of $\mathfrak{L}(s)$ such that $|\gamma_c - T| \leq \delta < 1/2$, then $\delta \ll (\log\log\log T)^{-1}$. To this end, let T be a large positive number and assume that $\mathfrak{L}(s)$ has no c-point such that $|\gamma_c - T| \leq \delta < 1/2$. Then, the function $f(s)$, as it was defined above is regular in the rectangle

$$\mathcal{R} := \{s \in \mathbb{C} : -(\sigma_0 + 1) \leq \sigma \leq \sigma_0 + 1, \quad |t - T| \leq \delta\}.$$

We set

$$N := \left\lfloor \frac{8\sigma_0}{\delta} \right\rfloor + 1, \tag{3.25}$$

and we consider the circles

$$\mathcal{K}_{m,n} := \left\{s \in \mathbb{C} : \left|s - \left(\sigma_0 - \frac{n\delta}{4} + iT\right)\right| = \frac{m\delta}{4}\right\};$$

we define the maximum value of f on them by

$$\mathbf{M}_{m,n} := \max_{s\in\mathcal{K}_{m,n}} |f(s)|$$

for every $n = 0,\ldots,N$ and $m = 1,2,3,4$. Observe that all circles $\mathcal{K}_{m,n}$ lie inside the strip $-(\sigma_0+1) \le \sigma \le \sigma_0+1$ and, in particular, in $\mathcal{R}$.

The absolute convergence of $\mathfrak{L}(s)$ in the half-plane $\sigma > 1$ implies that

$$\mathfrak{L}(s) \ll_\epsilon |t|^\epsilon, \ \ |t| \ge t_0 > 0,$$

uniformly in $\sigma \ge \sigma_0 > 1$. This and Lemma 3.6 yield that

$$\mathfrak{L}(s) \ll_\epsilon |t|^{(3/2+\sigma_0)d_{\mathfrak{L}}+\epsilon}, \ \ |t| \ge t_0 > 0,$$

uniformly in the strip $-(\sigma_0+1) \le \sigma \le \sigma_0+1$. We then deduce that

$$\Re(f(s)) = \log|g(s)| \le A_1 \log T, \tag{3.26}$$

uniformly in $\mathcal{R}$ whenever $T \gg 1$ is sufficiently large. If we apply now the Borel-Carathéodory inequality (Theorem A.20) to $f(s)$ on the circles $\mathcal{K}_{3,0}$ and $\mathcal{K}_{4,0}$, then

$$\mathbf{M}_{3,0} \le \frac{\frac{3}{2}\delta}{\delta-\frac{3}{4}\delta}\max_{s\in\mathcal{K}_{4,0}}\Re(f(s)) + \frac{\delta+\frac{3}{4}\delta}{\delta-\frac{3}{4}\delta}\left|f\left(\sigma_0+iT\right)\right|$$

and it follows from relations (3.24) and (3.26) that

$$\mathbf{M}_{3,0} \le 7A_1\log T + A_2. \tag{3.27}$$

It is easy to see that $\sigma_0 - \delta/4 + iT$ lies inside the bounded component of $\mathbb{C}\setminus\mathcal{K}_{3,0}$. Hence, an application of the Borel-Carathéodory inequality to $f(s)$ on the circles $\mathcal{K}_{3,1}$ and $\mathcal{K}_{4,1}$, the maximum modulus principle and relations (3.26) and (3.27) yield

$$\begin{aligned}\mathbf{M}_{3,1} &\le \frac{\frac{3}{2}\delta}{\delta-\frac{3}{4}\delta}\max_{s\in\mathcal{K}_{4,1}}\Re(f(s)) + \frac{\delta+\frac{3}{4}\delta}{\delta-\frac{3}{4}\delta}\left|f\left(\sigma_0-\delta/4+iT\right)\right| \\ &\le 7(A_1\log T + \mathbf{M}_{3,0}) \\ &\le \left(7+7^2\right)A_1\log T + 7^2A_2.\end{aligned}$$

Continuing this way we deduce that

$$\mathbf{M}_{3,n} \le (7+\cdots+7^n)\,A_1\log T + 7^{n+1}A_2 \le 7^nA_3\log T \tag{3.28}$$

for every $n = 0,\ldots,N$.

Next we apply Hadamard's three-circles theorem (Theorem A.19) to $f(s)$ on the circles $\mathcal{K}_{1,n}$, $\mathcal{K}_{2,n}$ and $\mathcal{K}_{3,n}$, that is,

$$\mathbf{M}_{2,n} \le \mathbf{M}_{1,n}^a\mathbf{M}_{3,n}^b$$

for every $n = 0,\ldots,N$, where $a = \log(3/2)/\log 3$ and $b = \log 2/\log 3$. Observe also that $\mathcal{K}_{1,n}$ lies inside the bounded component of $\mathbb{C}\setminus\mathcal{K}_{2,n-1}$ and, therefore, the maximum modulus principle implies that $\mathbf{M}_{1,n} \leq \mathbf{M}_{2,n-1}$ for every $n = 1, \ldots, N$. Hence,

$$\mathbf{M}_{2,n} \leq \mathbf{M}_{1,n}^{a} \mathbf{M}_{3,n}^{b} \leq \mathbf{M}_{2,n-1}^{a} \mathbf{M}_{3,n}^{b}$$

for every $n = 1, \ldots, N$. We then have that

$$\mathbf{M}_{2,1} \leq \mathbf{M}_{2,0}^{a} \mathbf{M}_{3,1}^{b}, \quad \mathbf{M}_{2,2} \leq \mathbf{M}_{2,1}^{a} \mathbf{M}_{3,2}^{b} \leq \mathbf{M}_{2,0}^{a^2} \mathbf{M}_{3,1}^{ab} \mathbf{M}_{3,2}^{b}, \ldots$$

and, eventually,

$$\mathbf{M}_{2,N} \leq \mathbf{M}_{2,0}^{a^N} \mathbf{M}_{3,1}^{a^{N-1}b} \mathbf{M}_{3,2}^{a^{N-2}b} \ldots \mathbf{M}_{3,N}^{b},$$

or, by (3.28),

$$\mathbf{M}_{2,N} \leq \mathbf{M}_{2,0}^{a^N} 7^{a^{N-1}b+2a^{N-2}b+\cdots+Nb} \left(A_3 \log T\right)^{a^{N-1}b+a^{N-2}b+\cdots+b}. \tag{3.29}$$

Since $a, b > 0$ and $a + b = 1$, it follows that

$$a^{N-1}b + 2a^{N-2}b + \cdots + Nb \leq N^2$$

and

$$a^{N-1}b + a^{N-2}b + \cdots + b = b(1 - a^N)/(1 - a) = 1 - a^N.$$

Moreover, in view of relation (3.24),

$$\mathbf{M}_{2,0}^{a^N} \leq \mathbf{M}_{2,0} \ll 1.$$

Thus, relation (3.29) becomes

$$\mathbf{M}_{2,N} \leq \ A_4 7^{N^2} \left(\log T\right)^{1-a^N}. \tag{3.30}$$

Now observe that relation (3.23) and Lemma 3.6 imply that

$$\mathfrak{L}(s) \gg |t|^{(1/2+\sigma_0)d_{\mathfrak{L}}}, \quad |t| \geq t_0 > 0,$$

uniformly in the strip $-(\sigma_0 + 1) \leq \sigma \leq -\sigma_0$ and, since $-(\sigma_0 + 1) \leq \sigma_0 - N\delta/4 \leq -\sigma_0$, it follows that

$$\mathbf{M}_{2,N} \geq \left| f\left(\sigma_0 - \frac{N\delta}{4} + iT\right) \right| \geq A_5 \log T,$$

where $T \gg 1$ is sufficiently large. Taking into account relation (3.30), we obtain that

$$A_5 (\log T)^{a^N} \leq A_4 7^{N^2}$$

or

$$\log\log T \leq \left(\frac{1}{a}\right)^N \left(\log \frac{A_4}{A_5} + N^2 \log 7\right).$$

Taking once more the logarithm yields that

$$\log\log\log T \ll N$$

and, by the definition (3.25) of N, the lemma follows. □

Corollary 3.1. *Let $\mathfrak{L} \in \mathcal{S} \setminus \{1\}$, $b > 0$ be a real number and c a complex number. Then there exists a subsequence of c-points $(\rho_{c,n_k})_{k\in\mathbb{N}}$, of $\mathfrak{L}(s)$, such that $\gamma_{c,n_k} = bk + o(1)$, as $k \to \infty$, and the sequence $(a\gamma_{c,n_{k^m}})_{k\in\mathbb{N}}$, is uniformly distributed mod 1 for every real number $a \notin b^{-1}\mathbb{Q}$ and every positive integer m.*

Proof. Let $(\rho_{c,n})_{n\in\mathbb{N}}$ be the sequence of c-points of $\mathfrak{L}(s)$. Then, for every $k \geq k' := \lfloor \exp(3)/b \rfloor$, Lemma 3.7 yields the existence of a positive integer n_k such that $\gamma_{c,n_k} = bk + O\left((\log\log\log(bk))^{-1}\right) = bk + o(1)$, as $k \to \infty$. We also set $\gamma_{c,n_k} = \gamma_{c,n_{k'}}$ for every positive integer $k < k'$. Now, if $a \notin b^{-1}\mathbb{Q}$ is a real number and m a positive integer, then the sequence $(abk^m)_{k\in\mathbb{N}}$, is uniformly distributed mod 1 (see Theorem A.4) and $\lim_{k\to\infty}(abk^m - a\gamma_{c,n_{k^m}}) = 0$. Hence, Theorem A.1 implies that the sequence $(a\gamma_{c,n_{k^m}})_{k\in\mathbb{N}}$ is uniformly distributed mod 1. □

We can now show how the preceding corollary generates infinitely many *good* sequences of ordinates of c-points of a given function from the Selberg class which satisfy the conditions of our universality theorems.

Theorem 3.5. *Let $\mathfrak{L} \in \mathcal{S} \setminus \{1\}$, $\alpha \in (0,1]$ be transcendental and c be a complex number. Then there exists a subsequence of c-points $(\rho_{c,n_k})_{k\in\mathbb{N}}$, of $\mathfrak{L}(s)$, such that $(\gamma_{c,n_k})_{k\in\mathbb{N}}$ is $\mathfrak{U}(\zeta,\alpha,1/2)$- and $\mathfrak{U}(L,1/2)$-universal.*

Proof. From Corollary 3.1 there exists exists a subsequence of c-points $(\rho_{c,n_k})_{k\in\mathbb{N}}$, of $\mathfrak{L}(s)$, such that $\gamma_{c,n_k} = 2\pi k + o(1)$, as $k \to \infty$, and the sequence $(a\gamma_{c,n_k})_{k\in\mathbb{N}}$, is uniformly distributed mod 1 for any positive number $a \notin (2\pi)^{-1}\mathbb{Q}$. This, the linear independence of the sets $\{\log p : p \in \mathbb{P}\}$ and $\{\log(m+\alpha) : m \in \mathbb{N}\}$ over $\mathbb{Q}$, respectively, and Corollary A.2 yield that the sequences

$$\left(\left(\gamma_{c,n_k}\frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}\right) \quad\text{and}\quad \left(\left(\gamma_{c,n_k}\frac{\log(m+\alpha)}{2\pi}\right)_{0\leq m\leq Q-1}\right), \quad k \in \mathbb{N},$$

are uniformly distributed mod 1 for any finite set of primes $\mathscr{M}$ and any positive integer Q.

Now let $H > 0$, $1/2 < \sigma_2 < \sigma_1 < 1$ and set

$$f_Q(\sigma,h,t) := \zeta(\sigma+ih+it;\alpha) - \zeta_Q(\sigma+ih+it;\alpha)$$

for any $\sigma_2 \leq \sigma \leq \sigma_1$, any $|h| \leq H$, any $t > 0$ and any $Q \in \mathbb{N}$. Then Lemma 2.1 implies that, for any sufficiently large integer $N \gg_{Q,H} 1$ with $|\gamma_{c,n_k} - 2\pi k| < 1/2$ for $k \geq N$, $T_0 = 2\pi N - 1/4$, $T = 2\pi N + 1$, $\mathscr{B} = \{\gamma_{c,n_k} : N \leq k \leq 2N\}$ and

$$\delta = \frac{1}{2} < \min_{N\leq m\neq k\leq 2N} |\gamma_{c,n_m} - \gamma_{c,n_k}|,$$

the following inequality holds:

$$\sum_{k=N}^{2N} |f_Q(\sigma,h,\gamma_{c,n_k})|^2 \leq 2\int_{2\pi N-1/4-H}^{4\pi N+3/4+H} |f_Q(\sigma,0,t)|^2\,\mathrm{d}t + \left[\int_{2\pi N-1/4-H}^{4\pi N+3/4+H} |f_Q(\sigma,0,t)|^2\mathrm{d}t \int_{2\pi N-1/4-H}^{4\pi N+3/4+H} |f_Q'(\sigma,0,t)|^2\mathrm{d}t\right]^{1/2} \tag{3.31}$$

for any $\sigma_2 \leq \sigma \leq \sigma_1$, any $|h| \leq H$ and any $Q \in \mathbb{N}$. Since the interval $[2\pi N - 1/4 - H, 4\pi N + 3/4 + H]$ is a subset of

$$\left[2\pi\left(\frac{N}{2}-\frac{1}{2}\right), 2\pi N\right] \cup \left[2\pi\left(N-\frac{1}{2}\right), 4\pi N\right] \cup \left[2\pi\left(2N-\frac{1}{2}\right), 8\pi N\right],$$

for sufficiently large integer $N \gg_{Q,H} 1$, we deduce from relations (2.5) and (2.6) of the previous chapter, and relation (3.31) that

$$\limsup_{N\to\infty} \frac{1}{N+1}\sum_{k=N}^{2N} \left|\zeta\left(\sigma + ih + i\gamma_{c,n_k}; \alpha\right) - \zeta_Q\left(\sigma + ih + i\gamma_{c,n_k}; \alpha\right)\right|^2 \ll_\alpha Q^{1-2\sigma_1}$$

or

$$\lim_{Q\to\infty}\limsup_{N\to\infty} \frac{1}{N+1}\sum_{k=N}^{2N} \left|\zeta\left(\sigma + ih + i\gamma_{c,n_k}; \alpha\right) - \zeta_Q\left(\sigma + ih + i\gamma_{c,n_k}; \alpha\right)\right|^2 = 0,$$

uniformly in $\sigma_2 \leq \sigma \leq \sigma_1$ and $|h| \leq H$. Therefore, the second condition of Theorem 3.2 is also satisfied and, thus, we deduce that the sequence $(\gamma_{c,n_k})_{k\in\mathbb{N}}$ is $\mathfrak{U}(\zeta, \alpha, 1/2)$-universal.

To prove that $(\gamma_{c,n_k})_{k\in\mathbb{N}}$, is $\mathfrak{U}(L, 1/2)$-universal we argue similarly as above and as we did in the case of Beatty sequences, by making suitable changes wherever needed. □

3.2.3 Polynomials and Monomials

In the last part of this section, we give an example regarding universality theorems with respect to sequences of numbers which are generated by polynomials of degree larger than one. In this case, the usual method of using Gallagher's lemma in order to estimate discrete mean-squares, the second condition of our universality theorems, through continuous ones, is not applicable anymore. Instead, we employ our approach from Chapter 2. The only drawback is we are not able to prove discrete universality theorems to the whole strip $\mathcal{D}$, but to a thinner one having a width depending on the degree of the polynomial in discussion.

Theorem 3.6. *Let $\alpha \in (0, 1]$ be a transcendental number and $d \geq 2$ be an integer. Then for almost all $\underline{a}, \underline{b} \in [0, +\infty)^d$, the sequences $(P_{\underline{a}}(n))_{n\in\mathbb{N}}$ and $(P_{\underline{b}}(n))_{n\in\mathbb{N}}$ are $\mathfrak{U}(\zeta, \alpha, \mathbf{S}(d))$- and $\mathfrak{U}(L, \mathbf{S}(d))$-universal, respectively, where $P_{\underline{a}}(x) = a_1x + \cdots + a_dx^d$ and $\mathbf{S}(d)$ is defined by* (2.19).

We break down the proof by showing, essentially, the two conditions of Theorems 3.1 and 3.2 in two seperate lemmas.

Lemma 3.8. *Let $\alpha \in (0, 1]$ be a transcendental number and $d \geq 2$ be an integer. For almost all $\underline{a}, \underline{b} \in [0, +\infty)^d$, the sequences*

$$\left(\left(P_{\underline{a}}(n)\frac{\log(k+\alpha)}{2\pi}\right)_{0\leq k\leq Q-1}\right) \quad \text{and} \quad \left(\left(P_{\underline{b}}(n)\frac{\log p}{2\pi}\right)_{p\in\mathscr{M}}\right), \quad n \in \mathbb{N},$$

are uniformly distributed mod 1 for any finite set of primes $\mathscr{M}$ and any positive integer Q, respectively.

Proof. Let

$$\underline{a} \in \mathcal{L}(\alpha) := [0,+\infty)^d \setminus \left\{ \left(\sum_{k=0}^{Q-1} \frac{m_k \log(k+\alpha)}{2\pi q} \right)^{-1} : q \in \mathbb{Z},\, Q \in \mathbb{N},\, \underline{m} \in \mathbb{Z}^Q \setminus \{\underline{0}\} \right\}^d$$

and

$$\underline{b} \in \mathcal{L} := [0,+\infty)^d \setminus \left\{ \frac{2\pi q}{\log r} : (q,r) \in \mathbb{N}_0 \times (\mathbb{Q}_{>0} \setminus \{1\}) \right\}^d .$$

It is not difficult to see that $\mathcal{L}$ and $\mathcal{L}(\alpha)$ are of full Lebesgue measure and, in order to prove that the sequences in the lemma are uniformly distributed modulo 1, it suffices to show that, for any lattice points $\underline{e} = (e_k)_{0 \le k \le Q-1} \in \mathbb{Z}^Q \setminus \{\underline{0}\}$, and $\underline{h} = (h_p)_{p \in \mathscr{M}} \in \mathbb{Z}^{\sharp \mathscr{M}} \setminus \{\underline{0}\}$, the sequences

$$\left(\frac{P_{\underline{a}}(n)}{2\pi} \sum_{k=0}^{Q-1} e_k \log(k+\alpha) \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left(\frac{P_{\underline{b}}(n)}{2\pi} \sum_{p \in \mathscr{M}} h_p \log p \right)_{n \in \mathbb{N}}$$

are uniformly distributed mod 1 (see Corollary A.2). But this follows from our choice of $\underline{a}$ and $\underline{b}$ and Theorem A.4. □

We recall here briefly the setting from Subsection 2.2.4. Let

$$\mathbf{S}(d) = \min_{0<\mu<\frac{d^2+d-2}{4d}} \max\{\mathbf{A}(\mu), 1-\mathbf{B}(d,\mu)\} \in (0,1),$$

be as in (2.19), μ be the number in the interval $(0,(d^2+d-2)/4d)$ for which the latter minimun is attained, $\sigma_0 \in (\mathbf{S}(d),1]$, $\epsilon \in (0, \mu d(\sigma_0 - \mathbf{S}(d)/2)$ arbitrary but fixed, and $\mathcal{F}(d,\alpha,\mu,\epsilon)$ as in (2.17).

Lemma 3.9. *Let $\alpha \in (0,1]$ be a transcendental number, $d \ge 2$ be an integer, $\sigma_0 \in (\mathbf{S}(d),1]$ and $H>0$. Then, for all $\underline{a} \in \mathcal{F}(d,\alpha,\mu,\epsilon)$,*

$$\lim_{Q\to\infty} \limsup_{N\to\infty} \frac{N}{2N} \sum_{n=1}^{N} \left| (\zeta - \zeta_Q)(\sigma + i(P_{\underline{a}}(n)+h);\alpha) \right|^2 = 0,$$

uniformly in $\mathbf{S}(d) < \sigma_0 \le \sigma \le 1$ and $|h| \le H$. Moreover, for $\alpha = 1$ and any $\underline{b} \in \mathcal{F}(d,1,\mu,\epsilon)$,

$$\lim_{w\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=N}^{2N} \left| \left(L - L_{\{p\in\mathbb{P}:p\le w\}} \right)(\sigma + i(P_{\underline{b}}(n)+h),\chi) \right|^2 = 0,$$

uniformly in $\mathbf{S}(d) < \sigma_0 \le \sigma \le 1$ and $|h| \le H$.

