Almost Completely Decomposable Groups of Type (1,2)

Dissertation zur Erlangung des
naturwissenschaftlichen Doktorgrades
der Bayerischen Julius–Maximilians–Universität
Würzburg

vorgelegt von

Ebru Solak

aus Ankara

Würzburg 2007

Eingereicht am: 30. Juli 2007

bei der Fakultät für Mathematik und Informatik

1. Gutachter: Prof. Dr. O. Mutzbauer
2. Gutachter: Prof. Dr. P. Müller

Wer den Weg ans Meer nicht weiß,
gehe nur dem Flusse nach

Sprichwort

Danksagung

Mein besonderer Dank gilt Herrn Prof. Dr. O. Mutzbauer für seine freundliche und hilfreiche Unterstützung während meiner Promotion. Ich danke meinen Eltern, Nurdan und Dogan Solak , meiner Schwester Nilay Solak und meinen Freunden, die mir während dieser ganzen Zeit Mut gemacht haben.

Würzburg, 30.Juli 2007 Ebru Solak

Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig angefertigt und dazu nur die angegebenen Quellen benutzt habe.

Würzburg, den 30. Juli 2007

Ebru Solak

Contents

1. Introduction

We wish to determine indecomposable local almost completely decomposable groups with a critical typeset in $(1, 2)$ configuration. As a $(1, 2)$-type configuration we understand an ordered set $\{\tau_1, \tau_2, \tau_3\}$ of three types with the single order relation $\tau_2 < \tau_3$. Arnold and Dugas, [2] and [6] obtained that local almost completely decomposable groups of type $(1, 2)$, briefly $(1, 2)$-groups, with fixed critical types τ_1, τ_2, τ_3, and regulator quotient of exponent at least p^7, allow infinitely many isomorphism types of indecomposable groups. It is not known if the exponent 7 is minimal, i.e., if there are only finitely many isomorphism types of such indecomposable groups with smaller exponent. We describe groups by representing matrices relative to the two main invariants of almost completely decomposable groups, namely the isomorphism types of the regulator and the regulator quotient, with the intention to show that there are only finitely many indecomposable $(1, 2)$-groups with those invariants. Note that representing matrices describe an almost completely decomposable group G as an extension of the regulator R by the regulator quotient G/R.

Since we are interested in indecomposable groups we may assume the group G to be p-reduced, and moreover, we can switch the groups within a near-isomorphism class. This last statement is due, first to the fact that nearly isomorphic groups coincide in these two invariants, and secondly to a theorem of Arnold [1, Corollary 12.9], saying that groups that are directly decomposable share this property with all nearly isomorphic groups.

All final results are collected in the Theorems 10.1 and 10.2. It is shown that indecomposable $(1, 2)$-groups with regulator quotient of exponent $\leq p^4$ are of rank ≤ 5. It is proved that there is an indecomposable group of rank 4 and there is an explicit test example of a group of rank 5. The latter group is not known to be indecomposable or not. Moreover, there are several isomorphism types of regulator quotients, also for higher exponents, for which there are no indecomposable $(1, 2)$-groups. In so far the remaining gap for regulator quotients of exponent p^5, p^6 gets smaller. But there are still a lot of open problems waiting for an answer whether for example there are finitely or infinitely many isomorphism types of indecomposable $(1, 2)$-groups for those exponents and a fixed critical typeset.

2. Preliminaries

Let $R = \bigoplus_{i=1}^n S_i x_i \subset \mathbb{Q} R$ be a *completely decomposable group*, completely decomposed, with rational groups $\mathbb{Z} \subset S_i \subset \mathbb{Q}$. We call this a *decomposition* of R and the set $X = (x_1, \ldots, x_n)$ a *decomposition basis of* R. Let m be a natural number. If $p^{-1} \notin S_i$ for all primes p dividing m and $pS_i \neq S_i$, then X is called an *m-decomposition basis for the given decomposition of* R. If additionally $S_i \subset S_j$ for $\mathrm{t}(S_i) \leq \mathrm{t}(S_j)$ where $\mathrm{t}(S_i)$ and $\mathrm{t}(S_j)$ denote the types of S_i and S_j respectively, then X is called an *m-Koehler basis for the given decomposition of* R. For each decomposition of R there exist such m-Koehler bases. Since a Koehler basis for the given decomposition of R has the form $(a_1 x_1, \ldots, a_n x_n)$, where the a_i are rational numbers, we can even realize any set $(T_1, \ldots, T_n)$ of rational groups T_i, where $T_i \cong S_i$ such that the properties of an m-Koehler basis hold for the T_i. A torsion-free abelian group is called *m-reduced*, for a natural number m, if there is no proper p-divisible subgroup for any prime p dividing m, or equivalently for the group R, there is $pS_i \neq S_i$ for all $1 \leq i \leq n$ and all p dividing m. A torsion-free abelian group G of finite rank is called *almost completely decomposable* if it has a completely decomposable subgroup, say R, of finite index. In particular, G is m-reduced if and only if R is m-reduced. The completely decomposable subgroups of an almost completely decomposable group G with minimal (finite) index are called *regulating subgroups*. The intersection of all the (finitely many) regulating subgroups of G is called the *regulator of* G. This regulator is a uniquely determined subgroup, that is known to be completely decomposable. The isomorphism types of the regulator and the regulator quotient are isomorphism invariants of an almost completely decomposable group. Note that the quotient of an almost completely decomposable group relative to some regulating subgroup is not an invariant.

Proposition 2.1. *Let m be a natural number and let*

$$R = \bigoplus_{i=1}^{n} S_i' x_i' = \bigoplus_{i=1}^{n} T_i' y_i'$$

be two direct decompositions of the completely decomposable m-reduced group R with $S_i' \cong T_i'$. Then there are two m-Koehler bases $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of R for the two decompositions of R, respectively, such that

(1) $R = \bigoplus_{i=1}^n S_i x_i = \bigoplus_{i=1}^n S_i y_i$, *where* $x_i \in S_i' x_i'$ *and* $y_i \in T_i' y_i'$,
(2) $x_i = \sum_{j=1}^n \rho_{i,j} y_j$, *where* $\rho_{i,j} \in \mathbb{Z}$,
(3) $S_i = \bigcap_{j=1}^n \rho_{i,j} S_j$, *and* $\rho_{i,j} = 0$ *if* $S_i \not\subset S_j$,
(4) $\det(\rho_{i,j})$ *is relatively prime to* m.

Proof. (1) is obvious, say $R = \bigoplus_{i=1}^n S_i' x_i' = \bigoplus_{i=1}^n S_i' y_i'$.
(2): There are rationals $\rho_{i,j}' \in S_i$ such that $x_i' = \sum_{j=1}^n \rho_{i,j}' y_j'$. If there is a prime divisor q of the denominator of some $\rho_{i,j}'$ with $qS_j' = S_j'$, then we replace y_j' by $y_j'' = q^{-t} y_j'$, where t is sufficiently big to change all $\rho_{i,j}'$ to $\rho_{i,j}''$ such that all the denominators of all $\rho_{i,j}''$ are relatively prime to q. Those changes of the basis elements y_j' do not change the coefficient groups S_j'. Thus $(y_1'', \ldots, y_n'')$ is still an m-Koehler basis, and we may assume that the least common multiple s of the denominators of all $\rho_{i,j}''$ in the expression $x_i' = \sum_{j=1}^n \rho_{i,j}'' y_j''$ has no prime divisor q for which there exist q-divisible coefficient groups S_j'. The least common multiple s of the denominators of all $\rho_{i,j}''$ is relatively prime to m, since we have m-Koehler bases.
Now we replace all y_j'' by $y_j = s_j^{-1} y_j''$, where s_j is a natural number such that $S_j y_j = S_j' s_j^{-1} y_j''$ with $q^{-1} \notin S_j$ for all primes q dividing the least common multiple s. This changes all coefficients $\rho_{i,j}''$ to integers $\rho_{i,j}^*$. Thus we obtain $R = \bigoplus_{i=1}^n S_i' x_i' = \bigoplus_{i=1}^n S_i y_i$. and $x_i' = \sum_{j=1}^n \rho_{i,j}^* y_j$. Doing the same with the m-Koehler basis $(x_1', \ldots, x_n')$, i.e., $S_j x_j = S_j' t_j^{-1} x_j'$ for suitable natural numbers t_j with $q^{-1} \notin S_j$ for all primes q dividing the least common multiple s, we get $R = \bigoplus_{i=1}^n S_i x_i = \bigoplus_{i=1}^n S_i' t_i^{-1} x_i' = \bigoplus_{i=1}^n S_i y_i$, and $x_i = \sum_{j=1}^n \rho_{i,j} y_j$, where $\rho_{i,j} = t_i^{-1} \rho_{i,j}^* \in S_j$, and since $\rho_{i,j}^* \in \mathbb{Z}$ we have $\rho_{i,j} \in \mathbb{Z}$. This shows (2).
(3): The equation follows by

$$\chi(x_i) = \chi(y_i) = \bigcap_{j=1}^n \chi(\rho_{i,j} y_j).$$

Consequently, $\rho_{i,j} = 0$ if $S_i \not\subset S_j$.
(4): To show that the determinant of the matrix $\rho = (\rho_{i,j})$ is relatively prime to m, observe that the adjoint ρ' of ρ satisfies $\rho\rho' = \det(\rho) E_n$, where E_n is the unit matrix. Then $\det(\rho) y_j = \sum_{i=1}^n \rho_{i,j}' x_i$, implying that $\det(\rho)$ is relatively prime to m. $\square$

The transition with the matrix ρ from one m-Koehler basis of R to another m-Koehler basis, as in Proposition 2.1, can be considered as an automorphism ρ of the divisible hull $\mathbb{Q}\, R$, defined as $\rho(x_i) =$

$\sum_{j=1}^{n} \rho_{i,j} y_j$. This automorphism is called an (R, m)-*automorphism.* An (R, m)-automorphism ρ preserves the divisible hulls of the type subgroups $R(\tau)$, i.e., $\rho(\mathbb{Q}\, R(\tau)) \subset \mathbb{Q}\, R(\tau)$ for all types τ.
An almost completely decomposable group G is called m-*local*, if the regulator quotient is a group of exponent dividing m.

Lemma 2.2. *Let G be an m-local, m-reduced almost completely decomposable group with regulator R. If ρ is an (R, m)-automorphism, then the group $H = R + \rho(G)$ is nearly isomorphic to G.*

Proof. By definition of ρ we have $\rho(G) \subset H$ with index relatively prime to m. Thus G, H are nearly isomorphic by [10, Theorem 9.2.4]. $\square$

Lemma 2.3. *Let G be an m-reduced almost completely decomposable group with a completely decomposable subgroup R such that G/R is a finite group of exponent dividing m. If $(x_1, \dots, x_n)$ is an m-decomposition basis of R, and if $(g'_1 + R, \dots, g'_r + R)$ is a basis of G/R, where the cyclic group $\mathbb{Z}(g'_j + R) \cong \mathbb{Z}_{k_j}$, i.e., k_j divides m, then there are representatives*

$$g_j = \frac{1}{k_j}\big(\alpha_{j,1} x_1 + \cdots + \alpha_{j,n} x_n\big) \in g'_j + R,$$

for $1 \le j \le r$, with integers $\alpha_{j,i}$. Moreover, the entry $\alpha_{j,i}$ is unique modulo k_j.

Proof. Let

$$g'_j = \frac{1}{k_j}\big(\alpha'_{j,1} x_1 + \cdots + \alpha'_{j,n} x_n\big),$$

where $\alpha'_{j,i} x_i \in R$. Then $\alpha'_{j,i} = \beta_{j,i}/\gamma_{j,i}$ is a fraction in canceled form, and the denominator $\gamma_{j,i}$ is relatively prime to m, since we have an m-decomposition basis. Let ρ be the least common multiple of all $\gamma'_{j,1}, \dots, \gamma'_{j,n}$, and let q, s be integers such that $q\rho = 1 + sk_j$. Then

$$g'_j = (q\rho - sk_j) g'_j = \frac{1}{k_j}\big(q\rho\alpha'_{j,1} x_1 + \cdots + q\rho\alpha'_{j,n} x_n\big) - s\big(\alpha'_{j,1} x_1 + \cdots + \alpha'_{j,n} x_n\big).$$

Since $\alpha'_{j,1} x_1 + \cdots + \alpha'_{j,n} x_n \in R$ and since all coefficients $q\rho\alpha'_{j,i} \in \mathbb{Z}$, the desired representative is $g_j = k_j^{-1}\big(q\rho\alpha'_{j,1} x_1 + \cdots + q\rho\alpha'_{j,n} x_n\big)$.
Moreover, if there are two different representatives g_j, g'_j with integer coefficients, then

$$g'_j - g_j = \frac{1}{k_j}\big((\alpha'_{j,1} - \alpha_{j,1}) x_1 + \cdots + (\alpha'_{j,n} - \alpha_{j,n}) x_n\big) \in R,$$

and $k_j^{-1}(\alpha'_{j,i} - \alpha_{j,i})x_i \in R$ for all $1 \leq i \leq n$. Thus $\alpha'_{j,i} \equiv \alpha_{j,i}$ modulo k_j for all $1 \leq i \leq n$, since G is m-reduced and since we have an m-decomposition basis. $\square$

Let G be an m-local, m-reduced almost completely decomposable group with regulator R and regulator quotient $G/R \cong \bigoplus_{h=1}^{r} \mathbb{Z}_{k_h}$. Let $S = \mathrm{diag}\big(k_h^{-1} \mid 1 \leq h \leq r\big)$ be a diagonal matrix corresponding to the isomorphism type of the regulator quotient. Relative to an m-decomposition basis $(x_1, \ldots, x_n)$ of R and a basis $(g_1 + R, \ldots, g_r + R)$ of the finite group G/R, there is an $r \times n$ integer matrix $\alpha = (\alpha_{j,i})$ as in Lemma 2.3. The matrix $S\alpha$ is called *representing matrix* of G relative to the two given bases. Note that $S\alpha$ is unique, if we choose the integer entries $0 \leq \alpha_{j,i} < k_j$ for all j, i.

Conversely, suppose that $R = \bigoplus_{i=1}^{n} S_i x_i$ with an m-decomposition basis $(x_1, \ldots, x_n)$. Let $S = \mathrm{diag}({k_1}^{-1}, \ldots, {k_r}^{-1})$ be a diagonal matrix and let $\alpha = (\alpha_{i,j})$ be an $r \times n$ integer matrix with $r \leq n$. Then the decomposition of R and $S\alpha$, both together, determine a unique group $G = \langle R, g_1, \ldots, g_r \rangle$ with $R \subset G \subset \mathbb{Q}\, R$, where

$$g_j = \frac{1}{k_j}\big(\alpha_{j,1}x_1 + \cdots + \alpha_{j,n}x_n\big),$$

for $1 \leq j \leq r$. Replacing the entries $\alpha_{j,i}$ by $\alpha'_{j,i}$, where $\alpha_{j,i} \equiv \alpha'_{j,i}$ mod k_j, will not change G. We therefore assume in general that all entries $\alpha_{j,i}$ satisfy $0 \leq \alpha_{j,i} < k_j$. In particular, we put $\alpha_{j,i} = 0$ if $\alpha_{j,i} \in k_j\,\mathbb{Z}$.

We need a well known fact on finite abelian groups. Let $\bar{G} = \bigoplus_{h=1}^{r} \mathbb{Z}\,\bar{g}_h$, where $\mathbb{Z}\,\bar{g}_h \cong \mathbb{Z}_{p^{k_h}}$, be a finite group of rank r and exponent m, with basis $(\bar{g}_1, \ldots, \bar{g}_r)$. Each automorphism of $\bar{G}$ allows a description by an integer matrix U with determinant relatively prime to m. Let $S = \mathrm{diag}\big(k_h^{-1} \mid 1 \leq h \leq r\big)$, corresponding to the isomorphism type of $\bar{G}$. Then the integer matrix U of size r and with determinant relatively prime to m describes an automorphism of $\bar{G}$ relative to the given basis if and only if there is an integer matrix U' such that $US = SU'$, cf. [7, Section 3.11, Theorem 3.15]. Clearly, the integer matrix U' is also of size r and has determinant relatively prime to m.

Proposition 2.4. *Let $S\alpha$ be the representing matrix of an m-local, m-reduced almost completely decomposable group G with regulator R*

relative to an m-Koehler basis of R and a basis of G/R. Let an automorphism of the regulator quotient be described by the integer matrix U with determinant relatively prime to m, and $US = SU'$. Let an (R,m)-automorphism ρ be described by the integer matrix $(\rho_{i,j})$. Then the group $H = R + \rho(G)$, that is nearly isomorphic to G, has a representing matrix

$$S\big(U'\alpha(\rho_{i,j})\big).$$

Proof. Let $S\alpha$ be the representing matrix of the group G relative to the m-Koehler basis $(x_1,\dots,x_n)$ of R and the basis $(g_1+R,\dots,g_r+R)$ of the regulator quotient, where the generators g_j are given as in Lemma 2.3 relative to the m-Koehler basis $(x_1,\dots,x_n)$. If the automorphism of the regulator quotient is given by U, then the new generators g'_j are given by the matrix $SU'\alpha$ relative to the m-Koehler basis $(x_1,\dots,x_n)$. Now switching to the new m-Koehler basis $(y_1,\dots,y_n)$ by ρ, as in Proposition 2.1, we get the representing matrix $S\big(U'\alpha(\rho_{i,j})\big)$ of H as desired. $\square$

We illustrate these basis transformations by an example.

Example. The two groups $G = \mathbb{Z}[11^{-1}]x + \mathbb{Z}[31^{-1}]y + \mathbb{Z}\,\frac{x+2y}{5}$ and $H = \mathbb{Z}[11^{-1}]x + \mathbb{Z}[31^{-1}]y + \mathbb{Z}\,\frac{x+y}{5}$ have the regulator $R = \mathbb{Z}[11^{-1}]x \oplus \mathbb{Z}[31^{-1}]y$. The groups G, H are nearly isomorphic, but not isomorphic. The regulator has the two 5-Koehler bases $\{x, 2y\}$ and $\{x, y\}$. There is no automorphism of the regulator $R = \mathbb{Z}[11^{-1}]x \oplus \mathbb{Z}[31^{-1}]y$ that maps those bases onto each other, since $11 \equiv 31 \equiv 1 \pmod 5$. Clearly, there is an (R,p)-automorphism, since 2 is a unit modulo 5. The choice of a new Koehler basis in general means that we change to another group. By way of contradiction assume that the restriction $\alpha|_R$ of an isomorphism $\alpha : G \longrightarrow H$ is an automorphism of R, thus it is given by a rational 2×2 matrix A of the form $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, relative to the basis (x,y) of $\mathbb{Q}\,x \oplus \mathbb{Q}\,y$. The matrix A is diagonal, since R is rigid. The entries $a \in \operatorname{Aut}\mathbb{Z}[11^{-1}]$ and $b \in \operatorname{Aut}\mathbb{Z}[31^{-1}]$, i.e., $a = 11^s$ and $b = 31^t$ for $s,t \in \mathbb{Z}$. The automorphism $\alpha|_R$ (or the isomorphism α) induces an automorphism $\overline{\alpha}$ of $5^{-1}R/R$ that is described by the 2×2 matrix $\overline{A} = \begin{pmatrix} \bar a & 0 \\ 0 & \bar b \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}_5)$ relative to the basis $(5^{-1}x + R, 5^{-1}y + R)$ of $5^{-1}R/R$. i.e., $\bar a = a + 5\,\mathbb{Z}, \bar b = b + 5\,\mathbb{Z} \in \mathbb{Z}/5\,\mathbb{Z}$. We have $\bar a = \bar b = 1$, since $11 \equiv 31 \equiv 1 \pmod 5$. Hence $\bar\alpha$ is the identity on $5^{-1}R/R$, thus $\bar{\alpha}(G/R) = G/R \neq H/R$, a contradiction. Thus G, H are not isomorphic.

