Coset Types and Tight Subgroups of Almost Completely Decomposable Groups

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The Road Not Taken

Two roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could
To where it bent in the undergrowth;

Then took the other, as just as fair,
And having perhaps the better claim,
Because it was grassy and wanted wear;
Though as for that the passing there
Had worn them really about the same,

And both that morning equally lay
In leaves no step had trodden black.
Oh, I kept the first for another day!
Yet knowing how way leads on to way,
I doubted if I should ever come back.

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I–
I took the one less traveled by,
And that has made all the difference.

– Robert Frost

Contents

1 Introduction and Basics

1.1 Overview of the Results

In this dissertation we show an extended version of the theorem of Bezout (Lemma 1.2), give a new criterion for the tightness of a completely decomposable subgroup (Theorem 2.1), derive some conditions under which a tight subgroup is regulating (Corollaries 2.2 and 2.3) and generalize a theorem of Campagna (Theorem 2.5). We give an example of an almost completely decomposable group, all of whose regulating subgroups do not have a quotient with minimal exponent (Example 2.6).

We show that among the types of elements of a coset modulo a completely decomposable group there exists a unique maximal type (Corollary 3.4) and define this type to be *the* coset type (Definition 3.7). We give criteria for tightness and regulating in term of coset types (Lemma 3.14) as well as a representation of the type subgroups using coset types (Lemma 3.15). We introduce the notion of reducible cosets (Definition 3.16) and show their key role for transitions from one completely decomposable subgroup up to another one containing the first one as a proper subgroup (Lemmata 3.19, 3.21 and 3.22).

We give an example of a tight, but not regulating subgroup which contains the regulator (Example 3.38).

We develop the notion of a fully single covered subset of a lattice (Definition 4.10), show that $\bigvee$-free implies fully single covered (Lemma 4.11), but not necessarily vice versa (Example 4.12), and we define an equivalence relation on the set of all finite subsets of a given lattice (Theorem 4.25). We develop some extension of ordinary Hasse diagrams, and apply the lattice theoretic results on the lattice of types and almost completely decomposable groups (Lemma 4.28).

1.2 Integers and Rational Groups

The following lemma was originally proven by K. Rogers (Univ. of Hawaii).

Lemma 1.1 *Let a, b, c be given, non-zero integers with $\gcd(a, b) = 1$. Then there exists an integer i such that $\gcd(a + ib, c) = 1$.*

Proof: Factor $c = c_1c_2$ such that $\gcd(b, c_2) = 1$ and every prime factor of c_1 is also a factor of b. By the Chinese Remainder Theorem the system

$$x \equiv a \pmod b, \qquad x \equiv 1 \pmod{c_2}$$

has a solution x. Then $x = a + ib$ for some integer i and $\gcd(x, b) = \gcd(a, b) = 1$. Hence $\gcd(x, c_1) = 1$, too. Q.E.D.

Lemma 1.2 (Extended Bezout) *Let a, b, c be three arbitrary integers. Then there exist two integers r, s such that $ar + bs = \gcd(a, b)$ and $\gcd(r, c) = 1$.*

Proof: Define $g = \gcd(a, b)$, $\bar{a} = a/g$, $\bar{b} = b/g$. Then $\gcd(\bar{a}, \bar{b}) = 1$ and by Bezout there exist two integers r_0, s_0 such that $r_0\bar{a} + s_0\bar{b} = 1$. Note that $\gcd(r_0, \bar{b}) = 1$.

Define $r_i = r_0 + i\bar{b}$ and $s_i = s_0 - i\bar{a}$ for all integer i. Then $r_i\bar{a} + s_i\bar{b} = r_0\bar{a} + s_0\bar{b} = 1$ and $r_ia + s_ib = \gcd(a, b)$.

By the previous Lemma there exists an i such that $\gcd(r_0 + i\bar{b}, c) = 1$, because $\gcd(r_0, \bar{b}) = 1$. With $r_i = r_0 + i\bar{b}$ the claim follows. Q.E.D.

The extended Bezout is applied in a diffent form in this paper: If a and m are two integers, then does there exist an integer b relatively prime to m such that $ab \equiv \gcd(a, m)$ modulo m? The extended Bezout answers this question affirmatively.

The following Lemma will be used mostly without reference.

Lemma 1.3 *Let $\mathbb{Z} \subseteq S \subseteq \mathbb{Q}$ be a rational group and let a, b be two integers. Then*

a) $\frac{a}{b} \in S$ *implies* $\frac{1}{b} \in S$ *for relatively prime a and b, and*

b) $\frac{1}{a} \in S$ *and* $\frac{1}{b} \in S$ *implies* $\frac{1}{\operatorname{lcm}(a,b)} \in S$*, and*

c) *if* $\frac{1}{p} \notin S$ *for all primes $p|a$, then* $S = \langle aS, 1\rangle$ *and* $a^{-1}S = \langle S, a^{-1}\rangle$.

Proof:

a) By Bezout there exist integers r, s such that $ra + sb = 1$. Then $r\frac{a}{b} + s = \frac{1}{b}$. With $1 \in S$ the claim follows.

b) By Bezout there exist integers r, s such that $ra + sb = \gcd(a, b)$. Then $r\frac{1}{b} + s\frac{1}{a} = \frac{\gcd(a,b)}{ab} = \frac{1}{\operatorname{lcm}(a,b)}$, because $\gcd(a, b)\operatorname{lcm}(a, b) = ab$.

c) Let $x \in S$. Then $x = \frac{y}{z}$ with $y \in \mathbb{Z}$ and $z \in \mathbb{N}$ and $\gcd(y,z) = 1$. Note that $\frac{1}{z} \in S$ if and only if $x = \frac{y}{z} \in S$ because of part a). So it suffices to show that $\frac{1}{z} \in \langle aS, 1\rangle$. Note that $\gcd(a,z) = 1$, because otherwise there would exist a prime p with $\frac{1}{p} \in S$ and $p|a$, contradicting the assumption. By Bezout there exist integers r, s such that $ra + sz = 1$. Then $r\frac{a}{z} + b = \frac{1}{z}$. As $\frac{a}{z} \in aS$ and $b \in \mathbb{Z}$ we have $\frac{1}{z} \in \langle aS, 1\rangle$ and $S = \langle aS, 1\rangle$. The second claim follows simply by division.

Q.E.D.

1.3 Almost Completely Decomposable Groups

For the sake of self-containment we define some basic notions. For more details see Mader's Book [1] Chapter 2. In particular we assume that the reader is familiar with types and type subgroups. Unless noted otherwise, we use the notation of [1]. We assume that all torsion free groups in this paper have finite rank unless noted otherwise.

Definition 1.4 *A torsion free abelian group is called* **completely decomposable** *if it is the direct sum of rational groups.*

Definition 1.5 *Let W be a completely decomposable group and $\{x_j\}_j$ be a decomposition basis of W.*

Then the basis is called **adjusted** *if $\mathrm{tp}^W(x_k) \leq \mathrm{tp}^W(x_l)$ implies $\chi^W(x_k) \leq \chi^W(x_l)$ for all k, l.*

For every completely decomposable group there exists an adjusted basis.

Definition 1.6 *A torsion free abelian group is called* **almost completely decomposable** *if it contains a completely decomposable group as subgroup of finite index.*

Definition 1.7 *A torsion free abelian group is a* **Butler** *group if it is the epimorphic image of a finite rank completely decomposable group.*

Every almost completely decomposable group is also a Butler group.

Definition 1.8 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Then W is called* **regulating** *if there exists no completely decomposable subgroup with smaller index.*

The concept of regulating subgroups is due to E. L. Lady.

Definition 1.9 *Let G be an almost completely decomposable group. The intersection of all regulating subgroups of G is called the* **regulator** *of G.*

Burkhardt introduced the regulator and has shown that the regulator of an almost completely decomposable group group is completely decomposable and has finite index.

Definition 1.10 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Then W is called* **tight** *if there exists no completely decomposable subgroup of G that is strictly larger than W.*

The notion of tight subgroups is due to Benabdallah, Mader and Ould-Beddi in [2].

The next definition is an alternative way of defining regulating subgroups in more generality for Butler groups and not as in Definition 1.8 only for almost completely decomposable groups.

Definition 1.11 *Let G be a Butler group containing a subgroup W of finite index. Let $W = \sum_{\tau \in T_{cr}} W_\tau$ with τ-homogeneous components W_τ. Then W is called* **regulating** *if $G(\tau) = W_\tau \oplus G^{\#}(\tau)$ for all critical types τ.*

The following Lemma is due to Mader (1965), quoted in [1] Lemma 1.1.3.

Lemma 1.12 *Let $M = K \oplus L$ be a direct decomposition of R-modules. Then*

$$\phi \mapsto L(1 + \phi) = \{x + x\phi : x \in L\}$$

defines a bijective correspondence between the maps of $\mathrm{Hom}_R(L, K)$ and the set of complementary summands of K in M. Furthermore, $1 + \phi : L \to L(1 + \phi)$ is an isomorphism.

2 Tight and Regulating Subgroups

2.1 Tight Criterion

Benabdallah, Mader and Ould-Beddi gave a criterion for tightness [2] Proposition 2.7.(2) which required the verification that all rank-1 summands of the subgroup in question were pure. Here we give a different criterion which requires to check the order of elements in a set derived from the type subgroups.

Lemma 2.1 *Let G be an almost completely decomposable group with completely decomposable subgroup W of finite index. Then W is not tight if and only if there exist a critical type τ and an element $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$ of prime order modulo W.*

Proof: "if" It suffices to show that $H := \langle W, g\rangle$ is completely decomposable. Split $g = \frac{1}{p}(a+b)$ where $a \in W_\tau$ and $b \in W^\#(\tau)$. Note that $\mathrm{ht}_p^G(a) = 0$, because otherwise $g \in G^\#(\tau)$. As $\mathrm{tp}(a) \leq \mathrm{tp}(b)$ and $\mathrm{ht}_p(a) \leq \mathrm{ht}_p(b)$ there exists a natural number k such that $\chi^G(a) \leq \chi^G(kb)$ and $\gcd(k,p) = 1$. Then by Bezout there exist two integers r and s such that $rp + sk = 1$. Note that $H = \langle W, g - rb\rangle$ because $rb \in W$. Write $g - rb = \frac{1}{p}(a + (1-rp)b) = \frac{1}{p}(a+skb)$. As $\chi^G(a) \leq \chi^G(skb)$ there exists a homomorphism $\varphi \in \mathrm{Hom}(W_\tau, W^\#(\tau))$ such that $\varphi(a) = skb$. Hence $p(g-rb) = a+skb = a(1+\varphi) \in W_\tau(1+\varphi) \subseteq W$. But then $W_\tau(1+\varphi)$ is not pure in H. As $|H/W| = p \in \mathbb{P}$ it is clear that $H = (W_\tau(1+\varphi))_*^H \oplus \bigoplus_{\sigma\neq\tau} W_\sigma$. As $1+\varphi$ is an isomorphism of W it is clear that $(W_\tau(1+\varphi))_*^H$ is completely decomposable and hence H, too.

"only if" If W is not tight, then by [2] Proposition 2.7 (2) there exists a rank-1 summand of W which is not pure. Assume then that $W = \bigoplus_j W_j$ where W_1 is not pure in G. Then there exists an element $g \in {W_1}_*^G \setminus W_1$ of prime order over W_1 and W. Set $\tau := \mathrm{tp}(g) = \mathrm{tp}(W_1)$ and note that $g \in G(\tau)$. Let $m_0 = \exp G/W$. Then $m_0 g \in W_1$ and $g \notin W_1$ imply $g \notin W_1 \oplus G^\#(\tau)$. Hence $g \in G(\tau) \setminus W_1 \oplus W^\#(\tau)$, as desired. Q.E.D.

As Corollary we obtain [2] Lemma 4.5.

Corollary 2.2 *Let G be an almost completely decomposable group containing a tight subgroup W such that $p(G/W) = 0$ for some prime p. Then W is regulating in G.*

Proof: Assume for contradiction that W was not regulating. Then there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^\#(\tau)$. Let $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. Then $g \notin W$ and by $p(G/W) = 0$ we have that g has prime order over W. By the previous lemma W is not tight then, a contradiction. Q.E.D.

We extend the previous corollary to a more general case. The proof is shortened significantly by an idea of Otto Mutzbauer.

Corollary 2.3 *Let G be an almost completely decomposable group containing a tight subgroup W such that $k(G/W) = 0$ for some square free integer k. Then W is regulating in G.*

Proof: Assume for contradiction that W was not regulating. Then there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^\#(\tau)$. Let $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. Let m be the order of g modulo W. Then m must be square free. Let $p_1, \ldots, p_k$ be the prime divisors of m. Then $\gcd(\frac{m}{p_1}, \ldots, \frac{m}{p_k}) = 1$ as m is square free. Hence $m = \sum_{i=1}^k \mu_i(\frac{m}{p_i})$ for suitable integers μ_i. Define $g_i := g(\frac{m}{p_i})$ and note that g_i has order p_i modulo W. As W is tight, we know that $g_i \in W_\tau \oplus g^\#(\tau)$ for all i, as all g_i have prime order. But $g = (\sum_i \mu_i(\frac{m}{p_i}))g = \sum_i \mu_i g_i$ and hence $g \in W_\tau \oplus G^\#(\tau)$, contradicting our assumption. Q.E.D.

The following corollary is also found in [2] Corollary 4.6.

Corollary 2.4 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W such that $k(G/W) = 0$ for some square free integer k. Then W is contained in a regulating subgroup V of G such that $k(G/V) = 0$.*

Proof: The completely decomposable group W is contained in some tight subgroup V of G. Note that $k(G/W) = 0$ implies $kG \subseteq W \subseteq V$ and hence $k(G/V) = 0$. So V is regulating by the previous corollary. Q.E.D.

2.2 Transitions to Regulating Subgroups

In [4] Theorem 2.5 Campagna has shown that the existence of a cyclic quotient implies the existence of a cyclic regulating quotient. We generalize this to the case of more than one generator and thus answer a question posed by Benabdalla, Mader and Ould-Beddi in [2] Question 4.1.(2).

Theorem 2.5 *Let $G = \langle W, g_1, \ldots, g_k\rangle$ be an almost completely decomposable group containing the completely decomposable group W of finite index. Then there exists a regulating subgroup V with $G = \langle V, g_1, \ldots, g_k\rangle$.*

Proof: Let W be not regulating. We will only show that there exists a completely decomposable subgroup W' with $G = \langle W', g_1, \ldots, g_k\rangle$ and $|G/W| > |G/W'|$. The claim follows by induction, as $|G/W|$ is finite.

We will first construct a completely decomposable subgroup W', then we show that $G = \langle W', g_1, \ldots, g_k\rangle$, and in the last section we show that W' has smaller index in G than W.

Let $W = \bigoplus_{\sigma\in T_{cr}} W_\sigma$ be a homogeneous decomposition of W. As W is not regulating, there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^\#(\tau)$. Hence there exists a $g \in G(\tau) \setminus (W_\tau \oplus G^\#(\tau))$. Note that also $g + W_\tau \subseteq G(\tau) \neq W_\tau \oplus G^\#(\tau)$. So every element in $g + W_\tau$ witnesses that W is not regulating.

For later purposes we are now selecting a special element from $g + W_\tau$. Define $\bar{W} = \bigoplus_{\sigma\neq\tau} W_\sigma$. Then $W = W_\tau \oplus \bar{W}$. As $\{g_i + W\}_{i=1}^k$ are generators of G/W we know that there exist integers $\lambda_1, \ldots, \lambda_k$ such that $g + W = \sum_{i=1}^k \lambda_i g_i + W$. Then $g - \sum_{i=1}^k \lambda_i g_i = \sum_{\sigma\in T_{cr}} v_\sigma \in W$ with $v_\sigma \in W_\sigma$. Our special element $h \in g + W_\tau$ is defined as follows: $h := g - v_\tau \in G(\tau) \setminus (W_\tau \oplus G^\#(\tau))$. As $g - (\sum_{i=1}^k \lambda_i g_i) - v_\tau \in \bar{W}$ we obtain

$$h \in \langle \bar{W}, g_1, \ldots, g_k\rangle.$$

The particular property of h is that the τ-component of h is solely a linear combination of the τ-components of the generators $\{g_i\}_i$. No element of W_τ is needed for that. This is helpful, because W_τ will be replaced later and we do not want the replacement to affect h.

Now we split h in two components a and b, which we will use as building bricks for the construction of a new τ-homogeneous component of W. Define $\varphi = |h + W|$ and let $h = \frac{1}{\varphi}(\sum_{\sigma\in T_{cr}} w_\sigma)$ with $w_\sigma \in W_\sigma$. As $h \in G(\tau)$ we know that $w_\sigma = 0$ for all $\sigma \not\geq \tau$. So we can set $a = w_\tau$, $b = \sum_{\sigma>\tau} w_\sigma$ and we obtain $h = \frac{1}{\varphi}(a + b)$. As $\mathrm{tp}(a) \leq \mathrm{tp}(b)$ and as φ is finite there exists an integer l such that

$$l^{-1}a \in W, \quad \chi^W(l^{-1}a) \leq \chi^W(b), \quad \mathrm{ht}_p^W(l^{-1}a) = 0 \text{ for all } p|\varphi.$$

Note that l is not a multiple of φ, because then we would have $\frac{1}{\varphi}a \in W_\tau$ and $h = \frac{1}{\varphi}a + \frac{1}{\varphi}b \in W_\tau \oplus G^\#(\tau)$, which cannot be.

