The Forbidden Pattern Approach to Concatenation Hierarchies

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Introduction

The study of the class of regular languages and its subclasses has a long tradition, starting with characterizations in terms of regular expressions, finite automata, finite monoids, equivalence relations and monadic second order logic [Kle56, Myh57, Ner58, Tra58, Büc60]. This thesis looks at the subclass formed by star–free languages, i.e., languages that can be described by regular expressions using only Boolean operations and concatenation. In particular, iteration (the Kleene star) is not allowed. Also for this subclass various characterizations have been obtained. A celebrated theorem by Schützenberger [Sch65] states that a regular language is star–free if and only if it can be recognized by an aperiodic finite monoid. Since a recognizing monoid can be computed from a given regular language, and since its aperiodicity can be effectively determined, this yields a decision algorithm for the membership problem of the class of star–free languages. Other characterizations in terms of permutation–free finite automata and first–order logic go back to the work of McNaughton and Papert [MP71], while a characterization by loop–free alternating finite automata is given in [SY00]. Moreover, several authors have drawn the connection to propositional temporal logic [Kam68, MP71, GPSS80, CPP93, EW96, TW96, Wil99]. Along these lines, further subclasses have attracted a lot of attention, among them the locally testable languages, the piecewise testable languages and their variations, which are matter of classical results [BS73, McN74, Sim75, BF80] as well as present work, e.g., [Tra99]. For an overview on star–free languages we refer to [Pin95, Pin96, Tho96].

Brzozowski and Cohen were the first to ask the following natural question [CB71]. Suppose we count the number of alternations in star–free expressions between Boolean operations on one hand and concatenation on the other hand. Given a star–free language L, what is the minimal number of such alternations needed to define L? The distinction between these two kinds of operations reflects that Boolean operations are combinatorial in nature while concatenation is a sequential operation. In fact, we can think of the number of unavoidable alternations as a natural complexity measure for star–free languages. The question whether there exists an algorithm that determines for a given language its alternation complexity became famous as the dot–depth problem.

If we start with a class of languages having a neglectable complexity in these terms, we obtain classes of more complex languages by taking repeatedly the closure of this class under Boolean operations and concatenation. This leads to so–called concatenation hierarchies that exhaust the class of star–free languages. Prominent examples are the dot–depth hierarchy, first studied in [CB71], and the Straubing–Thérien hierarchy [Str81, Thé81, Str85] which both formalize the dot–depth problem in terms of hierarchy classes: the dot–depth of a language L is just the minimal level in the dot–depth hierarchy that contains L. Both concatenation hierarchies are closely related to each other and it is known that there are languages of arbitrary dot–depth [BK78, Tho84].

A seemingly weaker form of the dot–depth problem is to solve the membership problem of a fixed level of such a hierarchy. Clearly, if this can be done for all levels in some uniform way, this also yields a solution to the dot–depth problem. A lot of effort has been invested in the last thirty years to cope with the levelwise membership problems. However, it seems to be a very difficult task and results are known only for some lower levels. The dot–depth problem was considered recently as one of the most important open questions on regular languages [Pin98]. To our knowledge, the membership problems of levels 1/2, 1 and 3/2 of both hierarchies are known to be decidable [Sim75, Kna83, Arf87, PW97, GS00d] while the question is open for any other level. Partial results are known for level 2 [Str88] and level 5/2 [GS00b] of the Straubing–Thérien hierarchy which are decidable if a two–letter alphabet is considered (among others, the latest results for level 3/2 of the dot–depth hierarchy and level 5/2 of the Straubing–Thérien hierarchy will be presented in this thesis). It should be noted that—due to the difficulties to find a solution to the dot–depth problem—some researchers tend to look for undecidability results. We do not follow these lines here, but think one should have this in mind.

Levelwise connections have been exposed from concatenation hierarchies to other fields of research, e.g., to finite model theory [Tho82, PP86], to the theory of finite semigroups [Str85, PW97], to complexity theory [HLS$^+$93] and others. Consequently, the dot–depth problem can be rephrased in these terms and various methods are at hand to attack the problem—however, none of them being successful in general so far. On the other hand, the dot–depth problem itself has stimulated a lot of research activities, resulting in new fundamental insights into the class of regular languages. To begin with, mention must be made of the algebraic theory of finite semigroups. Many of the results cited above have been obtained within this theory ([Pin96] gives an overview). For instance, in [Str85] it is shown that level n of the dot–depth hierarchy (for integer n) is decidable if and only if level n of the Straubing–Thérien hierarchy is decidable, which is a simplification of the dot–depth problem.