Remark 2.5. If an almost completely decomposable group G is decomposable, then there is a decomposition basis $(x_1, \ldots, x_n)$ of its regulator such that $G = \langle x_1, \ldots, x_s \rangle_* \oplus \langle x_{s+1}, \ldots, x_n \rangle_*$, the sum of two pure hulls in G. Then there is a basis of the regulator quotient such that a corresponding representing matrix is the direct sum of two matrices, i.e., it is a block diagonal matrix. By Proposition 2.1 the (R, m)-automorphisms allow to switch between arbitrary decompositions of the regulator. By Lemma 2.2 we get a nearly isomorphic group this way, and by Proposition 2.4 it is clear how to obtain a representing matrix of those groups. Thus for decomposition questions it is enough to use (R, m)-automorphisms and automorphisms of the regulator quotient.
The transition from one m-Koehler basis to another m-Koehler basis by an (R, m)-automorphism and from one basis of the regulator quotient to another one transforms a representing matrix of a group to the representing matrix of another group that is nearly isomorphic. We formulate this briefly by saying, that we get a *corresponding representing matrix*.

Let p be a prime. We call an integer m a *unit modulo p* if m is not divisible by p. An integer matrix is said to be *p-invertible* if its determinant is a unit modulo p. Two integer matrices of the same format are called *congruent modulo p^k* if all entries of the difference matrix are divisible by p^k. The *p-rank* of an integer matrix is the rank of the reduction of this matrix modulo p over the Galois field $\mathrm{GF}(p)$. A square integer matrix is p-invertible if and only if its reduction modulo p is invertible.

For later use we formulate an elementary result for integer matrices.

Lemma 2.6. *Let p be a prime, let r, n, k be natural numbers. For an integer matrix M of format $r \times n$ the following are equivalent:*

(1) *All matrices LMY, where L is a p-invertible lower triangular matrix and Y is p-invertible, have the property that every row has at least one entry that is a unit modulo p.*
(2) *$r \leq n$ and for all p-invertible matrices X there is a p-invertible matrix Y such that $XMY \equiv (E_r, 0)$ modulo p^k, where E_r denotes the unit matrix of size r.*
(3) *M has p-rank r.*

Proof. It is enough to show that (1) implies (2). We consider all matrices over the field $\mathbb{Z}_p$. Then the indicated property translates to "entries not 0" instead of "units modulo p". Thus M has rank r, i.e., $r \leq n$. Since the lower triangular matrices over a field describe the Gauß algorithm downwards, there is an invertible lower triangular matrix L and a permutation matrix Q such that LMQ has upper triangular form, cf. the LU-decomposition for matrices over fields. Thus there is an invertible matrix Y such that LMY has precisely r entries 1 along the main diagonal, and all other entries are 0. Now we consider the original integer matrices. For every p-invertible integer matrix X, the matrix XM has p-rank r and there is a column permutation P such that $XMP = (N, H)$, where N is p-invertible. Considering these matrices over the ring $\mathbb{Z}/p^k\mathbb{Z}$, the matrix N is invertible and with $Y = P\left(\begin{smallmatrix} N^{-1} & -N^{-1}H \\ 0 & E \end{smallmatrix}\right)$ we obtain the desired result. $\square$

3. $(\mathbf{1},\mathbf{2})$-GROUPS

A p-local, p-reduced almost completely decomposable group of type $(1,2)$ is briefly called a $(1,2)$-*group.* Now we specialize the notation to $(1,2)$-groups. The regulator quotient $G/R \cong \bigoplus_{h=1}^{f} \big(\mathbb{Z}_{p^{k_h}}\big)^{l_h}$, where $k = k_1 > \cdots > k_h \geq 1$, is a finite p-group of exponent p^k and rank $r = \sum_{h=1}^{f} l_h$. The regulator quotient has the hth *step* $\big(\mathbb{Z}_{p^{k_h}}\big)^{l_h}$. A basis of G/R is the union of the bases of those steps, where

$$\{g_j + R \mid \sum_{i=1}^{h-1} l_i < j \leq \sum_{i=1}^{h} l_i\}$$

is a basis of the hth step of the regulator quotient. The regulator is the direct sum $R = R_1 \oplus R_2 \oplus R_3$, where R_i is homogeneous of rank r_i, and $n = r_1 + r_2 + r_3$ is the rank of G, and the types of the R_i form a $(1,2)$-diagram. R is the regulator of G if and only if R_1 and $R_2 \oplus R_3$ are pure in G.

To obtain a representing matrix for the group G, we fix a p-Koehler basis $\big(x_1, \ldots, x_{r_1};\ y_1, \ldots, y_{r_2};\ z_1, \ldots, z_{r_3}\big)$ of the regulator R according to the given decomposition of R. Thus, if R is the regulator of G, and since G is p-reduced, the characteristics of the elements of a p-Koehler basis all have p-height $\chi_p(x_i) = \chi_p(y_i) = \chi_p(z_i) = 0$.

Let $(g_j + R \mid 1 \leq j \leq r)$ be a basis of the regulator quotient G/R,

$$g_j = p^{-k_h}\Big(\sum_{i=1}^{r_1} \alpha_{ji} x_i + \sum_{i=1}^{r_2} \beta_{ji} y_i + \sum_{i=1}^{r_3} \gamma_{ji} z_i\Big), \tag{3.1}$$

where the negative p-power in front is p^{-k_h} if $\sum_{i=1}^{h-1} l_i < j \leq \sum_{i=1}^{h} l_i$ for $1 \leq h \leq f$, according to the given decomposition of the regulator quotient.

By Lemma 2.3 the three matrices $\alpha = (\alpha_{ji}), \beta = (\beta_{ji}), \gamma = (\gamma_{ji})$ may be assumed to have integer entries, and they form a so called *section matrix* $\big(\alpha, \beta, \gamma\big)$ of overall format $r \times n$, whereas the single sections α, β, γ are of format $r \times r_1$, $r \times r_2$, $r \times r_3$, respectively. Let $S = \operatorname{diag}\big(p^{-k_h} E_{l_h} \mid 1 \leq h \leq f\big)$, where the unit matrices E_{l_h} are of size l_h and $k = k_1 > k_2 > \cdots > k_f \geq 1$ with the exponent $p^k = \exp(G/R)$ of the regulator quotient. The matrix $M = S(\alpha, \beta, \gamma)$ is a representing matrix of the group G. The section matrix (α, β, γ) is called *section part* of the representing matrix M. Clearly, the matrix S is given by the isomorphism type of the regulator quotient, i.e., S is unique for a given G. Moreover, S together with the section part of M determines G

up to isomorphism but the section part of a representing matrix is not unique.
The isomorphism type of the regulator of a $(1,2)$-group is given by the sequence $\big((r_1,\tau_1),(r_2,\tau_2),(r_3,\tau_3)\big)$, and the isomorphism type of the regulator quotient is given by the sequence $\big((k_h,l_h)\mid h=1,\ldots,f\big)$. A representing matrix $M=S(\alpha,\beta,\gamma)$ of such a group reflects all invariants of the isomorphism types of the regulator and the regulator quotient except of the specific critical types τ_1,τ_2,τ_3 of G.
A more precise description of the automorphisms of G/R and of (R,p)-automorphisms by matrices is necessary. By Proposition 2.1 an integer matrix V describing an (R,p)-automorphism has block structure according to the $(1,2)$-type constellation,

$$V=\begin{pmatrix} X_1 & 0 & 0\\ 0 & X_2 & X_4\\ 0 & 0 & X_3\end{pmatrix},$$

where X_1,X_2,X_3 are p-invertible matrices, and X_4 is arbitrary. For decomposition questions it is enough to consider (R,p)-automorphisms, cf. Remark 2.5.
The integer matrices describing automorphisms of finite abelian p-groups inherit a block structure by the block structure of the group. Specialized to our case let $l_1,\ldots,l_f$ be natural numbers. An integer matrix

$$M=(A_{ij})=\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1f}\\ A_{21} & A_{22} & \ldots & A_{2f}\\ \vdots & \vdots & \ddots & \vdots\\ A_{f1} & A_{f2} & \ldots & A_{ff}\end{pmatrix}$$

with *blocks* A_{ij} of format $l_i\times l_j$ and $\sum_{i=1}^f l_i = r$, is called a *block matrix of format* $(l_h)_h$. Note that the block matrix M is square of size r. Let $k=k_1>\cdots>k_h\geq 1$ with natural numbers k_h. A block matrix of format $(l_h)_h$ is called an $(l_h,k_h)_h$*-automorphism matrix* if all diagonal blocks A_{ii} are p-invertible and if $A_{ij}\in p^{k_j-k_i}\,\mathrm{M}(l_i\times l_j,\mathbb{Z})$ for all $i>j$. Note that an $(l_h,k_h)_h$-automorphism matrix is p-invertible and describes an automorphism of a finite p-group isomorphic to $\bigoplus_{h=1}^f\big(\mathbb{Z}_{p^{k_h}}\big)^{l_h}$ relative to some basis. Moreover, for a fixed sequence $(l_h,k_h)_h$, the set of all $(l_h,k_h)_h$-automorphism matrices forms a multiplicative group.

Let $H = \bigoplus_{i=1}^{r} \mathbb{Z}\, h_i \cong \bigoplus_{h=1}^{f} \big(\mathbb{Z}_{p^{k_h}}\big)^{l_h}$ be a finite p-group of exponent p^k and rank $r = \sum_{h=1}^{f} l_h$. Let $S = \operatorname{diag}\big(p^{-k_h} E_{l_h} \mid 1 \leq h \leq f\big)$, where the unit matrices E_{l_h} are of size l_h corresponding to the isomorphism type of H. The p-invertible integer matrix U of size r describes an automorphism of H relative to the given basis if and only if there is a p-invertible integer matrix U' such that $US = SU'$. In particular, $(l_h, k_h)_h$-automorphism matrices describe automorphisms of H, and if $U = (U_{ij})$, using block notation, then $U' = (U'_{ij})$, where all diagonal blocks U'_{ii} are p-invertible and $U'_{ij} \in p^{k_i - k_j}\, \mathrm{M}(l_i \times l_j, \mathbb{Z})$ for all $i < j$.

The following Lemma is a straightforward consequence of Proposition 2.4.

Lemma 3.1. *Let $S(\alpha, \beta, \gamma)$ be the representing matrix of a $(1,2)$-group with regulator R. Let an automorphism of the regulator quotient be given by the $(l_h, k_h)_h$-automorphism matrix U, and $US = SU'$.*
Let an (R,p)-automorphism be given by the p-invertible integer matrix $V = \left(\begin{smallmatrix} X_1 & 0 & 0 \\ 0 & X_2 & X_4 \\ 0 & 0 & X_3 \end{smallmatrix}\right)$. Then the corresponding representing matrix is

$$S(U'\alpha X_1, U'\beta X_2, U'\gamma X_3 + U'\beta X_4).$$

4. Properties of Representing Matrices

We collect some properties of representing matrices. In particular, we are interested in properties that are forced by the indecomposability of the group G, and the fact that R is the regulator.
If the regulator quotient of a group G has exponent p^k, then replacing the section part (α, β, γ) of a representing matrix by a section matrix $(\alpha', \beta', \gamma')$, that is congruent modulo p^k, will not change the group. More precisely, by Lemma 2.3, the entries $\alpha_{j,i}, \beta_{j,i}, \gamma_{j,i}$ of the section matrix $\big((\alpha_{j,i}), (\beta_{j,i}), (\gamma_{j,i})\big)$ are unique only modulo p^{k_h}, where k_h and j correspond to each other. We will in general replace the entries $\alpha_{j,i}, \beta_{j,i}, \gamma_{j,i}$ by 0 if they are in $p^{k_h}\,\mathbb{Z}$.
If there are row permutations and column permutations that change the section part (α, β, γ) into a block diagonal form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ modulo p^k, then the group is directly decomposable. In particular, included is the special case that there is no block B, i.e., the representing matrix is of the form $(A, 0)$ modulo p^k. Clearly, if the representing matrix has a 0-column modulo p^k, then the group has a direct summand of rank 1, a rational group. Groups without rational direct summands are called *clipped.*

Lemma 4.1. *If $S(\alpha, \beta, \gamma)$ is a representing matrix of a $(1,2)$-group, then there is a unit in each row of α and there is a unit in each row of (β, γ).*

Proof. If there is a row in α without unit, then $R_2 \oplus R_3$ is not pure in G. If there is a row in (β, γ) without unit, then R_1 is not pure in G. In either case the regulator criterion for R would be violated. $\square$

We state that the properties of a representing matrix $S(\alpha, \beta, \gamma)$ as in Lemma 4.1 will not get lost if a p-Koehler basis of the regulator is replaced by any other p-Koehler basis, and the same for some replacement of a basis of G/R by any other basis.

Lemma 4.2. *If $S(\alpha, \beta, \gamma)$ is a representing matrix of a clipped $(1,2)$-group, then the matrix α is (square) p-invertible. Moreover, there is a p-Koehler basis of R_1 and representatives of the basis elements of G/R such that $S(E, \beta, \gamma)$ is the corresponding representing matrix.*

Proof. Changing the p-Koehler basis of the regulator and the basis of the regulator quotient, translates by Lemma 3.1 for the first part α of the section matrix to the matrix $\alpha' = U'\alpha X_1$, where $US = SU'$ for some $(l_h, k_h)_h$-automorphism matrix U. The matrices U' form a group that contains the subgroup of p-invertible lower triangular matrices, and X_1 is p-invertible. Thus, by Lemma 2.6 there are U' and X_1 such that $U'\alpha X_1 \equiv (E_r, 0)$ modulo p^k. But since the group is clipped no 0-columns occur, hence α is square and p-invertible. Moreover, by Lemma 3.1 we may choose $X_1 = \alpha^{-1}$ changing the p-Koehler basis of R_1 and no basis changes in R_2, R_3. Then the corresponding representing matrix is congruent to $S(E, \beta, \gamma)$ modulo p^k. Finally there is a suitable choice of representatives in the basis elements of the regulator quotient such that precisely the unit matrix E_r is obtained as the first part of the section matrix. $\square$

The matrices β and γ are of format $r \times r_2$ and $r \times r_3$, respectively. They have a *step structure* $(l_h)_{h=1}^{f}$, i.e., $r = \sum_{h=1}^{f} l_h$, and the submatrix $\beta_h = (\beta_{i,j})$, where $\sum_{k=1}^{h-1} l_k < i \leq \sum_{k=1}^{h} l_k$ and $1 \leq j \leq r_2$, is called the hth *step* of β. Similarly γ has an $(l_h)_{h=1}^{f}$ step structure.

Lemma 4.3. *Let G be $(1,2)$-group with a representing matrix $S(E, \beta, \gamma)$. Suppose that all entries $\{\beta_{b,j} \mid 1 \leq j \leq r_2\} \subset p^t\,\mathbb{Z}$ of the bth row of β be contained in $p^t\,\mathbb{Z}$ for some $t \geq 0$. Suppose that $\beta_{b,j} = p^t \beta^*_{b,j} \in p^t\,\mathbb{Z} \backslash p^{t+1}\,\mathbb{Z}$ for some $1 \leq j \leq r_2$, i.e., $\beta^*_{b,j}$ is relatively prime to p. Then there is a p-decomposition basis of R_2 such that the bth row of β' of the corresponding representing matrix $S(E, \beta', \gamma)$ has the form $(0, \ldots, 0, p^t, 0, \ldots, 0)$ where the entry p^t is at the jth position. Moreover, for all $c \neq b$*

- $\beta'_{c,j} = \beta_{c,j} {\beta^*_{b,j}}^{-1}$,
- $\beta'_{c,l} = \beta_{c,l} - \beta_{c,j} {\beta^*_{b,j}}^{-1} \beta_{b,l}$ *for all $l \neq j$ and $l \leq r_2$,*

where ${\beta^*_{b,j}}^{-1} \in \mathbb{Z}$, ${\beta^*_{b,j}}^{-1} \beta^*_{b,j} \equiv 1 \pmod{p^k}$ and $p^k G \subset R$.
If $\beta_{b,j}$ is a unit, i.e., $t = 0$, then there is a p-Koehler basis of $R_2 \oplus R_3$ such that the bth row of (β', γ') of the corresponding representing matrix $S(E, \beta', \gamma')$ has the form $(0, 0, \ldots, 1, 0, \ldots, 0)$ with entry 1 at the jth position and for all $c \neq b$ there is additional

- $\gamma'_{c,l} = \gamma_{c,l} - \beta_{c,j} \beta_{b,j}^{-1} \gamma_{b,l}$ *for all $l \leq r_3$.*

In particular, $\beta_{c,j} \in p^s\,\mathbb{Z}$ if and only if $\beta'_{c,j} \in p^s\,\mathbb{Z}$.

Proof. Let $(x_1, \ldots, x_{r_1}; y_1, \ldots, y_{r_2}; z_1, \ldots, z_{r_3})$ be the p-Koehler basis of $R = R_1 \oplus R_2 \oplus R_3$ for the given representing matrix. There is a new

p-Koehler basis of R_2 with the element

$$y'_j \;= y_j + p^{-t}\sum_{\substack{l=1\\ l\neq j}}^{r_2}\beta_{b,l}y_l$$

instead of y_j. This changes the representing matrix to $S(E,\beta',\gamma)$ where the bth row of β' has the form $(0,\dots,0,p^t,0,\dots,0)$ and p^t is in the jth column. To avoid duplication in the proof we deal with only the case that $\beta_{b,j}$ is a unit. Then there is a new p-Koehler basis of $R_2\oplus R_3$ with the element

$$y'_j = \sum_{l=1}^{r_2}\beta_{b,l}y_l + \sum_{s=1}^{r_3}\gamma_{b,s}z_s$$

instead of y_j. This changes the representing matrix to $S(E,\beta',\gamma')$ where the bth row of (β',γ') has the form $(0,0,\dots,0,1,0,\dots,0)$ with entry 1 at the jth position. Denote an arbitrary row of (β,γ) with index c as a representing element of a generating element of G/R in the form

$$g_c = p^{-k'}\big(x_c + \sum_{l=1}^{r_2}\beta_{c,l}y_l + \sum_{s=1}^{r_3}\gamma_{c,s}z_s\big).$$

If we choose $(y'_1,\dots,y'_j,\dots,y'_{r_2};z_1,\dots,z_{r_3})$ where $y'_l = y_l$ for all $l\neq j$ and y'_j defined as above, as a new basis of $R_2\oplus R_3$, then modulo R

$$\begin{aligned} g_c &\equiv p^{-k'}[x_c + \sum_{\substack{l=1\\ l\neq j}}^{r_2}\beta_{c,l}y'_l + \sum_{s=1}^{r_3}\gamma_{c,s}z_s + \beta_{c,j}\beta_{b,j}^{-1}(y'_j - \sum_{\substack{l=1\\ l\neq j}}^{r_2}\beta_{b,l}y'_l - \sum_{s=1}^{r_3}\gamma_{b,s}z_s)]\\ &\equiv p^{-k'}[x_c + \beta_{c,j}\beta_{b,j}^{-1}y'_j + \sum_{\substack{l=1\\ l\neq j}}^{r_2}(\beta_{c,l} - \beta_{c,j}\beta_{b,j}^{-1}\beta_{b,l})y'_l + \sum_{s=1}^{r_3}(\gamma_{c,s} - \beta_{c,j}\beta_{b,j}^{-1}\gamma_{b,s})z_s]. \quad \square \end{aligned}$$

Lemma 4.4. *Let G be $(1,2)$-group with a representing matrix $S(E,\beta,\gamma)$. If in the bth row of γ there is a unit $\gamma_{b,j}$ for some $1\le j\le r_2$, then there is a p-Koehler basis of R_3 such that the new corresponding representing matrix is $S(E,\beta,\gamma')$ where the bth row of γ' has the form $(0,0,\dots,1,0,\dots,0)$ with entry 1 at the jth position. Moreover, for all $c\neq b$*

$$\gamma'_{c,j} = \gamma_{c,j}\gamma_{b,j}^{-1} \quad \textit{and} \quad \gamma'_{c,l} = \gamma_{c,l} - \gamma_{c,j}\gamma_{b,j}^{-1}\gamma_{b,l} \quad \textit{for all} \quad l\neq j,$$

where $\gamma_{b,j}^{-1}\in\mathbb{Z}$, $\gamma_{b,j}^{-1}\gamma_{b,j}\equiv 1 \pmod{p^k}$ and $p^kG\subset R$.
In particular, $\gamma_{c,j}\in p^s\,\mathbb{Z}$ if and only if $\gamma'_{c,j}\in p^s\,\mathbb{Z}$.