Proof. Let $\underline{a} = (a_1,\ldots,a_d) \in \mathcal{F}(d,\alpha,\mu,\epsilon) \setminus \{\underline{0}\}$. Then, there is $K_{\underline{a}} \in \mathbb{N}$ such that

$$\underline{a} \in \left[0, 2^d M_{\underline{a}} + H\right]^d \setminus \mathcal{G}(d,\alpha,\mu,\epsilon,2^d M_{\underline{a}}+H, K_{\underline{a}}).$$

Using the approximate functional equation for $\zeta(s;\alpha)$ from Theorem 2.4, we deduce that

$$(\zeta-\zeta_Q)(\sigma+it;\alpha)=\sum_{Q\le n\le t^{\mu}}\frac{1}{n^{\sigma+it}}+O_{\mu,\sigma_0}\left(t^{-\nu}\right),$$

for any $\sigma_0\le\sigma\le 1$ and $t\ge Q^{1/\mu}$. Then, for any sufficiently large $N\gg_{\underline{a},H}\max\left\{Q^{1/\mu},K_{\underline{\alpha}}\right\}$ and $t_{n,h}:=P_{\underline{a}}(n)+h\gg_{\underline{a},H}n$, $n\ge N$, we have that

$$\begin{aligned}\sum_{n=N}^{2N}\left|(\zeta-\zeta_Q)\left(\sigma+i(P_{\underline{a}}(n)+h);\alpha\right)\right|^2&=\sum_{n=N}^{2N}\left|\sum_{k\le t_{n,h}^{\mu}}\frac{1}{(k+\alpha)^{\sigma+it_{n,h}}}+O_{\mu,\sigma_0}\left(t_{n,h}^{-\nu}\right)\right|^2\\&\ll_{\mu,\sigma_0,\underline{a},H}S_N'+R_N'+N^{1-2\nu}+1,\end{aligned}\tag{3.32}$$

where

$$S_N':=\sum_{n=N}^{2N}\sum_{k\le t_{n,h}^{\mu}}\frac{1}{(k+\alpha)^{2\sigma}}\le\sum_{n=N}^{2N}\sum_{k\ge Q}\frac{1}{(k+\alpha)^{2\sigma}}\ll NQ^{1-2\sigma_0},\tag{3.33}$$

$$\begin{aligned}R_N'&:=\sum_{n=N}^{2N}\sum_{1\le\ell\ne k\le t_{n,h}^{\mu}}\frac{1}{((k+\alpha)(\ell+\alpha))^{\sigma}}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{it_{n,h}}\\&\ll\sum_{1\le\ell\ne k\le(P_{\underline{a}}(2N)+h)^{\mu}}\frac{1}{((k+\alpha)(\ell+\alpha))^{\sigma}}\left|\sum_{n\in\mathscr{B}(N,\underline{a},h,k,\ell)}\left(\frac{k+\alpha}{\ell+\alpha}\right)^{iP_{\underline{a}}(n)}\right|\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}\mathscr{B}(N,\underline{a},h,k,\ell)&:=\left\{N\le n\le 2N: P_{\underline{a}}(n)\ge\max\left\{(k+\alpha)^{1/\mu},(\ell+\alpha)^{1/\mu}\right\}-h\right\}\\&\subseteq\{N,\dots,2N\}.\end{aligned}$$

We may argue similarly as in (2.23)-(2.26) to prove that $R_N^*=o(N)$ as $N\to\infty$, where the implicit constant depends only on d,σ_0 and $\underline{a}$. Therefore, we obtain from (3.32)-(3.34) that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=N}^{2N}\left|(\zeta-\zeta_Q)\left(\sigma+i(P_{\underline{a}}(n)+h);\alpha\right)\right|^2\ll_{d,\sigma_0,\underline{\alpha},H}Q^{1-2\sigma_0}\tag{3.35}$$

and, thus,

$$\lim_{Q\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|(\zeta-\zeta_Q)\left(\sigma+i(P_{\underline{a}}(n)+h);\alpha\right)\right|^2=0,$$

uniformly in $\sigma_0\le\sigma\le 1$ and $|h|\le H$.

Now let $\underline{b} \in \mathcal{F}(d, 1, \mu, \epsilon)$ and $\varepsilon > 0$ be arbitrary. Then, the absolute convergence of $L_{\{p\in\mathbb{P}:p\leq w\}}(s)$ in the half-plane $\sigma > 0$ implies that for every $w > 0$ there is $T_w > 0$ with

$$L_{\{p\in\mathbb{P}:p\leq w\}}(\sigma + it, \chi) = \sum_{\substack{n\leq t^{\mu} \\ p|n\Rightarrow p\leq w}} \frac{\chi(n)}{n^{\sigma+it}} + O_{\sigma_0}(\varepsilon)$$

for any $\sigma_0 \leq \sigma \leq 1$ and $t \geq T_w$. Hence, using the approximate functional equation for $L(s, \chi)$ from Theorem 2.4, we deduce that

$$(L - L_{\{p\in\mathbb{P}:p\leq w\}})(\sigma + it, \chi) = \sum_{n\leq t^{\mu}}{}^{*} \frac{\chi(n)}{n^{\sigma+it}} + O_{d,\sigma_0}\left(\varepsilon + t^{-\nu}\right)$$

for any $\sigma_0 \leq \sigma \leq 1$ and $t \geq T_w$, where $\sum^*$ runs over the positive integers n satisfying $p \nmid n$ for all primes $p \leq w$.

The remaining part of the proof follows the same reasoning as for $\zeta(s; \alpha)$ with appropriate changes where needed. After the required calculations, we have that

$$\lim_{w\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\left(L - L_{\{p\in\mathbb{P}:p\leq w\}}\right)\left(\sigma + i(P_{\underline{b}}(n) + h), \chi\right)\right|^2 \ll_{d,\sigma_0,\underline{b},H} \varepsilon^2,$$

uniformly in $\sigma_0 \leq \sigma \leq 1$ and $|h| \leq H$. Since ε was arbitrarily chosen, the lemma follows. □

Proof of Theorem 3.6. The sets $\mathcal{L}(\alpha)\cap\mathcal{F}(d, \alpha, \mu, \sigma_0, \epsilon)$ and $\mathcal{L}\cap\mathcal{F}(d, 1, \mu, \sigma_0, \epsilon)$ as they were defined in the previous lemmas, are of full Lebesgue measure. Moreover, if $\underline{a} \in \mathcal{L}(\alpha) \cap \mathcal{F}(d, \alpha, \mu, \sigma_0, \epsilon)$ and $\underline{b} \in \mathcal{L} \cap \mathcal{F}(d, 1, \mu, \sigma_0, \epsilon)$, then the sequences $\left(P_{\underline{a}}(n)\right)_{n\in\mathbb{N}}$ and $\left(P_{\underline{b}}(n)\right)_{n\in\mathbb{N}}$ satisfy the assumptions of Theorems 3.1 and 3.2, respectively. Thus, $\left(P_{\underline{a}}(n)\right)_{n\in\mathbb{N}}$ is $\mathfrak{U}(\zeta, \alpha, \mathbf{S}(d))$- universal, while $\left(P_{\underline{b}}(n)\right)_{n\in\mathbb{N}}$ is $\mathfrak{U}(L, \mathbf{S}(d))$-universal. □

We can prove with the same machinery the analogous result for monomials:

Theorem 3.7. *Let $\alpha \in (0, 1]$ be a transcendental number and $d \geq 2$ be an integer. Then, for almost all $a \in [0, +\infty)$, the sequence $(P_a(n))_{n\in\mathbb{N}}$ is $\mathfrak{U}(\zeta, \alpha, \mathbf{S}_{mo}(d))$- and $\mathfrak{U}(L, \mathbf{S}_{mo}(d))$-universal, where $P_a(x) = ax^d$ and $\mathbf{S}_{mo}(d)$ is defined by* (2.29).

The restriction on the strip of universality, which is characterized by the left abscissae $\mathbf{S}(d)$ and $\mathbf{S}_{mo}(d)$, depending on which case we are working on, occurs in Lemma 3.9 (and the corresponding lemma when it is about monomials). The reason behind is that we are forced to use approximate functional equations for $\zeta(s; \alpha)$ and $L(s, \chi)$ which contain Dirichlet polynomials of short length but for which it is not known yet whether they hold for the whole strip $\mathcal{D}$. It is apparent that if we could widen the strip in Lemma 3.9, then it would follow immediately that we could widen the strip of universality, since Lemma 3.8 has no restrictions on that matter. Therefore, as we did in Section 2.3, we could give some conditional results, with respect to the Lindelöf hypothesis, which are now easy to prove, because we established the necessary background.

Theorem 3.8. *Under LH for $\zeta(s; \alpha)$ and GLH, Theorem 3.6 and Theorem 3.7 hold true with $\mathbf{S}(d) = \mathbf{S}_{mo}(d) = 1/2$, for any integer $d \geq 2$.*

3.3 A *Shortcut* via Euler Products

The main difference between $L(s,\chi)$ and $\zeta(s;\alpha)$ when $\alpha \neq 1/2, 1$, is that $L(s,\chi)$ can be expressed as an Euler product in the half-plane of absolute convergence of its Dirichlet series representation, that is, in the half-plane $\sigma > 1$. This already implies that any $L(s,\chi)$ is zero-free in the half-plane $\sigma > 1$, on the contrary to $\zeta(s;\alpha)$ when $\alpha \neq 1/2, 1$ for which it has been proved by Davenport and Heilbronn [13] and Cassels [12] that it has zeros in the aforementioned half-plane.

In our case, the existence of an Euler product allows a degree of flexibility when dealing with universality theorems, because one may use the logarithm of $L(s,\chi)$, that is,

$$\log L(s,\chi) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\chi(p)}{p^s}\right)^k = \sum_{k=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s \log n} \tag{3.36}$$

for every $s \in \mathbb{C}$ with $\sigma > 1$, where we chose the single-valued analytic branch of $\log L(s,\chi)$ on the half-plane $\sigma > 1$ such that

$$\lim_{\sigma\to+\infty} \log L(s,\chi) = 0.$$

Here $\Lambda : \mathbb{N} \to \mathbb{C}$ is the von Mangoldt-function

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The expression (3.36) is impractical when we wish to study the behaviour of $L(s,\chi)$ in $\mathcal{D}$. Nevertheless, one can extend the definition of $\log L(s,\chi)$ inside $\mathcal{D}$ in the following way: for each zero $\beta + i\gamma$ of $L(s,\chi)$ with $\beta > 1/2$, we remove the segment $[1/2 + i\gamma, \beta + i\gamma]$ from the half-plane $\sigma > 1/2$, as well as the segment $(1/2, 1]$. We call the resulting slit half-plane $\mathcal{H}_\chi$ and for every s with $1/2 < \sigma \leq 1$ we define $\log L(s,\chi)$ to be the value obtained from $\log L(2,\chi)$ by continuous variation along the line segments $[2, 2+it]$ and $[2+it, \sigma+it]$.

Of course the preceding process would be superfluous under *GRH*, but we are trying to avoid conditional results, which means that our knowledge regarding the distribution of the zeros of $L(s,\chi)$ inside $\mathcal{D}$ is rather poor. Still it is sufficient enough, with respect to zero-density estimates, to obtain an approximate functional equation for $\log L(s,\chi)$:

Theorem 3.9. *Let $\delta > 0$ be fixed and χ be a Dirichlet character mod q. Let also y be a function on $T > 0$ such that $y \to +\infty$ and $y = o\left(T^{\delta/2}\right)$ as $T \to +\infty$. Then, for all sufficiently large $T \gg_\delta 1$, there exists a set $\mathcal{J}(T) \subseteq [T, 2T]$ of measure $m(\mathcal{J}(T)) \ll_q T^{1-\delta} y (\log T)^{14}$ such that*

$$\log L(s+i\tau,\chi) = \sum_{n\leq y} \frac{\Lambda(n)\chi(n)}{n^{s+i\tau}\log n} + O_q\left(\frac{(\log y)^2 \log t}{y^{\delta/2}}\right)$$

for all $s \in \mathcal{R}(\delta, y) := \{s \in \mathbb{C} : 1/2 + 2\delta < \sigma \leq 1,\ |t| \leq y\}$ and $\tau \in [T, 2T] \backslash \mathcal{J}(T)$.

This is a generalization of a lemma due to Lamzouri *et al.* [49, Lemma 4.1] who proved it for $\zeta(s)$ whenever s lies in a disc with center $3/4 + it_0$ and radius $r < 1/4$. We will prove our theorem in the same manner as they did for $\zeta(s)$ by applying a result due to Granville and Soundararajan [27, Lemma 8.2] and a weak version of a theorem due to Montgomery [59, Theorem 12.1].

Lemma 3.10. *Let χ be a Dirichlet character mod q and $s = \sigma + it$ be a complex number with $\sigma > 1/2$ and $t \geq 4$. Let also y and σ_0 be real numbers satisfying $2 \leq y \leq t - 2$ and $1/2 \leq \sigma_0 < \sigma$. If $L(z, \chi)$ is zero-free in the rectangle $\{z \in \mathbb{C} : \sigma_0 \leq \Re z \leq 1, |\Im z - t| \leq y + 3\}$, then*

$$\log L(s, \chi) = \sum_{2 \leq n \leq y} \frac{\Lambda(n)\chi(n)}{n^s \log n} + O_q\left(\frac{y^{\sigma_1 - \sigma} \log t}{(\sigma_1 - \sigma_0)^2}\right),$$

where $\sigma_1 := \min\{(\sigma + \sigma_0)/2, \sigma_0 + 1/\log y\}$.

Proof. The proof follows in the same way as in [27, Lemma 8.2]. The additional assumption on y allows to avoid the possible pole of $L(s, \chi)$ at $s = 1$ in case $\chi = \chi_0$ is the principal character mod q. □

Lemma 3.11. *Let $N(\sigma, T, \chi)$ be the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ with $\sigma \leq \beta \leq 1$ and $0 \leq |\gamma| \leq T$, where $\sigma > 0$ and $T \geq 2$. If $1/2 \leq \sigma \leq 1$, then*

$$\sum_{\chi} N(\sigma, T, \chi) \ll (qT)^{3/2 - \sigma} (\log T)^{14},$$

where the summation runs over all Dirichlet characters χ mod q.

Proof. For a proof see [59, Theorem 12.1]. □

Proof of Theorem 3.9. Let $\sigma_0 := 1/2 + \delta$ and define $\mathcal{I}_j$, $j = 1, 2$, to be the sets of those $\tau \in [T + (j-1)y, 2T - (j-1)y]$ for which the rectangles

$$\{z \in \mathbb{C} : \sigma_0 \leq \Re z \leq 1, |\Im z - \tau| < jy + 3\}$$

are free of zeros of $L(z, \chi)$, respectively. Observe that $\mathcal{I}_2 \subseteq \mathcal{I}_1$ and $\tau + t \in \mathcal{I}_1$ for all $\tau \in \mathcal{I}_2$ and $s \in \mathcal{R}(\delta, y)$. Therefore, if we set $\sigma_1 := \sigma_0 + 1/\log y$ and take sufficiently large $T \gg_\delta 1$, it follows from our assumption on y and Lemma 3.10 that

$$\log L(s + i\tau, \chi) = \sum_{n \leq y} \frac{\Lambda(n)\chi(n)}{n^{s + i\tau} \log n} + O_q\left(\frac{(\log y)^2 \log t}{y^{\delta/2}}\right)$$

uniformly in $s \in \mathcal{R}(\delta, y)$ and $\tau \in \mathcal{I}_2$. If we set $\mathcal{J}(T) = [T, 2T] \setminus \mathcal{I}_2$, then it follows from the definition of $\mathcal{I}_2$, our assumption on y and Lemma 3.11 that $\mathrm{m}(\mathcal{J}(T)) \ll_q T^{1-\delta} y (\log T)^{14}$ for all sufficiently large $T \gg_\delta 1$. □

We are now ready to prove the last theorem of this chapter.

Theorem 3.10. *Let $d \geq 2$ be an integer and $\delta \in (0, 1/2)$. Let also $\chi_1, \ldots, \chi_J$ be pairwise non-equivalent Dirichlet characters. Then, for almost all $a \in [1, +\infty)$ the following holds:*

If $\mathcal{K} \subseteq \{s \in \mathbb{C} : 1/2 + 2\delta < \sigma < 1\}$ is a compact set with connected complement, z is a positive real number, ξ_p, $p \leq z$, are real numbers, $f_1, \ldots, f_J$ are continuous non-vanishing functions on $\mathcal{K}$ and analytic in its interior, and ε is a positive real number, then

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \begin{array}{c} \max\limits_{1\leq j\leq J} \max\limits_{s\in\mathcal{K}} \left| L\left(s + ian^d, \chi_j\right) - f_j(s) \right| < \varepsilon \\ \max\limits_{p\leq z} \left\| x_n \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\} > 0.$$

One could argue that what we are actually proving is that, for almost all $a \in [1, +\infty)$, the sequence $\left(an^d\right)_{n\in\mathbb{N}}$ is $\mathfrak{U}(L, 1/2 + 2\delta)$-universal. However, the latter definition was for sequences which satisfy the conditions of Theorems 3.1, 3.2 or 3.3. In our case we were unable to prove the condition on the discrete mean-squares for a strip with width that does not depend on the degree of the polynomial. On the other hand, we can prove the analogous condition for the logarithms of $L(s, \chi_j)$ and this will allows us to obtain the above universality theorem.

Proof of Theorem 3.10. By Mergelyan's theorem, we may assume without loss of generality that $f_1, \ldots, f_J$ are polynomials which are non-vanishing in $\mathcal{K}$. Therefore, we can find a bounded admissable set $\mathcal{R}$ satisfying $\mathcal{K} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{D}$ where $f_1, \ldots, f_J$ are non-vanishing. Let

$$a \in \mathcal{L} := [1, +\infty) \setminus \left\{ \frac{2\pi q}{\log r} : (q, r) \in \mathbb{N}_0 \times (\mathbb{Q}_{>0} \setminus \{1\}) \right\}.$$

Then, in view of Corollary A.2 and Theorem A.4 , it follows that the sequence

$$\left(an^{\tilde{d}} \frac{\log p}{2\pi} \right)_{p\in\mathscr{M}} \qquad n \in \mathbb{N},$$

is uniformly distributed mod 1 for any finite set of primes $\mathscr{M}$, where $\tilde{d} = \tilde{d}(d, \delta) \equiv 0 \pmod d$ will be determined later on. Therefore, it follows from Lemma 3.2 that there exist positive numbers c and v such that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \begin{array}{c} \max\limits_{1\leq j\leq J} \max\limits_{s\in\mathcal{K}} \left| L_{\{p\in\mathbb{P}\,:\, z<p\leq u\}}\left(s + ian^{\tilde{d}}, \chi_j\right) - f_j(s) \right| < \varepsilon \\ \max\limits_{p\leq z} \left\| an^{\tilde{d}} \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\} > c \tag{3.37}$$

for every $u \geq v$.

Let $\log L(s, \chi_1), \ldots, \log L(s, \chi_J)$ be defined as in the beginning of this section on $\bigcap_{j=1}^J \mathcal{H}_{\chi_j}$. Let also $\log f_1, \ldots, \log f_J$ denote the principal branches of the logarithms of $f_1, \ldots, f_J$ on $\mathcal{R}$, respectively. Then the latter relation and continuity implies that, if

$$\mathscr{A}(u,\varepsilon) := \left\{ n \in \mathbb{N} : \begin{array}{c} \max_{1\le j\le J} \max_{s\in\mathcal{K}} \left| \log L_{\{p\in\mathbb{P}\,:\,p\le u\}}\left(s + ian^{\tilde{d}}, \chi_j\right) - \log f_j(s)\right| < \dfrac{\varepsilon}{2} \\ \max_{p\le z} \left\| an^{\tilde{d}} \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\}$$

for all sufficiently small $\varepsilon > 0$, then

$$\liminf_{N\to\infty} \frac{\sharp\left(\mathscr{A}(u,\varepsilon)\cap[1,N]\right)}{N} > c \tag{3.38}$$

for every $u \ge v$.

Set $y := (\log T)^{4/\delta}$. Then it follows from Theorem 3.9 that, for all sufficiently large $T \gg_{\delta,\mathcal{R}} 1$, there exists a set $\mathcal{J}(T) \subseteq [T, 2T]$ with $\mathrm{m}(\mathcal{J}(T)) \ll T^{1-\delta/2}$ such that

$$\log L(s+i\tau, \chi_j) = \sum_{k\le y} \frac{\Lambda(k)\chi_j(k)}{k^{s+i\tau}\log k} + O\left(\frac{1}{\log T}\right)$$

for all $s \in \mathcal{R}$, $\tau \in [T, 2T]\setminus\mathcal{J}(T)$ and $j = 1, \ldots, J$. The implicit constants depend on $q_1, \ldots, q_J$, but we will not mention this dependence from now on.

Let N, n and M be positive integers such that $N \le n \le 2N$ and $\tilde{d} = \tilde{d}(d,\delta)$ be an integer multiple of d such that $\tilde{d}\delta > 4$. If we set

$$\mathcal{E}'(n,M,d,\delta) := \left\{ a \in [1,2M] : an^{\tilde{d}} \in \bigcup_{k=1}^{M} \mathcal{J}\left(kn^{\tilde{d}}\right) \right\} = f^{-1}\left(\bigcup_{k=1}^{M} \mathcal{J}\left(kn^{\tilde{d}}\right) \right),$$

where $f(x) := xn^{\tilde{d}}$ is a linear transformation on $\mathbb{R}$, then, for all sufficiently large integers $N \gg_{\delta,\mathcal{R}} 1$ and any positive integer M, we have

$$\mathrm{m}(\mathcal{E}'(n,M,d,\delta)) \ll \frac{1}{n^{\tilde{d}}} \sum_{k=1}^{M} \left(kn^{\tilde{d}}\right)^{1-\delta/2} \ll_{M,\delta} n^{-\tilde{d}\delta/2}.$$

Therefore, the set

$$\mathcal{G}'(K,M,d,\delta) := \bigcup_{N=K}^{\infty} \bigcup_{k=N}^{2N} \mathcal{E}'(k,M,d,\delta)$$

has Lebesgue measure

$$\mathrm{m}(\mathcal{G}'(K,M,d,\delta)) \ll_M K^{2-\tilde{d}\delta/2}$$

for all sufficiently large integers $K \gg_{\delta,\mathcal{R}} 1$ and any positive integer M. Now if we set $\mathcal{G}'(K,M) = [1,2M]$ for the remaining of the positive integers K and

$$\mathcal{F}'(d,\delta) := [1,+\infty) \setminus \left(\bigcup_{M=1}^{\infty} \bigcap_{K=1}^{\infty} \mathcal{G}'(K,M,d,\delta) \right),$$

then $\mathcal{F}'(d,\delta)$ is of full Lebesgue measure. Moreover, if $a \in \mathcal{F}'(d,\delta)$, then, by construction, there exists $K_0 = K_0(a,\delta,\mathcal{R}) > 0$ such that

$$\log L\left(s + ian^{\tilde{d}}, \chi_j\right) = \sum_{k \le y'} \frac{\Lambda(k)\chi_j(k)}{k^{s+ian^{\tilde{d}}}\log k} + O\left(\frac{1}{\log N}\right) \tag{3.39}$$

for any $s \in \mathcal{R}$, any integers $N \ge K_0$ and $N \le n \le 2N$ and any $j = 1,\ldots,J$, where $y' := \left(\tilde{d}\log(aN)\right)^{4/\delta}$.

