# Weighted uniform distribution related to primes and the Selberg Class

**Dissertationsschrift**
**zur Erlangung des naturwissenschaftlichen Doktorgrades der**
**Julius-Maximilians Universität Würzburg**

vorgelegt von


Martin Rehberg

aus

Friedberg


Würzburg 2019

| | |
|---|---|
| Referent: | Prof. Dr. habil. Jörn Steuding |
| Korreferent: | Prof. Dr. Kai Bruchlos |
| | Prof. Dr. habil. Jürgen Sander |

# Danksagung

An erster Stelle gilt mein Dank Herrn Prof. Dr. Steuding für die geduldige Anleitung und freundliche Unterstützung während der letzten drei Jahre. Ferner danke ich Herrn Prof. Dr. Bruchlos für die fruchtbaren Gespräche während meiner Tätigkeit als wissenschaftlicher Mitarbeiter an der Technischen Hochschule Mittelhessen, sowie Herrn Prof. Dr. Sander für sein Interesse an meiner Arbeit.

Darüber hinaus danke ich den Mitgliedern des Instituts für Mathematik der Julius-Maximilians-Universität Würzburg und des Fachbereichs Mathematik, Naturwissenschaften und Datenverarbeitung der Technischen Hochschule Mittelhessen, für die interessanten Diskussionen und die angenehme Arbeitsatmosphäre.

Der Abschnitt über $a$-Stellen in der vorliegende Arbeit beruht auf einer Kooperation der Philipps-Universität Marburg und der Julius-Maximilians-Universität Würzburg. Hierfür gilt mein besonderer Dank Frau Prof. Dr. Agricola und Herrn Prof. Dr. Friedrich.

Für den willkommenen sportlichen Ausgleich danke ich den Mitgliedern des Maschinenraum CrossFit in Ober-Mörlen und den Trainingspartnern des Dragos Wing Tsun in Gießen.

Mein letzter Dank gilt natürlich Familie und Freunden, insbesondere meiner Schwester Christina, für die Unterstützung und Motivation während der letzten Jahre.

Martin Rehberg
Friedberg, 2019

# Zusammenfassung

In der vorliegenden Arbeit werden verschiedene zahlentheoretisch interessante Folgen im Kontext der Theorie der Gleichverteilung modulo eins untersucht.

Im ersten Teil wird für positiv reelles $z \neq 1$ für die Folge $\big((2\pi)^{-1}(\log z)\gamma_a\big)$ eine Diskrepanzabschätzung hergeleitet, wobei $\gamma_a$ die positiven Imaginärteile der nichttrivialen $a$–Stellen der Riemannschen Zetafunktion durchlaufe: Sind die eingehenden Imaginärteile durch $T$ beschränkt, dann strebt für $T \to \infty$ die Diskrepanz der Folge $\big((2\pi)^{-1}(\log z)\gamma_a\big)$ wie $(\log\log\log T)^{-1}$ gegen Null. Der Beweis knüpft an das Vorgehen von Hlawka an, welcher eine Diskrepanzabschätzung für die Folge, in der die positiven Imaginärteile der nichttrivialen Nullstellen der Riemannschen Zetafunktion eingehen, ermittelte.

Der zweite Teil der Arbeit widmet sich einer Folge deren Wachstumsverhalten durch Primzahlen motiviert ist. Ist $\alpha \neq 0$ reell und $f$ eine logarithmisch wachsende Funktion, dann werden mehrere Bedingungen an $f$ angegeben, unter denen die Folge $(\alpha f(q_n))$ gleichverteilt modulo eins ist. Entsprechende Diskrepanzabschätzungen der Folgen werden angegeben. Die Folge reeller Zahlen $(q_n)$ ist selbst streng wachsend und die Bedingungen, die dabei an deren Zählfunktion $Q(x) = \#\{q_n \leq x\}$ gestellt werden, sind von Primzahlen und Primzahlen in arithmetischen Progressionen erfüllt. Als Anwendung ergibt sich, dass die Folge $\big((\log q_n)^K\big)$ für beliebiges $K > 1$ gleichverteilt modulo eins ist, etwa wenn die $q_n$ Primzahlen oder Primzahlen in arithmetischen Progessionen durchlaufen. Der Spezialfall das $q_n$ als die $n$te Primzahl $p_n$ gewählt wird, wurde von Too, Goto und Kano untersucht.

Im letzten Teil der Arbeit wird für irrationales $\alpha$ die Folge $(\alpha p_n)$ irrationaler Vielfacher von Primzahlen im Rahmen der gewichteten Gleichverteilung modulo eins untersucht. Nach einem Resultat von Vinogradov über Exponentialsummen ist diese Folge gleichverteilt modulo eins. Ein alternativer Beweis von Vaaler verwendet L–Funktionen. Dieser Ansatz wird im Kontext von Funktionen aus der Selberg–Klasse mit polynomiellem Eulerprodukt ausgebaut. Dabei werden zwei gewichtete Versionen des vinogradovschen Resultats gewonnen: Die Folge $(\alpha p_n)$ ist $(1 + \chi_D(p_n)) \log p_n$–gleichverteilt modulo eins, wobei $\chi_D$ den Legendre–Kronecker Charakter bezeichnet. Der Beweis verwendet die Dedekindsche Zetafunktion zum quadratischen Zahlkörper $\mathbb{Q}(\sqrt{D})$. Als Anwendung ergibt sich etwa für $D = -1$,

dass $(\alpha p_n)$ gleichverteilt modulo eins ist, wenn die durchlaufenen Primzahlen kongruent zu eins modulo vier sind. Unter zusätzlichen Bedingungen an die Funktionen aus der Selberg–Klasse lässt sich weiter zeigen, das die Folge $(\alpha p_n)$ auch $(\sum_{j=1}^{\nu_F} \alpha_j(p_n)) \log p_n$–gleichverteilt modulo eins, wobei die Gewichte in direktem Zusammenhang mit dem Eulerprodukt der Funktion stehen.

# Abstract


In the thesis at hand, several sequences of number theoretic interest will be studied in the context of uniform distribution modulo one.

In the first part we deduce for positive and real $z \neq 1$ a discrepancy estimate for the sequence $\big((2\pi)^{-1}(\log z)\gamma_a\big)$, where $\gamma_a$ runs through the positive imaginary parts of the nontrivial $a$–points of the Riemann zeta–function. If the considered imaginary parts are bounded by $T$, the discrepancy of the sequence $\big((2\pi)^{-1}(\log z)\gamma_a\big)$ tends to zero like $(\log\log\log T)^{-1}$ as $T \to \infty$. The proof is related to the proof of Hlawka, who determined a discrepancy estimate for the sequence containing the positive imaginary parts of the nontrivial zeros of the Riemann zeta–function.

The second part of this thesis is about a sequence whose asymptotic behaviour is motivated by the sequence of primes. If $\alpha \neq 0$ is real and $f$ is a function of logarithmic growth, we specify several conditions such that the sequence $(\alpha f(q_n))$ is uniformly distributed modulo one. The corresponding discrepancy estimates will be stated. The sequence $(q_n)$ of real numbers is strictly increasing and the conditions on its counting function $Q(x) = \#\{q_n \leq x\}$ are satisfied by primes and primes in arithmetic progessions. As an application we obtain that the sequence $\big((\log q_n)^K\big)$ is uniformly distributed modulo one for arbitrary $K > 1$, if the $q_n$ are primes or primes in arithmetic progessions. The special case that $q_n$ equals the $n$th prime number $p_n$ was studied by Too, Goto and Kano.

In the last part of this thesis we study for irrational $\alpha$ the sequence $(\alpha p_n)$ of irrational multiples of primes in the context of weighted uniform distribution modulo one. A result of Vinogradov concerning exponential sums states that this sequence is uniformly distributed modulo one. An alternative proof due to Vaaler uses L–functions. We extend this approach in the context of the Selberg class with polynomial Euler product. By doing so, we obtain two weighted versions of Vinogradov's result: The sequence $(\alpha p_n)$ is $(1+\chi_D(p_n))\log p_n$–uniformly distributed modulo one, where $\chi_D$ denotes the Legendre–Kronecker character. In the proof we use the Dedekind zeta-function of the quadratic number field $\mathbb{Q}(\sqrt{D})$. As an application we obtain in case of $D = -1$, that $(\alpha p_n)$ is uniformly distributed modulo one, if the considered primes are congruent to one modulo four. Assuming additional conditions on the functions from the Selberg class we prove that the sequence

$(\alpha p_n)$ is also $(\sum_{j=1}^{\nu_F} \alpha_j(p_n)) \log p_n$–uniformly distributed modulo one, where the weights are related to the Euler product of the function.

# Contents

**Danksagung** **3**

**Zusammenfassung** **6**

**Abstract** **8**

**Notations** **12**

**1 Introduction** **13**
1.1 Uniform distribution of sequences modulo one . . . . . . . . . 14
1.2 Zeta-functions . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
1.3 Motivation and Results . . . . . . . . . . . . . . . . . . . . . . 26

**2 A discrepancy estimate for $a$-points of the Riemann zeta-function** **37**
2.1 The Landau type formula . . . . . . . . . . . . . . . . . . . . 37
2.2 The discrepancy estimate . . . . . . . . . . . . . . . . . . . . 45

**3 Log-like functions and sequences with integal logarithm asymptotic** **49**
3.1 Proof of Theorem 1.3.3 . . . . . . . . . . . . . . . . . . . . . . 51
3.2 Proof of Theorem 1.3.4 . . . . . . . . . . . . . . . . . . . . . . 53
3.3 Proof of Theorem 1.3.5 . . . . . . . . . . . . . . . . . . . . . . 54

**4 Weighted uniform distribution of multiples of primes** **55**
4.1 Weights from $\mathcal{S}^{poly}$ . . . . . . . . . . . . . . . . . . . . . . . 57
4.2 Weights from the Dedekind zeta-function . . . . . . . . . . . . 58
4.3 Weights from the class $\mathcal{A}$ . . . . . . . . . . . . . . . . . . . 95

**Bibliography** **115**

# Notations

The used notation is standard in number theory (see [29]). The *Laundau symbols* $O(\cdot)$ and $o(\cdot)$ have their usual meaning, as well as the *Vinogradov symbols* $\ll$ and $\gg$.

| | |
|---|---|
| $f(x) \asymp g(x)$ | Means $f(x) \ll g(x)$ and $g(x) \ll f(x)$. |
| $f(x) \sim g(x)$ as $x \to x_0$ | Means $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, with $x_0$ possibly infinity. |
| $\operatorname{res}_{s=s_0} f(s)$ | The residue of $f(s)$ at $s = s_0$. |
| $\Gamma(s)$ | The Gamma–function, defined by $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $\operatorname{Re} s > 0$ and otherwise by analytic continuation. |
| $\zeta(s)$ | The Riemann zeta–function, defined by $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ for $\operatorname{Re} s > 1$ and otherwise by analytic continuation. |
| $N(T)$ | The number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the critical strip with $0 < \gamma \le T$. |
| $N(\sigma, T)$ | The number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the critical strip with $0 < \gamma \le T$ and $\sigma \le \beta$. |
| $\rho_a = \beta_a + i\gamma_a$ | A non–trivial $a$–point of $\zeta(s)$ with $\operatorname{Re} \rho_a = \beta_a$, $\operatorname{Im} \rho_a = \gamma_a$. |
| $N_a(T)$ | The number of $a$–points $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s)$ in the critical strip with $0 < \gamma_a \le T$. |
| $\Lambda(n)$ | The von Mangoldt function, defined by $\Lambda(n) = \log p$ if $n = p^m$ and zero otherwise. |
| $\sum_{n\le x}$ | The sum over all positive integers $n \le x$. |
| $\sum_{p\le x}$ | The sum over all primes $p \le x$. |
| $\sum_{d\mid n}$ | The sum over all positive divisors of $n$. |
| $\psi(x)$ | $= \sum_{n\le x} \Lambda(n)$. |
| $\vartheta(x)$ | $= \sum_{p\le x} \log p$. |
| $\pi(x)$ | $= \sum_{p\le x} 1$, the number of primes not exceeding $x$. |
| $\omega(n)$ | The number of distinct prime divisors of $n$. |

| | |
|---|---|
| $\mu(n)$ | The Möbius function, defined by $\mu(n) = (-1)^m$ if $n = p_1...p_m$ (different primes) and zero otherwise; and $\mu(1) = 1$. |
| $\varphi(n)$ | Euler's function, defined by $\varphi(n) = \#\{1 \leq m \leq n \ : \ m$ and $n$ are coprime$\}$. |
| $L(s, \chi)$ | The Dirichlet L–function associated to a Dirichlet character $\chi$, defined by $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-1}$ for $\mathrm{Re}\, s > 1$ and otherwise by analytic continuation. |
| $N(T, \chi)$ | The number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ in the critical strip with $\lvert\gamma\rvert \leq T$. |
| $N(\sigma, T, \chi)$ | The number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ in the critical strip with $\lvert\gamma\rvert \leq T$ and $\sigma \leq \beta$. |
| $\zeta_K(s)$ | The Dedekind zeta–function associated to an algebraic number field $K$, defined by $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$ for $\mathrm{Re}\, s > 1$ and otherwise by analytic continuation. |
| $N_K(T, \chi)$ | The number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s, \chi)$ in the critical strip with $0 \leq \gamma \leq T$. |
| $N_K(\sigma, T, \chi)$ | The number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s, \chi)$ in the critical strip with $\lvert\gamma\rvert \leq T$ and $\sigma \leq \beta$. |
| $\Lambda_K(n)$ | The generalised von Mangoldt function for a quadratic number field $K = \mathbb{Q}(\sqrt{D})$, defined by $\Lambda_K(n) = (1 + \chi_D(p)^m) \log p$ if $n = p^m$ and zero otherwise. |
| $\psi_K(x)$ | $= \sum_{n \leq x} \Lambda_K(n)$. |
| $\vartheta_K(x)$ | $= \sum_{p \leq x} (1 + \chi_D(p)) \log p$ for a quadratic number field $K = \mathbb{Q}(\sqrt{D})$. |
| $\mathcal{S}$ | The Selberg class, satisfying $(\mathcal{S}_1)$ – $(\mathcal{S}_5)$. |
| $\mathcal{S}^{\#}$ | The extended Selberg class, satisfying $(\mathcal{S}_1)$ – $(\mathcal{S}_3)$. |
| $\mathcal{S}^{poly}$ | The restricted Selberg class with polynomial Euler product. |
| $\mathcal{A}$ | The subclass of $\mathcal{S}^{poly}$, satisfying $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$. |
| $N_F^+(T)$ | The number of zeros $\rho = \beta + i\gamma$ of $F(s)$ in the critical strip with $0 \leq \gamma \leq T$. |
| $N_F^-(T)$ | The number of zeros $\rho = \beta + i\gamma$ of $F(s)$ in the critical strip with $-T \leq \gamma < 0$. |
| $N_F(\sigma, T, \chi)$ | The number of zeros $\rho = \beta + i\gamma$ of $F(s, \chi)$ in the critical strip with $\lvert\gamma\rvert \leq T$ and $\sigma \leq \beta$. |
| $\Lambda_F(n)$ | The generalised von Mangoldt function for $F \in \mathcal{S}^{poly}$, defined by $\Lambda_F(n) = (\log p) \sum_{j=1}^{\nu_F} (\alpha_j(p))^m$ if $n = p^m$ and zero otherwise. |
| $\psi_K(x)$ | $= \sum_{n \leq x} \Lambda_F(n)$. |
| $\vartheta_K(x)$ | $= \sum_{p \leq x} (\log p) \sum_{j=1}^{\nu_F} \alpha_j(p)$ for $F \in \mathcal{S}^{poly}$. |
| $[x]$ | The greatest integer not exceeding the real number $x$. |
| $\{x\}$ | $= x - [x]$, the fractional part of $x$. |

# 1 Introduction

The theory of uniformly distributed sequences modulo one is approximately one hundred years old and the subject of study is the distribution of fractional parts of sequences of real numbers. While such sequences are interesting objects on their own, since the property of being uniformly distributed modulo one is more restrictive than being dense, they also have applications in analytic number theory and numerical integration. For example, such sequences appear in the proof of Voronin's celebrated universality theorem, which, roughly speaking, states that any non-vanishing analytic function can be approximated uniformly by shifts of the Riemann zeta-function in the critical strip. On the other hand, by using uniformly distributed sequences modulo one, it is possible to improve Kronecker's famous approximation theorem. Kronecker's theorem itself can be applied to determine the least upper and greatest lower bound of the absolute value of $\zeta(\sigma + it)$ for any fixed $\sigma > 1$, where in the proof the sequence of logarithms of primes appears. Therefore, the study of sequences related to primes or the Riemann zeta-function in the context of uniformly distributed sequences is a natural one. In this thesis we investigate such sequences.

In Section 1.1 we recall the basic theory of uniformly distributed sequences and give some more information on the historical context. If no explicit sources are stated, the definitions and results can be found in the literature, i.e. in the books of Hlawka [26], Kuipers & Niederreiter [37], Drmota & Tichy [10] and Bugeaud [6].

Section 1.2 is about the Riemann zeta-function and its generalisations. If not stated otherwise, the used sources are the books of Davenport [8], Ivić [29], Karatsuba [34], Prachar [53] and Iwaniec & Kowalski [28]. The introductory part on the Selberg class mentions only what is needed and more details on this interesting topic can be found in Selberg's paper [61], the detailed introductions of Kaczorowski & Perelli [30], [31], [32] or Steuding's book [64].

The last introductory Section 1.3 states the results that are subject of this thesis and places them into context of what is already known in related lines of research. The corresponding proofs are content of the subsequent sections.

## 1.1 Uniform distribution of sequences modulo one

A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be *dense modulo one* if every interval of positive length in the unit inteval $[0,1)$ contains at least one element of the sequence of fractional parts $(\{x_n\})_{n\geq 1}$. Kronecker's approximation theorem [36] from 1884 states that the integer multiples $\alpha n$ of an irrational number $\alpha$ are dense modulo one. It seems natural to ask for more details, for example, to ask *how* the elements of the sequence are distributed modulo one: are there subintervals of the unit interval that contain only *a few* elements, while other subintervals contain *many*? The theorem of Kronecker and the resulting questions can be seen as the starting point of the theory of uniform distribution of sequences. Weyl's paper "*Über die Gleichverteilung von Zahlen mod. Eins*" [81] from 1916 was the first systematic treatment of uniformly distributed sequences.

A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be *uniformly distributed modulo one* (abbreviated u.d. mod 1) if for every pair $\alpha, \beta$ of real numbers with $0 \leq \alpha < \beta \leq 1$ the proportion of the fractional parts of the $x_n$ in the interval $[\alpha, \beta)$ exists and tends to its length in the following sense:

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \{x_n\} \in [\alpha, \beta)\}}{N} = \beta - \alpha. \tag{1.1}$$

There are several criteria to determine whether a sequence is uniformly distributed or not. We only render two of them, already contained in Weyl's mentioned paper: A sequence of real numbers $(x_n)_{n\geq 1}$ is uniformly distributed modulo one if, and only if, for any complex-valued Riemann integrable function on the closed unit interval $[0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx.$$

This criterion, that is related to a certain class of functions, may be interpreted as an early version of the celebrated Birkhoff pointwise ergodic theorem [3]. The second one, the so-called *Weyl-criterion*, is probably the most famous criterion in the theory of uniformly distributed sequences and relates the distribution property of the sequence to exponential sums: A sequence of real numbers $(x_n)_{n\geq 1}$ is uniformly distributed modulo one if, and only if,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i h x_n) = 0 \tag{1.2}$$

for all integers $h \neq 0$. Using the Weyl criterion it is not difficult to go beyond the theorem of Kronecker and to prove that the sequence $(n\alpha)_{n\geq 1}$ is u.d. mod 1 if, and only if, $\alpha$ is irrational. By this result of Weyl the theory of uniformly distributed sequences is rooted in number theory, especially diophantine approximation.

As a quantitative measure to study the distribution property of a sequence, Bergström [2] and van der Corput & Pisot [76] introduced the so-called *discrepancy* of the sequence: Let $x_1, ..., x_N$ be a finite sequence of real numbers. The number

$$\begin{aligned} D_N &= D_N\left(x_1, ..., x_N\right) \\ &= \sup_{0\le\alpha<\beta\le 1}\left|\frac{\#\{1\le n\le N : \{x_n\}\in[\alpha,\beta)\}}{N} - (\beta-\alpha)\right| \end{aligned}$$

is called the *discrepancy* of the given sequence. If $(x_n)_{n\ge1}$ is an infinite sequence (or a finite sequence containing at least $N$ terms), $D_N$ is meant to be the discrepancy of the first $N$ terms of the sequence. Since the discrepancy measures the deviation between the distribution of the sequence and the uniform distribution, a sequence is u.d. mod 1 if, and only if, its discrepancy tends to zero as $N$ tends to infinity. However, the convergence to zero can not be arbitrarily fast, since for every sequence of $N$ numbers we have $N^{-1} \le D_N \le 1$. One task of the theory is to determine upper and lower bounds for the discrepancy of special sequences. By a result of van Aardenne-Ehrenfest [75], for some positive constant $k$, every sequency satisfies

$$D_N \ge \frac{k\cdot\log\log\log N}{N}$$

for infinitely many $N$, but not for all. This lower bound has been improved by Roth [57] to

$$D_N \ge \frac{\sqrt{\log N}}{16N}$$

for infinitely many $N$, and by Schmidt [59] to

$$D_N \ge \frac{1}{66\log 4}\cdot\frac{\log N}{N}.$$

To obtain upper bounds for the discrepancy it is a fruitful method to estimate the discrepancy by means of exponential sums appearing in the Weyl criterion (1.2). Then techniques from analytic number theory to estimate these exponential sums can be applied. Two results in this direction are LeVeque's inequality [42] and the *theorem of Erdös and Turán* [11]. We only make use of the latter one: Let $x_1, ..., x_N$ be a finite sequence of real numbers and $m$ any positive integer. Then

$$D_N \le \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N}\exp\left(2\pi i h x_n\right)\right|.$$

In fact, we use a simplified version [37, equation (2.42)], which states that there exists an absolute constant $C$ such that

$$D_N \le C\cdot\left(\frac{1}{m}+\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}\exp\left(2\pi i h x_n\right)\right|\right), \tag{1.3}$$

where $m$ can be any positive integer.

There are several generalisations of the notion of uniform distributed sequences modulo one. The one we shall consider in the sequel was first studied by Tsuji [71] and is about uniform distribution with respect to weights. Tsuji's work was continued by Vaaler [72], [73], [74] and we discuss the contribution of Vaaler more detailed in the subsequent section *Motivation and Results*.

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers and $(a(n))_{n\geq 1}$ a sequence of positive real numbers. Further, let

$$A(N) = \sum_{n=1}^{N} a(n); \tag{1.4}$$

we shall refer to the $a(n)'s$ as our weights. A sequence $(x_n)_{n\geq 1}$ is said to be $a(n)$*–uniformly distributed modulo one* if for every pair $\alpha, \beta$ of real numbers with $0 \leq \alpha < \beta \leq 1$, we have

$$\lim_{N\to\infty} \frac{1}{A(N)} \sum_{\substack{n=1 \\ \alpha\leq\{x_n\}<\beta}}^{N} a(n) = \beta - \alpha \tag{1.5}$$

(provided that the limit exists and equals the right-hand side). If $a(n) = 1$ for every $n$, this leads to uniform distribution modulo one as introduced in (1.1). The Weyl criterion (1.2) generalises in the way, that the sequence $(x_n)_{n\geq 1}$ is $a(n)$–uniformly distributed modulo one if, and only if, for each integer $h \neq 0$

$$\lim_{N\to\infty} \frac{1}{A(N)} \sum_{n=1}^{N} a(n) \exp(2\pi i h x_n) = 0. \tag{1.6}$$

Similar to the classical proof in uniform distribution of sequences modulo one [37, p. 23], resp. [44, p. 480], one can use the generalised version of the Weyl criterion to prove that if $(x_n)_{n\geq 1}$ is $a(n)$–uniformly distributed modulo one, then also $(mx_n)_{n\geq 1}$ is $a(n)$–uniformly distributed modulo one, if $m$ is a non-zero integer.

Other general notions of uniform distribution by using different summability methods have been defined (see [37]). But for our purposes we consider uniform distribution with respect to a sequence of weights.

## 1.2 Zeta-functions

Let $s = \sigma + it$. Then for $\sigma > 1$ the *Riemann zeta-function* is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.7}$$

This series is converget for $\sigma > 1$, and the convergence is uniformly in every compact subset of the half plan $\sigma \geq 1 + \delta,\ \delta > 0$. In $s = 1$ the Riemann zeta-function $\zeta(s)$ has a simple pole. An important contribution of Euler to the theory of this function is its product formula

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.8}$$

for $\sigma > 1$, where the infinite product is over all primes. This formula, also called the analytic version of the fundamental theorem of arithmetic, implies that there are infinitely many primes, since in view of the simple pole at $s = 1$ the product can not converge. Euler's formula also demonstrates a close connection between the Riemann zeta-function and primes. Moreover, it implies that $\zeta(s) \neq 0$ for $\sigma > 1$.

Unless Euler, who considered the zeta-function as a function of a real variable, Riemann treated it as a complex function. He discovered that $\zeta(s)$ can be continued analytically to the whole complex plane, except for the simple pole at $s = 1$ with residue 1. Riemann also proved the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.9}$$

implying a certain symmetry around $s = \frac{1}{2}$. Since the Gamma-function has no zeros, but simple poles at $s = 0, -1, -2, ...$, the functional equation (1.9) shows the existence of *trivial zeros* of $\zeta(s)$ at $s = -2n,\ n \in \mathbb{N}$. The remaining zeros in the *critical strip* $0 \leq \sigma \leq 1$ are called *nontrivial.* Their distribution is part of the probably most famous conjecture in mathematics, the yet unsolved *Riemann hypothesis*, which states that all the nontrivial zeros of $\zeta(s)$ lie on the *critical line* $\sigma = \frac{1}{2}$.

The importance of the Riemann hypothesis lies in its connection to the distribution of primes. Hadamard and de la Vallée-Poussin proved (independently) the (*weak version* of the) *prime number theorem*

$$\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x \tag{1.10}$$

or equivalently $\vartheta(x) := \sum_{p \leq x} \log p \sim x$ resp. $\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x}$ as $x \to \infty$. Here $\psi(x)$ and $\vartheta(x)$ are *Chebyshev's functions* and $\Lambda(n)$ denotes the *von Mangoldt function.* The latter one arises from the logarithmic derivative

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (\sigma > 1), \tag{1.11}$$

being $\log p$ on prime powers $p^k$ $(k \geq 1)$ and zero otherwise. Vinogradov [79] and Korobov [35] independently obtained the best known error term

$$\psi(x) = x + O\left(x \exp\left(-C(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right),$$

and it can be shown that

$$\psi(x) - x \ll x^{\theta}(\log x)^2 \quad \Longleftrightarrow \quad \zeta(s) \neq 0 \ \text{ in } \ \sigma > \theta.$$

Let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 \leq \beta \leq 1,\ 0 < \gamma \leq T$ (counting multiplicities). Von Mangoldt proved

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + O\left(\log T\right) \tag{1.12}$$

for $T \to \infty$, which was already conjectured by Riemann. Nowadays (1.12) is called the *Riemann-von Mangoldt formula*. In particular, this formula implies that there are infinitely many zeros in the critical strip, and Hardy proved that infinitely many zeros are on the critical line. One should remark that $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$, a statement which is equivalent to the prime number theorem. Therefore by (1.9), the zeta-function does not vanish on the vertical line $\sigma = 0$, too.

The zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function in the strip $-1 \leq \sigma \leq 2$ also appear in the partial-fraction decomposition

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma - t| \leq 1}} \frac{1}{s - \rho} + O\left(\log |t|\right), \tag{1.13}$$

where $|t| \geq 2$.

Analogously to the distribution of zeros of the Riemann zeta-function one can ask about the distribution of its *a-points*, i.e. solutions of $\zeta(s) = a$ for some fixed complex number $a$. This line of investigation was initiated by Landau [40], who considered the study of the roots of $\zeta(s) = a$ of equal importance to the study of its zeros.

Landau [4] proved that for sufficiently large $n$, there is an $a$-point near every trivial zero $s = -2n$. These are called *trivial a-points*. Also for sufficiently large real part $\sigma$, there is a right half-plane which is free of $a$-points. The remaining ones in the vertical strip are called *nontrivial a-points*. Let $N_a(T)$ denote the number of $a$-points $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s)$ with $0 \leq \beta_a \leq 1,\ 0 < \gamma_a \leq T$ (counting multiplicities). Then for $T \to \infty$ the *generalised Riemann-von Mangoldt formula*

$$N_a(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e c_a}\right) + O\left(\log T\right) \tag{1.14}$$

holds, where $c_a = 1$ for $a \neq 1$ and $c_1 = 2$ (see [4]).

The Riemann zeta-function can be seen as the prototype of the several known zeta-functions. Another common function was introduced by Dirichlet in order to study the distribution of primes in arithmetic progessions.

Let $q \geq 1$ be a fixed integer. A *Dirichlet character* $\chi$ mod $q$ is an arithmetical function which is completely multiplicative, i.e. $\chi(mn) = \chi(m)\chi(n)$ for all integers $m$ and $n$, complex valued and satisfies the estimate $|\chi(n)| \leq 1$. Moreover, $\chi(n) \neq 0$ if $(n, q) = 1$, while $\chi(n) = 0$ if $(n, q) \geq 2$. There are $\varphi(q)$ characters mod $q$, where $\varphi$ denotes Euler's totient function (being the number of positive integers not exceeding $q$ which are relatively prime to $q$). Under pointwise multiplication, characters mod $q$ form an abelian group which is isomorphic to the multiplicative group of the reduced system of residues mod $q$. By $\chi_0$ we denote the *principal character*, defined by $\chi_0(n) = 1$ if $(n, q) = 1$ and zero otherwise. If $\chi$ mod $k$ is a character with $k$ dividing $q$, then $\chi$ *induces* a Dirichlet character $\psi$ mod $q$, defined by $\psi(n) = \chi(n)$ if $(n, q) = 1$ and $\psi(n) = 0$ if $(n, q) \geq 2$. A character mod $q$ which is not induced by any character mod $k$ with $k < q$ is said to be *primitive*. If $\chi$ mod $q$ is induced by another character, we denote the character inducing $\chi$ by $\chi^*$ mod $q^*$ and call $\chi$ *imprimitive*. Note that the principal character is not regarded as a primitive charcter and that every non-principal character is induced by a primitive one. We call two characters *non-equivalent*, if they are not induced by the same character. To a common modulus the characters are pairwise non-equivalent.

Let $\chi$ be a Dirichlet character mod $q$. The associated *Dirichlet L-function* is defined for $\sigma > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.15}$$

The Riemann zeta-function $\zeta(s)$ can be seen as the Dirichlet L-function to the principal character $\chi_0$ mod 1. It follows from $|\chi(n)| \leq 1$ that $L(s, \chi)$ is analytic in the half plane $\sigma > 1$ and by the complete mutiplicativity of its coefficients, there is an analogon to the Euler product formula (1.8), namely for $\sigma > 1$,

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{1.16}$$

Therefore $L(s, \chi) \neq 0$ for $\sigma > 1$. If $\chi$ mod $q$ is the principal character $\chi_0$ mod $q$, we have

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

implying that $L(s, \chi_0)$ has a simple pole at $s = 1$ arising from $\zeta(s)$. For $\chi \neq \chi_0$ the Dirichlet L-function $L(s, \chi)$ is regular. If $\chi$ mod $q$ is induced by $\chi^*$ mod $q^*$, (1.16) can we rewritten as

$$L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right) \tag{1.17}$$

Since $\chi^*(p) = 0$ for $p|q^*$, the primes that divide $q^*$ are not contained in the product in (1.17).

Like the Riemann zeta-function, any Dirichlet L-function can be continued analytically to the whole complex plane (except for a possible simple pole at $s = 1$) satisfying for primitive $\chi$ the functional equation

$$\left(\frac{\pi}{q}\right)^{-\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}} \left(\frac{\pi}{q}\right)^{-\frac{1-s+a}{2}} \Gamma\left(\frac{1-s+a}{2}\right) L(1-s,\overline{\chi}), \tag{1.18}$$

where $a = a(\chi) = 0$ if $\chi(-1) = 1$ and $a = a(\chi) = 1$ if $\chi(-1) = -1$; and $\tau(\chi)$ is the Gaussian sum

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) \exp\left(\frac{2\pi i n}{q}\right).$$

The functional equation (1.18) implies *trivial zeros* at $s = 0, -2, -4, \ldots$ if $\chi(-1) = 1$ and at $s = -1, -3, -5, \ldots$ if $\chi(-1) = -1$. The remaining zeros lie in the strip $0 \leq \sigma \leq 1$ and are called *nontrivial.*

For the logarithmic derivative of the Dirichlet L-function we obtain, for $\sigma > 1$,

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}. \tag{1.19}$$

For $1 \leq \ell \leq q$ with $(\ell, q) = 1$ define

$$\psi(x,\chi) = \sum_{n \leq x} \chi(n)\Lambda(n),$$
$$\psi(x; q, \ell) = \sum_{\substack{n \leq x \\ n \,\equiv\, \ell \ (\mathrm{mod}\ q)}} \Lambda(n).$$

The (*weak version* of the) *prime number theorem for arithmetic progessions* states that

$$\psi(x; q, \ell) \sim \frac{x}{\varphi(q)} \tag{1.20}$$

resp.

$$\pi(x; q, \ell) := \sum_{\substack{p \leq x \\ p \,\equiv\, \ell \ (\mathrm{mod}\ q)}} 1 \sim \frac{x}{\varphi(q) \log x}$$

hold as $x \to \infty$. A refinement shows that for each character $\chi$ mod $q$ there is a positive constant $c$ such that uniformly in $q \leq \exp\left(c\sqrt{\log x}\right)$

$$\psi(x,\chi) = m_\chi x - \frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \tag{1.21}$$

Here $m_\chi$ equals one if $\chi = \chi_0$ and is zero otherwise. So the term $x$ only appears if $\chi$ is the princpal character. The term $-\beta_1^{-1}x^{\beta_1}$ can be omitted unless there is a real character $\chi_1$ for which $L(s,\chi)$ has a (unique and simple) zero $\beta_1$ satisfying $\beta_1 > 1 - c'(\log x)^{-1}$, the so called *exceptional character* resp. *exceptional zero*. Also the constant $c'$ is absolute. Using

$$\psi(x;q,\ell) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(\ell)}\psi(x,\chi),$$

one obtains uniformly in $q \leq \exp\left(c\sqrt{\log x}\right)$ for the distribution of primes in arithmetic progessions

$$\psi(x;q,\ell) = \frac{x}{\varphi(q)} - \frac{\chi_1(\ell)}{\varphi(q)} \cdot \frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

This implies

$$\pi(x;q,\ell) = \varphi(q)^{-1} \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right),$$

which holds uniformly in $q \leq (\log x)^A$ for sufficienly large $A$. Note that the constant $c$ in the previous error terms may not be the same. The *generalised Riemann hypothesis*, which states that all nontrivial zeros of $L(s,\chi)$ have real part $\frac{1}{2}$, would imply that, for all $q \leq x$,

$$\psi(x;q,\ell) = \frac{x}{\varphi(q)} + O\left(x^{\frac{1}{2}}(\log x)^2\right),$$

resp.

$$\pi(x;q,\ell) = \varphi(q)^{-1} \int_2^x \frac{dt}{\log t} + O\left(x^{\frac{1}{2}} \log x\right).$$

Denoting by $N(T,\chi)$ the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ to a primitive character mod $q$ with $0 \leq \beta \leq 1$, $|\gamma| \leq T$ (counting multiplicities), there is a similar estimate to (1.12), namely

$$N(T,\chi) = \frac{T}{\pi} \log\left(\frac{qT}{2\pi}\right) - \frac{T}{2\pi} + O\left(\log(qT)\right) \tag{1.22}$$

as $T \to \infty$.

If $\chi$ mod $q$ is induced by $\chi^*$ mod $q^*$, the product formula (1.17) yields additional trivial zeros on the imaginary axis at

$$\frac{i(\arg \chi^*(p) + 2\pi m)}{\log p}, \tag{1.23}$$

where $p$ divides $q$, but does not divide $q^*$, and $m$ runs through all integers. In this case,

$$N(T,\chi) = \frac{T}{\pi} \log T + A(q)T + O\left(\log(qT)\right), \tag{1.24}$$

where $A(q)$ is real and $A(q) \ll \log(2q)$.

For $-1 \leq \sigma \leq 2$ the zeros $\rho = \beta + i\gamma$ of a L-function to a (primitive or imprimitive) character $\chi$ mod $q$ satisfying $|\gamma - t| \leq 1$ also appear in the partial-fraction decomposition

$$\frac{L'}{L}(s;\chi) = \sum_{\substack{\rho=\beta+i\gamma\\|\gamma-t|\leq 1}} \frac{1}{s-\rho} + O\left(\log(q|t|)\right), \tag{1.25}$$

where $|t| \geq 2$. If one excludes from the half-plane $\sigma \leq -1$ the points inside the disc $|s+m| \leq \frac{1}{2}$ for $m = 0, 1, 2, ...$, the logarithmic derivative can be estimated in terms of

$$\frac{L'}{L}(s;\chi) \ll \log\left(q(|s|+2)\right), \tag{1.26}$$

where $\chi \bmod q$ may be imprimitive.