Define $a' := l^{-1}a$ and $R := \{r \in \mathbb{Q} | ra' \in W_\tau\}$. Then obviously $Ra' \subseteq W_\tau$ and $\frac{1}{p} \notin R$ for all $p|\varphi$, because $\mathrm{ht}_p^W(a') = 0$ for all $p|\varphi$. Note that Ra' is a pure subgroup of the homogeneous group W_τ, so by [3] 86.8 we know that $W_\tau = Ra' \oplus U$ for some complement $U \subset W_\tau$. By [3] 86.7 we know that U is completely decomposable.

Define $\varphi^* := \gcd(\varphi, l)$. As l is not a multiple of φ, we know that φ^* is a proper divisor of φ. Note that $\frac{1}{\varphi^*}a \in W_\tau$, as $\varphi^*|l$. Together with $\frac{1}{\varphi^*}(a+b) = \frac{\varphi}{\varphi^*}h \in G$ we get that $\frac{1}{\varphi^*}b \in G$. By the extended Bezout (Lemma 1.2) there exist two integers r and s such that

$$\varphi^* = rl + s\varphi \quad \text{and} \quad \gcd(s, \varphi) = 1.$$

Define $w := r\frac{1}{\varphi}(a+b) + sa'$. Then $\varphi w = ra + rb + s\varphi a' = (rl + s\varphi)a' + rb = \varphi^* a' + rb$ and

$$w = \frac{1}{\varphi}rb + \frac{\varphi^*}{\varphi}a' = \frac{\varphi^*}{\varphi}\left(a' + r\frac{1}{\varphi^*}b\right).$$

The last term shows the intention of the process. The summand Ra' of W is to be rotated ($a' \mapsto a' + r\frac{1}{\varphi^*}b$) and shifted ($\frac{\varphi^*}{\varphi}$). We want to show that Ra' can be replaced by Rw to obtain a new completely decomposable subgroup of G.

We get $lw = lr\frac{1}{\varphi}(a+b) + lsa' = (\varphi^* - \varphi s)\frac{1}{\varphi}(a+b) + as = \frac{\varphi^*}{\varphi}(a+b) - s(a+b) + as = \frac{\varphi^*}{\varphi}(a+b) - sb$ and $\frac{l}{\varphi^*}w = \frac{1}{\varphi}(a+b) - s\frac{1}{\varphi^*}b$. So we can write

$$\frac{1}{\varphi}(a+b) = \frac{l}{\varphi^*}w + s\frac{1}{\varphi^*}b.$$

Note that for all primes $p|\varphi$ we have $\frac{1}{p} \notin R$ and hence $\mathrm{ht}_p^W(a') = 0 \leq \mathrm{ht}_p^G(\frac{1}{\varphi^*}b)$. This implies $\chi^W(a') \leq \chi^G(\frac{1}{\varphi^*}b)$ and hence $R\frac{1}{\varphi^*}b \subseteq G$. We have $\frac{\varphi}{\varphi^*}wR = (a' + r\frac{1}{\varphi^*}b)R \subseteq Ra' + Rr\frac{1}{\varphi^*}b \subseteq G(\tau)$. Also we know that $w \in G(\tau)$. By Lemma 1.3 c) we find $Rw \subseteq G(\tau)$. Now we define

$$W' := Rw \oplus U \oplus \bar{W}, \quad G' := \langle W', \frac{1}{\varphi^*}b, g_1, \ldots, g_k\rangle \quad \text{and} \quad H := \langle W', g_1, \ldots, g_k\rangle.$$

Then $W' \subseteq G$ and hence $G', H \subseteq G$. We want to show that $H = G$. This is done in two steps, first we show $G' = G$ and then $H = G'$.

Note that $Rb \subseteq W^\#(\tau) \subseteq W' \subseteq G'$. By definition we have $\frac{1}{\varphi^*}b \in G'$. Again by Lemma 1.3 c) we obtain $R\frac{1}{\varphi^*}b \subseteq G'$. We know that $Rw \subseteq W' \subseteq G'$

and that $\frac{\varphi}{\varphi^*} \in \mathbb{Z}$. Together we obtain $\frac{\varphi}{\varphi^*}Rw \subseteq G'$ and hence $R(a' + r\frac{1}{\varphi^*}b) \subseteq G'$. With $r\frac{1}{\varphi^*}b \subseteq G'$ together we get $Ra' \subseteq G'$. But then $W = Ra' \oplus U \oplus \bar{W} \subseteq G'$ and thus $G' = G$.

Remember that we chose h such that $h \in \langle \bar{W}, g_1 \ldots, g_k\rangle \subseteq H$. Hence $h = \frac{1}{\varphi}(a+b) = \frac{l}{\varphi^*}w + s\frac{1}{\varphi^*}b \in H$. Then $\frac{l}{\varphi^*}w \in \mathbb{Z}w \subseteq W' \subseteq H$ implies $s\frac{1}{\varphi^*}b \in H$. As $\gcd(s,\varphi) = 1$ we know that $\gcd(s,\varphi^*) = 1$ and by Lemma 1.3 b) we get that $\frac{1}{\varphi^*}b \in H$. Hence $H = G' = G = \langle W', g_1, \ldots, g_k\rangle$.

It now remains to show that the index of W' in G is smaller than the index of W. We will show this by defining a subgroup X that is contained in both W and W' and then calculating the index of X in W and W'.

Define $X = \varphi^* a' R \oplus U \oplus \bar{W}$. Obviously we have $X \subseteq W = Ra' \oplus U \oplus \bar{W}$. Note that $\varphi^* a' R = (\varphi w - rb)R$. Then $X = (\varphi w - rb)R \oplus U \oplus \bar{W}$. As $Rb \subseteq W^{\#}(\tau) \subseteq \bar{W}$ we have that $X = \varphi w R \oplus U \oplus \bar{W}$. Then obviously $X \subseteq W' = Rw \oplus U \oplus \bar{W}$. So $X \subseteq W \cap W'$. Now we calculate $|W : X|$ and $|W' : X|$. As $\frac{1}{p} \notin R$ for all $p|\varphi^*$ we have $\varphi^* = |R : \varphi^* R| = |W : X|$. As $\frac{1}{p} \notin R$ for all $p|\varphi$ we have $\varphi = |R : \varphi R| = |W' : X|$. With $|G : W||W : X| = |G : W'||W' : X|$ we get $|G : W'| = \frac{\varphi^*}{\varphi}|G : W|$ and hence $|G : W'| < |G : W|$.

Q.E.D.

In [2] there was also the question, whether the existence of a tight subgroup with a quotient of a given exponent implies the existence of a regulating quotient with an equal or smaller exponent (Question 4.1.(1)).

Example 2.6 *Let p, q, r, s, t, u be different primes. Let*

$$W = \mathbb{Q}^{(q)}x_1 \oplus \mathbb{Q}^{(q,r)}x_2 \oplus \mathbb{Q}^{(q,s)}x_3 \oplus \mathbb{Q}^{(t)}x_4 \oplus \mathbb{Q}^{(u)}x_5,$$

$$g_1 = \frac{1}{p^3}(px_1 + x_2 + x_3 + p^2 x_4), \quad g_2 = \frac{1}{p^3}(x_1 + x_5), \quad G = \langle W, g_1, g_2\rangle .$$

We claim that W is tight in G with $\exp(G/W) = p^3$ and that $\exp(G/V) = p^4$ for every regulating subgroup V of G. Hence the regulating quotients do not have minimal exponent, although they have minimal order.

The claims will be verified at the end of section 3.2.

What makes this example remarkable is the fact that it shows that regulating subgroups may have the minimal property with respect to index, but not necessarily with respect to exponent. Intersecting all tight subgroups with minimal index (that is regulating subgroups) yields the (index-)regulator. Similarly one could ask about the intersection of all tight subgroups with minimal exponent, which we call the **exponent-regulator**. Except for the obvious fact that the exponent-regulator is a characteristic subgroup, many properties are still open to research.

3 Coset Types and Reducible Cosets

3.1 Carriers and Cutted Elements

Definition 3.1 *Let W be a completely decomposable group with decomposition basis $\{x_j\}_j$ such that $W = \bigoplus_j S_j x_j$ for suitable $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$. Let $x \in \mathbb{Q}W = \bigoplus_j \mathbb{Q}x_j$ with $x = \sum_j \alpha_j x_j$. Then we define*

$$\mathrm{I}^{\{x_j\}_j}(x) := \{j | \alpha_j \neq 0\}$$

to be the **carrier** *of x with respect to the basis $\{x_j\}_j$,*

$$\mathrm{R}_W^{\{x_j\}_j}(x) := \{j | \alpha_j \notin S_j\}$$

to be the **real carrier** *of x with respect to W and its basis $\{x_j\}_j$, and*

$$\mathrm{Cut}_W^{\{x_j\}_j}(x) = \sum_{j \in \mathrm{R}_W^{\{x_j\}_j}(x)} \alpha_j x_j.$$

Remark 3.2 *Note that $x + W = \mathrm{Cut}_W(x) + W$, regardless of the choice of the basis, because $x - \mathrm{Cut}_W(x) = \sum_{j \notin \mathrm{R}_W(x)} \alpha_j x_j = \sum_{\alpha_j \in S_j} \alpha_j x_j \in W$.*

Lemma 3.3 *Let W be a completely decomposable group with decomposition basis $\{x_j\}_j$. Let $x, y \in m^{-1}W$ with $m \in \mathbb{Z}$ and $x + W = y + W$. Then the following hold (always using W and $\{x_j\}_j$ for all carriers and cuts):*

a) $\mathrm{I}(\mathrm{Cut}(x)) = \mathrm{R}(x) \subseteq \mathrm{I}(x)$,

b) $\mathrm{tp}(x) = \bigwedge_{j \in \mathrm{I}(x)} \mathrm{tp}(x_j)$,

c) $\mathrm{tp}(\mathrm{Cut}(x)) \geq \mathrm{tp}(x)$,

d) $\mathrm{I}(y) \supseteq \mathrm{R}(x)$,

e) $\mathrm{Cut}(\mathrm{Cut}(x)) = \mathrm{Cut}(x)$ *and* $\mathrm{R}(x) = \mathrm{I}(\mathrm{Cut}(x)) = R(\mathrm{Cut}(x))$,

f) $\mathrm{R}(x) = \mathrm{R}(y)$ *and* $\mathrm{tp}(\mathrm{Cut}(x)) = \mathrm{tp}(\mathrm{Cut}(y))$*,*

g) $\mathrm{R}(kx) \subseteq \mathrm{R}(x)$ *for all* $k \in \mathbb{Z}$*.*

Proof:

a) Follows from the definition of $\mathrm{Cut}(x)$ and from the fact that $\alpha_j \notin S_j$ implies $\alpha_j \neq 0$.

b) $\mathrm{tp}(x) \geq \bigwedge_{j\in \mathrm{I}(x)} \mathrm{tp}(x_j)$ is obvious, and the equality is due to the direct sum $x \in \bigoplus_j S_j x_j$.

c) Follows with a) and b).

d) Let $z := y - \mathrm{Cut}(x)$. Then $z \in W$ because $y+W = x+W = \mathrm{Cut}(x)+W$. We now compare $\mathrm{I}(y) = \mathrm{I}(z + \mathrm{Cut}(x))$ with $\mathrm{I}(\mathrm{Cut}(x)) = R(x)$. Let $z = \sum_j \beta_j x_j$ and $\mathrm{Cut}(x) = \sum_j \gamma_j x_j$. Then $\beta_j \in S_j$ and $\gamma_j \in (\mathbb{Q} \setminus S_j) \cup \{0\}$ for all j. If $j \in \mathrm{I}(\mathrm{Cut}(x))$, then $\gamma_j \in \mathbb{Q} \setminus S_j$ and $\gamma_j + \beta_j \in \mathbb{Q} \setminus S_j$ and $j \in \mathrm{I}(\mathrm{Cut}(x) + z)$. Hence $\mathrm{I}(\mathrm{Cut}(x) + z) \supseteq \mathrm{I}(\mathrm{Cut}(x))$.

e) From the definition of $\mathrm{Cut}(x)$ it is clear that $\mathrm{Cut}(\mathrm{Cut}(x)) = \mathrm{Cut}(x)$. Then $\mathrm{R}(x) = \mathrm{I}(\mathrm{Cut}(x)) = \mathrm{I}(\mathrm{Cut}(\mathrm{Cut}(x)) = \mathrm{R}(\mathrm{Cut}(x))$.

f) Let $x' = \mathrm{Cut}(x)$ and $y' = \mathrm{Cut}(y)$. Using d) and symmetry we get $\mathrm{I}(x') \supseteq \mathrm{R}(y')$ and $\mathrm{I}(y') \supseteq \mathrm{R}(x')$. By e) we know that $\mathrm{I}(x') = \mathrm{R}(x')$ and $\mathrm{I}(y') = \mathrm{R}(y')$. Hence $\mathrm{R}(x') = \mathrm{I}(x') \supseteq \mathrm{R}(y') = \mathrm{I}(y') \supseteq \mathrm{R}(x')$ and $\mathrm{R}(x') = \mathrm{R}(y')$. Again by e) we have $\mathrm{R}(x) = \mathrm{R}(x')$ and $\mathrm{R}(y) = \mathrm{R}(y')$ and hence $\mathrm{R}(x) = \mathrm{R}(y)$.

g) Let $x = \sum_{j=1}^n \alpha_j x_j$. Assume that $l \in \mathrm{R}(kx) = \{j \mid k\alpha_j \notin S_j\}$. Then $k\alpha_l \notin S_l$ and hence $\alpha_l \notin S_l$. So $l \in \mathrm{R}(x) = \{j \mid \alpha_j \notin S_j\}$ and $\mathrm{R}(kx) \subseteq \mathrm{R}(x)$.

Q.E.D.

Corollary 3.4 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let $\{x_j\}_j$ be a decomposition basis of W.*

Then for every coset of G modulo W there exists a type τ which is the unique maximal type contained in the typeset of the coset, and there exists a unique minimum carrier which is a subset of the carrier of any element in the coset.

Proof: Clearly $G \subseteq m^{-1}W$ for some $m \in \mathbb{Z}$. By part d) of the previous lemma we know that $\mathrm{R}(x)$ is the unique minimum carrier of that coset. With the help of part b) we know that smaller carrier implies greater or equal type. Hence a unique minimal carrier implies a unique maximal type. Q.E.D.

Corollary 3.5 *Let W be a completely decomposable group and $g + W$ a coset of finite order. Then all cosets that generate $\langle g + W\rangle$ have the same coset carrier (with respect to a fixed decomposition basis) and the same qualified coset type.*

Proof: Let $h + W \in \langle g + W\rangle$ be a coset such that $\langle h + W\rangle = \langle g + W\rangle$. Then also $g \in \langle h + W\rangle$ and both cosets are multiples of each other. Write $h+W = kg+W$ and $g+W = lh+W$ with $k, l \in \mathbb{Z}$. Then $g+W = klg+W$ and by Lemma 3.3 f) we know that $\mathrm{R}(g) = \mathrm{R}(klg)$. Likewise we obtain that $\mathrm{R}(h) = \mathrm{R}(kg)$. By Lemma 3.3 g) we know that $\mathrm{R}(g) \supseteq \mathrm{R}(kg) \supseteq \mathrm{R}(klg)$. Hence $\mathrm{R}(g) = \mathrm{R}(kg) = \mathrm{R}(klg) = \mathrm{R}(h)$ and $\mathrm{I}(g + W) = \mathrm{I}(h + W)$. Hence also $\mathrm{tp}(g + W) = \mathrm{tp}(h + W)$. Q.E.D.

Lemma 3.6 *Let $W = \bigoplus_{j=1}^{n} S_j x_j$ with m_0-basis $\{x_j\}_j$. Let $g+W$ be a coset of order m_0. Then there exists a representative $h \in g+W$ such that $h \in \frac{1}{m_0} \bigoplus_{j=1}^{n} \mathbb{Z}x_j$ and $\mathrm{I}^{\{x_j\}_j}(h) \subseteq \mathrm{I}^{\{x_j\}_j}(g)$.*

Proof: Write $g = \frac{1}{m_0} \sum_{j=1}^{n} \alpha_j x_j$ with $\alpha_j \in S_j$. Then $\alpha_j = \frac{\beta_j}{\gamma_j}$ with $\beta_j \in \mathbb{Z}$, $\gamma_j \in \mathbb{N}$ and $\gcd(\beta_j, \gamma_j) = 1$. Define $\delta = \mathrm{lcm}_{j=1}^{n} \lambda_j$. Split $\delta = \delta_1\delta_2$ such that $\gcd(m_0, \delta_2) = 1$ and $\{p \mid p|\delta_1\} \subseteq \{p \mid p|m_0\}$. Then $\gcd(\delta_1, \delta_2) = 1$ and there exists an integer s such that $s\delta_2 \equiv 1(m_0)$. Hence $s\delta_2 g \in g + W$ and $s\delta_2\alpha_j \in S_j$ is a fraction that has only prime divisors which are also prime divisors of m_0. Note that $\mathrm{I}^{\{x_j\}_j}(s\delta_2 g) = \mathrm{I}^{\{x_j\}_j}(g)$.