We turn to the logarithms of the truncated Euler products $L_{\{p\in\mathbb{P}:p\le w\}}(s,\chi_j)$ for $\sigma, w > 0$ and $j = 1,\ldots,J$. Choosing the single-valued analytic branch of $\log L_{\{p\in\mathbb{P}:p\le w\}}(s,\chi_j)$ on the half-plane $\sigma > 0$ such that

$$\lim_{\sigma\to+\infty} \log L_{\{p\in\mathbb{P}:p\le w\}}(s,\chi) = 0,$$

we have

$$\log L_{\{p\in\mathbb{P}:p\le w\}}(s,\chi_j) = -\sum_{p\le w}\log\left(1 - \frac{\chi(p)}{p^s}\right) = \sum_{\substack{k=1\\ p|k\Rightarrow p\le w}}^{\infty} \frac{\Lambda(k)\chi_j(k)}{k^s\log k}$$

for $\sigma, w > 0$ and any $j = 1,\ldots,J$. Then, the absolute convergence of the Dirichlet series on the right-hand side of the latter relation in the half-plane $\sigma > 0$ implies that, for every $w > 0$, there is $T_w > 0$ with

$$\log L_{\{p\in\mathbb{P}:p\le w\}}(s,\chi_j) = \sum_{\substack{k\le y'\\ p|k\Rightarrow p\le w}} \frac{\Lambda(k)\chi_j(k)}{k^s\log k} + O_{\sigma_0}(\varepsilon)$$

for any $\sigma_0 \le \sigma \le 1$ and $t \ge T_w$. Hence, using the approximate functional equation for $\log L(s,\chi)$ (3.39), we deduce that

$$\left(\log L - \log L_{\{p\in\mathbb{P}:p\le w\}}\right)\left(s + ian^{\tilde{d}}, \chi_j\right) = \sum_{\substack{k\le y'\\ p|k\Rightarrow p> w}} \frac{\Lambda(k)\chi_j(k)}{k^{s+ian^{\tilde{d}}}\log k} + O\left(\varepsilon + \frac{1}{\log N}\right)$$

for any $s \in \mathcal{R}$, any integers $N \ge K_0 + T_w$ and $N \le n \le 2N$ and any $j = 1,\ldots,J$.

Now we consider discrete moments. Let $j \in \{1,\ldots,J\}$ and $N \ge K_0 + T_w$ be an integer. Then

$$\sum_{n=N}^{2N}\left|\left(\log L - \log L_{\{p\in\mathbb{P}:p\le w\}}\right)\left(s + ian^{\tilde{d}}, \chi_j\right)\right|^2 \ll S'_N + N\left(\varepsilon + \frac{1}{\log N}\right)^2, \tag{3.40}$$

where

$$\begin{aligned} S'_N &= (N+1)\sum_{\substack{k\le y'\\ p|k\Rightarrow p>w}} \frac{\Lambda(k)\chi_j(k)}{k^{2\sigma}\log k} + \sum_{\substack{1\le k\ne\ell\le y'\\ p|k\Rightarrow p>w\\ p|\ell\Rightarrow p>w}} \frac{\Lambda(k)\Lambda(\ell)\chi_j(k)\chi_j(\ell)}{(k\ell)^\sigma\log k\log\ell}\sum_{n=N}^{2N}\left(\frac{k}{\ell}\right)^{an^{\tilde{d}}} \\ &\ll Nw^{-4\delta} + \sum_{1\le\ell<k\le y'}\frac{1}{(k\ell)^\sigma}\left[\left|\sum_{n=1}^{2N}\left(\frac{k}{\ell}\right)^{an^{\tilde{d}}}\right| + \left|\sum_{n=1}^{N}\left(\frac{k}{\ell}\right)^{an^{\tilde{d}}}\right|\right], \end{aligned} \tag{3.41}$$

uniformly in $s \in \mathcal{R}$. We bound the latter exponential sums from above by using the estimates from Subsection 2.2.5. In particular, if we assume without loss of generality that $\tilde{d} \geq 16$, then for $h = 1$, $\alpha = 1$, $\mu = 1/(4\tilde{d})$, $\epsilon = 1/12$ and $a \in \mathcal{F}_{mo}(\tilde{d}, 1, 1/(4\tilde{d}), 1/12)$, where $\mathcal{F}_{mo}(d, \alpha, \mu, \epsilon) \subseteq [0, +\infty)$ is defined as in Lemma 2.5 and is of full Lebesgue measure, it follows that there exist positive integers M_a and K_a such that $a \leq M_a$ and

$$\left|\sum_{n=1}^{N} \left(\frac{k}{\ell}\right)^{ian^{\tilde{d}}}\right|^2 \ll \frac{\left\lfloor \frac{M_a}{2\pi} \log \frac{k}{\ell} \right\rfloor + 1}{\log \frac{k}{\ell}} N^{3/2} \tag{3.42}$$

for every integer $N \geq K_a$ and any integers $1 \leq \ell < k \leq \left(\tilde{d} M_a N^{\tilde{d}}\right)^{1/(4\tilde{d})}$. Observe that

$$y' = \left(\tilde{d} \log(aN)\right)^{4/\delta} \ll_\delta \left(\tilde{d} M_a N^{\tilde{d}}\right)^{1/(4\tilde{d})}$$

as $N \to +\infty$. Therefore, if we choose an a from the set of full Lebesgue measure $\mathcal{F}'(d, \delta) \cap \mathcal{F}_{mo}(\tilde{d}, 1, 1/(4\tilde{d}), 1/12)$, then relations (3.41) and (3.42) yield that

$$\begin{aligned} S'_N &\ll N w^{-4\delta} + N^{3/4} \sum_{1 \leq \ell < k \leq y'} \frac{1}{(k\ell)^\sigma} \frac{\left\lfloor \frac{M_a}{2\pi} \log \frac{k}{\ell} \right\rfloor + 1}{\log \frac{k}{\ell}} \\ &\ll_{a,\delta} N w^{-4\delta} + N^{3/4} \left(\tilde{d} \log(aN)\right)^{(2-2\sigma)4/\delta} \log\left(\tilde{d} \log(aN)\right), \end{aligned} \tag{3.43}$$

uniformly in $s \in \mathcal{R}$ and for all sufficiently large integers $N \gg_\delta K_0 + K_a + T_w$.

It follows from relations (3.40) and (3.43) that, if a is an element of the set of full Lebesgue measure $\mathcal{F}'(d, \delta) \cap \mathcal{F}_{mo}(\tilde{d}, 1, 1/4\tilde{d}, 1/12)$, then

$$\limsup_{N \to \infty} \frac{1}{N+1} \sum_{n=N}^{2N} \left| \left(\log L - \log L_{\{p \in \mathbb{P} : p \leq w\}}\right) \left(s + ian^{\tilde{d}}, \chi_j\right) \right|^2 \ll_{a,\delta} \varepsilon^2, \tag{3.44}$$

unifromly in $s \in \mathcal{R}$ and for all sufficiently large $w \gg_{a,\delta} \varepsilon^{-1/(2\delta)}$. In view of Theorem A.16 we have

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left(\max_{s \in \mathcal{K}} \left| \left(\log L - \log L_{\{p \in \mathbb{P} : p \leq w\}}\right) \left(s + ian^{\tilde{d}}, \chi_j\right) \right| \right)^2 \ll_{\mathcal{R},\mathcal{K},a,\delta} \varepsilon^2$$

for all sufficiently large $w \gg_{a,\delta} \varepsilon^{-1/(2\delta)}$ and any $j = 1, \ldots, J$. Thus, if

$$\mathscr{B}(w, \varepsilon) := \left\{ n \in \mathbb{N} : \max_{1 \leq j \leq J} \max_{s \in \mathcal{K}} \left| \left(\log L - \log L_{\{p \in \mathbb{P} : p \leq w\}}\right) \left(s + ian^{\tilde{d}}, \chi_j\right) \right| < \frac{\varepsilon}{2} \right\}$$

for any sufficiently small positive number $\varepsilon \ll_{\delta,\mathcal{K},J} 1$, then

$$\liminf_{N \to \infty} \frac{\sharp(\mathscr{B}(w, \varepsilon) \cap [1, N])}{N} > 1 - \varepsilon \tag{3.45}$$

for all sufficiently large $w \gg_{a,\delta} \varepsilon^{-1/(2\delta)}$.

Assume now that a is an element of the set

$$\mathcal{L} \cap \mathcal{F}'(d,\delta) \cap \mathcal{F}_{mo}(\tilde{d}, 1, 1/(4\tilde{d}), 1/12)$$

which is of full Lebesgue measure. If we set

$$\mathscr{C}(\varepsilon) := \left\{ n \in \mathbb{N} : \begin{array}{c} \max\limits_{1\le j\le J} \max\limits_{s\in\mathcal{K}} \left| \log L\left(s + ian^{\tilde{d}}, \chi_j\right) - \log f_j(s)\right| < \varepsilon \\ \max\limits_{p\le z} \left\| x_n \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\},$$

then for any sufficiently small positive number $\varepsilon \ll_{\delta,\mathcal{K},J} c$, where c and v are as in (3.37), and any sufficiently large number $w \gg_{a,\delta} \varepsilon^{-1/(2\delta)} + v$, it follows from relations (3.38) and (3.45) and an application of the trianlge inequality, that

$$\liminf_{N\to\infty} \frac{\sharp(\mathscr{C}(\varepsilon) \cap [1,N])}{N} \ge \liminf_{N\to\infty} \frac{\sharp(\mathscr{A}(w,\varepsilon) \cap \mathscr{B}(w,\varepsilon) \cap [1,N])}{N} > 0.$$

Now observe that, if

$$\mathscr{C}'(\varepsilon) := \left\{ n \in \mathbb{N} : \begin{array}{c} \max\limits_{1\le j\le J} \max\limits_{s\in\mathcal{K}} \left| L\left(s + ian^{d}, \chi_j\right) - f_j(s)\right| < \varepsilon \\ \max\limits_{p\le z} \left\| x_n \dfrac{\log p}{2\pi} - \xi_p \right\| < \varepsilon \end{array} \right\},$$

then continuity and the assumption $\tilde{d} \equiv 0 \pmod{d}$ imply that $\mathscr{C}'(\varepsilon) \supseteq \mathscr{C}(\varepsilon)$, when $\varepsilon \ll_{\delta,\mathcal{K},J} c$ is a sufficiently small positive number. This concludes the proof of the theorem. □

Corollary 3.2. *Let $d \ge 2$ be an integer, $\alpha \in (0,1]$ be rational and $\delta \in (0,1/2)$. Then, for almost all $a \in [1,+\infty)$ the following holds:*

If $\mathcal{K} \subseteq \{s \in \mathbb{C} : 1/2 + 2\delta < \sigma < 1\}$ is a compact set with connected complement, f is a continuous function on $\mathcal{K}$ and analytic in its interior, and ε is a positive real number, then

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N : \max_{s\in\mathcal{K}} \left| \zeta\left(s + ian^d; \alpha\right) - f(s) < \varepsilon \right| \right\} > 0.$$

Proof. The proof is almost the same as the one of Theorem 3.3, only this time we use the previous theorem. □

Chapter 4

Hurwitz Zeta-Functions with Algebraic Parameter

In this chapter we study the value-distribution of $\zeta(s;\alpha)$ inside $\overline{\mathcal{D}}$, where α is an algebraic number in the interval $(0,1]$. We mainly incorporate ideas of Voronin and Good, who studied the case of $\zeta(s)=\zeta(s;1)$, to obtain a weak but effective universality result and a strong but ineffective one in vertical strips inside $\overline{\mathcal{D}}$, which width depends on the degree of α.

4.1 Effective and Ineffective Results

The question whether $\zeta(s;\alpha)$ with an algebraic irrational number $\alpha\in(0,1]$ is universal or not, remains, up to now, open. The main difficulty occurs from the fact that the numbers $\log(n+\alpha)$, $n\in\mathbb{N}$, could be linearly dependent over $\mathbb{Q}$ and, therefore, the machinery used for universality theorems is not applicable. To give an example, if $\alpha=(2\sqrt{2}-1)/2$, then $\log\alpha-3\log(\alpha+1)+2\log(\alpha+2)=0$.

Nevertheless, the majority of the mathematical community which is interested in the universality properties of zeta-functions expects that $\zeta(s;\alpha)$ with $\alpha\in\mathbb{A}_I$, where $\mathbb{A}_I$ will denote the set of algebraic irrational numbers in the unit interval $[0,1]$, should also be universal. For instance, Laurinčikas and Steuding [51] obtained limit theorems for $\zeta(s;\alpha)$, $\alpha\in\mathbb{A}_I$, which unfortunately are not sufficiently explicit for being used in a hypothetical proof of universality.

Moreover, there is the belief that such a result could follow by the already established methods for universality theorems and by a classical lemma due to Cassels [12], who proved that for any α algebraic irrational and every sufficiently large positive integer N, more than half of the numbers $\log(n+\alpha)$, $n=0,\dots,N$, are linearly independent over $\mathbb{Q}$. Indeed, Mishou [57] used Cassel's lemma to prove that for any $\alpha\in\mathbb{A}_I$, any $\varepsilon>0$ and any complex number z, there is $s\in\mathbb{C}$ with $\sigma>1$ such that $|\zeta(s;\alpha)-z|<\varepsilon$. This interesting result can be regarded as a forunner of a universality theorem. However, it is obtained in the half-plane $\sigma>1$ where the behaviour of such zeta-functions is, more or less, well-understood.

Almost all universality theorems that can be found in the literature have certainly one thing in common: they are ineffective. It is intriguing that, although these theorems provide us with sets of positive lower density, which elements can be used as vertical shifts of the zeta-function in interest to approximate almost any function in some suitable vertical strip, no information regarding upper bounds for the first such shift can be extracted.

The first one who obtained upper bounds for shifts of *weak universality* was Good [26], who studied the value-distribution of $\log\zeta(s)$. Good proved, among other things, that for any a from a suitable subset of $[1,2]$ of full Lebesgue measure, any integer $N>0$, any vector of complex numbers $\mathbf{a}=(a_0,\ldots,a_N)$, any $\sigma\in(1/2,1)$ and any $\varepsilon>0$, that there exists an effectively computable positive number $n_0=n_0(a,N,\mathbf{a},\sigma,\varepsilon)$ such that the system of inequalities

$$|\left(\log\zeta\right)^{(k)}(\sigma+ian)-a_k|<\varepsilon,\;\; k=0,\ldots,N,$$

has a positive integer solution $n\le n_0$. This is a quantitative version of Voronin's *weak universality* theorem [98] stating that, for $\sigma\in(1/2,1]$ and any integer $N>0$,

$$\overline{\{(\zeta(\sigma+in),\zeta'(\sigma+in),\ldots,\zeta^{(N)}(\sigma+in)) : n\in\mathbb{N}\}}=\mathbb{C}^{N+1}. \tag{4.1}$$

In 1989, Voronin [100] gave an alternative proof of Good's aforementioned result.

Our first theorem is an analogue of the preceding statements in the case of $\zeta(s;\alpha)$, $\alpha\in\mathbb{A}_I$. We postpone the definitions of the set $\mathcal{A}(Q,M)\subseteq\mathbb{A}_I$ and of the numbers $\mathbf{E}=\mathbf{E}(R,Q,\sigma)$ and $\mathbf{K}=\mathbf{K}(Q,M,\alpha)$ that appear in Theorem 4.1 to the next section, where we also explain how we came up with them (see (4.29), (4.5) and (4.30), respectively). We only point out that $\mathbb{A}_I\setminus\mathcal{A}(Q,M)$ is finite, for every choice of Q and M. Moreover, all constants appearing in this chapter, implicit or not, are effectively computable unless stated otherwise.

Theorem 4.1. *For every $\sigma\in(1/2,1]$, $N\in\mathbb{N}$, $A\in(0,1]$ and $d\ge3$, there exist positive numbers c_0, c_1, c_2 which depend on σ and N, $c_3=c_3(N,A)$, c_4, $c_5=c_5(N,d)$ and $\nu=\nu(d,N)$, such that the following is true:*

Let $\varepsilon>0$ and $\mathbf{a}:=(a_0,\ldots,a_n)\in\mathbb{C}^{N+1}$. Let also

$$R\ge c_0\,\varepsilon^{4/(1-2\sigma)}$$

and $Q_0\ge c_1R$ be positive integers satisfying the system of inequalities

$$c_2\left(|a_k|+A^{-1/2}\right)\le\mathbf{E}\left(\frac{\log\left(\frac{Q_0}{R+1}\right)}{2N\log Q_0}\right)^N k!(N-k)!\left(\log Q_0\right)^k,\;\; k=0,\ldots,N.$$

Then, for any $Q\ge c_3\left(Q_0+1/\varepsilon^8\right)$, $M\ge c_4\exp\left(2Q^2\right)$, $\alpha\in\mathcal{A}(Q,M)\cap[A,1]$ of degree $d(\alpha)\le d-1$, where

$$d+\frac{1}{2d}\le\frac{40}{267}\left(\frac{1}{3(1-\sigma)}\right)^{1/2} \tag{4.2}$$

and the left-hand side of the inequality is $+\infty$ for $\sigma=1$, and any

$$T\ge c_5\max\left\{\left(\mathbf{K}\exp\left((M+2)\exp\left(Q^2\right)\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}},\varepsilon^{-2\nu}\right\}, \tag{4.3}$$

there is $\tau \in [T, 2T]$ *with*

$$\left|\zeta^{(k)}(\sigma + i\tau; \alpha) - a_k\right| < \varepsilon, \qquad k = 0, \ldots, N. \tag{4.4}$$

Moreover, if $\mathcal{M}_T(\alpha, \sigma)$ *is the set of those* $\tau \in [T, 2T]$ *for which*

$$\left|\zeta^{(k)}(\sigma + i\tau; \alpha) - a_k\right| < \left(2\frac{Q^2+1}{Q^2-1}\right)^{1/2} \varepsilon, \qquad k = 0, \ldots, N,$$

then

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{m}\left(\mathcal{M}_T(\alpha, \sigma)\right) \geq \frac{1}{2} Q^{-2Q} \left(1 - Q^{-2}\right).$$

Observe that the theorem has meaning only when $\sigma \geq 1 - \xi$, where

$$\xi := \frac{2^8 \cdot 5^2}{3 \cdot 19^2 \cdot 89^2} \approx 0.000746,$$

as follows from (4.2) for $d = 3$.

Previously, we characterized Good's and Voronin's results as *weak universality* theorems for some reason. It can be easily seen that the universality of $\zeta(s)$ inside $\mathcal{D}$ implies relation (4.1) for any positive integer N and any $\sigma \in (1/2, 1)$ (but not for $\sigma = 1$). The steps of the proof are simple and we give them here briefly. If $\mathbf{a} = (a_0, \ldots, a_N)$ is a vector of complex numbers with $a_0 \neq 0$ and $\sigma \in (1/2, 1)$, then we can construct an entire function f which is zero-free and satisfies $f^{(k)}(\sigma) = a_k$ for all $k = 0, \ldots, N$. We then apply the universality theorem for $\zeta(s)$ to approximate f in a disc $D(\sigma, \delta)$ of fixed and sufficiently small radius $\delta > 0$ such that $D(\sigma, \delta) \subseteq \mathcal{D}$, we expand $\zeta(s)$ in its Taylor series in the prescribed disc and we use Cauchy's estimates for the function $\zeta(s + i\tau) - f(s)$.

The inverse process does not imply the universality theorem for $\zeta(s)$ in its full power. However, as it was noticed by Garunkštis *et al.* [21] it yields a weak form of the original theorem. Following their approach we will also prove a *weak universality* theorem for $\zeta(s; \alpha)$, $\alpha \in \mathbb{A}_I$.

Theorem 4.2. *Let* $1 - \xi \leq \sigma_0 \leq 1$, $s_0 = \sigma_0 + it_0$ *and* $f : \mathcal{K} \to \mathbb{C}$ *be continuous and analytic in the interior of* $\mathcal{K} = \{s \in \mathbb{C} : |s - s_0| \leq r\}$, *where* $r > 0$. *Let also* $0 < A < 1$ *and* $\varepsilon \in (0, |f(s_0)|)$. *Then, for all but finitely many algebraic irrationals* α *in* $[A, 1]$ *of degree at most* $d_0 - 1$, *where*

$$d_0 + \frac{1}{2d_0} = \frac{40}{267} \left(\frac{1}{3(1 - \sigma_0)}\right)^{1/2},$$

there exist real numbers $\tau \in [T, 2T]$ *and* $\delta = \delta(\varepsilon, f, T) > 0$ *such that*

$$\max_{|s - s_0| \leq \delta r} |\zeta(s + i\tau; \alpha) - f(s)| < 3\varepsilon,$$

whenever $T = T(\varepsilon, f, \alpha)$ *satisfies* (4.3). *The set of the exceptional* α *can be described effectively, while the dependence of* T *on* f *arises from the first* N *Taylor coefficients of* f *for sufficiently large* N.

While proving Theorem 4.2, we realized that our method can be applied also in the case of strong universality, that is, without "shrinking" the given compact set $\mathcal{K}$. However, effectivity is lost with respect to the finitely many exceptions of α and an upper bound for T.

Theorem 4.3. *Let $\mathcal{K} \subseteq \{s \in \mathbb{C} : 1-\xi \leq \sigma_0 < \sigma < 1\}$ be a compact set with connected complement and f be continuous on $\mathcal{K}$ and analytic in its interior. Let also $A \in (0,1]$ and $\varepsilon > 0$. Then for all but finitely many algebraic irrationals α in $[A,1]$ of degree at most $d_0 - 1$, where*

$$d_0 + \frac{1}{2d_0} = \frac{40}{267}\left(\frac{1}{3(1-\sigma_0)}\right)^{1/2},$$

there exist real numbers $\tau \in [T, 2T]$ such that

$$\max_{s\in\mathcal{K}} |\zeta(s+i\tau;\alpha) - f(s)| < 2\varepsilon$$

for every T sufficiently large. Moreover, the set of τ satisfying the latter inequality, has positive lower density.

The restriction on the strip of universality reminds us of the case of the Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$, where $\mathbb{K}$ is an algebraic number field over $\mathbb{Q}$. Reich [69], [71] proved that $\zeta_{\mathbb{K}}(s)$ is universal in the strip $\max\{1/2, 1-1/d\}$, where $d = [\mathbb{K} : \mathbb{Q}]$ is the degree of the number field.

4.2 Auxiliary Lemmas

The proofs of the theorems are rather long and technical. Therefore, we try to present large parts of them in this section as auxiliary lemmas.

We start with a modification of Good's Lemma 9 in [25] on the effective approximation of vectors of complex numbers by suitable twisted Dirichlet polynomials. An alternative option would be to follow Voronin's approach as in [43, Chapter 8, Section 2, Lemma 1]. Interestingly enough, we could not deduce a result for $\sigma = 1$ by the second way. And since also Good does not include the case of $\sigma = 1$, we shall add it in our proof.