The number of zeros is also the subject of another conjecture, the so-called grand density conjecture: Let $N(\sigma, T, \chi)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $\sigma \leq \beta$ and $|\gamma| \leq T$ (counting multiplicities). Then the *grand density conjecture* states that for $k \geq 1,\ Q \geq 1,\ T \geq 3$ and $\frac{1}{2} \leq \sigma \leq 1$

$$\sum_{\substack{q\leq Q\\(q,k)=1}} \sum_{\substack{\psi \pmod q\\ \psi \text{ primitive}}} \sum_{\xi \pmod k} N(\sigma, T, \xi\psi) \ll (kQ^2T)^{2(1-\sigma)} \left(\log(kQ^2T)\right)^C \tag{1.27}$$

where $C$ and the implied constant are absolute.

Dirichlet L-functions are of special interest to us since they also appear as factors of an algebraic motivated zeta-function, the so-called Dedekind zeta-function. This zeta-function was introduced by Dedekind [9] in order to study the multiplicative structure of number fields:

Let $K$ be an algebraic number field, i.e. a finite extension of $\mathbb{Q}$, and $\mathcal{O}_K$ its ring of integers. Then the associated *Dedekind zeta-function* is for $\sigma > 1$ defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1}, \tag{1.28}$$

where $\mathfrak{a}$ runs through all non-zero integral ideals of $\mathcal{O}_K$, $\mathfrak{p}$ runs through all prime ideals of $\mathcal{O}_K$ and $\mathrm{N}(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$. The (formal) equality in (1.28) is the analytic version of the fact that any non-zero integral ideal has a unique factorisation into prime ideals. Since the Dedekind zeta-function is contained in a larger class of functions which we introduce soon, we only sketch some of the results concerning the Dedekind zeta-function and refer for more details to the book of Narkiewicz [50].

If $r(n)$ denotes the number of integral ideals $\mathfrak{a}$ of norm $\mathrm{N}(\mathfrak{a}) = n$, the series in (1.28) can be rewritten as a Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s}, \tag{1.29}$$

and if $K = \mathbb{Q}$, we get the Riemann zeta-function $\zeta(s) = \zeta_{\mathbb{Q}}(s)$. Like the Riemann zeta-function, the Dedekind zeta-function can be continued analytically to the whole complex plane, except for a simple pole at $s = 1$. The residue at $s = 1$ is of special interest in algebraic number theory, since it contains information about the class number of the number field.

If $n = [K : \mathbb{Q}]$ is the degree of the number field, $r_1$ the number of real conjugate fields, $2r_2$ the number of complex conjugate fields and $d(K)$ the discriminant of $K$, the functional equation

$$A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s) = A^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_K(1-s), \tag{1.30}$$

holds, where

$$A = \frac{\sqrt{|d(K)|}}{2^{r_2} \pi^{\frac{n}{2}}}.$$

Of special interest for us is the case of a quadratic number field $K = \mathbb{Q}(\sqrt{D})$ for $D$ nonzero and squarefree. In this case,

$$\zeta_K(s) = \zeta(s) L(s, \chi_D), \tag{1.31}$$

where $\chi_D(n) = \left(\frac{D}{n}\right)$ is the *Legendre-Kronecker character* [7, Proposition 10.5.5]. For a prime $p$ we have $\left(\frac{D}{p}\right) = -1, 1$ or $0$, according to whether $p$ is inert, splits or ramifies in $K/\mathbb{Q}$.

We remark that the case of a quadratic number field is the simplest example of an abelian number field (i.e. a finite Galois extension $K$ of $\mathbb{Q}$ with abelian Galois group $\mathrm{Gal}(K/\mathbb{Q})$) and that (1.31) can be generalised to abelian number fields.

The Riemann zeta-function, Dirichlet L-functions and Dedekind zeta-functions are only three examples of a general class of functions introduced by Selberg [61]. This class, the so-called *Selberg class* $\mathcal{S}$, is defined by the following axioms:

($\mathcal{S}_1$) (*Dirichlet series*) Every $F \in \mathcal{S}$ is a Dirichlet series $F(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$, absolutely convergent for $\sigma > 1$.

($\mathcal{S}_2$) (*Analytic continuation*) There exists an integer $m_F \geq 0$ such that $(s-1)^{m_F} F(s)$ is an entire function of finite order.

($\mathcal{S}_3$) (*Functional equation*) $F$ satisfies a functional equation of the form
$$\Phi(s) = \omega\overline{\Phi(1-\overline{s})},$$
where
$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$
say, with $r \geq 0,\ Q > 0, \lambda_j > 0, \operatorname{Re}\mu_j \geq 0$ and $\omega$ satisfying $|\omega| = 1$ are parameters depending on $F$.

($\mathcal{S}_4$) (*Ramanujan conjecture*) $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

($\mathcal{S}_5$) (*Euler product*) $\log F(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$ for every $\sigma > 1$, where $b(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Besides the mentioned Riemann zeta-function, shifted Dirichlet L-functions (to a primitive Dirichlet character) and Dedekind's zeta-function to an algebraic number field, the Selberg class contains numerous L-functions appearing in number theory, for example Hecke's L-function (to a primitive Hecke character), L-functions associated with a holomorphic newform of a congruence subgroup of $SL(2,\mathbb{Z})$ (after suitable normalization) and the Rankin-Selberg convolution of two normalized holomorphic newforms. Some further functions belong conjecturally to the Selberg class, such as Artin L-functions and the $GL(n,K)$ automorphic L-functions.

The class of functions only satisfying axioms ($\mathcal{S}_1$), ($\mathcal{S}_2$) and ($\mathcal{S}_3$) is called the *extended Selberg class* $\mathcal{S}^{\#}$. Another class is defined by variation of the Euler product ($\mathcal{S}_5$). In general, this product can be rewritten in the standard form
$$F(s) = \prod_p F_p(s), \qquad \text{with} \qquad F_p(s) = \sum_{m=0}^{\infty} \frac{a\left(p^m\right)}{p^{ms}},$$
where the $F_p(s)$ are called the *local factors*. In fact, in all known cases of functions from the Selberg class the local factors are of *polynomial type*, i.e.
$$F_p(s) = \prod_{j=1}^{\nu_F} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \tag{1.32}$$
where $\nu_F > 0$ is an integer (not depending on $p$) and $|\alpha_j(p)| \leq 1$ for all $j$ and $p$. This is expected to hold for all functions from the Selberg class. The corresponding subclass, consisting of every function in $\mathcal{S}$ with polynomial Euler product, is denoted by $\mathcal{S}^{poly}$.

Selberg also introduced the notion of twisting: Given a function $F \in \mathcal{S}$ and a primitive Dirichlet character $\chi$, the *multiplicative twist* of $F$ by $\chi$ is defined by
$$F(s;\chi) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^s}. \tag{1.33}$$

It is a priori not clear if for a function $F \in \mathcal{S}$ its multiplicative twist is again an element of the Selberg class. Since $|\chi(n)| \leq 1$, the Ramanujan conjecture ($\mathcal{S}_4$) is clearly satisfied and for $F \in \mathcal{S}^{poly}$ we have by the multiplicativity of Dirichlet characters an Euler product of the form

$$F(s;\chi) = \prod_p \prod_{j=1}^{\nu_F} \left(1 - \frac{\alpha_j(p)\chi(p)}{p^s}\right)^{-1}. \tag{1.34}$$

But in this general context of functions, analytic continuation seems to be a difficult problem.

For $\sigma > 1$ the logarithmic derivative of $F \in \mathcal{S}$ has an absolutely convergent Dirichlet series representation

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} \qquad \text{with} \qquad \Lambda_F(n) = b(n)\log n, \tag{1.35}$$

where $\Lambda_F(n)$ is the *generalised von Mangoldt function* (supported on prime powers in view of ($\mathcal{S}_5$)). For $F \in \mathcal{S}^{poly}$ we have for prime powers

$$\Lambda_F(p^k) = (\log p) \sum_{j=1}^{\nu_F} (\alpha_j(p))^k. \tag{1.36}$$

Using the generalised von Mangoldt function, one can define for real $x \geq 0$ the *generalised Chebyshev $\psi$-function* associated with $F$ by

$$\psi_F(x) = \sum_{n \leq x} \Lambda_F(n). \tag{1.37}$$

It is expected (and motivated by (1.10), for example) that for every $F \in \mathcal{S}$ the *prime number theorem*

$$\psi_F(x) = m_F x + o(x), \qquad (x \to \infty) \tag{1.38}$$

holds, where $m_F$ from ($\mathcal{S}_2$) is the order of the pole at $s = 1$. Equivalent to the prime number theorem (1.38) is the non-vanishing on the vertical line $\operatorname{Re} s = 1$ (see [32, Theorem 1]):

$$\psi_F(x) = m_F x + o(x) \qquad \Longleftrightarrow \qquad F(1+it) \neq 0 \quad \forall t \in \mathbb{R}. \tag{1.39}$$

Some notions in the theory of the Selberg class are motivated by the classical theory of L-functions: The strip $0 \leq \sigma \leq 1$ is called the *critical strip*, while the line $\sigma = \frac{1}{2}$ is called the *critical line*. The zeros of $F(s)$ arising from the poles of $\gamma(s)$, i.e. $\rho = -\frac{k+\mu_j}{\lambda_j}$ with $k = 0, 1, \ldots$ and $j = 1, \ldots, r$ are called *trivial zeros* and are located in the half-plane $\sigma \leq 0$. Their multiplicity $m(\rho)$ is the number of pairs $(k, j)$ with $-\frac{k+\mu_j}{\lambda_j} = \rho$. The case $\rho = 0$ requires special attention, since by $(\mathcal{S}_3)$ and a possible pole of $F(s)$ at $s = 1$, its multiplicity is

$$m(0) = \#\{(k,j) : k + \mu_j = 0,\ k = 0, 1, \ldots \text{ and } j = 1, \ldots, r\} - m_F.$$

Remaining zeros of $F(s)$ are located in the critical strip and are called *nontrivial zeros*; the half-plane $\sigma > 1$ contains no zeros.

For $F \in \mathcal{S}$ with zeros $\rho = \beta + i\gamma$, there is a Riemann-von Mangoldt type formula: Let

$$N_F^+(T) = \{\rho : F(\rho) = 0,\ 0 \leq \beta \leq 1 \text{ and } 0 \leq \gamma \leq T\}$$

(counting multiplicities) and

$$N_F^-(T) = \{\rho : F(\rho) = 0,\ 0 \leq \beta \leq 1 \text{ and } -T \leq \gamma < 0\},$$

(counting multiplicities), then

$$N_F^+(T) = N_F^-(T) = \frac{d_F}{2\pi} T \log T + c_1 T + O(\log T), \tag{1.40}$$

where $d_F = 2\sum_{j=1}^{r} \lambda_j$ is the *degree* of $F$ and $c_1$ is a constant not depending on $F$.

For $-1 \leq \sigma \leq 2$ the zeros $\rho = \beta + i\gamma$ satisfying $|\gamma - t| \leq 1$ also appear in the partial-fraction decomposition

$$\frac{F'}{F}(s) = \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma - t| \leq 1}} \frac{1}{s - \rho} + O\left(\log |t|\right), \tag{1.41}$$

where $|t| \geq 2$ and the implied constant may depend on the data of $F$ (see [62, p. 839]).

## 1.3 Motivation and Results

The **first part** of this thesis is dedicated to the distribution of $a$-points of the Riemann zeta-function. It was Rademacher [54] in 1956 who noticed that the sequence of ordinates of the nontrivial zeros of the Riemann zeta-function is uniformly distributed modulo one, if one assumes the Riemann hypothesis. That this result holds unconditional, was proved by Elliott [12] in 1972. The precise statement is the following:

*Let $z \neq 1$ be a positive real number and $\gamma$ ranges through the set of positive imaginary parts of the nontrivial zeros of $\zeta(s)$ in ascending order. Then the sequence $\left(\frac{\log z}{2\pi}\gamma\right)_\gamma$ is uniformly distributed modulo one.*

Letting $\alpha = (2\pi)^{-1} \log z$, this implies that for all real $\alpha \neq 0$ the sequence $(\alpha\gamma)_\gamma$ is u.d. mod 1 and specially for $\alpha = 1$ the sequence of ordinates itself is u.d. mod 1.

Hlawka [27] in 1990 was the first to prove a discrepancy estimate for this sequence:

*Let* $z > 1$ *be a real number and* $D_T$ *the discrepancy of the sequence* $\left(\frac{\log z}{2\pi}\gamma\right)_\gamma$. *Then for* $T \to \infty$

$$D_T = O\left(\frac{\log z}{\log\log T}\right) \tag{1.42}$$

*and by assuming the Riemann hypothesis*

$$D_T = O\left(\frac{\log z}{\log T}\right).$$

In addition to the theorem of Erdös and Turán (1.3), one important part of his proof is an improvement of a result of Landau [38], whose theorem states that *for real* $x > 1$ *and* $T \to \infty$

$$\sum_{0<\gamma\leq T} x^\rho = -\Lambda(x)\frac{T}{2\pi} + O\left(\log T\right), \tag{1.43}$$

*where the sum is taken over all nontrivial zeros with imaginary part* $0 < \gamma \leq T$ *and the implied constant depends on* $x$. Since with $\rho$ also $1-\rho$ is a nontrivial zero, there is analogon of (1.43) for $0 < x < 1$, namely that *for* $T \to \infty$,

$$\sum_{0<\gamma\leq T} x^\rho = -x\Lambda\left(\frac{1}{x}\right)\frac{T}{2\pi} + O\left(\log T\right).$$

Hlawka determined the dependency on $x$ in the error term in (1.43), i.e. that $O(\log T)$ can be replaced by $O\left(x^2\log T\right)$. The therefore known $x$-dependency enabled him to choose the parameter $m$ in (1.3) best possible.

An improved discrepancy estimate was obtained by Fujii [17] in 1993, namely that (1.42) can be replaced by

$$D_T = O\left(\frac{\log\log T}{\log T}\right),$$

where the implied constant depends on $z$. Later, in 2002, Fujii [18] proved *that for every* $\varepsilon > 0$ *and all* $T > T_0$ *with* $T^{-\frac{1}{3}} \ll \log z \ll \log T$,

$$D_T = O\left(\frac{1}{\left(\frac{\log T}{\log z}\right)^{1-\varepsilon}}\right),$$

*where the implied constant depends on* $\varepsilon$*; and for all* $T > T_0$ *with* $\sqrt{\frac{\log T}{T\log\log T}} \ll \log z \ll \log T$,

$$D_T = O\left(\frac{\log\left(\frac{\log T}{\log z}\right)}{\frac{\log T}{\log z}}\right).$$

However, we are interested in the distribution of $a$-points of the Riemann zeta-function. It was Steuding [65] (resp. [66]) who in 2011 generalised the Landau type formula (1.43) with respect to an application in uniform distribution theory (but other generalisations are possible, see Fujii [14], [15], [16], Gonek [20], Koczorowski, Languasco & Perelli [33], and for the Selberg class M.R. & V.K. Murty [45] and in a recently submitted paper, Sourmelidis, Srichan & Steuding [63]). We remark that there is an inaccuracy in Steuding's paper [66], namely that the function $\zeta'(s)(\zeta(s)-a)^{-1}$ appearing in (1.44) has no Dirichlet series representation for $a = 1$. To state Steuding's result, we introduce some notation:

Let $x \neq 1$ be a positive real number, $n \geq 2$ an integer and $a$ a complex number. If $a \notin \{0, 1\}$, let $\alpha(x)$ for $x = n$ be the $n$th coefficient of the Dirichlet series

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n \geq 2} \frac{\alpha(n)}{n^s}, \tag{1.44}$$

and $\alpha(x) = 0$ otherwise. If $a = 0$, let $\Lambda(\frac{1}{x})$ for $x = \frac{1}{n}$ be the $n$th coefficient of the Dirichlet series

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \geq 2} \frac{\Lambda(n)}{n^s}, \tag{1.45}$$

and $\Lambda(\frac{1}{x}) = 0$ otherwise. Steuding's theorem states that *for* $\varepsilon > 0$, $a \neq 1$ *and* $T \to \infty$

$$\sum_{0 < \gamma_a \leq T} x^{\rho_a} = \left(\alpha(x) - x\Lambda\left(\frac{1}{x}\right)\right) \frac{T}{2\pi} + O\left(T^{\frac{1}{2}+\varepsilon}\right), \tag{1.46}$$

*where the implied constant depends on* $x$. In a master thesis [58], supervised by Steuding, the appearing error term in (1.46) was improved to $O\left((\log T)^2\right)$, where again the implied constant depends on $x$.

Using (1.46) and a theorem of Levinson [43], Steuding proved that the ordinates of nontrivial $a$-points are uniformly distributed modulo one, precisely:

*Let* $z \neq 1$ *be a positive real number,* $a \neq 1$ *and* $\gamma_a$ *ranges through the set of positive imaginary parts of the nontrivial* $a$*-points of* $\zeta(s)$ *in ascending order. Then the sequence* $\left(\frac{\log z}{2\pi}\gamma_a\right)_{\gamma_a}$ *is uniformly distributed modulo one.*

Analogously to the sequence of ordinates of zeros, this implies that for all real $\alpha \neq 0$ the sequence $(\alpha\gamma_a)_{\gamma_a}$ is u.d. mod 1 and specially for $\alpha = 1$ the sequence of ordinates itself is u.d. mod 1 for $a \neq 1$.

We remark that the aforementioned paper [63] contains a generalised result concerning subsequences of $a$-points of functions from the Selberg class.

The aim in the first part of this thesis is to prove for $a \neq 1$ a discrepancy estimate for nontrivial $a$-points, motivated by Hlawka's result (1.42). Therefore, we transfer Hlawka's approach to the case of $a$-points. In a first step, we determine the dependency on $x$ in the improved error term $O\left((\log T)^2\right)$

in (1.46). We adopt the notation for $\alpha(x)$ and $\Lambda(\frac{1}{x})$ with respect to (1.44) and (1.45) and prove what will be the major part of the first section of this thesis:

**Theorem 1.3.1** *Let $x \neq 1$ be a positive real number, $a \neq 1$ be a complex number and $B_a = 2 + \frac{1}{|a-1|}$. Then*

$$\sum_{0<\gamma_a \leq T} x^{\rho_a} = \left(\alpha(x) - x\Lambda\left(\frac{1}{x}\right)\right)\frac{T}{2\pi} + O\left(\max\{x^{-1}, x^{B_a}\}\,(\log T)^2\right),$$

*as $T \to \infty$.*

The resulting discrepancy estimate follows straightforward: We denote by $D_T$ the discrepancy of the sequence $\left(\frac{\log z}{2\pi}\gamma_a\right)_{\gamma_a}$, where $(\gamma_a)$ ranges through the set of positive imaginary parts of the nontrivial $a$-points of $\zeta(s)$ in ascending order, and obtain

**Theorem 1.3.2** *Let $z \neq 1$ be a positive real number, $a \neq 1$ be a complex number and $B_a = 2 + \frac{1}{|a-1|}$. Then, as $T \to \infty$,*

$$D_T = O\left(\frac{|\log z|}{\log\log\log T}\right) \tag{1.47}$$

*where the implied constant depends only on $B_a$.*

This result is part of [55], except for $a = 1$ in view of the mentioned inaccuracy, which was not corrected in [55]. In comparison with Hlawka's result (1.42), there is one additional iterated logarithm in the denominator in (1.47). In a paper to appear, Baluyot and Gonek [1] proved that this one more iterated logarithm can be omitted. They also consider the case $a = 1$.

The **second part** of this thesis is about sequences whose asymptotic behaviour is motivated by the asymptotic behaviour of primes. There are many results in the theory of uniformly distributed sequences related to primes (we only mention a few of them, for an extensive list see [67]): Let $p_n$ be the $n$th prime number in ascending order. Vinogradov [77], [78], [80] proved that for a fixed irrational number $\alpha$, the sequence $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1. For powers of primes, he obtained that $(p_n^\theta)_{n\geq 1}$ is u.d. mod 1 for every fixed non-integral $\theta > 1$. Stux [68] proved that this also holds for $0 < \theta < 1$, extending the result of Piatetski-Shapiro [52] for $\frac{2}{3} < \theta < 1$. These cases fit well in the general case of sequences $(f(p_n))_{n\geq 1}$, where $f$ is a polynomial or a more general function. The relation of this sequences to the Riemann zeta-function was studied by Schoißengeier [60].

We are interested in the general case $(\alpha f(p_n))_{n\geq 1}$, where $\alpha$ is a real constant and $f$ satisfies some additional growth condition. The case where

$f(x) = o\left((\log x)^K\right)$ and $K > 0$ has been studied by Too [70] and Goto & Kano [21], [22]. Their results imply that the sequence $\left((\log p_n)^K\right)_{n\geq 1}$ is u.d. mod 1 for $K > 1$, while Wintner [82] proved that $(\log p_n)_{n\geq 1}$ is not u.d. mod 1. It seems that the case $0 < K < 1$ has not yet been studied.

We extend the results of Too, Goto and Kano to a more general sequence, defined as follows: Let $(q_n)_{n\geq 1}$ be a sequence of real numbers, satisfying $1 < q_1 < q_2 < \ldots$ with $q_n \to \infty$ as $n \to \infty$. Assume further that the sequence $(q_n)_{n\geq 1}$ satisfies

$$Q(x) - c\int_2^x \frac{dt}{\log t} \ll \frac{x}{(\log x)^k} \tag{1.48}$$

for every $k > 1$, where $Q(x) := \sum_{q_n \leq x} 1$ and $c > 0$ is some fixed constant.

Condition (1.48) holds for the sequence of primes, where $c = 1$, and for the sequence of primes in an arithmetic progression, where $c = \varphi(q)^{-1}$ ($\varphi$ is Euler's totient function and $q$ the modulus). Sequences $(q_n)_{n\geq 1}$ related to primes have been studied by Grosswald and Schnitzer [23], who defined a corresponding Euler product and studied its relation to the Riemann zeta-function.

For the strictly increasing sequence $(q_n)_{n\geq 1}$ satisfying (1.48), we prove (as in the corresponding paper [56]) three theorems. The first one has an application to sequences containing logarithms.

**Theorem 1.3.3** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is twice differentiable with $f' > 0$,*
*(b.) $x^2 f''(x) \to -\infty$ as $x \to \infty$,*
*(c.) $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nonincreasing for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any non-zero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{-q_N^2 f''(q_N)}} + \frac{1}{(\log q_N)(-q_N^2 f''(q_N))}$$

*as $N \to \infty$.*

A comparison of Theorem 1.3.3 with [70, Theorem 3] shows that the *non-decreasing* condition is replaced by *non-increasing*. It was already remarked in [67] that this replacement is necessary.

An application of Theorem 1.3.3 to the function $f(x) = (\log x)^K$ with an arbitrary but fixed $K > 1$ yields that the sequence $\left((\log q_n)^K\right)_{n\geq 1}$ is u.d. mod 1 and generalises the former mentioned example of the sequence $\left((\log p_n)^K\right)_{n\geq 1}$.

A shorter proof of Wintner's result, that the sequence $(\log p_n)_{n\geq 1}$ is not u.d. mod 1, can be found in the book of Parent [51, Exercise 5.19]. Here only the asymptotic of the counting function of the sequence is of importance. Therefore, also the sequence $(\log q_n)_{n\geq 1}$ is not uniformly distributed modulo one if (1.48) is replaced by $Q(x) \sim x(\log x)^{-1}$, as $x \to \infty$. A sequence satisfying $Q(x) \sim x(\log x)^{-1}$ as $x \to \infty$ is, for example, the sequence where each $q_n$ fulfills $p_n \leq q_n \leq p_{n+1}$ (see [23], [47]).

The next two theorems are variations of Theorem 1.3.3:

**Theorem 1.3.4** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is twice differentiable with $f' > 0$,*
*(b.) $x^2 f''(x) \to \infty$ as $x \to \infty$,*
*(c.) $(\log x)^2 f''(x)$ is nonincreasing for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any non-zero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{q_N^2 f''(q_N)}}$$

*as $N \to \infty$.*

**Theorem 1.3.5** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is continuously differentiable,*
*(b.) $xf'(x) \to \infty$ as $x \to \infty$,*
*(c.) $(\log x) f'(x)$ is monotone for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any non-zero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \max\left\{\frac{1}{N}, \frac{1}{q_N f'(q_N)}\right\}$$

*as $N \to \infty$.*

It should be remarked that in view of [21, Theorem 1] in Theorem 1.3.5 a exchanging of conditions $(a.)$ and $(b.)$ by

*(a.')* $f$ is continuously differentiable and $f(x) \to \infty$ as $x \to \infty$,
*(b.')* $x\,|f'(x)| \to \infty$ as $x \to \infty$,

would lead to the same discrepancy estimate, where $f'(q_N)$ has to be replaced by $|f'(q_N)|$.

An interesting question that arises naturally in this context is, whether if in analogy to Vinogradov's result on the sequence $(\alpha p_n)_{n\geq 1}$, the sequence $(\alpha q_n)_{n\geq 1}$ with $p_n \leq q_n < p_{n+1}$ is also u.d. mod 1 for irrational $\alpha$. Obviously, in view of condition $(d.)$, none of our previous theorems can be applied to this case. By [6, Theorem 1.7], which requires that $\liminf_{n\to\infty}(q_{n+1}-q_n) > 0$, this sequence is u.d. mod 1 for almost all real numbers $\alpha$. But for prime numbers we know this result only to hold for irrational $\alpha$.

The **third part** of this thesis is dedicated to the study of the sequences $(\alpha p_n)_{n\geq 1}$ in the context of weighted uniform distribution. We are interested in a special case, first considered in Vaaler's PhD-thesis [72], where $(x_n)_{n\geq 1}$ and $(a(n))_{n\geq 1}$ satisfy the conditions

$(\mathcal{H}_1)$ $0 \leq x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ with $x_n \to \infty$ as $n \to \infty$,

$(\mathcal{H}_2)$ the series $\displaystyle\sum_{n=1}^{\infty} a(n)e^{-\delta x_n}$ converges if, and only if, $\delta > 0$.

In this case define for a complex variable $z$ with real part $\delta > 0$ the general Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{-zx_n}. \tag{1.49}$$

This series is analytic in the half-plane $\delta > 0$ and $f(z) \to \infty$ as $\delta \to 0+$.

Since in our investigation the coefficients of (1.49) are connected to ordinary Dirichlet series, it will be of advantage to use the complex variable $z$ for the general Dirichlet series (1.49) and the complex variable $s$ for the ordinary Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$.

Letting $a(n) = \Lambda(n)$ and $x_n = n$, Vaaler gave an alternative proof of Vinogradov's result that that the sequence $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1 for irrational $\alpha$, without using exponential sums and Weyl's criterion (1.2). Instead, his proof is based on his theorem [72, p. 35 ff.], [73]:

*Let $(x_n)_{n\geq 1}$ and $(a(n))_{n\geq 1}$ satisfy the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, and let $f(z)$ be as in (1.49). If $f(z)$ is meromorphic in an open set containing the closed half-plane $\delta \geq 0$ and has a pole of positive order at $z = 0$, then for any fixed real $\alpha > 0$ the following conditions are equivalent:*

*(i) the sequence $(\alpha x_n)_{n\geq 1}$ is $a(n)$-uniformly distributed modulo one,*

*(ii) for each integer $h \neq 0$,*

$$\lim_{\delta\to 0+} \frac{f(\delta + 2\pi i\alpha h)}{f(\delta)} = 0. \tag{1.50}$$

Hence his poof is based on the study of general Dirichlet series. We remark that Vaaler's paper [74] contains a reformulation of this theorem.

He also remarked in [72, p. 37], [73] the following *special case* of his theorem, that *if* (1.49) *is meromorphic in an open set containing the closed* *half-plane* $\delta \geq 0$, *then* (1.50) *holds if* (1.49) *has a pole of positive order at* $z = 0$ *and is analytic at the vertical line* $z = it$, $t \in \mathbb{R} \setminus \{0\}$. Considering the logarithmic derivative of the Riemann zeta-function, he proved that the sequence $(\alpha \log p_n)_{n\geq 1}$ is $p_n^{-1} \log p_n$-uniformly distributed modulo one for every $\alpha > 0$.

We generalise both results, i.e. the result for the sequence $(\alpha p_n)_{n\geq 1}$ and for $(\alpha \log p_n)_{n\geq 1}$, in the context of the polynomial Selberg class $\mathcal{S}^{poly}$. In a first step we obtain for the sequence of logarithms

**Lemma 1.3.6** *Let* $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$ *be a positive real number for every prime* $p$, $m_F \geq 1$ *and* $F(1 + it) \neq 0$ *for all* $t \in \mathbb{R}$. *Then the sequence* $(\alpha \log p_n)_{n\geq 1}$ *is* $A(p_n)p_n^{-1} \log p_n$*-uniformly distributed modulo one for every* $\alpha \neq 0$.

For the sequence of irrational multiples of primes we obtain two generalisations of Vaaler's result; one related to the Dedekind zeta-function of a quadratic number field $K = \mathbb{Q}(\sqrt{D})$ and one related to a subclass of $\mathcal{S}^{poly}$, which we will denote by $\mathcal{A}$. In both cases we study for a Dirichlet character $\chi$ mod $q$ the general Dirichlet series

$$f(z) = \sum_{n\geq 1} \Lambda_j(n) e^{-nz} \tag{1.51}$$

resp.

$$f(z; \chi) = \sum_{n\geq 1} \Lambda_j(n)\chi(n) e^{-nz}, \tag{1.52}$$

where $z = \delta + i\xi$, $\delta > 0$, $j \in \{K, F\}$. We use the index $K$ in the generalised von Mangoldt function in the number field case and the index $F$ in the case of the mentioned subclass of $\mathcal{S}^{poly}$. In the first case we prove the following

**Theorem 1.3.7** *Let* $\alpha$ *be irrational and* $\chi_D$ *the Legendre-Kronecker character of the quadratic number field* $K = \mathbb{Q}(\sqrt{D})$. *Then the sequence* $(\alpha p_n)_{n\geq 1}$ *is* $(1 + \chi_D(p_n)) \log p_n$*– u.d. mod 1, where* $p_1, p_2, ...$ *is the sequence of primes for which* $1 + \chi_D(p_n)$ *is positive.*

In his thesis [72, p. 12], Vaaler also studied the exchange of sequences of weights: Assume two sequences of weights are given, $a(n)$ and $b(n)$, say, and consider the cases

($\mathcal{W}_1$) $(x_n)_{n\geq 1}$ is $a(n)$–uniformly distributed modulo 1,

($\mathcal{W}_2$) $(x_n)_{n\geq 1}$ is $b(n)$–uniformly distributed modulo 1,

for a sequence $(x_n)_{n\geq 1}$ of real numbers. If ($\mathcal{W}_1$) implies ($\mathcal{W}_2$), we say that $b(n)$ is *stronger* than $a(n)$. Vaaler proved that *if* $a(n)$ *is a sequence of weights,* $\gamma(n)$ *a monotone sequence (either increasing or decreasing) of positive real numbers and if* *(i)* $\sum_{n=1}^{N} a(n)\gamma(n)$ *diverges as* $N \to \infty$,

*(ii)* $\gamma(N) \sum_{n=1}^{N} a(n) \ll \sum_{n=1}^{N} a(n)\gamma(n)$ *for all* $N \geq 1$,

*then* $a(n)\gamma(n)$ *is stronger than* $a(n)$. This implies in combination with Theorem 1.3.7

**Corollary 1.3.8** *Let* $\alpha$ *be irrational and* $\chi_D$ *the Legendre-Kronecker character of the quadratic number field* $K = \mathbb{Q}(\sqrt{D})$*. The sequence* $(\alpha p_n)_{n\geq 1}$ *is* $(1+\chi_D(p_n))$*– u.d. mod 1, where* $p_1, p_2, ...$ *is the sequence of primes for which* $1 + \chi_D(p_n)$ *is positive.*

As an application we obtain for $D = -1$, that $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1, if every prime number in this sequence is congruent to one modulo four. In case of $D = -3$, the corollary implies that $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1, if every prime number in the considered sequence is congruent to one modulo three. More examples can be obtained by considering the primes that split in the quadratic number field $\mathbb{Q}(\sqrt{D})$.

We emphasize that in the case of a quadratic number field no assumption on the corresponding Dedekind zeta-function is needed and the required infomation can be obtained from the underlying functions in (1.31). In fact, the regularity in the subsequent assumption $(\mathcal{A}_2)$ would not hold if we twist the Dedekind zeta-function with $\chi_D$.

The second generalisation of Vaaler's result is related to a subclass of $\mathcal{S}^{poly}$, where we suppose that $F$ satisfies the following *assumptions*

$(\mathcal{A}_1)$ $\Lambda_F(p^k)$ is a non-negative real number on prime powers $p^k, k \geq 1$.

$(\mathcal{A}_2)$ If $\chi$ is a primitive Dirichlet character, then the multiplicative twist $F(\cdot, \chi)$ of $F$ is an element of $\mathcal{S}^{poly}$ and is regular at $s = 1$.

$(\mathcal{A}_3)$ For $x \geq 2$,

$$\sum_{n\leq x} \chi(n)\Lambda_F(n) = m_F x - \frac{x^{\beta_1}}{\beta_1} + O\left(c_1 x \exp\left(-c_2\sqrt{\log x}\right)\right),$$

uniformly in $q \leq \exp\left(c_2\sqrt{\log x}\right)$, where $m_F$ is positive for $\chi = \chi_0$ and zero otherwise. The implied constant and $c_2 > 0$ are absolute, $c_1$ is a fixed constant which may depends on $F$, and $\beta_1$ ($\frac{1}{2} < \beta_1 < 1$) is the one exceptional zero of $F$, if it exists. If not, the term $-\beta_1^{-1}x^{\beta_1}$ can be omitted.

$(\mathcal{A}_4)$ For the multiplicative twist $F(\cdot, \chi)$ of $F$ let $N_F(\sigma, T, \chi)$ denote the number of non-trivial zeros $\beta + i\gamma$ with $\sigma \leq \beta$ and $|\gamma| \leq T$ (counted multiplicities). Then, for $\sigma \geq \frac{1}{2}$,

$$\sum_{\chi \pmod q} N_F(\sigma, T, \chi) \ll (qT)^{c(1-\sigma)+\varepsilon} \left(\log(qT)\right)^{\check{c}},$$

where $\frac{5}{2} - 2\varepsilon > c \geq 0, \check{c} \geq 0$ are some fixed constants and $\varepsilon > 0$.

We denote the subclass of $\mathcal{S}^{poly}$ satisfying assumptions $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$ by $\mathcal{A}$. Our main result in this context is

**Theorem 1.3.9** *Let $\alpha$ be irrational, $F \in \mathcal{A}$ and $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$. Then the sequence $(\alpha p_n)_{n\geq 1}$ is $A(p_n)\log p_n$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $A(\cdot)$ is positive.*

Exchanging the weights in combination with Theorem 1.3.9 yields

**Corollary 1.3.10** *Let $\alpha$ be irrational and $F \in \mathcal{A}$. Then $(\alpha p_n)_{n\geq 1}$ is $A(p_n)$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $A(\cdot)$ is positive.*

Among the yet known functions that are contained in $\mathcal{S}^{poly}$, only the Riemann zeta-function satisfies the assumptions of the class $\mathcal{A}$, where one has to assume that the grand density conjecture (1.27) holds, in order to satisfy $(\mathcal{A}_4)$. Therefore, assuming the grand density conjecture, our general approach yields the already known result that the sequence $(\alpha p_n)_{n\geq 1}$ is uniformly distributed modulo one for irrational $\alpha$. Since from Vaaler's proof [72] it is known that this result holds unconditionally if we take into account the zero-denity estimates [48, p. 143] instead of $(\mathcal{A}_4)$, this may indicate that the assumptions on the class $\mathcal{A}$ can be improved.

# 2 A discrepancy estimate for $a$-points of the Riemann zeta-function

The aim of this section is to prove the Landau type formula in Theorem 1.3.1 and the corresponding discrepancy estimate in Theorem 1.3.2. Therefore, we determine the dependency on $x$ in the improved error term $O\left((\log T)^2\right)$ in (1.46), which yields to

**Theorem 1.3.1** *Let $x \neq 1$ be a positive real number, $a \neq 1$ be a complex number and $B_a = 2 + \frac{1}{|a-1|}$. Then*

$$\sum_{0<\gamma_a \leq T} x^{\rho_a} = \left(\alpha(x) - x\Lambda\left(\frac{1}{x}\right)\right)\frac{T}{2\pi} + O\left(\max\{x^{-1}, x^{B_a}\}\left(\log T\right)^2\right),$$

*as $T \to \infty$.*

As a consequence of this theorem, we can prove the claimed discrepancy estimate.

**Theorem 1.3.2** *Let $z \neq 1$ be a positive real number, $a \neq 1$ be a complex number and $B_a = 2 + \frac{1}{|a-1|}$. Then, as $T \to \infty$,*

$$D_T = O\left(\frac{|\log z|}{\log\log\log T}\right)$$

*where the implied constant depends only on $B_a$.*

The proof of Theorem 1.3.1 is the content of the first section of this chapter. In the second section of the chapter we prove the discrepancy estimate.