Proof. Let $(z_1, z_2, \ldots, z_{r_3})$ be a p-Koehler basis of R_3. If $\gamma_{b,j}$ is a unit, then there is a new p-decomposition basis of R_3 with the element

$$z'_j = \sum_{s=1}^{r_3} \gamma_{b,s} z_s$$

instead of z_j. This changes the representing matrix to $S(E, \beta, \gamma')$ where the bth row of γ' has the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the entry 1 at the jth position. The new γ' is obtained by exactly the same rules as in the analogous proof of Lemma 4.3. Clearly, this transformation will not change β. □

A straightforward consequence of Lemma 4.3 and Lemma 4.4 is the following corollary.

Corollary 4.5. *Let G be a $(1,2)$-group with $S(E, \beta, \gamma)$ as representing matrix*

(1) *Suppose that $\beta_{b,j} = p^t$ where $t \geq 1$ and $j_1, \ldots, j_s \in [1, r_2]$ such that $\beta_{b,j_l} \in p^t\,\mathbb{Z} \setminus 0$. Replacing y_b by*

$$y'_b = y_b + p^{-t} \sum_{l=1}^{s} \beta_{b,j_l} y_l$$

changes only the entries of β with column indices j_l. In particular, if $\beta_{c,j_l} = 0$ for $c \neq b$, then $\beta'_{c,j_l} = -\beta_{c,j}\beta_{b,j_l}$.

(2) *Suppose that $\gamma_{b,j}$ is a p-unit and $j_1, \ldots, j_s \in [1, r_3]$ such that $\gamma_{b,j_l} \neq 0$. Replacing z_b by*

$$z'_b = z_b + \sum_{l=1}^{s} \gamma_{b,j_l} z_{j_l}$$

changes only the entries of γ with column indices j_l. In particular, if $\gamma_{c,j_l} = 0$ for $c \neq b$, then $\gamma'_{c,j_l} = -\gamma_{c,j}\gamma_{b,j_l}$.

A zero matrix with a rows and b columns is denoted by $0[a \times b]$. Recall that the unit matrix of size s is denoted by E_s. Let for a matrix β the subblock of β consisting of the rows with index $a \leq c \leq b$ be denoted by $\beta_{[a,b]}$.

Lemma 4.6. *Let G be $(1,2)$-group with a representing matrix $S(E, \beta, \gamma)$. Suppose that no entry of the subblock $\beta_{[a,b]}$ is a unit. If $\gamma_{[a,b]} = (0[(b-a) \times s], \delta)$, then there is a p-Koehler basis of R_3 and a basis of G/R such that the corresponding representing matrix (β', γ') has the following properties:*

- $\beta'_{[1,a-1]} = \beta_{[1,a-1]}$.
- *There is no unit in* $\beta'_{[a,b]}$.
- *The first* s *columns of* $\gamma_{[1,a-1]}$ *and* $\gamma'_{[1,a-1]}$ *are the same. If the last* $r_3 - s$ *columns of* γ *form the* 0*-matrix, then this part of* γ' *forms the* 0*-matrix.*
- *There is* $r_3 \geq s + b - a$ *and*

$$\gamma'_{[a,b]} = \Big(0[(b-a) \times s], E_{b-a}, 0[(b-a) \times \big(r_3 - (s+b-a)\big)]\Big).$$

Proof. Since there is no unit in $\beta_{[a,b]}$, all units of this block of (β, γ) are in $\gamma_{[a,b]}$. Assume that $\gamma_{a,j}$ is a unit for some $s < j \leq r_3$. By Lemma 4.4 there is a p-Koehler basis of R_3 such that the ath row of γ is changed to $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the entry 1 has the column index $s+1$. This transformation does not change β. Moreover, there is a basis of G/R such that $\gamma_{c,s+1} = 0$ for all $a+1 \leq c \leq b$. This basis transformation of G/R changes the entries of β with row indices c where $a+1 \leq c \leq b$. But there are no changes in $\beta_{[1,a-1]}$. Let $S(E, \beta'', \gamma'')$ be the new representing matrix. Then we have

$$\beta''_{[1,a-1]} = \beta_{[1,a-1]}.$$

Since there is no unit in $\beta''_{[a,b]}$ and by Lemma 4.4 the first s columns of $\gamma''_{[1,a-1]}$ and $\gamma_{[1,a-1]}$ are the same. Furthermore, by Lemma 2.6 and Lemma 4.1 the submatrix of the last $r_3 - s$ columns of $\gamma''_{[a,b]}$ has p-rank $b-a$. Thus $r_3 \geq s+b-a$ and there is a p-Koehler basis of R_3 and a basis of G/R such that we finally get

$$\gamma'_{[a,b]} = (0[(b-a) \times s], E_{b-a}, 0[(b-a) \times (r_3 - (s+b-a))]).$$

If the last $r_3 - s$ columns of γ form the 0-matrix, then by Lemma 4.4 this part of γ' forms the 0-matrix. $\square$

Lemma 4.7. *Let* $S(E, \beta, \gamma)$ *be a representing matrix of a* $(1,2)$*-group. Suppose that the entry* $\beta_{a,b} \in p^t\,\mathbb{Z} \backslash p^{t+1}\,\mathbb{Z}$, $t \geq 0$, *is in the* h*th block. Then there is a basis of* G/R *such that the corresponding representing matrix is* $S(E, \beta', \gamma')$ *with the property that*

(1) $\beta'_{c,b}$ *in the* s*th block,* $s \geq h$, *is either* $\beta'_{c,b} = 0$ *or* $\beta'_{c,b} \notin p^t\,\mathbb{Z}$,

(2) $\beta'_{c,b}$ *in the* s*th block,* $s < h$, *is either* $\beta'_{c,b} = 0$ *or* $\beta'_{c,b} \notin p^{k_s - k_h + t}\,\mathbb{Z}$.

In particular, if $\beta_{a,b}$ *is a unit, i.e.,* $t = 0$, *then for* $c > a$ *all* $\beta_{c,b} = 0$.

Proof. We may assume that $\beta_{a,b} = p^t$. Denote a row of (β, γ) with index c as representing element of a generating element of G/R in the form

$$g_c = p^{-k_{h'}}(x_c + \sum_{l=1}^{r_2} \beta_{c,l} y_l + \sum_{s=1}^{r_3} \gamma_{c,s} z_s).$$

For $c > a$ and $\beta_{c,b} \in p^t\,\mathbb{Z}$, in the sth block of β, $s \geq h$, we choose representatives for a new basis of G/R by

$$\begin{aligned} g'_c &= g_c - p^{k_h - k_s - t}\beta_{c,b} g_a, \\ g'_j &= g_j \qquad \text{for} \quad j \neq c. \end{aligned}$$

Then the representing matrix changes to $S(E, \beta', \gamma')$ where $\beta'_{c,b} = 0$. For $c < a$ and $\beta_{c,b} \in p^{k_s - k_h + t}\,\mathbb{Z}$, in the sth block of β, $s < h$, we choose representatives for a new basis of G/R by

$$\begin{aligned} g'_c &= g_c - p^{k_h - k_s - t}\beta_{c,b} g_a, \\ g'_j &= g_j \qquad \text{for} \quad j \neq c. \end{aligned}$$

Then the representing matrix changes to $S(E, \beta', \gamma')$ where $\beta'_{c,b} = 0$. $\square$

Lemma 4.8. *Let $S(E, \beta', \gamma)$ be a representing matrix of a $(1,2)$-group. If β' has a unit $\beta'_{a,b}$ in the hth block, then there is a p-Koehler basis of $R_2 \oplus R_3$ and a basis of the regulator quotient such that the first row of the hth block of (β, γ) of the corresponding representing matrix is $(0, \ldots, 1, 0, \ldots, 0)$, where the entry 1 is at the bth position and the bth column of (β, γ) is $(\beta_{1,b}, \ldots, \beta_{\sum_{i=1}^{h-1} l_i, b}, 1, 0, \ldots, 0)^t$.*
Moreover, if $\beta_{i,b}$ is an entry of β in the sth block, for $s \leq h$, and in the bth column, then either $\beta_{i,b} = 0$ or $\beta_{i,b} \notin p^{k_s - k_h}\,\mathbb{Z}$.

Proof. If $\beta'_{a,b}$ in the hth block of β' is a unit, then, by Lemma 4.3, there is a p-Koehler basis of $R_2 \oplus R_3$ such that the ath row of the hth block of the new representing matrix (β, γ) is $(0, \ldots, 1, 0, \ldots, 0)$, where the entry 1 is at the bth position. We may permute the ath row to the first row in the hth block by a change of basis of G/R, and the rest follows by Lemma 4.7. $\square$

5. Normal Form

Let $A = (A_{ij}) \in M(m \times n, \mathbb{Z})$ be a block matrix with blocks A_{ij}, where $A_{ij} \in M(u_i \times v_j, \mathbb{Z})$ and $u_1 + \cdots + u_{q_1} = m$ and $v_1 + \cdots + v_{q_2} = n$. $(A_{ij} \mid j)$ is the ith block row of the block matrix (A_{ij}), i.e., a matrix of format $u_i \times (\sum_j v_j)$.

Definition. Let $M = S(\alpha, \beta, \gamma)$ be a representing matrix of a $(1,2)$-group G where $S = \operatorname{diag}(p^{-k_h} E_{l_h} \mid h = 1, \dots, f)$. Then M is said to be in *normal form* of format $(k_h, l_h, m_h)_{h=1}^{f}$ if $\alpha = E_r$, where $r = \sum_{h=1}^{f} l_h$, the matrix β is of format $r \times r_2$, the matrix γ is of format $r \times r_3$ and β and γ are block matrices with block rows $\beta_1, \dots, \beta_f$ and $\gamma_1, \dots, \gamma_f$, respectively, i.e., $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_f \end{pmatrix}$ where β_h is of format $l_h \times r_2$ and $\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_f \end{pmatrix}$ where γ_h is of format $l_h \times r_3$ for all $h = 1, \dots, f$. Moreover, the block rows of β and γ have the following structure:

- $r_2 \geq \sum_{z=1}^{f} m_z$, and β_h has block rows $\beta_h^{(1)}$ and $\beta_h^{(2)}$ for $h = 1, \dots, f$ of format $m_h \times\ r_2$ and $(l_h -\ m_h) \times\ r_2$, respectively.
$$\beta_h^{(1)} = \Bigl(0\bigl[m_h \times \sum_{z=1}^{h-1} m_z\bigr], E_{m_h}, 0\bigl[m_h \times (r_2 - \sum_{z=1}^{h} m_z)\bigr]\Bigr),$$
$$\beta_h^{(2)} = \Bigl(0\bigl[(l_h - m_h) \times \sum_{z=1}^{h} m_z\bigr], p\beta_h'\Bigr),$$
where β_h' is of format $(l_h - m_h) \times (r_2 - \sum_{z=1}^{h} m_z)$.
If $m_h = 0$ or $l_h = m_h$, then one of these two block rows do not exist.
- $r_3 \geq \sum_{z=1}^{h}(l_z - m_z)$ and γ_h has block rows $\gamma_h^{(1)}$ and $\gamma_h^{(2)}$ for $h = 1, \dots, f$ of format $m_h \times\ r_3$ and $(l_h -\ m_h) \times\ r_3$, respectively.
$$\gamma_h^{(1)} = 0\bigl[m_h \times r_3\bigr],$$
$$\gamma_h^{(2)} = \Bigl(0\bigl[(l_h - m_h) \times \sum_{z=1}^{h-1}(l_z - m_z)\bigr], E_{(l_h - m_h)}, 0\bigl[(l_h - m_h) \times (r_3 - \sum_{z=1}^{h}(l_z - m_z))\bigr]\Bigr).$$
If $m_h = 0$, then $\gamma_h^{(1)}$ does not exist, and if $l_h = m_h$, then $\gamma_h^{(2)}$ does not exist.

Lemma 5.1. *Let G be a $(1,2)$-group with $G/R \cong \bigoplus_{h=1}^{f} (\mathbb{Z}_{p^{k_h}})^{l_h}$. Then there is a p-Koehler basis of R and a corresponding representing matrix $S(E, \beta, \gamma)$ such that the hth block of (β, γ) has the form*

$$(\beta_h, \gamma_h) = \left.\begin{array}{|c|c||c|c|} \hline E_{m_h} & 0 & \multicolumn{2}{c|}{0} \\ \hline 0 & p\eta_h & E_{l_h - m_h} & 0 \\ \hline \end{array}\right. ,$$

FIGURE 1

i.e., (β_h, γ_h) is in normal form. If $m_h = l_h$, then $(\beta_h, \gamma_h) = (E_{l_h}, 0)$. If $m_h = 0$, then $(\beta_h, \gamma_h) = (p\eta_h, E_{l_h}, 0)$.
Moreover, if G has a representing matrix $S(E, \beta', \gamma')$ with a zero row in the hth block of β', then there is also a zero row in the hth block of β.

Proof. Let G be represented by $S(E, \beta', \gamma')$ relative to a p-Koehler basis of R. If there is no unit in β'_h, then by Lemma 4.6, Figure 1 specifies to $(\beta_h, \gamma_h) = (p\eta_h, E_{l_h}, 0)$, i.e., $m_h = 0$ and the block row beginning with E_{m_h} does not exist.
Let a be minimal where $\sum_{u=1}^{h-1} l_u < a \leq \sum_{u=1}^{h} l_u$ with respect to that there is a unit in the ath row of β'_h. By Lemma 4.8 there is a p-Koehler basis of $R_2 \oplus R_3$ and a basis of G/R such that the first row of (β'_h, γ'_h) changes to $(1, 0, \ldots, 0)$ and the first column of β'_h changes to $(1, 0, \ldots, 0)^t$. There is possibly again a row with minimal index b where $a < b \leq \sum_{u=1}^{h} l_u$ such that there is a unit in the bth row of β'_h. Then again by Lemma 4.8 there is a p-Koehler basis of $R_2 \oplus R_3$ and a basis of G/R such that the second row of (β'_h, γ'_h) changes to $(0, 1, 0, \ldots, 0)$ and the second column of β'_h changes to $(0, 1, 0, \ldots, 0)^t$. We may continue with this procedure for β'_h as far as there are units. Then (β'_h, γ'_h) changes to (β''_h, γ''_h) where

$$(\beta''_h, \gamma''_h) = \begin{pmatrix} E_{m_h} & 0 & 0 \\ 0 & * & * \end{pmatrix}.$$

If $m_h = l_h$, then (β''_h, γ''_h) specifies to $(\beta_h, \gamma_h) = (E_{l_h}, 0)$. If $m_h < l_h$, then (β''_h, γ''_h) has the form

$$(\beta''_h, \gamma''_h) = \left(\begin{array}{cc|c} E_{m_h} & 0 & 0 \\ \hline 0 & p\delta^{(1)} & \delta^{(2)} \end{array}\right).$$

If there was a 0-row in β'_h, then this row is not changed by the above basis transformations and will occur in the matrix $p\delta^{(1)}$.

Since each row of the matrix (β, γ) has to contain a unit, forced by the regulator criterion, the matrix $\delta^{(2)}$ has at least that many columns as rows, i.e., $r_3 \geq l_h - m_h$. By Lemma 4.6, the matrix $\delta^{(2)}$ changes to $(E_{l_h - m_h}, 0)$. The unit matrix E_{m_h} and the corresponding rows and columns remain unchanged. Moreover, no new units occur in β_h''. Hence (β_h'', γ_h'') changes to (β_h, γ_h) where

$$(\beta_h, \gamma_h) = \left(\begin{array}{cc|cc} E_{m_h} & 0 & 0 & 0 \\ \hline 0 & p\eta_h & E_{l_h - m_h} & 0 \end{array} \right). \qquad \square$$

Lemma 5.2. *Let G be a $(1,2)$-group with $G/R \cong \bigoplus_{h=1}^{f} (\mathbb{Z}_{p^{k_h}})^{l_h}$, where $f \geq 2$. Then there is a p-Koehler basis of R and a corresponding representing matrix $S(E, \beta, \gamma)$ such that the hth and $(h+1)$th blocks of (β, γ) have the form as in Figure 2.*

$$\left(\begin{array}{c||c} \beta_h & \gamma_h \\ \hline \beta_{h+1} & \gamma_{h+1} \end{array} \right) = \begin{array}{|c|c|c||c|c|c|} \hline E_{m_h} & \multicolumn{2}{c||}{0} & \multicolumn{3}{c|}{0} \\ \hline & \multicolumn{2}{c||}{p\eta_h} & E_{l_h - m_h} & \multicolumn{2}{c|}{0} \\ \cline{2-6} & E_{m_{h+1}} & 0 & & \multicolumn{2}{c|}{0} \\ \cline{2-3} \cline{5-6} 0 & 0 & p\eta_{h+1} & 0 & E_{l_{h+1} - m_{h+1}} & 0 \\ \hline \end{array}$$

Figure 2

In particular,

- *if $m_h = 0$, then $r_3 \geq l_h$ and (β_h, γ_h) has the form*

$$(\beta_h, \gamma_h) = (p\eta_h, E_{l_h}, 0[l_h \times (r_3 - l_h)]).$$

- *If $m_h = l_h$, then $r_2 \geq l_h$ and (β_h, γ_h) has the form*

$$(\beta_h, \gamma_h) = (E_{l_h}, 0[l_h \times (r_2 - l_h)], 0[l_h \times r_3]).$$

- *If $m_{h+1} = 0$, then $r_3 \geq l_h + l_{h+1} - m_h$ and $(\beta_{h+1}, \gamma_{h+1})$ has the form*

$$\begin{aligned} (\beta_{h+1}, \gamma_{h+1}) \quad &= \quad (0[l_{h+1} \times m_h], p\eta_{h+1}, \\ & 0[l_{h+1} \times (l_h - m_h)], E_{l_{h+1}}, 0[l_{h+1} \times (r_3 - (l_h - m_h + l_{h+1}))]. \end{aligned}$$

- *If $m_{h+1} = l_{h+1}$, then $r_2 \geq m_h + l_{h+1}$ and $(\beta_{h+1}, \gamma_{h+1})$ has the form*

$(\beta_{h+1}, \gamma_{h+1}) = (0[l_{h+1} \times m_h], E_{l_{h+1}}, 0[l_{h+1} \times (r_2 - (m_h + l_{h+1}))], 0[l_{h+1} \times r_3]).$

Moreover, if G has a representing matrix $S(E, \beta', \gamma')$ with a zero row in the hth or in the $(h+1)$th block of β', then there is also a zero row in the hth or in the $(h+1)$th block of β, respectively.