Lemma 4.1. *For every $\sigma \in (1/2, 1]$ and $N \in \mathbb{N}$, there exist positive numbers C_0, C_1 and C_2, depending on σ and N, such that the following is true:*

Let $A \in (0,1]$, $\varepsilon > 0$ and $\boldsymbol{a} = (a_0, \ldots, a_N) \in \mathbb{C}^{N+1}$. Let also

$$R \geq C_0\, \varepsilon^{4/(1-2\sigma)}$$

and $Q_0 \geq C_1 R$ be integers satisfying the system of inequalities

$$C_2\left(|a_k| + A^{-1/2}\right) \leq \mathbf{E}(R, Q_0, \sigma)\left(\frac{\log\frac{Q_0}{R+1}}{2N\log Q_0}\right)^N k!(N-k)!\left(\log Q_0\right)^k,$$

$k = 0, \ldots, N$, *where*

$$\mathbf{E}(R, Q, \sigma) := \begin{cases} \dfrac{R^{1-\sigma}}{2^{3+\sigma}(1-\sigma)} \left[\left(\dfrac{Q}{R+1} \right)^{(1-\sigma)/(4N^3)} - 1 \right] & , \sigma \neq 1, \\ \\ \dfrac{\log \frac{Q}{R+1}}{2^5 N^3} & , \sigma = 1. \end{cases} \tag{4.5}$$

Then, for every $Q \geq Q_0$ *and* $\alpha \in [A, 1]$, *there exists* $\underline{\theta}_0 \in [0, 1]^Q$ *such that*

$$\left| \frac{\partial^k}{\partial s^k} \zeta_Q \left(s, \underline{\theta}_0, \alpha \right) \bigg|_{s=\sigma} - a_k \right| < \varepsilon, \quad k = 0, \ldots, N.$$

Proof. Let $R = R(\varepsilon, \sigma, N)$ be a positive integer which will be specified later on. We consider for every integer $Q > R$ the set of vectors

$$\mathcal{D}_{RQ} := \{ \mathbf{z} = (z_R, \ldots, z_{Q-1}) : |z_n| \leq 1, \, n = R, \ldots, Q-1 \}$$

and define the functions

$$(\mathbf{z}, \alpha) \longmapsto g_k(\mathbf{z}, \alpha) := \sum_{n=R}^{Q-1} z_n \frac{(-\log(n+\alpha))^k}{(n+\alpha)^\sigma}, \tag{4.6}$$

for every $(\mathbf{z}, \alpha) \in \mathcal{D}_{RQ} \times (0, 1]$ and $k = 0, \ldots, N$.

First we will determine for a given vector of complex numbers $(A_0, \ldots, A_N)$ an integer Q such that, for every $0 < \alpha \leq 1$, the system of equalities

$$g_k(\mathbf{z}, \alpha) = A_k, \;\; k = 0, \ldots, N, \tag{4.7}$$

has a solution $\mathbf{z}_\alpha \in \mathcal{D}_{RQ}$, that is, $(A_0, \ldots, A_N)$ belongs to the set

$$\mathcal{G} := \{ (g_0(\mathbf{z}, \alpha), \ldots, g_N(\mathbf{z}, \alpha)) : \mathbf{z} \in \mathcal{D}_{RQ} \}.$$

Observe that $\mathcal{G}$ is a closed convex subset of the complex Hilbert space $\mathbb{C}^{N+1}$ endowed with the inner product

$$\langle (x_0, \ldots, x_N), (y_0, \ldots, y_N) \rangle := \sum_{k=0}^{N} \Re(x_k \overline{y_k}).$$

Thus, in view of Theorem A.18 it is sufficient to show that for sufficiently large Q and for arbitrary $0 < \alpha \leq 1$ and non-zero $(\ell_0, \ldots, \ell_N) \in \mathbb{C}^{N+1}$, there is $\mathbf{z} \in \mathcal{D}_{RQ}$ such that

$$\sum_{k=0}^{N} \ell_k g_k(\mathbf{z}, \alpha) = \sum_{k=0}^{N} \ell_k A_k. \tag{4.8}$$

One can see that

$$\sum_{k=0}^{N} \ell_k g_k(\mathcal{D}_{RQ}, \alpha) = \left\{ z : |z| \leq V := \sum_{n=R}^{Q-1} \frac{1}{(n+\alpha)^\sigma} \left| \sum_{k=0}^{N} \ell_k (-\log(n+\alpha))^k \right| \right\}. \tag{4.9}$$

Indeed, the inclusion of the set on the left-hand side in the set on the right-hand side is obvious, while if $w = |w|\mathrm{e}(\phi)$ belongs to the disc described in the right-hand side of (4.9), we can choose $\mathbf{z} \in \mathcal{D}_{RQ}$ with

$$z_n = \frac{|w|}{V}\mathrm{e}\left(\phi - \arg\left(\sum_{m=0}^{N} \ell_m(-\log(n+\alpha))^m\right)\right)$$

such that

$$\sum_{k=0}^{N} \ell_k g_k(\mathbf{z},\alpha) = w.$$

Therefore, from (4.8) and (4.9) it is sufficient to show that, for sufficiently large Q and for arbitrary $0 < \alpha \leq 1$ and non-zero $(\ell_0,\dots,\ell_N) \in \mathbb{C}^{N+1}$,

$$\sum_{k=0}^{N} |\ell_k||A_k| \leq \sum_{n=R}^{Q-1} \frac{1}{(n+\alpha)^\sigma}\left|\sum_{k=0}^{N} \ell_k(-\log(n+\alpha))^k\right|. \tag{4.10}$$

Now, consider the polynomial

$$P(x) := \sum_{k=0}^{N}(-1)^k \ell_k x^k, \quad x \in \mathbb{R}, \tag{4.11}$$

and the following partition of the interval $[\log(R+\alpha), \log Q]$

$$x_k := \log(R+\alpha) + \frac{k}{N}\log\frac{Q}{R+\alpha}, \quad k = 0,\dots,N. \tag{4.12}$$

If we set in addition

$$G_k(x) := \prod_{\substack{m=0\\ m\neq k}}^{N}(x - x_m), \quad k = 0,\dots,N,$$

then it follows that

$$\left|G_k^{(j)}(0)\right| \leq \sum_{\substack{m_1=0\\ m_1\neq k}}^{N} \sum_{\substack{m_2=0\\ m_2\neq k,m_1}}^{N} \cdots \sum_{\substack{m_j=0\\ m_j\neq k,m_1,\dots,m_{j-1}}}^{N} \left|-x_{m_j}\right| \leq \frac{N!}{(N-j)!}(\log Q)^{N-j} \tag{4.13}$$

and

$$|G_k(x_k)| = \prod_{\substack{m=0\\ m\neq k}}^{N}\left|\frac{k-m}{N}\log\frac{Q}{R+\alpha}\right| = \left(\frac{1}{N}\log\frac{Q}{R+\alpha}\right)^N k!(N-k)! \tag{4.14}$$

for any $j,k = 0,\dots,N$. In view of Lagrange's interpolation theorem (see Theorem A.13)

$$P(x) = \sum_{k=0}^{N} \frac{P(x_k)}{G_k(x_k)} G_k(x),$$

and relations (4.13) and (4.14), we obtain

$$j!|\ell_j| = \left|P^{(j)}(0)\right| = \left|\sum_{k=0}^{N} \frac{P(x_k)G_k^{(j)}(0)}{G_k(x_k)}\right| \leq \sum_{k=0}^{N} \frac{|P(x_k)|N!(\log Q)^{N-j}}{k!(N-k)!(N-j)!} \left(\frac{N}{\log \frac{Q}{R+\alpha}}\right)^N$$

for $j = 0, \ldots, N$. Therefore,

$$\frac{1}{N+1}\left(\frac{\log \frac{Q}{R+\alpha}}{2N\log Q}\right)^N \sum_{j=0}^{N} j!|\ell_j|(N-j)!\,(\log Q)^j \leq \sum_{k=0}^{N} |P(x_k)|\,. \tag{4.15}$$

Let y_k, $k = 1, \ldots, N$, be such that $x_{k-1} \leq y_k \leq x_k$ and

$$|P(y_k)| = \max_{x\in[x_{k-1},x_k]} |P(x)| = \max_{x\in[-1,1]} \left|P\left(x\frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2}\right)\right|$$

for $k = 1, \ldots, N$. Markov's inequality (see Theorem A.14) states that

$$\max_{x\in[-1,1]} \left|\tilde{P}'(x)\right| \leq N^2 \max_{x\in[-1,1]} \left|\tilde{P}(x)\right|$$

for any $\tilde{P} \in \mathbb{C}[X]$ of degree at most N. Thus,

$$\begin{aligned}
\max_{x\in[x_{k-1},x_k]} |P'(x)| &= \max_{x\in[-1,1]} \left|P'\left(x\frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2}\right)\right| \\
&= \max_{x\in[-1,1]} \frac{2}{x_k - x_{k-1}} \left|\frac{\mathrm{d}}{\mathrm{d}x}P\left(x\frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2}\right)\right| \\
&\leq \frac{2N^2}{x_k - x_{k-1}} \max_{x\in[-1,1]} \left|P\left(x\frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2}\right)\right| \\
&= \frac{2N^2}{x_k - x_{k-1}}|P(y_k)|
\end{aligned} \tag{4.16}$$

for $k = 1, \ldots, N$. If we set now

$$\mathcal{I}_k := \left\{x \in [x_{k-1}, x_k] : |x - y_k| \leq S := \frac{\log \frac{Q}{R+\alpha}}{4N^3}\right\}, \quad k = 1, \ldots, N, \tag{4.17}$$

then relations (4.12), (4.16), (4.17) and the mean-value theorem imply that for every $x \in \mathcal{I}_k$ there is a ξ_x between the points x and y_k such that

$$|P(x)| \geq |P(y_k)| - |P(y_k) - P(x)| = |P(y_k)| - |P'(\xi_x)\,(y_k - x)| \geq \frac{|P(y_k)|}{2}$$

or

$$\max_{x\in[x_{k-1},x_k]} |P(x)| = |P(y_k)| \leq 2 \min_{x\in[x_{k-1},x_k]k} |P(x)| \tag{4.18}$$

for $k = 1, \ldots, N$. Since

$$x_k - x_{k-1} = \frac{\log \frac{Q}{R+\alpha}}{N}, \quad k = 1, \ldots, N,$$

at least one of the intervals $[y_k - S, y_k]$ and $[y_k, y_k + S]$ is contained in $\mathcal{I}_k$. We denote those intervals by

$$\mathcal{J}_k := [c_k, c_k + S], \quad k = 1, \dots, N. \tag{4.19}$$

Then, it follows from (4.11), (4.12), (4.18) and (4.19) that

$$\begin{aligned}
\sum_{n=R}^{Q-1} \frac{1}{(n+\alpha)^\sigma} \left| \sum_{k=0}^{N} \ell_k (-\log(n+\alpha))^k \right| &= \sum_{n=R}^{Q-1} \frac{|P(\log(n+\alpha))|}{(n+\alpha)^\sigma} \\
&\geq \sum_{k=1}^{N} \sum_{\log(n+\alpha)\in\mathcal{J}_k} \frac{|P(\log(n+\alpha))|}{(n+\alpha)^\sigma} \\
&\geq \sum_{k=1}^{N} \frac{|P(y_k)|}{2} \sum_{e^{c_k}\leq n+\alpha\leq e^{c_k+S}} \frac{1}{(2n)^\sigma}.
\end{aligned} \tag{4.20}$$

Observe that

$$\sum_{e^{c_k}\leq n+\alpha\leq e^{c_k+S}} \frac{1}{n^\sigma} \geq \begin{cases} \dfrac{e^{c_k(1-\sigma)}\left(e^{S(1-\sigma)} - 1\right)}{1-\sigma} + O\left(e^{-c_k}\right) & ,\sigma < 1, \\[2ex] \log \dfrac{e^{c_k+S}}{e^{c_k}} + O\left(e^{-c_k}\right) & ,\sigma = 1. \end{cases}$$

Since $c_k \geq \log R$ for $k = 1, \dots, N$, the definition of S yields that

$$\sum_{e^{c_k}\leq n+\alpha\leq e^{c_k+S}} \frac{1}{n^\sigma} \geq \begin{cases} \dfrac{R^{1-\sigma}}{2(1-\sigma)} \left[\left(\dfrac{Q}{R+1}\right)^{(1-\sigma)/(4N^3)} - 1\right] & ,\sigma < 1, \\[2ex] \dfrac{\log \frac{Q}{R+1}}{8N^3} & ,\sigma = 1, \end{cases}$$

for sufficienty large $R \gg 1$and $Q \geq C_1 R$, where $C_1 = C_1(\sigma, N)$. Recall that the right-hand side part of the latter inequality is equal to $2^{2+\sigma}\mathbf{E}(R, Q, \sigma)$. It follows now from relations (4.18) and (4.20) that

$$\begin{aligned}
\sum_{n=R}^{Q-1} \frac{1}{(n+\alpha)^\sigma} \left| \sum_{k=0}^{N} \ell_k (-\log(n+\alpha))^k \right| &\geq 2\mathbf{E}(R, Q, \sigma) \sum_{k=1}^{N} |P(y_k)| \\
&\geq \mathbf{E}(R, Q, \sigma) \sum_{k=0}^{N} |P(x_k)|.
\end{aligned} \tag{4.21}$$

Thus, in view of relations (4.15) and (4.20), if we choose $Q \geq C_1 R$ large enough so that the system of inequalities

$$|A_k| \leq \mathbf{E}(R, Q, \sigma) \left(\frac{\log \frac{Q}{R+1}}{2N \log Q} \right)^N k!(N-k)!\, (\log Q)^k, \quad k = 0, \dots, N, \tag{4.22}$$

is satisfied, then relation (4.10) holds for arbitrary $\alpha \in (0,1]$ and any non-zero vector $(\ell_0, \ldots, \ell_N) \in \mathbb{C}^{N+1}$. Hence, for every $\alpha \in (0,1]$ the system (4.7) has a solution $\mathbf{z}_\alpha \in \mathcal{D}_{RQ}$ as long as $Q \geq C_1 R$ satisfies (4.22).

If $U \gg_{\sigma,N} 1$ is large enough so that the functions

$$x \longmapsto \frac{(\log x)^k}{x^\sigma}, \quad k = 0, \ldots, N, \tag{4.23}$$

are decreasing in $[U, +\infty]$, then for every $R > U$, $\alpha \in [A,1]$ and $k = 0, \ldots, N$, we have that

$$\begin{aligned}\left|\sum_{n=0}^{R-1} \frac{(-1)^n(-\log(n+\alpha))^k}{(n+\alpha)^\sigma}\right| &\leq \frac{(-\log\alpha)^k}{\alpha} + \left|\sum_{n=1}^{U} \frac{(-1)^n\left(\log(n+\alpha)\right)^k}{(n+\alpha)^\sigma}\right| \\ &\leq \frac{(-\log\alpha)^k}{\alpha^\sigma} + \max_{y\in[0,1]}\left|\sum_{n=1}^{U} \frac{(-1)^n\left(\log(n+y)\right)^k}{(n+y)^\sigma}\right| \\ &\leq C_2 A^{-1/2},\end{aligned}$$

where $C_2 = C_2(\sigma, N) \geq 1$. Therefore, if we set

$$A_k = A_k(\alpha) := a_k - \sum_{n=0}^{R-1} \frac{(-1)^n(-\log(n+\alpha))^k}{(n+\alpha)^\sigma}, \quad k = 0, \ldots, N,$$

it follows from (4.22) that for every $\alpha \in [A,1]$ the system of equalities (4.7) has a solution $\mathbf{z}_\alpha \in \mathcal{D}_{RQ}$ as long as $Q \geq C_1 R$ satisfies the system of inequalities

$$C_2\left(|a_k| + A^{-1/2}\right) \leq \mathbf{E}(R,Q,\sigma)\left(\frac{\log\frac{Q}{R+1}}{2N\log Q}\right)^N k!(N-k)!(\log Q)^k, \quad k = 0, \ldots, N.$$

Since the right-hand side of these inequalities tends to infinity as $Q \to \infty$, the system is solvable for all sufficiently large Q.

Let $Q_0 \geq C_1 R$ be the smallest integer satisfying the aforementioned system, $Q \geq Q_0$ and $\alpha \in [A,1]$. Let also $\mathbf{z}_\alpha := (z_n)_{R \leq n \leq Q_0 - 1}$ be an element of $\mathcal{D}_{RQ_0}$ such that

$$g_k(\mathbf{z}_\alpha, \alpha) = A_k(\alpha), \quad k = 0, \ldots, N. \tag{4.24}$$

From Theorem A.17 we know that there are real numbers θ_n, $n = R, \ldots, Q-1$, such that

$$\begin{aligned}&\left\|\left(\sum_{n=R}^{Q_0-1} z_n \frac{(-\log(n+\alpha))^k}{(n+\alpha)^\sigma} - \sum_{n=R}^{Q-1} \frac{(-\log(n+\alpha))^k \,\mathrm{e}(\theta_n)}{(n+\alpha)^\sigma}\right)_{0\leq k\leq N}\right\|^2_{\mathbb{C}^{N+1}} \\ &\leq 4\sum_{n=R}^{Q-1}\left\|\left(\frac{(-\log(n+\alpha))^k}{(n+\alpha)^\sigma}\right)_{0\leq k\leq N}\right\|^2_{\mathbb{C}^{N+1}} \\ &\leq 4\sum_{k=0}^{N}\sum_{n=R}^{Q-1} \frac{(\log(n+1))^{2k}}{n^{2\sigma}} \\ &\ll_{\sigma,N} R^{1-2\sigma}.\end{aligned} \tag{4.25}$$

Let

$$R \gg_{\sigma,N} \left(U + \frac{1}{\varepsilon}\right)^{\frac{4}{2\sigma-1}} \gg_{\sigma,N} \left(\frac{1}{\varepsilon}\right)^{\frac{4}{2\sigma-1}}$$

be sufficiently large, with U being the number defined in (4.23), and set $\underline{\theta}_0 = (\theta_{0n})_{0\le n\le Q-1}$ to be

$$\theta_{0n} := \begin{cases} n/2, & 0 \le n \le R-1, \\ \theta_n, & R \le n \le Q-1. \end{cases}$$

Then (4.6), (4.24) and (4.25) yield

$$\begin{aligned} \left| \left. \frac{\partial^k}{\partial s^k} \zeta_Q\left(s, \underline{\theta}_0, \alpha\right) \right|_{s=\sigma} - a_k \right| &= \left| A_k(\alpha) - \sum_{n=R}^{Q-1} \frac{(-\log(n+\alpha))^k \, \mathrm{e}\left(\theta_n\right)}{(n+\alpha)^\sigma} \right| \\ &< \left| g_k(\mathbf{z}_\alpha, \alpha) - \sum_{n=R}^{Q_0-1} z_n \frac{(-\log(n+\alpha))^k}{(n+\alpha)^\sigma} \right| + \varepsilon \\ &= \varepsilon \end{aligned}$$

for $k = 0, \ldots, N$. □

Lemma 4.2. *For every $d \ge 3$ and $k \in \mathbb{N}_0$, there exists a positive number $\nu = \nu(d,k)$ such that*

$$\zeta^{(k)}(s;\alpha) = \sum_{n=0}^{\lfloor t^{1/d} \rfloor} \frac{(-\log(n+\alpha))^k}{(n+\alpha)^s} + O_{d,k}\left(t^{-\nu}\right), \quad t \ge t_1 > 0,$$

uniformly in $\mathbf{A}\left((d+1/(2d))^{-1}\right) \le \sigma \le 1$ and $0 < \alpha \le 1$, where

$$\mathbf{A}\left(\left(d + \frac{1}{2d}\right)^{-1}\right) = 1 - \theta \left(d + \frac{1}{2d}\right)^{-2}, \tag{4.26}$$

$\theta = 4/(27\eta^2)$ and $\eta = 4.45$.

Proof. Since $d \ge 3$, we have from Theorem 2.4 that $\mathbf{A}(\mu) = 1 - \theta\mu^2$ for any $0 < \mu \le 1/d$. In addition, there exists a positive number $\nu(d)$ such that

$$\zeta(s;\alpha) = \sum_{0 \le n \le t^{1/d}} \frac{1}{(n+\alpha)^s} + O_d\left(t^{-\nu}\right), \;\; t \ge t_1 > 1,$$

uniformly in $0 < \alpha \le 1$ and

$$\mathbf{A}\left(\frac{1}{d}\right) < \frac{1}{2}\left(\mathbf{A}\left(\left(d + \frac{1}{2d}\right)^{-1}\right) + \mathbf{A}\left(\frac{1}{d}\right)\right) \le \sigma \le 2.$$

Now the lemma follows by applying Cauchy's integral formula in the latter approximate functional equation for $\zeta(s;\alpha)$. □

Before proving the next lemma, we need to introduce some notation. The *naive height* of a complex polynomial $P(X)$, denoted by $H(P)$, is the maximum of the absolute values of its coefficients. If α is an algebraic number, then its *degree* and *height*, which we denote by $d(\alpha)$ and $H(\alpha)$, are defined to be the degree and the height of its minimal polynomial over $\mathbb{Z}$, respectively.