So WLOG we may assume that all prime divisors of γ_j are prime divisors of m_0. Define $e_j := |S_j/m_0S_j|$ and $\bar{e}_j := m_0/e_j$. With the m_0-basis property of $\{x_j\}_j$ we conclude that $S_j\bar{e}_j = S_j$. Note that $\alpha_j \in S_j$ implies $\frac{1}{\gamma_j} \in S_j$ and $\frac{1}{p} \in S_j$ for all prime divisors of γ_j. Hence γ_j contains only those prime divisors of m_0, for which S_j is divisible. In particular we have $\gcd(\gamma_j, e_j) = 1$.

For every β_j we define some $t_j \in \mathbb{Z}$ as follows: If $\beta_j = 0$ then set $t_j = 0$. If $\beta_j \neq 0$, then let t_j be an integer that solves $\beta_j + e_j t_j \equiv 0(\gamma_j)$. A solution exists, because $\gcd(\beta_j, \gamma_j) = 1$ and $\gcd(e_j, \gamma_j) = 1$. Note that

$\frac{1}{m_0}\frac{e_j t_j}{\gamma_j} = \frac{t_j}{\bar{e}_j\gamma_j} \in S_j$ as the denominator contains only primes, for which S_j is divisible.

Define $w := \sum_{j=1}^n \frac{t_j}{\bar{e}_j\gamma_j} x_j$. Then $w \in W$ and $g + w \in g + W$. We find $g + w = \frac{1}{m_0}\sum_{j=1}^n \left(\frac{\beta_j}{\gamma_j} + \frac{m_0 t_j}{\bar{e}_j\gamma_j}\right) x_j = \frac{1}{m_0}\sum_{j=1}^n \left(\frac{\beta_j + e_j t_j}{\gamma_j}\right) x_j \in \frac{1}{m_0}\bigoplus_{j=1}^n \mathbb{Z}x_j$. Note that $\mathrm{I}^{\{x_j\}_j}(g + w) \subseteq \mathrm{I}^{\{x_j\}_j}(g)$ as $t_j = 0$ if $\beta_j = 0$. Q.E.D.

3.2 Coset Types

Definition 3.7 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let $\{x_j\}_j$ be a decomposition basis of W. Let $x + W$ be a coset of G modulo W. Then*

$$\mathrm{tp}(x + W) := \mathrm{tp}(\mathrm{Cut}(x))$$

is called the **coset type** *of $x + W$ and*

$$\mathrm{I}(x + W) := \mathrm{R}(x) = \mathrm{I}(\mathrm{Cut}(x))$$

is called the **coset carrier** *of $x + W$. A coset type $\tau = \mathrm{tp}(x + W)$ is called* **true**, *if there exists a $j \in \mathrm{I}(x + W)$ with $\mathrm{tp}(x_j) = \tau$. Otherwise the coset type shall be called* **false**. *By* **qualified coset type** *we mean a coset type together with the additional property 'true' or 'false' given.*

Remark 3.8 *By Lemma 3.3f) , coset type and coset carrier are well defined. If the coset type is not a critical type, it cannot be a true coset type, because the components of x have only critical types.*

Lemma 3.9 *Let G be an almost completely decomposable group containing a completely decomposable group W of finite index. Let $g \in G$ and let $\{x_j\}_j$ and $\{y_j\}_j$ be two decomposition bases of W with $\langle x_j\rangle_*^W = \langle y_j\rangle_*^W$ for all j. Then*

$$\mathrm{I}^{\{x_j\}_j}(g) = \mathrm{I}^{\{y_j\}_j}(g), \ \ \mathrm{R}_W^{\{x_j\}_j}(g) = \mathrm{R}_W^{\{y_j\}_j}(g), \quad \text{and} \quad \mathrm{Cut}_W^{\{x_j\}_j}(g) = \mathrm{Cut}_W^{\{y_j\}_j}(g).$$

As a consequence,

$$\mathrm{tp}^{\{x_j\}_j}(g + W) = \mathrm{tp}^{\{y_j\}_j}(g + W), \quad \text{and} \quad \mathrm{I}^{\{x_j\}_j}(g + W) = \mathrm{I}^{\{y_j\}_j}(g + W).$$

Proof: Write $S_j x_j = \langle x_j \rangle_*^W = \langle y_j \rangle_*^W = S'_j y_j$. Then $x_j = \gamma_j y_j$ with $\gamma_j \in \mathbb{Q}, \gamma_j \neq 0, S_j \gamma_j = S'_j$, and $S_j, S'_j \subseteq \mathbb{Q}$. Let $m_0 := \exp G/W$ and write $g = \frac{1}{m_0} \sum_{j=1}^n \alpha_j x_j = \frac{1}{m_0} \sum_{j=1}^n \beta_j y_j$ with $\alpha_j \in S_j$ and $\beta_j \in S'_j$. Then $\alpha_j x_j = \beta_j y_j$ and $\alpha_j \gamma_j = \beta_j$. As $\gamma_j \neq 0$ for all j we get

$$\mathrm{I}^{\{x_j\}_j}(g) = \{j \mid \alpha_j \neq 0\} = \{j \mid \alpha_j \gamma_j \neq 0\} = \{j \mid \beta_j \neq 0\} = \mathrm{I}^{\{y_j\}_j}(g)$$

and

$$\mathrm{R}_W^{\{x_j\}_j}(g) = \{j \mid \alpha_j \notin S_j\} = \{j \mid \alpha_j \gamma_j \notin S_j \gamma_j\} = \{j \mid \beta_j \notin S'_j\} = \mathrm{R}_W^{\{y_j\}_j}(g).$$

Then

$$\mathrm{Cut}_W^{\{x_j\}_j}(g) = \sum_{\mathrm{R}_W^{\{x_j\}_j}(g)} \alpha_j x_j = \sum_{\mathrm{R}_W^{\{y_j\}_j}(g)} \beta_j y_j = \mathrm{Cut}_W^{\{y_j\}_j}(g),$$

$$\mathrm{tp}^{\{x_j\}_j}(g + W) = \mathrm{tp}(\mathrm{Cut}_W^{\{x_j\}_j}(g)) = \mathrm{tp}(\mathrm{Cut}_W^{\{y_j\}_j}(g)) = \mathrm{tp}^{\{y_j\}_j}(g + W),$$

and

$$\mathrm{I}^{\{x_j\}_j}(g + W) = \mathrm{R}_W^{\{x_j\}_j}(g) = \mathrm{R}_W^{\{y_j\}_j}(g) = \mathrm{I}^{\{y_j\}_j}(g + W).$$

Q.E.D.

Corollary 3.10 *Let G be an almost completely decomposable group containing a completely decomposable group W of finite index. Let $g + W$ be a coset of G modulo W. Then coset type and coset carrier depend only on the decomposition of W into rank-1 summands.*

Proof: Once W is decomposed $W = \bigoplus_{j=1}^n W_j$ where $\mathrm{rk} W_j = 1$, the degrees of freedom for choosing a corresponding decomposition basis have been greatly reduced. Any two sets $\{x_j\}_j$ and $\{y_j\}_j$ which are supposed to correspond to the given rank-1 decomposition of W, must have $\langle x_j \rangle_*^W = W_j = \langle y_j \rangle_*^W$. Hence by the previous Lemma, carriers and cuts are not affected when $\{x_j\}_j$ is changed into $\{y_j\}_j$ and vice versa. Hence the decomposition of W already determines uniquely carriers and cuts. Q.E.D.

Definition 3.11 *Let G be an almost completely decomposable group containing a completely decomposable group W of finite index. Let $W = \bigoplus_{j=1}^n W_j$ be a decomposition of W into rank-1 summands $\{W_j\}_j$. Let $g + W$ be a coset of G modulo W. Then we define*

$$\mathrm{I}^{\{W_j\}_j}(g + W) = \mathrm{I}^{\{x_j\}_j}(g + W), \quad \mathrm{tp}^{\{W_j\}_j}(g + W) = \mathrm{tp}^{\{x_j\}_j}(g + W)$$

where $\{x_j\}_j$ is an arbitrary basis of W such that $\langle x_j \rangle_*^W = W_j$ for all j. Welldefinedness follows with the previous corollary.

Lemma 3.12 *Let G be an almost completely decomposable group containing a completely decomposable group W of finite index. Let $g + W$ be a coset of G modulo W. Then the qualified coset type* $\mathrm{tp}(g+W)$ *true/false does not depend on a given decomposition of W.*

Proof: It suffices to show that the qualified coset type does not change if one component of W is rotated, as it is known that every transition from one decomposition to another can be broken up into 1-component rotations.

Let $W = \bigoplus_{j=1}^n W_j$ where $W_j = S_j x_j$ for some $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$. We assume WLOG that $\{x_j\}_j$ is an adjusted basis. WLOG we can assume that $g = \mathrm{Cut}^{\{W_j\}_j}(g)$ and that we rotate component number 1. Let $\sigma = \mathrm{tp}(W_1)$ and let $\tau = \mathrm{tp}^{\{x_j\}_j}(g + W)$. Define $T_{>\sigma} := \{j \mid \mathrm{tp}(W_j) > \sigma\}$ and similarly $T_{\not\geq\tau}$, etc. Write $\tilde{W} = \bigoplus_{j=2}^n W_j$ such that $W = \tilde{W} \oplus W_1$. We are now examining all possible complements of $\tilde{W}$ in W. By Lemma 1.12 we know that all possible complements of $\tilde{W}$ in W are of the form $W_1(1+\varphi)$ where $\mathrm{Hom}(W_1, \tilde{W}) = \bigoplus_{j=2}^n \mathrm{Hom}(W_1, W_j) = \bigoplus_{j\neq 1, j\in T_{\geq\sigma}} \mathrm{Hom}(W_1, W_j) = \mathrm{Hom}(W_1, \bigoplus_{j\neq 1, j\in T_{\geq\sigma}} W_j)$.

Set $W'_j = \begin{cases} W_1(1+\varphi) & j = 1 \\ W_j & j \neq 1 \end{cases}$ and $x'_j = \begin{cases} x_1(1+\varphi) & j = 1 \\ x_j & j \neq 1 \end{cases}$. Then $W = \bigoplus_{j=1}^n W'_j$ and $W'_j = S_j x'_j$, that is the same rational groups $\{S_j\}_j$ are used.

Write $g = \frac{1}{m_0}\sum_{j=1}^n \alpha_j x_j$. As $g = \mathrm{Cut}^{\{W_j\}_j}(g)$ we know that either $\alpha_j = 0$ or $m_0^{-1}\alpha_j \notin S_j$ for any given j. Write $x_1\varphi = \sum_{j=1}^n \beta_j x_j$. Then $\beta_j = 0$ for all $j \in T_{\not\geq\sigma}$ and for $j = 1$. So $x'_1 = x_1 + \sum_{j=2}^n \beta_j x_j$. We can write g in terms of the $\{x'_j\}_j$. Then $g = \frac{1}{m_0}\left[\alpha_1(x_1 + \sum_{j=2}^n \beta_j x_j) + \sum_{j=2}^n (\alpha_j - \alpha_1\beta_j)x_j\right] = \frac{1}{m_0}\left[\alpha_1 x'_1 + \sum_{j=2}^n (\alpha_j - \alpha_1\beta_j)x'_j\right]$. From this equation it is clear that $1 \in R^{\{W_j\}_j}(g) \Leftrightarrow m_0^{-1}\alpha_1 \notin S_j \Leftrightarrow 1 \in R^{\{W'_j\}_j}(g)$ and for all $l \in T_{\not\geq\sigma}$ that $l \in R^{\{W_j\}_j}(g) \Leftrightarrow l \in R^{\{W'_j\}_j}(g)$. In particular, if $\alpha_1 = 0$ then $R^{\{W_j\}_j}(g) = R^{\{W'_j\}_j}(g)$ and the qualified coset type remains unchanged. So it remains to examine the case $\alpha_1 \neq 0$. This implies $\tau \leq \sigma$. Note that with the new decomposition g has still a component of type σ. The indices in which $R^{\{W_j\}_j}(g)$ and $R^{\{W'_j\}_j}(g)$ could differ, all lie in $T_{\geq\sigma}$. Hence the coset types, being the intersections over the types of components that appear in the real carrier, cannot change, as in both cases there is a component of type σ present, making changes in components with types $> \sigma$ irrelevant for the type of the coset.

So it remains only to check, whether the true/false status changes. If $\tau < \sigma$ then a component of type τ is either present with respect to both decompositions or not present, so there is no change. If $\tau = \sigma$ then the coset type is true with respect to both decompositions, as both decompositions have a nonzero component of type σ.

So after the rotation has been performed, the qualified coset type is still the same as before. Q.E.D.

Lemma 3.13 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let τ be a critical type and $g = \mathrm{Cut}(g)$.*

Then $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$ if and only if $\mathrm{tp}(g + W) = \tau$ true critical.

Proof: Let $\{x_j\}_j$ be a decomposition basis of W. Write $g = \sum_{j=1}^{n} \alpha_j x_j$. Then for all j either $\alpha_j = 0$ or $\alpha_j \notin S_j$, because $g = \mathrm{Cut}(g)$.

"If" Assume $\mathrm{tp}(g + W) = \tau$ true critical. Then $\alpha_j \neq 0$ for some j with $\mathrm{tp}(x_j) = \tau$. For this j we know $\alpha_j \notin S_j$, hence $g \notin W_\tau \oplus G^\#(\tau)$. With $\mathrm{tp}(g) = \mathrm{tp}(\mathrm{Cut}(g)) = \tau$ we get $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$.

"Only If" Let $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. Then $\mathrm{tp}(g) = \tau = \mathrm{tp}(g + W)$. As $g \notin W_\tau \oplus G^\#(\tau)$, we know that there must be some nonzero component of type τ. Hence the coset type is true critical. Q.E.D.

Lemma 3.14 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Then there is the set of cosets of G modulo W with their respective coset types.*

a) *W is regulating in G if and only if there exists no true critical coset type.*

b) *W is tight in G if and only if there exists no true critical coset type among the cosets of prime order.*

Part a) is basically derived from the carrier condition defined in [6] Lemma 3.2.

Proof:

a) "Only if" Assume there was a true critical coset type $\tau = \mathrm{tp}(g+W) = \mathrm{tp}(\mathrm{Cut}(g)) \in T_{cr}$. Then $\mathrm{Cut}(g) \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$ by Lemma 3.13 and hence $G(\tau) \neq W_\tau \oplus G^\#(\tau)$ and W is not regulating.

"If" Assume W was not regulating. Then there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^\#(\tau)$. Choose $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. As $g - \mathrm{Cut}(g) \in W(\tau) \subseteq W_\tau \oplus G^\#(\tau)$, we have $\mathrm{Cut}(g) =: g' \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. By Lemma 3.13 we know that $\mathrm{tp}(g' + W) = \tau$ true critical.

b) "Only if" Assume that here was a true critical coset type $\tau = \mathrm{tp}(g + W) = \mathrm{tp}(\mathrm{Cut}(g) + W) \in T_{cr}$ with $p(g + W) \subseteq W$ for some prime p. Then $\mathrm{Cut}(g) + W = g + W$ and hence $p(\mathrm{Cut}(g) + W) \subseteq W$. By the previous lemma $\mathrm{Cut}(g) \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. As $p\mathrm{Cut}(g) \in W$ we conclude that W is not tight with Lemma 2.1.

"If" Assume W was not tight. Then by Lemma 2.1 there exists a critical type τ and an element $g \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$ and a prime p with $pg \in W$. As $g - \mathrm{Cut}(g) \in W(\tau) \subseteq W_\tau \oplus G^\#(\tau)$ we have $g' := \mathrm{Cut}(g) \in G(\tau) \setminus W_\tau \oplus G^\#(\tau)$. Also $pg' \in W$. With the previous lemma we get $\mathrm{tp}(g' + W) = \tau$ and $p(g' + W) \subseteq W$.

Q.E.D.

Lemma 3.15 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let τ be a critical type. Then*

$$G(\tau) + W = \bigcup \{g + W \mid \mathrm{tp}(g + W) \geq \tau\},$$

$$G^\#(\tau) + W = \bigcup \{g + W \mid \mathrm{tp}(g + W) > \tau \vee \mathrm{tp}(g + W) = \tau \text{ false }\}.$$

Proof: For the first equation let $h \in G(\tau) + W$. Then h can be split $h = g + w$ with $g \in G(\tau)$ and $w \in W$. Then $\mathrm{tp}(g) \geq \tau$ and hence $\mathrm{tp}(g + W) \geq \tau$. So $h \in g + W$ is contained in $\{g + W \mid \mathrm{tp}(g + W) \geq \tau\}$. Conversely assume that $h \in \{g + W \mid \mathrm{tp}(g + W) \geq \tau\}$. Then there exists a coset $g + W$ such that $h \in g + W$ and $\mathrm{tp}(g + W) \geq \tau$. This implies that there exists some element $f \in g + W = h + W$ with $\mathrm{tp}(f) \geq \tau$. Hence $f \in G(\tau)$ and with $h \in f + W$ we get $h \in G(\tau) + W$.