Proof. Let G be represented by $S(E, \beta', \gamma')$ with $S = \operatorname{diag}(p^{-k_h} E_{l_h} \mid h)$. Then by Lemma 5.1 there is a p-Koehler basis of $R_2 \oplus R_3$ and a basis of G/R such that (β', γ') changes to (β'', γ'') where the hth block row of (β'', γ'') has the form

$$(\beta''_h, \gamma''_h) = \left(\begin{array}{cc|cc} E_{m_h} & 0 & 0 & 0 \\ 0 & p\eta_h & E_{l_h - m_h} & 0 \end{array} \right).$$

Moreover, if $m_h = 0$, then $(\beta_h, \gamma_h) = (p\eta_h, E_{l_h}, 0[l_h \times (r_3 - l_h)])$ and if $m_h = l_h$, then $(\beta_h, \gamma_h) = (E_{l_h}, 0[l_h \times (r_2 - l_h)], 0[l_h \times r_3])$.
Hence we have

$$\left(\begin{array}{c|c} \beta''_h & \gamma''_h \\ \hline \beta''_{h+1} & \gamma''_{h+1} \end{array} \right) = \left(\begin{array}{cc|cc} E_{m_h} & 0 & 0 & 0 \\ 0 & p\eta_h & E_{l_h - m_h} & 0 \\ \hline \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array} \right).$$

Then there is a basis transformation of G/R such that $\delta_1 = 0[l_{h+1} \times m_h]$ and $\delta_3 = 0[l_{h+1} \times (l_h - m_h)]$. The hth block of β'' is unchanged. The matrix δ_2 is of format $l_{h+1} \times (r_2 - m_h)$ where $r_2 - m_h \geq l_{h+1}$ because of the regulator criterion. Then by Lemma 5.1 there is a p-Koehler basis of R_2 and a basis of G/R such that δ_2 changes to $\delta'_2 = \begin{pmatrix} E_{m_{h+1}} & 0 \\ 0 & p\mu_2 \end{pmatrix}$.
The matrix δ_4 in $(\beta''_{h+1}, \gamma''_{h+1})$ is of format $l_{h+1} \times (r_3 - (l_h - m_h))$ where $r_3 - (l_h - m_h) \geq l_{h+1} - m_{h+1}$ because of the regulator criterion. Hence by Lemma 5.1 the matrix (δ_2, δ_4) changes to

$$(\delta'_2, \delta'_4) = \left(\begin{array}{cc|cc} E_{m_{h+1}} & 0 & 0 & 0 \\ 0 & p\mu_2 & E_{l_{h+1} - m_{h+1}} & 0 \end{array} \right).$$

Thus the $(h+1)$th block of (β'', γ'') has the form

$$(\beta''_{h+1} \mid \gamma''_{h+1}) = \left(\begin{array}{ccc|ccc} 0 & E_{m_{h+1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & p\mu_2 & 0 & E_{l_{h+1} - m_{h+1}} & 0 \end{array} \right).$$

If $m_{h+1} = 0$, then

$(\beta_{h+1}, \gamma_{h+1}) = (0[l_{h+1} \times m_h, p\eta_{h+1}], 0[l_{h+1} \times (l_h - m_h)], E_{l_{h+1}}, 0[l_{h+1} \times (r_3 - (l_h - m_h + l_{h+1}))]).$

If $m_{h+1} = l_{h+1}$, then $r_2 \geq m_h + l_{h+1}$ by the regulator criterion and

$$(\beta_{h+1}, \gamma_{h+1}) = (0[l_{h+1} \times m_h], E_{l_{h+1}}, 0[l_{h+1} \times (r_2 - (m_h + l_{h+1}))], 0[l_{h+1} \times r_3]).$$

The matrix E_{m_h} in β_h, the unit matrix $E_{l_h - m_h}$ in γ_h and the corresponding rows and columns are not changed. Moreover, no new units occur in the hth block of (β'', γ''). If there was a zero row in β'_h, then this row is not changed and will occur in the matrix η_h. If the original β' has a 0-row in $(h+1)$th block it will remain unchanged and will occur in μ_2. Thus the new representing matrix is $S(E, \beta, \gamma)$ where the hth block and $(h+1)$th block of (β, γ) have the form as in Figure 2. $\square$

Lemma 5.3. *Let G be a $(1,2)$-group with $G/R \cong \bigoplus_{h=1}^{f} (\mathbb{Z}_{p^{k_h}})^{l_h}$. If $k_h - k_{h+1} = 1$ for some $h = 1, \dots, f-1$, then there is a p-Koehler basis of R and a basis of G/R such that the corresponding representing matrix is $S(E, \beta_*, \gamma_*)$ where*

$$\left(\begin{array}{c||c} \beta_*^h & \gamma_*^h \\ \hline \beta_*^{h+1} & \gamma_*^{h+1} \end{array} \right) = \begin{array}{|c|c|c|c|c||c|c|c|} \hline E_{m_h} & \multicolumn{4}{c||}{0} & \multicolumn{3}{c|}{0} \\ \hline & 0 & pD_1 & 0 & 0 & E_{l_h - m_h} & \multicolumn{2}{c|}{0} \\ \cline{2-8} & E_{m_{h+1}} & \multicolumn{3}{c||}{0} & & \multicolumn{2}{c|}{0} \\ \hline 0 & 0 & 0 & 0 & pD_2 & 0 & E_{l_{h+1} - m_{h+1}} & 0 \\ \hline \end{array},$$

with diagonal matrices pD_1 and pD_2. The matrices pD_1 and pD_2 contain only p-powers as entries. If $m_h = 0$ or $m_h = l_h$, then either the block row and block column containing the unit matrix E_{m_h} does not exist or the block row and block column containing pD_1 does not exist and if $m_{h+1} = 0$ or $m_{h+1} = l_{h+1}$, then either the block row and block column containing the unit matrix $E_{m_{h+1}}$ does not exist or the block row and block column containing pD_2 does not exist.

Proof. Let G be represented by $S(E, \beta, \gamma)$ and let (β^h, γ^h) and $(\beta^{h+1}, \gamma^{h+1})$ be the hth and $(h+1)$th blocks of (β, γ), respectively. Then by Lemma 5.2

$$M = \left(\begin{array}{c||c} \beta^h & \gamma^h \\ \hline \beta^{h+1} & \gamma^{h+1} \end{array} \right) = \left(\begin{array}{ccc|ccc} E_{m_h} & 0 & 0 & 0 & 0 & 0 \\ 0 & p\eta^{(1)} & p\eta^{(2)} & E & 0 & 0 \\ \hline 0 & E_{m_h+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & p\eta^{(3)} & 0 & E & 0 \end{array} \right).$$

Since $k_h - k_{h+1} = 1$ there is a basis transformation of G/R such that $p\eta^{(1)} = 0$, that does not change anything else.
Now we apply basis transformations of G/R only affecting the rows of $p\eta^{(2)}$ and $p\eta^{(3)}$. Those will neither change the rows of E_{m_h} nor of $E_{m_{h+1}}$. There are additional changes in γ that we consider later. Hence only the matrix $\psi = \begin{pmatrix} p\eta^{(2)} \\ p\eta^{(3)} \end{pmatrix}$ is treated by the adequate automorphisms of G/R.
Let $\big(\psi_{a,b} \mid a, b\big)$ be the ideal generated by all entries of ψ such that $\big(\psi_{a,b} \mid a, b\big) \subset p^{t_1}\,\mathbb{Z} \setminus p^{t_1+1}\,\mathbb{Z}$. If there is an entry $p\eta^{(2)}_{a,b}$ with $p\eta^{(2)}_{a,b}\,\mathbb{Z} \subset p^{t_1}\,\mathbb{Z} \setminus p^{t_1+1}\,\mathbb{Z}$, then by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 a basis of G/R such that the first row of $p\eta^{(2)}$ changes to $(p^{t_1}, 0, \ldots, 0)$ and the first column of ψ changes to $(p^{t_1}, 0, \ldots, 0)^t$, i.e., ψ changes to the form as in Figure 3. If there is no entry $p\eta^{(2)}_{a,b}$ with $p\eta^{(2)}_{a,b}\,\mathbb{Z} \subset p^{t_1}\,\mathbb{Z} \setminus p^{t_1+1}\,\mathbb{Z}$, then by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that the first row of $p\eta^{(3)}$ takes the form $(0, \ldots, 0, p^{t_1})$ and the last column of ψ changes to $(0, \ldots, 0, p^{t_1}, 0, \ldots, 0)^t$. This last conclusion follows by $k_h - k_{h+1} = 1$. Hence we get ψ as in Figure 4.
Because of notational reasons the p-powers that are in the first part of ψ are denoted by t_i and those in the right part of ψ are denoted by s_i where $i \geq 1$.

$$\psi = \begin{array}{|c|c|}
\hline
p^{t_1} & 0 \quad \cdots \quad 0 \\
\hline
\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & p\eta^{(4)} \\
\hline
\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & p\eta^{(5)} \\
\hline
\end{array}$$

FIGURE 3

$$\psi_1 = \begin{array}{|c|c|}
\hline
p\eta^{(6)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
0 \quad \cdots \quad 0 & p^{t_1} \\
\hline
p\eta^{(7)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
\end{array}$$

FIGURE 4

If we repeat this procedure with $\begin{pmatrix} p\eta^{(4)} \\ p\eta^{(5)} \end{pmatrix}$ or $\begin{pmatrix} p\eta^{(6)} \\ p\eta^{(7)} \end{pmatrix}$, respectively, then the first row and first column in the case of Figure 3 will not change. Also in the case of Figure 4 the first row of the second block of ψ_1 and the last column of ψ_1 will not change. Thus we obtain one of the cases as in Figure 5, 6 or 7,

$$\psi_2 = \begin{array}{|cc|c|}
\hline
p^{t_1} & 0 & 0 \;\; \dots \;\; 0 \\
0 & p^{t_2} & 0 \;\; \dots \;\; 0 \\
\cline{3-3}
0 & 0 & \\
\vdots & \vdots & p\nu^{(1)} \\
0 & 0 & \\
\hline
0 & 0 & \\
\vdots & \vdots & p\nu^{(2)} \\
0 & 0 & \\
\hline
\end{array}$$

FIGURE 5

$$\psi_3 = \begin{array}{|c|ccc|c|}
\hline
p^{t_1} & 0 & 0 \;\; \cdots & & 0 \\
\cline{2-4}
0 & & & & 0 \\
\vdots & & p\nu^{(3)} & & 0 \\
0 & & & & 0 \\
\hline
0 & 0 & \cdots & 0 & p^{s_1} \\
\hline
0 & & & & 0 \\
\vdots & & p\nu^{(4)} & & \vdots \\
0 & & & & 0 \\
\hline
\end{array}$$

FIGURE 6

$$\psi_4 = \begin{array}{|ccc|c|c|}
\hline
 & & & 0 & 0 \\
 & p\nu^{(5)} & & \vdots & \vdots \\
 & & & 0 & 0 \\
\hline
0 & \dots & 0 & 0 & p^{s_1} \\
0 & \dots & 0 & p^{s_2} & 0 \\
\hline
 & & & 0 & 0 \\
 & p\nu^{(6)} & & \vdots & \vdots \\
 & & & 0 & 0 \\
\hline
\end{array}$$

FIGURE 7

where $0 < t_1 \leq t_2$ and $0 < s_1 \leq s_2$.
Successively repeating this procedure on the submatrices $p\nu^{(j)}$ the original matrix ψ changes to ψ' where

$$\psi' = \begin{pmatrix} pD_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p\theta \end{pmatrix},$$

where pD_1 is a diagonal matrix and pD_1 has only p-powers, not equal to 1, as entries, and $p\theta = \begin{pmatrix} 0 & \dots & 0 & p^{s_1} \\ 0 & \dots & p^{s_2} & 0 \\ \vdots & \iddots & & \vdots \\ p^{s_u} & 0 & \dots & 0 \end{pmatrix}$ with $s_i \geq 1$ for $i = 1, \dots, u$.
There is a row permutation, i.e., a new basis of G/R, such that $p\theta$ changes to a diagonal matrix $pD_2 = \operatorname{diag}(p^{s_u}, \dots, p^{s_1})$. This basis transformation does not change the other blocks of β. But the unit matrix for α is changed. By Lemma 4.2 we obtain the unit matrix back without changing anything else. Hence ψ' changes to ψ'' where

$$\psi'' = \begin{pmatrix} pD_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & pD_2 \end{pmatrix}$$

with diagonal matrices pD_1 and pD_2. Furthermore, pD_1 and pD_2 have only p-powers as entries.

As mentioned before the above transformations do not change the the rows of the block matrices E_{m_h} and $E_{m_{h+1}}$. Clearly, the unit matrices in γ will change to invertible matrices U_1 and U_2, respectively. Moreover, we get

$$\left(\begin{array}{c|c} \beta_1^h & \gamma_1^h \\ \hline \beta_1^{h+1} & \gamma_1^{h+1} \end{array}\right) = \left(\begin{array}{ccccc|ccc} E_{m_h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & pD_1 & 0 & 0 & U_1 & \tau^{(1)} & 0 \\ \hline 0 & E_{m_{h+1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pD_2 & \tau^{(2)} & U_2 & 0 \end{array}\right),$$

with suitable matrices $\tau^{(1)}$ and $\tau^{(2)}$. Then by Corollary 4.5 there is a p-Koehler basis of R_3 such that $\tau^{(1)} = 0$ and $\tau^{(2)} = 0$, since U_1, U_2 were invertible. This basis transformation does not change β. Hence the matrix M changes to the claimed form of $\left(\begin{array}{c|c} \beta_*^h & \gamma_*^h \\ \hline \beta_*^{h+1} & \gamma_*^{h+1} \end{array}\right)$. $\square$

Lemma 5.4. *Let G be a $(1,2)$-group with representing matrix $S(E,\beta,\gamma)$. Then there is a p-Koehler basis of R and a basis of G/R such that the corresponding representing matrix is $S(E,\beta',\gamma')$ where*

$$(\beta_*,\gamma_*) = \left(\begin{array}{ccccc||ccccc} E_{m_1} & \multicolumn{4}{c||}{0} & \multicolumn{5}{c}{0} \\ & \multicolumn{4}{c||}{p\mu_1} & E_{l_1-m_1} & \multicolumn{4}{c}{0} \\ & E_{m_2} & \multicolumn{3}{c||}{0} & & \multicolumn{4}{c}{0} \\ & & \multicolumn{3}{c||}{p\mu_2} & & E_{l_2-m_2} & \multicolumn{3}{c}{0} \\ & & \ddots & & \vdots & & & & \ddots & \vdots \\ & & & E_{m_f} & 0 & & & & \multicolumn{2}{c}{0} \\ 0 & 0 & 0 & 0 & p\mu_f & 0 & 0 & 0 & E_{l_f-m_f} & 0 \end{array}\right)$$

FIGURE 8

- *If* $m_h = 0$, *then* $r_3 \geq \sum_{s=1}^{h}(l_s - m_s)$ *and*

$$\begin{aligned}(\beta_h, \gamma_h) &= (0[l_h \times \sum_{s=1}^{h} m_s], p\mu_h, \\ &\qquad 0[l_h \times \sum_{s=1}^{h-1}(l_s - m_s)], E_{l_h}, 0[l_h \times (r_3 - \sum_{s=1}^{h}(l_s - m_s))]),\end{aligned}$$

- *if* $m_h = l_h$, *then* $r_2 \geq \sum_{s=1}^{h} m_s$ *and*

$$(\beta_h, \gamma_h) = (0[l_h \times \sum_{s=1}^{h-1} m_s], E_{l_h}, 0[l_h \times (r_2 - \sum_{s=1}^{h} m_s)], 0[l_h \times r_3]).$$

Proof. By Lemma 4.7 and Lemma 5.1 the matrix (β, γ) can be transformed to the following matrix (β', γ') where $r_2 \geq m_1$ and $r_3 \geq l_1 - m_1$ since there is a unit in each row of (β', γ').

$$(\beta', \gamma') = \left[\begin{array}{c|c||c|c} E & 0 & \multicolumn{2}{c}{0} \\ \hline & p\mu_1 & E & 0 \\ \cline{2-4} & & & \\ & \beta_1 & & \gamma_1 \\ 0 & & 0 & \end{array}\right]$$

There are only zero matrices below the unit matrices in (β', γ') by Lemma 4.7. Now we apply Lemma 4.7 and Lemma 5.1 on the submatrix (β_1, γ_1). Then (β', γ') changes to the following matrix (β'', γ'') where $r_2 \geq m_1 + m_2$ and $r_3 \geq (l_1 - m_1) + (l_2 - m_2)$ since there is a unit in each row of (β'', γ'').

$$
(\beta'', \gamma'') =
\left.\begin{array}{|c|c|c||c|c|c|}
\hline
E & \multicolumn{2}{c||}{0} & \multicolumn{3}{c|}{0} \\
\hline
 & \multicolumn{2}{c||}{p\mu_1} & E & \multicolumn{2}{c|}{0} \\
\cline{2-3}\cline{5-6}
 & E & 0 & & \multicolumn{2}{c|}{0} \\
\cline{2-3}\cline{5-6}
 & & p\mu_2 & & E & 0 \\
\cline{3-3}\cline{5-6}
 & & \beta_2 & & & \gamma_2 \\
0 & 0 & & 0 & 0 & \\
\hline
\end{array}\right.
$$

Again by Lemma 4.7 there are only 0-matrices below the unit matrices in (β'', γ''). Moreover, by Lemma 4.6 the form of the first block does not change and the 0-matrices in (β', γ') remain unchanged. By induction, successively applying Lemma 4.7 and and Lemma 5.1, we get $r_2 \geq \sum_{s=1}^{h} m_s$ and $r_3 \geq \sum_{s=1}^{h}(l_s - m_s)$ and we obtain finally (β_*, γ_*) as in Figure 8. $\square$

6. General Decomposability

The basic technique to handle decompositions is the next, well known, lemma.

Lemma 6.1. ([8, 9.3]) *Let $A = B \oplus C$ be an abelian group with fully invariant subgroup F. Then $F = (F \cap B) \oplus (F \cap C)$.*

In general intersections and sums of fully invariant subgroups are fully invariant. Fully invariance is a transitive property. In torsion-free abelian groups pure hulls of fully invariant subgroups are fully invariant. In the context of an almost completely decomposable group G there are certain fully invariant groups that play an important role. The regulator R, the type subgroups $G(\tau), G^\sharp(\tau)$, and nG.
Also helpful for considering decompositions of almost completely decomposable groups is the following result.

Lemma 6.2. *Let n be a natural number, let A be a torsion-free abelian group, and $X, Y, X', Y', R \subset A$, where $X \subset X'$, $Y \subset Y'$, $X \oplus Y \subset_* R$ and X'/X and Y'/Y are torsion. Then*

$$n^{-1}R \cap (X' \oplus Y') = (n^{-1}X \cap X') \oplus (n^{-1}Y \cap Y').$$

Proof. Obviously $n^{-1}R \cap (X' \oplus Y') \supset (n^{-1}X \cap X') \oplus (n^{-1}Y \cap Y')$. Let $r = x' + y' \in n^{-1}R \cap (X' \oplus Y')$ in unique presentation in $X' \oplus Y'$. Then $nr = nx' + ny' \in R$ where $nx' \in X'$ and $ny' \in Y'$. On the other hand $nr \in \langle X \oplus Y \rangle_*^R = X \oplus Y$, since $X' \oplus Y' \subset \langle X \oplus Y \rangle_*^A$ and $X \oplus Y \subset_* R$. Hence $nr = x + y$ with unique representation in $X \oplus Y$. By $X \subset X'$ and $Y \subset Y'$ we get $x = nx'$, $y = ny'$, i.e., $nx' \in X$, $ny' \in Y$. Thus $x' \in n^{-1}X \cap X'$ and $y' \in n^{-1}Y \cap Y'$, such that $n^{-1}R \cap (X' \oplus Y') \subset (n^{-1}X \cap X') \oplus (n^{-1}Y \cap Y')$. □

For a decomposable $(1,2)$-group $G = H \oplus L$ with regulator $R = R_1 \oplus R_2 \oplus R_3$ we obtain by Lemma 6.1 and by the Dedekind modular law the following facts:

$$\begin{aligned}
R_1 &= (R_1 \cap H) \oplus (R_1 \cap L), \\
R_3 &= (R_3 \cap H) \oplus (R_3 \cap L), \\
R_2 \oplus R_3 &= \big[H \cap \big(R_2 \oplus (R_3 \cap L)\big)\big] \oplus \big[L \cap \big(R_2 \oplus (R_3 \cap H)\big)\big] \oplus R_3, \\
G/R_3 &\cong H/(H \cap R_3) \oplus L/(L \cap R_3) \quad \text{is a } (1,1)\text{-group}, \\
G/R_1 &\cong H/(H \cap R_1) \oplus L/(L \cap R_1) \quad \text{is completely decomposable.}
\end{aligned}$$

Moreover, we know that an indecomposable $(1,1)$-group has rank ≤ 2.