Let

$$\lambda : \mathbb{R} \to \mathbb{R}_+$$

be an infinitely differentiable function with $\operatorname{supp}(\lambda) \subseteq [-1,1]$ and $\int_{-\infty}^{+\infty} \lambda(x)\mathrm{d}x = 1$. We also assume that λ is bounded above by 1. If $Q \geq 2$ is an integer, we set $\delta := Q^{-2}$ and define the function

$$\underline{\theta} \longmapsto \Lambda_Q(\underline{\theta}) := \prod_{n=0}^{Q-1} \lambda\left(\frac{\theta_n}{\delta}\right),$$

for any $\underline{\theta} = (\theta_0, \ldots, \theta_{Q-1}) \in [-1,1]^Q$. Then, $\operatorname{supp}(\Lambda_Q) \subseteq [-1/2, 1/2]^Q$ and we can extend Λ_Q onto all $\mathbb{R}^Q$ by periodicity with period 1 in each of the variables θ_n, $n = 0, \ldots, Q-1$. The function

$$\theta \longmapsto \lambda\left(\frac{\theta}{\delta}\right)$$

extended to $\mathbb{R}$ by periodicity with period 1, has a Fourier expansion

$$\lambda\left(\frac{\theta}{\delta}\right) := \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}(n\theta),$$

where

$$c_0 = \delta \quad \text{and} \quad c_n = \int_0^1 \lambda\left(\frac{\theta}{\delta}\right) \mathrm{e}(-n\theta)\mathrm{d}\theta \ll \frac{1}{n^2\delta^2}, \;\; n \in \mathbb{Z} \setminus \{0\}. \tag{4.27}$$

The last relation follows from integrating twice by parts and the implicit constant depends only on our choice of λ. Thus, the Fourier expansion of Λ_Q is given by

$$\Lambda_Q(\underline{\theta}) := \sum_{\underline{m}} d_{\underline{m}} \mathrm{e}(\langle \underline{m}, \underline{\theta} \rangle),$$

where $\underline{m} = (m_0, \ldots, m_{Q-1}) \in \mathbb{Z}^Q$ and

$$d_{\underline{m}} := \prod_{n=0}^{Q-1} c_{m_n}.$$

We define for every $\underline{m} \in \mathbb{Z}^Q \setminus \{\underline{0}\}$ and $x \in \mathbb{R}$ the polynomials

$$Q^+_{\underline{m}}(x) := \prod_{\substack{n=0 \\ m_n > 0}}^{Q-1} (n+x)^{m_n} \quad \text{and} \quad Q^-_{\underline{m}}(x) := \prod_{\substack{n=0 \\ m_n < 0}}^{Q-1} (n+x)^{-m_n}.$$

Let $\hat{M} := \mathbb{Z}^Q \cap [-M\,,\, M\,]^Q$ and

$$\mathscr{P}(Q,M) := \left\{ P_{\underline{m}} = Q^+_{\underline{m}} - Q^-_{\underline{m}} : \underline{m} \in \hat{M} \setminus \{\underline{0}\} \right\}. \tag{4.28}$$

Observe that $\mathscr{P}(Q,M)$ is a set of non-zero integer polynomials of degree at most MQ and height bounded by a constant $\mathbf{H}(Q,M)$. We also define the set

$$\mathcal{A}(Q,M) = \mathcal{A}_1 \cup \mathcal{A}_2, \tag{4.29}$$

where

$$\mathcal{A}_1 := \{\alpha \in \mathbb{A}_I : d(\alpha) > MQ+1\},$$

$$\mathcal{A}_{2\underline{m}} := \left\{ \alpha \in \mathbb{A}_I \setminus \mathcal{A}_1 : \forall x,y \in \mathbb{N} \cap \left[0, \exp^2\left(Q^2\right)\right],\ \frac{Q^+_{\underline{m}}(\alpha)}{Q^-_{\underline{m}}(\alpha)} \neq \frac{x+\alpha}{y+\alpha} \right\},$$

for $\underline{m} \in \hat{M} \setminus \{0\}$, and

$$\mathcal{A}_2 := \bigcap_{\underline{m} \in \hat{M}\setminus\{\underline{0}\}} \mathcal{A}_{2\underline{m}}.$$

Finally, we consider the curve

$$\mathbb{R} \times (0,1] \ni (\tau,\alpha) \longmapsto \gamma_Q(\tau,\alpha) := \left(\frac{\log(n+\alpha)}{2\pi}\tau \right)_{0 \le n < Q}.$$

Lemma 4.3. *For any $k \in \mathbb{N}_0$ and $d \geq 3$, there exist positive numbers $C_3 = C_3(k)$, C_4 and $C_5(d,k)$, such that the following is true:*

Let $\varepsilon > 0$, $Q \geq C_3/\varepsilon^8$, $M \geq C_4 \exp(2Q^2)$, $\alpha \in \mathcal{A}(Q,M)$ and $d \geq d(\alpha)+1$. Then there exists positive number $\nu = \nu(d,k)$, such that if

$$T \geq C_5 \max\left\{ \left(\mathbf{K} \exp\left((M+2)\exp\left(Q^2\right)\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}}, \varepsilon^{-2\nu} \right\},$$

where

$$\mathbf{K} = \mathbf{K}(Q,M,\alpha) := \left[\mathbf{H}(Q,M)\,(MQ+2)\right]^{d(\alpha)-1} \left[H(\alpha)(d(\alpha)+1)^{1/2}\right]^{MQ+1}, \tag{4.30}$$

we have

$$\left| \frac{1}{\delta^Q T} \int_T^{2T} \Lambda(\gamma_Q(\tau,\alpha) - \underline{\theta}_1) \mathrm{d}\tau - 1 \right| < Q^{-2}$$

and

$$\begin{aligned} &\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \left| \zeta^{(k)}\left(\sigma + i\tau;\alpha\right) - \frac{\partial^k}{\partial s^k}\zeta_Q\left(s+i\tau,\underline{0},\alpha\right)\Big|_{s=\sigma} \right|^2 \mathrm{d}\tau \\ &\quad < \varepsilon^2 \int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \mathrm{d}\tau \end{aligned} \tag{4.31}$$

for any $\underline{\theta}_1 \in \mathbb{R}^Q$ and $\mathbf{A}\left((d+1/(2d))^{-1}\right) \leq \sigma \leq 1$, where $\mathbf{A}\left((d+1/(2d))^{-1}\right)$ is defined as in (4.26).

Proof. First, we will show that

$$\left| \frac{1}{\delta^Q T} \int_T^{2T} \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_1)\mathrm{d}\tau - 1 \right| < Q^{-2}$$

for suitable Q, α, T and any $\underline{\theta}_1 \in \mathbb{R}^Q$. The Fourier expansion of the function

$$\underline{\theta} \longmapsto \Lambda_Q(\underline{\theta} - \underline{\theta}_1)$$

is given by

$$\Lambda_Q(\underline{\theta} - \underline{\theta}_1) := \sum_{\underline{m}} h_{\underline{m}} \mathrm{e}(\langle \underline{m}, \underline{\theta} \rangle),$$

where $h_{\underline{0}} := \delta^Q$ and $h_{\underline{m}} := d_{\underline{m}} \mathrm{e}(\langle \underline{m}, -\underline{\theta}_1 \rangle)$, $\underline{m} \in \mathbb{Z}^Q$. For $M \in \mathbb{N}$,

$$\left| \sum_{\underline{m} \notin \hat{M}} h_{\underline{m}} \mathrm{e}(< \underline{m}, \underline{\theta} >) \right| \leq \sum_{\underline{m} \notin \hat{M}} |h_{\underline{m}}| \leq Q \left(\sum_{|n|>M} |c_n| \right) \left(\sum_{n=-\infty}^{+\infty} |c_n| \right)^{Q-1}. \tag{4.32}$$

From (4.27) we know that

$$\sum_{|n|>M} |c_n| \ll \frac{1}{\delta^2 M} \quad \text{and} \quad \sum_{n=-\infty}^{+\infty} |c_n| \leq \left(\frac{A}{\delta} \right)^2, \tag{4.33}$$

where $A > 1$ is an absolute costant. Therefore, from (4.32) we conclude that

$$\Lambda_Q(\underline{\theta} - \underline{\theta}_1) = \sum_{\underline{m} \in \hat{M}} h_{\underline{m}} \mathrm{e}(\langle \underline{m}, \underline{\theta} \rangle) + O\left(\frac{Q}{M} \left(\frac{A}{\delta} \right)^{2Q} \right). \tag{4.34}$$

Observe that by $\delta = Q^{-2}$ we have

$$Q \left(\frac{A}{\delta} \right)^{2Q} \leq Q \delta^Q \left(\frac{A}{\delta} \right)^{3Q} \leq \delta^Q Q \left(AQ \right)^{6Q} \ll \delta^Q \exp\left(Q^2 \right).$$

Hence, relation (4.34) can be written as

$$\Lambda_Q(\underline{\theta} - \underline{\theta}_1) = \sum_{\underline{m} \in \hat{M}} h_{\underline{m}} \mathrm{e}(\langle \underline{m}, \underline{\theta} \rangle) + O\left(\frac{\delta^Q \exp\left(Q^2 \right)}{M} \right). \tag{4.35}$$

In the sequel we use the notations

$$\ell(\tau) := \Lambda_Q \left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1 \right) \quad \text{and} \quad \tilde{n} := n + \alpha$$

in order to avoid extensive expressions. In view of (4.35), we have

$$\int_T^{2T} \ell(\tau)\mathrm{d}\tau = h_{\underline{0}} T + \sum_{\underline{m} \in \hat{M} \setminus \{\underline{0}\}} h_{\underline{m}} \int_T^{2T} \mathrm{e}(\langle \underline{m}, \gamma_Q(\tau,\alpha) \rangle) + O\left(\frac{T\delta^Q \exp\left(Q^2 \right)}{M} \right),$$

or

$$\frac{1}{\delta^Q T}\int_T^{2T}\ell(\tau)\mathrm{d}\tau = 1+\frac{1}{\delta^Q T}\sum_{\underline{m}\in\hat{M}\setminus\{\underline{0}\}} h_{\underline{m}}\int_T^{2T}\left(\frac{Q^+_{\underline{m}}(\alpha)}{Q^-_{\underline{m}}(\alpha)}\right)^{i\tau}\mathrm{d}\tau + O\left(\frac{\exp\left(Q^2\right)}{M}\right). \tag{4.36}$$

It follows from the definition of $h_{\underline{m}}$ and (4.33) that

$$\sum_{\underline{m}}|h_{\underline{m}}| \le \left(\frac{A}{\delta}\right)^{2Q} \le \delta^Q\left(AQ\right)^{6Q} \ll \delta^Q\exp\left(Q^2\right). \tag{4.37}$$

It also follows from (4.28) and (4.29) that if $\underline{m}\in\hat{M}\setminus\{\underline{0}\}$ and $\alpha\in\mathcal{A}(Q,M)$, then $P_{\underline{m}}(\alpha)=Q^+_{\underline{m}}(\alpha)-Q^-_{\underline{m}}(\alpha)\neq 0$. Thus,

$$\int_T^{2T}\left(\frac{Q^+_{\underline{m}}(\alpha)}{Q^-_{\underline{m}}(\alpha)}\right)^{i\tau}\mathrm{d}\tau \ll \left|\log\frac{Q^+_{\underline{m}}(\alpha)}{Q^-_{\underline{m}}(\alpha)}\right|^{-1} \ll \frac{\max\{Q^+_{\underline{m}}(\alpha),Q^-_{\underline{m}}(\alpha)\}}{\left|Q^+_{\underline{m}}(\alpha)-Q^-_{\underline{m}}(\alpha)\right|}. \tag{4.38}$$

Now Corollary A.3 yields that, for every $\underline{m}\in\hat{M}\setminus\{\underline{0}\}$ and $\alpha\in\mathcal{A}(Q,M)$,

$$\left|Q^+_{\underline{m}}(\alpha)-Q^-_{\underline{m}}(\alpha)\right| \ge \left[\mathbf{H}(Q,M)\left(MQ+1\right)\right]^{1-d(\alpha)}\left[H(\alpha)(d(\alpha)+1)^{1/2}\right]^{-MQ} > \mathbf{K}^{-1}. \tag{4.39}$$

Along with the estimate

$$\max\{Q^+_{\underline{m}}(\alpha),Q^-_{\underline{m}}(\alpha)\} \ll \prod_{n=1}^{Q} n^M \ll \exp\left(MQ^2\right), \tag{4.40}$$

we conclude from (4.36)-(4.40) that

$$\frac{1}{\delta^Q T}\int_T^{2T}\ell(\tau)\mathrm{d}\tau - 1 \ll \frac{\exp\left(Q^2\right)}{M}+\frac{\mathbf{K}\exp\left((M+1)Q^2\right)}{T}.$$

For $Q\gg 1$, $M\gg\exp\left(2Q^2\right)$, $\alpha\in\mathcal{A}(Q,M)$ and $T\gg\mathbf{K}\exp\left((M+2)Q^2\right)$, with suitable constants in $\gg$, we obtain

$$\left|\frac{1}{\delta^Q T}\int_T^{2T}\Lambda_Q(\gamma_Q(\tau,\alpha)-\underline{\theta}_1)\mathrm{d}\tau - 1\right| < Q^{-2} \tag{4.41}$$

We proceed now with the proof of relation (4.31). Let I denote the left-hand side of (4.31). Let also $\alpha\in\mathcal{A}(Q,M)$ and $d\ge d(\alpha)+1$. It follows from Lemma 4.2 that there exists positive number $\nu=\nu(d,k)$, such that

$$\zeta^{(k)}(s;\alpha)=\sum_{n=0}^{\lfloor t^{1/d}\rfloor}\frac{\left(-\log(n+\alpha)\right)^k}{(n+\alpha)^s}+O_{d,k}\left(t^{-\nu}\right),\quad t\ge t_1>0,$$

uniformly in $\mathbf{A}\left((d+1/(2d))^{-1}\right) \leq \sigma \leq 1$ and $0 < \alpha \leq 1$. By substituting this approximate functional equation in I for sufficiently large $T \gg_d Q$, and by setting $p(\tau) := \lfloor \tau^{1/d} \rfloor$, it follows that $I \ll I_1 + I_2$, where

$$I_1 = \int_T^{2T} \ell(\tau) \left| \sum_{n=Q}^{p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma+i\tau}} \right|^2 \mathrm{d}\tau \quad \text{and} \quad I_2 = \int_T^{2T} \ell(\tau)\, O_{d,k}\left(\tau^{-\nu}\right) \mathrm{d}\tau.$$

Thus, it suffices to prove the theorem for I_1 and I_2. We start by estimating I_1:

$$\begin{aligned} I_1 \leq & \left(\delta^Q + O\left(\frac{\delta^Q \exp\left(Q^2\right)}{M}\right)\right) \int_T^{2T} \left| \sum_{n=Q}^{p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma+i\tau}} \right|^2 \mathrm{d}\tau + \\ & + \left| \sum_{\underline{m} \in \hat{M} \setminus \{\underline{0}\}} h_{\underline{m}} \int_T^{2T} \mathrm{e}\left(\langle \underline{m}, \gamma_Q(\tau,\alpha)\rangle\right) \left| \sum_{n=Q}^{p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma+i\tau}} \right|^2 \mathrm{d}\tau \right| \\ \leq & \left[\delta^Q + O\left(\frac{\delta^Q \exp\left(Q^2\right)}{M}\right)\right] S_1 + \sum_{\underline{m} \in \hat{M} \setminus \{\underline{0}\}} \left|h_{\underline{m}}\right| \left|S_{2\underline{m}}\right|. \end{aligned} \tag{4.42}$$

We estimate each of the terms on the right-hand side of (4.42) seperately. By interchanging integration and summation we obtain

$$S_1 = \sum_{n=Q}^{p(2T)} \frac{(\log \tilde{n})^{2k}}{\tilde{n}^{2\sigma}} \int_{T_1}^{2T} \mathrm{d}\tau + \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log \tilde{n}_1 \log \tilde{n}_2)^k}{\tilde{n}_1^\sigma \tilde{n}_2^\sigma} \int_{T_2}^{2T} \left(\frac{\tilde{n}_2}{\tilde{n}_1}\right)^{i\tau} \mathrm{d}\tau,$$

where $T_1 = \max\left\{T, \tilde{n}^d\right\}$ and $T_2 = \max\left\{T, \tilde{n}_1^d, \tilde{n}_2^d\right\}$. Since $\alpha \in (0,1]$, $d \geq 3$ and $\sigma \geq \mathbf{A}(1/(d+1/(2d))) > 3/4$, we get

$$\sum_{n=Q}^{p(2T)} \frac{(\log \tilde{n})^{2k}}{\tilde{n}^{2\sigma}} \ll_k \sum_{n=Q}^{\infty} \frac{(\log n)^{2k}}{n^{3/2}} \ll_k Q^{-1/2} (\log Q)^{2k} \ll_k Q^{-1/4} \tag{4.43}$$

and

$$\sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log \tilde{n}_1 \log \tilde{n}_2)^k}{\tilde{n}_1^\sigma \tilde{n}_2^\sigma} \ll_k \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log p(2T))^{2k}}{(\tilde{n}_1 \tilde{n}_2)^{3/4}}. \tag{4.44}$$

Therefore,

$$\begin{aligned} S_1 & \ll_k Q^{-1/4} T + \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log p(2T))^{2k}}{(\tilde{n}_1 \tilde{n}_2)^{3/4}} \left| \int_{T_2}^{2T} \left(\frac{\tilde{n}_2}{\tilde{n}_1}\right)^{i\tau} \mathrm{d}\tau \right| \\ & \ll_k Q^{-1/4} T + (\log p(2T))^{2k} \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{1}{(\tilde{n}_1 \tilde{n}_2)^{3/4}} \left| \log \frac{\tilde{n}_2}{\tilde{n}_1} \right|^{-1} \\ & \ll_k Q^{-1/4} T + p(2T)^{1/2} (\log p(2T))^{1+2k} \\ & \ll_k Q^{-1/4} T + p(2T)^{3/4}. \end{aligned} \tag{4.45}$$

For the second sum we have by interchanging integration and summation

$$S_{2\underline{m}} = \sum_{n=Q}^{p(2T)} \frac{(\log \tilde{n})^{2k}}{\tilde{n}^{2\sigma}} \int_{T_1}^{2T} \left(\frac{Q_{\underline{m}}^+(\alpha)}{Q_{\underline{m}}^-(\alpha)} \right)^{i\tau} \mathrm{d}\tau + $$
$$+ \sum_{Q \le n_1 \ne n_2 \le p(2T)} \frac{(\log \tilde{n}_1 \log \tilde{n}_2)^k}{\tilde{n}_1^\sigma \tilde{n}_2^\sigma} \int_{T_2}^{2T} \left(\frac{Q_{\underline{m}}^+(\alpha)\tilde{n}_2}{Q_{\underline{m}}^-(\alpha)\tilde{n}_1} \right)^{i\tau} \mathrm{d}\tau. \tag{4.46}$$

Here we consider two subcases, depending on whether $\alpha \in \mathcal{A}_1$ or $\alpha \in \mathcal{A}_2$. It follows from the definitions in (4.28) and (4.29) that, if $\underline{m} \in \hat{M} \setminus \{\underline{0}\}$ and $\alpha \in \mathcal{A}_1$, then

$$Q_{\underline{m}}^+(\alpha) - Q_{\underline{m}}^-(\alpha) \ne 0 \quad \text{and} \quad Q_{\underline{m}}^+(\alpha)\tilde{n}_2 - Q_{\underline{m}}^-(\alpha)\tilde{n}_1 \ne 0. \tag{4.47}$$

Thus, applying Corollary A.3, it follows similar as in (4.38)-(4.40) that

$$\int_{T_1}^{2T} \left(\frac{Q_{\underline{m}}^+(\alpha)}{Q_{\underline{m}}^-(\alpha)} \right)^{i\tau} \mathrm{d}\tau \ll \mathbf{K} \exp\left(MQ^2\right) \tag{4.48}$$

and

$$\int_{T_1}^{2T} \left(\frac{Q_{\underline{m}}^+(\alpha)\tilde{n}_2}{Q_{\underline{m}}^-(\alpha)\tilde{n}_1} \right)^{i\tau} \mathrm{d}\tau \ll \frac{\max\{Q_{\underline{m}}^+(\alpha)\tilde{n}_2, Q_{\underline{m}}^-(\alpha)\tilde{n}_1\}}{\left|Q_{\underline{m}}^+(\alpha)\tilde{n}_2 - Q_{\underline{m}}^-(\alpha)\tilde{n}_1\right|} \ll \mathbf{K}\, p(2T)^{d(\alpha)} \exp\left(MQ^2\right). \tag{4.49}$$

From relations (4.43), (4.44), (4.46), (4.48) and (4.49) we obtain

$$\begin{aligned} S_{2\underline{m}} &\ll_k \left(Q^{-1/4} + p(2T)^{1/2+d(\alpha)} (\log p(2T))^{2k} \right) \mathbf{K} \exp\left(MQ^2\right) \\ &\ll_k \left(Q^{-1/4} + p(2T)^{3/4+d(\alpha)} \right) \mathbf{K} \exp\left(MQ^2\right). \end{aligned} \tag{4.50}$$

If now $\alpha \in \mathcal{A}_2$, then the second condition of relation (4.47) may not be satisfied. However, by the construction of the set $\mathcal{A}_2$ this can not happen too often. Indeed, for every $\underline{m} \in \hat{M} \setminus \{\underline{0}\}$, the equation

$$\frac{Q_{\underline{m}}^+(\alpha)}{Q_{\underline{m}}^-(\alpha)} = \frac{x+\alpha}{y+\alpha}$$

has at most one solution in the positive integers, $(x_{\underline{m}}, y_{\underline{m}})$ say, with $x_{\underline{m}} \ne y_{\underline{m}}$, as follows from the irrationality of α. In case such a solution does not exist in the set $(\mathbb{N} \cap [Q, +\infty))^2$, the estimate for $S_{2\underline{m}}$ is the same as in (4.50). If it exists, then we have to add in (4.50) the term

$$\frac{(\log(x_{\underline{m}}+\alpha))^k (\log(y_{\underline{m}}+\alpha))^k}{(x_{\underline{m}}+\alpha)^\sigma (y_{\underline{m}}+\alpha)^\sigma} T,$$

where $x_{\underline{m}}$ and $y_{\underline{m}}$ are both greater than Q and at least one of them is greater than $\left(\exp\left(Q^2\right)\right)^2$. Therefore, for sufficiently large $Q \gg_k 1$, the additional term is bounded above by

$$\frac{T}{\exp\left(Q^2\right) Q^{1/2}}.$$

In view of the preceding and (4.37), (4.42), (4.45) and (4.50), we conclude that

$$I_1 \ll_k \left[\delta^Q + O\left(\frac{\delta^Q \exp\left(Q^2\right)}{M}\right)\right]\left(Q^{-1/4}T + p(2T)^{3/4}\right) + \\ + \delta^Q \exp\left(Q^2\right)\left[\left(Q^{-1/4} + p(2T)^{3/4+d(\alpha)}\right)\mathbf{K}\exp\left(MQ^2\right) + \frac{T}{\exp\left(Q^2\right) Q^{1/2}}\right]$$

or

$$I_1 \ll_k Q^{-1/4}\left[2 + \frac{\exp\left(Q^2\right)}{M} + \frac{\mathbf{K}\exp\left((M+1)Q^2\right)}{T}\right]\delta^Q T + \\ + \left[1 + \frac{\exp\left(Q^2\right)}{M} + \mathbf{K}\exp\left((M+1)Q^2\right)\right]\delta^Q p(2T)^{3/4+d(\alpha)}. \tag{4.51}$$

Observe that

$$p(2T)^{3/4+d(\alpha)} \ll_{d,k} T^{\frac{3+4(d(\alpha)-d)}{4d}} T.$$

Then, for $Q \gg_k 1/\varepsilon^8$, $M \gg \exp\left(2Q^2\right)$, $\alpha \in \mathcal{A}(Q,M)$, $d \geq d(\alpha)+1$ and

$$T \gg_{d,k} \left(\mathbf{K}\exp\left((M+2)Q^2\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}}, \tag{4.52}$$

with suitable constants in $\gg$, we deduce from (4.41) and (4.51) that

$$I_1 < \frac{\varepsilon^2}{2}\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \mathrm{d}\tau \tag{4.53}$$

for every $\mathbf{A}\left(\left(d + 1/(2d)\right)^{-1}\right) \leq \sigma \leq 1$ and $\underline{\theta}_1 \in \mathbb{R}^Q$.