For the second equation we need to assume a decomposition basis $\{x_j\}_j$ of W. Let $h \in G^\#(\tau) + W$. Then h can be split $h = g + w$ with $g \in G^\#(\tau)$ and $w \in W$. Then $\mathrm{I}_W^{\{x_j\}_j}(g) \subseteq T_{>\tau}$. Hence also $\mathrm{I}^{\{x_j\}_j}(g + W) \subseteq T_{>\tau}$. Hence either $\mathrm{tp}(g + W) > \tau$ or $\mathrm{tp}(g + w) = \tau$ false, as there is no component of type τ in the coset carrier of $g + W$. So $h \in g + W$ is contained in $\{g + W \mid \mathrm{tp}(g + W) > \tau \vee \mathrm{tp}(g + W) = \tau \text{ false }\}$.

Conversely assume that $h \in \{g+W \mid \mathrm{tp}(g+W) > \tau \vee \mathrm{tp}(g+W) = \tau \text{ false}\}$. Then there exists a coset $g+W$ such that $h \in g+W$ and either $\mathrm{tp}(g+W) > \tau$ or $\mathrm{tp}(g+W) = \tau$ false. This implies that there exists some element $f \in g+W = h+W$ with either $\mathrm{tp}(f) > \tau$ or $\mathrm{tp}(f) = \tau$ and $\mathrm{I}_W^{\{x_j\}_j}(f) \subseteq T_{>\tau}$. Hence $f \in G^{\#}(\tau)$ and with $h \in f+W$ we get $h \in G^{\#}(\tau)+W$. Q.E.D.

Proof: (Of Example 2.6) Let $\tau_i := \mathrm{tp}(x_i)$. Note that $G/W = \langle g_1+W\rangle \oplus \langle g_2+W\rangle$. So we can easily determine the cosets of prime order in G/W. These are exactly $\lambda_1 p^2 g_1 + \lambda_2 p^2 g_2 + W$ with $\lambda_i \in \mathbb{Z}$ and not both multiples of p. Then $\lambda_1 p^2 g_1 + \lambda_2 p^2 g_2 + W = \lambda_1 \frac{1}{p}(x_2+x_3) + \lambda_2 \frac{1}{p}(x_1+x_5) + W$. Obviously there is no true critical coset type among these cosets, so by Lemma 3.14 we find that W is tight in G. But as $pg_1 + W = \frac{1}{p^2}(px_1+x_2+x_3)+W$ has coset type τ true, we conclude with Lemma 3.14 that W is not regulating.

Define $V = \mathbb{Q}^{(q)}px_1 \oplus \mathbb{Q}^{(q,r)}x_2 \oplus \mathbb{Q}^{(q,s)}x_3 \oplus \mathbb{Q}^{(t)}x_4 \oplus \mathbb{Q}^{(u)}x_5$, and $g_3 = x_1$. Then $G = \langle V, g_1, g_2, g_3\rangle$. We rewrite $V = \mathbb{Q}^{(q)}(px_1+x_2+x_3) \oplus \mathbb{Q}^{(q,r)}x_2 \oplus \mathbb{Q}^{(q,s)}x_3 \oplus \mathbb{Q}^{(t)}x_4 \oplus \mathbb{Q}^{(u)}x_5$,. Setting $x_1' := px_1+x_2+x_3$ we get $x_1 = \frac{1}{p}(x_1' - x_2 - x_3)$ and

$$V = \mathbb{Q}^{(q)}x_1' \oplus \mathbb{Q}^{(q,r)}x_2 \oplus \mathbb{Q}^{(q,s)}x_3 \oplus \mathbb{Q}^{(t)}x_4 \oplus \mathbb{Q}^{(u)}x_5,$$

$$g_1 = \frac{1}{p^3}(x_1'+p^2x_4), \quad g_2 = \frac{1}{p^3}\left(\frac{1}{p}(x_1'-x_2-x_3)+x_5\right) = \frac{1}{p^4}(x_1'-x_2-x_3+px_5),$$

$$g_3 = \frac{1}{p}(x_1'-x_2-x_3), \quad G = \langle V, g_1, g_2, g_3\rangle.$$

Note that $\frac{1}{p^2}x_1' = pg_1 - x_4 \in V$. So define $x_1'' = \frac{1}{p^2}x_1'$ and define

$$V' = \mathbb{Q}^{(q)}x_1'' \oplus \mathbb{Q}^{(q,r)}x_2 \oplus \mathbb{Q}^{(q,s)}x_3 \oplus \mathbb{Q}^{(t)}x_4 \oplus \mathbb{Q}^{(u)}x_5.$$

As $V' \supseteq V$ we know that $G = \langle V', g_1, g_2, g_3\rangle$. Rewriting the generators we obtain

$$g_1 = \frac{1}{p^3}(p^2x_1''+p^2x_4) = \frac{1}{p}(x_1''+x_4), \quad g_2 = \frac{1}{p^4}(p^2x_1''-x_2-x_3+px_5),$$

$$g_3 = \frac{1}{p}(p^2x_1'-x_2-x_3) = \frac{1}{p}(-x_2-x_3)+px_1'', \quad G = \langle V, g_1, g_2, g_3\rangle.$$

Note that $p^3g_2 + W = g_3 + W$ and hence $G = \langle V', g_1, g_2\rangle$. We now have to check whether V' is regulating in G. Note that $G/V' = \langle g_1+V'\rangle \oplus \langle g_2+V'\rangle$.

It is clear that no linear combination of g_1 and g_2 will yield a coset type τ_2 or τ_3, as they cannot be seperated - a coset carrier contains either both or none of them.

There is no coset with a carrier that is equal to $\{4\}$, because in order to obtain such a coset carrier one would have to take a nontrivial multiple of g_1 and remove the x_1 component by adding a suitable multiple of g_2. But the price for removing the x_1 component would be components of type τ_2, τ_3 and τ_5 and hence there is no way to obtain a coset carrier equal to $\{4\}$. The same argumentation works for coset type τ_5.

Now it remains to check, whether there exists a coset with type τ_1 true, that is a coset with carrier either $\{1\}, \{1,2\}, \{1,3\}$, or $\{1,2,3\}$. Note that carrier $\{2,3\}$ yields coset type τ_1 false. If a linear combination of g_1 and g_2 contains a nontrivial contribution of g_1, then we have necessarily component 4 in the carrier. So only multiples of g_2 could have coset type τ_1. The problem with g_2 is the component of type τ_5. We have to multiply g_2 with p^3 to get rid of the x_5 component. But $p^3 g_2 + W = \frac{1}{p}(x_2 + x_3)$ which corresponds to coset type τ_1 false. Hence V' is regulating in G and $\exp(G/V') = p^4$.

It remains to show that all regulating subgroups have the same exponent. For this purpose we will determine the regulator and see that there are only p different regulating subgroups that have all the same exponent.

We have seen in the preceeding paragraph that the only critical coset type was $\mathrm{tp}(p^3 g_2 + W) = \tau$ false and that of its nontrivial multiples. So by Lemma 3.15 we have $G^\#(\tau_i) = W^\#(\tau_i)$ for all $i = 2, \dots, 5$. Hence the Burkhardt invariants for these types are equal to 1.
Then $G^\#(\tau_1) = \langle W^\#(\tau_1), p^3 g_2\rangle$ and $\beta_1 := \exp \frac{G^\#(\tau_1}{W^\#(\tau)} = p$. Hence we obtain the regulator $R(G) = V' = \mathbb{Q}^{(q)} p x_1'' \oplus \mathbb{Q}^{(q,r)} x_2 \oplus \mathbb{Q}^{(q,s)} x_3 \oplus \mathbb{Q}^{(t)} x_4 \oplus \mathbb{Q}^{(u)} x_5$. Note that all regulating subgroups are of the form

$$R_k = \mathbb{Q}^{(q)} \left(x_1'' + k\frac{1}{p}(x_2 + x_3) \right) \oplus \mathbb{Q}^{(q,r)} x_2 \oplus \mathbb{Q}^{(q,s)} x_3 \oplus \mathbb{Q}^{(t)} x_4 \oplus \mathbb{Q}^{(u)} x_5$$

with respective generators

$$g_{1k} = \frac{1}{p^2}(p x_1'' + k x_2 + k x_3 + p x_4), \quad g_{2k} = \frac{1}{p^4}(p^2 x_1'' + (pk-1)x_2 + (pk-1)x_3 + p x_5).$$

Note that g_{2k} has order p^4 for all k as $\mathrm{ht}_p(kp-1) = 0$ for all k. It does not matter that g_{1k} and g_{2k} are no longer linearly independent for $k \neq 0$.

The same argumentation that was used to show that $V' = R_0$ is regulating, works for R_k in general. So all R_k are indeed regulating and have exponent p^4 in G.

Q.E.D.

3.3 Reducible Cosets

Definition 3.16 *A coset is called* **reducible** *if the coset and all its nontrivial multiples have the same true critical coset type.*

Lemma 3.17 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let $g + W$ be a reducible coset. Then*

a) all cosets $kg + W$ where $kg \notin W$ and $k \in \mathbb{Z}$ are reducible with the same coset type, and

b) if K is completely decomposable with $W \subseteq K \subseteq \langle W, g \rangle$, then g+K is reducible with the same coset type.

Proof:

a) The multiples of $kg + W$ are a subset of the multiples of $g + W$, so the "reducible" property carries over.

b) Define $l := |\langle g + W \rangle|$. As $\langle W, g \rangle / W$ is cyclic, we can write $K = \langle W, mg \rangle$ for some $m \in \mathbb{Z}$ with $m|l$. Set $n := l/m$ and write

$$K = \bigcup_{i=0}^{n-1} (img + W).$$

$$rg+K = rg+\bigcup_{i=0}^{n-1}(img+W) = \bigcup_{i=0}^{n-1}(rg+img+W) = \bigcup_{i=0}^{n-1}((r+im)g+W).$$

For all r with $rg \notin W$ we have to show that $\mathrm{tp}(rg + K) = \mathrm{tp}(g + W) =: \tau$ true critical. Note that $rg \in K$ is equivalent to $m|r$. So assuming $rg \notin K$ we know that $m \not| r$. Hence $m \not| r + im$ for all i. As $m|l$ we get $l \not| r + im$ for all i. So $(r + im)g + W \neq W$ for all i. Hence $\mathrm{tp}((r + im)g + W) = \tau$ true critical for all i, as $(r + im)g + W$ is not a 'trivial' multiple in the sense of Definition 3.16. So we get $\mathrm{tp}(rg + K) = \max_i\{\mathrm{tp}((r + im)g + W)\} = \tau$ true critical.

Q.E.D.

Lemma 3.18 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index.*

If $g+W$ is reducible then there exists a decomposition of W such that $\mathrm{R}(g) = \mathrm{I}(g+W)$ *contains only one element.*

Proof: Let $W = \bigoplus_{j=1}^n W_j$ with $\mathrm{rk} W_j = 1$. Let $m_1 := |g+W|$ and let $\{x_j\}_j$ be an adjusted m_1-basis of W such that $W_j = S_j x_j$ for suitable $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$. Let $\tau = \mathrm{tp}(g+W)$.

It is possible that $m_1 g = \sum_{j=1}^n g_j$ with $g_j \in W_j$ has more than one nonzero component of type τ. We will first show that this case can be dealt with by redecomposing $W_\tau = \bigoplus_{j \in T_{=\tau}} W_j$ into different rank-1 summands. Set $\tilde{W} = \bigoplus_{j \notin T_{=\tau}} W_j$. Then $W = W_\tau \oplus \tilde{W}$ and we can write $m_1 g = g_\tau + \tilde{g}_\tau$ with $g_\tau \in W_\tau$ and $\tilde{g}_\tau \in \tilde{W}$. As W_τ is a τ-homogeneous group, we know by [3] 86.x that $\langle g_\tau \rangle_*^{W_\tau}$ is a direct summand of W_τ. The complement of $\langle g_\tau \rangle_*^{W_\tau}$ in W_τ is again completely decomposable, so we have a new decomposition of W_τ, and subsequently of W, into rank-1 components such that g has only one component of type τ, that is $g \in \langle g_\tau \rangle_*^{W_\tau}$. So WLOG we can assume that $\mathrm{I}^{\{x_j\}_j}(g) \cap T_{=\tau} = \{1\}$.

By Lemma 3.6 we can WLOG assume that $g = \frac{1}{m_1} \sum_{j=1}^n \alpha_j x_j$ with $\alpha_j \in \mathbb{Z}$. As $g+W$ is reducible we know for all $m_1 \not| l$ that $1 \in \mathrm{I}^{\{x_j\}_j}(lg+W)$. In particular this implies $\gcd(\alpha_1, m_1) = 1$, as otherwise $\mathrm{I}^{\{x_j\}_j}(\frac{m_1}{\gcd(\alpha_1, m_1)} g + W) \not\ni 1$, contradicting $1 \in \mathrm{I}^{\{x_j\}_j}(lg+W)$ for all $m_1 \not| l$.

Let $h := \mathrm{Cut}_W^{\{x_j\}_j}(g)$. Then we can write $h = \frac{1}{m_1} \sum_{j=1}^n \beta_j x_j$. As $\mathrm{tp}(g+W) = \mathrm{tp}(h+W) = \tau$ and $\mathrm{I}^{\{x_j\}_j}(g+W) = \mathrm{I}^{\{x_j\}_j}(h)$, we know that $\beta_j = 0$ for all j with $\mathrm{tp}(x_j) \not\geq \tau$. Note also that $\alpha_1 = \beta_1$ and hence $\gcd(\beta_1, m_1) = 1$.

Define $\bar{x} := \sum_{j=2}^n \beta_j x_j$ Then $\mathrm{tp}(\bar{x}) \geq \tau$ and by the adjusted basis property we have $\chi^W(\bar{x}) \geq \bigwedge_{\beta_j \neq 0, j \neq 1} \chi^W(x_j) \geq \chi^W(x_1)$. Note also that $\bar{x} \in W^{\#}(\tau) \subseteq W$. We can write $h = \frac{1}{m_1}(\beta_1 x_1 + \bar{x})$.

The problem at the moment is that it could happen that $\chi^W(\bar{x}) \not\geq \chi^W(\beta_1 x_1)$. Let γ be an integer such that $\gamma \beta_1 \equiv 1 (m_1)$. Define $\delta := \frac{\gamma\beta_1 - 1}{m_1}$. Then $\delta \in \mathbb{Z}$ and $\beta_1 | \delta m_1 + 1$. Let $h' := h + \delta \bar{x} \in h + W = g + W$. We get

$$h' = \frac{1}{m_1}(\beta_1 x_1 + \bar{x} + \delta \bar{x} m_1) = \frac{1}{m_1} \beta_1 (x_1 + \gamma \bar{x}).$$

As $\chi^W(\gamma\bar{x}) \geq \chi^W(x_1)$ there exists a homomorphism $\varphi \in \mathrm{Hom}(W_1, W^{\#}(\tau))$ with $x_1\varphi = \gamma\bar{x}$. Set $W = W_1(1+\varphi) \oplus \bigoplus_{j=2}^n W_j$. Then $h' = \frac{1}{m_1}\beta_1 x_1(1+\varphi)$ has only one component with respect to the new decomposition of W. Q.E.D.

Lemma 3.19 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index.*

If $g+W$ is reducible then $\langle W, g\rangle$ is completely decomposable.

Proof: As $g+W$ is reducible there exists a decomposition of W such that $\mathrm{R}(g)$ contains only one element, by Lemma 3.18. So we have WLOG $W = \bigoplus_{j=1}^n W_j$ and $\mathrm{R}^{\{W_j\}_j}(g) = \{1\}$. Let $h := \mathrm{Cut}^{\{W_j\}_j}(g)$. Then $h \in \mathbb{Q}W_1$ and $\langle W_1, h\rangle$ is a rational group. Hence $\langle W_1, h\rangle \oplus \bigoplus_{j=2}^n W_j = \langle W, h\rangle = \langle W, g\rangle$ is completely decomposable. Q.E.D.

Proof: (Alternative version) Assume $g+W$ is a reducible coset. Let $l := |\langle g+W\rangle|$ and let $1 = l_0 < l_1 < \ldots < l_n = l$ be a chain of integers such that l_i/l_{i-1} is prime. Define $K_i := \langle W, l_i g\rangle$. Then $W = K_n \subseteq K_{n-1} \subseteq \ldots \subseteq K_0 = \langle W, g\rangle$ where $|K_i/K_{i+1}|$ is prime. We will show by induction that K_0 is completely decomposable.