The next corollary displays how Lemma 6.2 can be used.

Corollary 6.3. *Let $G = H \oplus L$ be a direct decomposable almost completely decomposable group with regulator R. Then for non-negative integers i, j*

$$p^{-i}R \cap (p^jG + R) = \big(p^{-i}R \cap H \cap (p^jG + R)\big) \oplus \big(p^{-i}R \cap L \cap (p^jG + R)\big).$$

There are some simple constellations in a representing matrix that allow to read off direct summands of rank $2, 3, 4, 5$, respectively.

Proposition 6.4. *Let G be a $(1,2)$-group with $S(E,\beta,\gamma)$ as a representing matrix.*

(1) *If*

$$(\beta,\gamma) = \left[\begin{array}{ccccccc||ccc} & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \end{array}\right]$$

where 1 is at position (i,j) in β, then $\langle x_i, y_j\rangle_*$ is a direct summand of rank 2.

If

$$(\beta,\gamma) = \left[\begin{array}{ccc||ccccccc} & & & & & & 0 & & & \\ & & & & & & \vdots & & & \\ & & & & & & 0 & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & 0 & & & \\ & & & & & & \vdots & & & \\ & & & & & & 0 & & & \end{array}\right]$$

where 1 is at position (i,j) in γ, then $\langle x_i, z_j\rangle_*$ is a direct summand of rank 2.

(2) *If*

$$(\beta,\gamma) = \left[\begin{array}{ccccccc||ccccccc} & & & 0 & & & & & & & 0 & & & \\ & & & \vdots & & & & & & & \vdots & & & \\ & & & 0 & & & & & & & 0 & & & \\ 0 & \cdots & 0 & p^l & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & & & & & 0 & & & \\ & & & \vdots & & & & & & & \vdots & & & \\ & & & 0 & & & & & & & 0 & & & \end{array}\right]$$

where $p^l \neq 1$ is at position (i,j) in β, then $\langle x_i, y_j, z_i\rangle_*$ is a direct summand of rank 3.

(3) *If*

$$(\beta,\gamma)=\left(\begin{array}{ccccccc||ccccccc}
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & \\
0 & \cdots & 0 & p^l & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & &
\end{array}\right)$$

where $p^l \neq 1$ *is at position* (i_1, j) *and* 1 *at position* (i_2, j)*, both in* β*, then* $\langle x_{i_1}, x_{i_2}, y_j, z_{i_1}\rangle_*$ *is a direct summand of rank* 4*.*

(4) *If*

$$(\beta,\gamma)=\left(\begin{array}{ccccccc||cccccccccc}
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
0 & \cdots & 0 & p^m & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
0 & \cdots & 0 & p^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & &
\end{array}\right)$$

where $p^m \neq 1$ *is at position* (i_1, j)*, and* $p^n \neq 1$*, at position* (i_2, j) *in* β*, then* $\langle x_{i_1}, x_{i_2}, y_j, z_{i_1}, z_{i_2}\rangle_*$ *is a direct summand of rank* 5*.*

Lemma 6.5. *Let* G *be a* $(1,2)$*-group with representing matrix* $S(E,\beta,\gamma)$*. If there is a zero row in* β*, then* G *is decomposable.*

Proof. Let $S(E,\beta,\gamma)$ be the representing matrix of G. By Lemma 5.4 the matrix (β,γ) has the form as in Figure 8. Now assume that the ith row, that is in the hth block β_h of β, is zero. This 0-row occurs in the second part of the hth block , i.e., it is in $p\mu_h$ in Figure 8. Then for $j = i - \sum_{s=1}^{h} m_s$, there is a direct summand $\langle x_i, z_j\rangle_*$ of G of rank 2, c.f. Proposition 6.4. Thus G is decomposable. $\square$

Lemma 6.6. *Let* G *be an indecomposable* $(1,2)$*-group with a representing matrix* $S(E,\beta,\gamma)$*. If the entry* $\beta_{a,b}$ *in the* h*th block* β_h *is a unit, then*

there is an $s < h$ *such that the* s*th step* β_s *has an entry* $\beta_{c,b} \notin p^{k_s - k_h}\,\mathbb{Z}$ *in the same column* b.
In particular, no entry of the first block β_1 *of the representing matrix of an indecomposable group is a unit, and if* $k = k_2 + 1$*, then also no entry of the second block* β_2 *of the representing matrix is a unit.*

Proof. Let the entry $\beta_{a,b}$ in the hth block β_h be a unit. If $\beta_{c,b} \in p^{k_s - k_h}\,\mathbb{Z}$ for all entries in the blocks β_s of β where $s < h$, then by Proposition 6.4 $\langle x_a, y_b \rangle_*$ is a direct summand, contradicting the indecomposability of G. The statements for the first and the second blocks of β are immediate consequences. $\square$

7. About Decompositions for Regulator quotient of Exponent p^k

Theorem 7.1. *An indecomposable* $(1,2)$*-group with homocyclic regulator quotient has rank* 3*. In particular, there is a natural number* $l < k$ *for an indecomposable* $(1,2)$*-group* G *with homocyclic regulator quotient of exponent* p^k*,* $k \geq 2$*, such that* G *has the normal form*

$$G = \big[\langle x\rangle_* \oplus \langle y\rangle_* \oplus \langle z\rangle_*\big] + p^{-k}\,\mathbb{Z}(x + p^l y + z).$$

Proof. Let G be the indecomposable group with regulator R. By Lemma 5.4, and by Lemma 6.6 and since the regulator quotient is homocyclic, a representing matrix has the form $p^{-k}(E, p\eta, E)$. By Lemma 3.1 and the elementary divisor theorem there is a basis of the regulator quotient and a p-decomposition basis of R_2 such that the corresponding $p\eta$ has non-zero entries only on the main diagonal, i.e., $p\eta = \operatorname{diag}(p^{i_1}, \ldots, p^{i_s}, 0, \ldots)$ where $1 \leq i_1 \leq \cdots \leq i_s < p^k$. If, in particular, $k = 1$, then $p\eta = 0$ and G has a direct summand of rank 2 and is decomposable. Thus, $k \geq 2$. By Proposition 6.4 a 0-line in $p\eta$ causes a direct summand of rank 2. This is a contradiction since G is an indecomposable $(1,2)$-group, hence at least of rank 3. Thus $p\eta$ is square and each row of the representing matrix displays a direct summand of rank 3, already in the desired normal form with an entry p^l, $1 \leq l < k$. Those summands of rank 3 are indecomposable, since an easy application in this special case shows that such a summand is clipped. □

Theorem 7.2. *A* $(1,2)$*-group with regulator quotient isomorphic to* $(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_{p^{k-1}})^{l_2}$ *is decomposable and the direct sum of rational groups, of indecomposable* $(1,1)$*-groups of rank* 2*, and of indecomposable* $(1,2)$*-group with homocyclic regulator quotient of rank* 3*.*

Proof. By Lemma 5.3 we obtain a normal form for the representing matrix. If we omit the obvious direct summands of rank 1 and of rank 2, then we get a representing matrix of the form $S(E, p\eta, E)$, where η is a square diagonal matrix with p-power entries on the main diagonal. Each row of the representing matrix displays a direct summand of rank 3, already in the desired normal form for groups with homocyclic regulator quotient. Those summands of rank 3 are indecomposable by Lemma 7.1. □

Theorem 7.3. *An indecomposable* $(1,2)$*-group with regulator quotient isomorphic to* $(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_p)^{l_2}$ *with* $k \geq 3$ *is of rank* 4.
In particular, there is a natural number $l < k-1$ *for an indecomposable* $(1,2)$*-group* G *with the above regulator quotient such that* G *has the normal form*

$$G = \big[\langle x_1\rangle_* \oplus \langle x_2\rangle_* \oplus \langle y\rangle_* \oplus \langle z\rangle_*\big] + p^{-k}\,\mathbb{Z}(x_1 + p^l y + z) + p^{-1}\,\mathbb{Z}(x_2 + y).$$

Proof. By Lemma 5.2 we have a normal form of the representing matrix of the group G as in Figure 2. Omitting all the obvious direct summands of rank ≤ 3, and using that the entries in the second block of β are either p-units or 0 we get the normal form:

$$(\beta,\gamma) = \left[\begin{array}{cc|c} p\eta_1 & p\eta_2 & E \\ E & 0 & 0 \end{array}\right].$$

By Lemma 3.1 we may even assume that the block matrix $(p\eta_1, p\eta_2)$ has p-powers on the main diagonal and all other entries are 0. By Proposition 6.4 and since G is indecomposable, the matrix $p\eta_1$ is a square diagonal matrix of size l_1, and $p\eta_2 = 0$. So we end up with the normal form:

$$(\beta,\gamma) = \left[\begin{array}{c|c} p\eta_1 & E \\ E & 0 \end{array}\right].$$

Thus $l_1 = l_2$ and $\langle x_1, x_{l_1+1}, y_1, z_1\rangle_*$ is a direct summand of G of rank 4. Thus $l_1 = l_2 = 1$ and G is of rank 4 and in the desired normal form. Note for $k < 3$ this normal form can be simplified to $p\eta_1 = 0$.
It remains to prove that this group G is indecomposable. First note that the group G is clipped as shown below. The matrix $(\beta,\gamma) = \begin{pmatrix} p & 1 \\ 1 & 0 \end{pmatrix}$ is p-invertible. By Lemma 3.1 all basis transformations transform (β,γ) into an equivalent matrix that is also p-invertible and does not contain a 0-line. A rational direct summand of type as x_1 is equivalent to the existence of a 0-row of β, and the existence of a rational direct summand of one of the both other types is equivalent to the existence of a 0-column of (β,γ). Hence G is clipped.
Second, if $G = H \oplus L$, then we may assume that the regulator quotient of H is p^k. All possible representatives g of an element in the regulator quotient of order p^k are up to some unit factor and modulo the regulator of the form

$$g = p^{-k}(x_1 + p^l y + z) + ap^{-1}(x_2 + y) = p^{-k}\big(x_1 + ap^{k-1}x_2 + (p^l + ap^{k-1})y + z\big).$$

All those elements g have non-zero coefficients in $\mathbb{Q}\langle x_1, x_2\rangle$ and since $p^l + ap^{k-1}$ is never a unit, there is no decomposition of $R_2' \oplus R_3 = R_2 \oplus R_3$ such that $g \in \langle R_1 \oplus R_2' \rangle_*$. Thus H is not of rank 2, it must be at least of rank 3. Hence, since G is clipped, it is indecomposable. $\square$

8. Decomposability for Regulator quotient $(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_{p^{k-1}})^{l_2} \oplus (\mathbb{Z}_{p^{k-1}})^{l_3}$ with $k \leq 6$

Lemma 8.1. *Let G be a $(1,2)$-group with regulator quotient*

$$G/R \simeq \ (\mathbb{Z}_{p^m})^s \oplus \mathbb{Z}_{p^n},$$

where $m > n$. Let $S(E,\beta,\gamma)$ be the representing matrix of G with $(\beta,\gamma) = \left(\begin{array}{c|c} p^iE & \\ \hline p^f\mu & E \end{array}\right)$ where $i > f$ and $\mu = (\mu_j \mid j)$ and at least one entry μ_j is a unit. Then there is a p-decomposition basis of R and a basis of G/R such that the representing matrix changes to $S(E,\beta',\gamma')$ where

$$(\beta',\gamma') = \left(\begin{array}{cccc|c} & p^iE & & & \\ \hline p^f & 0 & \dots & 0 & E \end{array}\right).$$

Proof. Let $(x_1,\dots,x_{s+1}; y_1,\dots,y_s; z_1,\dots,z_{s+1})$ be the p-Koehler basis of R. Let $(g_1+R,\dots,g_s+R,h+R)$ be a basis of G/R where

- $g_k = p^{-m}(x_k + p^iy_k + z_k)$ for $1 \leq k \leq s$ and
- $h = p^{-n}(x_{s+1} + p^f\sum_{j=1}^{s}\mu_jy_j + z_{s+1})$, where $\mu_j \in \mathbb{Z}$ for $j = 1,\dots,s$.

We may assume that μ_1 is a unit, say $\mu_1 = 1$. Then there is a new p-Koehler basis of R_2 with the element

$$y_1' = y_1 + \sum_{l=2}^{s}\mu_ly_l.$$

Then by Lemma 4.3 the matrix β changes to the matrix β'' where the first row of β'' has the form $(p^i, -p^i\mu_2,\dots,-p^i\mu_s)$ and the last row of β'', the matrix $p^f\mu$, has the form $(p^f,0,\dots,0)$. All the other rows remain unchanged. Now we choose the new basis of G/R as $(g_1'+R, g_2+R,\dots,g_s+R,h)$ where

$$g_1' = g_1 + \mu_2g_2 + \cdots + \mu_sg_s$$

as a new basis of G/R such that β'' changes to $\beta' = \left(\begin{array}{cccc} & p^iE & & \\ \hline p^f & 0 & \dots & 0 \end{array}\right)$ and the matrix γ changes to an upper triangular matrix. But then by Lemma 4.4 there is a basis of R_3, using the diagonal entries as pivots, such that again the unit matrix in γ is reestablished. Hence the claimed form of (β,γ) is obtained. $\square$

Theorem 8.2. *Let G be a $(1,2)$-group with regulator quotient*

$$G/R \simeq (\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_{p^{k-1}})^{l_2} \oplus (\mathbb{Z}_{p^{k-2}})^{l_3}$$

of exponent $\leq p^6$. Then G is decomposable.

Proof. Assume that G is indecomposable with a representing matrix $S(E,\beta,\gamma)$. If there is a unit in the first two blocks of β, then by Lemma 6.6 the group G has a direct summand of rank 2. By Lemma 6.5 the same is true if there is a 0-row in β. Since the group is in particular clipped, there is no 0-column in β, and by Lemma 5.3 the matrix (β,γ) has the following normal form:

$$(\beta,\gamma) = \begin{array}{|c|c|c||c|c|c|} \hline pD_1 & 0 & 0 & E & 0 & 0 \\ \hline 0 & pD_2 & 0 & 0 & E & 0 \\ \hline X & Y & Z & 0 & 0 & W \\ \hline \end{array}$$

where D_1, D_2 are diagonal matrices with p-powers (possibly 1) on the diagonal of size h_1, h_2, respectively. The unit matrices in γ are already used to create the 0-blocks in the block row $(X, Y, Z, 0, 0, W)$, and it is used that β has no 0-row.

G has rank ≥ 7 forced by the regulator quotient. In the following we change the representing matrix without changing the respective letters indicating the relevant blocks.

By the Gauss algorithm downward we may assume that the columns of β with index $\leq h_1 + h_2$, i.e., columns with pD_1, pD_2 above, have the following property:

Property: *If $pd \in p^s\,\mathbb{Z} \backslash p^{s+1}\,\mathbb{Z}$ is an entry in pD_1, pD_2, on the diagonal, respectively, then $x \in (p^{s-1}\,\mathbb{Z} \backslash p^s\,\mathbb{Z}) \cup \{0\}$ for all entries x in the same column of X, Y, respectively.*	$(*)$

If there is an entry of X or Y, say x, that does not have this property, then by Lemma 4.7 there is a basis of G/R, using the entry in pD_1,pD_2 as pivot, such that x changes to 0. This will change the last block row $(0, 0, W)$ in γ. But by Lemma 4.8 there is a basis of $R_2 \oplus R_3$ such that again the original form $(0, 0, W)$ is obtained. Either there is a unit in (X, Y, Z) and we take it as a pivot, or there is no unit in a row of (X, Y, Z), but then by the regulator criterion there is a unit in W that is used as a pivot. Hence we may assume that each column of β with index $\leq h_1 + h_2$ has Property $(*)$.

If there is a unit in (Y, Z) with position (i, j) in β, then by Lemma 4.8 there is a basis of $R_2 \oplus R_3$ such that the ith row of (β, γ) changes to

$(0,\dots,0,1,0,\dots)$ where the entry 1 is at position j and by Lemma 4.7 there is a basis of G/R such that the jth column of β changes to $(0,\dots,0,1,0,\dots,0)^t$. But then by Proposition 6.4 there is a direct summand of rank 2, contradicting the assumption that G is indecomposable. Hence the entries of the matrices Y and Z are in $p\,\mathbb{Z}$. Moreover, by row and column permutations the matrix pD_1 changes to $\left(\begin{smallmatrix} pE & 0 \\ 0 & p^2D_1 \end{smallmatrix}\right)$, where pE is of size h_3. This changes the unit matrix in the first block of γ, but by column permutations in γ again the unit matrix is reestablished, without changing (β,γ) elsewhere. By Property $(*)$ there are units in all columns of X corresponding to pE, i.e., with index $\leq h_3$, and there is no unit in the columns of X with index $> h_3$. Otherwise, by Proposition 6.4, using such a unit as pivot, there is a direct summand of rank 2, contradicting the assumption that G is indecomposable. By Lemma 4.8 there is a p-Koehler basis of $R_2 \oplus R_3$ and by Lemma 4.7 there is a basis of G/R such that (β,γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c|c||c|c|c|c|}
\hline
pE & \multicolumn{3}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
0 & p^2D_1 & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
\multicolumn{2}{|c|}{0} & pD_2 & 0 & \multicolumn{2}{c|}{0} & E & 0 \\
\hline
E & pX_1 & pY_1 & pZ_1 & \multicolumn{4}{c|}{0} \\
\hline
0 & pX_2 & pY_2 & pZ_2 & \multicolumn{3}{c|}{0} & E \\
\hline
\end{array}$$

If there is a p-unit in D_2, at position (i,j) in β, then the ith row of β is of the form $(0,\dots,0,p,0,\dots,0)$ where p has column index j and by Lemma 4.7 there is a basis of G/R such that the jth column of β changes to $(0,\dots,0,p,0,\dots,0)^t$ where p is at position (i,j). But then $\langle x_i, y_j, z_i\rangle_*$ is a direct summand of rank 3 by Proposition 6.4, contradicting the assumption that G is indecomposable. Hence the entries of the matrices pD_2 are in $p^2\,\mathbb{Z}$.