Finally, we estimate I_2 by

$$I_2 \ll_{d,k} T^{-\nu}\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \mathrm{d}\tau. \tag{4.54}$$

The lemma now follows from (4.52)-(4.54). □

If in the latter lemma we take $k = 0$ and s instead of σ, where s will range over a rectangle inside the critical strip, then by minor modifications in the above proof where necessary, we obtain the following

Lemma 4.4. *For every $H \geq 0$ and $d \geq 3$, there exist positive constants C_3', C_4' and $C_5' = C_5'(H,d)$, such that the following is true:*

Let $\varepsilon > 0$, $Q \geq C_3'/\varepsilon^8$, $M \geq C_4' \exp(2Q^2)$, $\alpha \in \mathcal{A}(Q,M)$ and $d \geq d(\alpha)+1$. Then there exists a positive number $\nu = \nu(d)$, such that, if

$$T \geq C_5' \max\left\{ \left(\mathbf{K} \exp\left((M+2)\exp\left(Q^2\right)\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}}, \varepsilon^{-2\nu} \right\},$$

we have

$$\left| \frac{1}{\delta^Q T} \int_T^{2T} \Lambda(\gamma_Q(\tau,\alpha) - \underline{\theta}_1) \mathrm{d}\tau - 1 \right| < Q^{-2}$$

and

$$\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \left|\left(\zeta - \zeta_Q\right)(s+i\tau;\alpha)\right|^2 \mathrm{d}\tau < \varepsilon^2 \int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_1\right) \mathrm{d}\tau$$

for any $\underline{\theta}_1 \in \mathbb{R}^Q$ and $s \in \overline{\mathcal{R}}$, where $\mathcal{R}$ is the open rectangle with vertices $\sigma_1 \pm iH$, $\sigma_2 \pm iH$ and $\mathbf{A}(1/(d+1/(2d))) \leq \sigma_1 < \sigma_2 \leq 1$.

4.3 Proofs of the Main Results

We are now able to finish the proofs of the theorems from the first section.

Proof of Theorem 4.1. Let σ, N, A, ε, $\mathbf{a}$, R and Q_0 be as in Lemma 4.1. Then, for every $Q \geq Q_0$ and $\alpha \in [A,1]$, the system of inequalities

$$\left| \frac{\partial^k}{\partial s^k} \zeta_Q\left(s,\underline{\theta},\alpha\right)\Big|_{s=\sigma} - a_k \right| < \frac{\varepsilon}{4}, \qquad k = 0,\ldots,N,$$

has a solution $\underline{\theta}_0 = \underline{\theta}_0(\alpha)$. If we take $\delta = Q^{-2}$, then the inequality

$$|\theta_n - \theta_{0n}| \leq \delta \tag{4.55}$$

implies that

$$\begin{aligned}\left| \frac{\partial^k}{\partial s^k} \left(\zeta_Q\left(s,\underline{\theta},\alpha\right) - \zeta_Q\left(s,\underline{\theta}_0,\alpha\right)\right)\Big|_{s=\sigma} \right| &\leq \sum_{n=0}^{Q-1} \frac{\left(\log(n+\alpha)\right)^k \left|\mathrm{e}(\theta_n) - \mathrm{e}(\theta_{0n})\right|}{(n+\alpha)^\sigma} \\ &\ll \frac{1}{\alpha^\sigma} \delta Q \log^N Q \\ &\ll A^{-1/2} Q^{-1} \log^N Q \\ &\ll_{N,A} Q^{-1/2}\end{aligned}$$

for $k = 0,\ldots,N$. Thus, the system of inequalities

$$\left| \frac{\partial^k}{\partial s^k} \zeta_Q\left(s,\underline{\theta},\alpha\right)\Big|_{s=\sigma} - a_k \right| < \frac{\varepsilon}{2} < \left(2\frac{Q^2+1}{Q^2-1}\right)^{1/2} \frac{\varepsilon}{2}, \qquad k = 0,\ldots,N, \tag{4.56}$$

is satisfied whenever $\alpha \in [A,1]$, $Q \gg_{N,A} Q_0 + 1/\varepsilon^4$ and (4.55) holds. On the other hand Lemma 4.3 yields, for every $Q \geq C_3(N)/\varepsilon^8$, $M \geq C_4 \exp(2Q^2)$, $\alpha \in \mathcal{A}(Q,M) \cap [A,1]$ and $d \geq d(\alpha)+1$, the existence of a positive number $\nu(d,N)$ such that, for every

$$T \geq C_5(d,N) \max\left\{ \left(\mathbf{K} \exp\left((M+2)\exp\left(Q^2\right)\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}}, \varepsilon^{-2\nu} \right\},$$

we have

$$\int_T^{2T} \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_0)\mathrm{d}\tau \geq \delta^Q \left(1 - Q^{-2}\right) T \tag{4.57}$$

and

$$\begin{aligned}
&\sum_{k=0}^{N} \int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha) - \underline{\theta}_0\right) \left| \zeta^{(k)}\left(\sigma + i\tau;\alpha\right) - \frac{\partial^k}{\partial s^k}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma} \right|^2 \mathrm{d}\tau \\
&< \sum_{k=0}^{N} \frac{\varepsilon^2}{4(N+1)} \int_T^{2T} \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_0)\mathrm{d}\tau \\
&= \frac{\varepsilon^2}{4} \int_T^{2T} \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_0)\mathrm{d}\tau \\
&< \frac{\varepsilon^2}{4}\delta^Q \left(1 + Q^{-2}\right) T
\end{aligned} \tag{4.58}$$

for $\mathbf{A}(1/(d+1/(2d))) \leq \sigma \leq 1$.

Let $Q \gg_{N,A} (Q_0 + C_3(N)/\varepsilon^8)$, $\mathbf{A}(1/(d+1/(2d))) \leq \sigma \leq 1$ and assume that there is no solution τ in $[T,2T]$ for the system of inequalities (4.4). Then, for every $\tau \in [T,2T]$, there is a $k_\tau \in \{0,\ldots,N\}$ such that

$$\begin{aligned}
&\sum_{k=0}^{N} \left| \zeta^{(k)}\left(\sigma + i\tau;\alpha\right) - \frac{\partial^k}{\partial s^k}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma} \right|^2 \\
&\quad \geq \left| \zeta^{(k_\tau)}\left(\sigma + i\tau;\alpha\right) - \frac{\partial^{k_\tau}}{\partial s^{k_\tau}}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma} \right|^2 \\
&\quad \geq \frac{1}{2}\left| \zeta^{(k_\tau)}\left(\sigma + i\tau;\alpha\right) - a_{k_\tau} \right|^2 - \left| a_{k_\tau} - \frac{\partial^{k_\tau}}{\partial s^{k_\tau}}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma} \right|^2 \\
&\quad \geq \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{4} \\
&\quad = \frac{\varepsilon^2}{4},
\end{aligned}$$

as follows from (4.56). However, this contradicts (4.58).

Now let

$$\mathcal{U}_T(\alpha) := \left\{ \tau \in [T,2T] : \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_0) \neq 0 \right\}. \tag{4.59}$$

By definition Λ_Q is bounded above by 1. This and (4.57) imply that

$$\mathrm{m}\left(\mathcal{U}_T(\alpha)\right) \geq \delta^Q \left(1 - Q^{-2}\right) T. \tag{4.60}$$

If $\mathcal{M}_T(\alpha, \sigma)$ is the set of those $\tau \in \mathcal{U}_T(\alpha)$ for which the system of inequalities

$$\left|\zeta^{(k)}\left(\sigma + i\tau; \alpha\right) - a_k\right| < \left(2\frac{Q^2+1}{Q^2-1}\right)^{1/2} \varepsilon, \qquad k = 0, \ldots, N,$$

is satisfied, then relations (4.56)-(4.60) yield that

$$\mathrm{m}\left(\mathcal{M}_T(\alpha, \sigma)\right) \geq \frac{1}{2}\delta^Q \left(1 - Q^{-2}\right) T.$$

For if that was not true, we would have

$$\sum_{k=0}^{N} \left|\zeta^{(k)}\left(\sigma + i\tau; \alpha\right) - \frac{\partial^k}{\partial s^k}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma}\right|^2 \geq \frac{\varepsilon^2}{2}\frac{Q^2+1}{Q^2-1}$$

for every τ in the set of positive measure $\mathcal{U}_T(\alpha) \setminus \mathcal{M}_T(\alpha, \sigma)$. It then would follow from (4.56), (4.57) and (4.60) that

$$\begin{aligned}
&\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau, \alpha) - \underline{\theta}_0\right) \sum_{k=0}^{N} \left|\zeta^{(k)}\left(\sigma + i\tau; \alpha\right) - \frac{\partial^k}{\partial s^k}\zeta_Q\left(s + i\tau, \underline{0}, \alpha\right)\Big|_{s=\sigma}\right|^2 \mathrm{d}\tau \\
&\geq \frac{\varepsilon^2}{2}\frac{Q^2+1}{Q^2-1} \int_{\mathcal{U}_T(\alpha)\setminus\mathcal{M}_T(\alpha,\sigma)} \Lambda_Q\left(\gamma_Q(\tau, \alpha) - \underline{\theta}_0\right) \mathrm{d}\tau \\
&\geq \frac{\varepsilon^2}{2}\frac{Q^2+1}{Q^2-1} \left[\int_{\mathcal{U}_T(\alpha)} \Lambda_Q\left(\gamma_Q(\tau, \alpha) - \underline{\theta}_0\right) \mathrm{d}\tau - \mathrm{m}\left(\mathcal{M}_T(\alpha, \sigma)\right)\right] \\
&> \frac{\varepsilon^2}{4}\delta^Q \left(1 + Q^{-2}\right) T,
\end{aligned}$$

which contradicts (4.58). □

Proof of Theorem 4.2. Beginning with the Taylor series of f,

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(s_0)}{k!}(s - s_0)^k,$$

valid for $s \in \mathcal{K}$, we observe, by Cauchy's formula

$$f^{(k)}(s_0) = \frac{k!}{2\pi i} \int_{|s-s_0|=r} \frac{f(s)}{(s - s_0)^k}\mathrm{d}s,$$

that $\left|f^{(k)}(s_0)\right| \leq k! M r^{-k}$, where $M := \max_{|s-s_0|=r} |f(s)|$. Fixing a number $\delta_0 \in (0, 1)$, we get

$$\left|\frac{f^{(k)}(s_0)}{k!}(s - s_0)^k\right| \leq M\delta_0^k$$

for $|s-s_0|\leq\delta_0 r$. If $\varepsilon\in(0,|f(s_0)|)$, we can find $N=N(\delta_0,\varepsilon,M)$ such that

$$\Sigma_1 := \left|f(s)-\sum_{k=0}^{N}\frac{f^{(k)}(s_0)}{k!}(s-s_0)^k\right| < \varepsilon,$$

for $|s-s_0|\leq\delta_0 r$.

Now let $\delta\in(0,\delta_0)$. Then, of course, the latter inequality holds in particular for s satisfying $|s-s_0|\leq\delta r$. Now we apply Theorem 4.1 with $a_k=f^{(k)}(s_0)$, $k=0,\ldots,N$. Then, for $\alpha\in\mathcal{A}(Q,M)\cap[A,1]$ of degree at most d_0-1, and T satisfying relation (4.3), there exists $t_1\in[T,2T]$ such that

$$|\zeta^{(k)}(\sigma_0+it_1;\alpha)-f^{(k)}(s_0)|<\varepsilon,\ \ k=0,\ldots,N.$$

Thus,

$$\begin{aligned}\Sigma_2 &:= \left|\sum_{k=0}^{N}\frac{\zeta^{(k)}(\sigma_0+it_1;\alpha)}{k!}(s-s_0)^k-\sum_{k=0}^{N}\frac{f^{(k)}(s_0)}{k!}(s-s_0)^k\right|\\ &< \varepsilon\sum_{k=0}^{N}\frac{(\delta r)^k}{k!}\\ &< \varepsilon\exp(\delta r),\end{aligned}$$

for $|s-s_0|\leq\delta_0 r$. Now write $\tau=t_1-t_0$, then $1+it_1=s_0+i\tau$.

Next we use the Taylor expansion for $\zeta(s;\alpha)$ on the shifted disk $\mathcal{K}+i\tau$. For this purpose we need to exclude the simple pole at $s=1$; hence we also request $T>r$. Under this assumption we have

$$\zeta(s+i\tau;\alpha)=\sum_{k=0}^{\infty}\frac{\zeta^{(k)}(s_0+i\tau;\alpha)}{k!}(s-s_0)^k$$

for $s\in\mathcal{K}$. Let $M(\tau):=\max_{|s-s_0|=r}|\zeta(s+i\tau;\alpha)|$. Then, again by Cauchy's formula,

$$\left|\frac{\zeta^{(k)}(s_0+i\tau;\alpha)}{k!}(s-s_0)k\right|\leq M(\tau)\delta^k$$

for $|s-s_0|\leq\delta_0 r$. Hence,

$$\begin{aligned}\Sigma_3 &:= \left|\zeta(s+i\tau;\alpha)-\sum_{k=0}^{N}\frac{\zeta^{(k)}(s_0+i\tau;\alpha)}{k!}(s-s_0)^k\right|\\ &= \left|\sum_{k>N}\frac{\zeta^{(k)}(s_0+i\tau;\alpha)}{k!}(s-s_0)^k\right|\\ &\leq M(\tau)\frac{\delta^N}{1-\delta},\end{aligned}$$

for $|s-s_0|\leq\delta_0 r$. In combination with the above estimates this yields

$$|\zeta(s+i\tau;\alpha)-f(s)| \ \leq\ \Sigma_1+\Sigma_2+\Sigma_3<\varepsilon+\varepsilon\exp(\delta r)+M(\tau)\frac{\delta^N}{1-\delta}$$

for $|s-s_0| \le \delta r$. Now we choose $\delta > 0$ such that $M(\tau)\frac{\delta^N}{1-\delta} = \varepsilon(2-\exp(\delta r))$; this choice is possible since the left hand side tends to zero as $\delta \to 0$ while the right hand side tends to $\varepsilon > 0$, resp. the left hand side tends to infinity but the right hand side remains bounded as $\delta \to 1$. This proves the corollary. □

Proof of Theorem 4.3. Let $\mathcal{R}$ be an open rectangle in the strip

$$\frac{3}{4} < 1-\xi \le \sigma_0 \le \sigma \le 1$$

with vertices $\sigma_1 \pm iH$, $\sigma_2 \pm iH$, where $H \gg_{\mathcal{K}} 1$ such that $K \subseteq \mathcal{R}$. Let also Q_0 be the ineffective constant given in Lemma 3.4 for

$$R \gg \left(\frac{4}{\varepsilon A^{3/4}}\right)^2 .$$

Then, for every $Q > Q_0$ and $\alpha \in [A,1]$, there are real numbers $\theta_{0n} = \theta_{0n}(\alpha)$ for which

$$\max_{s\in\mathcal{K}} \left| f(s) - \sum_{n=0}^{R-1} \frac{1}{(n+\alpha)^s} - \sum_{n=R}^{Q-1} \frac{\mathrm{e}(\theta_{0n})}{(n+\alpha)^s} \right| \ll R^{-1/2} \ll \frac{\varepsilon}{4}.$$

If we take $\theta_{0n} = 0$ for all $0 \le n \le R-1$ and $\delta^{-1} = Q^2 > QR$, then the inequality

$$|\theta_n - \theta_{0n}| \le \delta \tag{4.61}$$

implies that

$$\max_{s\in\mathcal{K}} |\zeta_Q(s,\underline{\theta},\alpha) - \zeta_Q(s,\underline{\theta}_0,\alpha)| \le \sum_{n=0}^{Q-1} \frac{|\mathrm{e}(\theta_n) - \mathrm{e}(\theta_{0n})|}{(n+\alpha)^{\sigma_1}} \le \frac{\delta Q}{\alpha^{\sigma_1}} < \frac{1}{RA^{3/4}} \ll \frac{\varepsilon}{4}.$$

Thus, by increasing if necessary R with respect to A and ε, we have

$$\max_{s\in\mathcal{K}} |f(s) - \zeta_Q(s,\underline{\theta},\alpha)| < \frac{\varepsilon}{2} < \left(2\frac{Q^2+1}{Q^2-1}\right)^{1/2} \frac{\varepsilon}{2} \tag{4.62}$$

whenever $A \le \alpha \le 1$ and (4.61) is satisfied.

On the other hand Lemma 4.4 yields, for every

$$Q \ge C_3' \varepsilon_0^8 = C_3' \left(\frac{\sqrt{2\pi(\sigma_2-\sigma_1)H}}{d(\mathcal{K},\partial\mathcal{G})\varepsilon}\right)^8 ,$$

$M \ge C_4' \exp(2Q^2)$, $\alpha \in \mathcal{A}(Q,M) \cap [A,1]$ and $d \ge d(\alpha)+1$, the existence of a positive number $\nu(d)$ such that, for every

$$T \ge C_5'(H,d) \max\left\{ \left(\mathbf{K} \exp\left((M+2)\exp\left(Q^2\right)\right)\right)^{\frac{4d}{4(d-d(\alpha))-3}}, \varepsilon_0^{-2\nu} \right\},$$

we have

$$\int_T^{2T} \Lambda_Q(\gamma_Q(\tau,\alpha) - \underline{\theta}_0)\mathrm{d}\tau \ge \delta^Q \left(1-Q^{-2}\right) T \tag{4.63}$$

and

$$\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha)-\underline{\theta}_0\right)\left|\left(\zeta-\zeta_Q\right)(s+i\tau;\alpha)\right|^2 \mathrm{d}\tau < \frac{\varepsilon_0^2}{4}\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha)-\underline{\theta}_0\right)\mathrm{d}\tau$$

for every $s \in \overline{\mathcal{R}}$. Then, Theorem A.16 yields that

$$\begin{aligned}
&\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha)-\underline{\theta}_0\right)\left(\max_{s\in\mathcal{K}}\left|\zeta\left(s+i\tau;\alpha\right)-\zeta_Q\left(s+i\tau,\underline{0},\alpha\right)\right|\right)^2 \mathrm{d}\tau \\
&\leq \frac{\pi}{d(\mathcal{K},\partial\mathcal{G})^2}\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha)-\underline{\theta}_0\right)\iint_G \left|\left(\zeta-\zeta_Q\right)(s+i\tau;\alpha)\right|^2 \mathrm{d}\sigma\mathrm{d}t\mathrm{d}\tau \\
&< \frac{\varepsilon^2}{4}\int_T^{2T} \Lambda_Q\left(\gamma_Q(\tau,\alpha)-\underline{\theta}_0\right)\mathrm{d}\tau \\
&< \frac{\varepsilon^2}{4}\delta^Q\left(1+Q^{-2}\right)T.
\end{aligned} \tag{4.64}$$

Now the theorem follows by arguing similarly as in the proof of Theorem 4.1. For the sake of completeness we repeat briefly the proof of the positive lower density.

Let $Q \gg Q_0 + C_3'\varepsilon_0^8$, $M \gg \exp\left(2Q^2\right)$ and $\alpha \in \mathcal{A}(Q,M)\cap[A,1]$ be of degree at most d_0-1. If $\mathcal{N}_T(\alpha)$ is the set of those $\tau \in \mathcal{U}_T(\alpha)$, where $\mathcal{U}_T(\alpha)$ was defined in (4.59), such that the inequality

$$\max_{s\in\mathcal{K}}\left|\zeta(s+i\tau;\alpha)-f(s)\right| < \left(2\frac{Q^2+1}{Q^2-1}\right)^{1/2}\varepsilon < 2\varepsilon$$

is satisfied, then it follows from (4.62)-(4.64) that

$$\mathrm{m}\left(\mathcal{N}_T(\alpha)\right) \geq \frac{1}{2}\delta^Q\left(1-Q^{-2}\right)T.$$

The only difference to the proof of Theorem 4.1 is that here the number Q_0, and in consequence the number Q too, is ineffectively computable. □

Chapter 5

Hypertranscendence

We use Ostrowski's theorem to prove the hypertranscendence of ordinary Dirichlet series, whose coefficients are given by some arithmetical function with specific properties. At first we consider arithmetical functions which are almost-periodic in the sense of Bohr and Besicovitch, while in the second part we study zeta-functions which arise from Beatty sets.

5.1 A Passage from Arithmetic to Analysis

Let $\mathbb{C}[X_0, \ldots, X_n]$ be the set of all polynomials of $n+1$ variables which have complex coefficients. A holomorphic function is called hypertranscendental if it does not satisfy a non-trivial algebraic differential equation. More precisely, a holomorphic function $f : \mathcal{U} \to \mathbb{C}$ defined on a non-empty open set $\mathcal{U} \subseteq \mathbb{C}$ is called *hypertranscendental*, if for any relation

$$P\left(f(s), f'(s), \ldots, f^{(n)}(s)\right) = 0,$$

where $P \in \mathbb{C}[X_0, \ldots, X_n]$, that holds identically for $s \in \mathcal{U}$, we have $P \equiv 0$.