We may assume that regular languages are presented by deterministic finite automata (DFA, for short). Note that there are standard algorithms to pass from a representation by automata, expressions and semigroups one to another. Consider again the class of star–free languages. It is shown in [MP71] that a language L is star–free if and only if the minimal DFA $\mathcal{M}$ accepting L is permutation–free. The latter means that there is no word w that induces a cycle of the following type in the transition graph of $\mathcal{M}$: there are pairwise distinct states $s_1, s_2, \ldots, s_m$ for some $m \geq 2$ such that $\mathcal{M}$ moves on input w from s_i to s_{i+1} for all $1 \leq i \leq m$ (with $s_{m+1} =_{\text{def}} s_1$). So w induces a non–trivial permutation on a subset of states of $\mathcal{M}$. This type of cycle is in fact an example of a forbidden pattern (i.e., a forbidden subgraph) in the transition graph of a DFA, and by the cited result, this forbidden pattern characterizes the class of star–free languages. It is known that the problem to decide from a DFA whether it has this pattern in its transition graph is complete for PSPACE [CH91].

We carry over in this thesis the forbidden pattern approach to subclasses of star–free languages, i.e., we look for results of the type "*L belongs to the class $\mathcal{C}$ if and only if an accepting* DFA *does not have pattern $\mathbb{P}$ in its transition graph.*" The major advantage of such a characterization is that it implies decidability of the membership problem of $\mathcal{C}$ because the existence of a certain subgraph can be effectively verified if a transition graph is given (at least in all reasonable cases). But such a result says even more: it reflects the effect of language operations in the structure of automata. One can understand a forbidden pattern as *the* particular property that cannot be expressed due to the limited resources of the characterized class. In general, forbidden pattern characterizations are far from being easy to achieve. However, we consequently follow this approach in connection with concatenation hierarchies.

There are more examples of forbidden pattern characterizations in the literature. In [Ste85a] a levelwise characterization of dot–depth one languages is given, and in [CPP93] conditions on certain varieties of finite semigroups are translated to forbidden patterns in order to obtain decision procedures for these classes. In [Wil98, Wil99] syntactical fragments of temporal logic are investigated, using forbidden patterns to determine their expressive power. Finally, the known decidability results of level 1/2 and level 3/2 of the Straubing–Thérien hierarchy from [Arf91], and of level 1/2 of the dot–depth hierarchy, are given in [PW97] in a forbidden pattern manner. We come back to these results throughout the exposition.

Before we give a chapter overview, we want to make a general bibliographic remark. The clarification of definitions in Section 1.2 and the results of Chapters 4, 5 and 6 were obtained by the author in joint work with Christian Glaßer, Würzburg. Chapters 2 and 3 and the remaining parts of Chapter 1 are due to the author.

Overview

The thesis has two major parts. Besides the introductory Chapter 1, the larger first part comprises Chapters 2 to 4 which share a common viewpoint: we parameterize the relation between the dot–depth hierarchy and the Straubing–Thérien hierarchy in terms of the number of consecutively specified letters in a language definition (taking up an idea from [Sim72, Str85]). Using this together with the forbidden pattern approach, we look in the first part at the fine structure of level 3/2 in these hierarchies. Hereby we provide alternative proofs of virtually all previously known decidability results for concatenation hierarchies, we significantly refine these results and obtain a complete overview on the structure of the lower levels. And we make progress on the dot–depth problem by proving the decidability of level 3/2 of the dot–depth hierarchy.

Chapters 5 and 6 form the second part. Here we generalize from what we have achieved so far and develop an abstract theory of forbidden patterns. We apply this theory to the dot–depth hierarchy and the Straubing–Thérien hierarchy and show a lower bound result for the dot–depth problem. And we prove the decidability of level 5/2 of the Straubing–Thérien hierarchy in case of a two–letter alphabet.

Each chapter starts with references to the main theorems. There is a discussion section at the end of every chapter with more bibliographic notes and an outlook to further research.