If there is a row pivot in Z_2 for the block row (pX_2, pY_2, pZ_2), at position (i,j) in β, then by Corollary 4.5 there is a p-Koehler basis of R_2 such that the ith row of (β,γ) changes to $(0,\dots,0,p,0,\dots,0,1,0,\dots,0)$ where p is at position (i,j) and 1 is at position (i,l) where $l = i - h_3$. Moreover, by Lemma 4.7 there is a basis of G/R such that the jth column of β changes to $(0,\dots,0,p,0,\dots,0)^t$ where p has row index i. This changes the submatrix $M = \gamma([l_1{+}l_2{+}1, l_1{+}l_2{+}h_3], [l_1{+}l_2{+}1, r_3])$, and the unit matrix in the last block of γ. But then by Corollary 4.5 there is a p-Koehler basis of R_3 such that again the unit matrix in the last block of γ is obtained. Furthermore, by Lemma 4.8 there is a p-Koehler basis of $R_2 \oplus R_3$, using the entries of the unit matrix in the

last block of β as pivots, such that again M changes to 0. This changes the submatrix $M' = \gamma([1, h_3], [l_1 + l_2 + 1, r_3])$ in the first block of γ. By Corollary 4.5 there is p-Koehler basis of R_3 such that again the M' changes to 0. All the other rows and columns remain unchanged. Then $\langle x_i, y_j, z_i\rangle_*$ is a direct summand of G of rank 3 by Proposition 6.4, contradicting the assumption that G is indecomposable. Hence there is no row pivot in pZ_2 for the block row (pX_2, pY_2, pZ_2), in particular the entries of pZ_2 are in $p^2\,\mathbb{Z}$.
Similarly, if there is a row pivot in pZ_1 for (pX_1, pY_1, pZ_1), at position (i, j) in β, then by Corollary 4.5 there is a p-Koehler basis of R_2 such that the ith row of (β, γ) changes to $(0, \ldots, 0, 1, 0, \ldots, 0, p, 0, \ldots, 0)$ where p is in pZ_1, at position (i, j) in β, and 1 is at position (i, l) where $l = i - (l_1 + l_2)$. Furthermore, by Lemma 4.7 there is a basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p, 0, \ldots, 0)^t$ where p has row index i. This changes the unit matrix in the last block of β and the submatrix $N = \beta([l_1 + l_2 + h_3 + 1, r], [1, h_3])$ below E in β. Then by Corollary 4.5 there is a p-Koehler basis of R_2 such that again the unit matrix in the last block of β is obtained. This changes pE in the first block of β, but by Lemma 8.1 again pE is reestablished. By Lemma 4.3 the new entries of N are all divisible by p since the entries of pZ_2 are in $p^2\,\mathbb{Z}$. Then by Lemma 4.7 there is a basis of G/R, using the entries of pE as pivots, such that again the original N is obtained. This causes some changes in the submatrix $N' = \gamma([l_1 + l_2 + h_3 + 1, r], [1, h_3])$ in the last block of γ. By Corollary 4.5 there is a p-decomposition basis of R_3, using the entries of E in γ as pivots, such that again N' changes to 0. But then by Proposition 6.4 there is a direct summand of rank 5, contradicting the assumption that G is indecomposable of rank ≥ 7. Hence there is no row pivot in pZ_1 for (pX_1, pY_1, pZ_1) i.e., the entries of pZ_1 are in $p^2\,\mathbb{Z}$. By the same arguments above there is no row pivot in pY for the block row (pX, pY, pZ), in particular the entries of pY are in $p^2\,\mathbb{Z}$. Hence all the row pivots of the block row (pX, pY, pZ) are in pX. Moreover, there is no zero row in X. Otherwise, since the row pivots of (pX, pY, pZ) are in pX this row of (Y, Z) is also 0. Then by Proposition 6.4 there is a direct summand of rank ≤ 4, contradicting the assumption that G is indecomposable. By Property $(*)$, and since the entries of Y are in $p^2\,\mathbb{Z}$, the entries of p^2D_2 are in $p^3\,\mathbb{Z}$. Hence

(β, γ) changes to

$$
(\beta,\gamma) = \begin{array}{|c|c|c|c||c|c|c|c|}
\hline
pE & 0 & 0 & 0 & E & 0 & 0 & 0 \\
\hline
0 & p^2D_1 & 0 & 0 & 0 & E & 0 & 0 \\
\hline
0 & 0 & p^3D_2 & 0 & 0 & 0 & E & 0 \\
\hline
E & pX_1 & p^2Y_1 & p^2Z_1 & 0 & 0 & 0 & 0 \\
\hline
0 & pX_2 & p^2Y_2 & p^2Z_2 & 0 & 0 & 0 & E \\
\hline
\end{array}
$$

If the regulator quotient is of exponent p^4, then $p^2Y = 0$ and $p^2Z = 0$. But then by Proposition 6.4 there is a direct summand of rank ≤ 3. Hence these matrices do not exist, contradicting the given regulator quotient. Hence the group G with regulator quotient of exponent p^4 is decomposable.

Now let the exponent of the regulator quotient be p^5 or p^6. By Lemma 4.7 there is a basis of G/R such that the entries in each column of pX_1 that have higher p-power divisors than the entries in the same column of pX_2 change to 0. The remaining entries in pX_1 are automatically column pivots. Then we permute the smallest p-power divisor in pX_1, at position $(1, 1)$ in pX_1. By Corollary 4.5 there is a p-Koehler basis of R_2 and since the entries in pX_1 are column pivots by Lemma 4.7 there is a basis of G/R such that pX changes to

$$
pX = \begin{array}{|c|ccc|}
\hline
p^{t_1} & 0 & \cdots & 0 \\
\hline
0 & & & \\
\vdots & & pX' & \\
0 & & & \\
\hline
\end{array}
$$

We repeat the same procedure with the matrix pX'. The first row and the first column of the matrix pX will not change. Thus we obtain

$$
pX = \begin{array}{|cc|ccc|}
\hline
p^{t_1} & 0 & 0 & \dots & 0 \\
0 & p^{t_2} & 0 & \dots & 0 \\
\hline
0 & 0 & & & \\
\vdots & \vdots & & pX'' & \\
0 & 0 & & & \\
\hline
\end{array}
$$

.

Successively repeating this procedure on the submatrices, and using that pX has no 0-row and no 0-column the matrix pX changes to $\begin{pmatrix} pX_1 \\ pX_2 \end{pmatrix} = \begin{pmatrix} pD_3 & 0 \\ 0 & p\theta \end{pmatrix}$. This changes the submatrix $H = \beta([l_1 + l_2 + h_3 + 1, r], [1, h_3]$ below E in β, but does not change the property that all the pivots are in pX. The new entries of H are all divisible by p. By Lemma 4.7 there is a basis of G/R, using the entries of pE as pivots, such that again H changes to 0. This causes changes in the matrix $H' = \gamma([l_1 + l_2 + h_3 + 1, r], [1, h_3])$. But then by Corollary 4.5 there is a basis of R_3, using the entries of E in the last block of γ as pivots, such that again H' changes to 0. All the other rows and columns remain unchanged. Moreover, by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that $p\theta$ changes to pD_4, where D_4 is a diagonal matrix with p-powers on the diagonal. This will change the matrix p^2D_1 and the unit matrix in the last block of γ, but by Property $(*)$ there is a basis of G/R such that again p^2D_1 is reestablished. By Corollary 4.5 there is a basis of R_3 such that again the unit matrix in γ is obtained. Hence the matrix pX changes to the matrix $\begin{pmatrix} pD_3 & 0 \\ 0 & pD_4 \end{pmatrix}$ and (β, γ) has the following form:

$$(\beta,\gamma) = \begin{array}{|c|c|c|c|c||c|c|c|c|}
\hline
pE & \multicolumn{4}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
0 & \multicolumn{2}{c|}{p^2D_1} & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
\multicolumn{3}{|c|}{0} & p^3D_2 & 0 & \multicolumn{2}{c|}{0} & E & 0 \\
\hline
E & pD_3 & 0 & p^2Y_1 & p^2Z_1 & \multicolumn{4}{c|}{0} \\
\hline
0 & 0 & pD_4 & p^2Y_2 & p^2Z_2 & \multicolumn{3}{c|}{0} & E \\
\hline
\end{array}$$

By Corollary 4.5 there is a basis of R_2, using the entries of the unit matrix in β as pivots, such that the first row of (E, pD_3) changes to $(1, 0, \ldots, 0)$. This changes the entry at position $(1,1)$ in the submatrix $T = \beta([1, h_3], [h_3 + 1, l_1])$ in the first block of β. By Property $(*)$ if the entry in p^2D_1, at position (i,j) in β, where $i = j = h_3 + 1$, is in $p^s\,\mathbb{Z} \setminus p^{s+1}\,\mathbb{Z}$, then the entry in pD_3, at position (l, j) in β, with $l = l_1 + l_2 + 1$, is in $p^{s-1}Z \setminus p^s\,\mathbb{Z}$. Thus by Lemma 4.3 the new entry at position $(1,1)$ in the matrix T, is in $p^s\,\mathbb{Z} \setminus p^{s+1}\,\mathbb{Z}$. Hence by Lemma 4.7 there is a basis of G/R, using the entries of p^2D_1 as pivots, such that the jth column of β changes to $(0, \ldots, 0, p^s, 0, \ldots, 0)^t$ where p^s has row index i. This will change the submatrix $T' = \gamma([1, h_3], [h_3+1, l_1])$ in the first block of γ. By Corollary 4.5 there is a basis of R_3 such that again T' changes to 0. All the other rows and columns remain unchanged. But then $\langle x_i, y_j, z_i \rangle_*$ is a direct summand of rank 3 by Proposition 6.4, contradicting the assumption that G is indecomposable of rank ≥ 7. Hence the unit matrix in the last block of β and the corresponding columns do not exist and (β, γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c||c|c|c|}
\hline
p^2D_1 & \multicolumn{2}{c||}{0} & E & \multicolumn{2}{c|}{0} \\
\hline
0 & p^3D_2 & 0 & 0 & E & 0 \\
\hline
pX & p^2Y & p^2Z & \multicolumn{2}{c|}{0} & E \\
\hline
\end{array}$$

Now we show that there is no zero row in p^2Z. Assume that the ith row of p^2Z is zero. If the ith row of p^2Y is also zero, then by Corollary 4.5 there is a p-Koehler basis of R_2 such that the ith row of β changes to $(0, \ldots, 0, p^l, 0, \ldots, 0)$ where p^l is the pivot of the ith row of β, that is in pX and at position (i,j) in β. This changes p^2D_1 but by Property $(*)$, there is a basis of G/R such that again the matrix p^2D_1 is obtained. Moreover, there is a basis of G/R such that the jth column changes to $(0,\ldots,0,p^{l+1},0,\ldots,0,p^l,0,\ldots,0)^t$, where p^{l+1} is in p^2D_1 and at position (j,j) in β, and p^l is at position (i,j) in β. This changes only the unit matrix in the last block of γ to a lower triangular matrix. By Corollary 4.5 there is a p-Koehler basis of R_3 such that again the unit matrix in the last block of γ is reestablished. Then by Proposition 4.5 there is a direct summand of rank 5, contradicting the assumption that G is indecomposable of rank ≥ 7. Hence, if the ith row of p^2Z is zero, then there is at least one non-zero entry in the ith row of p^2Y. Assume that the entry p^l in p^2Y, at position (i,j) in β, is the row pivot of the ith row of p^2Y. By the assumption that the ith row of p^2Z is zero and by choosing a new basis of R_2 the ith row of (p^2Y,p^2Z) changes to $(0,\ldots,0,p^l,0,\ldots,0)$ where p^l has column index j. This changes p^3D_2, but by Property $(*)$ there is a basis of G/R such that again p^3D_2 is reestablished. Moreover, there is a basis of G/R such that the jth column of β changes to $(0,\ldots,0,p^l,0,\ldots,0)^t$ where p^l has row index i. This changes the submatrix $A=\beta([l_1+1,l_1+l_2],[1,l_1])$ in the second block of β, and causes some changes in γ which are not important for our result. By Lemma 4.7 there is a basis of G/R, using the entries of p^2D_1 as pivots, such that again the matrix A changes to 0. But then there is a 0-row in the second block of β, i.e., by Proposition 6.4 a direct summand of rank 2, contradicting the assumption that G is indecomposable. Thus there is no 0-row in Z and since G is clipped there is no 0-column in Z. Thus, and by the assumption that the given regulator quotient is of exponent ≤ 6, by Lemma 4.3 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that p^2Z changes to the matrix $\begin{pmatrix} p^2E & 0 \\ 0 & p^3E \end{pmatrix}$. This causes some changes in the unit matrix in the last block of γ, but then there is a p-Koehler basis of R_3 such that again the original γ is obtained. Thus (β,γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|c|c|}
\hline
p^2D_1 & \multicolumn{3}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
0 & p^3D_2 & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
pX_1 & p^2Y_1 & p^2E & 0 & \multicolumn{2}{c|}{0} & E & 0 \\
\hline
pX_2 & p^2Y_2 & 0 & p^3E & \multicolumn{3}{c|}{0} & E \\
\hline
\end{array}$$

By Corollary 4.5 there is a p-decomposition basis of R_2 such that p^2Y_1 changes to 0. All the other rows and columns remain unchanged. Since all the row pivots of the block row (pX_1,p^2Y_1,p^2E) are in pX_1 the entries of pX_1 are in $(p\,\mathbb{Z}\setminus p^2\,\mathbb{Z})\cup\{0\}$. Thus, and since there is no zero row in pX_1 by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that pX_1 changes to $(pE, 0)$. This will change p^2D_1, and the matrix p^2E to a lower triangular matrix and the unit matrix in γ. But by Property $(*)$ there is a basis of G/R such that again p^2D_1 is obtained and by Corollary 4.5 there is a basis of R_2 such that again p^2E is reestablished. Moreover, by Corollary 4.5 there is a basis of R_3 such that again the unit matrix in γ is obtained. Thus (β, γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c|c|c||c|c|c|c|}
\hline
\multicolumn{2}{|c|}{p^2D_1} & \multicolumn{3}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
\multicolumn{2}{|c|}{0} & p^3D_2 & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
pE & 0 & 0 & p^2E & 0 & \multicolumn{2}{c|}{0} & E & 0 \\
pX_2 & pX_2' & p^2Y_2 & 0 & p^3E & \multicolumn{3}{c|}{0} & E \\
\hline
\end{array}$$

where pE is of size h_4.
If the given regulator quotient is of exponent p^5, then p^3E is the 0-matrix and the second part of the last block of (β, γ) does not exist. But then by Proposition 6.4 there is a direct summand of rank 6. Hence pE and the corresponding rows and columns do not exist, contradicting the given regulator quotient. Thus the group G with the given regulator quotient of exponent p^5 is decomposable.
Now let the regulator quotient be of exponent p^6. By Corollary 4.5 there is a basis of R_2 and by Lemma 4.7 there is a basis of G/R such that (pX_2, pX_2') changes to $\begin{pmatrix} 0 & pE & 0 \\ p^2X_2'' & 0 & p^2X_3 \end{pmatrix}$ where pE is of size h_5. This changes p^2D_1, the matrix p^3E gets a lower triangular matrix and the unit matrix $B = \gamma([l_1 + l_2 + h_4 + 1, r], [l_1 + l_2 + h_4 + 1, r_3])$. By Property $(*)$, there is a basis of G/R such that again p^2D_1 is obtained. Moreover, there is a p-Koehler basis of R_2, using the diagonal entries of the lower triangular matrix as pivots, such that again the matrix p^3E is reestablished, without changing (β, γ) elsewhere. By Corollary 4.5 there is a basis of R_3, using the diagonal entries of B as pivots, such that again the matrix B changes to unit matrix. Thus (β, γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c|c|c|c||c|c|c|c|}
\hline
\multicolumn{3}{|c|}{p^2D_1} & \multicolumn{3}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
\multicolumn{3}{|c|}{0} & p^3D_2 & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
pE & \multicolumn{2}{c|}{0} & 0 & p^2E & 0 & \multicolumn{2}{c|}{0} & E & 0 \\
0 & pE & 0 & p^2Y_2 & 0 & p^3E & \multicolumn{3}{c|}{0} & E \\
p^2X_2'' & 0 & p^2X_3 & & & & \multicolumn{3}{c|}{} & \\
\hline
\end{array}$$

Moreover, there is a basis of G/R, with pivots in pE above, such that the submatrix pX_2'' changes to 0. This will change the submatrices $F = \beta([l_1+l_2+h_4+h_5+1, r], [l_1+l_2+1, l_1+l_2+h_4])$ below p^2E and $F' = \gamma([l_1+l_2+h_4+h_5+1, r], [l_1+l_2+1, l_1+l_2+h_4])$ below E in γ. The new entries of F are divisible by p^2. By Corollary 4.5 there is a p-Koehler basis of R_2, using the entries of p^3E, such that the entries of F that are in $p^3\,\mathbb{Z}$ change to 0. Hence the new entries of F are $\in (p^2\,\mathbb{Z} \setminus p^3\,\mathbb{Z}) \cup \{0\}$. But since all the row pivots of the block row (pX, p^2Y, p^2Z) are in pX, the matrix F is zero. Furthermore, by Corollary 4.5 there is a basis of R_3, using the entries of E in the last block of γ as pivots, such that F' changes to 0. Thus (β,γ) has the following form:

$$(\beta,\gamma) = \left(\begin{array}{cccccc||cccc}
\multicolumn{3}{c}{p^2D_1} & \multicolumn{3}{c||}{0} & E & \multicolumn{3}{c}{0} \\
\hline
\multicolumn{3}{c}{0} & p^3D_2 & \multicolumn{2}{c||}{0} & 0 & E & \multicolumn{2}{c}{0} \\
\hline
pE & \multicolumn{2}{c}{0} & 0 & p^2E & 0 & \multicolumn{2}{c}{0} & E & 0 \\
\hline
0 & pE & 0 & p^2Y_2 & 0 & p^3E & \multicolumn{3}{c}{0} & E \\
 & 0 & p^2X_3 & & & & & & &
\end{array}\right)$$

Then by Proposition 6.4 there is a direct summand of rank 6. Hence $pE = \beta([l_1+l_2+1, l_1+l_2+h_4], [1, h_4])$ and the corresponding rows and columns do not exist and (β,γ) has the following form:

$$(\beta,\gamma) = \left(\begin{array}{cccc||ccc}
\multicolumn{2}{c}{p^2D_1} & \multicolumn{2}{c||}{0} & E & \multicolumn{2}{c}{0} \\
\hline
\multicolumn{2}{c}{0} & p^3D_2 & 0 & 0 & E & 0 \\
\hline
pE & 0 & p^2Y_2 & p^3E & \multicolumn{2}{c}{0} & E \\
0 & p^2X_3 & p^2Y_2' & & & &
\end{array}\right)$$

Since there is no 0-row and no 0-column in p^2X_3, and since the entries of p^2X_3 are in $(p^2\,\mathbb{Z} \setminus p^3\,\mathbb{Z}) \cup \{0\}$ by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that p^2X_2'' changes to p^2E. This will change p^2D_1, and p^3E and the unit matrix in the last block of γ. But by Property $(*)$ there is a basis of G/R such that again p^2D_1 is obtained. By Corollary 4.5 there is a basis of R_2 such that again p^3E is obtained and a basis of R_3 such that again the unit matrix in γ is reestablished. Again by Corollary 4.5 there is a p-Koehler basis, using the entries of p^3E as pivots, such that the entries of p^2Y_2' that are in $p^3\,\mathbb{Z}$ change to 0. All the other rows and columns remain unchanged. Hence the entries of $p^2Y_2 \in (p^2\,\mathbb{Z} \setminus p^3\,\mathbb{Z}\{0\}$. Since all the row pivots for $(p^2X_3, p^2Y_2', 0, p^3E)$ are in p^2X_3 the matrix p^2Y_2'

is 0. Thus (β, γ) has the following form:

$$
(\beta,\gamma)=
\begin{array}{|c|c|c|c||c|c|c|}
\hline
\multicolumn{2}{|c|}{p^2D_1} & \multicolumn{2}{c||}{0} & E & \multicolumn{2}{c|}{0} \\
\hline
\multicolumn{2}{|c|}{0} & p^3D_2 & 0 & 0 & E & 0 \\
\hline
pE & 0 & p^2Y_2 & & & & \\
\cline{1-3}
0 & p^2E & 0 & \multirow{-2}{*}{p^3E} & \multicolumn{2}{c|}{\multirow{-2}{*}{0}} & \multirow{-2}{*}{E} \\
\hline
\end{array}
$$

But then by Proposition 6.4 there is a direct summand of rank 6. Hence the matrix p^2E and the corresponding rows and columns do not exist and (β, γ) has the following form:

$$
(\beta,\gamma)=
\begin{array}{|c|c|c||c|c|c|}
\hline
p^2D_1 & \multicolumn{2}{c||}{0} & E & \multicolumn{2}{c|}{0} \\
\hline
0 & p^3D_2 & 0 & 0 & E & 0 \\
\hline
pE & p^2Y_2 & p^3E & \multicolumn{2}{c|}{0} & E \\
\hline
\end{array}
$$