## 2.1 The Landau type formula

Before we state the proof of Theorem 1.3.1, we establish two lemmas which we will be needed for the proof. The first lemma yields an estimate for the coefficients of the Dirichlet series (1.44):

**Lemma 2.1.1** *Let $a \neq 1$ be a fixed complex number and $B_a = 2 + \frac{1}{|a-1|}$. Then for every integer $n \geq 2$,*

$$\alpha(n) \ll n^{B_a}.$$

**Proof**
Consider the Dirichlet series

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n \geq 2} \frac{\alpha(n)}{n^s} \tag{2.1}$$

whose abscissa of absolute convergence $\sigma_a$ is less than or equal to $B_a = 2 + \frac{1}{|a-1|}$ (this has already been proved in [65] and we give some more details in the proof of Theorem 1.3.1). It is known (see [24], [64]) that the abscissa of absolute convergence satisfies

$$\sigma_a = \limsup_{N \to \infty} \frac{\log(\sum_{n=2}^{N} |\alpha(n)|)}{\log N},$$

resp.

$$\sigma_a = \limsup_{N \to \infty} \frac{\log(\sum_{n=N+1}^{\infty} |\alpha(n)|)}{\log N},$$

according to whether $\sum_{n=2}^{\infty} |\alpha(n)|$ is divergent or not. This implies in the latter case $\sum_{n \geq N+1} |\alpha(n)| \leq N^{B_a} + \varepsilon$ for every positive $\varepsilon$ and every sufficiently large $N$. Therefore,

$$|\alpha(N+1)| = \sum_{n \geq N+1} |\alpha(n)| - \sum_{n \geq N+2} |\alpha(n)| \leq N^{B_a}.$$

The case when $\sum_{n=2}^{\infty} |\alpha(n)|$ is divergent can be treated analogously, implying $\alpha(n) \ll n^{B_a}$ for $n \geq 2$ and $a \neq 1$. In case of $a = 0$, the Dirichlet series (2.1) is the logarithmic derivative of the Riemann zeta-function. Its coefficients are given by $-\Lambda(n)$ $(n \geq 2)$, the negative of von Mangoldt's function, and are of size $\log n \ll n^{B_a}$. □

The second lemma we need establishs an estimate for $\zeta'(s)(\zeta(s) - a)^{-1}$ in some vertical strip:

**Lemma 2.1.2** *Let $a$ be a fixed complex number and $s = \sigma + it$ a complex number with $-1 \leq \sigma \leq 2$ and $t \geq 1$. Then*

$$\frac{\zeta'(s)}{\zeta(s) - a} = O\left((\log t)^2\right),$$

*as $t \to \infty$.*

**Proof**
We use from [19] that for a fixed complex number $a$ and $s = \sigma + it$ with $-1 \leq \sigma \leq 2,\ |t| \geq 1$ the partial fraction decomposition

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{|t-\gamma_a|\leq 1} \frac{1}{s - \rho_a} + O\left(\log\left(|t| + 1\right)\right), \tag{2.2}$$

holds, where the sum is taken over all $a$-points $\rho_a = \beta_a + i\gamma_a$ satisfying $|t - \gamma_a| \leq 1$. In view of the generalised Riemann-von Mangoldt formula (1.14), we obtain that the number of summands in (2.2) is

$$\#\{\rho_a : |t - \gamma_a| \leq 1\} = N_a(t+1) - N_a(t-1) + O(1) = O\left(\log t\right).$$

From [19, equation (11)]) it is also known that for the summands in (2.2) for sufficiently large $t$ the estimate

$$\frac{1}{s - \rho_a} \ll \log t$$

holds, implying the upper bound $O\left((\log t)^2\right)$ for (2.2) for sufficiently large $t \geq 1$. $\square$

We are now in a position to prove the Landau type formula we need for the discrepancy estimate.

**Proof of Theorem 1.3.1**
We can adopt parts of Steuding's proof from [65] (resp. [66]). Assume $a \neq 1$. Since $\zeta(s) - a$ has a convergent Dirichlet series representation for sufficiently large $\sigma$, there exists a half-plane $\sigma > B_a$ which is free of $a$-points. To compute such an abscissa $B_a$ explicitly, we assume $\sigma > 1$ and estimate

$$|\zeta(s) - 1| \leq \sum_{n\geq 2} n^{-\sigma} < \int_1^\infty u^{-\sigma} du = \frac{1}{\sigma - 1}.$$

Thus,

$$\zeta(s) - a \neq 0 \qquad \text{for} \qquad \sigma > 1 + \frac{1}{|a-1|}.$$

As a consequence of Landau's theorem [39, Satz 5], with $\zeta(s) - a$ also $(\zeta(s) - a)^{-1}$ has a convergent Dirichlet series expansion in the same half-plane. Multiplying with the convergent Dirichlet series for $\zeta'(s)$, i.e.

$$\zeta'(s) = \sum_{n\geq 2} \frac{\log n}{n^s}$$

for $\sigma > 1$, yields the convergent Dirichlet series expansion

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n\geq 2} \frac{\alpha(n)}{n^s}.$$

In case of $a = 0$ this equals the logarithmic derivative of the zeta-function

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\geq 2} \frac{\Lambda(n)}{n^s},$$

so that in case of $a = 0$ we have $\alpha(n) = -\Lambda(n)$. We observe that both series have no constant term since the series for $\zeta'(s)$ has no constant term. The abscissa of convergence and the abscissa of absolute convergence of an ordinary Dirichlet series differ by at most one (see [24, Theorem 9]), so the abscissa of absolute convergence of the Dirichlet series (2.1) is less than or equal to $B_a = 2 + |a-1|^{-1}$ for $a \neq 1$. In view of the generalised Riemann-von Mangoldt formula (1.14), for any positive $T_0$ we can find some $T \in [T_0, T_0+1)$ such that the distance between $T$ and the nearest ordinate $\gamma_a$ is bounded by $(\log T)^{-1}$. Let $b := 1 + (\log T)^{-1}$; then only finitely many (nontrivial) $a$-points lie to the left of the vertical line $\sigma = 1-b$. Since the logarithmic derivative of $\zeta(s) - a$ has simple poles at each $a$-point with residue equal to the order, the residue theorem implies

$$\sum_{0<\gamma_a\leq T} x^{\rho_a} = \frac{1}{2\pi i}\int_{\mathcal{R}} x^s \frac{\zeta'(s)}{\zeta(s)-a}ds + O(1), \tag{2.3}$$

where $\mathcal{R}$ denotes the counterclockwise oriented rectangle with vertices $B_a + i, B_a + iT, 1-b+iT, 1-b+i$. The error term arrises from a possible contribution of $a$-points on the left of the vertical line $\sigma = 1-b$ and the area between $\mathcal{R}$ and the real axis. We rewrite the appearing integral in (2.3) as

$$\begin{aligned}\int_{\mathcal{R}} x^s \frac{\zeta'(s)}{\zeta(s)-a}ds &= \left\{\int_{B_a+i}^{B_a+iT} + \int_{B_a+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i} + \int_{1-b+i}^{B_a+i}\right\} x^s \frac{\zeta'(s)}{\zeta(s)-a}ds \\ &=: \sum_{j=1}^{4} I_j,\end{aligned}$$

say, and estimate each integral individually. In $I_1$ we interchange the order of summation and integration and obtain

$$I_1 = \sum_{n\geq 2} \alpha(n) i \int_1^T \left(\frac{x}{n}\right)^{B_a+it} dt.$$

If $x = n$,

$$\int_1^T \left(\frac{x}{n}\right)^{B_a+it} dt = T-1,$$

while for $x \neq n$,

$$\int_1^T \left(\frac{x}{n}\right)^{B_a+it} dt \ll \left(\frac{x}{n}\right)^{B_a} \frac{1}{\left|\log\left(\frac{x}{n}\right)\right|}.$$

Therefore,

$$I_1 = \alpha(x)iT + O\left(|\alpha(x)|\right) + O\left( x^{B_a} \Big\{ \sum_{\substack{n \\ 2 \le n < x}} + \sum_{n>x} \Big\} \frac{|\alpha(n)|}{n^{B_a} \left|\log\left(\frac{x}{n}\right)\right|} \right).$$

Lemma 2.1.1 yields an estimate for the first error term containing $\alpha(x)$. For real $x > 2$ let $c_x = x - 1$ for $x \in \mathbb{N}^{>2}$ and $c_x = [x]$ for $\mathbb{R}^{>2} \setminus \mathbb{N}$. Then

$$\left|\log\left(\frac{x}{n}\right)\right|^{-1} \le \left|\log\left(\frac{x}{c_x}\right)\right|^{-1}$$

and by Lemma 2.1.1

$$x^{B_a} \sum_{\substack{n \\ 2 \le n < x}} \frac{|\alpha(n)|}{n^{B_a} \left|\log\left(\frac{x}{n}\right)\right|} = \chi_{\{x>2\}}(x) O\left( \frac{x^{2B_a+1}}{2^{B_a} \left|\log\left(\frac{x}{c_x}\right)\right|} \right),$$

where $\chi$ denotes the characteristic function. For the remaining error term we note that $n > x$ implies $\left|\log\left(\frac{x}{n}\right)\right| = \log\left(\frac{n}{x}\right)$ and the lower bound for the possible values of $n$ yields

$$\log\left(\frac{1}{x}\left([x]+1\right)\right) \le \log\left(\frac{n}{x}\right).$$

Therefore,

$$x^{B_a} \sum_{n>x} \frac{|\alpha(n)|}{n^{B_a} \left|\log\left(\frac{x}{n}\right)\right|} \le \frac{x^{B_a}}{\log\left(\frac{1}{x}\left([x]+1\right)\right)} \sum_{n>x} \frac{|\alpha(n)|}{n^{B_a}},$$

where the appearing series $\sum_{n>x} |\alpha(n)|\, n^{-B_a}$ converges. So for $I_1$ we obtain, after case-by-case analysis for $c_x$,

$$I_1 = \alpha(x)iT + O\left( \frac{x^{2B_a+1}}{\log\left(\frac{1}{x}\left([x]+1\right)\right)} \right). \tag{2.4}$$

To estimate $I_2$ we apply Lemma 2.1.2 on the interval $\left[-(\log T)^{-1}, 2\right]$ and use the convergence of the Dirichlet series for $\zeta(s)(\zeta(s)-a)^{-1}$ on $(2, B_a]$ to obtain

$$I_2 = -\int_{-\frac{1}{\log T}}^{B_a} x^{\sigma+iT} \frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)-a} d\sigma \ll (\log T)^2 \int_{-\frac{1}{\log(T)}}^{B_a} x^{\sigma} d\sigma.$$

Estimating the integral on the right-hand side trivially, yields

$$I_2 \ll \chi_{\{x>1\}}(x)(\log T)^2 x^{B_a} + \chi_{\{0<x<1\}}(x)(\log T)^2 x^{-\frac{1}{\log T}}. \tag{2.5}$$

To investigate the integral $I_3$ we seperate it as follows:

$$I_3 = -\left\{\int_{1-b+i}^{1-b+it_0} + \int_{1-b+it_0}^{1-b+iT}\right\} x^s \frac{\zeta'(s)}{\zeta(s)-a}ds := I_{3,1} + I_{3,2},$$

where $t_0 \in (1, T)$. Again we estimate the first integral trivially, which leads to $I_{3,1} \ll x^{-\frac{1}{\log T}}$, whereas the integral $I_{3,2}$ will be seperated further: Since there are only finitely many $a$-points on the left of the vertical line $\sigma = 1-b$ and $T$ is increasing, it is possible to choose $t_0$ such that there is no $a$-point on the boundary and inside the rectangle with vertices $1-b+it_0, 1-b+iT, -1+iT, -1+it_0$. Then the integrand is holomorphic on this rectangle and by Cauchy's theorem,

$$I_{3,2} = -\left\{\int_{1-b+it_0}^{-1+it_0} + \int_{-1+it_0}^{-1+iT} + \int_{-1+iT}^{1-b+iT}\right\} x^s \frac{\zeta'(s)}{\zeta(s)-a}ds =: J_0 + J_1 + J_2,$$

say. For the integrals $J_0$ and $J_2$ we get by Lemma 2.1.2 and trivial estimation of the remaining integrals

$$J_0 \ll \chi_{\{x>1\}}(x) x^{-\frac{1}{\log(T)}} + \chi_{\{0<x<1\}}(x)x^{-1}$$

resp.

$$J_2 \ll \chi_{\{x>1\}}(x)(\log T)^2 x^{-\frac{1}{\log T}} + \chi_{\{0<x<1\}}(x)(\log T)^2 x^{-1}.$$

To estimate $J_1$ we use some ideas from Garunkštis and Steuding [19]: By their result [19, Lemma 4] there exists a $t_0$ such that $\left|\frac{a}{\zeta(s)}\right| < \frac{1}{2}$ for $s = -1+it$ for all $t \geq t_0$. For such values of $t$ we may expand into a geometric series

$$\frac{\zeta'(s)}{\zeta(s)-a} = \frac{\zeta'}{\zeta}(s) \cdot \frac{1}{1-\frac{a}{\zeta(s)}} = \frac{\zeta'}{\zeta}(s)\left(1 + \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k\right).$$

Therefore,

$$-J_1 = \int_{-1+it_0}^{-1+iT} x^s \frac{\zeta'}{\zeta}(s)ds + \int_{-1+it_0}^{-1+iT} x^s \frac{\zeta'}{\zeta}(s)\sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k ds =: l_1 + l_2.$$

We first consider $l_1$ and use that the functional equation (1.9) can be rewritten as

$$\zeta(s) = \Delta(s)\zeta(1-s), \qquad \text{where} \qquad \Delta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right),$$

implying

$$\frac{\zeta'}{\zeta}(s) = \frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s) \quad \text{with} \quad \frac{\Delta'}{\Delta}(\sigma+it) = -\log\left(\frac{t}{2\pi}\right) + O\left(\frac{1}{t}\right),$$

valid for $t > 1$ (see [19, equation (6)]). Using this formula, we get

$$l_1 = ix^{-1}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2}\int_{t_0}^{T}(xn)^{it}dt + ix^{-1}\int_{t_0}^{T}x^{it}\left(-\log\left(\frac{t}{2\pi}\right)+O\left(\frac{1}{t}\right)\right)dt$$
$$=: l_{1,1} + l_{1,2}.$$

For the integral in $l_{1,1}$ we find for $x = \frac{1}{n}$,

$$\int_{t_0}^{T}(xn)^{it}dt = T - t_0,$$

while for $n \neq \frac{1}{x}$,

$$\int_{t_0}^{T}(xn)^{it}dt \ll \frac{1}{|\log(xn)|}.$$

Therefore,

$$l_{1,1} = ix\Lambda\left(\frac{1}{x}\right)T + O\left(\Lambda\left(\frac{1}{x}\right)x\right) + O\left(\frac{1}{x}\Big\{\sum_{\substack{n\\1\le n<\frac{1}{x}}} + \sum_{n>\frac{1}{x}}\Big\}\frac{\Lambda(n)}{n^2\,|\log(xn)|}\right).$$

In view of $\Lambda\left(\frac{1}{x}\right) \le |\log x|$,

$$O\left(x\Lambda\left(\frac{1}{x}\right)\right) = O\left(x\,|\log x|\right).$$

For real $\frac{1}{x} > 1$ let $\tilde{c}_x = \frac{1}{x} - 1$ for $\frac{1}{x} \in \mathbb{N}^{>1}$ and $\tilde{c}_x = \left[\frac{1}{x}\right]$ for $\frac{1}{x} \in \mathbb{R}^{>1} \setminus \mathbb{N}$. Then

$$\frac{1}{|\log(xn)|} \le \frac{1}{|\log(x\tilde{c}_x)|}$$

and

$$\frac{1}{x}\sum_{\substack{n\\1\le n<\frac{1}{x}}}\frac{\Lambda(n)}{n^2\,|\log(xn)|} = \chi_{\{\frac{1}{x}>1\}}(x)O\left(\frac{\log\left(\left[\frac{1}{x}\right]\right)}{x^2\,|\log(x\tilde{c}_x)|}\right).$$

For the remaining error term we note that $n > \frac{1}{x}$ implies $|\log(xn)| = \log(xn)$ and the lower bound for the possible values of $n$ yields

$$\log\left(x\left(\left[\frac{1}{x}\right]+1\right)\right) \le \log(xn).$$

Therefore,

$$\frac{1}{x}\sum_{n>\frac{1}{x}}\frac{\Lambda(n)}{n^2\,|\log(xn)|} \le \frac{1}{x\log\left(x\left(\left[\frac{1}{x}\right]+1\right)\right)}\sum_{n>\frac{1}{x}}\frac{\Lambda(n)}{n^2},$$

where the series on the right-hand side converges. So we obtain

$$l_{1,1} = ix\Lambda\left(\frac{1}{x}\right)T + O\left(x\left|\log x\right|\right) + \chi_{\{\frac{1}{x}>1\}}(x)O\left(\frac{\log\left(\left[\frac{1}{x}\right]\right)}{x^2\left|\log\left(x\tilde{c}_x\right)\right|}\right) + \\ + O\left(\frac{1}{x\log\left(x\left(\left[\frac{1}{x}\right]+1\right)\right)}\right).$$

For $l_{1,2}$ we use $t_0 \in (1,T)$ to estimate the second integral and get

$$l_{1,2} = -ix^{-1}\int_{t_0}^{T} x^{it}\log\left(\frac{t}{2\pi}\right)dt + O\left(\frac{\log T}{x}\right).$$

Applying partial integration, we obtain

$$\int_{t_0}^{T} x^{it}\log\left(\frac{t}{2\pi}\right)dt = \left[\log\left(\frac{t}{2\pi}\right)\frac{e^{it\log x}}{i\log x}\right]_{t=t_0}^{t=T} - \int_{t_0}^{T}\frac{e^{it\log x}}{it\log x}dt \\ \ll \frac{\log T}{\left|\log x\right|}$$

which yields

$$l_{1,2} = O\left(\frac{\log T}{x\left|\log x\right|}\right) + O\left(\frac{\log T}{x}\right).$$

The bounds for $l_{1,1}$ and $l_{1,2}$ imply an estimate for $l_1$ and it remains to estimate $l_2$. We note that by [5, Satz 2.6.1] for the complex number $s = -1 + it$ with $t \geq t_0 \in (1,T)$ the logarithmic derivative of the zeta function fulfills

$$\frac{\zeta'}{\zeta}(s) = O\left(\log t\right).$$

By [19, Lemma 4] we also have for the fixed complex number $a$, $\sigma < 0$ and $|t| \geq 2$

$$\frac{a}{\zeta(\sigma+it)} \ll \frac{\log t}{t^{\frac{1}{2}-\sigma}}.$$

Therefore,

$$l_2 \ll \frac{T\log T}{x}\sum_{k\geq 1}\left(\frac{\log T}{T^{\frac{3}{2}}}\right)^k \ll \frac{T^{-\frac{1}{2}}\left(\log T\right)^2}{x}.$$

In view of

$$I_3 = I_{3,1} + \left(J_0 - \left(\left(l_{1,1} + l_{1,2}\right) + l_2\right) + J_2\right)$$

we obtain

$$I_3 = -ix\Lambda\left(\frac{1}{x}\right)T + \chi_{\{x>1\}}(x)O\left((\log T)^2 x^{-\frac{1}{\log T}}\right) \\ + \chi_{\{0<x<1\}}(x)O\left((\log T)^2 x^{-1}\right).$$

For the integral $I_4$ we use from [66, p. 689] the estimate $I_4 \ll 1+x^{B_a}$. These estimates, in combination with the estimate for $I_1$ and $I_2$, resp. (2.4) and (2.5), imply for (2.3) that

$$\sum_{0<\gamma_a\leq T} x^{\rho_a} = \left(\alpha(x) - x\Lambda\left(\frac{1}{x}\right)\right)\frac{T}{2\pi} + O\left(\max\{x^{-1}, x^{B_a}\}(\log T)^2\right).$$

□

## 2.2 The discrepancy estimate

Let $a \neq 1$ be a complex number, $x \neq 1$ a positive real number and $\hat{B}_a$ be the upper bound for the real parts of the $a$-points, i.e. $\hat{B}_a = 1 + \frac{1}{|a-1|}$ for $a \neq 1$. As in [66], we use that

$$\exp(y) - 1 = \int_0^y \exp(t)dt \ll |y| \max\{1, \exp(y)\}$$

implies

$$\left|x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}\right| \leq x^{\beta_a}\left|e^{\left(\frac{1}{2}-\beta_a\right)\log x} - 1\right| \leq \left|\beta_a - \frac{1}{2}\right| \max\{x^{\frac{1}{2}}, x^{\hat{B}_a}\}\,|\log x|\,.$$

In addition, from Levinson's paper [43] (cf. [66, equation (32)]) it is known that

$$\sum_{0<\gamma_a\leq T}\left|\beta_a - \frac{1}{2}\right| \ll \frac{T\log T}{\log\log T}$$

as $T \to \infty$. Therefore,

$$\sum_{0<\gamma_a\leq T} x^{\frac{1}{2}+i\gamma_a} = \sum_{0<\gamma_a\leq T} x^{\beta_a+i\gamma_a} + O\left(\max\{x^{\frac{1}{2}}, x^{\hat{B}_a}\}\,|\log x|\,\frac{T\log T}{\log\log T}\right).$$

Let $\kappa(x)$ be $\alpha(x)$ if $x = n$ for some integer $n \geq 2$, $-x\Lambda\left(\frac{1}{x}\right)$ if $x = \frac{1}{n}$ for some integer $n \geq 2$, and zero otherwise. Then Theorem 1.3.1 implies

$$\begin{aligned}\sum_{0<\gamma_a\leq T} x^{\frac{1}{2}+i\gamma_a} &= \kappa(x)\frac{T}{2\pi} + O\left(\max\{x^{-1}, x^{B_a}\}(\log T)^2\right) + \\ &\quad + O\left(\max\{x^{\frac{1}{2}}, x^{\hat{B}_a}\}\,|\log x|\,\frac{T\log T}{\log\log T}\right) \\ &= \kappa(x)\frac{T}{2\pi} + O\left(\max\{x^{-1}, x^{B_a}\}\frac{T\log T}{\log\log T}\right).\end{aligned}$$

Multiplication with $x^{-\frac{1}{2}}$ and letting $x = z^h$, where $z \neq 1$ is a positive real number and $h$ is a positive integer, yields

$$\sum_{0<\gamma_a\leq T} z^{ih\gamma_a} = \frac{\kappa(z^h)}{2\pi z^{\frac{h}{2}}}T + O\left(\max\{z^{-\frac{3}{2}h}, z^{h(B_a-\frac{1}{2})}\}\frac{T\log T}{\log\log T}\right)$$

resp.

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} e^{ih(\log z)\gamma_a} = \frac{\kappa(z^h)}{2\pi z^{\frac{h}{2}}} \cdot \frac{T}{N_a(T)} + \\ + O\left(\max\{z^{-\frac{3}{2}h}, z^{h(B_a-\frac{1}{2})}\} \frac{T\log T}{N_a(T)\log\log T}\right).$$

Taking into account the generalised Riemann-von Mangoldt formula (1.14),

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} e^{ih(\log z)\gamma_a} = \frac{\kappa(z^h)}{2\pi z^{\frac{h}{2}}} \cdot \frac{1}{\log T} + \\ + O\left(\max\{z^{-\frac{3}{2}h}, z^{h(B_a-\frac{1}{2})}\} \frac{1}{\log\log T}\right).$$

We want to determine an upper bound for the terms on the right side. In view of $(\log\log T)^{-1} \geq (\log T)^{-1}$ for $T \geq 3$, we consider the terms containing $z^h$:

If $z \geq 2$ is an integer $n$, say, then so is $z^h = n^h$ and by Lemma 2.1.1

$$\frac{\kappa(z^h)}{z^{\frac{h}{2}}} \ll n^{h(B_a-\frac{1}{2})}.$$

Up to a constant factor this is the $z$-term in our previous $O(\cdot)$-term.

If $z$ is the reciprocal of an integer $n \geq 2$, then $z^h = \frac{1}{n^h}$ and

$$\frac{\kappa(z^h)}{z^{\frac{h}{2}}} = -\frac{\Lambda(n^h)}{n^{\frac{h}{2}}} \ll \frac{\log(n^h)}{n^{\frac{h}{2}}},$$

while $z^{-\frac{3}{2}h} = n^{\frac{3}{2}h}$. Therefore, in the second case,

$$\frac{\kappa(z^h)}{z^{\frac{h}{2}}} \ll z^{-\frac{3}{2}h}.$$

As a common bound we obtain

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} e^{ih(\log z)\gamma_a} \ll \max\{z^{-\frac{3}{2}h}, z^{h(B_a-\frac{1}{2})}\} \cdot \frac{1}{\log\log T}.$$

Substituting this in the simplified version (1.3) of the inequality of Erdös and Turán yields

$$D_T \ll \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} e^{ih(\log z)\gamma_a} \right| \\ \ll \frac{1}{m} + \frac{1}{\log\log T} \sum_{h=1}^{m} \frac{\max\{z^{-\frac{3}{2}h}, z^{h(B_a-\frac{1}{2})}\}}{h}.$$

For further estimations, we use from Hlawka's paper [27] that for real $\omega > 1$ we have

$$\sum_{h=1}^{m} \frac{\omega^h}{h} \ll \frac{\omega^m}{m}. \tag{2.6}$$

In case of $z > 1$ this implies for $\omega = z^{B_a - \frac{1}{2}}$ that

$$D_T \ll \frac{1}{m} + \frac{1}{\log\log T} \cdot \frac{z^{m(B_a - \frac{1}{2})}}{m}.$$

By letting

$$m := \left[ \frac{\log\log\log T}{(B_a - \frac{1}{2}) \log z} \right] + 1$$

we obtain for $T \to \infty$

$$D_T \ll \frac{(B_a - \frac{1}{2}) \log z}{\log\log\log T} \ll \frac{\log z}{\log\log\log T},$$

which is Theorem 1.3.2 for $z > 1$, with an implied constand to depend on $B_a$.

If on the other hand $0 < z < 1$, then $z^{-1} > 1$ and for $\omega = z^{-\frac{3}{2}}$ we obtain from (2.6)

$$D_T \ll \frac{1}{m} + \frac{1}{\log\log T} \cdot \frac{z^{-\frac{3}{2}m}}{m}.$$

We let

$$m := \left[ -\frac{2}{3} \cdot \frac{\log\log\log T}{\log z} \right] + 1$$

and obtain

$$D_T \ll \frac{-\log z}{\log\log\log T} \ll \frac{|\log z|}{\log\log\log T}$$

as $T \to \infty$, which is Theorem 1.3.2 for $0 < z < 1$. □

# 3 Log-like functions and sequences with integal logarithm asymptotic

In this section we prove our Theorems 1.3.3, 1.3.4 and 1.3.5. Therefore, recall that $(q_n)_{n\geq 1}$ is a sequence of real numbers, satisfying $1 < q_1 < q_2 < \ldots$ with $q_n \to \infty$ as $n \to \infty$. Further, we assume that the sequence $(q_n)_{n\geq 1}$ satisfies the asymptotic (1.48), i.e.

$$Q(x) - c\int_2^x \frac{dt}{\log t} \ll \frac{x}{(\log x)^k}$$

for every positive $k > 1$, where $Q(x) = \sum_{q_n \leq x} 1$ and $c > 0$ is some fixed constant. For this sequence we prove in a first step

**Theorem 1.3.3** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is twice differentiable with $f' > 0$,*
*(b.) $x^2 f''(x) \to -\infty$ as $x \to \infty$,*
*(c.) $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nonincreasing for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any nonzero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{-q_N^2 f''(q_N)}} + \frac{1}{(\log q_N)(-q_N^2 f''(q_N))} \tag{3.1}$$

*as $N \to \infty$.*

The proof of this theorem is based on the version (1.3) of the theorem of Erdös and Turán. To estimate the occurring exponential sum, we rewrite the sum by calculus for Stieltjes integrals and estimate the arrising exponential integrals. In this sense, the subsequent Theorem 1.3.4 and Theorem 1.3.5 are variations of Theorem 1.3.3, resulting from different estimates for exponential integrals.

**Theorem 1.3.4** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is twice differentiable with $f' > 0$,*
*(b.) $x^2 f''(x) \to \infty$ as $x \to \infty$,*
*(c.) $(\log x)^2 f''(x)$ is nonincreasing for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any nonzero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n \geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{q_N^2 f''(q_N)}} \tag{3.2}$$

*as $N \to \infty$.*

**Theorem 1.3.5** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a, \infty) \to (0, \infty)$ satisfy the conditions*
*(a.) $f$ is continuously differentiable,*
*(b.) $x f'(x) \to \infty$ as $x \to \infty$,*
*(c.) $(\log x) f'(x)$ is monotone for sufficiently large $x$,*
*(d.) $f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*
*Then, for any nonzero real constant $\alpha$, the sequence $(\alpha f(q_n))_{n \geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \max\left\{\frac{1}{N}, \frac{1}{q_N f'(q_N)}\right\} \tag{3.3}$$

*as $N \to \infty$.*

The following estimates will be used to prove Theorem 1.3.3 and Theorem 1.3.4 and can be found in the book of Zygmund [83, Lemma 10.2, Lemma 10.3, p. 225].

**Lemma 3.0.1** *Let $G(x)$ be a positive decreasing function.*
*(a.) If $F''(x) < 0$, $F'(x) \geq 0$ and $\frac{G'(x)}{F''(x)}$ is monotone, then*

$$\left|\int_a^b G(x) e^{2\pi i F(x)} dx\right| \leq 4 \max_{a \leq x \leq b} \left\{\frac{G(x)}{|F''(x)|^{\frac{1}{2}}}\right\} + \max_{a \leq x \leq b} \left\{\left|\frac{G'(x)}{F''(x)}\right|\right\}.$$

*(b.) If $F''(x) > 0$ and $F'(x) \geq 0$, then*

$$\left|\int_a^b G(x) e^{2\pi i F(x)} dx\right| \leq 4 \max_{a \leq x \leq b} \left\{\frac{G(x)}{F''(x)^{\frac{1}{2}}}\right\}.$$

For the remaining Theorem 1.3.5 we use from the book of Titchmarsh [69, Lemma 4.3]

**Lemma 3.0.2** *Let $F(x)$ and $G(x)$ be real functions, $\frac{G(x)}{F'(x)}$ monotone and $\frac{F'(x)}{G(x)} \geq m > 0$, or $\frac{F'(x)}{G(x)} \leq -m < 0$. Then*

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq \frac{4}{m}.$$

As remarked in the introductory section on uniform distribution of sequences modulo one, the property of a sequence to be u.d. mod 1 is equivalent to the property that its discrepancy tends to zero. Therefore, it is enough to prove each of the equations (3.1), (3.2) and (3.3) and to ensure that these tend to zero. Moreover, by [37, Ex. 2.4] it is enought to prove these estimates for $\alpha > 0$ and if we replace $f$ by $\frac{1}{\alpha} f$, we see that it is enough to prove each of these estimates for $\alpha = 1$.

## 3.1 Proof of Theorem 1.3.3

Condition $(b.)$ implies that $f''(x) < 0$ for sufficiently large $x$. Therefore, we may assume that for $x \geq a$ we have $f''(x) < 0$ and that both $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ from condition $(c.)$ are nonincreasing for $x \geq a$. By condition $(b.)$ and $(d.)$, the term on the right-hand side in (3.1) tends to zero for $N$ tending to infinity. So it is enough to verify the stated discrepancy estimate (3.1) to prove that the sequence $(f(q_n))_{n \geq n_0}$ is uniformly distributed modulo one. We apply the theorem of Erdös and Turán in the version

$$D_N \ll \frac{1}{m} + \sum_{h=n_0}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=n_0}^{N} e^{2\pi i h f(q_n)} \right|, \tag{3.4}$$

where $m$ is an arbitrary positive integer to be specified later. To estimate the appearing exponential sum, we rewrite it by calculus for Stieltjes integrals. Let $q_0 := \frac{q_{n_0}+a}{2}$ and $\chi(n)$ be the characteristic function of the sequence $(q_n)_{n \geq 1}$, i.e. $\chi(n) = 1$ if $n$ is a member of the sequence $(q_n)_{n \geq 1}$ and zero otherwise. Then $\sum_{n \leq x} \chi(n) = Q(x)$. Further, let $R^*(x) := Q(x) - L^*(x)$, $L^*(x) := c \int_{q_0}^{x} \frac{dt}{\log t}$ for $c > 0$ constant, $x \geq q_0$ and $\int_a^b := \int_{(a,b]}$. Using integration by parts,

$$\begin{aligned} E(n_0, N; q_n) := \sum_{n=n_0}^{N} e^{2\pi i h f(q_n)} &= \sum_{q_0 < m \leq q_N} e^{2\pi i h f(m)} \chi(m) \\ &= \int_{q_0}^{q_N} e^{2\pi i h f(x)} \, \mathrm{d}\, Q(x) \\ &= Q(q_N) e^{2\pi i h f(q_N)} - Q(q_0) e^{2\pi i h f(q_0)} \\ &\quad - \int_{q_0}^{q_N} (L^*(x) + R^*(x)) \, \mathrm{d}\, e^{2\pi i h f(x)}. \end{aligned}$$

Again integration by parts yields

$$\int_{q_0}^{q_N} L^*(x)\,\mathrm{d}\,e^{2\pi i h f(x)} = L^*(q_N)e^{2\pi i h f(q_N)} - L^*(q_0)e^{2\pi i h f(q_0)} - c\int_{q_0}^{q_N} (\log x)^{-1} e^{2\pi i h f(x)} dx$$

and

$$\int_{q_0}^{q_N} R^*(x)\,\mathrm{d}\,e^{2\pi i h f(x)} = 2\pi i h \int_{q_0}^{q_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx.$$

Using $R^*(x) = Q(x) - L^*(x)$, we can rewrite the exponential sum in terms of

$$\begin{aligned} E(n_0, N; q_n) = &\Big(R^*(q_N)e^{2\pi i h f(q_N)} - R^*(q_0)e^{2\pi i h f(q_0)}\Big) + \\ &+ c\int_{q_0}^{q_N} (\log x)^{-1} e^{2\pi i h f(x)} dx \\ &- 2\pi i h \int_{q_0}^{q_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx \\ =:&\, I_1 + I_2 + I_3. \end{aligned}$$

We estimate each term $I_i$ $(i = 1, 2, 3)$ individually: Our assumption (1.48) implies

$$R^*(x) = Q(x) - c\int_{q_0}^{x} \frac{dt}{\log t} \ll \frac{x}{(\log x)^k} \tag{3.5}$$

for every $k > 1$, and therefore

$$I_1 \ll \frac{q_N}{(\log q_N)^{1+K}} \tag{3.6}$$

for $K > 0$ as $N \to \infty$. Estimate (3.5) can also be applied in combination with condition $(a.)$ to get the upper bound

$$I_3 \ll \frac{q_N h f(q_N)}{(\log q_N)^{1+K}}. \tag{3.7}$$

We remark that, in view of $f > 0$ and $h$ being a positive integer, (3.7) for sufficiently large $N$ is also an estimate of (3.6). To estimate $I_2$ we use Lemma 3.0.1 $(a.)$ and obtain

$$\begin{aligned} I_2 &\ll \max_{q_0 \le x \le q_N} \left\{ \frac{4}{|h(\log x)^2 f''(x)|^{\frac{1}{2}}} + \left| \frac{1}{hx(\log x)^2 f''(x)} \right| \right\} \\ &\ll \frac{1}{(\log q_N)\,(-hf''(q_N))^{\frac{1}{2}}} + \frac{1}{q_N(\log q_N)^2\,(-hf''(q_N))} \end{aligned}$$

as $N \to \infty$. This estimate, with (3.7), yields in (3.4) for $N \to \infty$

$$D_N \ll \frac{1}{m} + \frac{1}{N(\log q_N)\,(-f''(q_N))^{\frac{1}{2}}} + \frac{1}{Nq_N(\log q_N)^2\,(-f''(q_N))} + \\ + \frac{mq_N f(q_N)}{N(\log q_N)^{1+K}}.$$

We compare the first and the last term and let

$$m := \left[\left(N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)}\right)^{\frac{1}{2}}\right].$$

This choice of $m$, in combination with $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$ and $Q(q_N) = N$, leads to the discrepancy estimate (3.1). □

## 3.2 Proof of Theorem 1.3.4

Condition ($a$.) of the previous theorem remains unchangend. In view of condition ($b$.) we can assume that $f''(x) > 0$ for sufficiently large $x$. Moreover, by condition ($c$.), we may assume that for $x \geq a$ we have $f''(x) > 0$ and that $(\log x)^2 f''(x)$ is nonincreasing for $x \geq a$. We prove the discrepancy estimate (3.2) by using (3.4). For $N \to \infty$ the right-hand side of (3.2) surely tends to zero in view of conditions ($b$.) and ($d$.), so the discrepany estimate (3.2) implies that the sequence $(f(q_n))_{n \geq n_0}$ is u.d. mod 1. Since the determined estimates (3.6) and (3.7) for $I_1$ and $I_3$ still hold in our considered setting, it remains to estimate the integral $I_2$. Lemma 3.0.1 ($b$.) yields in this case

$$I_2 \ll \max_{q_0 \leq x \leq q_N} \left\{ \frac{1}{(h(\log x)^2 f''(x))^{\frac{1}{2}}} \right\} \ll \frac{1}{(h(\log q_N)^2 f''(q_N))^{\frac{1}{2}}}$$

and together with (3.6) and (3.7) we obtain in (3.4)

$$D_N \ll \frac{1}{m} + \frac{mq_N f(q_N)}{N(\log q_N)^{1+K}} + \frac{1}{\left(q_N^2 f''(q_N)\right)^{\frac{1}{2}}},$$

where we used $Q(q_N) = N$ and $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$ to obtain the third term. After comparing the first and second term in this estimate for the discrepancy, we let

$$m := \left[\left(N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)}\right)^{\frac{1}{2}}\right],$$

implying

$$D_N \ll \left(\frac{f(q_N)}{(\log q_N)^K}\right)^{\frac{1}{2}} + \frac{1}{\left(q_N^2 f''(q_N)\right)}.$$

As already remarked, this tends to zero as $N \to \infty$. □

## 3.3 Proof of Theorem 1.3.5

With respect to condition (*b.*) we can assume that $f'(x) > 0$ for sufficiently large $x$. Further, we may assume that for $x \geq a$ we have $f'(x) > 0$ and that $(\log x)f'(x)$ is monotone for $x \geq a$. In view of conditions (*b.*) and (*d.*), the right-hand side of (3.3) tends to zero as $N \to \infty$, implying that the sequence $(f(q_n))_{n\geq n_0}$ is u.d. mod 1. We adopt the previous estimates (3.6) and (3.7) for $I_1$ and $I_3$ and apply Lemma 3.0.2 to estimate the intgral $I_2$:

$$I_2 \ll \frac{1}{h} \cdot \max\left\{1, \frac{1}{[(\log q_N)f'(q_N)]}\right\}.$$

Combining this with (3.6) and (3.7) we get for (3.4)

$$\begin{aligned} D_N &\ll \frac{1}{m} + \max\left\{\frac{1}{N}, \frac{1}{N\,[(\log q_N)f'(q_N)]}\right\} + \frac{m q_N f(q_N)}{N\,(\log q_N)^{1+K}} \\ &\ll \frac{1}{m} + \max\left\{\frac{1}{N}, \frac{1}{[q_N f'(q_N)]}\right\} + \frac{m q_N f(q_N)}{N\,(\log q_N)^{1+K}}. \end{aligned}$$

Note that in the second argument in the maximum we used $Q(q_N) = N$ and $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$. After comparing the first and the last term in the discrepancy estimate, we let

$$m := \left[\left(N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)}\right)^{\frac{1}{2}}\right],$$

implying

$$D_N \ll \left(\frac{f(q_N)}{(\log q_N)^K}\right)^{\frac{1}{2}} + \max\left\{\frac{1}{N}, \frac{1}{[q_N f'(q_N)]}\right\}.$$

The right-hand side tends to zero as $N \to \infty$ in view of (*b.*) and (*d.*). □

# 4 Weighted uniform distribution of multiples of primes

The aim of this section is to prove our main theorems in the context of weighted uniform distribution modulo one, Theorem 1.3.7 and 1.3.9, as well as the corresponding Corollaries 1.3.8 and 1.3.10. Therefore we consider the sequence of irrational multiples of primes with weights related to the Dedekind zeta-function of a quadratic number field resp. the restricted class $\mathcal{A} \subset \mathcal{S}^{poly}$.