The study of differential independence for Dirichlet series has a long history. The first result in the literature is from 1887 and is due to Hölder [35] who proved that Euler's Gamma-function Γ is hypertranscendental. In his famous address at the 1900 International Congress for Mathematicians in Paris, Hilbert [34] stated that $\zeta(s)$ is hypertranscendental and the proof is based on the hypertranscendence of the Gamma-function and the functional equation of $\zeta(s)$.

A detailed proof of Hilbert's statement for the hypertanscendence of $\zeta(s)$ was given by Stadigh (cf. [62]) in his PhD thesis. Later, Ostrowski [62] showed a much more general theorem. Before stating it, we need to inroduce some notation. In the sequel we denote by p_n the n-th prime and, for any positive integer y, we set

$$\mathscr{M}_y := \left\{p_1^{k_1} \cdots p_y^{k_y} : k_1, \ldots, k_y \in \mathbb{N}_0\right\}.$$

If $L(s; a)$ is an ordinary Dirichlet series, that is,

$$L(s; a) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a : \mathbb{N} \to \mathbb{C}$ is an arithmetical fucntion, let $\sigma(L)$ be the abscissa of absolute convergence of $L(s;a)$, that is,

$$\sigma(L) = \begin{cases} \limsup\limits_{N\to\infty} \dfrac{1}{\log N} \log\left(\sum\limits_{n=1}^{N} |a(n)|\right), & \text{if } \sum_{n\geq 1} |a(n)| \text{ diverges,} \\ \limsup\limits_{N\to\infty} \dfrac{1}{\log N} \log\left(\sum\limits_{n\geq N} |a(n)|\right), & \text{if } \sum_{n\geq 1} |a(n)| \text{ converges.} \end{cases} \tag{5.1}$$

The latter formula can be found in [91, 9.15].

Assume that μ is a non-negative integer, $h_0 < h_1 < \cdots < h_\mu$ are real numbers and $\nu_0, \nu_1, \ldots, \nu_\mu$ are non-negative integers with

$$\boldsymbol{\nu} := \sum_{j=0}^{\mu} (\nu_j + 1).$$

Finally, we define $\mathscr{D}$ to be the class of all ordinary Dirichlet series $L(s;a)$ satisfying the following two conditions:

i. $\sigma(L) < \infty$;
ii. there exists no positive integer y such that $\{n \in \mathbb{N} : a(n) \neq 0\} \subseteq \mathscr{M}_y$.

With the above notation, which will be kept for the rest of the chapter, Ostrowski proved

Theorem 5.1. *Let $L(s;a) \in \mathscr{D}$ and $P \in \mathbb{C}[X_1, \ldots, X_{\boldsymbol{\nu}}]$ be a non-zero polynomial. Then,*

$$P(L(s+h_0;a), L'(s+h_0;a), \ldots, L^{(\nu_0)}(s+h_0;a), L(s+h_1;a), \\ \ldots, L^{(\nu_1)}(s+h_1;a), \ldots, L(s+h_\mu;a), \ldots, L^{(\nu_\mu)}(s+h_\mu;a)) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\sigma + h_0 > \sigma(L)$. In particular, for $\mu = 0$ and $h_0 = 0$, $L(s;a)$ is hypertranscendental.

Reich [72] improved Ostrowski's theorem, where he substituted the non-zero polynomial P, with a continuous and *locally non-trivial* function $\Phi : \mathbb{C}^{\boldsymbol{\nu}} \to \mathbb{C}$, that is, a function such that for every non-empty open set $\mathcal{U} \subseteq \mathbb{C}^{\boldsymbol{\nu}}$, the restriction of Φ to $\mathcal{U}$ is not identically zero. Recently, Nagoshi [61] proved a generalization of Reich's result:

Theorem 5.2. *Let $L(s;a) \in \mathscr{D}$, N be a non-negative integer and $\Phi_N : \mathbb{C}^{\boldsymbol{\nu}} \to \mathbb{C}$ be a continuous and locally non-trivial function. When $N \geq 1$, let $\Phi_0, \ldots, \Phi_{N-1} : \mathbb{C}^{\boldsymbol{\nu}} \to \mathbb{C}$ be continuous functions. Then,*

$$\sum_{n=0}^{N} s^n \Phi_n(L(s+h_0;a), L'(s+h_0;a), \ldots, L^{(\nu_0)}(s+h_0;a), L(s+h_1;a), \\ \ldots, L^{(\nu_1)}(s+h_1;a), \ldots, L(s+h_\mu), \ldots, L^{(\nu_\mu)}(s+h_\mu;a)) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\sigma + h_0 > \sigma(L)$.

Corollary 5.1. *Assume that $L(s;a) \in \mathscr{D}$. Moreover, let N and M be non-negative integers and $\Phi_0, \ldots, \Phi_{N-1}, \Phi_N : \mathbb{C}^{M+1} \to \mathbb{C}$ be defined as in Theorem 5.2. If*

$$\sum_{n=0}^{N} s^n \Phi_n(L(s;a), L'(s;a), \ldots, L^{(M)}(s;a)) = 0$$

holds identically for $s \in \mathbb{C}$ with $\sigma > \sigma(L)$, then $\Phi_n \equiv 0$ for every $0 \leq n \leq N$.

The latter corollary can be considered as a weak version of the functional independence described in Chapter 1. In the sequel we identify two classes of Dirichlet series $L(s;a)$ which are not (yet) proven to have the universality property, as subsets of $\mathscr{D}$. Although, this approach does not lead to universality or even functional independence in the sense of Voronin, the simple property (ii) that we ask from an arithmetical function a to have, will allow us to prove, using only elementary methods, that $L(s;a) \in \mathscr{D}$ and, thus, it satisfies Corollary 5.1.

5.2 Dirichlet Series with Almost Periodic Coefficients

In this section we study ordinary Dirichlet series $L(s;a)$ which arise from almost periodic arithmetical functions $a : \mathbb{N} \to \mathbb{C}$. We refer to Section A.5 for an introduction to the notion of almost periodicity. We present only the necessary definitions needed to state our results. Let

$$\mathcal{A} := \operatorname{span}_{\mathbb{C}} \left\{ t \longmapsto \mathrm{e}(\beta t) : \beta \in [0,1) \right\}.$$

An arithmetical function a is called *uniformly almost periodic* if, for any $\varepsilon > 0$, there exists $P \in \mathcal{A}$ such that $\|a - P\|_\infty < \varepsilon$, where $\|a\|_\infty := \sup_{n\in\mathbb{N}} |a(n)|$. We denote the set of all these functions by $\mathcal{A}_u$.

Moreover, if $q \in [1, +\infty)$, then an arithmetical function a is called B^q*-almost periodic* if, for any $\varepsilon > 0$, there exists $P \in \mathcal{A}$ such that $\|a - P\|_q < \varepsilon$, where

$$\|a\|_q := \left(\limsup_{x\to\infty} \frac{1}{x} \sum_{n\leq x} |a(n)|^q \right)^{1/q}.$$

We denote the set of all these functions by $\mathcal{A}^q$.

Lemma 5.1. *Let $\mathcal{A}_0^q := \{a \in \mathcal{A}^q : ||a||_q > 0\}$, $q \geq 1$. Then, $\mathcal{A}_u \setminus \{0\} \subseteq \bigcap_{q\geq 1} \mathcal{A}_0^q$.*

Proof. For $a \in \mathcal{A}_u \setminus \{0\}$, there exists a positive integer n_0 such that $a(n_0) \neq 0$, and a trigonometric polynomial $P \in \mathcal{A}$ such that

$$\sup_{n\in\mathbb{N}} |a(n) - P(n)| < \frac{\varepsilon}{3}, \tag{5.2}$$

where ε is a fixed positive number satisfying $\varepsilon < |a(n_0)|$. It follows from Theorem A.22 that the set $\mathcal{E}(P, \varepsilon/3) \cap \mathbb{Z}$, where

$$\mathcal{E}\left(P, \frac{\varepsilon}{3}\right) := \left\{ \tau \in \mathbb{R} : \sup_{x\in\mathbb{R}} |P(x+\tau) - P(x)| < \frac{\varepsilon}{3} \right\},$$

is relatively dense. Thus, there exists $\ell > 0$ such that any interval of the real line of length ℓ contains at least one integer from the set $\mathcal{E}(P, \varepsilon/3)$. Hence, for any $k \in \mathbb{N}_0$, there is an integer $\tau_k \in [n_0 + 2k\ell, n_0 + (2k+1)\ell]$ such that

$$\sup_{x\in\mathbb{R}} |P(x+\tau_k) - P(x)| < \frac{\varepsilon}{3}. \tag{5.3}$$

Then, relations (5.2) and (5.3) yield that

$$\sup_{n\in\mathbb{N}} |a(n+\tau_k) - a(n)| < \varepsilon$$

and, thus,

$$|a(n_0+\tau_k)| > |a(n_0)| - \varepsilon > 0$$

for every $k \in \mathbb{N}_0$. Since $\tau_k \leq n_0 + (2k+1)\ell$ and $\tau_{k+1} - \tau_k \geq 2(k+1)\ell - (2k+1)\ell > 0$, it follows that

$$\frac{1}{n_0+\tau_k} \sum_{n=1}^{n_0+\tau_k} |a(n)|^q \geq \frac{1}{2n_0 + (2k+1)\ell} \sum_{n=0}^{k} |a(n_0+\tau_n)|^q > \frac{(k+1)\left(|a(n_0)| - \varepsilon\right)^q}{2n_0 + (2k+1)\ell}$$

for every $k \in \mathbb{N}_0$. Therefore,

$$||a||_q \geq \left(\lim_{k\to\infty} \frac{1}{n_0+\tau_k} \sum_{n=1}^{n_0+\tau_k} |a(n)|^q \right)^{1/q} = \frac{|a(n_0)| - \varepsilon}{(2\ell)^{1/q}} > 0$$

and this completes the proof. □

Theorem 5.3. *If $a \in \mathcal{A}_0^q$ for some $q \geq 1$, then $L(s;a) \in \mathscr{D}$.*

Proof. For $a \in \mathcal{A}^q$, there exists a triginometric polynomial $R \in \mathcal{A}$, such that

$$\|a\|_q > 0 \ \text{ and } \ \|a - R\|_q < 1.$$

The first condition implies that the series $\sum_{n\geq 1} |a_n|$ is divergent, while the second one, by Minkowski's inequality (Theorem A.12), that

$$\frac{1}{N} \sum_{n=1}^{N} |a(n)|^q \ll \max_{x\in\mathbb{R}} |R(x)|$$

for all $N \in \mathbb{N}$. In view of the formula (5.1) for the abscissa of absolute convergence $\sigma(L)$ and Hölder's inequality (Theorem A.11), it follows that

$$\begin{aligned}
\sigma(L) &= \limsup_{N\to\infty} \frac{1}{\log N} \log \left(\sum_{n=1}^{N} |a(n)| \right) \\
&\leq \limsup_{N\to\infty} \frac{1}{\log N} \log \left(N^{1-1/q} \left(\sum_{n=1}^{N} |a(n)|^q \right)^{1/q} \right) \\
&= 1.
\end{aligned}$$

Now assume that $a^* := \{n \in \mathbb{N} : a(n) \neq 0\} \subseteq \mathscr{M}_y$ for some positive integer y. The assumption $\|a\|_q > 0$ implies that a^* is infinite. Since $a \in \mathcal{A}_0^q$, for the positive number $\|a\|_q$, there exists a trigonometric polynomial $P \in \mathcal{A}$ such that

$$\left(\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} |a(n) - P(n)|^q\right)^{1/q} = \|a - P\|_q < \|a\|_q. \tag{5.4}$$

Using Minkowski's inequality we obtain that

$$\begin{aligned}
\left(\sum_{n\le x} |a(n) - P(n)|^q\right)^{1/q} &\ge \left(\sum_{n\le x} \mathbb{1}_{a^*}(n)|a(n) - P(n)|^q\right)^{1/q} \\
&\ge \left(\sum_{n\le x} \mathbb{1}_{a^*}(n)|a(n)|^q\right)^{1/q} - \left(\sum_{n\le x} \mathbb{1}_{a^*}(n)|P(n)|^q\right]^{1/q} \\
&\ge \left(\sum_{n\le x} |a(n)|^q\right)^{1/q} - L\left(\sum_{n\le x} \mathbb{1}_{\mathscr{M}_y}(n)\right)^{1/q},
\end{aligned}$$

where $|P(x)| \le L$ for all $x \in \mathbb{R}$. Thus,

$$\|a\|_q \le \|a - P\|_q + L\left(\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} \mathbb{1}_{\mathscr{M}_y}(n)\right)^{1/q}. \tag{5.5}$$

Recalling the definition of $\mathcal{M}_y$, it follows that, for suffciently large $x \gg_y 1$,

$$\sum_{n\le x} \mathbb{1}_{\mathscr{M}_y}(n) = \sum_{\substack{k_1=0 \\ \sum_{i=1}^{y} k_i \log p_i \le \log x}}^{\infty} \cdots \sum_{k_y=0}^{\infty} 1 \le \sum_{k_1=0}^{\left\lfloor \frac{\log x}{\log p_1}\right\rfloor} \cdots \sum_{k_y=0}^{\left\lfloor \frac{\log x}{\log p_y}\right\rfloor} 1 \ll_y (\log x)^y \prod_{i=1}^{y} \frac{1}{\log p_i}.$$

Therefore,

$$\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} \mathbb{1}_{\mathscr{M}_y}(n) = 0$$

and, thus, from (5.5)

$$\|a\|_q \le \|a - P\|_q,$$

which contradicts inequality (5.4). Hence, there is no positive integer y such that $a^* \subseteq \mathcal{M}_y$. □

Theorem 5.3 can not be generalized to the whole space $\mathcal{A}^q$. Consider, for example, the function $\mathbb{1}_A : \mathbb{N} \to \{0, 1\}$ where $A = \{2^n : n \in \mathbb{N}\}$. Then, $\mathbb{1}_A$ belongs to $\mathcal{A}^q$ with $\|\mathbb{1}_A\|_q = 0$ for any $q \ge 1$, since

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} \mathbb{1}_A(n) = 0.$$

Taking the corresponding Dirichlet series

$$f(s) := L(s; \mathbb{1}_A) = \sum_{n=1}^{\infty} \frac{\mathbb{1}_A(n)}{n^s} = \sum_{n=0}^{\infty} \frac{1}{2^{ns}} = \frac{2^s}{2^s - 1},$$

one can prove that it satisfies the non-trivial algebraic differential equation

$$f' + \log 2(f^2 - f) = 0$$

and, thus, it is not hypertranscendental.

However, the set $\{a \in \mathcal{A}^q : ||a||_q = 0\}$ contains also arithmetical functions with far more interesting properties. In [79] it is proved that for suitable multiplicative functions a with zero mean-value, the corresponding Dirichlet series $L(s; a)$ has the universality property, in the sense of Voronin, inside some strip of the complex plane which depends on a. In the case of a periodic arithmetical function a, the value-distribution of the corresponding Dirichlet series $L(s; a)$ is much better understood. We refer to [86] for an exposition of relevant results, where universality results are obtained as well.

5.3 Beatty Zeta-Functions

For given positive real numbers α, β, the associated *Beatty set* or *Beatty sequence* is defined by

$$\mathscr{B}(\alpha, \beta) := \{\lfloor \alpha n + \beta \rfloor : n \in \mathbb{N}\},$$

where it is understood that the elements of $\mathscr{B}(\alpha, \beta)$ are to be enumerated in increasing order. Often enough only the case $\beta = 0$ is considered and, thus, sometimes $\mathscr{B}(\alpha, 0)$ is called *homogeneous*. Otherwise, $\mathscr{B}(\alpha, \beta)$ is called *inhomogeneous* Beatty set. We refer to [88] and the references cited there for a survey on Beatty sets and some generalisations thereof. We will only present a very special feature of Beatty sets, as well as the prime number theorem analogue for such sets.

Lemma 5.2. *Suppose that $\alpha \geq 1$ and $\beta \geq 0$ are real numbers. Then an integer m is an element of $\mathscr{B}(\alpha, \beta)$ if and only if*

$$\frac{m}{\alpha} \in \left(\frac{\beta - 1}{\alpha}, \frac{\beta}{\alpha} \right] \mod 1.$$

Proof. This follows immediately from the definition of $\mathscr{B}(\alpha, \beta)$. □

Theorem 5.4. *Suppose that $\alpha \geq 1$ and $\beta \geq 0$ are real numbers. Then,*

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sharp \left\{ p \leq x \text{ prime} : \frac{m}{\alpha} \in \left(\frac{\beta - 1}{\alpha}, \frac{\beta}{\alpha} \right] \mod 1 \right\} = \frac{1}{\alpha}$$

The proof of the theorem follows from a classical theorem due to Vinogradov (see [39, p. 489]:

Theorem 5.5. *Let p_n denote the n-th prime and suppose that $\alpha > 0$ is irrational. Then the sequence $(\{p_n/\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed mod 1.*

We consider polynomials $P \in \mathbb{Z}[X]$ of positive degree satisfying $P(\mathbb{N}) \subseteq \mathbb{N}$ and we want to investigate the more sophisticated Dirichlet series, which we also call *Beatty zeta-function*,

$$L(s; P, \mathscr{B}(\alpha, \beta)) := \sum_{n=1}^{\infty} \frac{1}{P(\lfloor \alpha n + \beta \rfloor)^s}.$$

Further we will assume that $\lfloor \alpha + \beta \rfloor \geq 1$ to have everything well-defined.

Theorem 5.6. *Let α, β be positive real numbers with $\lfloor \alpha+\beta \rfloor \geq 1$ and $P \in \mathbb{Z}[X]$ be a non-constant polynomial with $P(\mathbb{N}) \subseteq \mathbb{N}$. Then $L(s; P, \mathscr{B}) \in \mathscr{D}$.*

Proof. Since P is a non-constant polynomial, it follows that

$$P(\lfloor \alpha n + \beta \rfloor) \gg \lfloor \alpha n + \beta \rfloor \gg n$$

as $n \to \infty$. Therefore, $L(s; P, \mathscr{B}(\alpha, \beta))$ is absolutely convergent for $\sigma > 1$. This allows us to rearrange the terms of the latter series in the following manner

$$L(s; P, \mathscr{B}(\alpha, \beta)) = \sum_{n=1}^{\infty} \frac{a(m)}{m^s},$$

where $a(m) = \sharp\{n \in \mathbb{N} : P(\lfloor \alpha n + \beta \rfloor) = m\}$. Thus, $L(s; P, \mathscr{B})$ is an ordinary Dirichlet series with $\sigma(L) \leq 1$.

Now assume that $a^* := \{n \in \mathbb{N} : a(n) \neq 0\} \subseteq \mathscr{M}_y$ for some positive integer y. Let $P(x) = a_k x^k + \ldots + a_1 x + a_0$ be the given polynomial with integer coefficients and

$$\tau = \begin{cases} {p_1}^d p_2 \ldots p_y + 1, & \text{if } P(\lfloor \beta \rfloor) = 0, \\ \\ |P(\lfloor \beta \rfloor)| {p_1}^d p_2 \ldots p_y, & \text{if } P(\lfloor \beta \rfloor) \neq 0, \end{cases}$$

where $d \in \mathbb{N}$ is sufficiently large so that $P(\lfloor \alpha n \tau + \beta \rfloor) > P(\lfloor \beta \rfloor)$ for all $n \in \mathbb{N}$. Recall that p_n denotes the n-th prime. We consider two cases. If α is a positive rational number, that is, if $\alpha = r/q$ for some positive integers r and q, then we set $n_0 = q\tau$ and we observe that

$$P(\lfloor \alpha n_0 + \beta \rfloor) = a_k(r\tau + \lfloor \beta \rfloor)^k + \ldots + a_1(r\tau + \lfloor \beta \rfloor) + a_0 = \ell r \tau + P(\lfloor \beta \rfloor),$$

where ℓ is a strictly positive integer by our choice of d. But then

$$P(\lfloor \alpha n_0 + \beta \rfloor) = \begin{cases} \ell r({p_1}^d p_2 \ldots p_y + 1), & \text{if } P(\lfloor \beta \rfloor) = 0, \\ \\ |P(\lfloor \beta \rfloor)|(\ell r {p_1}^d p_2 \ldots p_y \pm 1), & \text{if } P(\lfloor \beta \rfloor) \neq 0, \end{cases}$$

which clearly implies that $P(\lfloor \alpha n_0 + \beta \rfloor) \notin M_y$. Thus, the assumption of the existence of such y is not true in case α is a positive rational number.

If α is a positive irrational number, we employ the theory of continued fractions (see [32, Chapter X]). Let $[a_0, a_1, a_2 \dots]$ be the continued fraction expansion of α and

$$\frac{r_n}{q_n} := [a_0, a_1, \dots, a_n],$$

$n \in \mathbb{N}$, its n-the convergent. Then

$$0 < \alpha - \frac{r_{2n}}{q_{2n}} < \frac{1}{q_{2n}^2} \leq \frac{1}{2nq_{2n}} \tag{5.6}$$

for all $n \in \mathbb{N}$. Hence, if we take a sufficiently large positive integer n such that

$$\frac{1}{2n} < \frac{1 - (\beta - \lfloor \beta \rfloor)}{\tau}, \tag{5.7}$$

then relations (5.6) and (5.7) yield that

$$0 < \alpha q_{2n}\tau + \beta - (r_{2n}\tau + \lfloor \beta \rfloor) < 1$$

or, equivalently,

$$\lfloor \alpha q_{2n}\tau + \beta \rfloor = r_{2n}\tau + \lfloor \beta \rfloor.$$

If we set $n_0 = q_{2n}\tau$, then

$$P(\lfloor \alpha n_0 + \beta \rfloor) = P(r_{2n}\tau + \lfloor \beta \rfloor) = \ell r_{2n}\tau + P(\lfloor \beta \rfloor),$$

where $\ell \in \mathbb{N}$, and the proof follows as in the case of α being a positive rational number. □

If $P(x) = x$, $\alpha \geq 1$ and $\beta \geq 0$, we can actually derive more information about the Beatty zeta-function

$$L(s; \mathscr{B}(\alpha, \beta)) := \sum_{n=1}^{\infty} \frac{1}{\lfloor \alpha n + \beta \rfloor} = \sum_{n=1}^{\infty} \frac{\mathbb{1}_{\mathscr{B}(\alpha,\beta)}(n)}{n^s}, \quad \sigma > 1. \tag{5.8}$$

From Theorem 5.4 it follows that, if $p_n^{\mathscr{B}}$ is the n-th prime in $\mathscr{B}(\alpha, \beta)$, then

$$p_n^{\mathscr{B}} \sim \alpha n \log n. \tag{5.9}$$

This yields an interesting result regarding the *measure of hypertranscendence* of $L(s; \mathscr{B}(\alpha, \beta))$.