Obviously K_n is completely decomposable. Now assume that K_i was completely decomposable. Then by the previous lemma $g+K_i$ is reducible and $l_{i-1}g + K_i$ has the prime order l_i/l_{i-1} and is reducible. So modulo K_i we have a coset with true critical coset type of prime order. Let $\mathrm{tp}(l_{i-1}g+W) = \tau$ true critical. Then $g_{i-1} := \mathrm{Cut}(l_{i-1}g)$ has type τ and $\mathrm{tp}(g_{i-1}+W) = \tau$ true critical. Hence $g_{i-1} \in G(\tau) \setminus W_\tau \oplus G^{\#}(\tau)$. Hence $K_{i-1} = \langle K_i, g_{i-1}\rangle$ is completely decomposable. Induction finally yields K_0 is completely decomposable. Q.E.D.

Remark 3.20 *The converse of Lemma 3.19 is not true, as the following example shows: Let $W = \mathbb{Q}^{(2)}a \oplus \mathbb{Q}^{(3)}b$ and $g = \frac{1}{5}a + \frac{1}{7}b$. Note that $\mathrm{Cut}_W^{\{a,b\}}(g) = g$ and $\mathrm{tp}(g+W) = \mathrm{tp}(g) = \mathrm{tp}(\mathbb{Z})$, which must be a false coset type. Hence $g+W$ is not reducible. But $G = \langle W, g\rangle = \langle W, 5g, 7g\rangle = \langle W, a+\frac{5}{7}b, b+\frac{7}{5}a\rangle = \langle W, \frac{1}{7}b, \frac{1}{5}a\rangle = \mathbb{Q}^{(2)}\frac{1}{5}a \oplus \mathbb{Q}^{(3)}\frac{1}{7}b$ is completely decomposable.*

Lemma 3.21 *Let $U \subseteq V$ be two completely decomposable groups with $|V : U| = p$ prime. Then there exists a decomposition $U = \bigoplus_{j=1}^n U_j$ with $\mathrm{rk}U_j = 1$ such that $V = \frac{1}{p}U_1 \oplus \bigoplus_{j=1}^n U_j$. In particular, every nontrivial coset of V/U is reducible.*

Proof: As U is not tight in V, by [2] Proposition 2.7 (2) there exists a rank-1 summand of U that is not pure in V. Call this summand U_1. The complement of U_1 in U is completely decomposable by [3] 86.7. So we have $U = \bigoplus_{j=1}^n U_i$ and $W := \langle U_1\rangle_*^V \oplus \bigoplus_{j=2}^n U_j$. Then $U \subseteq W \subseteq V$ with $U \neq W$. As $|V/U|$ is prime, this implies $W = V$. Note that $|V/U| = p$ implies that $\left|\frac{\langle U_1\rangle_*^V}{U_1}\right|$ divides p. As $\langle U_1\rangle_*^V \neq U_1$ we find that $\langle U_1\rangle_*^V = \frac{1}{p}U_1$. As $\mathrm{I}^{\{U_j\}_j}(g) = \{1\} = \mathrm{R}^{\{U_j\}_j}(g)$ we have that $\mathrm{tp}(g+U) = \mathrm{tp}(U_1)$ true. As the other nontrivial cosets of V/U are generators, too, they have the same qualified coset type by Corollary 3.5. Hence every nontrivial coset is reducible. Q.E.D.

Lemma 3.22 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let $g_1 + W$ and $g_2 + W$ be reducible cosets of the same type τ that have relatively prime orders modulo W.*

Then there exists a reducible coset $h + W$ of type τ such that $\langle g_1, g_2, W\rangle = \langle h, W\rangle$.

Proof: Let $o_i := |g_i + W|$ and $W = \bigoplus_{j=1}^n S_j x_j$ with $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$. WLOG we assume that $g_i = \mathrm{Cut}_W^{\{x_j\}_j}(g_i)$. Write $o_i g_i = \bar{g}_i = \sum_{j=1}^n \alpha_{ij} x_j$ and $o_i g_{i\tau} = \sum_{j\in T_{=\tau}} \alpha_{ij} x_j$. Note that $\mathrm{ht}_p^W(\bar{g}_i) = 0$ for all primes $p|o_i$, because otherwise the order o_i would not be correct.

As g_i+W is reducible of type τ, we know that every nontrivial multiple must have a nonzero component of type τ. That is $lg_i \notin W$ implies $lg_{i\tau} \notin W$ for all $l \in \mathbb{Z}$. As W is a direct sum, we get that $lg_i \in W$ implies $lg_{i\tau} \in W$ for all $l \in \mathbb{Z}$. Hence $lg_i \in W$ if and only if $lg_{i\tau} \in W$. Note that we have $\gcd(o_1, o_2) = 1$.

Now set $h := g_1 + g_2$. We claim that $h + W$ is reducible of type τ and that $\langle W, h\rangle = \langle W, g_1, g_2\rangle$.

a) For the latter see $\langle W, h\rangle = \langle W, o_1 h, o_2 h\rangle = \langle W, o_1 g_1 + o_1 g_2, o_2 g_1 + o_2 g_2\rangle = \langle W, o_1 g_2, o_2 g_1\rangle = \langle W, \frac{o_1}{o_2}\bar{g}_2, \frac{o_2}{o_1}\bar{g}_1\rangle = \langle W, \frac{1}{o_2}\bar{g}_2, \frac{1}{o_1}\bar{g}_1\rangle = \langle W, g_2, g_1\rangle$. For the equations note that $o_1 g_1 \in W$ and that for reduced fractions $\frac{a}{b} \in S \subseteq \mathbb{Q}$ implies $\frac{1}{b} \in S$.

b) Obviously $\mathrm{tp}^G(h) \geq \tau$ as $\mathrm{tp}(g_i + W) = \tau$ and g_1, g_2 are cut. So it remains to show that all nontrivial multiples of h have a τ-component that does not lie in W. It is easily seen that $|h + W| = o_1 o_2$ as $\langle g_1 + W\rangle \cap \langle g_2 + W\rangle = 0 + W$. We have to show that $l(g_{1\tau} + g_{2\tau}) \in$

W only if $lh \in W$. So assume that $l(g_{1\tau} + g_{2\tau}) \in W$. Note that $\langle g_{1\tau} + W\rangle \cap \langle g_{2\tau} + W\rangle = 0 + W$ as the orders are relatively prime. Hence there are no "interactions" of $g_{i\tau}$ and $g_{2\tau}$ and we can say: $l(g_{1\tau} + g_{2\tau}) \in W$ implies $lg_{1\tau} \in W \wedge lg_{2\tau} \in W$ which implies $lg_1 \in W$ and $lg_2 \in W$ which implies $lh = l(g_1 + g_2) \in W$ as desired.

Q.E.D.

Lemma 3.23 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index.*

If $g+W$ and $h+W$ are reducible with $\mathrm{tp}(g+W) < \mathrm{tp}(h+W)$, then $g+\langle W, h\rangle$ is reducible of the same order and type as $g + W$.

Proof: By Lemma 3.18 we can choose a decomposition basis of W such that $\mathrm{I}(h + W) = \{1\}$. Write $\tau = \mathrm{tp}(g + W)$. WLOG we may assume that $g = \mathrm{Cut}(g)$. As $g + W$ is reducible, we know that every nontrivial multiple of $g+W$ has a component of type τ. The transition from $g+W$ to $g + \langle W, h\rangle$ will affect only component number 1, as basis element number 1 is the only basis element with a new coefficient set. So components of type τ are not affected and hence every nontrivial multiple of $g + \langle W, h\rangle$ has a component of type τ. So $g + \langle W, h\rangle$ is reducible of type τ. Q.E.D.

3.4 Order Carrier

Definition 3.24 *A coset is called* **primary reducible** *if it is reducible and has prime power order.*

Definition 3.25 *A set R of primary reducible cosets is called* **simultaneously reducible** *if $g + \langle W, h \mid h + W \in S\rangle$ is reducible for all $g + W \in R$ and all $S \subseteq R$.*

Definition 3.26 *Assume that a decomposition basis $\{x_j\}_j$ of W is given. Let $g + W$ be a primary reducible coset of type τ and order p^k.*

Then the **order carrier** *of $g + W$ is defined as*

$$\mathrm{J}^{\{x_j\}_j}(g + W) = \mathrm{I}^{\{x_j\}_j}(p^{k-1}g + W) \cap T_{=\tau}.$$

Lemma 3.27 *Assume that a decomposition basis $\{x_j\}_j$ of W is given with $W = \bigoplus_{j=1}^n S_j x_j$ for suitable $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$. Let $g + W$ be a primary reducible coset of type τ with order p^k. Assume that $g = \frac{1}{p^k}\sum_{j=1}^n \alpha_j x_j$ with $\alpha_j \in S_j$.*

Then

$$J^{\{x_j\}_j}(g+W) = \{j | \mathrm{tp}(x_j) = \tau, \mathrm{ht}_p^{S_j}(\alpha_j) = 0\}.$$

Proof: "$\subseteq$". Let $l \in \mathrm{J}^{\{x_j\}_j}(g+W) = \mathrm{I}^{\{x_j\}_j}(p^{k-1}g+W) \cap T_{=\tau}$. It is obvious that $\mathrm{tp}(x_l) = \tau$ then. We also know that $p^{k-1}\frac{1}{p^k}\alpha_l x_l \notin W$. Hence $\mathrm{ht}_p^W(\alpha_l x_l) = 0 = \mathrm{ht}_p^{S_l x_l}(\alpha_l x_l) = \mathrm{ht}_p^{S_l}(\alpha_l)$.

"$\supseteq$" Let $l \in \{j | \mathrm{tp}(x_j) = \tau, \mathrm{ht}_p^{S_j}(\alpha_j) = 0\}$. Then $\mathrm{tp}(x_l) = \tau$ and $l \in T_{=\tau}$. Note that $\mathrm{ht}_p^{S_l}(\alpha_l) = 0 = \mathrm{ht}_p^{S_l x_l}(\alpha_l x_l) = \mathrm{ht}_p^W(\alpha_l x_l)$ and hence $p^{k-1}\frac{1}{p^k}\alpha_l x_l \notin W$. So $l \in \mathrm{I}^{\{x_j\}_j}(p^{k-1}g + W)$. Q.E.D.

Lemma 3.28 *The order carrier of a primary reducible coset is never the empty set.*

Proof: As all nontrivial multiples of a primary reducible coset of type τ must have a coset type τ true critical, we know that all coset carriers of nontrivial multiples must have a component of type τ. If the order carrier was the empty set, then there is one multiple without a component of type τ, a contradiction. Q.E.D.

Lemma 3.29 *Let G be an almost completely decomposable group containing a completely decomposable subgroup W of finite index. Let $m_0 = \exp G/W$ and $W = \bigoplus_{j=1}^n S_j x_j$ with adjusted m_0-basis $\{x_j\}_j$. Let $g \in G$ and let $g + W$ be a primary reducible coset of order p^k.*

If $l \in \mathrm{J}^{\{x_j\}_j}(g+W)$ then there exists an adjusted basis $\{x'_j\}_j$ of W such that

a) $\mathrm{I}^{\{x'_j\}_j}(g+W) = \{l\}$,

b) *$\{x'_j\}_j$ is an adjusted m_0-basis with $W = \bigoplus_{j=1}^k S_j x'_j$,*

c) *$x'_j = x_j$ for all $j \neq l$,*

d) *$x'_l = x_l(1+\varphi)$ with $\varphi \in \mathrm{Hom}(S_l x_l, \bigoplus_{j \neq l} S_j x_j)$*

e) *$\mathrm{ht}_q^W(x_l \varphi) \geq 0$ for all $q | m_0$ and $q \neq p$,*

f) *in particular for every primary reducible coset* $h + W$ *of order* q^m*,* $q \neq p$*, we have* $\mathrm{J}^{\{x_j\}_j}(h+W) = \mathrm{J}^{\{x'_j\}_j}(h+W)$.

Proof: It is obvious that $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$ for all j. Assume WLOG that $g = \mathrm{Cut}_W^{\{x_j\}_j}(g)$. By Lemma 3.6 we know that there exists a representative $h \in g + W$ such that $h = \frac{1}{p^k}\sum_{j=1}^n \alpha_j x_j$ with $\alpha_j \in \mathbb{Z}$ and $\mathrm{I}^{\{x_j\}_j}(h) \subseteq \mathrm{I}^{\{x_j\}_j}(g)$. As the carrier of $g = \mathrm{Cut}_W^{\{x_j\}_j}(g)$ must be minimal in $g + W$ we get $\mathrm{I}^{\{x_j\}_j}(h) = \mathrm{I}^{\{x_j\}_j}(g)$ and hence $h = \mathrm{Cut}_W^{\{x_j\}_j}(h)$.

Let $\tau = \mathrm{tp}(g+W) = \mathrm{tp}(h+W)$. As $g + W$ is primary reducible, the order carrier is not the empty set by Lemma 3.28. So let $l \in \mathrm{J}^{\{x_j\}_j}(h + W) = \mathrm{J}^{\{x_j\}_j}(g+W)$. Using Lemma 3.27 we know that $\mathrm{tp}(x_l) = \tau$ and $\mathrm{ht}_p^{S_l}(\alpha_l) = 0$. As there exists an element with finite p-height in S_l we can conclude that $pS_l \neq S_l$. Together with the p-basis property we get $\frac{1}{p} \notin S_l$. Hence $\mathrm{ht}_p^W(\alpha_l x_l) = 0$.

Let $h_l := \alpha_l x_l$ and $\bar{h}_l := \sum_{j \neq l} \alpha_j x_j$. Then $p^k h = h_l + \bar{h}_l$. Let $t = \prod\{q \mid q | m_0, p \neq q\}$. Then $\gcd(t,p) = 1$. As $\gcd(\alpha_l, p) = 1$ there exists an integer γ such that $\gamma\alpha_l \equiv 1 (p^k)$ and $t|\gamma$. Define $\delta := \frac{\gamma\alpha_l - 1}{p^k}$. Then $\delta \in \mathbb{Z}$ and $\alpha_l | \delta p^k + 1$. Set $f := h + \delta\bar{h}_l \in h + W = g + W$. We get $f = \frac{1}{p^k}(\alpha_l x_l + \bar{h}_l + \delta p^k \bar{h}_l) = \frac{\alpha_l}{p^k}(x_l + \gamma\bar{h}_l)$.

As $\{x_j\}_j$ is an adjusted basis and as $\alpha_j = 0$ for all $j \notin T_{\geq\tau}$ we get $\chi^W(\lambda\bar{h}_l) \geq \chi^W(\bar{h}_l) = \chi^W(\sum_{j \in T_{\geq\tau}} \alpha_j x_j) \geq \bigwedge_{j \in T_{\geq\tau}} \chi^W(x_j) \geq \chi^W(x_l)$. Hence there exists a homomorphism $\varphi \in \mathrm{Hom}(S_l x_l, \bigoplus_{j \neq l} S_j x_j)$ with $x_l\varphi = \gamma\bar{h}_l$. Set $x'_j = \begin{cases} x_j & j \neq l \\ x_l(1+\varphi) & j = l \end{cases}$. Then $f = \frac{1}{p^k}(\alpha_l x'_l)$ and $W = \bigoplus_{j=1}^n S_j x'_j$ and $\mathrm{I}^{\{x_j\}_j}(f) = \{l\} = \mathrm{I}^{\{x_j\}_j}(g+W)$, showing a).

For e) note that $x_l\varphi = \gamma\bar{h}_l$ and hence $\mathrm{ht}_p^W(x_l\varphi) \geq \mathrm{ht}_p^{\mathbb{Z}}(\gamma)$. Then $t|\gamma$ yields the claim.

For claim f) let $h + W$ be a primary reducible coset of order q^m with $q \neq p$. Then $q|t$ and hence $q|\gamma$. Write $h = \frac{1}{q^m}\sum_{j=1}^n \beta_j x_j$ with $\beta_j \in S_j$. By Lemma 3.6 we may WLOG assume that $\beta_j \in \mathbb{Z}$. Set $x_l\varphi = \sum_{j=1}^n \mu_j x_j$ with $\mu_j \in S_j$. Then $\mu_l = 0$. As $x_l\varphi = \gamma\bar{h}_l$ with $\bar{h}_l \in W$ we get that $\gamma^{-1}\mu_j \in S_j$ for all $j \neq l$. Hence $\mathrm{ht}_q^{S_j}(\mu_j) \geq 1$ for all $j \neq l$.

Writing h in terms of the new basis $\{x'_j\}_j$ we get, using $x_l = x'_l - x_l\varphi$, that

$$h = \frac{1}{q^m}\left(\beta_l x'_l - \beta_l x_l\varphi + \sum_{j=1}^k \beta_j x'_j\right) = \frac{1}{q^m}\left(\beta_l x'_l + \sum_{j=1}^k [\beta_j - \beta_l\mu_j] x'_j\right).$$

We know that $\beta_j - \beta_l\mu_j \in S_j$ for all $j \neq l$. As $\mathrm{ht}_p^{S_j}(\mu_j) \geq 1$ and $\beta_l \in \mathbb{Z}$ we obtain $\mathrm{ht}_p^{S_j}(\beta_l\mu_j) \geq 1$ for all $j \neq l$. Note that $\mathrm{ht}_p^X(b) > \mathrm{ht}_p^X(a)$ implies that $\mathrm{ht}_p^X(a+b) = \mathrm{ht}_p^X(a)$ for arbitrary $a, b \in X$. With this in mind we get

$$\begin{array}{lll} & \lambda_\in \mathrm{J}^{\{x_j\}_j}(h+W) & \\ \Leftrightarrow & \mathrm{tp}(x_\lambda) = \mathrm{tp}(h+W) \quad \wedge & \mathrm{ht}_p^{S_\lambda}(\beta_\lambda) = 0 \\ \Leftrightarrow & \mathrm{tp}(x'_\lambda) = \mathrm{tp}(h+W) \quad \wedge & \mathrm{ht}_p^{S_\lambda}(\beta_\lambda - \beta_l\mu_\lambda) = 0 \\ \Leftrightarrow & \lambda \in \mathrm{J}^{\{x'_j\}_j}(h+W) & \end{array}$$

Q.E.D.