However, in the first section we prove for real multiples of logarithms of primes with weights from the local factors (1.32) of the Euler product of a function from $\mathcal{S}^{poly}$ the following

**Lemma 1.3.6** *Let $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$ be a positive real number for every prime $p$, $m_F \geq 1$ and $F(1+it) \neq 0$ for all $t \in \mathbb{R}$. Then the sequence $(\alpha \log p_n)_{n\geq 1}$ is $A(p_n)p_n^{-1}\log p_n$-uniformly distributed modulo one for every $\alpha \neq 0$.*

Then we consider irrational multiples of primes: For a quadratic number field $K = \mathbb{Q}(\sqrt{D})$ we study the general Dirichlet series (1.51), i.e.

$$f(z) = \sum_{n\geq 1} \Lambda_K(n) e^{-nz},$$

where $\Lambda_K$ is the generalised von Mangoldt function. We prove in a first step that for irrational $\alpha > 0$ and every nonzero integer $h$,

$$\lim_{\delta\to 0+} \frac{f(\delta + 2\pi i\alpha h)}{f(\delta)} = 0.$$

Then we consider the related function

$$g(z) = \sum_{p} (1 + \chi_D(p))(\log p) e^{-pz},$$

where $\chi_D$ is the Legendre-Kronecker character, and obtain as a consequence of the limit behaviour of $f$ that also

$$\lim_{\delta\to 0+} \frac{g(\delta + 2\pi i\alpha h)}{g(\delta)} = 0.$$

This implies by the corresponding theorem of Vaaler (p. 32) resp. (1.50)

**Theorem 1.3.7** *Let $\alpha$ be irrational and $\chi_D$ the Legendre-Kronecker character of the quadratic number field $K = \mathbb{Q}(\sqrt{D})$. Then the sequence $(\alpha p_n)_{n\geq 1}$ is $(1+\chi_D(p_n))\log p_n$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $1+\chi_D(p_n)$ is positive.*

Vaaler's result on the replacement of weights (see p. 34 resp. the subsequent Lemma 4.2.15) in combination with Theorem 1.3.7 yields

**Corollary 1.3.8** *Let $\alpha$ be irrational and $\chi_D$ the Legendre-Kronecker character of the quadratic number field $K = \mathbb{Q}(\sqrt{D})$. Then the sequence $(\alpha p_n)_{n\geq 1}$ is $(1+\chi_D(p_n))$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $1+\chi_D(p_n)$ is positive.*

As remarked in the introduction, this implies that for $D = -1$, the sequence $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1, if every prime number in this sequence is congruent to one modulo four. In case of $D = -3$, the corollary implies that $(\alpha p_n)_{n\geq 1}$ is u.d. mod 1, if every prime number in the considered sequence is congruent to one modulo three. More examples can be obtained by considering the primes that split in $\mathbb{Q}(\sqrt{D})$.

The proofs of Theorem 1.3.7 and Corollary 1.3.8 are the subject of the second section of this chapter.

In the third section we consider the subclass $\mathcal{A}$ of the Selberg class with polynomial Euler product $\mathcal{S}^{poly}$, introduced on page 34. We prove that

$$\lim_{\delta\to 0+} \frac{f(\delta + 2\pi i\alpha h)}{f(\delta)} = 0$$

holds, where

$$f(z) = \sum_{n\geq 1} \Lambda_F(n) e^{-nz}.$$

Here $\Lambda_F$ is the generalised von Mangoldt function arrising from the logarithmic derivative (1.35). Analogously to the case of a quadratic number field, we study the function

$$g(z) = \sum_{p} A(p)(\log p) e^{-pz},$$

where $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$ is the sum of the coefficients of the Euler product with local factors (1.32), and get

$$\lim_{\delta\to 0+} \frac{g(\delta + 2\pi i\alpha h)}{g(\delta)} = 0.$$

This yields in this general context

**Theorem 1.3.9** *Let $\alpha$ be irrational, $F \in \mathcal{A}$ and $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$. Then the sequence $(\alpha p_n)_{n\geq 1}$ is $A(p_n)\log p_n$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $A(\cdot)$ is positive.*

As a consequence we obtain by exchange of weights

**Corollary 1.3.10** *Let $\alpha$ be irrational and $F \in \mathcal{A}$. Then $(\alpha p_n)_{n\geq 1}$ is $A(p_n)$– u.d. mod 1, where $p_1, p_2, \ldots$ is the sequence of primes for which $A(\cdot)$ is positive.*

We want to point out that some parts of the proofs of Theorem 1.3.7 and 1.3.9, resp. Corollary 1.3.8 and 1.3.10, are similar. This is due to the fact that the Dedekind zeta-function of an algebraic number field is a member of the Selberg class $\mathcal{S}^{poly}$.

## 4.1 Weights from $\mathcal{S}^{poly}$

As remarked in the introductory part on uniform distribution of sequences modulo one, a consequence of (1.6) is that $\alpha > 0$ can be replaced by $-\alpha$. Therefore, we may assume $\alpha$ to be positive. Let $F \in \mathcal{S}^{poly}$. By (1.32), resp. (1.35) and (1.36), the logarithmic derivative of $F$ is

$$-\frac{F'}{F}(s) = \sum_p \sum_{k\geq 1} \left( \sum_{j=1}^{\nu_F} \alpha_j(p)^k \right) (\log p) p^{-ks},$$

for $\sigma > 1$. Separating the sum for $k = 1$ yields

$$\sum_p A(p) \log(p) p^{-1} e^{-s \log p} =$$

$$= -\frac{F'}{F}(s+1) - \sum_p \sum_{k\geq 2} \left( \sum_{j=1}^{\nu_F} \alpha_j(p)^k \right) (\log p) p^{-k(s+1)},$$

where $A(p) = \sum_{j=1}^{\nu_F} \alpha_j(p)$. The series on the right-hand side is for $\sigma > -1$ bounded by $\sum_p \frac{\log p}{p^{2(\sigma+1)}}$. By partial summation

$$\sum_{p\leq x} \frac{\log p}{p^{2(\sigma+1)}} = x^{-2(\sigma+1)} \sum_{p\leq x} \log p + 2(\sigma+1) \int_2^x \left( \sum_{p\leq y} \log p \right) y^{-2\sigma-3} dy.$$

The prime number theorem in the form $\vartheta(x) = \sum_{p\leq x} \log p \sim x$, as $x \to \infty$, implies that the series is dominated by

$$\lim_{x\to\infty} \left( x^{1-2(\sigma+1)} + 2(\sigma+1) \int_2^x y^{-2(\sigma+1)} dy \right),$$

converging for $\sigma > -\frac{1}{2}$. By assumption, $A(p)$ is a positive real number, $m_F \geq 1$ and $F(1+it) \neq 0$ for all $t \in \mathbb{R}$. Therefore, the function

$$f(s) = \sum_{n=1}^{\infty} A(p_n)(\log p_n)p_n^{-1}e^{-s\log p_n}$$

is analytic on the imaginary axis, except for a simple pole at $s = 0$. This implies Lemma 1.3.6 by the mentioned special case of Vaaler's theorem (see page 32). □

## 4.2 Weights from the Dedekind zeta-function

Let $K = \mathbb{Q}(\sqrt{D})$ be a fixed quadratic number field, where $D$ is a nonzero and squarefree integer. From (1.31) it is known that

$$\zeta_K(s) = \zeta(s)L(s,\chi_D),$$

where $\chi_D(n) = \left(\frac{D}{n}\right)$ is the Legendre-Kronecker character. We remark that $\chi_D$ is its own inverse in the group of characters.

Moreover, from (1.31) in combination with (1.8) and (1.16), we deduce

$$\zeta_K(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\left(1 - \frac{\chi_D(p)}{p^s}\right)^{-1}.$$

Its logarithmic derivative is

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n^s}. \tag{4.1}$$

The generalised von Mangoldt function is supported on prime powers, i.e.

$$\Lambda_K(p^m) = (\log p)\,(1 + \chi_D(p)^m) \tag{4.2}$$

and zero otherwise (see also (1.32), (1.35) and (1.36)). Since the Dedekind zeta-function does not vanish on the line $\sigma = 1$ (see [50, p. 343]) and has a simple pole at $s = 1$, the equivalence (1.39) implies

$$\psi_K(x) := \sum_{n\leq x} \Lambda_K(n) = x + o(x) \tag{4.3}$$

as $x \to \infty$ (one may consult for the Tauberian theorem [46] resp. [50]).

According to (1.51) and (1.52), define for a Dirichlet character $\chi$ mod $q$ the general Dirichlet series

$$f(z) = \sum_{n\geq 1} \Lambda_K(n)e^{-nz} \tag{4.4}$$

resp.

$$f(z;\chi) = \sum_{n\geq 1} \Lambda_K(n)\chi(n)e^{-nz}, \tag{4.5}$$

where $z = \delta + i\xi,\ \delta > 0$. The proof of Theorem 1.3.7 is subdivided into several parts. Our first step is to determine an asymptotic formula for $f(\delta)$:

**Theorem 4.2.1** *The series $f(\delta) = \sum_{n\geq 1} \Lambda_K(n)e^{-\delta n}$ converges if and only if $\delta > 0$. For $\delta \to 0+$ we have $f(\delta) \sim \delta^{-1}$.*

**Proof**
In view of (4.2) we get $\Lambda_K(n) \ll \log n$ and therefore

$$f(\delta) = \sum_{n\geq 1} \Lambda_K(n)e^{-\delta n} \ll \sum_{n\geq 1} (\log n)e^{-\delta n}.$$

Since the series on the right-hand side converges for $\delta > 0$, so does $f(\delta)$. For $\delta = 0$, formula (4.3) implies that the series

$$f(0) = \sum_{n\geq 1} \Lambda_K(n)$$

diverges. To prove the second part of the theorem, we apply partial integration to obtain for the partial sum

$$\begin{aligned}\sum_{n\leq x} \Lambda_K(n)e^{-\delta n} &= \int_1^x e^{-\delta y}\,\mathrm{d}\,\psi_K(y) = e^{-\delta y}\psi_K(y)\Big|_{y=1}^x + \ \delta\int_1^x e^{-\delta y}\psi_K(y)dy \\ &= e^{-\delta x}\psi_K(x) + \ \delta\int_0^x e^{-\delta y}\psi_K(y)dy,\end{aligned}$$

where we used that $\Lambda_K$ is supported on prime powers. In view of $\delta^{-1} = \delta\int_0^\infty ye^{-\delta y}dy$, we get for $x \to \infty$

$$\sum_{n\leq x} \Lambda_K(n)e^{-\delta n} = e^{-\delta x}\psi_K(x) + \delta^{-1} + \delta\int_0^x (\psi_K(y) - y)\, e^{-\delta y}dy.$$

Applying (4.3) yields

$$f(\delta) = \delta^{-1} + o\left(\delta\int_0^\infty ye^{-\delta y}dy\right) = \delta^{-1} + o(\delta^{-1})$$

or, $f(\delta) \sim \delta^{-1}$ as $\delta \to 0+$. □

In the next step of the proof of Theorem 1.3.7 we establish an upper bound for $f(\delta)^{-1}f(\delta + 2\pi i\alpha)$, whose further investigation will be the major part of the proof.

**Theorem 4.2.2** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b, q) = 1$, and let $z_0 = \delta + 2\pi i \left(\alpha - \frac{b}{q}\right)$. Denote by $\chi^*$ mod $q^*$ the primitive character inducing $\chi$ mod $q$. Then*

$$f(\delta)^{-1}|f(\delta+2\pi i\alpha)| \ll \frac{\sqrt{q}}{\varphi(q)} + \delta \log\left(\delta^{-1}\right)\sqrt{q}\log q + \\ + \delta \frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0,\chi_D^*}} |f(z_0;\chi^*)|,$$

*where $\chi_D^*$ denotes the character induced by $\chi_D$ or the character $\chi_D$ itself, if it appears among the characters modulo $q$.*

**Proof**
By Theorem 4.2.1,

$$f(\delta)^{-1} \ll \delta, \quad \text{or} \quad f(\delta) \ll \delta^{-1} \tag{4.6}$$

for sufficiently small $\delta > 0$. Also by Theorem 4.2.1,

$$f(\delta+2\pi i\alpha) = \sum_{n\geq 1} \Lambda_K(n) e^{-(\delta+2\pi i\alpha)n}$$

converges absolutely for $\delta > 0$ and by estimate (4.6),

$$|f(\delta+2\pi i\alpha)| \leq f(\delta) \ll \delta^{-1}. \tag{4.7}$$

If we consider characters mod $q$ we observe that for $q = 1$ or $q = 2$, there is no non-principal character. Therefore by (4.6) and (4.7),

$$f(\delta)^{-1}\,|f(\delta+2\pi i\alpha)| \ll \varphi(q)^{-1}$$

and the desired estimate holds. Now assume $q \geq 3$. Then (4.6) and splitting of the series yields

$$\begin{aligned} f(\delta)^{-1}\,|f(\delta+2\pi i\alpha)| &\ll \delta\,|f(\delta+2\pi i\alpha)| \\ &\ll \delta \left| \sum_{\substack{n\geq 1\\(n,q)=1}} \Lambda_K(n) e^{-n(\delta+2\pi i\alpha)} \right| + \delta \left| \sum_{\substack{n\geq 1\\(n,q)>1}} \Lambda_K(n) e^{-n(\delta+2\pi i\alpha)} \right| \\ &\ll \delta \left| \sum_{\substack{h=1\\(h,q)=1}}^{q-1} e^{-2\pi i\left(\frac{bh}{q}\right)} \sum_{\substack{n=1\\ n\equiv h \pmod q}}^{\infty} \Lambda_K(n) e^{-n\left(\delta+2\pi i\left(\alpha-\frac{b}{q}\right)\right)} \right| + \\ &\quad + \delta \sum_{p\mid q} \log p \sum_{m=1}^{\infty} e^{-\delta p^m}, \end{aligned}$$

where we used $\Lambda_K(p^k) \ll \log p$ to obtain the estimate for the second series.

To rewrite the first term, we use a character relation which states that for $(h,q)=1$,

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(h)\chi(n) = \begin{cases} 1, & \text{if } n \equiv h \pmod q, \\ 0, & \text{otherwise,} \end{cases}$$

(see [5, p. 34]), and get for $z_0 = \delta + 2\pi i\left(\alpha - \frac{b}{q}\right)$

$$\sum_{\substack{n=1 \\ n\equiv h \pmod q}}^{\infty} \Lambda_K(n) e^{-nz_0} = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(h) f(z_0;\chi).$$

For the second term in the estimate of $f(\delta)^{-1}\,|f(\delta+2\pi i\alpha)|$ we use that the summand is monotone non-increasing and therefore

$$\sum_{m=1}^{\infty} e^{-\delta p^m} \le \int_0^{\infty} e^{-\delta p^x}\,dx.$$

This yields

$$f(\delta)^{-1}\,|f(\delta+2\pi i\alpha)| \ll \delta \left| \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} e^{-2\pi i\left(\frac{bh}{q}\right)} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(h) f(z_0;\chi) \right| + \\ + \delta \sum_{p|q} \log p \int_0^{\infty} e^{-\delta p^x}\,dx.$$

For the first term on the right we recall that for a Gaussian sum

$$\tau(\chi) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \chi(h) e^{2\pi i\left(\frac{h}{q}\right)}$$

we have (by [8, p. 65])

$$\tau(\overline{\chi})\chi(b) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \overline{\chi}(h) e^{2\pi i\left(\frac{bh}{q}\right)}$$

whenever $(b,q)=1$. Thus

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} e^{-2\pi i\left(\frac{bh}{q}\right)} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(h) f(z_0;\chi) = \\ = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \tau(\overline{\chi})\chi(-b) f(z_0;\chi).$$

Substituting $y = \delta p^x$ yields for the integral in the above estimate of $f(\delta)^{-1} |f(\delta + 2\pi i\alpha)|$

$$\sum_{p|q} \log p \int_0^\infty e^{-\delta p^x} dx = \sum_{p|q} 1 \int_\delta^\infty e^{-y} y^{-1} dy,$$

and therefore

$$f(\delta)^{-1} |f(\delta + 2\pi i\alpha)| \ll$$
$$\ll \delta\varphi(q)^{-1} \left| \sum_{\chi \pmod q} \tau(\overline{\chi})\chi(-b) f(z_0;\chi) \right| + \delta \sum_{p|q} \int_\delta^\infty e^{-y} y^{-1} dy.$$

We remark that in case of $\chi = \chi_0$, the Gaussian sum is a Ramanujan sum, i.e.

$$\tau(\chi_0 \mod q) = \sum_{\substack{a \bmod q \\ (a,q)=1}} \exp\left(2\pi i \frac{a}{q}\right) = c_q(1),$$

and that

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right) \frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)}$$

(see [5, p. 20]) implies $|\tau(\chi_0 \bmod q)| = |\mu(q)| \le 1$, while in general $|\tau(\chi)| \le \sqrt{q}$ (see [8, p. 66]). We use these estimates to obtain

$$\sum_{\chi \pmod q} \tau(\overline{\chi})\chi(-b) f(z_0;\chi) =$$
$$= \tau(\overline{\chi_0})\chi_0(-b) \sum_{n\ge 1} \Lambda_K(n)\chi_0(n) e^{-nz_0} +$$
$$+ \tau(\overline{\chi_D^*})\chi_D^*(-b) \sum_{n\ge 1} \Lambda_K(n)\chi_D^*(n) e^{-nz_0} +$$
$$+ \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0, \chi_D^*}} \tau(\overline{\chi})\chi(-b) f(z_0;\chi),$$

and get therefore

$$\sum_{\chi \pmod q} \tau(\overline{\chi})\chi(-b) f(z_0;\chi) \ll (1+\sqrt{q}) \sum_{n\ge 1} \Lambda_K(n) e^{-n\delta} + \sqrt{q} \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0, \chi_D^*}} |f(z_0;\chi)| \tag{4.8}$$

as an estimate for the sum over characters.

For the integral in our estimate of $f(\delta)^{-1} |f(\delta + 2\pi i\alpha)|$ we observe that (by substituting $x = y^{-1}$),

$$\int_\delta^\infty e^{-y} y^{-1} dy = \int_0^1 x^{-1} e^{-\frac{1}{x}} dx + \int_1^{\frac{1}{\delta}} x^{-1} e^{-\frac{1}{x}} dx \ll \log\left(\delta^{-1}\right),$$

and with $\omega(q) := \sum_{p|q} 1 \ll \log q$ (see [25, p. 471]), we get

$$\sum_{p|q} \int_{\delta}^{\infty} e^{-y} y^{-1} dy \ll (\log q) \log \left(\delta^{-1}\right).$$

Therefore,

$$\begin{aligned} f(\delta)^{-1} \left| f(\delta + 2\pi i \alpha) \right| \ll \\ \ll \delta \frac{1+\sqrt{q}}{\varphi(q)} \sum_{n \geq 1} \Lambda_K(n) e^{-n\delta} + \delta \frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left| f(z_0; \chi) \right| + \\ + \delta (\log q) \log \left(\delta^{-1}\right). \end{aligned}$$

If $\chi^*$ mod $q^*$ is the primitive character inducing $\chi$ mod $q$, then

$$\begin{aligned} |f(z_0;\chi)| &= \left| \sum_{n \geq 1} \Lambda_K(n) \chi^*(n) e^{-nz_0} - \sum_{\substack{n \geq 1 \\ (n,q) > 1}} \Lambda_K(n) \chi^*(n) e^{-nz_0} \right| \\ &\ll |f(z_0;\chi^*)| + \sum_{p|q} \log p \sum_{m=1}^{\infty} e^{-\delta p^m} \\ &\ll |f(z_0;\chi^*)| + (\log q) \log \left(\delta^{-1}\right). \end{aligned}$$

Substituting this in our upper bound we obtain

$$\begin{aligned} f(\delta)^{-1} |f(\delta + 2\pi i \alpha)| \ll \\ \ll \frac{\sqrt{q}}{\varphi(q)} + \delta \log \left(\delta^{-1}\right) \sqrt{q} (\log q) + \delta \frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} |f(z_0, \chi^*)| \end{aligned}$$

by (4.6) and that the number of characters modulo $q$ is $\varphi(q)$. □

To estimate the sum over characters modulo $q$, we need to establish two theorems that enable us to determine the logarithmic derivative of the twisted Dedekind zeta-function. As a member of $\mathcal{S}^{poly}$, the multiplicative twist of the Dedekind zeta-function by a primitive Dirichlet character $\chi^*$ mod $q^*$ is by (1.29) and (1.33) defined as

$$\zeta_K(s;\chi^*) = \sum_{n=1}^{\infty} \frac{r(n)\chi^*(n)}{n^s}, \tag{4.9}$$

where $r(n)$ denotes the number of integral ideals $\mathfrak{a}$ of norm $\mathrm{N}(\mathfrak{a}) = n$. Moreover, the twisted Euler product is by (1.31) and (1.34) a product of two Dirichlet L-functions, i.e.

$$\zeta_K(s;\chi^*) = L(s;\chi^*) L(s;\chi^*\chi_D), \tag{4.10}$$

where $\chi_D \bmod |D|$ is the Legendre-Kronecker character. We remark that the product $\chi^*\chi_D$ of the two characters $\chi^* \bmod q^*$ and $\chi_D \bmod |D|$ is a character modulo $\mathfrak{q} := q^*|D|$ (see [34, p. 108]). Logarithmic differentiation of (4.10) implies

$$\frac{\zeta_K'}{\zeta_K}(s;\chi^*) = \frac{L'}{L}(s;\chi^*) + \frac{L'}{L}(s;\chi^*\chi_D). \tag{4.11}$$

**Theorem 4.2.3** *Let $m \geq 2$ be an integer and $\chi^* \bmod q^*$ a primitive Dirichlet character. There exists a sequence of real numbers $T_m = T_m(\chi^*)$ with $m < T_m < m+1$, such that for $-1 \leq \sigma \leq 2$*

$$\frac{\zeta_K'}{\zeta_K}(\sigma \pm iT_m;\chi^*) \ll (\log(\mathfrak{q}T_m))^2,$$

*where $\mathfrak{q} = q^*|D|$ and the implied constant does not depend on $m$ and $\mathfrak{q}$.*

**Proof**
The proof is related to the one for Dirichlet L-functions [53, p. 226-227], but shall be stated for the sake of completeness: Let $\gamma_{01} \leq \gamma_{02} \leq ... \leq \gamma_{0j}$ be the ordinates of the non-trivial zeros of $\zeta_K(s;\chi^*)$ with $m < t < m+1$ in the upper half-plane, where as usual $s = \sigma + it$. Denote by $N_K(T,\chi^*)$ the number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s;\chi^*)$ with $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq T$ (counted with multiplicities). Then (4.10) implies

$$N_K(T,\chi^*) = N_1(T,\chi^*) + N_2(T,\chi^*\chi_D),$$

where $N_1(T,\chi^*)$ resp. $N_2(T,\chi^*\chi_D)$ denotes the number of zeros $\rho = \beta + i\gamma$ of $L(s;\chi^*)$ resp. $L(s;\chi^*\chi_D)$ with $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq T$ (counted with multiplicities). By (1.24),

$$N_K(T,\chi^*) = \frac{T}{\pi}\log T + (A(q) + A(\mathfrak{q}))T + O(\log T)$$

which yields that the number of zeros with imaginary part $T < \gamma \leq T+1$ is

$$N_K(T+1,\chi^*) - N_K(T,\chi^*) \ll \log T \tag{4.12}$$

for $T \geq 2$. Therefore,

$$\begin{aligned} j = j(m) &= \#\{\text{ordinates of non-trivial zeros in } m < t < m+1\} \\ &\ll \log m. \end{aligned}$$

Now, if we divide the interval $(m, m+1)$ into $j+1$ equal parts, there is at least one subinterval containing none of the $\gamma_{0i}$ $(i = 1, ..., j)$. Let $T_m$ be the center of such an subinterval. Each subinterval is of length $\gg (\log m)^{-1}$, so $|T_m - \gamma_{0i}| \gg (\log m)^{-1}$ for all $i = 1, ..., j$. Therefore,

$$|T_m - \gamma_0| \gg \frac{1}{\log m} \tag{4.13}$$

if $\gamma_0$ is the ordinate of an arbitrary zero of $\zeta_K(s;\chi^*)$. Taking into account the partial-fraction decomposition (1.25) related to $\chi^*$ and $\chi^*\chi_D$ in combination with (4.11),

$$\frac{\zeta_K'}{\zeta_K}(s;\chi^*) = \sum_{\substack{\rho_0 \\ |\gamma_0 - t| \leq 1}} \frac{1}{s-\rho_0} + O\left(\log(\mathfrak{q}|t|)\right),$$

where the sum is taken over all zeros $\rho_0 = \beta_0 + i\gamma_0$ of $\zeta_K(s;\chi^*)$ in the strip $-1 \leq \sigma \leq 2$. Using our previous obeservations (4.12) and (4.13),

$$\begin{aligned}\frac{\zeta_K'}{\zeta_K}(\sigma + iT_m;\chi^*) &= \sum_{\substack{\rho_0 \\ |\gamma_0 - T_m| \leq 1}} \frac{1}{\sigma + iT_m - \rho_0} + O\left(\log(\mathfrak{q}T_m)\right) \\ &\ll \sum_{\substack{\rho_0 \\ |\gamma_0 - T_m| \leq 1}} \log m + \log(\mathfrak{q}T_m) \ll \left(\log(\mathfrak{q}T_m)\right)^2,\end{aligned}$$

for $m < T_m < m+1$. Since the Riemann-von Mangoldt formula (1.24) also holds in the lower half-plane, we get analogously

$$\frac{\zeta_K'}{\zeta_K}(\sigma - iT_m;\chi^*) \ll \left(\log(\mathfrak{q}T_m)\right)^2,$$

completing the proof of the theorem. □

We also need an estimate for the logarithmic derivative (4.11) to the left of the vertical line $\sigma = -1$.

**Theorem 4.2.4** *Let $\chi^* \bmod q^*$ be a primitive Dirichlet character. Exclude from the half-plane $\sigma \leq -1$ the points inside the disc $|s+m| \leq \frac{1}{2}$ for $m = 0, 1, 2, \ldots$. Then in the remaining area,*

$$\frac{\zeta_K'}{\zeta_K}(s;\chi^*) \ll \log(\mathfrak{q}|s|),$$

*for $|s| \geq 2$.*

**Proof**
This is an immediate consequence of (4.11) and (1.26) for $|s| \geq 2$. □

In the next step of the proof of Theorem 1.3.7, we split the sum over characters appearing in Theorem 4.2.2. For this purpose, we introduce some notation.

**Definition 4.2.5** *Let $\chi^* \bmod q^*$ be a primitive Dirichlet character. Define $\mathcal{F}(\chi^*)$ to be the set of all non-trivial zeros $\rho = \beta + i\gamma$, $\beta > 0$ of the multiplicative twist of $\zeta_K$, i.e. $\zeta_K(s;\chi^*)$ in (4.9). If we write a sum over $\rho \in \mathcal{F}(\chi^*)$, we write this with the meaning that a zero of multiplicity $N$ appears $N$-times in the sum.*

**Theorem 4.2.6** *Let $\chi^* \bmod q^*$ be a primitive Dirichlet character with $q^* \geq 3$ and $\chi_D \bmod |D|$ the Legendre-Kronecker character. Further, let $\mathfrak{q} = q^*|D|$ be the modulus of $\chi^*\chi_D$ and $\mathfrak{q}^*$ the modulus of the primitive character $(\chi^*\chi_D)^*$ inducing $\chi^*\chi_D$. Then there exists entire functions*

$$h^{(1)}(z;\chi^*) = \sum_{n=0}^{\infty} \kappa_n(\chi^*) z^n$$

*and*

$$h^{(2)}(z;\chi^*) = \sum_{\substack{n\in\mathbb{Z}\\ n^*\neq 0}} \sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}} \kappa(n^*) z^{-n^*}$$

*where $n^* = n^*(D,p) = \frac{i(\arg(\chi^*\chi_D)^*(p)+2\pi n)}{\log p}$ for $p \mid \mathfrak{q}$, $p \nmid \mathfrak{q}^*$, such that for $z = \delta + i\xi$, $\delta > 0$, we have*

$$f(z;\chi^*) = -\sum_{\rho\in\mathcal{F}(\chi^*)} \Gamma(\rho) z^{-\rho} + h^{(1)}(z;\chi^*) + h^{(2)}(z;\chi^*).$$

*Moreover, if $\sum'$ denotes the sum over all primes with $(\chi^*\chi_D)^*(p) \neq 1$ and $\sum''$ the sum over all primes with $(\chi^*\chi_D)^*(p) = 1$, the coefficients are given by*

$$\begin{aligned}\kappa_n(\chi^*) = \frac{(-1)^n}{n!}\Bigg\{ &\log\left(\frac{q^*}{2\pi}\right) + \log\left(\frac{\mathfrak{q}^*}{2\pi}\right) + 2\frac{\Gamma'}{\Gamma}(1+n) + \\ &+ \frac{L'}{L}(1+n;\overline{\chi^*}) + \frac{L'}{L}(1+n;\overline{(\chi^*\chi_D)^*}) + \\ &+ \sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}}{}' \frac{(\chi^*\chi_D)^*(p)p^n\log p}{(\chi^*\chi_D)^*(p)p^n - 1} + \sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*\\ n\neq 0}}{}'' \frac{p^n \log p}{p^n - 1} + \frac{1}{2}\sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*\\ n=0}}{}'' \log p \Bigg\}\end{aligned}$$

*and*

$$\kappa(n^*) = -\frac{\pi}{\sin(\pi n^*)\Gamma(1-n^*)p^{n^*}}.$$

*They satisfy*

$$\kappa_n(\chi^*) \ll \frac{1+\log(1+n)+p_1^n}{n!}$$

*and*

$$\kappa(n^*) \ll |n|^{\frac{1}{2}-|n|},$$

*where $p_1$ is the largest prime that divides $\mathfrak{q}$, but not $\mathfrak{q}^*$.*

**Proof**
Applying Mellin transform (see [49, p. 144]) in the half-plane of absolute

convergence yields, in combination with (4.1),

$$\begin{aligned} f(z;\chi^*) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds \\ &= \lim_{T\to\infty}\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds, \end{aligned}$$

where $z=\delta+i\xi,\ \delta>0$. To apply the residue theorem, we discuss the poles of the integrand: It is well-known that $\Gamma(s)$ has simple poles at $s=-n,\ n\in\mathbb{N}_0$, while the logarithmic derivative of the twisted Dedekind zeta-function, i.e. (4.11), has

no pole at $s=1$ by (4.10), since the principal character $\chi_0$ is not considered to be primitive, and $\chi^*\neq\chi_D^*$ in view of the summation condition in Theorem 4.2.2;

simple poles at the non-trivial zeros in $\mathcal{F}(\chi^*)$, as well as simple poles at the trivial zeros in the left half-plane at $s=-n,\ n\in\mathbb{N}_0$ and on the imaginary axis, in view of (4.10), (1.17) and (1.23).

So the integrand has singularities at $\mathcal{F}(\chi^*)\dot\cup\mathcal{V}$, where

$$\begin{aligned} \mathcal{V} = &\Big\{-n:\ n\in\mathbb{N}_0\Big\}\cup \\ &\cup\ \Big\{n^*=\frac{i(\arg(\chi^*\chi_D)^*(p)+2\pi n)}{\log p}:p\mid\mathfrak{q},\ p\nmid\mathfrak{q}^*,\ n=0,\pm1,\pm2,...\Big\}. \end{aligned}$$

For $N\in\mathbb{N}$ and $T>1$ consider the rectangular area $\mathcal{R}$ with vertices $2\pm iT,\ -N-\hat\delta\pm iT$. Since the trivial zeros in the left half-plane $\sigma<0$ are at negative integers, we can choose $\hat\delta>0$ such that no trivial zero is on the vertical line joining $-N-\hat\delta-iT$ and $-N-\hat\delta+iT$. W.l.o.g. $0<\hat\delta<1$ (in fact, $\hat\delta=\frac12$ would be sufficient, but in view of the later generalisation we prefer to work with an unspecified $\hat\delta$ between zero and one). Moreover, we choose $T>1$ in a way to fulfill Theorem 4.2.3 and Theorem 4.2.4. Applying residue calculus yields

$$\begin{aligned} &\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds = \\ &=\frac{1}{2\pi i}\left\{\int_{2-iT}^{-N-\hat\delta-iT}+\int_{-N-\hat\delta-iT}^{-N-\hat\delta+iT}+\int_{-N-\hat\delta+iT}^{2+iT}\right\}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds \\ &\quad+\sum_{s_0\in\mathcal{R}^\circ\cap(\mathcal{V}\cup\mathcal{F}(\chi^*))}\mathrm{res}_{s=s_0}\,\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right). \end{aligned}$$

We estimate each integral on the right-hand side individually: Since by assumption $z=\delta+i\xi,\ \delta>0$, we can write $z=re^{i\theta}$ with $r>0$ and $|\theta|<\frac{\pi}{2}$.