There is no 0-row and no 0-column in p^2Y_2. Otherwise there is a direct summand of rank 6. Hence p^2Y_2 changes to p^2E. This will change p^3D_2, and the matrices pE and p^3E to a lower triangular matrix. But by Property $(*)$, there is a basis of G/R such that again p^3D_2 is obtained. There is a basis of R_2 such that again the matrices p^2E and p^3E are obtained. Hence (β, γ) has the form:

$$
(\beta,\gamma)=
\begin{array}{|c|c|c||c|c|c|}
\hline
p^2D_1 & \multicolumn{2}{c||}{0} & E & \multicolumn{2}{c|}{0} \\
\hline
0 & p^3D_2 & 0 & 0 & E & 0 \\
\hline
pE & p^2E & p^3E & 0 & 0 & E \\
\hline
\end{array}
$$

There is a p-Koehler basis of R_2 such that the ith row of (β, γ) where $i = l_1 + l_2 + 1$ changes to $(p, \ldots, 0, \ldots, 0, p^3, \ldots, 0, 1, \ldots, 0)$ where p is at position $(i, 1)$, and p^3 is at position (i, j) where $j = i$ and 1 is in γ, at position (i, l) where $l = i$. This changes only the entry at position $(1, 1)$ in the submatrix $H = \beta([1, l_1], [l_1 + 1, l_1 + 1_2])$ in the first block of β. The new entry at position $(1, 1)$ in H, and at position $(1, l_1 + 1)$ in β, is divisible by p^3. Hence there is a basis of G/R, using the entries of p^3D_2 as pivots, such that again the submatrix H changes to 0. Then by Proposition 6.4 there is a direct summand of rank 3, contradicting the assumption that G is indecomposable. Hence p^3D_2 and the corresponding rows and columns do not exist, contradicting the given regulator quotient. Thus the group G with the given regulator quotient of exponent p^6 is decomposable. $\square$

9. (1, 2)-Groups with Regulator quotient of Exponent $\leq p^4$

Theorem 9.1. *There is no indecomposable* $(1,2)$*-group with regulator quotient of exponent* p*. An indecomposable* $(1,2)$*-group with regulator quotient* G/R *of exponent* p^2 *is of rank* 3*.*

Proof. Let G be a $(1,2)$-group with (homocyclic) regulator quotient of exponent p. By Theorem 7.1 the group G is decomposable. If the regulator quotient $G/R \cong (\mathbb{Z}_{p^2})^{l_1} \oplus (\mathbb{Z}_p)^{l_2}$, then by Theorem 7.2 the group G is decomposable. Hence $G/R \cong (\mathbb{Z}_{p^2})^l$, i.e., the regulator quotient is homocyclic. Then by Theorem 7.1 the group G has rank 3 if it is indecomposable. □

Theorem 9.2. *An indecomposable* $(1,2)$*-group with regulator quotient of exponent* p^3 *is of rank* 3 *or* 4*.*

Proof. We discuss the different isomorphism types of the regulator quotient G/R. If $G/R \cong (\mathbb{Z}_{p^3})^l$ or $G/R \cong (\mathbb{Z}_{p^3})^{l_1} \oplus (\mathbb{Z}_{p^2})^{l_2}$, then G is of rank 3 by Theorem 7.1 and by Theorem 7.2. The case $G/R \cong (\mathbb{Z}_{p^3})^{l_1} \oplus (\mathbb{Z}_{p^2})^{l_2} \oplus (\mathbb{Z}_p)^{l_3}$ cannot happen by Theorem 8.2. There is only left the case $G/R \cong (\mathbb{Z}_{p^3})^{l_1} \oplus (\mathbb{Z}_p)^{l_2}$, and by Theorem 7.3 the rank of G is then 4. □

Let G be a $(1,2)$-group with a representing matrix $S(E,\beta,\gamma)$ and with regulator quotient G/R of exponent p^4. If the group G is indecomposable with regulator quotient $G/R \simeq (\mathbb{Z}_{p^4})^{l_1}$, then by Theorem 7.1 the group G has rank 3. By Theorem 7.2 there is no indecomposable $(1,2)$-group G with the regulator quotient $G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^3})^{l_2}$. If G is indecomposable with the regulator quotient $G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^1})^{l_2}$, then by Theorem 7.3 the group G has rank 4. Moreover, if the regulator quotient $G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^3})^{l_2} \oplus (\mathbb{Z}_{p^2})^{l_3}$, then by Theorem 8.2 the group G is decomposable. Now we will discuss the remaining isomorphism types of the regulator quotient G/R of exponent p^4.

Theorem 9.3. *An indecomposable* $(1,2)$*-group* G *with regulator quotient* $G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^2})^{l_2}$ *is of rank* ≤ 5*.*

Proof. Let $S(E,\beta,\gamma)$ be the representing matrix of G. If there is a unit in the first block of β, then by Lemma 4.3, by Lemma 4.7 and by Proposition 6.4 the group G has a direct summand of rank 2. By Lemma 6.5 the same is true if there is a 0-row in β. In the following we successively change the bases of R and of G/R, but to simplify notation we will use the same letters for β,γ and all occurring submatrices in (β,γ), choosing new bases of R or of G/R. By Lemma 5.2 there is a p-Koehler basis of R and a basis of G/R such that (β,γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|c|c|}\hline pE & \multicolumn{3}{c||}{0} & E & 0 & 0 & 0 \\ \hline & p^2E & \multicolumn{2}{c||}{0} & 0 & E & 0 & 0 \\ \cline{2-8} 0 & 0 & p^3E & 0 & 0 & 0 & E & 0 \\ \hline \hline \delta & \eta & \zeta & \rho & \multicolumn{3}{c|}{0} & \phi_1 \\ \hline \end{array}$$

with block matrices δ, η, ζ, ρ and ϕ_1.
By Lemma 4.3, by Lemma 4.7 and Proposition 6.4 if there is a unit in η, at position (i,j) in β, then $\langle x_i, y_j\rangle_*$ is a direct summand of rank 2. Hence all entries of η are in $p\mathbb{Z}$. The same holds for the matrices ζ and ρ. Moreover, there is no zero column in δ,η,ζ and ρ. Otherwise by Proposition 6.4 there is a direct summand of rank ≤ 3.
By Lemma 4.3 there is a basis of R_2 and by Lemma 4.7 there is a basis of G/R such that the matrix δ changes to $\begin{pmatrix} E & 0 \\ 0 & p\delta \end{pmatrix}$ where E is of size h_3. This will change pE but by Lemma 8.1 again pE is reestablished. Hence (β,γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c|c||c|c|c|c|}\hline \multicolumn{2}{|c|}{pE} & \multicolumn{3}{c||}{0} & E & 0 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{} & p^2E & \multicolumn{2}{c||}{0} & 0 & E & 0 & 0 \\ \cline{3-9} \multicolumn{2}{|c|}{0} & 0 & p^3E & 0 & 0 & 0 & E & 0 \\ \hline \hline E & 0 & p\eta & p\zeta & p\rho & \multicolumn{3}{c|}{0} & \phi_1 \\ \hline 0 & p\delta & p\mu_1 & p\mu_2 & p\mu_3 & \multicolumn{3}{c|}{0} & \phi_2 \\ \hline \end{array}$$

where pE is of size h_1.
By Lemma 4.8 there is a basis of $R_2 \oplus R_3$, using the entries of the identity matrix in the second block of β as pivots, such that ϕ_1 changes to 0 and $p\eta$, $p\zeta$ and $p\rho$ remain unchanged. This will change the submatrix $Y=\gamma([1,h_1]\times[r_3-l_1,r_3])$. But then by Corollary 4.5 there is a basis of R_3, using the entries of E_{h_1} in the first part of the first block of γ as pivots, again Y changes to 0 without changing γ somewhere else. By the regulator condition there is a unit in each row of ϕ_2. Since G is clipped there is no zero row in ϕ_2. Hence by Lemma 4.4 there is a new basis of R_3 and by Lemma 4.7 there is a new basis of G/R such that ϕ_2 changes to E without changing γ elsewhere. Moreover, by Lemma 4.7 there is a basis of G/R, using the entries of pE as pivots, such that $p\delta$ changes to 0. This will change the submatrix $A=\beta([l_1+h_3+1],[1,h_3])$ below E in β, and the submatrix $A'=\gamma([l_1+h_3+1,r],[1,h_1])$. By Lemma 4.7 there is a basis of G/R, using the entries of E in β as pivots, such that again A changes to 0. By Lemma 4.3 there is a basis of R_3, using the entries of E in the last block of γ as pivots, such that again

the submatrix A' changes to 0. But then there is a direct summand of rank 3, contradicting that G is indecomposable. Hence $p\delta$ and the corresponding columns do not exist and (β, γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|c|c|}\hline pE & \multicolumn{3}{c||}{0} & E & 0 & 0 & 0 \\ \hline & p^2E & \multicolumn{2}{c||}{0} & 0 & E & 0 & 0 \\ \cline{2-8} 0 & 0 & p^3E & 0 & 0 & 0 & E & 0 \\ \hline\hline E & p\eta & p\zeta & p\rho & \multicolumn{3}{c|}{0} & 0 \\ \hline 0 & p\mu_1 & p\mu_2 & p\mu_3 & \multicolumn{3}{c|}{0} & E \\ \hline \end{array}$$

Recall that pE is of size h_1 and let p^2E be of size h_2. Assume that the matrix μ_2 has a p-unit, at position (i, j) in β. Since the entries of the submatrix $p\mu_2 \in (p\,\mathbb{Z} \setminus p^2\,\mathbb{Z}) \cup 0$, there is a p-Koehler basis of R_2 such that the ith row of (β, γ) changes to $(0, \ldots, 0, p, 0 \ldots, 1, 0, \ldots, 0)$ where p is at position (i, j) and in the matrix $p\mu_2$, and the entry 1 is at position (i, l) in γ where $l = i - h_1$. This causes some changes in the block row of β with p^3E that are not important for our result. Moreover, by Lemma 4.7 there is a new basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p, 0, \ldots, 0)^t$ where the entry p is in the ith row of β. This changes the submatrix $M = \gamma([h_1+h_2+1, l_1], [r_3-l_1, r_3])$ and the submatrix $N = \gamma([l_1+1, l_1+h_1], [r_3-l_1, r_3])$. By Corollary 4.5 there is a new basis of R_2 such that M changes to 0 and by Lemma 4.8 there is a new basis of $R_2 \oplus R_3$ such that again the matrix N changes to 0. All the other rows and columns remain unchanged. Then $\langle x_i, y_j, z_l\rangle_*$ is a direct summand of rank 3 by Proposition 6.4. Hence $p\mu_2 = 0$ and by the same arguments also $p\mu_3 = 0$. Thus (β, γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|c|c|}\hline pE & \multicolumn{3}{c||}{0} & E & 0 & 0 & 0 \\ \hline & p^2E & \multicolumn{2}{c||}{0} & 0 & E & 0 & 0 \\ \cline{2-8} 0 & 0 & p^3E & 0 & 0 & 0 & E & 0 \\ \hline\hline E & p\eta & p\zeta & p\rho & \multicolumn{3}{c|}{0} & 0 \\ \hline 0 & p\mu_1 & 0 & 0 & \multicolumn{3}{c|}{0} & E \\ \hline \end{array}$$

If μ_1 has a p-unit, at position (i, j) in β, then by Lemma 4.3 there is a p-decomposition basis of R_2 such that the ith row of (β, γ) changes to $(0, \ldots, 0, p, 0, \ldots, 0, 1, 0, \ldots, 0)$ where p is in the matrix $p\mu_1$, at position (i, j) in β, and the entry 1 is at position (i, l) in γ where $l = i-h_1$. This will change p^2E, but by Lemma 8.1 again the matrix p^2E is obtained. Moreover, there is a basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p^2, 0, \ldots, 0, p, 0, \ldots, 0)^t$ where p^2 has row index j and p has row index i. This will change the submatrix $N = \gamma([l_1 + 1, l_1 + h_1], [r_3 - l_1, r_3])$. But then by Lemma 4.8 there is a basis of $R_2 \oplus R_3$, using the entries of E_{h_1} in the second block of β as pivots, such that again N changes to 0. This causes changes only in the submatrix $Y = \gamma([1, h_1], [r_3 - l_1, r_3])$ in the first block of γ. By Corollary 4.5 there is a basis of R_3 such that again Y changes to 0. All the other rows and columns remain unchanged. Then $\langle x_j, x_i, y_j, z_j, z_l \rangle$ is a direct summand of rank 5 by Proposition 6.4, contradicting the assumption that G is indecomposable. Hence $p\mu_1 = 0$. But then by Proposition 6.4 the group G has a direct summand of rank 2, contradicting the assumption. Thus the matrix $p\mu_1$ and the corresponding rows and columns do not exist and (β, γ) has the following form:

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|c|}\hline pE & \multicolumn{3}{c||}{0} & E & 0 & 0\\ \cline{2-7} 0 & p^2E & \multicolumn{2}{c||}{0} & 0 & E & 0\\ \cline{2-7} & 0 & p^3E & 0 & 0 & 0 & E\\ \hline\hline E & p\eta & p\zeta & p\rho & \multicolumn{3}{c|}{0}\\ \hline\end{array}$$

If the matrix ρ has a p-unit, at position (i, j) in β, then by Corollary 4.5 there is a p-Koehler basis of R_2 such that the ith row of β changes to $(0, \ldots, 0, 1, 0, \ldots, 0, p, 0, \ldots, 0)$ where 1 has column index $i - l_1$ and p has column index j. Moreover, by Lemma 4.7 there is a new basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p, 0, \ldots, 0)^t$ where p has the row index i. This will change the unit matrix in the second block of β. But by Corollary 4.5 there is a new basis of R_2 such that again this unit matrix is obtained. This will change the matrix pE in β. But by Lemma 8.1 again the matrix pE is obtained. This causes changes in E_{h_1} in the first block of γ. By Corollary 4.5 there is a basis of R_3 such that this unit matrix in the first block of β is reestablished without changing γ elsewhere. Then $\langle x_a, x_i, y_a, y_j, z_a \rangle$ is a direct summand of rank 5 by Proposition 6.4, where $a = i - l_1$. Hence $p\rho$ and the corresponding rows and columns do not exist and (β, γ) has the following form:

$$(\beta,\gamma)=\begin{array}{|c|c|c||c|c|c|}\hline pE & \multicolumn{2}{c||}{0} & E & 0 & 0\\ \cline{2-6} 0 & p^2E & 0 & 0 & E & 0\\ \cline{2-6} & 0 & p^3E & 0 & 0 & E\\ \hline\hline E & p\eta & p\zeta & \multicolumn{3}{c|}{0}\\ \hline\end{array}$$

If there is a p-unit, in ζ, at position (i, j) in β, then by Corollary 4.5 there is a p-Koehler basis of R_2 and by Lemma 4.7 there is a basis of G/R such that the ith row of β changes to $(0, \ldots, 0, 1, 0, \ldots, 0, p, 0, \ldots, 0)$ where 1 has column index $i - l_1$ and p has column index j. This will change the matrix p^3E and the submatrix $X = \beta([h_1 + h_2 + 1, l_1], [h_1 + 1, h_1 + h_2])$. By Lemma 8.1 again the matrix p^3E is obtained. The new entries of X are divisible by p^3 by Lemma 4.3. Then there is a basis of G/R such that again X changes to 0 without changing β elsewhere. But this will change the submatrix $Y = \gamma([h_1+h_2+1, l_1], [h_1+1, h_1+h_2])$. Then by Corollary 4.5 there is a basis of R_3, using the entries of E in γ as pivots, such that Y changes to 0, without changing (β, γ) somewhere else. Moreover, by Lemma 4.7 there is a new basis of G/R such that the jth column of β changes to $(0, \dots, 0, p, 0, \dots, 0)^t$. This will change the submatrix $T = \beta([h_1 + h_2 + 1, l_1], [1, h_1])$. The new entries of T are all divisible by p^2 by Lemma 4.3. Then there is a basis of G/R such that again T changes to 0 without changing β somewhere else. This causes some changes in γ that do not play an important role for the result. But then there is a zero row in the submatrix $\beta([h_1+h_2+1, l_1], [1, r_2])$, i.e., a direct summand of rank 2 by Proposition 6.4, contradicting the assumption that G is indecomposable. Hence the block matrix p^3E and the corresponding rows and columns do not exist and (β, γ) has the following form:

$$(\beta,\gamma) = \begin{array}{|c|c||c|c|} \hline pE & 0 & E & 0 \\ 0 & p^2E & 0 & E \\ \hline \hline E & p\eta & 0 & 0 \\ \hline \end{array}$$

There is a basis of R_2, using the entries of E in β as pivots, such that $p\eta$ changes to 0. This will change the submatrix $B = \beta([1, h_1], [h_1+1, l_1])$. The new entries of T are divisible by p^2. But then by Lemma 4.7 there is a basis of G/R, using the entries of p^2E as pivots, such that again B changes to 0. This causes changes in $B' = \gamma([1, h_1], [h_1 + 1, l_1])$. By Corollary 4.5 there is a basis of R_3, using the entries of E in γ, such that again B' changes to 0. But then there is a direct summand of rank ≤ 4. Since G is indecomposable the part (β, γ) has either the form $\left(\begin{array}{c|c} p & 1 \\ 1 & 0 \end{array}\right)$ or $\left(\begin{array}{c|cc} p^2 & 1 & 0 \\ p & 0 & 1 \end{array}\right)$, i.e., the group G has rank ≤ 5 if it is indecomposable. $\square$

Theorem 9.4. *Let G be a $(1,2)$-group with regulator quotient*

$$G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^3})^{l_2} \oplus (\mathbb{Z}_{p^1})^{l_3}.$$

Then G is decomposable.

Proof. Assume that G is indecomposable with a representing matrix $S(E, \beta, \gamma)$. Then by Lemma 6.6 there is no unit in the first two blocks of β and by Lemma 5.3 the part (β, γ) of the representing has the normal form

$$(\beta,\gamma) = \begin{array}{|c|c|c||c|c|c|}
\hline
pD_1 & 0 & 0 & E & 0 & 0 \\
\hline
0 & 0 & pD_2 & 0 & E & 0 \\
\hline
\mu & \nu & \eta & 0 & 0 & \rho \\
\hline
\end{array}$$

with diagonal matrices pD_1 and pD_2. Note that the group G with the given regulator quotient is of rank ≥ 6. In the following we successively change the bases of R and of G/R, but to simplify notation we will use the same letters for β, γ and all occurring submatrices in (β, γ).

The entries of μ,ν,η and ρ are units or zero and no 0-column exists in these matrices. Otherwise there is a direct summand of rank ≤ 3.

If there is a unit in ν, at position (i, j) in β, then $\langle x_i, y_j\rangle_*$ is a direct summand of rank 2 by Lemma 4.3, by Lemma 4.7 and by Proposition 6.4, contradicting the assumption that G is indecomposable. Hence $\nu = 0$. But this contradicts the fact that there is no 0-column in ν. Hence ν and the corresponding columns do not exist. Applying row permutations and column permutations to the matrix (β, γ) changes (β, γ) to

$$(\beta,\gamma) = \begin{array}{|c|c|c|c|c||c|c|c|}
\hline
pE & 0 & 0 & \multicolumn{2}{c||}{} & & & \\
0 & p^2E & 0 & \multicolumn{2}{c||}{0} & E & 0 & 0 \\
0 & 0 & p^3E & \multicolumn{2}{c||}{} & & & \\
\hline
\multicolumn{3}{|c|}{} & pE & 0 & & & \\
\multicolumn{3}{|c|}{0} & & & 0 & E & 0 \\
\multicolumn{3}{|c|}{} & 0 & p^2E & & & \\
\hline
\mu & \zeta & \sigma & \eta & \psi & \multicolumn{2}{c|}{0} & \rho \\
\hline
\end{array}$$

By the same arguments as above the matrices σ and ψ, and the corresponding columns do not exist. Thus (β, γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c||c|c|c|c|}
\hline
pE & \multicolumn{2}{c||}{0} & E & \multicolumn{3}{c|}{0} \\
\hline
0 & p^2E & 0 & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
0 & 0 & pE & 0 & 0 & E & 0 \\
\hline
\mu & \zeta & \eta & \multicolumn{3}{c|}{0} & \rho \\
\hline
\end{array}$$

where pE in the first block of β is of size h_1.