Lemma 3.30 *Let $W = \bigoplus_{j=1}^n S_jx_j$ be a completely decomposable group with adjusted p-basis $\{x_j\}_j$. Let $g+W$ be a primary reducible coset of order p^k.*

If there exists an element $\bar{x}_l \in W$ such that $W = \langle \bar{x}_l\rangle_*^W \oplus \bigoplus_{j\neq l} S_jx_j$ and $\mathrm{Cut}_W^{\{x_j\}_j}(g) \in \mathbb{Q}\bar{x}_l$, then $l \in \mathrm{J}^{\{x_j\}_j}(g+W)$.

Proof: As $\langle \bar{x}_l\rangle_*^W$ and S_lx_l have the same type, they are isomorphic and hence $\langle \bar{x}_l\rangle_*^W = S_lx'_l$ for some $x'_l \in \langle \bar{x}_l\rangle_*^W$. Let $\tau = \mathrm{tp}(g+W) = \mathrm{tp}(x_l) = \mathrm{tp}(x'_l)$. Set $\tilde{W} = \bigoplus_{j\neq l} S_jx_j$. So both S_jx_j and $S_jx'_j$ are complements of $\tilde{W}$ in W. By Lemma 1.12 we know that there exists an homomorphism $\varphi \in \mathrm{Hom}(S_lx'_l, \tilde{W})$ such that $S_jx_l = S_lx'_l(1+\varphi)$. As $\mathrm{Cut}_W^{\{x_j\}_j}(g) \in \mathbb{Q}x_l$ we know that there exists an $h \in g+W$ such that $p^kh \in S_lx'_l$ and $|h + S_lx'_l| = p^k$. So $h = \frac{1}{p^k}\alpha_lx'_l$ for some $\alpha_l \in S_l$. As $\{x_j\}_j$ is a p-basis, we know that either $pS_l = S_l$ or $\frac{1}{p} \notin S_l$. But as $h \notin W$ we can eliminate the case $pS_l = S_l$. Hence $\mathrm{ht}_p^W(x'_l) = 0$. As $\mathrm{ht}_p^W(\alpha_lx'_l) = 0$ for order reasons, we find $\mathrm{ht}_p^{S_l}(\alpha_l) = 0$. Now we write h in terms of $\{x_j\}_j$ solely. $h = \frac{1}{p^k}\alpha_lx'_l = \frac{\alpha_l}{p^k}(x_l - x'_l\varphi) = \frac{\alpha_l}{p^k}(x_l - \bar{x})$ where $\bar{x} \in \tilde{W}$ with $\mathrm{tp}(\bar{x}) \geq \tau$ and $\chi^W(\bar{x}) \geq \chi^W(x_l)$. So $\mathrm{tp}(x_l) = \tau$ and $\mathrm{ht}_p^{S_l}(\alpha_l) = 0$ and by Lemma 3.27 we know $l \in \mathrm{J}^{\{x_j\}_j}(g+W)$. Q.E.D.

Lemma 3.31 *Let $W = \bigoplus_{j=1}^n S_jx_j$ with $\mathbb{Z} \subseteq S_j \subseteq \mathbb{Q}$ and adjusted p-basis $\{x_j\}_j$. Assume that $g+W$ is primary reducible of type τ and order p^k. Then*

$$\mathrm{J}^{\{x_j\}_j}(g+W) = \left\{ l \,\middle|\, \exists x'_l : W = \langle x'_l\rangle_*^W \oplus \bigoplus_{j\neq l} S_jx_j, \mathrm{Cut}^{\{x_j\}_j}(g) \in \mathbb{Q}x'_l \right\}.$$

Proof: Lemma 3.29 shows "$\subseteq$" and Lemma 3.30 shows "$\supseteq$". Q.E.D.

So this Lemma actually guarantees the existence of a stacked basis for the two completely decomposable groups W and $\langle W, g\rangle$.

3.5 Completely Decomposable Subgroups

Lemma 3.32 *Let $A_1 \subseteq A_2 \subseteq A_3$ be three completely decomposable groups with $|A_3 : A_2| = p$ and $|A_2 : A_1| = q$ where p and q are different primes. Then there exists a completely decomposable group A_4 with $A_1 \subseteq A_4 \subseteq A_3$ and $|A_4 : A_1| = p$ and $|A_3 : A_4| = q$.*

Proof: By Lemma 3.21 we may WLOG assume that we can write

$$A_1 = \bigoplus_{j=1}^{n} S_j x_j, \qquad A_2 = \frac{1}{q} S_1 x_1 \oplus \bigoplus_{j=2}^{n} S_j x_j$$

such that $\{x_j\}_j$ is a pq-basis of A_1. Then $\frac{1}{q} \notin S_1$ and $A_2 = \langle A_1, \frac{1}{q} x_1 \rangle$. Choose a $g \in A_3 \setminus A_2$. WLOG we may assume that $g = \mathrm{Cut}_{A_2}^{\{x_j\}_j}(g)$. Then $\mathrm{I}^{\{x_j\}_j}(g) = \mathrm{R}_{A_2}^{\{x_j\}_j}(g) = \mathrm{I}^{\{x_j\}_j}(g + A_2)$. As $A_3 = \langle A_2, g \rangle$ we see that $|g + A_2| = p = |qg + A_2|$ and $A_3 = \langle A_2, qg \rangle$, because p and q are relatively prime. Furthermore we have $\mathrm{I}^{\{x_j\}_j}(g + A_2) = \mathrm{I}^{\{x_j\}_j}(qg + A_2)$ and that $qg = \mathrm{Cut}_{A_2}^{\{x_j\}_j}(qg)$. As $A_2 \supseteq A_1$ we also know that $qg = \mathrm{Cut}_{A_1}^{\{x_j\}_j}(qg)$ and thus $\mathrm{I}^{\{x_j\}_j}(qg + A_2) = \mathrm{I}^{\{x_j\}_j}(qg + A_1)$. So $qg + A_1$ and $qg + A_2$ have the same qualified coset type.

As A_2 is strictly smaller than A_3, we know that A_2 is not tight in A_3. By Lemma 3.14 we conclude that at least one coset type of the cosets of A_3/A_2 must be true critical. As all nontrivial coset types of A_3/A_2 are generators, we know that all of them have the same qualified coset type. So $\mathrm{tp}(qg + A_1)$ is true critical. Define $A_4 = \langle qg, A_1 \rangle$. If we can show that $|qg + A_1| = p$, then $qg + A_1$ is reducible and A_4 is completely decomposable.

As $|A_3/A_1| = pq$, every element of A_3 has order either $1, p, q$, or pq over A_1. Hence every element of qA_3 has an order of p or 1 modulo a_1. As $qg \notin A_2 \supseteq A_1$ we conclude that $|qg + A_1| = p$ and hence that A_4 is completely decomposable. Note that $|A_3 : A_4| = q$ and $|A_4 : A_1| = p$. Q.E.D.

Corollary 3.33 *Let $W \subseteq V$ be completely decomposable groups with finite quotient V/W. Define $V_p := \{g \in V \mid |g + W| \text{ is a } p\text{−power}\}$ for some prime p. Then V_p is completely decomposable.*

Proof: The quotient group W/V is finite abelian. Note that V_p is a group, as cosets $g + V$ of prime power order are closed under addition. Furthermore we see that V_p/V is the p-component of W/V. Let p^k be the greatest power of p that divides $|W/V|$. Hence if we find a group $V \subseteq S \subseteq W$ with $|S/V| =$

p^k, then we know that $S = V_p$. We will give a completely decomposable group that has index p^k over V.

As V is not tight in W there exists a coset $g + V$ of prime order with $\mathrm{tp}(g + V)$ true critical. Hence $g + V$ is reducible and $V_1 := \langle V, g\rangle$ is completely decomposable. With repeated application of that Lemma we obtain a chain of completely decomposable subgroups:

$$V = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_l = W$$

with $|W_{i+1}/W_i|$ prime.

Whenever there exists an index i such that $|W_i/W_{i-1}| \neq p$ and $|W_{i+1}/W_i| = p$, apply the previous Lemma to obtain a completely decomposable W_i' such that $|W_i'/W_{i-1}| = p$ and $|W_{i+1}/W_i'| \neq p$. Repeated application of this Lemma will eventually yield a chain whose first k steps have index p. So $|W_k/V| = p^k$ and $W_k = V_p$ is completely decomposable. Q.E.D.

Lemma 3.34 *Let $A_1 \subseteq A_2 \subseteq A_3$ be three completely decomposable groups with $|A_3 : A_2| = p = |A_2 : A_1|$ with p prime. Let $A_3 = \langle A_2, g_2\rangle$ and $A_2 = \langle A_1, g_1\rangle$ with $\mathrm{tp}(g_2 + A_2) > \mathrm{tp}(g_1 + A_1)$. Then there exists a completely decomposable group A_4 with $A_1 \subseteq A_4 \subseteq A_3$ and $|A_4 : A_1| = p = |A_3 : A_4|$ and an element g such that $A_4 = \langle A_1, g\rangle$ and $A_3 = \langle A_4, g_1\rangle$ and $A_3 = \langle A_2, g\rangle$.*

Proof: By Lemma 3.21 we may WLOG assume that we can write

$$A_1 = \bigoplus_{j=1}^{n} S_j x_j, \qquad A_2 = \frac{1}{p} S_1 x_1 \oplus \bigoplus_{j=2}^{n} S_j x_j$$

such that $\{x_j\}_j$ is a p-basis of A_1. Then $\frac{1}{p} \notin S_1$ and $A_2 = \langle A_1, \frac{1}{p} x_1\rangle$. Choose a $g \in A_3 \setminus A_2$ such that $g = \mathrm{Cut}_{A_2}^{\{x_j\}_j}(g)$. Then $\mathrm{I}^{\{x_j\}_j}(g) = \mathrm{R}_{A_2}^{\{x_j\}_j}(g) = \mathrm{I}^{\{x_j\}_j}(g + A_2)$.

As $A_2 \supseteq A_1$ we also know that $g = \mathrm{Cut}_{A_1}^{\{x_j\}_j}(g)$ and thus $\mathrm{I}^{\{x_j\}_j}(g + A_2) = \mathrm{I}^{\{x_j\}_j}(g + A_1)$. So $g + A_1$ and $g + A_2$ have the same qualified coset type.

As $\mathrm{tp}(g) > \mathrm{tp}(x_1)$ we know that $1 \notin \mathrm{I}^{\{x_j\}_j}(g)$ and hence that $|g + A_1| = |g + A_2| = p$, as component 1 is the only component in which A_1 and A_2 differ.

As A_2 is strictly smaller than A_3, we know that A_2 is not tight in A_3. By Lemma 3.14 we conclude that among the cosets of A_3 modulo A_2 there exists at least one coset with a true critical coset type. As all nontrivial cosets of A_3 modulo A_2 generate A_3/A_2, we know that all of them have the same qualified coset type. So $\mathrm{tp}(g+A_1) = \mathrm{tp}(g+A_2)$ is true critical. Define $A_4 = \langle g, A_1\rangle$. As $|g + A_1| = p$ we know that $g + A_1$ is reducible and A_4 is completely decomposable. Q.E.D.

The following conjectures arose naturally during the course of the research. They are the starting point for further research on coset types.

Conjecture 3.35 *Let W be a completely decomposable group and let $g + W$ and $h + W$ be two primary reducible cosets. If $\mathrm{tp}(g + W) \neq \mathrm{tp}(h + W)$ or $|g + W|$ is relatively prime to $|h + W|$, then $g + \langle W, h\rangle$ is reducible of the same type as $g + W$ and of the same order.*

Conjecture 3.36 *Let W be a completely decomposable group and R a set of primary reducible cosets modulo W. Partition the set R into $R = \bigcup_{\tau,p} R_{\tau,p}$ where $R_{\tau,p} = \{g + W \in R \mid \mathrm{tp}(g + W) = \tau, |g + W| = p^k, k \in \mathbb{Z}\}$. Let $\{x_j\}_j$ be a decomposition basis of W. Then R is simultaneously reducible if and only if $\langle g + W \mid g + W \in R_{\tau,p}\rangle$ contains only cosets with coset type τ true critical for all critical types τ and primes p.*

Conjecture 3.37 *Let W be a completely decomposable group and let $V \supseteq W$ with finite quotient. Then V is completely decomposable if and only if there exists a set of simultaneously reducible cosets which generate V/W.*

3.6 Tight Subgroups

We conclude the chapter on coset types by an example that shows that tight sugbroups can appear above the regulator under certain circumstances.

Example 3.38 *Let $2, q, r, s, t$ be five different primes. Let $W = \mathbb{Q}^{(q)}a \oplus \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d$ and $g_1 = \frac{1}{2}(b + c)$ and $g_2 = \frac{1}{4}(c + d)$. Define $G = \langle W, g_1, g_2\rangle$. Note that there is a quite obvious direct decomposition of $G = \mathbb{Q}^{(q)}a \oplus (G \cap \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d)$.*

Our first task is to check, whether W is regulating in G. This is done by examining the coset types of the G/W. Note first that g_1 and g_2 are linearly independent modulo W, so G/W has order 8. With some calculations we finally obtain:

$$\mathrm{tp}(\lambda_1 g_1 + \lambda_2 g_2 + W) = \begin{cases} \mathrm{tp}(0) & \lambda_i g_i \in W \quad \forall i \\ \mathrm{tp}(a)\ \text{false} & \text{otherwise} \end{cases}$$

So there is no true critical coset type and hence W is regulating.

We write τ_a for $\mathrm{tp}(a)$ and likewise $\tau_b, \tau_c, \ldots$. Note that $G^{\#}(\tau) = 0$ for all $\tau \neq \tau_a$. It is easy to verify that $G^{\#}(\tau_a) = \langle W^{\#}(\tau_a), g_1, g_2\rangle$. As the critical typeset of $G^{\#}(\tau_a)$ is an antichain we also know that $R(G^{\#}(\tau_a)) = W^{\#}(\tau_a)$. So the Burkhardt invariant $\beta_{\tau_a} = \exp\left|G^{\#}(\tau_a)/W^{\#}(\tau_a)\right| = 4$.

We obtain the regulator $R(G) = \mathbb{Q}^{(q)}4a \oplus \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d$. The index of the regulator $R(G)$ in G is $4 * 8 = 32$. We have $G = \langle R(G), g_1, g_2, a\rangle$ by Lemma 1.3 c). As $\chi^{R(G)}(4a) = \chi^{\mathbb{Q}^{(q)}}(1) \leq \chi^{W}(b+c)$ we can write $R(G) = \mathbb{Q}^{(q)}(4a+b+c) \oplus \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d$. Define $a' = a + \frac{1}{2}g_1 = a + \frac{1}{4}(b+c)$. Then $G = \langle R(G), g_1, g_2, a' - \frac{1}{4}(b+c)\rangle$ with $R(G) = \mathbb{Q}^{(q)}4a' \oplus \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d$.

Note that $2a' + R(G)$ is a coset of order 2 modulo $R(G)$, and that $\mathrm{tp}(2a' + R(G)) = \tau_a$ true. Hence $2a' + R(G)$ is reducible. Set $V := \langle R(G), 2a'\rangle = \mathbb{Q}^{(q)}2a' \oplus \mathbb{Q}^{(q,r)}b \oplus \mathbb{Q}^{(q,s)}c \oplus \mathbb{Q}^{(q,t)}d$. Trivially we get $G = \langle V, g_1, g_2, a' - \frac{1}{4}(b+c)\rangle$. As $V/R(G)$ has order 2, we find that G/V has order 16. So V cannot be regulating.

We claim that V is tight. To verify this claim we have to check whether there exist cosets of prime order that have a true critical coset type (Lemma 3.14). Consider a coset $h + V = \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3(a' - \frac{1}{2}g_1) + V$. First note that the only possible coset types are τ_a false and τ_a true. Assume that the given coset $h + V$ had coset type τ_a true. We will show that then $|h + V| = 4$, in particular $h + V$ does not have prime order. Let $h + V$ have true critical coset type. Then λ_3 must be odd, as $2(a' - \frac{1}{2}g_1) + V = g_1 + V$ and there would be no component of type τ_a in the coset carrier. With λ_3 odd, we check whether $2h + V = V$ or not. We get $2h + V = (2\lambda_1)g_1 + (2\lambda_2)g_2 + \lambda_3 g_1 + V = (2\lambda_2)g_2 + \lambda_3 g_1 + V \neq V$ for odd λ_3. Hence $h + V$ does not have order 2, and all cosets with coset type τ_a true have order 4. So V is tight.