By Theorem 4.2.3 and Theorem 4.2.4 we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-N-\hat{\delta}+iT}^{2+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds=\\
&\quad=\frac{1}{2\pi i}\int_{-N-\hat{\delta}}^{-1}\Gamma(\sigma+iT)\left(-\frac{\zeta_K'}{\zeta_K}(\sigma+iT;\chi^*)\right)\left(re^{i\theta}\right)^{-\sigma-iT}d\sigma+\\
&\quad\quad+\frac{1}{2\pi i}\int_{-1}^{2}\Gamma(\sigma+iT)\left(-\frac{\zeta_K'}{\zeta_K}(\sigma+iT;\chi^*)\right)\left(re^{i\theta}\right)^{-\sigma-iT}d\sigma\\
&\quad\ll\int_{-N-\hat{\delta}}^{-1}|\Gamma(\sigma+iT)|\log\left(\mathfrak{q}|\sigma+iT|\right)r^{-\sigma}e^{T|\theta|}d\sigma+\\
&\quad\quad+\int_{-1}^{2}|\Gamma(\sigma+iT)|\left(\log(\mathfrak{q}T)\right)^2r^{-\sigma}e^{T|\theta|}d\sigma.
\end{aligned}$$

Stirling's formula [13, p. 204]

$$\Gamma(s)=\sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}e^{H(s)} \tag{4.14}$$

for $|\arg(s)|<\pi$ and $H(s)\to 0$ as $s\to\infty$, implies

$$\Gamma(\sigma+iT)\ll T^{\sigma-\frac{1}{2}}e^{-\frac{\pi T}{2}} \tag{4.15}$$

as $T\to\infty$ (see [34, p. 45]). By (4.15) we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-N-\hat{\delta}+iT}^{2+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds\ll\\
&\quad\ll e^{\left(|\theta|-\frac{\pi}{2}\right)T}T^{-\frac{1}{2}}\log\left(\mathfrak{q}|N+\hat{\delta}+iT|\right)\int_{-N-\hat{\delta}}^{-1}T^{\sigma}r^{-\sigma}d\sigma+\\
&\quad\quad+e^{\left(|\theta|-\frac{\pi}{2}\right)T}T^{-\frac{1}{2}}\left(\log(\mathfrak{q}T)\right)^2\int_{-1}^{2}T^{\sigma}r^{-\sigma}d\sigma\\
&\quad\ll e^{\left(|\theta|-\frac{\pi}{2}\right)T}T^{\frac{7}{2}}.
\end{aligned}$$

In view of $|\theta|<\frac{\pi}{2}$, the right-hand side tends to zero as $T\to\infty$. The integral along the path $\left[2-iT,-N-\hat{\delta}-iT\right]$ can be treated similar, where we use $\overline{\Gamma(s)}=\Gamma(\overline{s})$ and $\log(\mathfrak{q}|N+\hat{\delta}+iT|)$ has to be replaced by $\log(\mathfrak{q}|N+\hat{\delta}-iT|)$. Thus,

$$\begin{aligned}
f(z;\chi^*)=\lim_{T\to\infty}\Bigg\{&\frac{1}{2\pi i}\int_{-N-\hat{\delta}-iT}^{-N-\hat{\delta}+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds+\\
&+\sum_{s_0\in\mathcal{R}^\circ\cap(\mathcal{V}\cup\mathcal{F}(\chi^*))}\mathrm{res}_{s=s_0}\,\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)\Bigg\}.
\end{aligned}$$

By the functional equation $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ and (4.15),

$$\begin{aligned}\Gamma\left(-N-\hat{\delta}+it\right) &= \frac{\Gamma\left(-\hat{\delta}+it\right)}{\left(-N-\hat{\delta}+it\right)\ldots\left(-1-\hat{\delta}+it\right)} \\ &\ll (N!)^{-1}\left|\Gamma\left(-1-\hat{\delta}+it\right)\right| \\ &\ll (N!)^{-1}\, t^{-\frac{1}{2}-\hat{\delta}} e^{-\frac{\pi t}{2}}\end{aligned}$$

for $t \to \infty$. In view of

$$|\Gamma(\tilde{\alpha}+it)| = |\Gamma(\tilde{\alpha}-it)| \tag{4.16}$$

for $\tilde{\alpha} \in \mathbb{R}$, the estimate also holds for $\Gamma\left(-N-\hat{\delta}-it\right)$. Therefore, by Theorem 4.2.4 and $z = re^{i\theta}$, $|\theta| < \frac{\pi}{2}$,

$$\begin{aligned}&\lim_{T\to\infty}\left|\frac{1}{2\pi i}\int_{-N-\hat{\delta}-iT}^{-N-\hat{\delta}+iT}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)ds\right| = \\ &= \lim_{T\to\infty}\left|\frac{1}{2\pi}\int_{-T}^{T}\Gamma(-N-\hat{\delta}+it)z^{N+\hat{\delta}-it}\left(-\frac{\zeta_K'}{\zeta_K}\left(-N-\hat{\delta}+it;\chi^*\right)\right)dt\right| \\ &\ll \int_{-\infty}^{\infty}\left|\Gamma(-N-\hat{\delta}+it)\right|\log\left(\mathfrak{q}|N+\hat{\delta}+it|\right)r^{N+\hat{\delta}}e^{|\theta t|}dt \\ &\ll \frac{r^{N+\hat{\delta}}}{N!}\int_0^{\infty}e^{\left(|\theta|-\frac{\pi}{2}\right)t}\log\left(\mathfrak{q}|N+\hat{\delta}+it|\right)t^{-\frac{1}{2}-\hat{\delta}}dt \\ &\ll \frac{r^{N+\hat{\delta}}}{N!}\int_0^{\infty}te^{\left(|\theta|-\frac{\pi}{2}\right)t}dt \ll \frac{r^{N+\hat{\delta}}}{N!},\end{aligned}$$

which tends to zero as $N \to \infty$. Thus, for $\delta > 0$, we have

$$f(z;\chi^*) = \sum_{s_0\in\mathcal{V}\cup\mathcal{F}(\chi^*)}\operatorname{res}_{s=s_0}\Gamma(s)z^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right). \tag{4.17}$$

Recalling our observation on possible poles,

$$f(z;\chi^*) = -\sum_{\rho\in\mathcal{F}(\chi^*)}\Gamma(\rho)z^{-\rho} + \sum_{s\in\mathcal{V}}\kappa_s(\chi^*)z^{-s},$$

where $\kappa_s(\chi^*)$ denotes the residue arising from $\Gamma(s)$ and $-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)$ at $s \in \mathcal{V}$. We note that the series $\sum \kappa_s(\chi^*)z^{-s}$ converges and

$$\begin{aligned}\sum_{s\in\mathcal{V}}\kappa_s(\chi^*)z^{-s} &= \sum_{n\in\mathbb{N}_0}\left(\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)\right)z^n + \\ &+ \sum_{n^*\neq 0}\left(\operatorname{res}_{s=n^*}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)\right)z^{-n^*}.\end{aligned}$$

To compute the first residue, we apply (4.11) to obtain

$$\begin{aligned}\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right) &= \\ &= \operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;\chi^*)\right) + \operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;\chi^*\chi_D)\right).\end{aligned}$$

We use the asymmetric version of the functional equation (1.18) for Dirichlet L-functions to a primitive character (see [8, p. 69]), i.e.

$$L(1-s;\chi^*) = \varepsilon(\chi^*)2^{1-s}\pi^{-s}(q^*)^{s-\frac{1}{2}}\cos\left(\frac{\pi}{2}(s-a)\right)\Gamma(s)L(s;\overline{\chi^*}),$$

where $\varepsilon(\chi^*)$ is a constant of absolute value one and $a = a(\chi^*)$ is defined as for (1.18). Logarithmic differentiation and a change of variables, i.e. $s \to 1-s$, yields

$$\begin{aligned}-\frac{L'}{L}(s,\chi^*) = \log\left(\frac{q^*}{2\pi}\right) - \frac{\pi}{2}\tan\left(\frac{\pi}{2}(1-s-a)\right) + \\ + \frac{\Gamma'}{\Gamma}(1-s) + \frac{L'}{L}(1-s,\overline{\chi^*}).\end{aligned}$$

We also require

$$\Gamma(s) = \frac{\pi}{\sin(\pi s)\Gamma(1-s)}, \tag{4.18}$$

to get

$$\begin{aligned}\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;\chi^*)\right) &= \\ = \operatorname{res}_{s=-n}\frac{\pi}{\sin(\pi s)\Gamma(1-s)}&\left(\log\left(\frac{q^*}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1-s) + \frac{L'}{L}(1-s,\overline{\chi^*})\right) + \\ + \operatorname{res}_{s=-n}\frac{-\pi^2}{2\sin(\pi s)\Gamma(1-s)}&\tan\left(\frac{\pi}{2}(1-s-a)\right).\end{aligned}$$

Since

$$\operatorname{res}_{s=-n}\frac{\pi}{\sin(\pi s)} = (-1)^n, \tag{4.19}$$

$\Gamma(1+n) = n!$ for $n \in \mathbb{N}_0$ and $L(1+it,\overline{\chi^*}) \neq 0$ for all $t \in \mathbb{R}$ (see [53, Satz 4.1]), we get for the first residue

$$\begin{aligned}\operatorname{res}_{s=-n}\frac{\pi}{\sin(\pi s)\Gamma(1-s)}\left(\log\left(\frac{q^*}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1-s) + \frac{L'}{L}(1-s,\overline{\chi^*})\right) &= \\ = \frac{(-1)^n}{n!}\left(\log\left(\frac{q^*}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1+n) + \frac{L'}{L}(1+n,\overline{\chi^*})\right).\end{aligned}$$

For the function

$$-\frac{\pi^2}{2\sin(\pi s)\Gamma(1-s)}\tan\left(\frac{\pi}{2}(1-s-a)\right)$$

we consider two cases for $s = -n$ resp. $-s - a = -(a-n)$:

If $a - n$ is odd, then $1 + n - a$ is even and therefore divisible by 2. Since $\tan(k\pi) = 0$ for $k \in \mathbb{Z}$, $\tan(\pi \cdot \frac{1-s-a}{2})$ has a zero at $s = -n$, which cancels the simple pole of $\frac{1}{\sin(\pi s)}$. So the pole is removable and the residue is zero.

If $a - n$ is even, then $\frac{\pi}{2}(1 + n - a) = \frac{\pi}{2} + \frac{\pi}{2}(n - a)$, which is $\frac{\pi}{2}$ plus an integer multiple of $\pi$. So $\tan\left(\frac{\pi}{2}(1 - s - a)\right)$ has a simple pole at $s = -n$, arising from the cosine-function, and

$$\frac{\tan\left(\frac{\pi}{2}(1-s-a)\right)}{\sin(\pi s)}$$

has a double pole at $s = -n$. Let $a - n = 2k$ and $s = w - n$, then

$$\frac{\tan\left(\frac{\pi}{2}(1-s-a)\right)}{\sin(\pi s)} = \frac{\tan\left(\frac{\pi}{2}(1-w-2k)\right)}{\sin(\pi w + 2\pi k - \pi a)}.$$

Using $\tan(x + \pi) = \tan x,\ \sin(x + 2\pi) = \sin x,\ \sin(x + \pi) = -\sin x$ in the first step, and $\tan(\frac{\pi}{2} - x) = \cot x$ in the second,

$$\begin{aligned}\frac{\tan\left(\frac{\pi}{2}(1-w-2k)\right)}{\sin(\pi w + 2\pi k - \pi a)} &= (-1)^a \frac{\tan\left(\frac{\pi}{2} - \frac{\pi}{2}w\right)}{\sin(\pi w)} = (-1)^a \frac{\cot\left(\frac{\pi}{2}w\right)}{\sin(\pi w)} \\ &= (-1)^a \frac{\cos\left(\frac{\pi}{2}w\right)}{\sin(\pi w)\sin\left(\frac{\pi}{2}w\right)}.\end{aligned}$$

Since $\sin(\pi w)$ and $\sin\left(\frac{\pi}{2}w\right)$ are both odd functions, their product is an even function. Note that $\cos\left(\frac{\pi}{2}w\right)$ is also an even function and that the quotient of two even functions is even again. Therefore, only even powers appear in the Laurent series and the residue at $s = -n$ of the double pole is zero, i.e.

$$\operatorname{res}_{s=-n} \frac{-\pi^2}{2\sin(\pi s)\Gamma(1-s)} \tan\left(\frac{\pi}{2}(1-s-a)\right) = 0.$$

Thus,

$$\begin{aligned}&\operatorname{res}_{s=-n} \Gamma(s) \left( -\frac{L'}{L}(s;\chi^*) \right) = \\ &= \frac{(-1)^n}{n!} \left( \log\left(\frac{q^*}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1+n) + \frac{L'}{L}(1+n, \overline{\chi^*}) \right).\end{aligned}$$

Logarithmis differentiation of (1.17) yields

$$\frac{L'}{L}(s;\chi^*\chi_D) = \frac{L'}{L}(s;(\chi^*\chi_D)^*) + \sum_{\substack{p|\mathfrak{q} \\ p\nmid \mathfrak{q}^*}} \frac{(\chi^*\chi_D)^*(p)\log p}{p^s - (\chi^*\chi_D)^*(p)}, \tag{4.20}$$

and implies

$$\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;\chi^*\chi_D)\right)=\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;(\chi^*\chi_D)^*)\right)+$$
$$+\operatorname{res}_{s=-n}\Gamma(s)\left(-\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)}\right),$$

where $(\chi^*\chi_D)^* \mod \mathfrak{q}^*$ is the primitive character inducing $\chi^*\chi_D \mod \mathfrak{q}$. Analogously to our previous observation for the logarithmic derivative of the L-function to the primitive character $\chi^* \bmod q^*$, we get

$$\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{L'}{L}(s;(\chi^*\chi_D)^*)\right)=$$
$$=\frac{(-1)^n}{n!}\left(\log\left(\frac{\mathfrak{q}^*}{2\pi}\right)+\frac{\Gamma'}{\Gamma}(1+n)+\frac{L'}{L}(1+n,\overline{(\chi^*\chi_D)^*})\right).$$

For the sum of residues, arising from (4.20), we use (4.18) and $\Gamma(1+n)=n!$, to get

$$\operatorname{res}_{s=-n}\Gamma(s)\left(-\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)}\right)=$$
$$=\frac{1}{n!}\operatorname{res}_{s=-n}\frac{\pi}{\sin(\pi s)}\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{(\chi^*\chi_D)^*(p)-p^s}$$
$$=\frac{1}{n!}\sideset{}{'}\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{(\chi^*\chi_D)^*(p)-p^{-n}}\operatorname{res}_{s=-n}\frac{\pi}{\sin(\pi s)}+$$
$$+\frac{1}{n!}\sideset{}{''}\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n\neq 0}}\frac{\log p}{1-p^{-n}}\operatorname{res}_{\substack{s=-n\\ n\neq 0}}\frac{\pi}{\sin(\pi s)}+\sideset{}{''}\sum_{\substack{p\mid\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n=0}}\log p\cdot\operatorname{res}_{s=0}\frac{\pi}{\sin(\pi s)(1-p^{-s})}.$$

From (4.19) we know the first residues, and from the Laurent-series expansion at $s=0$, i.e.

$$\frac{\pi}{\sin(\pi s)(1-p^{-s})}=-\frac{1}{\log p}s^{-2}+\frac{1}{2}s^{-1}-\frac{(\log p)^2+2\pi^2}{12\log p}+\frac{\pi^2}{12}s+O(s^2),$$

we get the remaining residue. Therefore,

$$\operatorname{res}_{s=-n}\Gamma(s)\left(-\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)}\right)=$$
$$=\frac{(-1)^n}{n!}\left\{\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}{}'\frac{(\chi^*\chi_D)^*(p)\log p}{(\chi^*\chi_D)^*(p)-p^{-n}}+\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n\neq 0}}{}''\frac{\log p}{1-p^{-n}}+\frac{1}{2}\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n=0}}{}''\log p\right\},$$

implying

$$\begin{aligned}\kappa_n(\chi^*)&=\operatorname{res}_{s=-n}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)\\&=\frac{(-1)^n}{n!}\left\{\log\left(\frac{q^*}{2\pi}\right)+\log\left(\frac{\mathfrak{q}^*}{2\pi}\right)+2\frac{\Gamma'}{\Gamma}(1+n)+\right.\\&\quad+\frac{L'}{L}(1+n;\overline{\chi^*})+\frac{L'}{L}(1+n;\overline{(\chi^*\chi_D)^*})+\\&\quad\left.+\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}{}'\frac{(\chi^*\chi_D)^*(p)p^n\log p}{(\chi^*\chi_D)^*(p)p^n-1}+\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n\neq 0}}{}''\frac{p^n\log p}{p^n-1}+\frac{1}{2}\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*\\ n=0}}{}''\log p\right\}.\end{aligned}$$

To compute the residue at $n^*\neq 0$ on the imaginary axis, we use (4.11) and (4.20), i.e.

$$\frac{\zeta_K'}{\zeta_K}(s;\chi^*)=\frac{L'}{L}(s;\chi^*)+\frac{L'}{L}(s;(\chi^*\chi_D)^*)+\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)},$$

and observe that the first and second appearing residue both are zero, since $\Gamma(s)$ has its poles at negative integers and zero, resp. $\chi^*$ and $(\chi^*\chi_D)^*$ are primitive and the logarithmic derivatives of the associated L-functions have poles at negative integers and zero. Therefore we obtain

$$\operatorname{res}_{s=n^*\neq 0}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)=$$
$$=\operatorname{res}_{s=n^*\neq 0}\Gamma(s)\left(-\sum_{\substack{p|\mathfrak{q}\\ p\nmid\mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)}\right).$$

Again by (4.18),

$$\operatorname{res}_{s=n^*\neq 0}\Gamma(s)\left(-\sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)\log p}{p^s-(\chi^*\chi_D)^*(p)}\right)=$$
$$=\frac{-\pi}{\sin(\pi n^*)\Gamma(1-n^*)}\sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}}(\chi^*\chi_D)^*(p)\log p\cdot\operatorname{res}_{s=n^*\neq 0}\frac{1}{p^s-(\chi^*\chi_D)^*(p)},$$

where

$$\operatorname{res}_{s=n^*\neq 0}\frac{1}{p^s-(\chi^*\chi_D)^*(p)}=\frac{1}{p^{n^*}\log p}.$$

Hence, for the residue at $n^*\neq 0$ we get

$$\kappa(n^*)=\operatorname{res}_{s=n^*\neq 0}\Gamma(s)\left(-\frac{\zeta_K'}{\zeta_K}(s;\chi^*)\right)=-\frac{\pi}{\sin(\pi n^*)\Gamma(1-n^*)p^{n^*}}.$$

To prove that

$$\sum_{n\in\mathbb{N}_0}\kappa_n(\chi^*)z^n$$

is entire, we estimate its coefficients: Since $\mathfrak{q}=q^*|D|$ and $\mathfrak{q}^*\leq\mathfrak{q}$,

$$\log\left(\frac{q^*}{2\pi}\right)+\log\left(\frac{\mathfrak{q}^*}{2\pi}\right)\ll\log\mathfrak{q}\ll 1,$$

By

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O\left(\frac{1}{|s|}\right)\tag{4.21}$$

for $|\arg(s)|<\pi$ (see [34, p. 45]),

$$2\cdot\frac{\Gamma'}{\Gamma}(1+n)\ll\log(1+n).$$

For $n\geq 1$, we get by [5, Satz 2.6.2],

$$\frac{L'}{L}(1+n;\overline{\chi^*})\ll 1$$

resp.

$$\frac{L'}{L}(1+n;\overline{(\chi^*\chi_D)^*})\ll 1.$$

Using $\omega(\mathfrak{q})\ll\log\mathfrak{q}$,

$$\sideset{}{'}\sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}}\frac{(\chi^*\chi_D)^*(p)p^n\log p}{(\chi^*\chi_D)^*(p)p^n-1}\ll\omega(\mathfrak{q})p_1^n\log p_1\ll p_1^n\log\mathfrak{q}\ll p_1^n,$$

where $p_1 = \max\{p : \; p \mid \mathfrak{q}, p \not| \; \mathfrak{q}^*\}$ is fixed. Analogously,

$$\sum_{\substack{p|\mathfrak{q} \\ p \not| \; \mathfrak{q}^* \\ n \neq 0}}'' \frac{p^n \log p}{p^n - 1} \ll \omega(\mathfrak{q}) p_1^n \log p_1 \ll p_1^n \log \mathfrak{q} \ll p_1^n$$

and

$$\sum_{\substack{p|\mathfrak{q} \\ p \not| \; \mathfrak{q}^* \\ n = 0}}'' \log p \ll \omega(\mathfrak{q}) \log p_1 \ll \log \mathfrak{q} \ll 1.$$

Therefore,

$$\kappa_n(\chi^*) \ll \frac{1}{n!}\{1 + \log(1+n) + p_1^n\}. \tag{4.22}$$

The formula of Cauchy-Hadamard in combination with this estimate for the coefficients implies that $\sum_{n \in \mathbb{N}_0} \kappa_n(\chi^*) z^n$ is an entire function.

For $n^* = \frac{i(\arg(\chi^* \chi_D)^*(p) + 2\pi n)}{\log p}$, with $p \mid \mathfrak{q}, \; p \not| \; \mathfrak{q}^*$ and $n \in \mathbb{Z}$, we note that

$$\sum_{n^* \neq 0} \kappa(n^*) z^{-n^*} = \sum_{\substack{p|\mathfrak{q} \\ p \not| \; \mathfrak{q}^*}} z^{-\frac{i \arg(\chi^* \chi_D)^*(p)}{\log p}} \sum_{\substack{n \in \mathbb{Z} \\ n^* \neq 0}} \kappa(n^*) z^{\frac{2\pi i n}{\log p}}.$$

To estimate the coefficient $\kappa(n^*)$, we remark that

$$\iota = \min_{n^*}\{|\sin(\pi n^*)|\}$$

is nonzero, since $n^* \neq 0$ is on the imaginary axis. Moreover,

$$p^{n^*} = \exp(i(\arg(\chi^* \chi_D)^*(p) + 2\pi n))$$

is of absolute value one, and by (4.14) and $1 - n^* \asymp n$, we get

$$\kappa(n^*) \ll \frac{1}{|\Gamma(1-n^*)|} \ll |n|^{\frac{1}{2} - |n|},$$

for $n \in \mathbb{Z}$, where the implied constant may depend on $\iota$. The formula of Cauchy-Hadamard in combination with this estimate for the coefficients implies that $\sum_{n^* \neq 0} \kappa(n^*) z^{-n^*}$ is an entire function. □

We adopt the notation of the previous Theorem 4.2.6 and consider the sum over characters containing the functions $h^{(j)}, \; j = 1, 2$.

**Lemma 4.2.7** *Let $\chi^* \bmod q^*$ be the primitive character inducing $\chi \bmod q$. Then for $|z| < 1$ and $j = 1, 2$*

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} |h^{(j)}(z; \chi^*)| \ll (\log(|D|q))^2.$$

**Proof**

If $q=1$ or $q=2$, there is no non-principal character mod $q$ and the estimate holds trivially. We assume $q \geq 3$. If $\chi^* \bmod q^*$ induces $\chi \bmod q$ and $\chi \neq \chi_0$, then $q^* \geq 3$. Hence the previous Theorem 4.2.6 can be applied and for $|z| < 1$ we get

$$\left|h^{(1)}\left(z;\chi^*\right)\right| \leq \sum_{n=0}^{\infty} |\kappa_n(\chi^*)|.$$

From the proof of the bound for the coefficients $\kappa_n(\chi^*)$ in Theorem 4.2.6 we know

$$\sum_{n=1}^{\infty} |\kappa_n(\chi^*)| \ll \sum_{n=1}^{\infty} \frac{\log \mathfrak{q} + \log(1+n) + p_1^n}{n!} \ll \log \mathfrak{q} \ll \log(|D|q),$$

since $\mathfrak{q} = q^*|D|$ and $q^* \leq q$, as well as

$$|\kappa_0(\chi^*)| \ll \log(|D|q) + \left|\frac{L'}{L}\left(1;\overline{\chi^*}\right)\right| + \left|\frac{L'}{L}\left(1;\overline{(\chi^*\chi_D)^*}\right)\right|.$$

Therefore,

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} |h^{(1)}\left(z;\chi^*\right)| \ll \log(|D|q) + \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left|\frac{L'}{L}\left(1;\overline{\chi^*}\right)\right| + $$
$$+ \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left|\frac{L'}{L}\left(1;\overline{(\chi^*\chi_D)^*}\right)\right|.$$

The sums over characters containing the logarithmic derivative of an L-function to a primitive character can be treated similarly and we start with the former one: Let $\psi(y,\chi^*) = \sum_{n \leq y} \chi^*(n)\Lambda(n)$. Then

$$\left|\frac{L'}{L}\left(1;\overline{\chi^*}\right)\right| = \left|\int_1^{\infty} y^{-1}\,\mathrm{d}\,\psi(y,\overline{\chi^*})\right| \leq \int_1^{\infty} y^{-2}\left|\psi(y,\overline{\chi^*})\right| dy.$$

Suppose that $\chi_1$ is the single real character modulo $q$, if it exists, for which $L(s;\chi_1)$ has a real zero $\beta_1$ satisfying $\beta_1 > 1 - c'(\log q)^{-1}$ for a certain positive constant $c'$ which does not depend on $q$. Then, from (1.21), for $\chi \neq \chi_0$, $q \leq \exp(\sqrt{\log y})$, and for a certain positive constant $c$ which does not depend on $q$,

$$\psi(x,\chi_1) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right),$$

resp.

$$\psi(x,\chi) = O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

for $\chi \neq \chi_1$. Since $q^* \leq q$, we get for $q \leq \exp(\sqrt{\log y})$ and $\chi \neq \chi_1$

$$\psi(x, \chi^*) = O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \tag{4.23}$$

In the half-plane $\sigma > 0$, the L-functions $L(s; \chi_1)$ and $L(s; \chi_1^*)$ have the same zeros, and we have for $q \leq \exp(\sqrt{\log y})$

$$\psi(x, \chi_1^*) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \tag{4.24}$$

We remark that $q \leq \exp\left(\sqrt{\log y}\right)$ implies $\exp\left(\log^2 q\right) \leq y$, and with (4.23), (4.24) it follows that

$$\begin{aligned}
&\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left| \frac{L'}{L}\left(1; \overline{\chi^*}\right) \right| \ll \\
&\ll \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left\{ \int_1^{\exp\left((\log q)^2\right)} + \int_{\exp\left((\log q)^2\right)}^{\infty} \right\} y^{-2} |\psi(y, \overline{\chi^*})| dy \\
&\ll \int_1^{\exp\left((\log q)^2\right)} y^{-1} dy + \frac{1}{\varphi(q)} \int_{\exp\left((\log q)^2\right)}^{\infty} \frac{y^{\beta_1 - 2}}{\beta_1} dy + \\
&\qquad + \int_{\exp\left((\log q)^2\right)}^{\infty} y^{-1} \exp\left(-c\sqrt{\log y}\right) dy,
\end{aligned}$$

where we used $\psi(y; \chi^*) \ll y$. We estimate each integral on the right side individually:

$$\int_1^{\exp\left((\log q)^2\right)} y^{-1} dy \ll (\log q)^2$$

and since $\beta_1 - 1 < 0$,

$$\int_{\exp\left((\log q)^2\right)}^{\infty} \frac{y^{\beta_1 - 2}}{\beta_1} dy \ll \frac{1}{\beta_1(1 - \beta_1)}.$$

For the remaining integral we obtain

$$\int_{\exp\left((\log q)^2\right)}^{\infty} y^{-1} \exp\left(-c\sqrt{\log y}\right) dy \ll \log q,$$

since $q \geq 3$. Therefore,

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left| \frac{L'}{L}\left(1; \overline{\chi^*}\right) \right| \ll (\log q)^2 + \frac{1}{\varphi(q)} \cdot \frac{1}{\beta_1(1 - \beta_1)}.$$

From [8, p. 99] we may use

$$\beta_1 < 1 - \frac{c''}{\sqrt{q}\,(\log q)^2}$$

with an computable constant $c'' > 0$, to get with $\varphi(q)^{-1} \ll q^{-1}\log q$

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left| \frac{L'}{L}\left(1; \overline{\chi^*}\right) \right| \ll (\log q)^2 + \frac{1}{\varphi(q)} \cdot \sqrt{q}\,(\log q)^2 \ll (\log q)^2\,,$$

where $\chi^* \bmod q^*$ is the primitive character inducing $\chi \bmod q$. Analogously,

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \left| \frac{L'}{L}\left(1; \overline{(\chi^*\chi_D)^*}\right) \right| \ll (\log \mathfrak{q})^2\,,$$

where $(\chi^*\chi_D)^* \mod \mathfrak{q}^*$ is the primitive character inducing $\chi^*\chi_D \mod \mathfrak{q}$. Since $\mathfrak{q} = q^*|D|$ and $q^* \leq q$,

$$\begin{aligned} \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} |h^{(1)}(z; \chi^*)| &\ll \log(|D|q) + (\log q)^2 + (\log(q^*|D|))^2 \\ &\ll (\log(|D|q))^2\,. \end{aligned}$$

Now consider for $j = 2$

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} |h^{(2)}(z; \chi^*)|, \quad \text{where} \quad h^{(2)}(z;\chi^*) = \sum_{\substack{n \in \mathbb{Z} \\ n^* \neq 0}} \sum_{\substack{p | \mathfrak{q} \\ p \nmid \mathfrak{q}^*}} \kappa(n^*) z^{-n^*}.$$

Again for $q = 1$ or $q = 2$, the bound of the lemma holds trivially. For $|z| < 1$, Theorem 4.2.6 implies

$$\begin{aligned} &\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} |h^{(2)}(z; \chi^*)| \ll \\ &\ll \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{p | \mathfrak{q} \\ p \nmid \mathfrak{q}^*}} \left| z^{-\frac{i \arg(\chi^*\chi_D)^*(p)}{\log p}} \right| \sum_{\substack{n \in \mathbb{Z} \\ n^* \neq 0}} |\kappa(n^*)| \cdot \left| z^{\frac{2\pi i}{\log p}} \right|^n \\ &\ll \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{p | \mathfrak{q} \\ p \nmid \mathfrak{q}^*}} \sum_{\substack{n \in \mathbb{Z} \\ n^* \neq 0}} |\kappa(n^*)| \\ &\ll \log(|D|q), \end{aligned}$$

where we used that
$$\sum_{\substack{p|\mathfrak{q}\\ p\nmid \mathfrak{q}^*}} 1 \leq \omega(\mathfrak{q}) \ll \log(|D|q).$$
□

As a corollary of the former Theorems 4.2.2 and 4.2.6, in combination with Lemma 4.2.7, we get

**Corollary 4.2.8** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b,q) = 1$, and let $z_0 = \delta + 2\pi i\left(\alpha - \frac{b}{q}\right)$. Denote by $\chi^*$ mod $q^*$ the primitive character inducing $\chi$ mod $q$. Then for $|z_0| < 1$,*
$$f(\delta)^{-1}|f(\delta+2\pi i\alpha)| \ll \frac{\sqrt{q}}{\varphi(q)} + \delta\log\left(\delta^{-1}\right)\sqrt{q}\left(\log(e+|D|q)\right)^2 + \\ + \delta\frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} \sum_{\rho\in\mathcal{F}(\chi^*)} \left|\Gamma(\rho)z_0^{-\rho}\right|,$$
*where $\chi_D^*$ denotes the character induced by $\chi_D$ or the character $\chi_D$ itself, if it appears among the characters modulo $q$.*

**Proof**
From Theorem 4.2.2 we know
$$f(\delta)^{-1}|f(\delta+2\pi i\alpha)| \ll \frac{\sqrt{q}}{\varphi(q)} + \delta\log\left(\delta^{-1}\right)\sqrt{q}\log q + \\ + \delta\frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} |f(z_0;\chi^*)|,$$
where the sum over characters $\chi \neq \chi_0, \chi_D^*$ mod $q$ is empty for $q \leq 2$. For $q \geq 3$, Theorem 4.2.6 and Lemma 4.2.7 yield for $z = z_0$
$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} |f(z_0;\chi^*)| \ll \\ \ll \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} \left| - \sum_{\rho\in\mathcal{F}(\chi^*)} \Gamma(\rho)z_0^{-\rho} + h^{(1)}(z_0;\chi^*) + h^{(2)}(z_0;\chi^*)\right| \\ \ll \frac{1}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} \left| \sum_{\rho\in\mathcal{F}(\chi^*)} \Gamma(\rho)z_0^{-\rho}\right| + \left(\log(|D|q)\right)^2.$$

We observe that for sufficiently small $\delta > 0$,

$$\delta \log\left(\delta^{-1}\right)\sqrt{q}\log q + \delta\sqrt{q}\left(\log(|D|q)\right)^2 \ll \delta\log\left(\delta^{-1}\right)\sqrt{q}\left(\log(e+|D|q)\right)^2,$$

which proves the corollary. □

To estimate the double series over characters and non-trivial zeros in Corollary 4.2.8, we write $z_0 = \delta + 2\pi i\left(\alpha - \frac{b}{q}\right) = re^{i\theta}$ with $\delta > 0$, $r < 1$ and $|\theta| < \frac{1}{2}$. Further let $\tilde{D} = \frac{\pi}{2} - |\theta|$ and observe

$$\delta r^{-1} = \cos\theta = \sin\left(\frac{\pi}{2} - |\theta|\right) = \sin\tilde{D} \le \tilde{D} \tag{4.25}$$

First we consider non-trivial zeros with $|\gamma| > 3$:

**Lemma 4.2.9** *Let $q \ge 3$ be an integer. Then*

$$\sum_{\substack{\chi \pmod q\\ \chi\ne\chi_0,\chi_D^*}} \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}} |\Gamma(\rho)z_0^{-\rho}| \ll$$
$$\ll \delta^{-1}\left\{r^{\frac{1}{2}}q + r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{5}{7}} + r^{-\frac{1}{2}}\delta^{\frac{1}{2}}\right\}\left(\log(|D|r\delta^{-1}q)\right)^{15},$$

*where $r = |z_0| < 1$, $\delta = \operatorname{Re} z_0$ and $\delta > 0$ is sufficiently small.*

**Proof**

In view of (4.10), i.e. $\zeta_K(s;\chi^*) = L(s;\chi^*)L(s;\chi^*\chi_D)$, the nontrivial zeros $\rho = \beta + i\gamma$, $\beta > 0$ of $\zeta_K(s;\chi^*)$ are exactly the nontrivial zeros of $L(s;\chi^*)$ and $L(s;(\chi^*\chi_D)^*)$. If $\rho = \beta + i\gamma$, $|\gamma| > 3$ and $z_0 = re^{i\theta}$, then by (4.15),

$$\Gamma(\beta + i\gamma) \ll |\gamma|^{\beta-\frac{1}{2}}e^{-\frac{\pi}{2}|\gamma|},$$

we obtain

$$\Gamma(\rho)z_0^{-\rho} \ll e^{-\tilde{D}|\gamma|}|\gamma|^{\beta-\frac{1}{2}}r^{-\beta}.$$

If $\psi$ is a primitive Dirichlet character and $L(\rho;\psi) = 0$, the functional equation (1.18) implies $L(1-\rho;\overline{\psi}) = 0$. Therefore

$$\sum_{\substack{\chi \pmod q\\ \chi\ne\chi_0,\chi_D^*}} \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}} |\Gamma(\rho)z_0^{-\rho}| \ll \sum_{\substack{\chi \pmod q\\ \chi\ne\chi_0,\chi_D^*}} \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3,\ \beta\ge\frac{1}{2}}} e^{-\tilde{D}|\gamma|}|\gamma|^{\beta-\frac{1}{2}}r^{-\beta}.$$

We use partial integration to obtain

$$\log|\gamma|\int_{\frac{1}{2}}^{\beta}|\gamma|^{\sigma}r^{-\sigma}d\sigma = r^{-\beta}|\gamma|^{\beta} - r^{-\frac{1}{2}}|\gamma|^{\frac{1}{2}} + \log r\int_{\frac{1}{2}}^{\beta}|\gamma|^{\sigma}r^{-\sigma}d\sigma,$$

resp.

$$r^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta}|\gamma|^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{|\gamma|}{r}\right)d\sigma = |\gamma|^{\beta-\frac{1}{2}}r^{-\beta}.$$

This yields for $\beta \geq \frac{1}{2}$ and $|\gamma| > 3$

$$\begin{aligned} e^{-\tilde{D}|\gamma|}|\gamma|^{\beta-\frac{1}{2}}r^{-\beta} &= e^{-\tilde{D}|\gamma|}r^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta} e^{-\tilde{D}|\gamma|}|\gamma|^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{|\gamma|}{r}\right)d\sigma \\ &= \int_{|\gamma|}^{\infty}\tilde{D}e^{-\tilde{D}t}r^{-\frac{1}{2}}dt + \int_{\frac{1}{2}}^{\beta}\int_{|\gamma|}^{\infty} -\frac{\partial}{\partial t}\left\{e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\right\}dtd\sigma \\ &\leq \int_{|\gamma|}^{\infty}\tilde{D}e^{-\tilde{D}t}r^{-\frac{1}{2}}dt + \int_{\frac{1}{2}}^{\beta}\int_{|\gamma|}^{\infty}\tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)dtd\sigma, \end{aligned}$$

since

$$\begin{aligned} &-\frac{\partial}{\partial t}\left\{e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\right\} = \\ &\quad = \tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right) - e^{-\tilde{D}t}\left(\sigma-\frac{1}{2}\right)t^{\sigma-\frac{3}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right) \\ &\quad - e^{-\tilde{D}t}t^{\sigma-\frac{3}{2}}r^{-\sigma}. \end{aligned}$$

Let $N_K(\sigma, T, \chi^*)$ denote the number of zeros $\rho \in \mathcal{F}(\chi^*)$ such that $\sigma \leq \beta$ and $|\gamma| \leq T$ (counting multiplicity). Then

$$\begin{aligned} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| > 3}} |\Gamma(\rho) z_0^{-\rho}| \ll \int_3^{\infty} \tilde{D}e^{-\tilde{D}t}r^{-\frac{1}{2}}\left\{\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} N_K\left(\frac{1}{2}, t, \chi^*\right)\right\}dt + \\ + \int_{\frac{1}{2}}^{1}\int_3^{\infty}\tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\left\{\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} N_K(\sigma, t, \chi^*)\right\}dtd\sigma. \end{aligned}$$

We remark that in view of $N_K\left(\frac{1}{2}, t, \chi^*\right) \asymp N_K(t, \chi^*)$, a Riemann-von Mangoldt type formula can be applied to estimate the first term on the right side. However, since this will only affect the occuring logarithmic term, which will be comparably small to the occuring linear term, we use a zero-density estimate to treat both terms on the right side similarly.