As we have aready defined in the beginning of this chapter, a holomorphic function $f : U \to \mathbb{C}$ is hypertranscendental if for any relation

$$P\left(f(s), f'(s), \dots, f^{(n)}(s)\right) = 0,$$

where $P \in \mathbb{C}[X_0, \dots, X_n]$, that holds identically for $s \in \mathcal{U}$, we have $P \equiv 0$. The term "hypertranscendence" is chosen for a good reason. That is because we ask from a function to have a property that exceeds the one of transcendental real numbers, like e and π. And as in the case of transcendental numbers, one may ask "how far" is f from being *algebraic*, in the sense of finding lower bounds for the quantity

$$\inf_{s\in U} \left| P\left(f(s), f'(s), \ldots, f^{(n)}(s)\right) \right|$$

if $P \not\equiv 0$ and $n \in \mathbb{N}$. Of course we can say, alternatively, that such bounds measure the hypertranscendence of f.

Such problems were first investigated by Popken [65, 66, 67, 68]. His main object of research was transcendence of real numbers or arithmetical functions. However, in those references one can find yet another example of how the properties of an arithmetical function a affect the value-distribution of the associated Dirichlet series $L(s;a)$. For a more detailed account on this special connection that an arithmetical function a share with $L(s;a)$, we refer to [45] and [46]. In particular, we are going to make use of the following theorem which is proved in [45, Corollary 3.3]:

Theorem 5.7. *Let $L(s;a) = \sum_{n\geq 1} a(n)n^{-s}$ be a Dirichlet series, r be a positive integer and $P \in \mathbb{C}[X_1, \ldots, X_r]$ be a non-zero polynomial of total degree g. If there is a set of $r+1$ primes $\{p_1 < \cdots < p_{r+1}\}$ such that $a(p_i) \neq 0$, $i = 1, \ldots, r+1$, then*

$$\left| P\left(L(\cdot;a), L'(\cdot;a), \ldots, L^{(r)}(\cdot;a)\right) \right| \geq \left(\frac{1}{p_{r+1}}\right)^g,$$

where the Dirichlet series, their derivatives and operations are considered formally.

The latter theorem, the definition (5.8) and relation (5.9) yield the following theorem regarding the measure of hypertranscendence of $L(s;\mathscr{B}(\alpha,\beta))$:

Theorem 5.8. *Let $\alpha \geq 1$ and $\beta \geq 0$. Then, for any positive integer r and any non-zero polynomial $P \in \mathbb{C}[X_1, \ldots, X_r]$ of total degree g,*

$$\left| P\left(L\left(s;\mathscr{B}(\alpha,\beta)\right), L'\left(s;\mathscr{B}(\alpha,\beta)\right), \ldots, L^{(r)}\left(s;\mathscr{B}(\alpha,\beta)\right)\right) \right| \gg_{\sigma_0} \left(\frac{1}{\alpha r \log r}\right)^g,$$

uniformly in $\sigma \geq \sigma_0 > 1$.

Appendix A

A.1 Uniformly Distributed Sequences

The material presented here can be found in [48]. We begin with the definition of a uniformly distributed sequence.

Definition A.1. *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for every* $0 \le a < b \le 1$ *the following relation holds:*

$$\lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : \{x_n\} \in [a,b)\}}{N} = b - a$$

Already from the definition, one can prove

Theorem A.1. *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is u.d. mod 1, and if* $(y_n)_{n\in\mathbb{N}}$ *is a sequence with the property* $\lim_{n\to\infty}(x_n - y_n) = \alpha$*, a real constant, then* $(y_n)_{n\in\mathbb{N}}$ *is u.d. mod 1.*

Proof. For a proof see [48, Chapter 1, Theorem 1.2]. □

Uniformly distributed sequences have two remarkable properties which are described in the next two theorems.

Theorem A.2. *The sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is u.d. mod 1 if for every real-valued continuous function* f *defined on* $[0,1]$ *the following relation holds:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\mathrm{d}x.$$

Proof. For a proof see [48, Chapter 1, Theorem 1.1]. □

Corollary A.1. *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is u.d. mod 1 if for every complex-valued Riemann-integrable function* f *on* $\mathbb{R}$ *with period 1 the following relation holds:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}x.$$

Proof. For a proof see [48, Chapter 1, Corollary 1.2]. □

Theorem A.3 (Weyl's Criterion)**.** *The sequence $(x_n)_{n\in\mathbb{N}}$ is u.d. mod 1 if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}\left(hx_n\right)=0,$$

for all integers $h\neq 0$.

Proof. For a proof see [48, Chapter 1, Theorem 2.1]. □

The latter theorem implies for example that the sequence $(\alpha n)_{n\in\mathbb{N}}$ is u.d. mod 1, if α is an irrational number. In fact, the following theorem holds true.

Theorem A.4. *Let $P_{\underline{a}}(x)=a_1x+\cdots+a_dx^d$ be a polynomial of degree $d\geq 1$ with real coefficients. If a_j is irrational for some $j=1,\ldots,d$, then the sequence $(P_{\underline{a}}(n))_{n\in\mathbb{N}}$ is u.d mod 1. If a_d is irrational, but a_k/a_d is rational for $k=1,\ldots,d$, then the sequence $\left(\lfloor P_{\underline{a}}(n)\rfloor\,\theta\right)_{n\in\mathbb{N}}$ is u.d. mod 1 if and only if $1,a_d,\theta a_d$ are linearly independent over $\mathbb{Q}$. Otherwise, the sequence $\left(\lfloor P_{\underline{a}}(n)\rfloor\,\theta\right)_{n\in\mathbb{N}}$ is u.d. mod 1 for every irrational θ.*

Proof. For the first part of the lemma see [48, Chapter 1, Theorem 3.2], while for the second part [11, Theorem 2]. □

We also give the multidimensional setting of uniformly distributed sequences.

Definition A.2. *A sequence of vectors of real numbers $\underline{x}_n=(x_{n1},\ldots,x_{n\ell})$, $n\in\mathbb{N}$, is said to be u.d. mod 1 in $\mathbb{R}^\ell$ if for every $0\leq a_j<b_j\leq 1$, $j=1,\ldots,\ell$, the following relation holds:*

$$\lim_{N\to\infty}\frac{1}{N}\sharp\left\{1\leq n\leq N:\{\underline{x}_n\}\in\prod_{j=1}^{\ell}[a_j,b_j]\right\}=\prod_{j=1}^{\ell}(b_j-a_j).$$

Theorem A.5. *The sequence of vectors of real numbers $(\underline{x}_n)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}^\ell$ if for every complex-valued Riemann-integrable function f defined on $[0,1]^\ell$ the following relation holds:*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\{\underline{x}_n\})=\int_{[0,1]^\ell}f(\underline{x})\mathrm{d}\underline{x}.$$

Proof. For a proof see [48, Chapter 1, Theorem 6.1]. □

Theorem A.6. *The sequence of vectors of real numbers $(\underline{x}_n)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}^\ell$ if and only if for every lattice point $\underline{h}\in\mathbb{Z}^\ell$, $\underline{h}\neq\underline{0}$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}\left(\langle\underline{h},\underline{x}_n\rangle\right)=0.$$

Proof. For a proof see [48, Chapter 1, Theorem 6.2]. □

Corollary A.2. *The sequence of vectors of real numbers* $(\underline{x}_n)_{n\in\mathbb{N}}$ *is u.d. mod 1 in* $\mathbb{R}^\ell$ *if and only if for every lattice point* $\underline{h} \in \mathbb{Z}^\ell$, $\underline{h} \neq \underline{0}$, *the sequence of real numbers* $(\langle \underline{h}, \underline{x}_n \rangle)_{n\in\mathbb{N}}$ *is u.d. mod 1 in* $\mathbb{R}$.

Proof. For a proof see [48, Chapter 1, Theorem 6.3]. □

Lastly, we present a discrete analogue of a theorem which can be found in [43, Appendix, §8, Theorem 3]. The proof is straightforward and so we omit it.

Theorem A.7. *Suppose that the sequence* $(\underline{x}_n)_{n\in\mathbb{N}}$ *is u.d. mod 1 in* $\mathbb{R}^N$. *Let* $\mathcal{L}$ *be a closed and Jordan measurable subset of* $[0,1]^N$ *and let* $\mathscr{F}$ *be a family of complex-valued continuous functions defined on* $\mathcal{L}$. *If* $\mathscr{F}$ *is uniformly bounded and equicontinuous, then the following relation holds uniformly with respect to* $f \in \mathscr{F}$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{\underline{x}_n\}) = \int_{\mathcal{L}} f(\underline{x}) \mathrm{d}\underline{x}.$$

A.2 Useful Formulas and Inequalities

Theorem A.8 (Stirling's Formula). *If* σ_1, σ_2 *are real numbers with* $\sigma_1 \leq \sigma_2$, *then*

$$|\Gamma(s)| = (2\pi)^{1/2} |t|^{\sigma-1/2} \exp\left(\frac{-\pi|t|}{2}\right) \left(1 + O_{\sigma_1,\sigma_2}\left(\frac{1}{|t|}\right)\right), \quad |t| \geq t_0 > 0,$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$.

Proof. For a proof see [89, Chapter II, Corollary 0.13]. □

Theorem A.9 (Perron's Formula). *Let* c, y *and* T *be positive real numbers. If we set*

$$\delta(y) := \begin{cases} 0, & 0 < y < 1, \\ \frac{1}{2}, & y = 1, \\ 1, & y > 1 \end{cases} \qquad \text{and} \quad I(y,T) := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^z \frac{\mathrm{d}z}{z},$$

then

$$|I(y,T) - \delta(y)| < \begin{cases} y^c \min\left(1, \dfrac{1}{T|\log y|}\right), & y \neq 1, \\ \dfrac{c}{T}, & y = 1. \end{cases}$$

Proof. For a proof see [38, Lemma 12.1]. □

Theorem A.10 (Abel's Summation Formula). *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers. If*

$$A(t) := \sum_{n \leq t} a_n, \quad t > 0$$

and $f \in C^1([1,x])$, then

$$\sum_{1\le n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\mathrm{d}t.$$

Proof. The proof follows immediately from an application of the formula for partial integration of Stieltjes-integrals:

$$\sum_{1\le n\le x} a_n f(n) = \int_{1^-}^x f(t)\mathrm{d}A(t) = \Big[A(t)f(t)\Big]_{1^-}^x - \int_1^x A(t)f'(t)\mathrm{d}t.$$

□

Theorem A.11 (Hölder's Inequality). *Let X be a measure space with measure μ and $p,q \in (1,+\infty)$ be such that $1/p + 1/q = 1$. Then*

$$\int_X |fg|\mathrm{d}\mu \le \left(\int_X |f|^p\mathrm{d}\mu\right)^{1/p}\left(\int_X |g|^q\mathrm{d}\mu\right)^{1/q} \tag{A.1}$$

for any measurable functions $f,g : X \to \mathbb{C}$.

Proof. For a proof see [16, Theorem B.15]. □

Theorem A.12 (Minkowski's Inequlaity). *Let X be a measure space with measure μ and $p \in (1,+\infty)$. Then*

$$\left(\int_X |f+g|^p\mathrm{d}\mu\right)^{1/p} \le \left(\int_X |f|^p\mathrm{d}\mu\right)^{1/p} + \left(\int_X |g|^p\mathrm{d}\mu\right)^{1/p} \tag{A.2}$$

for any measurable functions $f,g : X \to \mathbb{C}$.

Proof. For a proof see [16, Theorem B.16]. □

A.3 Polynomials and Polynomial Inequalities

The first two theorems can be found in the homonym book of Borwein and Erdélyi [8], where they are presented as exercises. Lagrange's interpolation theorem is given in [8, Chapter 1, Section 1, E.6], while Markov's inequality in [8, Chapter 5, Section 2, E.2].

Theorem A.13 (Lagrange's Interpolation Theorem). *Let z_i and y_i, $i = 0,\dots,n$, be arbitrary complex numbers except that the z_i must be pairwise distinct. Then the polynomial*

$$P(z) = \sum_{k=0}^n y_k \prod_{\substack{i=0\\ i\ne k}}^n \frac{z-z_j}{z_k - z_j}, \quad z \in \mathbb{C},$$

is called the Lagrange interpolation polynomial and it is the unique polynomial with complex coefficients and of degree at most n such that

$$P(z_i) = y_i,$$

for every $i = 0,\dots,n$.

Theorem A.14 (Markov's Inequality). *Suppose that $P \in \mathbb{C}[X]$ is of degree at most n and it satisfies*

$$\left| P\left(\cos \frac{j\pi}{n}\right) \right| \leq 1,$$

for every $j = 1, \ldots, n$. Then

$$\max_{x \in [-1,1]} \left| P^{(m)}(x) \right| \leq \frac{n^2 (n^2 - 1)(n^2 - 2^2) \ldots \left(n^2 - (m-1)^2\right)}{(2m-1)!}$$

for every $m = 1, \ldots, n$.

The next theorem originates from a work of Güting [28].

Theorem A.15. *Let $P(X)$ and $Q(X)$ be non-constant integer polynomials of degree n and m, respectively. Denote by α a zero of $Q(X)$ of order t. If $P(\alpha) \neq 0$, then*

$$|P(\alpha)| \geq (n+1)^{1-m/t}(m+1)^{-n/(2t)} H(P)^{1-m/t} H(Q)^{-n/t} (\max\{1, |\alpha|\})^n$$

Proof. For a proof see [10, Theorem A.1]. □

From Theorem A.15 we obtain

Corollary A.3. *Let $P(X)$ be a non-zero integer polynomial of degree n and $\alpha \in (0,1]$ an algebraic number of degree $d(\alpha)$ and height $H(\alpha)$. If $P(\alpha) \neq 0$, then*

$$|P(\alpha)| \geq (n+1)^{1-d(\alpha)}(d(\alpha)+1)^{-n/2} H(P)^{1-d(\alpha)} H(\alpha)^{-n}.$$

A.4 Facts from Hilbert Space Theory and Function Theory

Let $\mathcal{R} \subseteq \mathbb{C}$ be a bounded domain. The Bergman space $A^2(\mathcal{R})$ is the set of all analytic functions $f : \mathcal{R} \to \mathbb{C}$ which are square integrable on G, endowed with its natural norm

$$||f||^2_{A^2(\mathcal{R})} := \iint_{\mathcal{R}} |f(\sigma + it)|^2 \mathrm{d}\sigma \mathrm{d}t.$$

Theorem A.16. *Let $\mathcal{R}$ be a bounded domain. If $f \in A^2(\mathcal{R})$ and $z \in \mathcal{R}$, then*

$$|f(z)| \leq \frac{\sqrt{\pi}}{d(z, \partial\mathcal{R})} ||f||_{A^2(\mathcal{R})},$$

where $d(z, \mathcal{R}) = \min\{|z - w| : w \in \partial\mathcal{R}\}$.

Proof. For a proof see [15, Chapter 1, Theorem 1]. □

Theorem A.17. *Let $x_1, \dots, x_n$ be elements of a complex Hilbert space $\mathcal{H}$ and let $a_1, \dots, a_n$ be complex numbers with $|a_j| \leq 1$ for $1 \leq j \leq n$. Then there exist complex numbers $b_1, \dots, b_n$ with $|b_j| = 1$ for $1 \leq j \leq n$, satisfying the inequality*

$$\left|\left|\sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n} b_j x_j\right|\right|_{\mathcal{H}}^2 \leq 4 \sum_{j=1}^{n} ||x_j||_{\mathcal{H}}^2$$

Proof. For a proof see [86, Lemma 5.2]. □

Theorem A.18. *Let X be a locally convex vector space. Let $K \subseteq X$ be a closed convex set, and suppose that $z \in X \setminus K$. Then there exists a continuous linear functional $\ell \in X^*$ and a constant $c \in \mathbb{R}$ such that $\ell(y) \leq c < \ell(x)$ for all $y \in K$.*

Proof. For a proof see [16, Theorem 8.73]. □

Theorem A.19 (Hadamard's Three-Circles Theorem)**.** *Let $s_0 \in \mathbb{C}$ and f be an analytic function, regular for $r_1 \leq |s - s_0| \leq r_3$. Then for every $r_1 < r_2 < r_3$*

$$\max_{|s-s_0|=r_2} |f(s)| \leq \left(\max_{|s-s_0|=r_1} |f(s)|\right)^a \left(\max_{|s-s_0|=r_3} |f(s)|\right)^b,$$

where

$$a = \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \quad \text{and} \quad b = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}.$$

Proof. For a proof see [91, Section 5.3]. □

Theorem A.20 (Borel-Carathéodory Inequality)**.** *Let $s_0 \in \mathbb{C}$ and f be an analytic function, regular for $|s - s_0| \leq R$. Then for every $0 < r < R$*

$$\max_{|s-s_0|=r} |f(s)| \leq \frac{2r}{R-r} \max_{|s-s_0|=R} \Re(f(s)) + \frac{R+r}{R-r}|f(s_0)|.$$

Proof. For a proof see [91, Section 5.5]. □

Theorem A.21 (Mergelyan's Theorem)**.** *Let $\mathcal{K} \subseteq \mathbb{C}$ be a compact set with connected complement. Let also f be a continuous function on $\mathcal{K}$ and analytic in its interior. Then for every $\varepsilon > 0$ there exists a polynomial $P \in \mathbb{C}[X]$ such that*

$$\max_{z \in \mathcal{K}} |f(z) - P(z)| < \varepsilon.$$

Proof. For a proof see [74, Chapter 20]. □

A.5 Spaces of Almost Periodic Functions

The theory of almost-periodic functions was pioneered by Bohr and developed by himself, Besicovitch, Bochner, Stepanoff, Weyl and many others. We recommend Besicovitch's book [4] on a survey to the different kinds of almost periodic functions. However, since we focus on properties of arithmetical functions, [78] is also of great importance.

Definition A.3. *A set $\mathcal{E} \subseteq \mathbb{R}$ is said to be relatively dense if there exists a number $\ell > 0$ such that any interval of length ℓ contains at least one number of the set $\mathcal{E}$.*

Definition A.4. *A continuous function $f : \mathbb{R} \to \mathbb{C}$ is called uniformly almost-periodic if, for every $\varepsilon > 0$, the set*

$$\mathcal{E}(f, \varepsilon) := \left\{ \tau \in \mathbb{R} : \sup_{x \in \mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon \right\}$$

is relatively dense.

Observe that periodic functions belong to this more general class of functions.

Theorem A.22. *Let $f : \mathbb{R} \to \mathbb{C}$ be a uniformly almost periodic function. Then, for any $\varepsilon > 0$, the set $\mathcal{E}(f, \varepsilon) \cap \mathbb{Z}$ is relatively dense.*

Proof. For a proof see the theorem in [4, Chapter I, §11, 4°]. □

Let

$$\mathcal{A} := \operatorname{span}_{\mathbb{C}} \{ t \longmapsto \mathrm{e}(\beta t) : \beta \in [0, 1) \}$$

denote the set of all trigonometric polynomials with complex coefficients. Bohr [33] proved the following remarkable theorem:

Theorem A.23. *A continuous function $f : \mathbb{R} \to \mathbb{C}$ is uniformly almost periodic if and only if there is a sequence of trigonometric polynomials $P_n \in \mathcal{A}$, $n \in \mathbb{N}$, such that*

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |P_n(x) - f(x)| = 0.$$

Bohr's work inspired many mathematicians to study the set $\mathcal{A}$ (or subsets of $\mathcal{A}$) endowed with various norms or semi-norms. We turn our attention to the uniform norm and the Besicovitch semi-norms. It is straightforward to confirm the following lemma by using Minkowski's inequlaity (Theorem A.12).

Lemma A.1. *Let $q \geq 1$. The functionals*

$$\| \cdot \|_\infty, \| \cdot \|_q : \mathcal{A} \to \mathbb{R}$$

defined by the formula

$$\|P\|_\infty = \sup_{n \in \mathbb{N}} |P(n)|$$

and

$$\|P\|_q = \left(\limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |P(n)|^q \right)^{1/q}$$

for any $P \in \mathcal{A}$, are semi-norms. Moreover, $\| \cdot \|_\infty$ is a norm.

Now we can naturally introduce the concept of almost periodicity in the case of arihmetical functions $f : \mathbb{N} \to \mathbb{C}$:

Definition A.5. *We define*

$$\mathcal{A}_u = \|\cdot\|_\infty - closure\ of\ \mathcal{A}$$

to be the space of uniformly almost periodic arithmetical functions and for $q \geq 1$,

$$\mathcal{A}^q = \|\cdot\|_q - closure\ of\ \mathcal{A}$$

to be the space of B^q-almost periodic arithmetical functions.

It can be seen that $(\mathcal{A}^u, \|\cdot\|_\infty)$ and $(\mathcal{A}^q, \|\cdot\|_q)$ are Banach spaces and

$$\mathcal{A} \subseteq \mathcal{A}_u \subseteq \mathcal{A}^r \subseteq \mathcal{A}^q \subseteq \mathcal{A}^1$$

for any $1 \leq q \leq r$. The latter follows from the inequality $\|f\|_r \leq \|f\|_q$ for $r \leq q$ and f an arithmetical function. Moreover, if $f \in \mathcal{A}^1$, then its mean-value

$$M(f) := \lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} f(n)$$

exists. We refer to [78, Chapter VI] for the aforementioned statements. In our case we will need the following

Theorem A.24. *Let f and g_k, $k \in \mathbb{N}$, be B^q-almost periodic arithmetical functions for some $q \geq 1$ such that*

$$\lim_{k\to\infty} \|g_k - f\|_q = 0.$$

Then

$$\lim_{k\to\infty} M(g_k) = M(f).$$

Proof. We know that $M(f)$ and $M(g_k)$, $k \in \mathbb{N}$, exist and

$$|M(g_k) - M(f)| = \lim_{x\to\infty} \frac{1}{x} \left| \sum_{n\leq x} (g_k(n) - f(n)) \right| \leq \|g_k - f\|_1 \leq \|g_k - f\|_q.$$

Taking $k \to \infty$ the theorem follows. □

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