In the previous example one might wonder why g_2 has been carried through all calculations without ever playing a significant role. At first glance it might as well have been omitted. But the purpose of g_2 lies solely in increasing the Burkhardt invariant to 4 and thus lowering the regulator beyond "good".

Note that $g_1 + R(G)$ has no proper divisors in $G/R(G)$ in the sense that there exists no element of greater order in $G/R(G)$ which has $g_1 + R(G)$ as a multiple. The element is hence not "rooted" deeper in the group. We believe that this very "weakness" gave rise to the construction of a tight but not regulating subgroup above the regulator. Generalizing this example we come to this conjecture.

Conjecture 3.39 *Let W be a regulating subgroup of the almost completely decomposable group G. If there exists a critical type τ and an element $h + W(\tau) \in G(\tau)/W(\tau)$ such that the exponent of $G(\tau)/W(\tau)$ is strictly larger than the order of the greatest cyclic subgroup containing $h + W$, then there exists a non regulating tight subgroup containing the regulator.*

The basic idea for a proof is again the same as in the previous example, but a more subtle approach will be necessary. Every subgroup V has possibly many tight groups above which need not have the same index in G. Let U be a tight subgroup with $U \subseteq V$ and $|G/U|$ minimal among all the tight subgroups. Then we call $|U/V|$ the *reserve* of V in G, in the sense that this number gives an idea about what space is left above, what can be obtained by shifting optimally. With this definition in mind we can look at the example and note that there was a transition that reduced the index of the subgroup by 2, but has reduced the reserve by 4, which basically meant that there was no regulating subgroup above anymore. A proof for the conjecture will have to formalize the notion of the *reserve* and will have to trace the reserve during any transition. This notion in mind we believe that a proof of the conjecture might take several pages, but is strightforward in some sense.

Another idea is to determine the amount of information that is given with coset carriers. We have seen that being given the finite quotient group G/W and the respective coset types for every coset of the quotient is enough information to determine whether the subgroup is regulating or tight. If the finite quotient group G/W and all coset carriers with respect to some decomposition basis were given, does this suffice to determine the regulator quotient or the regulating index or some other given invariant of the group G? Does it suffice in case that W is the regulator or in case that W is the Core of G. Note that the amount of information increases if the quotient is enlarged suitably. This topic is related to the normal form question for almost completely decomposable groups.

4 *Closure

4.1 *Closures of Subsets of Lattices

We will work with lattices and subsets therof. The subsets we will use will never be considered as potential lattices but solely as subsets of lattices.

Even if a subset of a lattice happens to be a lattice itself, we will only recognize it as a subset and $\vee$ or $\wedge$ refer to the join or meet operation of the superset lattice. Hence an intersection of two elements of a subset may not be in the subset, but it is nevertheless defined and there is no ambiguity which element is meant.

In this section we are showing results on lattices in general, but we have typesets of almost completely decomposable groups in mind as a particular application. This particular application gave rise to development of the notion of "fully single covered" subsets of a lattice. There are properties of a critical typeset that go beyond the partial ordering of the typeset and which are not depicted in usual Hasse diagrams.

Definition 4.1 *Let T be a finite subset of a lattice $\mathcal{T}$. Let $\bar{T}$ denote the closure of T with respect to intersections. Define the* ***closure** *of T in $\mathcal{T}$ by*

$$^*T = \{x \in \bar{T} \mid \exists y \in T : y \leq x\}.$$

The finiteness of T implies the finiteness of $\bar{T}$ and $^*T$.

Definition 4.2 *Let $x, y \in T$ with $x \neq y$ where T is a finite subset of a lattice $\mathcal{T}$. Then x is called a* **cover** *of y in T if $x \geq z \geq y$ for some $z \in T$ implies either $x = z$ or $y = z$.*

Note that the covers of the element x are exactly the minimal elements in $\{y \mid y > x\}$. Hence two covers of the same element are necessarily incomparable.

Definition 4.3 *Let T be a finite subset of a lattice $\mathcal{T}$ and let $^*T$ be its *closure in $\mathcal{T}$. Then we will call those elements of $\mathcal{T}$* **critical** *which are elements of T, too.*

Lemma 4.4 *Let T be a finite subset of a lattice $\mathcal{T}$. Let $^*T$ be the *closure of T in $\mathcal{T}$. Then an element x of $^*T$ has the following properties:*

a) *We can write $x = \bigwedge S$ for some $S \subseteq T$.*

b) *If there exist more that one cover of x in $^*T$, then intersecting any two of them yields x.*

If x is noncritical we also have the following:

c) *There exist at least two covers of x in $^*T$.*

d) *There exists an element of $^*T$ that is covered by x in $^*T$.*

Proof:

a) Direct consequence of the *closure.

b) Let y_1, y_2 be two different covers of x in $^*T$. Define $y = y_1 \wedge y_2$. As y_1 and y_2 cover the same element, they are incomparable and $y \neq y_1$. As y is the intersection we get $y < y_1$ and $y \geq x$. So alltogether we get $x \leq y < y_1$ and as y_1 is a cover of x in $^*T$ we have $x = y$.

c) Assume for contradiction that there was only one cover y of x in $^*T$. Hence for every element $z > x$ we have that $z \geq y$. By a) we can write $x = \bigwedge S$ with $S \subseteq T$. Note that S can only contain elements greater than x. So $y \leq z$ for all $z \in S$. But then $y \leq \bigwedge S = x$, a contradiction.

d) As there exists an element $v \in T$ with $v < x$, this is obvious.

Q.E.D.

Lemma 4.5 *Let T be a finite subset of a lattice $\mathcal{T}$. Let $^*T$ be the *closure of T. If $^*T$ contains a lower bound for two of its elements, then it contains their intersection.*

Proof: Let $y_1, y_2 \in {}^*T$ and let $x \in {}^*T$ be their lower bound. Then there exists a $v \in T$ with $v \leq x$ and we get $y_1 \wedge y_2 \geq x \geq v$. Write $y_1 = \bigwedge S_1$ and $y_2 = \bigwedge S_2$ with $S_1, S_2 \subseteq T$. Then $v \leq y_1 \wedge y_2 = \bigwedge(S_1 \cup S_2)$ and hence $y_1 \wedge y_2 \in {}^*T$. Q.E.D.

Definition 4.6 *Let T be a finite subset of a lattice. Then T is called*

a) $\bigvee$-free, if $\{y \in T \mid y > x\}$ is a chain for all $x \in T$.

b) antichain, if all elements of T are maximal in T.

Lemma 4.7 *Let T be a finite subset of a lattice. If T is $\bigvee$-free then every nonmaximal element of T has exactly one cover in T. If T is a chain then every nonmaximal element of T has exactly one cover and no other element has the same cover.*

Proof: Let $x \in T$ be nonmaximal. Since $S = \{y \in T \mid y > x\}$ is a finite chain we know that its minimal element is the only cover of x in T.

If T is a chain, then the elements can be written $t_1 < t_2 < \cdots < t_n$ and the claim is obvious. Q.E.D.

Lemma 4.8 *Let T be a finite subset of a lattice $\mathcal{T}$. Let $^*T$ be the *closure of T in $\mathcal{T}$. If T is $\bigvee$-free, then $T = {}^*T$.*

Proof: Trivially we have $^*T \geq T$. Now assume for contradiction that $^*T \neq T$. Hence there exists some noncritical element $y \in {}^*T \setminus T$. By Lemma 4.4 we know that y has two incomparable covers s_1 and s_2. By the definition of the *closure there exists some $v \in T$ such that $v \leq y$, but then $v < s_1, s_2$ and the elements greater than v do not form a chain. That contradicts the $\bigvee$-freeness of T. Q.E.D.

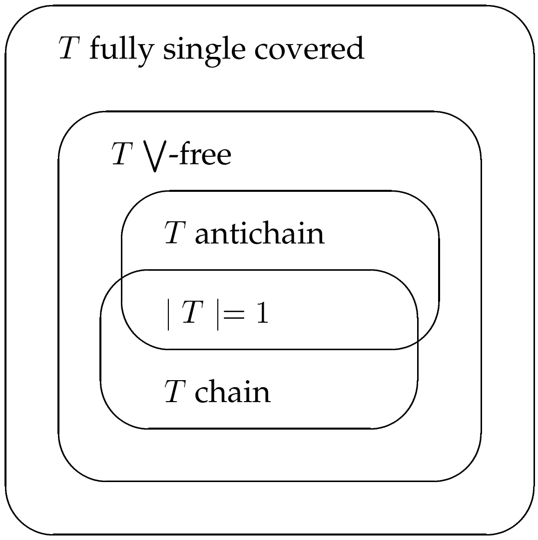
Definition 4.9 *Let T be a finite subset of a lattice $\mathcal{T}$. Let $^*T$ be the *closure of T in $\mathcal{T}$. An element of T is called* ***closure single covered** *if it has exacly one cover in $^*T$.*

Definition 4.10 *Let T be a finite subset of a lattice $\mathcal{T}$. Then T is* **fully single covered** *if all nonmaximal elements of T are *closure single covered.*

Lemma 4.11 *Let T be a finite subset of a lattice. If T is $\bigvee$-free then T is fully single covered.*

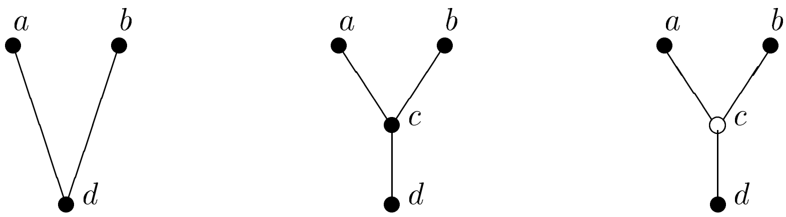
Proof: Let $\mathcal{T}$ be the superset lattice of T and let $^*T$ be the *closure of T in $\mathcal{T}$. As T is $\bigvee$-free we know by Lemma 4.8 that $T = {}^*T$, and by Lemma 4.7 that all nonmaximal elements of T have only one cover in T and thus also in $^*T$. Together we find that all nonmaximal elements of T have exactly one cover in $^*T$. Q.E.D.

We can visualize the classes of subsets of a lattice as follows:



We show that $\bigvee$-free and fully single covered are different properties.

Example 4.12 *Let $\mathcal{T}$ be a lattice with elements a, b, c, d such that $a \wedge b = c$ and $d < c$ while a and b are incomparable. Define $T = \{a, b, d\}$. Then $\bar{T} = \{a, b, d, a \wedge b\} = \{a, b, c, d\}$. This yields the *closure $^*T = \{a, b, c, d\}$. Then c is the only cover of d in $^*T$. As a and b are maximal in T and d is single covered in $^*T$, we find that T is fully single covered. Obviously T is not $\bigvee$-free.*



T $\qquad$ $\bar{T}$ $\qquad$ $^*T$

Lemma 4.13 *Let T be a fully single covered finite subset of a lattice $\mathcal{T}$. Let $^*T$ be the *closure of T in $\mathcal{T}$. Let $S \subseteq T$. Then $\bigwedge S \notin T$ if and only if S has at least two elements minimal in S.*

Proof: "If": If S had only one minimal element, then this element would be $\bigwedge S$ and hence $\bigwedge S \in T$.

"Only if": Now assume that S contains more than one minimal element. Define $s = \bigwedge S$. As minimal elements are incomparable we have $s \neq r$ for all minimal $r \in S$. So this yields $s < r$ for all $r \in S$. It remains to show that $s \notin T$. Assume for contradiction that $s \in T$. Then s is *closure single covered in $^*T$ and hence let $x > s$ be the single unique cover of s in $^*T$. Then $x \leq r$ for all $r \in S$ and we have $x \leq \bigwedge S$. But this contradicts $x > s$. Hence $s \notin T$, as desired. Q.E.D.

For a poset S let $\min S$ denote the set of minimal elements of the poset. We write $|\min S|$ to denote the cardinality of $\min S$.

Lemma 4.14 *Let T be a finite subset of a lattice $\mathcal{T}$. Assume for every subset $S \subseteq T$ that $\bigwedge S \in T$ implies $|\min S| = 1$. Then T is fully single covered.*

Proof: We begin with some prior considerations: Let $x, y \in {}^*T$ be two incomparable elements of the *closure $^*T$ of T in $\mathcal{T}$. Then there exist two subsets $X, Y \subseteq T$ such that $x = \bigwedge X$ and $y = \bigwedge Y$. Note that X and Y are not necessarily uniquely defined with this property. We have $x \wedge y = \bigwedge(X \cup Y)$ and claim that $X \cup Y$ must have more than one minimal element.

Assume for contradiction that there exists some single unique minimal element z of $X \cup Y$. Then $z = \bigwedge(X \cup Y)$. WLOG assume that $z \in X$. Then z is also the single unique minimal element of X. Hence $x = \bigwedge X = z$. As $z = \bigwedge(X \cup Y) \leq \bigwedge Y = y$ we have that $x = z \leq y$ which contradicts the incomparability of x and y. Hence $|\min(X \cup Y)| > 1$.

Now comes the actual proof: Assume for every subset $S \subseteq T$ that $\bigwedge S \in T$ implies $|\min S| = 1$. Let $\tau \in T$ and assume that τ is not maximal, as we do not have to verify anything for maximal elements. We want to show that τ is single covered in $^*T$. Assume for contradiction that τ had more that one cover in $^*T$. Let $x \neq y$ be any two of the covers of τ in $^*T$. Then by Lemma 4.4 b) we get $\tau = x \wedge y$. Note that x and y are incomparable. We can find $X, Y \subseteq T$ such that $x = \bigwedge X$ and $y = \bigwedge Y$. By our prior considerations we know that $(X \cup Y)$ has more than one minimal element. Set $S = X \cup Y$. Then $\tau = \bigwedge S \in T$ and $|\min S| > 1$. But as $\bigwedge S \in T$ we also get that $|\min S| = 1$, by our starting assumption. That is a contradiction, and hence τ has only one cover in $^*T$. So T is fully single covered. Q.E.D.

Lemma 4.15 *Let T be a finite subset of a lattice $\mathcal{T}$. Then the following are equivalent:*

a) *T is fully single covered,*

b) $\forall\ S \subseteq T: \quad (|\min S| > 1) \Rightarrow \bigwedge S \notin T$

c) $\forall\ S \subseteq T: \quad (|\min S| > 1) \Leftrightarrow \bigwedge S \notin T$

Proof: a)⇒b) was shown in Lemma 4.13.

b)⇒a) can be derived from Lemma 4.14 by negating the statements and reversing the direction of the implication.

c)⇒b) is trivial.

b)⇒c) comes from the fact that the intersection of a set with only one minimal element is exactly that minimal element. Q.E.D.

Lemma 4.16 *Subsets of fully single covered finite subsets of a lattice are fully single covered, too.*

So the property "fully single covered" inherits to subsets. This is not obvious! What might happen is that the unique single cover, that an element x of T had in ${}^*T$, is no longer in ${}^*A$ although $x \in A \subset T$.

Proof: Let T be a finite fully single covered subset of the lattice $\mathcal{T}$. Let A be a subset of T. Let ${}^*T$ and ${}^*A$ denote the *closures of T and A, respectively. Likewise let $\bar{T}$ and $\bar{A}$ be the closures of T and A, respectively, with respect to meets. Then ${}^*T \supseteq {}^*A$ because of $\bar{T} \supseteq \bar{A}$ and $T \supseteq A$. Let $S = \{z \in A \mid z > x\}$ for some $x \in A$. If S is empty, then x is maximal and we have nothing to show. Now we have two cases, according to the number of minimal elements in S.

Let S have only one minimal element r. Every noncritical element $t \in {}^*A$ with $t > x$ is the intersection of critical elements in A by Lemma 4.4 a). Write $t = \bigwedge B$ with $B \subseteq A$. As $t > x$ we find that $B \subseteq S$. So $b \geq r$ for every $b \in B$. Hence $t = \bigwedge B \geq r = \bigwedge S$. Thus every element $t > x$ is greater or equal to r. Hence r is the unique cover of x in ${}^*T$. As $S \subseteq A$ we find that $r \in^* A$. Then r is also the single unique cover of x in ${}^*A$.

Now assume that $S \subseteq A$ has more than one minimal element. Then $r = \bigwedge S \notin T$ by Lemma 4.13, as T is fully single covered. As $A \subseteq T$ we get that $r \notin A$, either. Note that $r \in {}^*A$, though. So $\bigwedge S = r \geq x$ implies $r > x$, as $x \in {}^*A$. We claim that r is the single unique cover of x in ${}^*A$. Thus we take an arbitrary element $y > x$ with $y \in^* A$. We have to show that $y \geq r$. As $y \in {}^*A$ we can apply Lemma 4.4 a) to obtain a set $B \subseteq A$ such that $y = \bigwedge B$. As $y > x$ we have $b > x$ for every $b \in B$. So $B \subseteq S$ and hence $\bigwedge B = y \geq r = \bigwedge S$, as desired.