In view of (4.10) resp. our observation that the nontrivial zeros $\rho = \beta + i\gamma$, $\beta > 0$ of $\zeta_K(s;\chi^*)$ are exactly the nontrivial zeros of $L(s;\chi^*)$ and $L(s;(\chi^*\chi_D)^*)$, we get

$$\sum_{\chi \ (\mathrm{mod}\ q)} N_K(\sigma, t, \chi^*) \leq \sum_{\chi \ (\mathrm{mod}\ q)} N_1(\sigma, t, \chi^*) + \sum_{\chi \ (\mathrm{mod}\ q)} N_2(\sigma, t, (\chi^*\chi_D)^*), \tag{4.26}$$

where $N_1(\sigma, T, \chi^*)$ resp. $N_2(\sigma, T, (\chi^*\chi_D)^*)$ denotes the number of zeros $\rho \in \mathcal{F}(\chi^*)$ arising from $L(s;\chi^*)$ resp. $L(s;(\chi^*\chi_D)^*)$ satisfying $\sigma \leq \beta$ and $|\gamma| \leq T$

(counting multiplicity). For the first sum on the right we use from [48, p. 143] that for $\chi$ mod $q$ induced by $\chi^*$ mod $q^*$ and $t \geq 2$ the estimate

$$\sum_{\chi \pmod q} N_1(\sigma, t, \chi^*) \ll (qt)^{\frac{5-4\sigma}{3}} (\log(qt))^{14}$$

holds for $\frac{1}{2} \leq \sigma \leq \frac{5}{7}$, and

$$\sum_{\chi \pmod q} N_1(\sigma, t, \chi^*) \ll (qt)^{\frac{5}{2}(1-\sigma)} (\log(qt))^{14}$$

holds for $\frac{5}{7} \leq \sigma \leq 1$. For the second sum on the right in (4.26) we observe that

$$\frac{\#\{\chi \mod q\}}{\#\{\chi^*\chi_D \mod q|D|\}} = \frac{\varphi(q)}{\varphi(q|D|)} \leq 1.$$

If $q$ and $|D|$ are coprime, the estimate is a consequence of the multiplicativity of $\varphi$; in case of $(q, |D|) \neq 1$, it follows from $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

This implies

$$\sum_{\chi \pmod q} N_2(\sigma, t, (\chi^*\chi_D)^*) \leq \sum_{\chi^*\chi_D \pmod{q|D|}} N_2(\sigma, t, (\chi^*\chi_D)^*).$$

To estimate the last sum we use again [48, p. 143] and by (4.26) we get for $t \geq 2$ and $\frac{1}{2} \leq \sigma \leq \frac{5}{7}$

$$\sum_{\chi \pmod q} N_K(\sigma, t, \chi^*) \ll (qt)^{\frac{5-4\sigma}{3}} (\log(|D|qt))^{14}, \tag{4.27}$$

whereas for $t \geq 2$ and $\frac{5}{7} \leq \sigma \leq 1$

$$\sum_{\chi \pmod q} N_K(\sigma, t, \chi^*) \ll (qt)^{\frac{5}{2}(1-\sigma)} (\log(|D|qt))^{14}. \tag{4.28}$$

Using (4.27), we estimate the first of the previous integrals by

$$\int_3^\infty \tilde{D} e^{-\tilde{D}t} r^{-\frac{1}{2}} \Bigg\{ \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} N_K\left(\frac{1}{2}, t, \chi^*\right) \Bigg\} dt \ll$$
$$\ll \tilde{D} r^{-\frac{1}{2}} q \int_3^\infty e^{-\tilde{D}t} t \, (\log(|D|qt))^{14} \, dt.$$

The substitution $\tau = \tilde{D}t$ yields

$$\begin{aligned} r^{-\frac{1}{2}} q\tilde{D} \int_3^\infty e^{-\tilde{D}t} t \left(\log(|D|qt)\right)^{14} dt \ll \\ \ll r^{-\frac{1}{2}} q\tilde{D}^{-1} \left\{ \int_{3\tilde{D}}^1 + \int_1^\infty \right\} e^{-\tau} \tau \left( \log\left( \frac{|D|q}{\tilde{D}} \tau \right) \right)^{14} d\tau \\ \ll r^{-\frac{1}{2}} q\tilde{D}^{-1} \left\{ \left( \log\left( \frac{|D|q}{\tilde{D}} \tau \right) \right)^{14} + \int_1^\infty e^{-\tau} \tau \left( \log\left( \frac{|D|q}{\tilde{D}} \tau \right) \right)^{14} d\tau \right\}, \end{aligned}$$

where we estimated the first integral trivially. By repeated partial integration we notice that for a function satisfying $f(\tau) \ll \tau^{2+\varepsilon}$ with $\varepsilon > 0$, we have

$$\int_1^\infty e^{-\tau} f(\tau) d\tau \ll f(1). \tag{4.29}$$

Using this for the second integral yields

$$\begin{aligned} \int_3^\infty \tilde{D} e^{-\tilde{D}t} r^{-\frac{1}{2}} \left\{ \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} N_K\left(\frac{1}{2}, t, \chi^*\right) \right\} dt \ll r^{-\frac{1}{2}} q\tilde{D}^{-1} \left( \log\left( \frac{|D|q}{\tilde{D}} \right) \right)^{14} \\ \ll r^{\frac{1}{2}} \delta^{-1} q \left( \log(|D| r \delta^{-1} q) \right)^{14}, \end{aligned}$$

where we used in the last step that (4.25) implies $\tilde{D}^{-1} \le \delta^{-1} r$. The remaining double integral can be treated similar: By Fubini's theorem, (4.27) and (4.28),

$$\begin{aligned} \int_{\frac{1}{2}}^1 \int_3^\infty \tilde{D} e^{-\tilde{D}t} t^{\sigma - \frac{1}{2}} r^{-\sigma} \log\left(\frac{t}{r}\right) \left\{ \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0, \chi_D^*}} N_K(\sigma, t, \chi^*) \right\} dt d\sigma \ll \\ \ll \int_3^\infty \tilde{D} e^{-\tilde{D}t} \log\left(\frac{t}{r}\right) t^{-\frac{1}{2}} \left(\log(|D|qt)\right)^{14} \times \\ \times \left\{ \int_{\frac{1}{2}}^{\frac{5}{7}} \left(\frac{t}{r}\right)^\sigma (qt)^{\frac{5-4\sigma}{3}} d\sigma + \int_{\frac{5}{7}}^1 \left(\frac{t}{r}\right)^\sigma (qt)^{\frac{5}{2}(1-\sigma)} d\sigma \right\} dt. \end{aligned}$$

By partial integration,

$$\int_3^\infty \tilde{D}e^{-\tilde{D}t}\log\left(\frac{t}{r}\right)t^{-\frac{1}{2}}\left(\log(|D|qt)\right)^{14}\times$$
$$\times\left\{\int_{\frac{1}{2}}^{\frac{5}{7}}\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5-4\sigma}{3}}\,d\sigma+\int_{\frac{5}{7}}^{1}\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5}{2}(1-\sigma)}\,d\sigma\right\}dt$$
$$\ll\int_3^\infty \tilde{D}e^{-\tilde{D}t}t^{-\frac{1}{2}}\left(\log(|D|qt)\right)^{14}\times$$
$$\times\left\{\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5-4\sigma}{3}}\Big|_{\sigma=\frac{1}{2}}^{\frac{5}{7}}+\frac{4}{3}\log(qt)\int_{\frac{1}{2}}^{\frac{5}{7}}\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5-4\sigma}{3}}\,d\sigma\right.$$
$$\left.+\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5}{2}(1-\sigma)}\Big|_{\sigma=\frac{5}{7}}^{1}+\frac{5}{2}\log(qt)\int_{\frac{5}{7}}^{1}\left(\frac{t}{r}\right)^{\sigma}(qt)^{\frac{5}{2}(1-\sigma)}\,d\sigma\right\}dt$$
$$\ll\int_3^\infty \tilde{D}e^{-\tilde{D}t}t^{-\frac{1}{2}}\left(\log(|D|qt)\right)^{14}\log(qt)\left(r^{-\frac{1}{2}}qt^{\frac{3}{2}}+r^{-\frac{5}{7}}q^{\frac{5}{7}}t^{\frac{10}{7}}+r^{-1}t\right)dt$$
$$\ll\int_3^\infty \tilde{D}e^{-\tilde{D}t}\left(\log(|D|qt)\right)^{15}\left(r^{-\frac{1}{2}}qt+r^{-\frac{5}{7}}q^{\frac{5}{7}}t^{\frac{13}{14}}+r^{-1}t^{\frac{1}{2}}\right)dt.$$

As before we substitute $\tau=\tilde{D}t$, estimate the occurring first integral trivial and use for the occurring second integral (4.29), to obtain

$$\int_3^\infty \tilde{D}e^{-\tilde{D}t}\left(\log(|D|qt)\right)^{15}\left(r^{-\frac{1}{2}}qt+r^{-\frac{5}{7}}q^{\frac{5}{7}}t^{\frac{13}{14}}+r^{-1}t^{\frac{1}{2}}\right)dt\ll$$
$$\ll\left(r^{-\frac{1}{2}}q\tilde{D}^{-1}+r^{-\frac{5}{7}}q^{\frac{5}{7}}\tilde{D}^{-\frac{13}{14}}+r^{-1}\tilde{D}^{-\frac{1}{2}}\right)\left(\log\left(|D|q\tilde{D}^{-1}\right)\right)^{15}$$
$$\ll\left(r^{\frac{1}{2}}\delta^{-1}q+r^{\frac{3}{14}}\delta^{-\frac{13}{14}}q^{\frac{5}{7}}+r^{-\frac{1}{2}}\delta^{-\frac{1}{2}}\right)\left(\log\left(|D|r\delta^{-1}q\right)\right)^{15},$$

where we used again in the last step that (4.25) implies $\tilde{D}^{-1}\le\delta^{-1}r$. Therefore,

$$\int_{\frac{1}{2}}^{1}\int_3^\infty \tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\left\{\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}N_K(\sigma,t,\chi^*)\right\}dt\,d\sigma\ll$$
$$\ll\left(r^{\frac{1}{2}}\delta^{-1}q+r^{\frac{3}{14}}\delta^{-\frac{13}{14}}q^{\frac{5}{7}}+r^{-\frac{1}{2}}\delta^{-\frac{1}{2}}\right)\left(\log\left(|D|r\delta^{-1}q\right)\right)^{15}$$

and

$$\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}}|\Gamma(\rho)z_0^{-\rho}|\ll$$
$$\ll\delta^{-1}\left\{r^{\frac{1}{2}}q+r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{5}{7}}+r^{-\frac{1}{2}}\delta^{\frac{1}{2}}\right\}\left(\log\left(|D|r\delta^{-1}q\right)\right)^{15}.$$

□

Now we consider the remaining case of non-trivial zeros with $|\gamma|\le 3$:

**Lemma 4.2.10** *Let $q \geq 3$ be an integer. Then*

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3}} |\Gamma(\rho) z_0^{-\rho}| \ll$$

$$\ll \delta^{-1} \left\{ r^{\frac{1}{2}} q + r^{\frac{3}{14}} \delta^{\frac{1}{14}} q^{\frac{5}{7}} + r^{-\frac{1}{2}} \delta^{\frac{1}{2}} \right\} (\log(|D|q))^{15},$$

*where $r = |z_0| < 1$, $\delta = \operatorname{Re} z_0$ and $\delta > 0$ is sufficiently small.*

**Proof**
For $\rho = \beta + i\gamma$, we split the proof in the two cases $\beta \geq \frac{1}{2}$ and $\beta < \frac{1}{2}$. If $|\gamma| \leq 3$ and $\beta \geq \frac{1}{2}$, then $\Gamma(\rho) z_0^{-\rho} \ll r^{-\beta}$. Moreover, we use

$$r^{-\beta} = r^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta} \frac{\log(r^{-1})}{r^{\sigma}} d\sigma,$$

to get

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \frac{1}{2} \leq \beta}} |\Gamma(\rho) z_0^{-\rho}| \ll$$

$$\ll \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \frac{1}{2} \leq \beta}} \left( r^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta} r^{-\sigma} \log(r^{-1}) d\sigma \right)$$

$$\ll r^{-\frac{1}{2}} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} N_K\left(\frac{1}{2}, 3, \chi^*\right) +$$

$$+ \int_{\frac{1}{2}}^{\beta} r^{-\sigma} \log(r^{-1}) \left\{ \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} N_K(\sigma, 3, \chi^*) \right\} d\sigma.$$

As in the proof of the previous lemma, we use the zero-density estimates (4.27) and (4.28) to obtain

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \frac{1}{2} \leq \beta}} |\Gamma(\rho) z_0^{-\rho}| \ll$$

$$\ll r^{-\frac{1}{2}} q \log^{14}(|D|q) + \log(r^{-1}) (\log(|D|q))^{14} \int_{\frac{1}{2}}^{\frac{5}{7}} r^{-\sigma} q^{\frac{5-4\sigma}{3}} d\sigma +$$

$$+ \log(r^{-1}) (\log(|D|q))^{14} \int_{\frac{5}{7}}^{1} r^{-\sigma} q^{\frac{5}{2}(1-\sigma)} d\sigma.$$

By partial integration,

$$\log(r^{-1})\,(\log(|D|q))^{14}\int_{\frac{1}{2}}^{\frac{5}{7}} r^{-\sigma} q^{\frac{5-4\sigma}{3}}\,d\sigma +$$

$$+\log(r^{-1})\,(\log(|D|q))^{14}\int_{\frac{5}{7}}^{1} r^{-\sigma} q^{\frac{5}{2}(1-\sigma)}d\sigma =$$

$$= (\log(|D|q))^{14}\left\{ r^{-\sigma} q^{\frac{5-4\sigma}{3}}\Big|_{\sigma=\frac{1}{2}}^{\frac{5}{7}} + \frac{4}{3}\log(q)\int_{\frac{1}{2}}^{\frac{5}{7}} r^{-\sigma} q^{\frac{5-4\sigma}{3}}\,d\sigma\right\} +$$

$$+ (\log(|D|q))^{14}\left\{ r^{-\sigma} q^{\frac{5}{2}(1-\sigma)}\Big|_{\sigma=\frac{5}{7}}^{1} + \frac{5}{2}\log(q)\int_{\frac{5}{7}}^{1} r^{-\sigma} q^{\frac{5}{2}(1-\sigma)}d\sigma\right\}$$

$$\ll (\log(|D|q))^{14}\log(q)\left(r^{-\frac{1}{2}}q + r^{-\frac{5}{7}}q^{\frac{5}{7}} + r^{-1}\right)$$

$$\ll (\log(|D|q))^{15}\left(r^{-\frac{1}{2}}q + r^{-\frac{5}{7}}q^{\frac{5}{7}} + r^{-1}\right)$$

and therefore,

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|\le 3,\ \frac{1}{2}\le\beta}} |\Gamma(\rho) z_0^{-\rho}| \ll \delta^{-1}\left(\delta r^{-\frac{1}{2}}q + \delta r^{-\frac{5}{7}}q^{\frac{5}{7}} + \delta r^{-1}\right)(\log(|D|q))^{15}.$$

We observe that $\delta = \operatorname{Re} z_0 \le |z_0| = r$ implies $\delta r^{-\frac{5}{7}} \le r^{\frac{3}{14}}\delta^{\frac{1}{14}}$ and $\delta r^{-1} \le r^{-\frac{1}{2}}\delta^{\frac{1}{2}}$, such that

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|\le 3,\ \frac{1}{2}\le\beta}} |\Gamma(\rho) z_0^{-\rho}| \ll \delta^{-1}\left(r^{\frac{1}{2}}q + r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{5}{7}} + r^{-\frac{1}{2}}\delta^{\frac{1}{2}}\right)(\log(|D|q))^{15}.$$

This proves the lemma for $\frac{1}{2} \le \beta$.

In the second case, where $|\gamma| \le 3$ and $\beta < \frac{1}{2}$, we use that $\Gamma(s) - s^{-1}$ is bounded on the rectangular area $0 \le \sigma \le \frac{1}{2},\ |t| \le 3$. Therefore,

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|\le 3,\ \beta<\frac{1}{2}}} |\Gamma(\rho) z_0^{-\rho}| \ll \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|\le 3,\ \beta<\frac{1}{2}}} |\rho|^{-1} r^{-\frac{1}{2}}.$$

Using

$$|\rho|^{-1} \ll \log q \ll \log(|D|q)$$

from [8, p. 122] and the zero-density estimate (4.27),

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|\le 3,\ \beta<\frac{1}{2}}} |\rho|^{-1} r^{-\frac{1}{2}} \ll \log(|D|q) r^{-\frac{1}{2}} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0,\chi_D^*}} N_K\left(\frac{1}{2}, 3, \chi^*\right)$$

$$\ll \delta^{-1}\left(\delta r^{-\frac{1}{2}}\right) q\,(\log(|D|q))^{15}.$$

Since $\delta r^{-1} \leq 1$ implies $\delta r^{-\frac{1}{2}} \leq r^{\frac{1}{2}}$,

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \beta < \frac{1}{2}}} |\rho|^{-1} r^{-\frac{1}{2}} \ll \delta^{-1} r^{\frac{1}{2}} q \left(\log(|D|q)\right)^{15},$$

which is already contained in the bound in case of $\frac{1}{2} \leq \beta$. Combination of both cases for $\beta$ proves the lemma. □

Taking into account Lemma 4.2.9 and Lemma 4.2.10 in Corollary 4.2.8, we get

**Corollary 4.2.11** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b, q) = 1$, and let $z_0 = \delta + 2\pi i \left(\alpha - \frac{b}{q}\right)$. Then for $r = |z_0| < 1$,*

$$f(\delta)^{-1} |f(\delta + 2\pi i \alpha)| \ll \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e + |D|q)\right)^2 + q^{-\frac{1}{2}} \left(\log(e+q)\right)^{16} + \\ + \left(r^{\frac{1}{2}} q^{\frac{1}{2}} + r^{\frac{3}{14}} \delta^{\frac{1}{14}} q^{\frac{3}{14}}\right) \left(\log(e + |D| r \delta^{-1} q)\right)^{16}.$$

**Proof**

Lemma 4.2.9 and Lemma 4.2.10 give an upper bound for

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0, \chi_D^*}} \sum_{\rho \in \mathcal{F}(\chi^*)} \left|\Gamma(\rho) z_0^{-\rho}\right|$$

in Corollary 4.2.8 if $q \geq 3$. For $q \leq 2$, the sum in empty. We use that $\varphi(q)^{-1} \ll \frac{\log(e+q)}{q}$ and $1 \leq r\delta^{-1}$, to get

$$\begin{aligned} f(\delta)^{-1} |f(\delta + 2\pi i \alpha)| &\ll \\ &\ll \frac{\sqrt{q}}{\varphi(q)} + \delta \log(\delta^{-1}) \sqrt{q} \left(\log(e + |D|q)\right)^2 + \\ &\quad + \frac{\sqrt{q}}{\varphi(q)} \left(r^{\frac{1}{2}} q + r^{\frac{3}{14}} \delta^{\frac{1}{14}} q^{\frac{5}{7}} + r^{-\frac{1}{2}} \delta^{\frac{1}{2}}\right) \left(\log(|D| r \delta^{-1} q)\right)^{15} \\ &\ll q^{-\frac{1}{2}} \log(e+q) + \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e + |D|q)\right)^2 + \\ &\quad + \left(r^{\frac{1}{2}} q^{\frac{1}{2}} + r^{\frac{3}{14}} \delta^{\frac{1}{14}} q^{\frac{3}{14}}\right) \left(\log(e + |D| r \delta^{-1} q)\right)^{16} + \\ &\quad + \left(r \delta^{-1} q\right)^{-\frac{1}{2}} \left(\log(e + |D| r \delta^{-1} q)\right)^{16}. \end{aligned}$$

Recall that $|D|$ is fixed. To estimate the last term, observe that

$$u^{-\frac{1}{2}} \left(\log(e + |D|u)\right)^{16}$$

is bounded and strictly decreasing for sufficiently large $u \geq 1$. Since $1 \leq |D|$ and $1 \leq r\delta^{-1}$ we have $q \leq |D| r\delta^{-1} q$, and therefore

$$\left(r\delta^{-1}q\right)^{-\frac{1}{2}} \left(\log(e + |D| r\delta^{-1} q)\right)^{16} \ll q^{-\frac{1}{2}} \left(\log(e + q)\right)^{16}.$$

Moreover,

$$q^{-\frac{1}{2}} \log(e+q) \leq q^{-\frac{1}{2}} \left(\log(e+q)\right)^{16}$$

which proves the estimate of the corollary. □

For rational approximation of $\alpha$ we use the well-known approximation theorem of Dirichlet, see for example Landau [41, Satz 159] or Hardy & Wright [25, Theorem 36]:

Let $\alpha$ and $Q \geq 1$ be real. Then there exist integers $b$ and $q$ such that $(b, q) = 1,\ 1 \leq q \leq Q$ and $\left|\alpha - \frac{b}{q}\right| < q^{-1} Q^{-1}$.

**Theorem 4.2.12** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small (but at least $\delta < \frac{1}{50}$). Let $Q = \delta^{-\frac{2}{3}}$ and let $b$ and $q$ be integers such that $(b, q) = 1,\ 1 \leq q \leq Q$ and $\left|\alpha - \frac{b}{q}\right| < q^{-1} Q^{-1}$. Then*

$$f(\delta)^{-1} |f(\delta + 2\pi i \alpha)| \ll \delta^{\frac{1}{7}} \left(\log(e + |D|\delta^{-1})\right)^{16} + q^{-\frac{1}{2}} \left(\log(e+q)\right)^{16}.$$

**Proof**

Let $Q = \delta^{-\lambda}$, where $\frac{1}{2} < \lambda < 1$. Suppose that $b$ and $q$ are choosen such that $(b, q) = 1,\ 1 \leq q \leq Q$ and $|\alpha - \frac{b}{q}| < q^{-1} Q^{-1}$. We look at two seperate cases:

If $\delta^{\lambda - 1} \leq q \leq \delta^{-\lambda}$, then

$$\begin{aligned}
\delta \leq r = |z_0| &= \left(\delta^2 + 4\pi^2 \left|\alpha - \frac{b}{q}\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\delta^2 + 4\pi^2 q^{-2} \delta^{2\lambda}\right)^{\frac{1}{2}} \leq \left(\delta^2 + 4\pi^2 \delta^{2-2\lambda} \delta^{2\lambda}\right)^{\frac{1}{2}} = \delta \left(1 + 4\pi^2\right)^{\frac{1}{2}} \ll \delta,
\end{aligned}$$

resp. $\delta \leq r \ll \delta$. This implies for the last term in Corollary 4.2.11

$$\begin{aligned}
&\left(r^{\frac{1}{2}} q^{\frac{1}{2}} + r^{\frac{3}{14}} \delta^{\frac{1}{14}} q^{\frac{3}{14}}\right) \left(\log(e + |D| r\delta^{-1} q)\right)^{16} \ll \\
&\qquad \ll \left(\delta^{\frac{1-\lambda}{2}} + \delta^{\frac{4-3\lambda}{14}}\right) \left(\log(e + |D|\delta^{-1})\right)^{16}.
\end{aligned}$$

If $1 \leq q \leq \delta^{\lambda - 1}$ then

$$\begin{aligned}
\delta \leq r = |z_0| &= \left(\delta^2 + 4\pi^2 \left|\alpha - \frac{b}{q}\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\delta^2 + 4\pi^2 q^{-2} \delta^{2\lambda}\right)^{\frac{1}{2}} \leq \left(\delta^2 + 4\pi^2 \delta^{2\lambda}\right)^{\frac{1}{2}} = \delta^{\lambda} \left(\delta^{2-2\lambda} + 4\pi^2\right)^{\frac{1}{2}} \\
&\leq \delta^{\lambda} \left(1 + 4\pi^2\right)^{\frac{1}{2}} \ll \delta^{\lambda},
\end{aligned}$$

resp. $\delta \le r \ll \delta^\lambda$. In this case, we get for the last term in Corollary 4.2.11

$$\left(r^{\frac{1}{2}}q^{\frac{1}{2}} + r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{3}{14}}\right)\left(\log(e+|D|r\delta^{-1}q)\right)^{16} \ll \\ \ll \left(\delta^{\lambda-\frac{1}{2}} + \delta^{\frac{3\lambda-1}{7}}\right)\left(\log(e+|D|\delta^{-1})\right)^{16}.$$

So either in the first or the second case,

$$\left(r^{\frac{1}{2}}q^{\frac{1}{2}} + r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{3}{14}}\right)\left(\log(e+|D|r\delta^{-1}q)\right)^{16} \ll \\ \ll \left(\delta^{\frac{1-\lambda}{2}} + \delta^{\frac{4-3\lambda}{14}} + \delta^{\lambda-\frac{1}{2}} + \delta^{\frac{3\lambda-1}{7}}\right)\left(\log(e+|D|\delta^{-1})\right)^{16}.$$

For the exponents,

$$\sup_{\frac{1}{2}<\lambda<1}\left\{\min\left(\frac{1-\lambda}{2}, \frac{4-3\lambda}{14}, \lambda-\frac{1}{2}, \frac{3\lambda-1}{7}\right)\right\} = \frac{1}{7}$$

occures at $\lambda = \frac{2}{3}$, and yields

$$\left(r^{\frac{1}{2}}q^{\frac{1}{2}} + r^{\frac{3}{14}}\delta^{\frac{1}{14}}q^{\frac{3}{14}}\right)\left(\log(e+|D|r\delta^{-1}q)\right)^{16} \ll \delta^{\frac{1}{7}}\left(\log(e+|D|\delta^{-1})\right)^{16}. \tag{4.30}$$

Moreover, $\lambda = \frac{2}{3}$ implies $Q = \delta^{-\frac{2}{3}}$, and if $\delta$ is at least $0 < \delta < \frac{1}{50}$, we have $Q > 1$ and $r = |z_0| < 1$. Therefore, Corollary 4.2.11 can be applied and in combination with (4.30)

$$f(\delta)^{-1}|f(\delta+2\pi i\alpha)| \ll \delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+|D|q)\right)^2 + q^{-\frac{1}{2}}\left(\log(e+q)\right)^{16} + \\ + \delta^{\frac{1}{7}}\left(\log(e+|D|\delta^{-1})\right)^{16}.$$

We observe that $q \le Q = \delta^{-\frac{2}{3}}$ implies

$$\delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+|D|q)\right)^2 \ll \delta^{\frac{1}{7}}\left(\log(e+|D|\delta^{-1})\right)^{16},$$

which completes the proof of the theorem. □

As an immediate consequence of Theorem 4.2.12 we obtain

**Corollary 4.2.13** *Let $\alpha$ be a irrational. Then*

$$\lim_{\delta\to 0+} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| = 0.$$

**Proof**
We adopt the notation of Theorem 4.2.12, where $\left|\alpha - \frac{b}{q}\right| < q^{-1}Q^{-1} = q^{-1}\delta^{\frac{2}{3}}$. If $\delta \to 0+$ we must have $q \to \infty$. Thus

$$\delta^{\frac{1}{7}}\left(\log(e+|D|\delta^{-1})\right)^{16} + q^{-\frac{1}{2}}\left(\log(e+q)\right)^{16}$$

tends to zero as $\delta \to 0+$. □

We investigate the related function

$$g(z) = \sum_{p} (1 + \chi_D(p))(\log p)e^{-pz},$$

where $z = \delta + i\xi,\ \delta > 0$. For this function we prove

**Theorem 4.2.14** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small (but at least $\delta < \frac{1}{50}$). Further, let $Q = \delta^{-\frac{2}{3}}$ and let $b$ and $q$ be such that $(b, q) = 1,\ 1 \le q \le Q$, and $\left|\alpha - \frac{b}{q}\right| < q^{-1}Q^{-1}$. Then*

$$g(\delta)^{-1}|g(\delta + 2\pi i\alpha)| \ll \delta^{\frac{1}{7}} \left(\log(e + |D|\delta^{-1})\right)^{16} + q^{-\frac{1}{2}} \left(\log(e + q)\right)^{16}.$$

**Proof**
First we verify for the generalised Chebyshev $\vartheta$-function

$$\vartheta_K(x) := \sum_{p \le x} (1 + \chi_D(p)) \log p \tag{4.31}$$

the asymptotic

$$\vartheta_K(x) = x + o(x) \qquad (x \to \infty). \tag{4.32}$$

Recalling (4.2) and (4.3),

$$\begin{aligned} \psi_K(x) &= \sum_{n \le x} \Lambda_K(n) = \sum_{p^m \le x} (1 + \chi_D(p^m)) \log p \\ &= \sum_{m=1}^{\infty} \sum_{p \le x^{\frac{1}{m}}} (1 + \chi_D(p)) \log p = \sum_{m=1}^{\infty} \vartheta_K\left(x^{\frac{1}{m}}\right). \end{aligned}$$

This is a finite sum and in fact empty if $x^{\frac{1}{m}} < 2$, implying $m > \frac{\log x}{\log 2} = \log_2 x$. Therefore,

$$\psi_K(x) = \sum_{m \le \log_2 x} \sum_{p \le x^{\frac{1}{m}}} (1 + \chi_D(p)) \log p = \sum_{m \le \log_2 x} \vartheta_K\left(x^{\frac{1}{m}}\right).$$

Since $1 + \chi_D(n) \ll 1$, we get $\vartheta_K(x) \ll \sum_{p \le x} \log p \ll x \log x$ and for the difference

$$\begin{aligned} 0 \le \psi_K(x) - \vartheta_K(x) &= \sum_{2 \le m \le \log_2 x} \vartheta_K\left(x^{\frac{1}{m}}\right) \\ &\ll \sum_{2 \le m \le \log_2 x} x^{\frac{1}{m}} \log\left(x^{\frac{1}{m}}\right) \ll (\log_2 x) x^{\frac{1}{2}} \log(x^{\frac{1}{2}}) \ll x^{\frac{1}{2}} (\log x)^2. \end{aligned}$$

So

$$0 \leq \frac{\psi_K(x)}{x} - \frac{\vartheta_K(x)}{x} \ll x^{-\frac{1}{2}}(\log x)^2,$$

where the right side tends to zero as $x \to \infty$, implying

$$\lim_{x\to\infty}\left(\frac{\psi_K(x)}{x} - \frac{\vartheta_K(x)}{x}\right) = 0. \tag{4.33}$$

Therefore, if one of the limits $\lim_{x\to\infty}\frac{\psi_K(x)}{x}$ or $\lim_{x\to\infty}\frac{\vartheta_K(x)}{x}$ exitsts, so does the other and both limits are equal. We know from (4.3) that $\lim_{x\to\infty}\frac{\psi_K(x)}{x} = 1$, implying $\lim_{x\to\infty}\frac{\vartheta_K(x)}{x} = 1$ resp. (4.32) by (4.33).

Using (4.32) and that $\vartheta_K(x)$ vanishes for $0 \leq x \leq 1$, one can prove as in Theorem 4.2.1 that

$$g(\delta) \sim \delta^{-1} \qquad (\delta \to 0+). \tag{4.34}$$

We rewrite

$$\begin{aligned} f(z) &= \sum_{n\geq 1}\Lambda_K(n)e^{-nz} \\ &= \sum_p (1+\chi_D(p))(\log p)e^{-pz} + \sum_p\sum_{m=2}^{\infty}(1+\chi_D(p^m))(\log p)e^{-p^m z} \\ &= g(z) + E(z), \end{aligned}$$

say. In view of (4.34) it remains to estimate $E(\delta)$. We use the representation as a Stieltjes integral,

$$E(\delta) = \int_1^{\infty} e^{-\delta y}\,\mathrm{d}\,U(y),$$

where

$$\begin{aligned} U(y) &= \sum_{\substack{p^m\leq y\\ 2\leq m}}(1+\chi_D(p^m))\log p = \sum_{m=2}^{\infty}\sum_{p\leq y^{\frac{1}{m}}}(1+\chi_D(p))\log p \\ &= \sum_{m=2}^{\infty}\vartheta_K\left(y^{\frac{1}{m}}\right). \end{aligned}$$

The inner sum $\vartheta_K\left(y^{\frac{1}{m}}\right) = \sum_{p\leq y^{\frac{1}{m}}}(1+\chi_D(p))\log p$ is finite and if we assume that it is non-empty, there are primes satisfying $2 \leq p \leq y^{\frac{1}{m}}$, i.e. $\frac{\log y}{\log 2} \geq m$, or $m \ll \log y$. We also note that for all $m \geq 3$, $\vartheta_K\left(y^{\frac{1}{m}}\right) \ll \vartheta_K\left(y^{\frac{1}{3}}\right)$. Since $\vartheta_K\left(y^{\frac{1}{m}}\right) = 0$ for $\left[\frac{\log y}{\log 2}\right] < m$, we get with (4.32)

$$\begin{aligned} U(y) &= \vartheta_K\left(y^{\frac{1}{2}}\right) + \sum_{m=3}^{\infty}\vartheta_K\left(y^{\frac{1}{m}}\right) = \vartheta_K\left(y^{\frac{1}{2}}\right) + O\left((\log y)\vartheta_K\left(y^{\frac{1}{3}}\right)\right) \\ &\ll y^{\frac{1}{2}} + (\log y)y^{\frac{1}{3}} \ll y^{\frac{1}{2}}. \end{aligned}$$

Applying partial integration

$$\begin{aligned} E(\delta) &= \int_1^\infty e^{-\delta y}\,\mathrm{d}\,U(y) = e^{-\delta y}U(y)\Big|_{y=1}^\infty + \delta \int_1^\infty U(y)e^{-\delta y}dy \\ &= \delta \int_1^\infty U(y)e^{-\delta y}dy \end{aligned}$$

where the first term vanishs in view of $\vartheta_K(1) = 0$ and $U(y) \ll y^{\frac{1}{2}}$. Using $\int_1^\infty y^{\frac{1}{2}} e^{-\delta y}dy \ll \delta^{-\frac{3}{2}}$ (which can be verified by substituting $x = \delta y$) we obtain

$$E(\delta) \ll \delta \int_1^\infty y^{\frac{1}{2}} e^{-\delta y}dy \ll \delta^{-\frac{1}{2}}. \tag{4.35}$$

Since $g(\delta)^{-1} \sim \delta \sim f(\delta)^{-1}$ for $\delta \to 0+$ and $f(z) = g(z) + E(z)$, we get by (4.35) and Theorem 4.2.12

$$\begin{aligned} g(\delta)^{-1}|g(\delta + 2\pi i\alpha)| &\ll f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| + f(\delta)^{-1}E(\delta) \\ &\ll \delta^{\frac{1}{7}} \left(\log(e + |D|\delta^{-1})\right)^{16} + q^{-\frac{1}{2}} \left(\log(e+q)\right)^{16} + \delta^{\frac{1}{2}} \\ &\ll \delta^{\frac{1}{7}} \left(\log(e + |D|\delta^{-1})\right)^{16} + q^{-\frac{1}{2}} \left(\log(e+q)\right)^{16}. \end{aligned}$$

□

Now we can prove our main theorem in the context of weights related to the Dedekind zeta-function of a quadratic number field:

**Proof of Theorem 1.3.7:**
Let $\alpha > 0$ be irrational. Analogue to the proof of Corollary 4.2.13, we obtain from Theorem 4.2.14

$$\lim_{\delta \to 0+} g(\delta)^{-1}|g(\delta + 2\pi i\alpha)| = 0.$$

This implies that for every nonzero integer $h$

$$\lim_{\delta \to 0+} \frac{g(\delta + 2\pi i\alpha h)}{g(\delta)} = 0,$$

which was to prove by (1.50). We also know $g(\delta) \sim \delta^{-1}$ for $\delta \to 0+$ from (4.34). Therefore, Vaaler's theorem (cf. p. 32) implies that $(\alpha p_n)_{n\geq 1}$ is $(1 + \chi_D(p_n))\log p_n$– u.d. mod 1, where $p_1, p_2, ...$ is the sequence of primes for which $1 + \chi_D(p_n)$ is positive. As remarked in the introductory part on uniform distribution of sequences modulo one, $\alpha$ can be replaced by $-\alpha$, which yields Theorem 1.3.7 for irrational $\alpha$. □

To prove that Theorem 1.3.7 implies Corollary 1.3.8, we need Vaaler's result on the replacement of weights (p. 34). For the sake of completeness we state his result as an lemma and recall its proof from [72].

**Lemma 4.2.15** *Let $a(n)$ be a sequence of weights and let $\gamma(n)$ be a monotone sequence (either increasing or decreasing) of positive real numbers. Suppose that*

*(i)* $\sum_{n=1}^{N} a(n)\gamma(n)$ *diverges for* $N \to \infty$,

*(ii)* $\gamma(N) \sum_{n=1}^{N} a(n) \ll \sum_{n=1}^{N} a(n)\gamma(n)$ *for all* $N \geq 1$.