If there is a unit in ζ, at position (i, j) in β, then by Lemma 4.8 there is a p-Koehler basis of $R_2 \oplus R_3$ such that the ith row of (β, γ) changes to $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 has the column index j. This changes the matrix p^2E and the submatrices $X = \beta([h_1 + 1, l_1], [1, h_1])$ and $Y = \beta([h_1 + 1, l_1], [l_1 + 1, r_2])$ and $Z = \gamma([h_1 + 1, l_1], [l_1 + 1, r_3])$. By Lemma 8.1 again the matrix p^2E is obtained. By Corollary 4.5 there is a p-Koehler basis of R_3 such that again Z changes to 0, without changing γ somewhere else. The new entries of X any Y are divisible by p^2 by Lemma 4.3. Then by Lemma 4.7 there is a basis of G/R such that again the matrices X and Y change to 0. This causes changes in $T = \gamma([h_1+1, l_1], [1, h_1])$ and in $V = \gamma([h_1+1, l_1], [l_1+1, r_3-h_1])$. But by Corollary 4.5 there is a basis of R_3 such that again T and V change to 0, without changing (β, γ) elsewhere. Moreover, by Lemma 4.7 there is a basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p^2, 0, \ldots, 0, 1, 0, \ldots, 0)^t$ where the entry p^2 is at position (j, j) and 1 is at position (i, j) in β. All the other rows and columns of (β, γ) remain unchanged. But then $\langle x_j, x_i, y_j, z_i\rangle_*$ is a direct summand of rank 4 by Proposition 6.4, contradicting the assumption that G is an indecomposable group of rank ≥ 6. Hence the matrix ζ and its corresponding columns do not exist. Thus (β, γ) has the following form:

$$(\beta,\gamma) = \begin{array}{|c|c||c|c|c|}
\hline
pE & 0 & E & \multicolumn{2}{c|}{0} \\
\hline
0 & pE & 0 & E & 0 \\
\hline
\mu & \eta & \multicolumn{2}{c|}{0} & \rho \\
\hline
\end{array}$$

By the same arguments as above for the non-existence of ζ also the matrix η and the corresponding columns do not exist. But then the second block of β is zero, i.e., by Proposition 6.5 there is a direct summand of rank 2. Hence the second block of (β, γ) does not exist, contradicting the given regulator quotient. $\square$

Theorem 9.5. *Let G be a $(1, 2)$-group with regulator quotient*

$$G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^2})^{l_2} \oplus (\mathbb{Z}_{p^1})^{l_3}.$$

Then G is decomposable.

Proof. Assume that G is indecomposable with a representing matrix $S(E, \beta, \gamma)$. By Lemma 6.6 there is no unit in the first block of β. Moreover, there is no zero row in β. Otherwise, by Lemma 6.5 there is a direct summand of rank 2. By Lemma 5.2 there is a p-Koehler basis of R and a basis of G/R such that (β, γ) changes to

$$(\beta,\gamma) = \begin{array}{|c|c|c|c||c|c|}
\hline
pE & \multicolumn{3}{c||}{0} & & \\
\cline{1-4}
0 & p^2E & \multicolumn{2}{c||}{0} & E & 0 \\
\cline{1-4}
\multicolumn{2}{|c|}{0} & p^3E & 0 & & \\
\hline
\mu & \nu & \eta & \zeta & 0 & \phi_1 \\
\hline
\delta & \sigma & \kappa & \rho & 0 & \phi_2 \\
\hline
\end{array}$$

Note that the group G with the given regulator quotient is of rank ≥ 5. In the following we successively change the bases of R and of G/R, but to simplify notation we will use the same letters for β, γ and all occurring submatrices in (β, γ), choosing new bases of R or of G/R. By Proposition 6.4 if there is a unit in ζ, at position (i, j) in β, then $\langle x_i, y_j\rangle_*$ is a direct summand of rank 2. Hence all entries of ζ are in $p\,\mathbb{Z}$. The same holds for ν and η, i.e., all entries of ν and η are in $p\,\mathbb{Z}$.
If there is a unit in ρ, at position (i, j) in β, then by Lemma 4.8 there is a new p-Koehler basis of $R_2 \oplus R_3$ such that the ith row of (β, γ) changes to $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 has column index j and by Lemma 4.7 there is a new basis of G/R such that the jth column of β changes to $(0, \ldots, 0, 1, 0, \ldots, 0)^t$ where 1 has row index i. Then $\langle x_i, y_j\rangle_*$ is a direct summand of rank 2 by Proposition 6.4, contradicting the assumption that G is indecomposable. Since the entries of ρ are units or zeros, $\rho = 0$. The same holds for κ. Thus (β, γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|}
\hline
pE & \multicolumn{3}{c||}{0} & & \\
\cline{1-4}
0 & p^2E & \multicolumn{2}{c||}{0} & E & 0 \\
\cline{1-4}
\multicolumn{2}{|c|}{0} & p^3E & 0 & & \\
\hline
\mu & p\nu & p\eta & p\zeta & 0 & \phi_1 \\
\hline
\delta & \sigma & 0 & 0 & 0 & \phi_2 \\
\hline
\end{array}$$

where pE is of size h_1 and p^2E is of size h_2.
If there is a unit in σ, at position (i, j) in β, then there is a p-Koehler basis of $R_2 \oplus R_3$ such that the ith row changes to $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 has column index j. This changes p^2E, and the submatrices $X = \beta([h_1+1, h_1+h_2], [1, h_1])$ and $Y = \gamma([h_1+1, h_1+h_2], [h_1+h_2+1, r_3])$. The new entries of X are all divisible by p^2 by Lemma 4.3. Then there is a new basis of G/R such that X changes to 0. By Lemma 8.1 again the matrix p^2E is obtained and by Corollary 4.5 there is a new basis of R_3 such that Y changes to 0. Moreover, there is a basis of G/R such that the jth column of β changes to $(0, \ldots, 0, p^2, 0, \ldots, 0, \ldots, 0, 1, 0, \ldots, 0)^t$ where p^2 has the row index j and 1 has the row index i. All the other rows and columns remain unchanged. Then by Proposition 6.4 there is a direct summand of rank ≤ 4, contradicting the assumption that G is an indecomposable group of rank ≥ 5. Thus, and since the entries of σ are units or zero the matrix $\sigma = 0$. Hence (β, γ) has the following

form

$$(\beta,\gamma)=\begin{array}{|c|c|c|c||c|c|}\hline pE & \multicolumn{3}{c||}{0} & \multicolumn{1}{c|}{} & \\ \cline{1-4} 0 & p^2E & \multicolumn{2}{c||}{0} & E & 0 \\ \cline{1-4} \multicolumn{2}{|c|}{0} & p^3E & 0 & & \\ \hline \mu & p\nu & p\eta & p\zeta & 0 & \phi_1 \\ \hline \delta & 0 & 0 & 0 & 0 & \phi_2 \\ \hline \end{array}$$

Since the entries of μ are units or zero by Corollary 4.5 there is a basis of R_2 and by Lemma 4.7 there is a basis of G/R such that μ changes to $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$. This will change pE, but by Lemma 8.1 again pE is obtained. Hence (β,γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c|c||c|c|}\hline \multicolumn{2}{|c|}{pE} & \multicolumn{3}{c||}{0} & & \\ \cline{1-5} \multicolumn{2}{|c|}{0} & p^2E & \multicolumn{2}{c||}{0} & E & 0 \\ \cline{1-5} \multicolumn{3}{|c|}{0} & p^3E & 0 & & \\ \hline E & 0 & & & & & \\ 0 & 0 & p\nu & p\eta & p\zeta & 0 & \phi_1 \\ \hline 0 & \delta & 0 & 0 & 0 & 0 & \phi_2 \\ \hline \end{array}$$

By the same arguments as above for the matrix σ, also $\delta=0$. But then the last block of β is 0. By Proposition 6.4 there is a direct summand of rank 2, contradicting the assumption that G is indecomposable. Thus, the matrix δ does not exist, contradicting the given regulator quotient. $\square$

Theorem 9.6. *Let G be a $(1,2)$-group with regulator quotient*

$$G/R \simeq (\mathbb{Z}_{p^4})^{l_1} \oplus (\mathbb{Z}_{p^3})^{l_2} \oplus (\mathbb{Z}_{p^2})^{l_3} \oplus (\mathbb{Z}_{p^1})^{l_4}.$$

Then G is decomposable.

Proof. Assume that G is indecomposable with a representing matrix $S(E,\beta,\gamma)$. By Lemma 6.6 there is no unit in the first two blocks of β. By Lemma 5.3 the matrix (β,γ) has the normal form

$$(\beta,\gamma)=\begin{array}{|c|c|c||c|c|c|}\hline pD_1 & 0 & 0 & E & 0 & 0 \\ \hline 0 & pD_2 & 0 & 0 & E & 0 \\ \hline \mu & \nu & \phi & 0 & 0 & \kappa \\ \hline \zeta & \nu & \rho & 0 & 0 & \tau \\ \hline \end{array}$$

with diagonal matrices pD_1 and pD_2.
Note that the group G with the given regulator quotient is of rank ≥ 8. In the following we successively change the bases of R and of G/R, but to simplify notation we will use the same letters for β,γ and all

occurring submatrices in (β,γ). By row permutations and column permutations the matrix (β,γ) changes to

$$(\beta,\gamma)=\left(\begin{array}{ccc|cc|c||c|c|c}
pE & 0 & 0 & & & & & & \\
0 & p^2E & 0 & \multicolumn{2}{c|}{0} & 0 & E & 0 & 0\\
0 & 0 & p^3E & & & & & & \\ \hline
 & & & pE & 0 & & & & \\
\multicolumn{3}{c|}{0} & & & 0 & 0 & E & 0\\
 & & & 0 & p^2E & & & & \\ \hline
\mu & \delta & \theta & \eta & \sigma & \phi & 0 & 0 & \kappa \\ \hline
\zeta & \psi & \omega & \nu & \varsigma & \rho & 0 & 0 & \tau
\end{array}\right)$$

There is no unit in δ,θ,η,σ,ϕ, ω,ς and ρ. Otherwise by Proposition 6.4 there is a direct summand of rank 2. Hence all entries of δ,θ,η,σ and ϕ are in $p\mathbb{Z}$. Thus, and since the entries of ω and ς and ρ are units or zero, the submatrix $\omega=0$, and the submatrices $\varsigma=0$ and $\rho=0$.
There is a p-Koehler basis of R_2 and a basis of G/R such that the matrix μ changes to $\begin{pmatrix} E & 0 \\ 0 & p\mu \end{pmatrix}$ where E is of size h_4. This changes pE in the first block of β, but by Lemma 8.1 again pE is reestablished. Moreover, there is a p-decomposition basis of $R_2 \oplus R_3$, and a basis of G/R , using the entries of E in β, such that (β,γ) changes to

$$(\beta,\gamma)=\left(\begin{array}{cc|c|c|c|c|c||c|c|c}
\multicolumn{2}{c|}{pE} & 0 & 0 & & & & & & \\
\multicolumn{2}{c|}{0} & p^2E & 0 & \multicolumn{2}{c|}{0} & 0 & E & 0 & 0\\
\multicolumn{2}{c|}{0} & 0 & p^3E & & & & & & \\ \hline
 & & & & pE & 0 & & & & \\
\multicolumn{4}{c|}{0} & & & 0 & 0 & E & 0\\
 & & & & 0 & p^2E & & & & \\ \hline
E & 0 & p\delta & p\theta & p\eta & p\sigma & p\phi & 0 & 0 & 0 \\
0 & p\mu & p\delta' & p\theta' & p\eta' & p\sigma' & p\phi' & 0 & 0 & \kappa' \\ \hline
0 & \zeta & \psi & 0 & \nu & 0 & 0 & 0 & 0 & \tau
\end{array}\right)$$

where pE and p^2E in the first block of β are of size h_1 and h_2 respectively, and the matrix pE in the second block of β is of size h_3.
By the regulator condition there is a unit in each row of κ'. Hence there is a p-Koehler basis of R_3 and a basis of G/R such that κ' changes to $(E,0)$. Moreover, there is a basis of G/R, using the entries of pE as pivots, such that $p\mu$ changes to 0. This changes the 0-matrix below E in β, and the submatrix $X=\gamma([l_1+l_2+h_4+1, l_1+l_2+l_3],[1,h_1])$. There is a basis of G/R, using the entries of E in β as pivots, such that again the 0-matrix below E is obtained. There is a basis of R_3, using the entries of $\kappa'=(E,0)$ such that X changes to 0. Furthermore, if there is a unit in ψ, at position (i,j) in β, then by Lemma 4.8 there is

a p-Koehler basis of $R_2 \oplus R_3$ such that the ith row of (β, γ) changes to $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 has column index j. By Corollary 4.5 this causes changes in the block row $(0, p^2E, 0, E, 0)$, in the first block of (β, γ). There are also changes in the third block of (β, γ) but they are not important for our result. By Lemma 8.1 again the matrix p^2E is obtained. By Corollary 4.5 the new entries of the submatrices $Y = \beta([h_1{+}1, h_1{+}h_2], [1, h_1])$, and $Z = \beta([h_1{+}1, h_1{+}h_2], [l_1{+}1, l_1{+}h_3])$ are divisible by p^2. Then by Lemma 4.7, using the entries of pE in the first block of β as pivots, again the matrix Y changes to 0. Similarly, by Lemma 4.7, using the entries of pE in the second block of β as pivots, Z changes to 0. This will change the first block of γ. But then by Corollary 4.5 there is a p-Koehler basis of R_3 such that again the first block of γ is reestablished, without changing (β, γ) elsewhere. Moreover, there is a basis of G/R such that the jth column of β changes to $(0, \dots, 0, p^2, 0, \dots, 0, 1, 0, \dots, 0)^t$ without changing (β, γ) elsewhere. But then $\langle x_j, x_i, y_j, z_j \rangle_*$ is a direct summand of rank 4 by Proposition 6.4, contradicting the assumption that G is indecomposable of rank ≥ 8. Hence $\psi = 0$. Thus, and by the same arguments as above $\nu = 0$. Hence (β, γ) changes to

$$(\beta,\gamma)=\begin{array}{|c|c|c|c|c|c|c||c|c|c|}
\hline
\multicolumn{2}{|c|}{pE} & \multicolumn{2}{c|}{0 \quad\; 0} & \multicolumn{2}{c|}{} & & & & \\
\multicolumn{2}{|c|}{0} & \multicolumn{2}{c|}{p^2E \quad 0} & \multicolumn{2}{c|}{0} & 0 & E & 0 & 0 \\
\multicolumn{2}{|c|}{0} & \multicolumn{2}{c|}{0 \quad p^3E} & \multicolumn{2}{c|}{} & & & & \\
\hline
\multicolumn{4}{|c|}{} & \multicolumn{2}{c|}{pE \quad\; 0} & & & & \\
\multicolumn{4}{|c|}{0} & \multicolumn{2}{c|}{0 \quad p^2E} & 0 & 0 & E & 0 \\
\hline
E & 0 & p\delta & p\theta & p\eta & p\sigma & p\phi & 0 & 0 & 0 \\
\hline
0 & 0 & p\delta' & p\theta' & p\eta' & p\sigma' & p\phi' & 0 & 0 & \kappa' \\
\hline
0 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau \\
\hline
\end{array}$$

If there is a unit in ζ, then by Proposition 6.4 there is a direct summand of rank ≤ 5. Hence the matrix $\zeta = 0$. But then by Proposition 6.4 there is a direct summand of rank 2. Thus the matrix τ does not exist, contradicting the given regulator quotient. $\square$

10. Conclusions and open questions

We collect all results.

Theorem 10.1. *Indecomposable* $(1,2)$*-groups with regulator quotient of exponent* p^2 *are of rank* 3.
Indecomposable $(1,2)$*-groups with regulator quotient of exponent* p^3 *are of rank* 3 *or of rank* 4.
Indecomposable $(1,2)$*-groups with regulator quotient of exponent* p^4 *are of rank* ≤ 5. *This boundary is sharp if and only if the test example* G_4, *given below, is indecomposable. In particular, the regulator quotient of indecomposable* $(1,2)$*-groups with regulator quotient of exponent* p^4 *is either isomorphic to* $\mathbb{Z}_{p^4}^{l_1} \oplus \mathbb{Z}_{p^2}^{l_2}$ *or homocyclic,* $\mathbb{Z}_{p^4}^{l}$.
Indecomposable $(1,2)$*-groups with homocyclic regulator quotient of exponent* $\geq p^2$ *are of rank* 3.
Indecomposable $(1,2)$*-groups with regulator quotient of exponent* $\geq p^3$ *and isomorphic to*

$$(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_p)^{l_2}$$

are of rank 4.

Theorem 10.2. *There is no indecomposable* $(1,2)$*-group with regulator quotient of exponent* p.
There is no indecomposable $(1,2)$*-group with regulator quotient of exponent* p^k *and isomorphic to*

$$(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_{p^{k-1}})^{l_2}.$$

There is no indecomposable $(1,2)$*-group with regulator quotient of exponent* $\leq p^6$ *and isomorphic to*

$$(\mathbb{Z}_{p^k})^{l_1} \oplus (\mathbb{Z}_{p^{k-1}})^{l_2} \oplus (\mathbb{Z}_{p^{k-2}})^{l_3}.$$

Open Questions.
(1) Test example G_4. The estimation $\operatorname{rank} G \leq 5$ for the maximal rank of an indecomposable (1,2)-groups G with regulator quotient of exponent p^4 is not known to be sharp. For this it remains to prove that the group G_4, explicitly given by its representing matrix

$$S(E,\beta,\gamma) = \begin{pmatrix} p^{-4} & 0 \\ 0 & p^{-2} \end{pmatrix} \left(\begin{array}{cc||c||cc} 1 & 0 & p^2 & 1 & 0 \\ \hline 0 & 1 & p & 0 & 1 \end{array} \right).$$

is indecomposable.

(2) There are some isomorphism types of the regulator quotient in the cases of exponent p^5 and p^6, which are not dealt with. It is not clear if the same arguments apply as before.

References

[1] D. Arnold, *Finite rank torsion-free abelian groups and rings*, Lecture Notes 931, Springer-Verlag, Berlin Heidelberg New York 1982.

[2] D. Arnold, Abelian Groups and Representations of Partially Ordered Sets, CMS Advanced Books in Mathematics, Springer-Verlag, New York, 2000.

[3] D. Arnold, M. Dugas, Finite rank Butler groups with small typesets, in Abelian Groups and Modules, Trends in Math. (1998), 107-119.

[4] J. L. Brenner. The linear homogeneous group III. Ann. Math. (2) (1960), 71:210-223.

[5] M. Dugas, BCD-groups with type set (1,2), Forum Mathematicum 13, (2000), 143-148.

[6] M. Dugas, BR-groups with critical type set $(1, 2)$, Forum. Math. **13**, (2001), 143-148.

[7] N. Jacobson, *Basic Algebra* I, W. H. Freeman and Company, Yale (1974).

[8] L. Fuchs, Infinite Abelian Groups, Vols. I and II, Academic Press (1970,1973).

[9] K.-J. Krapf and O. Mutzbauer. Classification of almost completely decomposable groups. In Abelian Groups and Modules, Proceedings of the Udine Conference 1984, volume 287 of CISM Courses and Lecture Notes, Springer Verlag (1984), 151-161.

[10] A. Mader, Almost Completely Decomposable Groups, Gordon Breach (2000).

[11] O. Mutzbauer. Normal forms of matrices with applications to almost completely decomposable groups. Trends in Mathematics, Birkhäuser Verlag Basel/Switzerland (1999), 121-134.