Chapter One

This chapter has an introductory character. Fix some finite alphabet A with $|A| \geq 2$, denote by $\mathrm{Pol}(\mathcal{C})$ the closure of a language class $\mathcal{C}$ under finite union and concatenation and let $\mathrm{BC}(\mathcal{C})$ denote its closure under Boolean operations. We define the classes $\mathcal{B}_{n/2}$ of the dot–depth hierarchy (DDH, for short) and the classes $\mathcal{L}_{n/2}$ of the Straubing–Thérien hierarchy (STH, for short) as follows:

$$
\begin{array}{llll}
\mathcal{B}_{1/2} =_{\mathrm{def}} \mathrm{Pol}(\{\, \{w\} \mid w \in A^+ \,\} \cup \{A^+\}) & & \mathcal{L}_{1/2} =_{\mathrm{def}} \mathrm{Pol}(\{\, A^*aA^* \mid a \in A \,\}) & \\
\mathcal{B}_{n+1} =_{\mathrm{def}} \mathrm{BC}(\mathcal{B}_{n+1/2}) & \text{for } n \geq 0 & \mathcal{L}_{n+1} =_{\mathrm{def}} \mathrm{BC}(\mathcal{L}_{n+1/2}) & \text{for } n \geq 0 \\
\mathcal{B}_{n+3/2} =_{\mathrm{def}} \mathrm{Pol}(\mathcal{B}_{n+1}) & \text{for } n \geq 0 & \mathcal{L}_{n+3/2} =_{\mathrm{def}} \mathrm{Pol}(\mathcal{L}_{n+1}) & \text{for } n \geq 0
\end{array}
$$

Starting with two different classes for level 1/2, the higher levels of both hierarchies are defined via the alternating application of the same closure operations. We immediately obtain from the definitions that both hierarchies are mutually comparable by inclusion. However, other definitions of the DDH and STH can be found in the literature, e.g., in [PW97] and we show that these coincide with ours (up to the empty word). As a consequence, we may carry over known closure properties and normal form results from [Arf91, PW97, Gla98].

Then we recall the dot–depth problem in terms of hierarchy classes, state the correspondence to first–order logic from [Tho82, PP86] and draw connections to complexity theory via the leaf language approach to define complexity classes [BCS92, Ver93, HLS$^+$93].

If we compare $\mathcal{L}_{1/2}$ with $\mathcal{B}_{1/2}$ we see that we can specify two or more consecutive letters in one case while only one letter may be fixed in the other. This observation leads in a natural way to a parametrization of the relation between STH and DDH in terms of the maximal block length $k+1$ for $k \geq 0$ that is allowed to specify. It is useful in this context to look at words by taking together each $k+1$ consecutive letters. We call the sequence of these blocks the k-decomposition of a word (k indicates the number of overlapping letters). Then we introduce classes $\mathcal{B}_{1/2,k}$ for $k \geq 0$ such that $\mathcal{L}_{1/2} = \mathcal{B}_{1/2,0}$ and $\mathcal{B}_{1/2}$ is the union over all $\mathcal{B}_{1/2,k}$. For fixed $k \geq 0$, we take $\mathcal{B}_{1/2,k}$ as level 1/2 of a concatenation hierarchy and define for $n \geq 1$ classes $\mathcal{B}_{n/2,k}$ just as in the case of the DDH and STH. It holds for all $n \geq 1$ that $\mathcal{L}_{n/2} = \mathcal{B}_{n/2,0}$ and $\mathcal{B}_{n/2}$ is the union over all $\mathcal{B}_{n/2,k}$ for $k \geq 0$. The idea of looking at a parameterization in terms of block lengths is from [Sim72] for dot–depth one and from [Str85] for the general case.

Finally, we turn in this chapter to deterministic finite automata. We make precise what we understand under a forbidden pattern $\mathbb{P}$ in a transition graph and what we mean by $\mathcal{FP}(\mathbb{P})$: it is the class of all regular languages L such that there exists some DFA accepting L which does *not* have $\mathbb{P}$ in its transition graph. Results of the type '$\mathcal{C} = \mathcal{FP}(\mathbb{P})$' for language classes $\mathcal{C}$ are called forbidden pattern characterizations.

Chapter Two

We contribute in this chapter to the study of languages having dot–depth one. To our knowledge, $\mathcal{B}_1$ is the highest level of the DDH which is closed under Boolean operations and which is known to be decidable. The decidability of $\mathcal{B}_1$ was first shown with an algebraic approach in [Kna83]. From characterizations via families of equivalence relations [Sim72] several subhierarchies in $\mathcal{B}_1$ can be derived. One can show that the family of classes $\mathcal{B}_{1,k}$ for $k \geq 0$ is the so–called γ–hierarchy [Brz76], first studied in [Sim72]. The decision algorithms for $\mathcal{B}_1$ have been investigated in [Ste85a, Ste85b] and the membership problem is known to be complete for NL [CH91]. Besides the connections to first–order logic and complexity theory there is also a relation to Boolean circuits, recently studied in [MPT00]. But it is not only its location in the DDH that makes $\mathcal{B}_1$ an interesting class to look at. It can also be viewed as the natural generalization of the two notions of local and piecewise testability which both have gained much attention [BS73, McN74, Sim75, TW85, BP89, Str94]. In fact, $\mathcal{B}_{1,0} = \mathcal{L}_1$ is the class of piecewise testable languages, shown to be decidable in [Sim75].