*Then $a(n)\gamma(n)$ is stronger than $a(n)$.*

**Proof**
Assume $(x_n)_{n\geq 1}$ is $a(n)$-u.d. mod 1. For each integer $h \neq 1$, let

$$A_h(N) = \sum_{n=1}^{N} a(n) e^{2\pi i h x_n}.$$

Note that by (1.6) the quotient $A(N)A_h(N) \to 0$ as $N \to \infty$, where $A(N)$ is the sum of weights (1.4). We prove that

$$\left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \left(\sum_{n=1}^{N} a(n)\gamma(n) e^{2\pi i h x_n}\right) \tag{4.36}$$

tends to zero as $N \to \infty$. Partial summation yields that (4.36) is equal to

$$\left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \left(A_h(N)\gamma(N) - \sum_{n=1}^{N} A_h(n)\left(\gamma(n+1) - \gamma(n)\right)\right) =$$

$$= \left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} A_h(N)\gamma(N)$$

$$- \left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \sum_{n=1}^{N-1} \frac{A_h(n)}{A(n)} \left(A(n)\left(\gamma(n+1) - \gamma(n)\right)\right).$$

By assumption *(ii)* we get for the first term

$$\left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} A_h(N)\gamma(N) \ll \left(\gamma(N)\sum_{n=1}^{N} a(n)\right)^{-1} A_h(N)\gamma(N)$$

$$\ll \frac{A_h(N)}{A(N)},$$

where the right side tends to zero as $N \to \infty$. For the other term we note that $\gamma(n+1) - \gamma(n)$ is of one sign for all $n \geq 1$, and that $|A(n)^{-1}A_h(n)| \to 0$ as $n \to \infty$. Therefore, since by *(i)* the sum $\sum_{n=1}^{N} a(n)\gamma(n)$ diverges as $N \to \infty$, it suffices to show that

$$\left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \sum_{n=1}^{N-1} A(n)\left|\gamma(n+1) - \gamma(n)\right| \ll 1.$$

Again by summation by parts,

$$\sum_{n=1}^{N} a(n)\gamma(n) = A(N)\gamma(N) - \sum_{n=1}^{N-1} A(n)\left(\gamma(n+1) - \gamma(n)\right),$$

implying

$$\sum_{n=1}^{N-1} A(n)\left|\gamma(n+1) - \gamma(n)\right| \leq A(N)\gamma(N) + \sum_{n=1}^{N} a(n)\gamma(n).$$

Thus

$$\begin{aligned}\left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \sum_{n=1}^{N-1} A(n)\left|\gamma(n+1) - \gamma(n)\right| \leq \\ \leq 1 + \left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} A(N)\gamma(N) \\ = 1 + \left(\sum_{n=1}^{N} a(n)\gamma(n)\right)^{-1} \gamma(N)\sum_{n=1}^{N} a(n) \ll 1\end{aligned}$$

by *(ii)*, which completes the proof. □

Now we can prove Corollary 1.3.8:

**Proof of Corollary 1.3.8:**
By partial sumation and (4.32), we observe that

$$\sum_{2\leq p\leq x} (1 + \chi_D(p)) = \frac{\vartheta_K(x)}{\log x} + \int_2^x \frac{\vartheta_K(t)}{t\,(\log t)^2} dt \sim \frac{x}{\log x}$$

as $x \to \infty$. Let $a(n) = (1 + \chi_D(p_n)) \log p_n$ and $\gamma(n) = (\log p_n)^{-1}$. Then

$$\sum_{n=1}^{N} a(n)\gamma(n) = \sum_{n=1}^{N} (1 + \chi_D(p_n))$$

diverges as $N \to \infty$, while for all $N \geq 1$

$$\begin{aligned}\gamma(N)\sum_{n=1}^{N} a(n) &= \frac{1}{\log p_N} \sum_{n=1}^{N} (1 + \chi_D(p_n)) \log p_n \\ &\leq \sum_{n=1}^{N} (1 + \chi_D(p_n)) = \sum_{n=1}^{N} a(n)\gamma(n).\end{aligned}$$

Therfore, $a(n)\gamma(n) = 1 + \chi_D(p_n)$ is stronger than $a(n) = (1 + \chi_D(p_n)) \log p_n$ and the corollary follows from Lemma 4.2.15 and Theorem 1.3.7. □

## 4.3 Weights from the class $\mathcal{A}$

For $F \in \mathcal{A}$ and a Dirichlet character $\chi$ mod $q$, let

$$f(z) = \sum_{n\geq 1} \Lambda_F(n) e^{-nz} \tag{4.37}$$

resp.

$$f(z;\chi) = \sum_{n\geq 1} \Lambda_F(n)\chi(n) e^{-nz}, \tag{4.38}$$

where $z = \delta + i\xi,\ \delta > 0$. Here $\Lambda_F$ denotes the generalised von Mangoldt function from (1.35), which is supported on prime powers (see also (1.36)).

As already remarked, some parts in the proofs in the case of weights from the class $\mathcal{A}$ are similar to the case of weights from the Dedekind zeta-function. Therefore, we focus in the following proofs on the parts that need to be adapted.

We notice that, for $\chi$ beeing the principal character $\chi_0$ mod 1, assumption $(\mathcal{A}_3)$ implies

$$\psi_F(x) = m_F x + o(x) \tag{4.39}$$

as $x \to \infty$. Thus $(\mathcal{A}_3)$ implies the prime number theorem for the class $\mathcal{A}$ with positive $m_F$.

Using (4.39), we get as an generalisation of Theorem 4.2.1

**Theorem 4.3.1** *The series $f(\delta) = \sum_{n\geq 1} \Lambda_F(n) e^{-\delta n}$ converges if and only if $\delta > 0$. For $\delta \to 0+$ we have $f(\delta) \sim m_F \delta^{-1}$.*

**Proof**
By (1.36), and since by definition $|\alpha_j(p)| \leq 1$ for all $j$ and $p$, we obtain

$$\Lambda_F(p^k) \ll \nu_F \log p \ll_F \log p, \tag{4.40}$$

where the implied constant may depend on the integer $\nu_F > 0$.

Note that from now on we allow the implied constant to depend on some constant which may depends on $F$, even if this is not explicitely stated.

Using estimate (4.40) and the prime number theorem (4.39), the proof is analogous to the proof of Theorem 4.2.1. □

The next theorem establishs an upper bound for $f(\delta)^{-1}|f(\delta + 2\pi i\alpha)|$, similar to the one in Theorem 4.2.2. In contrast to the Dedekind zeta-function of a quadratic number fields, it is a priori not clear which character has to be excluded such that the twist of $F \in \mathcal{A}$ is regular at $s = 1$, or if such a character exists.

**Theorem 4.3.2** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b, q) = 1$, and let $z_0 = \delta + 2\pi i \left(\alpha - \frac{b}{q}\right)$. Denote by $\chi^*$ mod $q^*$ the primitive character inducing $\chi$ mod $q$. Then*

$$f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \varphi(q)^{-1} + \delta \log\left(\delta^{-1}\right)\sqrt{q}\log q + \\ + \delta \frac{\sqrt{q}}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |f(z_0; \chi^*)|.$$

**Proof**
Assumption $(\mathcal{A}_1)$ and Theorem 4.3.1 imply $f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \varphi(q)^{-1}$ for $q = 1$ and $q = 2$, as in the proof of Theorem 4.2.2.

In case of $q \geq 3$, we use $(\mathcal{A}_1)$ and (4.40) to obtain

$$f(\delta)^{-1}\,|f(\delta + 2\pi i\alpha)| \ll \\ \ll \delta \left| \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} e^{-2\pi i\left(\frac{bh}{q}\right)} \sum_{\substack{n=1 \\ n \equiv h \pmod q}}^{\infty} \Lambda_F(n) e^{-n\left(\delta + 2\pi i\left(\alpha - \frac{b}{q}\right)\right)} \right| + \\ + \delta \sum_{p|q} \log p \sum_{m=1}^{\infty} e^{-\delta p^m}.$$

We proceed as in the proof of Theorem 4.2.2, where instead of (4.8) we get

$$\sum_{\chi \pmod q} \tau(\overline{\chi})\chi(-b) f(z_0; \chi) = \\ = \tau(\overline{\chi_0})\chi_0(-b) \sum_{n \geq 1} \Lambda_F(n)\chi_0(n) e^{-nz_0} + \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\overline{\chi})\chi(-b) f(z_0; \chi) \\ \ll \sum_{n \geq 1} \Lambda_F(n) e^{-n\delta} + \sqrt{q} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |f(z_0; \chi)|$$

and remark that this estimate leads to the different estimate in the theorem to prove, in comparison with Theorem 4.2.2. □

To continue we need some bounds related to the logarithmic derivative. Note that the trivial zeros of the multiplicative twist of $F$ are by assumption $(\mathcal{A}_2)$ at $\rho = -\frac{k + \mu_j}{\lambda_j}$ and the inequality $-1 \leq \operatorname{Re} \rho \leq 0$ implies

$$-\operatorname{Re} \mu_j \leq k \leq \lambda_j - \operatorname{Re} \mu_j,$$

where $j = 1, ..., r$ and $r \in \mathbb{N}$ is fixed. Since only finitely many integers $k \geq 0$ can satisfy this inequality,

$$\mathcal{M} := \max\{|\operatorname{Im} \rho| : \rho \text{ is a trivial zero of } F(\cdot, \chi),\ \operatorname{Re} \rho \geq -1\}$$

exists.

We remark that, unlike Dirichlet L-functions, the zeros of $F \in \mathcal{S}^{poly}$ may not be symmetric to the real line. Therefore we consider in the analogon of Theorem 4.2.3 two cases, corresponding to the upper half-plane and the lower half-plane.

**Theorem 4.3.3** *Let* $\chi^*$ mod $q^*$ *be a primitive Dirichlet character and* $m \geq \max\{2, \mathcal{M}\}$ *an integer. Then there exists a sequence of real numbers* $T_m = T_m(\chi^*)$ *with* $m < T_m < m+1$, *such that for* $-1 \leq \sigma \leq 2$

$$\frac{F'}{F}(\sigma + iT_m; \chi) \ll (\log(q^* T_m))^2 .$$

**Proof**

Consider the ordinates of the non-trivial zeros of $F(s, \chi^*)$ with $m < t < m+1$ in the upper half-plane. Assumption $(\mathcal{A}_2)$ implies that (1.40) can be applied, to obtain

$$N_F^+(T+1) - N_F^+(T) \ll \log T,$$

which is the analogon of (4.12) in the upper half-plane. We continue as in the proof of Theorem 4.2.3, where we use that $(\mathcal{A}_2)$ implies that the partial-fraction decomposition (1.41) holds in the class $\mathcal{A}$, i.e.

$$\frac{F'}{F}(s; \chi^*) = \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} + O\left(\log |t|\right);$$

the sum is taken over all non-trivial zeros $\rho = \beta + i\gamma$ of $F(\cdot, \chi^*)$ such that $|\gamma - t| \leq 1$ and the implied constant may depend on the data of $F(\cdot, \chi^*)$. □

In view of (1.40) we get by a similar reasoning a sequence $T_m = T_m(\chi^*)$ such that in the lower half-plane

$$\frac{F'}{F}(\sigma - iT_m; \chi^*) \ll (\log(q^* T_m))^2 , \tag{4.41}$$

where $-1 \leq \sigma \leq 2$.

Moreover, we also need an estimate for the logarithmic derivative to the left of the vertical line $\sigma = -1$.

**Theorem 4.3.4** *Let* $\chi^*$ mod $q^*$ *be a primitive Dirichlet character and denote by* $\rho$ *the trivial zeros of* $F(\cdot, \chi^*)$. *Exclude from the half-plane* $\sigma \leq -1$ *the points lying inside the disc* $|s + \rho| \leq \frac{1}{2}$. *Then in the remaining area we have for* $s = \sigma + iT$

$$\frac{F'}{F}(s; \chi^*) \ll \log |s|,$$

*if* $|T|$ *is sufficiently large.*

**Proof**

Since by assumption $(\mathcal{A}_2)$ the multiplicative twist with a primitive character is in $\mathcal{S}^{poly}$, the functional equation $(\mathcal{S}_3)$ implies

$$F(s;\chi^*) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma\left(\lambda_j(1-s)+\overline{\mu_j}\right)}{\Gamma\left(\lambda_j s+\mu_j\right)} \cdot \overline{F\left(1-\overline{s};\chi^*\right)}$$

with $Q>0,\ \lambda_j>0,\ \operatorname{Re}\mu_j \geq 0$ and $|\omega|=1$. For $\sigma<0$ we have $1-\sigma>1$, implying for $s=\sigma+iT$ that $1-\overline{s}=(1-\sigma)+iT$ is in the right half-plane with real part greater than one. In this half-plane $F\left(1-\overline{s},\chi^*\right)$ has an absolutely convergent Dirichlet-series expansion by $(\mathcal{S}_1)$ (of the form (1.33)), and

$$\overline{F\left(1-\overline{s};\chi^*\right)} = \sum_{n\geq 1} \overline{a(n)\chi^*(n)} n^{-(1-s)}.$$

So differentiation of this series with respect to $s$ is possible. Logarithmic differentiation and $(\mathcal{A}_1)$ yield

$$\begin{aligned}\frac{F'}{F}(s;\chi^*) = &- 2\log Q - \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s)+\overline{\mu_j}\right)\\ &- \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j s+\mu_j\right) - \frac{F'}{F}(1-s;\overline{\chi^*}).\end{aligned}$$

The logarithmic derivative $\frac{F'}{F}(1-s;\overline{\chi^*})$ is absolutely convergent for $\sigma<0$, and since $Q>0$ is fixed, we have

$$-2\log Q - \frac{F'}{F}(1-s;\overline{\chi^*}) \ll 1.$$

By (4.21),

$$\begin{aligned}&\sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s)+\overline{\mu_j}\right) \ll\\ &\qquad \ll \sum_{j=1}^{r} \log\left|\lambda_j(1-s)+\overline{\mu_j}\right| + O\left(\left|\lambda_j(1-s)+\overline{\mu_j}\right|^{-1}\right).\end{aligned}$$

For suffiently large $|T|$, we get $\log\left|\lambda_j(1-s)+\overline{\mu_j}\right| \ll \log|s|$ and $|\lambda_j(1-s)+\overline{\mu_j}|^{-1}=O(1)$. Thus

$$\sum_{j=1}^{r} \log\left|\lambda_j(1-s)+\overline{\mu_j}\right| + O\left(\left|\lambda_j(1-s)+\overline{\mu_j}\right|^{-1}\right) \ll \log|s|.$$

The seconds sum over the logarithmic derivative of the Gamma-function can be estimated in the same way. □

We adopt the notation of Definition 4.2.5 in the context of the class $\mathcal{A}$ and continue with the analogon of Theorem 4.2.6.

**Definition 4.3.5** *Let* $\chi^* \bmod q^*$ *be a primitive Dirichlet character. Define* $\mathcal{F}(\chi^*)$ *to be the set of all non-trivial zeros* $\rho = \beta + i\gamma$, $\beta > 0$ *of the multiplicative twist of* $F \in \mathcal{A}$, *i.e. (1.33). If we write a sum over* $\rho \in \mathcal{F}(\chi^*)$, *we write this with the meaning that a zero of multiplicity* $N$ *appears* $N$*-times in the sum.*

**Theorem 4.3.6** *Let* $\chi^* \bmod q^*$ *be a primitive Dirichlet character with* $q^* \geq 3$. *There exists entire functions*

$$h^{(1)}(z;\chi^*) = \sum_{n=0}^{\infty} \kappa_n(\chi^*) z^n$$

*and*

$$h^{(2)}(z;\chi^*) = \sum_{n\geq 0}\sum_{j=1}^{r} \kappa_{n,j}(\chi^*) z^{\frac{n+\mu_j}{\lambda_j}}$$

*such that for* $z = \delta + i\xi$, $\delta > 0$, *we have*

$$f(z;\chi^*) = -\sum_{\rho\in\mathcal{F}(\chi^*)} \Gamma(\rho) z^{-\rho} + h^{(1)}(z;\chi^*) + h^{(2)}(z;\chi^*).$$

*If* $\sum'$ *denotes the sum over all* $j = 1, ..., r$ *with* $-\lambda_j n + \mu_j \notin \mathbb{Z}$ *and* $\sum''$ *the sum over all* $j = 1, ..., r$ *with* $-\lambda_j n + \mu_j \in \mathbb{Z}$, *the coefficients are given by*

$$\kappa_n(\chi^*) = \frac{(-1)^n}{n!}\left\{2\log Q + \frac{F'}{F}\left(1+n;\overline{\chi^*}\right) + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1+n) + \overline{\mu_j}\right) + \right.$$
$$\left. + \sum_{j=1}^{r}{}' \lambda_j \frac{\Gamma'}{\Gamma}\left(-\lambda_j n + \mu_j\right) + \sum_{j=1}^{r}{}'' \lambda_j \frac{\Gamma'}{\Gamma}\left(1 + \lambda_j n - \mu_j\right)\right\}$$

*resp.*

$$\kappa_{n,j}(\chi^*) = \frac{\pi\lambda_j}{\sin\left(\pi\cdot\frac{n+\mu_j}{\lambda_j}\right)\Gamma\left(1+\frac{n+\mu_j}{\lambda_j}\right)}.$$

*They satisfy*

$$\kappa_n(\chi^*) \ll \frac{1+\log(1+n)}{n!}$$

*and*

$$\kappa_{n,j}(\chi^*) \ll \frac{1}{n^{\frac{1}{2}+n} e^n}.$$

**Proof**

For the first part of the proof we proceed as in the proof of Theorem 4.2.6. Note that in case of the class $\mathcal{A}$, the logarithmic derivative $\frac{F'}{F}(s;\chi^*)$ has simple poles at $s_j = -\frac{n+\mu_j}{\lambda_j}$ in the left half-plane, i.e. $\operatorname{Re} s_j < 0$, with $n = 0, 1, ...$ and $j = 1, ..., r$ , since $F(s;\chi^*) \in \mathcal{S}^{poly}$ has trivial zeros of order

$m(\rho)$ at $\rho = -\frac{n+\mu_j}{\lambda_j}$. Observe that the assumption of trivial zeros on the imaginary axis implies by the function equation $(\mathcal{S}_3)$ zeros on the vertical line $\sigma = 1$, contradicting the prime number theorem (4.39) by (1.39);

simple poles at the non-trivial zeros inside the strip $0 < \sigma < 1$ with residue equal to the order of the zero. Note that again by (1.39), there are no zeros on the vertical line $\sigma = 1$ and therefore the functional equation $(\mathcal{S}_3)$ implies that there are no zeros on the vertical line $\sigma = 0$.

By assumption $(\mathcal{A}_2)$, no pole arises at $s = 1$. Therefore, the integrand in

$$f(z;\chi^*) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \Gamma(s) z^{-s} \left(-\frac{F'}{F}(s;\chi^*)\right) ds$$

has singularities at $\mathcal{F}(\chi^*)\dot\cup\mathcal{V}$, where

$$\mathcal{V} = \left\{-n :\ n \in \mathbb{N}_0\right\} \cup \left\{s_j = -\frac{n+\mu_j}{\lambda_j} : \substack{n=0,1,\dots \text{ and } j=1,\dots,r \\ \lambda_j>0 \text{ real},\ \mu_j\in\mathbb{C},\ \operatorname{Re}\mu_j\geq 0}\right\}.$$

Consider the rectangular area $\mathcal{R}$ from the proof of Theorem 4.2.6, with vertices $2 \pm iT$, $-N - \hat\delta \pm iT$ and $N \in \mathbb{N}$, $T > 1$. Since the number of points (poles) in $\mathcal{V} \cap \mathcal{R}$ is finite (by a similar reasoning as the one stated before Theorem 4.3.3), we can choose $\hat\delta > 0$ such that there is no trivial zero on the vertical line joining $-N - \hat\delta - iT$ and $-N - \hat\delta + iT$. W.l.o.g. $0 < \hat\delta < 1$. Choose $T > 1$ in a way to fulfill Theorem 4.3.3, equation (4.41) and Theorem 4.3.4. By the same reasoning as in the proof of Theorem 4.2.6 we obtain the analogon of (4.17), i.e.

$$f(z;\chi^*) = \sum_{s_0\in\mathcal{V}\cup\mathcal{F}(\chi^*)} \operatorname{res}_{s=s_0} \Gamma(s) z^{-s} \left(-\frac{F'}{F}(s;\chi^*)\right), \tag{4.42}$$

resp.

$$f(z;\chi^*) = -\sum_{\rho\in\mathcal{F}(\chi^*)} \Gamma(\rho) z^{-\rho} + \sum_{s\in\mathcal{V}} \kappa_s(\chi^*) z^{-s},$$

where $\kappa_s(\chi^*)$ denotes the residue arising from $\Gamma(s)$ and $-\frac{F'}{F}(s;\chi^*)$ at $s \in \mathcal{V}$. Logarithmic differentiation of the functional equation $(\mathcal{S}_3)$ implies, as in the proof of Theorem 4.3.4,

$$\kappa_s(\chi^*) = \operatorname{res}_{s\in\mathcal{V}} \Gamma(s) \Bigg( 2\log Q + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s) + \overline{\mu_j}\right) + \\ + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j s + \mu_j\right) + \frac{F'}{F}\left(1-s;\overline{\chi^*}\right) \Bigg).$$

First we determine the residues at $s = -n$ for $n \in \mathbb{N}_0$:

$$\operatorname{res}_{s=-n} 2\Gamma(s)\log Q = \frac{(-1)^n}{n!} \cdot 2\log Q,$$

and by $(\mathcal{A}_2)$

$$\mathrm{res}_{s=-n}\,\Gamma(s)\frac{F'}{F}\left(1-s;\overline{\chi^*}\right)=\frac{(-1)^n}{n!}\cdot\frac{F'}{F}\left(1+n;\overline{\chi^*}\right).$$

Since $\lambda_j>0$ is real and $\mathrm{Re}(\overline{\mu_j})\geq 0$, the number $\lambda_j(1+n)+\overline{\mu_j}$ cannot be a negative integer or zero. Therefore,

$$\mathrm{res}_{s=-n}\,\Gamma(s)\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s)+\overline{\mu_j}\right)=\frac{(-1)^n}{n!}\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}\left(\lambda_j(1+n)+\overline{\mu_j}\right).$$

Now we consider two cases: If $-\lambda_j n+\mu_j$ is not a negative integer or zero, there are only the simple poles from the Gamma-function and

$$\mathrm{res}_{s=-n}\,\Gamma(s)\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}\left(\lambda_j s+\mu_j\right)=\frac{(-1)^n}{n!}\sum_{j=1}^{r}{}'\,\lambda_j\frac{\Gamma'}{\Gamma}\left(-\lambda_j n+\mu_j\right).$$

On the other hand, if $-\lambda_j n+\mu_j$ is a negative integer or zero, there are double poles. We use (4.18), i.e.

$$\Gamma(s)=\frac{\pi}{\Gamma(1-s)\sin(\pi s)},$$

which yields by logarithmic differentiation

$$\frac{\Gamma'}{\Gamma}(s)=\frac{\Gamma'}{\Gamma}(1-s)-\pi\frac{\cos(\pi s)}{\sin(\pi s)}.$$

Multiplying these equations implies, by (4.19) and $1+\lambda_j n-\mu_j\geq 1$,

$$\begin{aligned}&\mathrm{res}_{s=-n}\,\Gamma(s)\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j)=\\&\quad=\frac{(-1)^n}{n!}\cdot\frac{\Gamma'}{\Gamma}(1+\lambda_j n-\mu_j)-\frac{\pi^2}{n!}\,\mathrm{res}_{s=-n}\,\frac{\cos(\pi(\lambda_j s+\mu_j))}{\sin(\pi s)\sin(\pi(\lambda_j s+\mu_j))}\end{aligned}$$

We observe that for $s=-n$ and $-\lambda_j n+\mu_j$ beeing negative integers or zero, $\sin(\pi s)$ and $\sin(\pi(\lambda_j s+\mu_j))$ are both odd functions and $\cos(\pi(\lambda_j s+\mu_j))$ is an even funtion. Since the product of two odd functions is even, and the quotient of two even functions is even again, only even powers appear in the Laurent series expansion, implying

$$\mathrm{res}_{s=-n}\,\frac{\cos(\pi(\lambda_j s+\mu_j))}{\sin(\pi s)\sin(\pi(\lambda_j s+\mu_j))}=0.$$

This yields

$$\begin{aligned}&\mathrm{res}_{s=-n}\,\Gamma(s)\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}\left(\lambda_j s+\mu_j\right)=\\&\quad=\frac{(-1)^n}{n!}\sum_{j=1}^{r}{}'\,\lambda_j\frac{\Gamma'}{\Gamma}\left(-\lambda_j n+\mu_j\right)+\frac{(-1)^n}{n!}\sum_{j=1}^{r}{}''\,\lambda_j\frac{\Gamma'}{\Gamma}(1+\lambda_j n-\mu_j).\end{aligned}$$

In case of $s \in \mathcal{G} := \mathcal{V} \setminus \{-n :\ n \in \mathbb{N}_0\}$, we use (4.18) to obtain

$$\kappa_s(\chi^*) = \operatorname{res}_{s\in\mathcal{G}} \frac{\pi}{\sin(\pi s)\Gamma(1-s)} \left( 2\log Q + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s) + \overline{\mu_j}\right) + \right.$$
$$\left. + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j s + \mu_j\right) + \frac{F'}{F}\left(1-s;\overline{\chi^*}\right) \right).$$

By definition, $s \in \mathcal{G}$ is not an integer, so $\sin \pi s \neq 0$ and $1 - s = 1 + \frac{n+\mu_j}{\lambda_j}$ is not an integer. Therefore

$$\operatorname{res}_{s\in\mathcal{G}} \frac{2\pi \log Q}{\sin(\pi s)\Gamma(1-s)} = 0$$

and by $(\mathcal{A}_2)$

$$\operatorname{res}_{s\in\mathcal{G}} \frac{\pi}{\sin(\pi s)\Gamma(1-s)} \cdot \frac{F'}{F}\left(1-s;\overline{\chi^*}\right) = 0.$$

Since $\lambda_j(1-s) + \overline{\mu_j} = \lambda_j + n + 2\operatorname{Re}\mu_j > 0$,

$$\operatorname{res}_{s\in\mathcal{G}} \frac{\pi}{\sin(\pi s)\Gamma(1-s)} \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j(1-s) + \overline{\mu_j}\right) = 0.$$

For the last residue to calculate, we obtain for $s_j = -\frac{n+\mu_j}{\lambda_j}$

$$\begin{aligned}
&\operatorname{res}_{s\in\mathcal{G}} \frac{\pi}{\sin(\pi s)\Gamma(1-s)} \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j s + \mu_j\right) = \\
&= \frac{\pi\lambda_j}{\sin(\pi s_j)\Gamma(1-s_j)} \operatorname{res}_{s=s_j} \frac{\Gamma'}{\Gamma}\left(\lambda_j s + \mu_j\right) = \frac{-\pi\lambda_j}{\sin(\pi s_j)\Gamma(1-s_j)} \\
&= \frac{\pi\lambda_j}{\sin\left(\pi \cdot \frac{n+\mu_j}{\lambda_j}\right)\Gamma\left(1 + \frac{n+\mu_j}{\lambda_j}\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{s\in\mathcal{S}} \kappa_s(\chi^*) z^{-s} &= \sum_{n=0}^{\infty} \kappa_n(\chi^*) z^n + \sum_{s\in\mathcal{G}} \kappa_s(\chi^*) z^{-s} \\
&= \sum_{n=0}^{\infty} \kappa_n(\chi^*) z^n + \sum_{n\geq 0} \sum_{j=1}^{r} \kappa_{n,j}(\chi^*) z^{\frac{n+\mu_j}{\lambda_j}},
\end{aligned}$$

with $\kappa_n(\chi^*)$ and $\kappa_{n,j}(\chi^*)$ as stated in the theorem.

To estimate the coefficients, we use the absolute convergence of the logarithmic derivative of the twisted function (1.35), $(\mathcal{A}_2)$ and (4.21), to get for sufficiently large $n$

$$\begin{aligned}\kappa_n(\chi^*) &\ll \frac{1}{n!}\left\{1+\sum_{j=1}^{r}\lambda_j\log|\lambda_j(1+n)+\overline{\mu_j}|+\sum_{j=1}^{r}{}'\lambda_j\log|\lambda_j n-\mu_j|+\right.\\ &\quad\left.+\sum_{j=1}^{r}{}''\lambda_j\log|1+\lambda_j n-\mu_j|\right\}\\ &\ll \frac{1}{n!}\left\{1+\sum_{j=1}^{r}\lambda_j\log(1+n)\right\}\\ &\ll \frac{1}{n!}\{1+\log(1+n)\},\end{aligned}$$

where the implied constant may depend on the data of $F(\cdot,\chi^*)$. The formula of Cauchy-Hadamard in combination with the estimate of the coefficients implies that $\sum_{n=0}^{\infty}\kappa_n(\chi^*)z^n$ is an entire function.

To estimate the coefficients $\kappa_{n,j}(\chi^*)$ let

$$\iota=\min_{j=1,\dots,r}\left\{\left|\sin\left(\pi\cdot\frac{n+\mu_j}{\lambda_j}\right)\right|\right\}$$

and notice that by definition of $\mathcal{G}$ the number $\frac{n+\mu_j}{\lambda_j}$ is not an integer. Therefore $\iota$ is nonzero and by (4.14) we get for sufficiently large $n$

$$\kappa_{n,j}(\chi^*)\ll\frac{1}{\left|\Gamma\left(1+\frac{n+\mu_j}{\lambda_j}\right)\right|}\ll\frac{1}{n^{\frac{1}{2}+n}e^n}.$$

In view of

$$\sum_{n\geq 0}\sum_{j=1}^{r}\kappa_{n,j}(\chi^*)z^{\frac{n+\mu_j}{\lambda_j}}=\sum_{j=1}^{r}z^{\frac{\mu_j}{\lambda_j}}\sum_{n\geq 0}\kappa_{n,j}(\chi^*)\left(z^{\frac{1}{\lambda_j}}\right)^n,$$

again the formula of Cauchy-Hadamard and the estimate of the coefficients imply that $h^{(2)}(z;\chi^*)$ is an entire function. $\square$

Using the notation of the previous Theorem 4.3.6, we estimate the sum over characters containing the functions $h^{(1)}$ and $h^{(2)}$.

**Lemma 4.3.7** *Let* $\chi^*\bmod q^*$ *be the primitive character inducing* $\chi\bmod q$. *Then for* $|z|<1$ *and* $j=1,2$

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\neq\chi_0}}|h^{(j)}(z;\chi^*)|\ll(\log q)^2.$$

**Proof**
In case of $q = 1$ or $q = 2$, the estimate is trivial. If $q \geq 3$, the proof is similar to the proof of Lemma 4.2.7:

In view of $(\mathcal{A}_2)$ it is reasonable that the modulus $q^*$ appears in the functional equation $(\mathcal{S}_3)$, i.e. $Q = q^* c$ for a fixed constant $c = c(F^*)$. This is known to holds for Dirichlet L-functions and for classical automorphic L-functions whose twists are again in the Selberg class [30, p. 147]. Therefore,

$$\sum_{n=1}^{\infty} |\kappa_n(\chi^*)| \ll \log q$$

and

$$|\kappa_0(\chi^*)| \ll \log q + \left| \frac{F'}{F}\left(1; \overline{\chi^*}\right) \right|,$$

implying

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |h^{(1)}\left(z; \chi^*\right)| \ll \log q + \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \left| \frac{F'}{F}\left(1; \overline{\chi^*}\right) \right|.$$

Let $\psi(y, \chi^*) = \sum_{n \leq y} \chi^*(n) \Lambda_F(n)$. By $(\mathcal{A}_1)$,

$$\left| \frac{F'}{F}\left(1; \overline{\chi^*}\right) \right| \leq \int_1^{\infty} y^{-2} \left| \psi(y, \overline{\chi^*}) \right| dy.$$

Using assumption $(\mathcal{A}_3)$, the integral can be estimated as in the proof of Lemma 4.2.7, which yields

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |h^{(1)}\left(z; \chi^*\right)| \ll (\log q)^2 + \frac{1}{\varphi(q)} \cdot \frac{1}{\beta_1(1 - \beta_1)}.$$

Since $\beta_1$ is fixed, if it exists, and $0 < \beta_1 < 1$, $0 < 1 - \beta_1 < 1$, we get with $\varphi(q)^{-1} \ll \frac{\log q}{q}$ the estimate

$$\frac{1}{\varphi(q)} \cdot \frac{1}{\beta_1(1 - \beta_1)} \ll (\log q)^2.$$

This implies

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |h^{(1)}\left(z; \chi^*\right)| \ll (\log q)^2$$

and proves the lemma for $j = 1$.

In case of $j = 2$ Theorem 4.3.6 implies for $|z| < 1$

$$\left|h^{(2)}(z;\chi^*)\right| = \left|\sum_{j=1}^{r} z^{\frac{\mu_j}{\lambda_j}} \sum_{n\geq 0} \kappa_{n,j}(\chi^*)\left(z^{\frac{1}{\lambda_j}}\right)^n\right|$$
$$\ll \sum_{j=1}^{r}\left|z^{\frac{\mu_j}{\lambda_j}}\right|\sum_{n\geq 0}|\kappa_{n,j}(\chi^*)|\cdot\left|\left(z^{\frac{1}{\lambda_j}}\right)\right|^n$$
$$\ll \sum_{j=1}^{r}\sum_{n\geq 0}|\kappa_{n,j}(\chi^*)| \ll 1$$

and yields

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} |h^{(2)}(z;\chi^*)| \ll 1.$$

□

From Theorem 4.3.2, 4.3.6 and Lemma 4.3.7 we get

**Corollary 4.3.8** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b,q) = 1$, and let $z_0 = \delta + 2\pi i\left(\alpha - \frac{b}{q}\right)$. Denote by $\chi^*$ mod $q^*$ the primitive character inducing $\chi$ mod $q$. Then*

$$f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \varphi(q)^{-1} + \delta\log\left(\delta^{-1}\right)\sqrt{q}\,(\log(e+q))^2 +$$
$$+\,\delta\frac{\sqrt{q}}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}}\ \sum_{\rho\in\mathcal{F}(\chi^*)}\left|\Gamma(\rho)z_0^{-\rho}\right|.$$

We consider in the next step the non-trivial zeros with $|\gamma| > 3$. Write $z_0 = \delta + 2\pi i\left(\alpha - \frac{b}{q}\right) = re^{i\theta}$ with $\delta > 0$, $r < 1$ and $|\theta| < \frac{1}{2}$ and let $\tilde{D} = \frac{\pi}{2} - |\theta|$. Recall the estimate (4.25).