In both cases we have that $x \in A$ has a unique single cover in ${}^*A$ and hence A is fully single covered, too. Q.E.D.

Lemma 4.17 *Let A, B be fully single covered finite subsets of a lattice such that every element of A is incomparable to every element of B, then $A \cup B$ is fully single covered, too.*

Proof: Let $x \in A \cup B = T$. WLOG assume that $x \in A$. If x is maximal in A then x is also maximal in $A \cup B$ and we are done. So assume from now on that x is not maximal in A. We are looking for a cover of x in ${}^*T$. Let $S = \{z \in T \mid z > x\}$ and note that $S \subseteq A$, because $S \cap B \neq \emptyset$ would violate the incomparability condition for elements in A and B. Hence every element in ${}^*T$ that is greater than x, is an intersection of elements of A. So every element of ${}^*T$ that is greater than x is also in ${}^*A$. Then it is clear that the single unique cover of x in ${}^*A$ is also the single unique cover of x in ${}^*T$. Q.E.D.

We now want to reverse the process of taking the *closure and ask the question which sets S have a *closure that is equal to some given closure ${}^*T$.

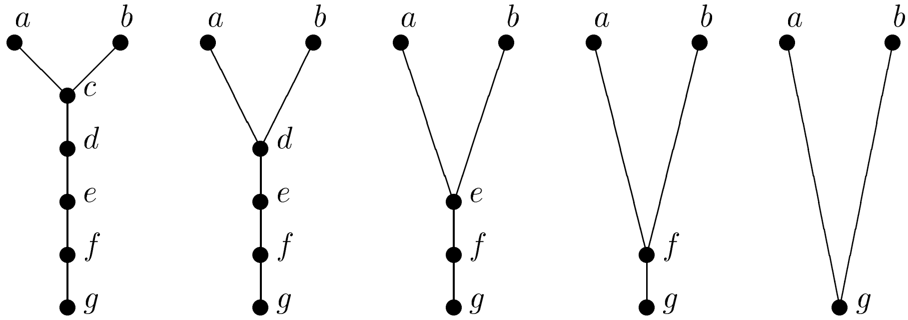
Definition 4.18 *Let T be a subset of a lattice. Then define*

$$^{\#}T := \{x \in T \mid x \text{ is maximal, minimal or single covered in } T\}.$$

We call $^{\#}T$ the **reverse *closure** *of T.*

Note that this reverse closure does not satisfy $^{\#}(^{\#}T) = {}^{\#}T$ as the following example shows.

Example 4.19 *Let a, b, c, d, e, f, g be distinct elements of a lattice such that $c > d > e > f > g$ and $a \wedge b = c$. Let $T = \{a, b, c, d, e, f, g\}$. Then $^{\#}T = \{a, b, d, e, f, g\}$ and $^{\#}(^{\#}T) = \{a, b, e, f, g\}$ and $^{\#}(^{\#}(^{\#}T)) = \{a, b, f, g\}$ and $^{\#}(^{\#}(^{\#}(^{\#}T))) = \{a, b, g\}$. Note that the reverse *closure always removes intermediate, non single covered elements. But during this operation, some of the surviving elements might have lost their single unique cover and thus may be removed in a following reverse *closure. The following figure helps visualizing the reverse *closure:*



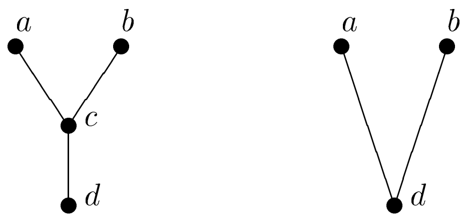
T $\quad$ ${}^{\#}T$ $\quad$ ${}^{\#}({}^{\#}T)$ $\quad$ ${}^{\#}({}^{\#}({}^{\#}T))$ $\quad$ ${}^{\#}({}^{\#}({}^{\#}({}^{\#}T)))$

Lemma 4.20 *Let T be a finite subset of a lattice $\mathcal{T}$. Let ${}^{*}T$ be the *closure of T in $\mathcal{T}$. Then ${}^{\#}({}^{*}T) \subseteq T$.*

Proof: Maximal and minimal elements of ${}^{*}T$ must be critical, because they cannot have come from taking the *closure. By Lemma 4.4 c) we see that single covered elements in ${}^{*}T$ must be critical. As an element of ${}^{\#}({}^{*}T)$ is either minimal/maximal or single covered by definition, we have ${}^{\#}({}^{*}T) \subseteq T$. Q.E.D.

The following example shows that ${}^{\#}({}^{*}T)$ can be strictly smaller than T.

Example 4.21 *Let a, b, c, d be distinct elements of a lattice such that $c > d$ and $a \wedge b = c$. Define $T = \{a, b, c, d\}$ and note that ${}^{*}T = T$, because T is closed with respect to intersections. As a direct consequence we have ${}^{\#}({}^{*}T) = {}^{\#}T$. But we have ${}^{\#}T = \{a, b, d\} = {}^{\#}({}^{*}T)$ whereas $T = \{a, b, c, d\}$.*



$T = {}^{*}T$ $\quad$ ${}^{\#}T = {}^{\#}({}^{*}T)$

Lemma 4.22 *Let T be a finite subset of a lattice $\mathcal{T}$. Let ${}^{*}T$ be the *closure of T in $\mathcal{T}$. Let ${}^{\#}({}^{*}T)$ be the reverse *closure of ${}^{*}T$. Then ${}^{*}({}^{\#}({}^{*}T)) = {}^{*}T$.*

Proof: Let $S := {}^{\#}({}^*T)$. We want to show that ${}^*S = {}^*T$. This is done in three steps. First we show that $\max {}^*S = \max {}^*T$ and $\min {}^*S = \min {}^*T$, then we show that also the single covered elements of ${}^*S$ and ${}^*T$ are the same, and finally we extend the equality to the remaining elements. Note that $\max$ and $\min$ refer to the subsets of maximal and minimal elements of a given poset.

We know that $\max {}^*T = \max T$, $\min {}^*T = \min T$, $\max {}^*S = \max S$ and $\min {}^*S = \min S$. Let $x \in \max {}^*T$. By the definition of the reverse *closure we can also find $x \in {}^{\#}({}^*T) = S$ and we know that it is maximal in S. Now conversely assume that $x \in \max S$. If $x \notin \max {}^*T$ then there would exist an $y \in \max {}^*T$ with $y > x$ and we would get $y \in \max S$, contradicting the maximality of x in S. Hence $\max S = \max {}^*T$ and thus $\max {}^*S = \max S = \max {}^*T = \max T$. Similarly we get $\min S = \min {}^*T$ and thus $\min {}^*S = \min S = \min {}^*T = \min T$.

The set of single covered elements of ${}^*T$ is contained in S by the definition of the reverse *closure, and is also contained in ${}^*S$, as ${}^*S \supseteq S$. Conversely let $x \in {}^*S$ be single covered. Then by Lemma 4.4 c) we know that x must be critical, that is $x \in S$. As $S \subseteq {}^*T$, we also have $x \in {}^*T$.

Now let $x \in {}^*T$. There exists a subset $R \subseteq T$ with $x = \bigwedge R$, and an element $y \in \min T$ with $y \leq x$. We have to show that $x \in {}^*S$. Note that $y \in \min T$ implies that $y \in \min S = \min {}^*S$. WLOG we can assume that R contains only maximal elements or elements that are single covered in ${}^*T$, by the following recursion: As x is not critical, it is neither single covered nor maximal, and hence it is the intersection of any two of its covers. Examine one of those two covers. It is either single covered or maximal or again the intersection of two of its covers.... As ${}^*T$ is finite, this recursion must end. So R contains only single covered or maximal elements. With the previous paragraph we obtain $R \subseteq S \subseteq {}^*S$. As $R \subseteq S$ and $y \in S$ we conclude that $x = \bigwedge R \in {}^*S$. So ${}^*T \subseteq {}^*S$.

Conversely assume $x \in {}^*S$. There exists a subset $R \subseteq S$ with $x = \bigwedge R$, and an element $y \in \min S$ with $y \leq x$. We have to show that $x \in {}^*T$. Note that $y \in \min S$ implies that $y \in \min T = \min {}^*T$. WLOG we can assume that R contains only maximal elements or elements that are single covered in ${}^*T$, by the same recursion as in the previous paragraph. We obtain $R \subseteq {}^*T$. (Note that we do not yet obtain $R \subseteq T$ as the analogy to the previous paragraph would suggest, because there is some asymmetry.) So we have that $x = \bigwedge R$ with $R \subseteq {}^*T$ and $y \leq x$ with $y \in T$. If R contains an element $r \in {}^*T$ which is not critical, that is $r \notin T$, then there exists a subset $R' \subseteq T$

such that $r = \bigwedge R'$, and we can replace r by R'. So WLOG we can assume that $R \subseteq T$. Hence $x \in {}^*T$, as desired. So ${}^*S \subseteq {}^*T$.

Together we obtain ${}^*S = {}^*T$. Q.E.D.

Lemma 4.23 *Let T, S be two finite subsets of a lattice $\mathcal{T}$ such that $T \subseteq S \subseteq {}^*T$ where ${}^*T$ is the *closure of T in $\mathcal{T}$. Then ${}^*S = {}^*T$.*

Proof: The minimal elements of S are exactly the minimal elements of T, because the *closure cannot generate additional minimal elements. Likewise we have that the maximal elements of S are exactly the maximal elements of T. Elements of S are intersections of elements in T, because $S \subseteq {}^*T$. Elements of ${}^*S$ are intersections of elements of S. Hence elements of ${}^*S$ are intersections of elements of T. This is equal to writing ${}^*S \subseteq \bar{T}$, where $\bar{T}$ is the closure of T with respect to meet.

Let $x \in {}^*S \subseteq \bar{T}$. Then there exists some $y \in \min S$ with $y \leq x$. As $y \in T$ we get $x \in {}^*T$. Conversely let $x \in {}^*T$. Then there exists some $y \in \min T = \min S$ with $y \leq x$. As x is the intersection of elements in T it is also the intersection of elements of S. Hence $x \in {}^*S$. So we have got ${}^*S = {}^*T$. Q.E.D.

Lemma 4.24 *Let T be a finite subset of a lattice $\mathcal{T}$ and let ${}^*T$ be the *closure of T in $\mathcal{T}$. Then there is at most one fully single covered set between ${}^\#({}^*T)$ and ${}^*T$.*

Proof: Let ${}^\#({}^*T) \subseteq R \subset S \subseteq {}^*T$ with $R \neq S$. By Lemma 4.23 and Lemma 4.22 we know that ${}^*R = {}^*S = {}^*T$. Assume that R is fully single covered. Then every element in ${}^*R \setminus R$ is noncritical and has two covers in ${}^*R$ by Lemma 4.4 c). As $R \subset S \subseteq {}^*R = {}^*T$, we have that $S \cap ({}^*R \setminus R) \neq \emptyset$. So there exist elements in S which are not single covered in ${}^*A$. So S is not fully single covered. Q.E.D.

Theorem 4.25 *The *closure and the reverse *closure define an equivalence relation on the set of all finite subsets of a given lattice $\mathcal{T}$ by:*

Two subsets R and S are equivalent, if and only if ${}^\#({}^*R) \subseteq S \subseteq {}^*R$.

Proof: We have to verify that the relation ($\equiv$) is reflexive, symmetric, and transitive.

Reflexivity is due to $R \subseteq {}^*R$ and ${}^\#({}^*R) \subseteq R$ for all R by Lemma 4.20.

By Lemma 4.23 and Lemma 4.22 we can obtain that ${}^*S = {}^*R$ and hence ${}^\#({}^*S) = {}^\#({}^*R)$. This implies symmetry.

Now for transitivity. Let $R \equiv S$ and $S \equiv T$. By the symmetry we can assume that $R \equiv S, T$. Using Lemmata 4.23 and 4.22 we obtain that $^*R = {}^*S = {}^*T$ and $^\#(^*R) = {}^\#(^*S) = {}^\#(^*T)$. Hence $^\#(^*T) \subseteq S \subseteq T$. Hence S and T are equivalent. Q.E.D.

4.2 Extended Hasse Diagrams

We use some modified form of a Hasse diagram to make the structure of finite subsets of lattices visible. We will assume that the reader is familiar with ordinary Hasse diagrams.

Extended Hasse diagrams are only used on subsets of a lattice, which offer an additional property that their elements can either have or not have. In particular we think of *closures, whose elements can be critical or not. An **extended Hasse diagram** of a *closure is a Hasse diagram of the elements of the *closure where critical elements get solid black circles and noncritical elements get empty circles.

Usually we will start with some given subset of a lattice, and then we apply some sort of closure on that given subset. The elements of the original subset are considered to be critical elements and will be depicted using solid black circles, whereas the other elements are represented by empty circles. Unless otherwise noted we will use the *closure. For example see the figure at Example 4.12. In the rightmost diagram the element c is not critical and hence gets an empty circle.

4.3 ACD/Butler Groups with Fully Single Covered Typeset

We now apply the lattice theoretic results to the lattice of types and see, what conclusions can be derived from the properties of the underlying critical typeset. Two well known results are:

Lemma 4.26 *An almost completely decomposable group whose critical typeset is a chain is completely decomposable.*

Lemma 4.27 *An almost completely decomposable group whose critical typeset is $\bigvee$-free has a regulating regulator.*

We note that there is some entity between the critical typeset of a group and the (whole) typeset of the group. We call the *closure of the critical typeset the **extended critical typeset**.

The notion of *closure single covered has also some implications of corresponding type subgroups.

Lemma 4.28 *Let G be a butler group with critical typeset T. If a critical type $\tau \in T$ is single covered in $^*T$ then $G^{\#}(\tau) = G^*(\tau)$.*

Proof: Let σ be the single unique cover of τ in the extended critical typeset of G. We know that $G(\sigma)$ is pure in G, and if we can show that $G(\sigma) = G^*(\tau)$ then this implies that $G^*(\tau)$ is pure and we are done. Note the obvious fact that $G(\sigma) \supseteq G(\rho)$ if $\rho \geq \sigma > \tau$. Using $T^{>\tau} = T^{\geq\sigma}$ we get $G^*(\tau) = \sum_{\rho \in T^{>\tau}} G(\rho) = \sum_{\rho \in T^{\geq\sigma}} G(\rho) = G(\sigma)$. Q.E.D.

Lemma 4.29 *Let C be a pure subgroup of the almost completely decomposable group G. If the typeset of G is fully single covered, then the critical typeset of C is fully single covered, too.*

Proof: Note that the critical typeset of C is a subset of the critical typeset of G. With Lemma 4.16 the claim follows. Q.E.D.

Lemma 4.30 *Let $G = A \oplus B$ be an almost completely decomposable group with direct summands A and B. If the critical typeset of G is fully single covered, then the critical typesets of A and B are fully single covered, too.*

Proof: As A and B are pure subgroups of G, the claim follows with the previous lemma. Q.E.D.

Lemma 4.31 *Let A, B be almost completely decomposable groups with fully single covered critical typeset. Let every critical type of A be incomparable to every critical type of B. Then the critical typeset of $A \oplus B$ is fully single covered.*

Proof: Application of Lemma 4.17. Q.E.D.

In [5] Definition 3.1 a) John E. Koehler introduced a notion that is now called a Koehler-basis.

Definition 4.32 *A decomposition basis $\{x_j\}_j$ of a completely decomposable group W is called a* **Koehler-basis** *of W if $\chi^W(x_i) \wedge \chi^W(x_j) = \chi^W(x_k)$ whenever $\mathrm{tp}^W(x_i) \wedge \mathrm{tp}^W(x_j) = \mathrm{tp}^W(x_k)$ for some integers i, j, k.*

Note that a Koehler-basis is always an adjusted basis. The proof of the existence of an adjusted basis for every completely decomposable group can be modified to obtain a statement on the existence of a Koehler-basis for the same group.

Note that completely decomposable groups with fully single covered critical typeset have the additional property that a decomposition basis is adjusted if and only if it is a Koehler-basis. So for the fully single covered case the two notions coincide.

The converse is not true. If for some almost completely decomposable group the notions of adjusted basis and Koehler-basis coincide, then this does not imply that the critical typeset is fully single covered. Every adjusted basis of $\mathbb{Q}^{(2,3)}a \oplus \mathbb{Q}^{(2,5)}b \oplus \mathbb{Q}^{(3,5)}c \oplus 30\mathbb{Z}d$ is necessarily a Koehler-basis, but it is not fully single covered, as $\mathrm{tp}(\mathbb{Z})$ has three covers in the extended critical typeset.

In my opinion, the following points deserve further investigation:

a) Does there exist another inverse of the *closure that would allow a strengthening of Lemma 4.24 such that "at most one" can be replaced by "exactly one". This would allow to take the unique fully single covered subset (which would have to exist then) as a canonical representative of an equivalence class defined in Theorem 4.25.

b) The fact that $G^{\#}(\tau) = G^*(\tau)$ for all critical types τ should be exploited somehow. Note that in this case the numerical invariants ϵ_τ as defined in [1] Definition 4.5.6 are all equal to 1. Some counting problems should profit from this simplification.

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