We contribute in two ways. First, we look at the classes $\mathcal{B}_{1/2,k}$ for $k \geq 0$. We recall a generalization of the subword relation introduced in [Ste85a] and prove that these relations $\preceq_k$ for $k \geq 0$ have a fundamental property: $\preceq_k$ is a well partial order on A^+. This generalizes the well–known result of this type for the usual subword relation from [Hig52]. Then we show that $\mathcal{B}_{1/2,k}$ is the class of all order ideals of $(A^+, \preceq_k)$ and we provide a forbidden pattern characterization: it holds that $\mathcal{B}_{1/2,k} = \mathcal{FP}(\mathbb{B}_{1/2,k})$ for a certain pattern $\mathbb{B}_{1/2,k}$. With the latter we show how the forbidden pattern $\mathbb{L}_{1/2}$ for $\mathcal{L}_{1/2}$ turns in a natural way into the forbidden pattern $\mathbb{B}_{1/2}$ for $\mathcal{B}_{1/2}$ as k increases. We will observe and exploit this mechanism again in Chapters 3 and 4.

Second, we consider the classes $\mathcal{B}_{1,k}$ for $k \geq 0$. We restate the main result from [Ste85a] which gives various characterizations and which we refine in the following way. It is shown in [Ste85a] that a language L belongs to $\mathcal{B}_{1,k}$ if and only if L induces only a finite number of alternations in $\preceq_k$-chains. We prove that the maximal number of such alternations with respect to L determines the location of L in the Boolean hierarchy over $\mathcal{B}_{1/2,k}$. This has the mentioned finiteness condition as a corollary, and we use our characterization to obtain strictness and decidability results for the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ (note that the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ exhausts $\mathcal{B}_{1,k}$). The decidability of these classes has been independently studied using a logical approach in [Sel01]. Such results are also known for the Boolean hierarchy over $\mathcal{B}_{1/2}$ [Gla99] and taking them into account we obtain a complete overview over the Boolean structure of $\mathcal{B}_1$. In particular, we identify a landscape that allows to study the question whether or not there exists a trade–off in $\mathcal{B}_1$ between the parameter k on one hand and Boolean operations on the other hand. Both can be understood as a measure of the descriptional complexity of dot–depth one languages. Finally, we derive a forbidden pattern characterization of $\mathcal{B}_1$ and show that $\mathcal{B}_1 = \mathcal{FP}(\mathbb{B}_1)$ for some pattern $\mathbb{B}_1$.

Chapter Three

This chapter deals with deterministic languages and restricted temporal logic. Let us recall right deterministic languages from [Eil76], see also [BF80, Pin86, CPP93]. A language is called right deterministic if it is a finite union of languages $A_0^* a_1 A_1^* \cdots a_n A_n^*$ with $A_i \subseteq A$ and $a_i \notin A_i$. We adapt with little modifications the notion of right deterministic languages to k-decompositions of words and introduce right k-deterministic languages. For fixed $k \geq 0$ we prove a forbidden pattern characterization of the class $\mathcal{D}_k^{\text{right}}$ of right k-deterministic languages and show that $\mathcal{D}_k^{\text{right}} = \mathcal{FP}(\mathbb{D}_k^{\text{rev}})$ for some pattern $\mathbb{D}_k^{\text{rev}}$.

As it turns out, there are close connections to restricted temporal logic (RTL, for short). The latter is a fragment of temporal logic (more precisely: propositional linear–time temporal logic), a formalism to describe events occurring over time. The ability of temporal logic to express temporal properties has been recently investigated and surveyed in [Wil99], see also [Wil98]. There the expressive power of fragments obtained by omitting one or the other of the usual temporal operators *next* ($\mathbf{X}$), *eventually* ($\mathbf{F}$) and *until* ($\mathbf{U}$) have been studied. In case of restricted temporal logic RTL the use of $\mathbf{U}$ is not allowed.