**Lemma 4.3.9** *Let $q \geq 3$ be an integer. Then*

$$\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}}\ \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}} |\Gamma(\rho)z_0^{-\rho}| \ll$$
$$\ll \delta^{-1}\left\{r^{-\frac{1}{2}+\varepsilon}\delta^{\frac{1}{2}-\varepsilon}q^{\varepsilon} + r^{\frac{1}{2}(c-1)+\varepsilon}\delta^{1-\left(\frac{c}{2}+\varepsilon\right)}q^{\frac{c}{2}+\varepsilon}\right\}\left(\log(r\delta^{-1}q)\right)^{\check{c}+1},$$

*where $r = |z_0| < 1$, $\delta = \operatorname{Re} z_0$ and $\delta > 0$ is sufficiently small.*

**Proof**
We proceed as in the proof of Lemma 4.2.9 to obtain, using the notation introduced in $(\mathcal{A}_4)$,

$$\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}} |\Gamma(\rho)z_0^{-\rho}| \ll \int_3^\infty \tilde{D}e^{-\tilde{D}t}r^{-\frac{1}{2}}\Bigg\{\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} N_F\left(\frac{1}{2},t,\chi^*\right)\Bigg\}dt+$$
$$+\int_{\frac{1}{2}}^1\int_3^\infty \tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\Bigg\{\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} N_F(\sigma,t,\chi^*)\Bigg\}dtd\sigma.$$

To estimate the first integral on the right hand side, we use $(\mathcal{A}_4)$, substitute $\tau=\tilde{D}t$ and apply (4.29), to obtain

$$\int_3^\infty \tilde{D}e^{-\tilde{D}t}r^{-\frac{1}{2}}\Bigg\{\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} N_F\left(\frac{1}{2},t,\chi^*\right)\Bigg\}dt \ll$$
$$\ll \left(q\tilde{D}^{-1}\right)^{\frac{c}{2}+\varepsilon} r^{-\frac{1}{2}}\left(\log\left(q\tilde{D}^{-1}\right)\right)^{\check{c}} \ll \left(r\delta^{-1}q\right)^{\frac{c}{2}+\varepsilon} r^{-\frac{1}{2}}\left(\log(r\delta^{-1}q)\right)^{\check{c}}$$
$$\ll \left(r^{\frac{1}{2}(c-1)+\varepsilon}\delta^{-\left(\frac{c}{2}+\varepsilon\right)}q^{\frac{c}{2}+\varepsilon}\right)\left(\log(r\delta^{-1}q)\right)^{\check{c}},$$

where we used that (4.25) implies $\tilde{D}^{-1}\leq r\delta^{-1}$. Analogously to the proof of Lemma 4.2.9 we get for the second integral

$$\int_{\frac{1}{2}}^1\int_3^\infty \tilde{D}e^{-\tilde{D}t}t^{\sigma-\frac{1}{2}}r^{-\sigma}\log\left(\frac{t}{r}\right)\Bigg\{\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} N_F(\sigma,t,\chi^*)\Bigg\}dtd\sigma \ll$$
$$\ll \int_3^\infty \tilde{D}e^{-\tilde{D}t}\left(\log(qt)\right)^{\check{c}+1}\left(r^{-1}q^{\varepsilon}t^{\frac{1}{2}+\varepsilon}+r^{-\frac{1}{2}}q^{\frac{c}{2}+\varepsilon}t^{\frac{c}{2}+\varepsilon}\right)dt$$
$$\ll \left(r^{-1}q^{\varepsilon}\tilde{D}^{-\left(\frac{1}{2}+\varepsilon\right)}+r^{-\frac{1}{2}}q^{\frac{c}{2}+\varepsilon}\tilde{D}^{-\left(\frac{c}{2}+\varepsilon\right)}\right)\left(\log(q\tilde{D}^{-1})\right)^{\check{c}+1}$$
$$\ll \left(r^{-\frac{1}{2}+\varepsilon}\delta^{-\left(\frac{1}{2}+\varepsilon\right)}q^{\varepsilon}+r^{\frac{1}{2}(c-1)+\varepsilon}\delta^{-\left(\frac{c}{2}+\varepsilon\right)}q^{\frac{c}{2}+\varepsilon}\right)\left(\log(r\delta^{-1}q)\right)^{\check{c}+1},$$

where we used again $\tilde{D}^{-1}\leq r\delta^{-1}$. Combining both estimates yields

$$\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \sum_{\substack{\rho\in\mathcal{F}(\chi^*)\\ |\gamma|>3}} |\Gamma(\rho)z_0^{-\rho}| \ll$$
$$\ll \left(r^{-\frac{1}{2}+\varepsilon}\delta^{-\left(\frac{1}{2}+\varepsilon\right)}q^{\varepsilon}+r^{\frac{1}{2}(c-1)+\varepsilon}\delta^{-\left(\frac{c}{2}+\varepsilon\right)}q^{\frac{c}{2}+\varepsilon}\right)\left(\log(r\delta^{-1}q)\right)^{\check{c}+1}$$
$$\ll \delta^{-1}\left(r^{-\frac{1}{2}+\varepsilon}\delta^{\frac{1}{2}-\varepsilon}q^{\varepsilon}+r^{\frac{1}{2}(c-1)+\varepsilon}\delta^{1-\left(\frac{c}{2}+\varepsilon\right)}q^{\frac{c}{2}+\varepsilon}\right)\left(\log(r\delta^{-1}q)\right)^{\check{c}+1}$$

and completes the proof. □

In case of non-trivial zeros with $|\gamma|\leq 3$ we get

**Lemma 4.3.10** *Let $q \geq 3$ be an integer. Then*

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3}} |\Gamma(\rho) z_0^{-\rho}| \ll$$
$$\ll \delta^{-1} \left\{ r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} + r^{-\frac{1}{2}+\varepsilon} \delta^{\frac{1}{2}-\varepsilon} q^{\varepsilon} \right\} \left(\log(r\delta^{-1}q)\right)^{\check{c}+1},$$

*where $r = |z_0| < 1$, $\delta = \operatorname{Re} z_0$ and $\delta > 0$ is sufficiently small.*

**Proof**
We adopt the reasoning from the proof of Lemma 4.2.10 and split the proof in the two cases $\beta \geq \frac{1}{2}$ and $\beta < \frac{1}{2}$. By the zero-density estimate $(\mathcal{A}_4)$ we get in the first case

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \beta \geq \frac{1}{2}}} |\Gamma(\rho) z_0^{-\rho}| \ll \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \beta \geq \frac{1}{2}}} \left( r^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta} \frac{\log(r^{-1})}{r^{\sigma}} d\sigma \right)$$
$$\ll r^{-\frac{1}{2}} \left\{ \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} N_F\left(\frac{1}{2}, 3, \chi^*\right) \right\} +$$
$$+ \int_{\frac{1}{2}}^{1} r^{-\sigma} \log(r^{-1}) \left\{ \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} N_F(\sigma, 3, \chi^*) \right\} d\sigma$$
$$\ll r^{-\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} (\log q)^{\check{c}} + \log(r^{-1}) (\log q)^{\check{c}} \int_{\frac{1}{2}}^{1} r^{-\sigma} q^{c(1-\sigma)+\varepsilon} d\sigma$$
$$\ll \left(\delta^{-1} r^{\frac{1}{2}}\right) q^{\frac{c}{2}+\varepsilon} (\log q)^{\check{c}} + (\log q)^{\check{c}+1} \left( r^{-1} q^{\varepsilon} + r^{-\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} \right),$$

where we used $r^{-1} \leq \delta^{-1}$ to obtain the last estimate. Using $\delta \leq r$ we estimate the third summand in brackets

$$\delta^{-1} r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} (\log q)^{\check{c}} + (\log q)^{\check{c}+1} \left( r^{-1} q^{\varepsilon} + r^{-\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} \right) \ll$$
$$\ll \delta^{-1} \left( r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} + r^{-1} \delta q^{\varepsilon} + r^{-\frac{1}{2}} \delta q^{\frac{c}{2}+\varepsilon} \right) (\log q)^{\check{c}+1}$$
$$\ll \delta^{-1} \left( r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} + r^{-1} \delta q^{\varepsilon} \right) (\log q)^{\check{c}+1}$$
$$\ll \delta^{-1} \left( r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} + r^{-\frac{1}{2}+\varepsilon} \delta^{\frac{1}{2}-\varepsilon} q^{\varepsilon} \right) (\log q)^{\check{c}+1},$$

where we used in the last estimate that $\delta r^{-1} \leq 1$ exponentiated with $\frac{1}{2} + \varepsilon$ implies $\delta r^{-1} \leq \delta^{\frac{1}{2}-\varepsilon} r^{-\frac{1}{2}+\varepsilon}$.

Now let $|\gamma| \leq 3$ and $\beta < \frac{1}{2}$. Since there is no non-trivial zero on the vertical line $\sigma = 0$ and the number of non-trivial zeros in the considered rectangular area is finite, $|\rho|^{-1} \ll \log q$. By $(\mathcal{A}_4)$ and $\delta \leq r$,

$$\begin{aligned}\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \beta < \frac{1}{2}}} |\Gamma(\rho) z_0^{-\rho}| &\ll \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \sum_{\substack{\rho \in \mathcal{F}(\chi^*) \\ |\gamma| \leq 3,\ \beta < \frac{1}{2}}} |\rho|^{-1} r^{-\frac{1}{2}} \\ &\ll (\log q) r^{-\frac{1}{2}} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} N_F\left(\frac{1}{2}, 3, \chi^*\right) \\ &\ll r^{-\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} (\log q)^{\check{c}+1} \\ &\ll \delta^{-1} r^{\frac{1}{2}} q^{\frac{c}{2}+\varepsilon} (\log q)^{\check{c}+1}.\end{aligned}$$

This term is already contained in the previous estimate. We note that $1 \leq r\delta^{-1}$ implies $(\log q)^{\check{c}+1} \leq \left(\log(r\delta^{-1}q)\right)^{\check{c}+1}$ and therefore the estimate which was to prove. □

Analogously to the proof of Corollary 4.2.11 we get

**Corollary 4.3.11** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small. Further, let $b$ be an integer, $q$ a positive integer, satisfying $(b, q) = 1$, and let $z_0 = \delta + 2\pi i \left(\alpha - \frac{b}{q}\right)$. Then for $r = |z_0| < 1$*

$$\begin{aligned}f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \delta \log(\delta^{-1}) q^{\frac{1}{2}} (\log(e+q))^2 + q^{-\frac{1}{2}+\varepsilon} (\log(e+q))^{\check{c}+2} + \\ + \left\{ r^{\frac{1}{2}(c-1)+\varepsilon} \delta^{1-(\frac{c}{2}+\varepsilon)} q^{\frac{1}{2}(c-1)+\varepsilon} + r^{\frac{1}{2}} q^{\frac{1}{2}(c-1)+\varepsilon} \right\} \left(\log(e + r\delta^{-1}q)\right)^{\check{c}+2}.\end{aligned}$$

For rational approximation of $\alpha$ we use again the approximation theorem of Dirichlet.

**Theorem 4.3.12** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small (but at least $\delta < \frac{1}{50}$). Further, let*

$$Q = \begin{cases} \delta^{-\frac{2}{3}} & \text{if } 2 - 2\varepsilon \leq c < \frac{5}{2} - 2\varepsilon, \\ \delta^{-\frac{c+2\varepsilon}{2c-1+4\varepsilon}} & \text{if } 1 < c \leq 2 - 2\varepsilon, \\ \delta^{-\frac{1+2\varepsilon}{1+4\varepsilon}} & \text{if } c = 1, \\ \delta^{-\frac{c+2\varepsilon}{c+4\varepsilon}} & \text{if } 0 \leq c < 1, \end{cases}$$

*and let $b$ and $q$ be such that $(b, q) = 1,\ 1 \leq q \leq Q$, and $\left|\alpha - \frac{b}{q}\right| < q^{-1}Q^{-1}$. Then for a fixed $\kappa$ satisfying $0 < \kappa \leq \frac{1}{6}$ we have*

$$f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \delta^{\kappa} \left(\log\left(e + \delta^{-1}\right)\right)^{\check{c}+3} + q^{-\frac{1}{2}} (\log(e+q))^{\check{c}+2}.$$

**Proof**

As in the proof of Theorem 4.2.12, let $Q = \delta^{-\lambda}$, where $\frac{1}{2} < \lambda < 1$. Suppose that $b$ and $q$ are choosen such that $(b, q) = 1$, $1 \leq q \leq Q$ and $|\alpha - \frac{b}{q}| < q^{-1}Q^{-1}$. We consider two cases:

*Case (i)*: If $\delta^{\lambda-1} \leq q \leq \delta^{-\lambda}$ then $\delta \leq r \ll \delta$. Using these estimates we obtain for the sum in brackets in Corollary 4.3.11 for $c > 1$

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \delta^{\frac{1}{2}-\lambda(\frac{c-1}{2}+\varepsilon)}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}.$$

Analogously we get for $c = 1$

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \delta^{\frac{1}{2}-\lambda\varepsilon}\left(\log(e+\delta^{-1})\right)^{\check{c}+2},$$

while for $0 \leq c < 1$ we get

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \delta^{\frac{1}{2}+(1-\lambda)(\frac{c-1}{2})-\lambda\varepsilon}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}.$$

*Case (ii)*: If $1 \leq q \leq \delta^{\lambda-1}$ then $\delta \leq r \ll \delta^{\lambda}$. Now we obtain for the sum in brackets in Corollary 4.3.11 for $c > 1$ the estimate

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \left\{\delta^{(2\lambda-1)(\frac{c-1}{2}+\varepsilon)+\frac{2-c}{2}-\varepsilon} + \delta^{(\lambda-1)(\frac{c-1}{2}+\varepsilon)+\frac{\lambda}{2}}\right\}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}.$$

For $c = 1$ we get

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \left\{\delta^{\frac{1}{2}+2(\lambda-1)\varepsilon} + \delta^{\frac{\lambda}{2}+(\lambda-1)\varepsilon}\right\}\left(\log(e+\delta^{-1})\right)^{\check{c}+2},$$

and for $0 \leq c < 1$

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \left\{\delta^{\frac{1}{2}+2(\lambda-1)\varepsilon} + \delta^{\frac{\lambda}{2}+(\lambda-1)\varepsilon}\right\}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}.$$

Combining the estimates from case $(i)$ and case $(ii)$ yields for $c > 1$

$$\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)} + r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll$$
$$\ll \left\{\delta^{\frac{1}{2}-\lambda(\frac{c-1}{2}+\varepsilon)} + \delta^{(2\lambda-1)(\frac{c-1}{2}+\varepsilon)+\frac{2-c}{2}-\varepsilon} + \delta^{(\lambda-1)(\frac{c-1}{2}+\varepsilon)+\frac{\lambda}{2}}\right\}\times$$
$$\times\left(\log(e+\delta^{-1})\right)^{\check{c}+2}.$$

For $f_1(\lambda) = \frac{1}{2} - \lambda(\frac{c-1}{2} + \varepsilon)$, $f_2(\lambda) = (2\lambda - 1)(\frac{c-1}{2} + \varepsilon) + \frac{2-c}{2} - \varepsilon$ and $f_3(\lambda) = (\lambda - 1)(\frac{c-1}{2} + \varepsilon) + \frac{\lambda}{2}$,

$$\sup_{\substack{\frac{1}{2}<\lambda<1 \\ 2-2\varepsilon\leq c<\frac{5}{2}-2\varepsilon}} \Big\{ \min\left(f_1, f_2, f_3\right) \Big\} = \frac{1}{2} - \frac{c-1+2\varepsilon}{3}$$

and occures at $\lambda = \frac{2}{3}$, while

$$\sup_{\substack{\frac{1}{2}<\lambda<1 \\ 1<c\leq 2-2\varepsilon}} \Big\{ \min\left(f_1, f_2, f_3\right) \Big\} = \frac{1}{2} - \frac{(c+2\varepsilon)(c-1+2\varepsilon)}{2(2c-1+4\varepsilon)}$$

occures at $\lambda = \frac{c+2\varepsilon}{2c-1+4\varepsilon}$.

For $2 - 2\varepsilon \leq c < \frac{5}{2} - 2\varepsilon$ we choose $Q = \delta^{-\frac{2}{3}}$ and restict $\delta$ to $0 < \delta < \frac{1}{50}$. Then $r = |z_0| < 1$ and from Corollary 4.3.11

$$\begin{aligned} f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| &\ll q^{-\frac{1}{2}+\varepsilon} \left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 + \\ &\quad + \delta^{\frac{1}{2}-\frac{c-1+2\varepsilon}{3}} \left(\log(e+\delta^{-1})\right)^{\check{c}+2} \\ &\ll q^{-\frac{1}{2}} \left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 + \\ &\quad + \delta^{\kappa} \left(\log(e+\delta^{-1})\right)^{\check{c}+2} \end{aligned}$$

for $0 < \kappa \leq \frac{1}{6}$. Since $q \leq Q = \delta^{-\frac{2}{3}}$ we have

$$\delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 \ll \delta^{\frac{2}{3}} \left(\log(e+\delta^{-1})\right)^{\check{c}+3} \ll \delta^{\kappa} \left(\log(e+\delta^{-1})\right)^{\check{c}+3}$$

and therefore

$$f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| \ll \delta^{\kappa} \left(\log(e+\delta^{-1})\right)^{\check{c}+3} + q^{-\frac{1}{2}} \left(\log\left(e+q\right)\right)^{\check{c}+2}$$

for $2 - 2\varepsilon \leq c < \frac{5}{2} - 2\varepsilon$ and $0 < \kappa \leq \frac{1}{6}$.

For $1 < c \leq 2 - 2\varepsilon$ let $Q = \delta^{-\frac{c+2\varepsilon}{2c-1+4\varepsilon}}$ and again restict $\delta$ to $0 < \delta < \frac{1}{50}$. Then $r = |z_0| < 1$ and from Corollary 4.3.11 we get

$$\begin{aligned} f(\delta)^{-1}|f(\delta + 2\pi i\alpha)| &\ll q^{-\frac{1}{2}+\varepsilon} \left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 \\ &\quad + \delta^{\frac{1}{2}-\frac{(c+2\varepsilon)(c-1+2\varepsilon)}{2(2c-1+4\varepsilon)}} \left(\log(e+\delta^{-1})\right)^{\check{c}+2} \\ &\ll q^{-\frac{1}{2}} \left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 \\ &\quad + \delta^{\frac{1}{6}} \left(\log(e+\delta^{-1})\right)^{\check{c}+2}. \end{aligned}$$

Since $q \leq Q = \delta^{-\frac{c+2\varepsilon}{2c-1+4\varepsilon}}$ we have

$$\begin{aligned} \delta \log(\delta^{-1}) q^{\frac{1}{2}} \left(\log(e+q)\right)^2 &\ll \delta^{1-\frac{1}{2}\left(\frac{c+2\varepsilon}{2c-1+4\varepsilon}\right)} \log(\delta^{-1}) \left(\log(e+\delta^{-\frac{c+2\varepsilon}{2c-1+4\varepsilon}})\right)^2 \\ &\ll \delta^{\frac{1}{2}} \left(\log(e+\delta^{-1})\right)^3. \end{aligned}$$

This implies

$$\begin{aligned} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| &\ll \delta^{\frac{1}{6}}\left(\log(e+\delta^{-1})\right)^{\check{c}+3}+q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \\ &\ll \delta^{\kappa}\left(\log(e+\delta^{-1})\right)^{\check{c}+3}+q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \end{aligned}$$

for $1<c\leq 2-2\varepsilon$ and $0<\kappa\leq\frac{1}{6}$ and completes the proof in case of $c>1$.

Now we consider the case $c=1$. Using our previous estimates from case $(i)$ and case $(ii)$, we get

$$\begin{aligned} &\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)}+r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll \\ &\ll\left\{\delta^{\frac{1}{2}-\lambda\varepsilon}+\delta^{\frac{1}{2}+2(\lambda-1)\varepsilon}+\delta^{\frac{\lambda}{2}+(\lambda-1)\varepsilon}\right\}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}. \end{aligned}$$

Note that for $g_1(\lambda)=\frac{1}{2}-\lambda\varepsilon$, $g_2(\lambda)=\frac{1}{2}+2(\lambda-1)\varepsilon$ and $g_3(\lambda)=\frac{\lambda}{2}+(\lambda-1)\varepsilon$,

$$\sup_{\substack{\frac{1}{2}<\lambda<1\\ c=1}}\left\{\min\left(g_1,g_2,g_3\right)\right\}=\frac{1}{2}-\frac{\varepsilon(1+2\varepsilon)}{1+4\varepsilon}$$

and occures at $\lambda=\frac{1+2\varepsilon}{1+4\varepsilon}$. We choose $Q=\delta^{-\frac{1+2\varepsilon}{1+4\varepsilon}}$ and again restict $\delta$ to $0<\delta<\frac{1}{50}$. Then $r=|z_0|<1$ and from Corollary 4.3.11 we get

$$\begin{aligned} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| &\ll q^{-\frac{1}{2}+\varepsilon}\left(\log\left(e+q\right)\right)^{\check{c}+2}+\delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^{2}+ \\ &\quad+\delta^{\frac{1}{2}-\frac{\varepsilon(1+2\varepsilon)}{1+4\varepsilon}}\left(\log(e+\delta^{-1})\right)^{\check{c}+2} \\ &\ll q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2}+\delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^{2}+ \\ &\quad+\delta^{\frac{1}{2}}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}. \end{aligned}$$

Since $q\leq Q=\delta^{-\frac{1+2\varepsilon}{1+4\varepsilon}}$ we have

$$\begin{aligned} \delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^{2} &\ll \delta^{1-\frac{1}{2}\left(\frac{1+2\varepsilon}{1+4\varepsilon}\right)}\log(\delta^{-1})\left(\log(e+\delta^{-\frac{1+2\varepsilon}{1+4\varepsilon}})\right)^{2} \\ &\ll \delta^{\frac{1}{2}}\left(\log(e+\delta^{-1})\right)^{3}. \end{aligned}$$

This implies for $0<\kappa\leq\frac{1}{6}$

$$\begin{aligned} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| &\ll \delta^{\frac{1}{2}}\left(\log(e+\delta^{-1})\right)^{\check{c}+3}+q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \\ &\ll \delta^{\kappa}\left(\log(e+\delta^{-1})\right)^{\check{c}+3}+q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \end{aligned}$$

and completes the proof for $c=1$.

Our last case to treat is $0\leq c<1$. Again by our previous estimates from case $(i)$ and case $(ii)$, we get

$$\begin{aligned} &\left\{r^{\frac{1}{2}(c-1)}r^{\varepsilon}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\delta^{1-(\frac{c}{2}+\varepsilon)}+r^{\frac{1}{2}}q^{\frac{1}{2}(c-1)}q^{\varepsilon}\right\}\left(\log(e+qr\delta^{-1})\right)^{\check{c}+2} \ll \\ &\ll\left\{\delta^{\frac{1}{2}+(1-\lambda)(\frac{c-1}{2})-\lambda\varepsilon}+\delta^{\frac{1}{2}+2\varepsilon(\lambda-1)}+\delta^{\frac{\lambda}{2}+(\lambda-1)\varepsilon}\right\}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}. \end{aligned}$$

For $h_1(\lambda) = \frac{1}{2} + (1-\lambda)(\frac{c-1}{2}) - \lambda\varepsilon$, $h_2(\lambda) = \frac{1}{2} + 2\varepsilon(\lambda - 1)$ and $h_3(\lambda) = \frac{\lambda}{2} + (\lambda-1)\varepsilon$,

$$\sup_{\substack{\frac{1}{2}<\lambda<1\\ 0\le c<1}} \left\{ \min\left(h_1, h_2, h_3\right) \right\} = \frac{1}{2} - \varepsilon\left(\frac{1+2\varepsilon}{c+4\varepsilon}\right)$$

and occures at $\lambda = \frac{c+2\varepsilon}{c+4\varepsilon}$. We choose $Q = \delta^{-\frac{c+2\varepsilon}{c+4\varepsilon}}$ and restict $\delta$ to $0 < \delta < \frac{1}{50}$. Then $r = |z_0| < 1$ and from Corollary 4.3.11 we get

$$\begin{aligned} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| &\ll q^{-\frac{1}{2}+\varepsilon}\left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^2 \\ &\quad + \delta^{\frac{1}{2}-\varepsilon\left(\frac{1+2\varepsilon}{c+4\varepsilon}\right)}\left(\log(e+\delta^{-1})\right)^{\check{c}+2} \\ &\ll q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} + \delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^2 \\ &\quad + \delta^{\frac{1}{6}}\left(\log(e+\delta^{-1})\right)^{\check{c}+2}. \end{aligned}$$

Since $q \le Q = \delta^{-\frac{c+2\varepsilon}{c+4\varepsilon}}$ we have

$$\begin{aligned} \delta\log(\delta^{-1})q^{\frac{1}{2}}\left(\log(e+q)\right)^2 &\ll \delta^{1-\frac{1}{2}\left(\frac{c+2\varepsilon}{c+4\varepsilon}\right)}\log(\delta^{-1})\left(\log(e+\delta^{-\frac{c+2\varepsilon}{c+4\varepsilon}})\right)^2 \\ &\ll \delta^{\frac{1}{6}}\left(\log(e+\delta^{-1})\right)^3. \end{aligned}$$

This implies

$$\begin{aligned} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| &\ll \delta^{\frac{1}{6}}\left(\log(e+\delta^{-1})\right)^{\check{c}+3} + q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \\ &\ll \delta^{\kappa}\left(\log(e+\delta^{-1})\right)^{\check{c}+3} + q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2} \end{aligned}$$

for $0 \le c < 1$ and $0 < \kappa \le \frac{1}{6}$. □

The last theorem implies

**Corollary 4.3.13** *Let $\alpha$ be irrational. Then*

$$\lim_{\delta\to 0+} f(\delta)^{-1}|f(\delta+2\pi i\alpha)| = 0.$$

We continue with the study of the related function

$$g(z) = \sum_p A(p)(\log p)e^{-pz},$$

where $A(p) = \sum_{j=1}^{\nu_F}\alpha_j(p)$ is the sum of the coefficients from the local factors (1.32) of the Euler product. For this function we obtain

**Theorem 4.3.14** *Let $\alpha$ be a real number and $\delta > 0$ sufficiently small (but at least $\delta < \frac{1}{50}$). Further, let $Q$ be as in Theorem 4.3.12 and let $b$ and $q$ be such that $(b,q) = 1$, $1 \le q \le Q$, and $\left|\alpha - \frac{b}{q}\right| < q^{-1}Q^{-1}$. Then for a fixed $\kappa$ satisfying $0 < \kappa \le \frac{1}{6}$ we have*

$$g(\delta)^{-1}|g(\delta+2\pi i\alpha)| \ll \delta^{\kappa}\left(\log\left(e+\delta^{-1}\right)\right)^{\check{c}+3} + q^{-\frac{1}{2}}\left(\log\left(e+q\right)\right)^{\check{c}+2}.$$

**Proof**

We proceed as in the proof of Theorem 4.2.14. The Chebyshev $\vartheta$-function for $\mathcal{S}^{poly}$ is defined by

$$\vartheta_F(x) = \sum_{p \le x} A(p) \log p.$$

In view of $|\alpha_j(p)| \le 1$ for all $j$ and $p$, $A(p) \ll 1$. The analogons of (4.32), (4.34) and (4.35), i.e. $\vartheta_F(x) = m_F x + o(x)$ as $x \to \infty$, $g(\delta) \sim \delta^{-1}$ as $\delta \to 0+$, and

$$E(\delta) = \sum_p \sum_{m=2}^{\infty} A(p^m)(\log p) e^{-p^m \delta} \ll \delta^{-\frac{1}{2}},$$

imply the theorem. □

Using Theorem 4.3.14, the proof of Theorem 1.3.9 is similar to the proof of Theorem 1.3.7.

Vaaler's result on weights, i.e. Lemma 4.2.15 with $a(n) = A(p_n) \log p_n$ and $\gamma(n) = (\log p_n)^{-1}$, implies Corollary 1.3.10. Again the proof is similar to the proof of Corollary 1.3.8.

# Bibliography


[1] Baluyot, S., Gonek, S.M.: Explicit formulae and discrepancy estimates for $a$-points of the Riemann zeta-function, *Pacific J. Math.* (to appear).

[2] Bergström, V.: Einge Bemerkungen zur Theorie der diophantischen Approximation, *Fysiogr. Sälsk. Lund. Förh.* **6**, no. 13 (1936), 1-19.

[3] Birkhoff, G.D.: Proof of the ergodic theorem, *Proc. Nat. Acad. Sci. USA* **17** (1931), 656-660.

[4] Bohr, H., Landau, E., Littlewood, J.E.: Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$, *Bull. de l'Acad. royale de Belgique* (1913), 3-35.

[5] Brüdern, J.: *Einführung in die analytische Zahlentheorie*, Springer, Berlin, 1995.

[6] Bugeaud, Y.: *Distribution modulo one and diophantine approximation*, Cambridge University Press, Cambridge, 2012.

[7] Cohen, H.: *Number Theory, Volume II: Analytic and Modern Tools*, Springer, New York, 2007.

[8] Davenport, H.: *Multiplicative Number Theory*, Graduate Texts in Mathematics, 3. Edition, Springer, New York, 2000.

[9] Dedekind, R.: *Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers*, in: Festschrift der Technischen Hochschule in Braunschweig zur Säkularfeier des Geburtstages von C.F. Gauß, Braunschweig (1877), 1-55; Gesammelte Werke, vol. 1, Vieweg, Braunschweig (1930), 104-158.

[10] Drmota, M., Tichy, R.F.: *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics, vol. 1651, Springer, Berlin, 1997.

[11] Erdös, P., Turán, P.: On a problem in the theory of uniform distribution I, II, *Indag. Math* **10** (1948), 370-378, 406-413.

[12] Elliott, P.D.T.A.: The Riemann zeta function and coin tossing, *J. reine angew. Math.* **254** (1972), 100-109.

[13] Freitag, E., Busam, R.: *Funktionentheorie 1*, 3. Auflage, Springer, Berlin, 2000.

[14] Fujii, A.: On a theorem of Landau, *Proc. Japan Acad.* **65** (1989), 51-54.

[15] Fujii, A.: On a theorem of Landau II, *Proc. Japan Acad.* **66** (1990), 291-296.

[16] Fujii, A.: *Uniform distribution of the zeros of the Riemann zeta-function and the mean value theorem of Dirichlet L-functions*, in: Analytic number Theory, Lecture Notes in Mathematics, vol. 1434, Springer, New York (1990), 103-125.

[17] Fujii, A.: Some problems on Diophantine approximation in the theory of the Riemann zeta function III, *Comment Math. Univ. St. Pauli* **42** (1993), no. 2, 161-187.

[18] Fujii, A.: On the Discrepancy Estimates of the Zeros of the Riemann Zeta Function, *Comment. Math. Univ. St. Pauli* **51** (2002), no. 1, 19-52.

[19] Garunkštis, R., Steuding, J.: On the roots of the equation $\zeta(s) = a$, *Abh. Math. Sem. Univ. Hambg.* **84** (2014), 1-15.

[20] Gonek, S.M.: An explicit formula of Landau and its application to the theory of the zeta-function, *Contemp. Math.* **143** (1993), 395-413.

[21] Goto, K., Kano, T.: Uniform Distribution of Some Special Sequences, *Proc. Japan Acad.* **61** (1985), ser. A, no. 3, 83-86.

[22] Goto, K., Kano, T.: Remarks to our Former Paper "Uniform Distribution of Some Special Sequences", *Proc. Japan Acad.* **68** (1992), ser. A, no. 10, 348-350.

[23] Grosswald, E., Schnitzer, F.J.: A Class of modified $\zeta$ and L-Functions, *Pacific Journal of Math.* **74** (1978), no. 2, 357-364.

[24] Hardy, G.H., Riez, M.: *The general theory of Dirichlet's series*, Cambridge University Press, Cambridge, 1915.

[25] Hardy, G.H., Wright, E.: *Theory of Numbers*, 6. Edition, Oxford University Press, Oxford, 2008.

[26] Hlawka, E.: *Theorie der Gleichverteilung*, Bibliographisches Institut, Mannheim, 1979.

[27] Hlawka, E.: *Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen*, Österr. Akad. Wiss., Math.-Naturw. Kl. Abt. II, 1975, 459-471; in: Selecta (Springer Collected Works in Mathematics), P. Gruber et al. (Eds.), Springer, Berlin, 1990, 385-397.

[28] Iwaniec, H., Kowalski, E.: *Analytic Number Theory*, American Mathematical Society, Providence, 2004.

[29] Ivić, A.: *The Riemann Zeta-Function*, Dover Publ., New York, 1985.

[30] Kaczorowski, J.: *Axiomatic Theory of L-Functions: The Selberg Class*, in: Analytic Number Theory, Lectures given at the C.I.M.E. Summer School, Cetraro 2002, A. Perelli et al. (Eds.), vol. 1891: Lecture Notes in Mathematics, Springer, 2006, 133-209.

[31] Kaczorowski, J., Perelli, A.: *The Selberg class: A survey*, in: Number Theory in Progress, Proceedings of the International Conference in Honor of the 60th Birthday of Andrej Schinzel, Zakopane 1997, vol. 2: Elementary and Analytic Number Theory, De Gruyter, 1999, 953-992.

[32] Kaczorowski, J., Perelli, A.: On the prime number theorem for the Selberg class, *Archiv der Mathematik* **80** (2003), 255-263.

[33] Kaczorowski, J., Languasco, A., Perelli, A.: A note on Landau's formula, *Funct. Approx. Comment.* **28** (2000), 173-186.
[34] Karatsuba, A.: *Basic Analytic Number Theory*, 2. Edition, Springer, Berlin, 2012.
[35] Korobov, N.M.: Estimates of trigonometrical sums and their application, *Uspehi Mat. Nauk* **13** (1958), 185-192.
[36] Kronecker, L.: Näherungsweise ganzzahlige Auflösung linearer Gleichungen, *Monatsber. Königlich. Preuss. Akad. Wiss. Berlin* (1884), 1179-1193, 1271-1299.
[37] Kuipers, L., Niederreiter, H.: *Uniform Distribution of Sequences*, Wiley, New York, 1974.
[38] Landau, E.: Über die Nullstellen der Zetafunktion, *Math. Ann.* **71** (1912), 548-564.
[39] Landau, E.: Über den Wertevorrat von $\zeta(s)$ in der Halbebene $\sigma > 1$, *Nachr. Ges. Wiss. Göttingen* **36** (1933), 81-91.
[40] Landau, E.: *Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion*, International Congress of Mathematics, Cambridge: August 1912; in: Edmund Landau – Collected Works, vol.5, 240-255, Essen: Thales Verlag, 1986.
[41] Landau, E.: *Vorlesungen über Zahlentheorie - Aus der elementaren Zahlentheorie*, Chelsea Publishing, New York (1950).
[42] LeVeque, W.J.: An inequality connected with Weyl's criterion for uniform distribution, *Proc. Symp. Pure Math.* **8** (1965), 22-33.
[43] Levinson, N: Almost all roots of $\zeta(s) = a$ are arbitrarily close to $\sigma = \frac{1}{2}$, *Proc. Nat. Acad. Sci. U.S.A* **72** (1975), 1322-1324.
[44] Murty, M.R.: *Problems in Analytic Number Theory*, 2. Edition, Springer, New York, 2008.
[45] Murty, M.R., Murty, V.K.: Strong multiplicity one for Selberg's class, *C.R. Acd. Sci. Paris Sér. I Math.* **319** (1994), 315-320.
[46] Murty, M.R., Murty, V.K.: *Non-vanishing of L-Functions and Applications*, Springer, Basel, 1997.
[47] Müller, H.: Über eine Klasse modifizierter $\zeta$- und L-Funktionen, *Arch. Math.* **36** (1981), 157-161.
[48] Montgomery, H.L.: *Topics in Multiplicative Number Theory*, Springer, Berlin, 1971.
[49] Montgomery, H.L., Vaughan, R.C.: *Multiplicative Number Theory - Classical Theory*, Cambridge University Press, Cambridge, 2007.
[50] Narkiewicz, W.: *Elementary and Analytic Theory of Algebraic Numbers*, 3. Edition, Springer, Berlin, 2004.
[51] Parent, D.P.: *Exercises in Number Theory*, Springer, New York, 1984.
[52] Piatetski-Shapiro, I.: On the distribution of prime numbers in sequences of the form $[f(n)]$, *Math. Sbnornik* **33 (75)** (1953), 559-566.
[53] Prachar, K.: *Primzahlverteilung*, Springer, Berlin, 1957.

[54] Rademacher, H.A.: *Fourier Analysis in Number Theory, Symposium on Harmonic Analysis and Related Integral Transforms*, Cornell University, Ithaca, 1956; in: Collected Papers of Hans Rademacher, vol. 2, 434-458, Camebridge: Massachusetts Insitute of Technology, 1974.

[55] Rehberg, M.: *A discrepancy estimate for the a-points of the Riemann zeta function*, in: Anal. Probab. Methods Number Theory, Proceedings of the Sixth International Conference in Honour of J. Kubilius, Palanga 2016, A. Dubickas et al. (Eds.), TEV, Vilnius 2017, 165-178.

[56] Rehberg, M.: Log-like functions and uniform distribution modulo one, *Unif. Distrib. Theory* **13** (2018), no. 2, 93-101.

[57] Roth, K.F.: On irregularities of distribution, *Mathematika* **1** (1954), 73-79.

[58] Schmid, K.: *Gleichverteilung modulo Eins im Zusammenhang mit den Lösungen der Gleichung $\zeta(s) = a$*, Master thesis, University of Würzburg, 2015.

[59] Schmidt, W.M.: Irregularities of distribution VII, *Act. Arith.* **21** (1972), 45-50.

[60] Schoißengeier, J.: The Connection Between the Zeros of the $\zeta$-Function and sequences $(g(p))$, $p$ Prime, mod 1, *Mh. Math.* **87** (1979), 21-52.

[61] Selberg, A.: *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory, Maiori, 1989, E. Bombiere et al. (Eds.), Università di Salerno, 1992, 367-385.

[62] Smajlović, L.: On Li's criterion for the Riemann hypothesis for the Selberg class, *Journal of Number Theory* **130** (2010), 828-851.

[63] Sourmelidis, A., Srichan, T., Steuding, J.: On the Vertical Distribution of Values of $L$-functions in the Selberg Class (submitted).

[64] Steuding, J.: *Value-Distribution of L-Functions*, Springer, Berlin, 2007.

[65] Steuding, J.: *The roots of the equation $\zeta(s) = a$ are uniformly distributed modulo one*, in: Anal. Probab. Methods Number Theory, Proceedings of the Fifth International Conference in Honour of J. Kubilius, Palanga 2011, A. Laurincikas et al. (Eds.), TEV, Vilnius 2012, 243-249.

[66] Steuding, J.: *One Hundred Years Uniform Distribution Modulo One and Recent Applications to Riemann's Zeta-Function*; in: Topics in Mathematical Analysis and Applications, T.M. Rassias et al. (Eds.), Berlin: Springer, 2014, 659-698.

[67] Strauch, O.-Porubský, Š.: *Distribution of Sequences: A Sampler* (2016), eBook, https://math.boku.ac.at/udt/books/MYBASISNew.pdf (10/07/2019, 3 p.m.)

[68] Stux, I.E.: On the uniform distribution of prime powers, *Comm. Pure Appl. Math.* **27** (1974), 729-740.

[69] Titchmarsh, E.C.: *The Theory of the Riemann Zeta-Function*, Oxford University Press, 2. Edition, New York, 1986.

[70] Too, Y.-H.: On the Uniform Distribution Modulo One of Some Log-like Sequences, *Proc. Japan Acad.* **68** (1992), ser. A, no. 9, 269-272.

[71] Tsuji, M.: On the uniform distibution of numbers mod. 1, *Journal Math. Soc. Japan* **4** (1952), 313-322.

[72] Vaaler, J.D.: *Some results on uniform distribution with weights*, PhD thesis, University of Illinois, 1974.

[73] Vaaler, J.D.: *A Tauberian Theorem Related to Weyl's Criterion*, in: Répartition Modulo 1, Actes du Colloque de Marsilles-Luminy 1974, G. Rauzy et al. (Eds.), vol. 475: Lecture Notes in Mathematics, Springer (1975), 253-258.

[74] Vaaler, J.D.: A Tauberian Theorem Related to Weyl's Criterion, *Journal of Number Theory* **9** (1977), 71-78.

[75] Van Aardenne-Ehrenfest, T.: Proof of the impossibility of a just distribution of an infinite sequence of points over an interval, *Indag. Math.* **7** (1945), 71-76.

[76] Van der Corput, J.G., Pisot, C.: Sur la discrépance mudulo un, *Indag. Math.* **1** (1939), 260-269.

[77] Vinogradov, I.M.: The representation of odd numbers as a sum of three primes, *Dokl. Akad. Nauk SSSR Ser. Mat.* **15** (1937), 291-294.

[78] Vinogradov, I.M.: On the evaluation of trigonometric sums with prime numbers, *Izv. Akad. Nauk SSSR Ser. Mat.* **12** (1948), 225-248.

[79] Vinogradov, I.M.: A new estimate for the function $\zeta(1+it)$, *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 161-164.

[80] Vinogradov, I.M.: *The Method of Trigonometrical Sums in the Theory of Numbers*, Dover Publications, New York, 2004.

[81] Weyl, H.: Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77** (1916), 313-352.

[82] Wintner, A.: On the cyclical distribution of the logarithms of the prime numbers, *Quart. J. Math.* **6** (1935), no. 1, 65-68.

[83] Zygmund, A.: *Trigonometric Series*, 3. Edition, Cambridge University Press, London, 2002.