Several proofs are known for the fact that formulas involving all three operators together with Boolean connectives (interpreted over finite words) yield the star–free languages [Kam68, MP71, GPSS80, CPP93, Wil99]. A natural hierarchy of star–free languages emerges from counting the nesting depth in $\mathbf{U}$. This until–hierarchy was introduced and shown to be strict in [EW96] while its decidability goes back to [TW96]. Interestingly, this is an example of a strict hierarchy exhausting the class of star–free languages with decidable membership problems. However, there is a family of languages in $\mathcal{B}_{3/2}$ separating all levels of the until–hierarchy, so the necessity of a large nesting depth in $\mathbf{U}$ to define languages does not imply a large dot–depth [EW96].

Note that RTL is just the zero level of the until–hierarchy. Effective characterizations in terms of forbidden patterns for RTL are known from [CPP93]. In case **X** is also forbidden this was done in [CPP93, EW96]. Observe also that in the latter case we are not allowed to specify the *next* event while in the former the unrestricted use of **X** is possible. A natural way to further classify RTL is to restrict the nesting depth in the next operator **X**. We introduce the so–called next hierarchy. It formalizes in terms of hierarchy classes the question of how many nested uses of **X** are needed to express a certain property in restricted temporal logic. Then we prove that the languages in level k of the next hierarchy are exactly the right k-deterministic languages. As the main result of this chapter we show that the following concepts to define languages in fact coincide:

(1) L is definable by an RTL formula having next depth at most k.
(2) L is a finite union of right k-deterministic languages.
(3) Any DFA accepting L does not have pattern $\mathbb{D}_k^{\mathrm{rev}}$ in its transition graph.

The third statement allows to give concise proof of strictness and decidability results for the next hierarchy. Moreover, we see that our generalized deterministic languages are exactly the languages definable in restricted temporal logic. We also investigate in detail the relation of the next hierarchy to the DDH and STH. At the end of the chapter we come back to complexity theory and show how languages definable in restricted temporal logic and the complexity class Δ_2^{p} are related.

Chapter Four

We turn to level 3/2 of the DDH and show its decidability which answers an open question from [Pin96, PW97]. To obtain this result we recall from Chapter 2 how one can prove an effective characterization of $\mathcal{B}_{1/2}$ using the forbidden pattern $\mathbb{B}_{1/2,k}$ for $\mathcal{B}_{1/2,k}$ together with a bound on k in the size of a given DFA. This is fairly easy in case of $\mathcal{B}_{1/2}$ and something similar can be observed in case of right k-deterministic languages. We follow this approach one more time and consider the classes $\mathcal{B}_{3/2,k}$ for fixed $k \geq 0$.

As a first step we carry over the normal form result for $\mathcal{L}_{3/2} = \mathcal{B}_{3/2,0}$ known from [Arf91] to $\mathcal{B}_{3/2,k}$ for arbitrary k. One of the main technical contributions in this chapter is the proof of a forbidden pattern characterization of $\mathcal{B}_{3/2,k}$, i.e., we show that $\mathcal{B}_{3/2,k} = \mathcal{FP}(\mathbb{B}_{3/2,k})$ for a certain pattern $\mathbb{B}_{3/2,k}$. It implies the decidability of $\mathcal{B}_{3/2,k}$ for fixed $k \geq 0$ and enables us to prove the strictness of the hierarchy of classes $\mathcal{B}_{3/2,k}$ for $k = 0, 1, 2 \ldots$. Since we encounter in case $k = 0$ level 3/2 of the STH we provide as a by–product another proof of the decidability result for this class. Note that the previous proofs in [Arf91] and [PW97] use deep results from [Has83] and [Sim90], respectively.

With help of our generalization to arbitrary k we identify a single forbidden pattern $\mathbb{B}_{3/2}$ that must occur if k is large in comparison to the alphabet size and the size of the DFA. This pattern characterizes $\mathcal{B}_{3/2}$ and implies the announced result which extends the known decidability results for the DDH. It has consequences in first–order logic and the algebraic theory of finite semigroups. At the beginning of the chapter we develop a combinatorial tool that allows to partition words of arbitrary length into factors of bounded length such that every second factor u leads to a loop with label u in a given DFA.

Chapter Five

We follow the idea of forbidden pattern characterizations in a more general way and develop a method for a uniform definition of hierarchies via iterated patterns in transition graphs. Based on the previous result for $\mathcal{B}_{3/2}$ we observe how the forbidden pattern $\mathbb{B}_{1/2}$ characterizing $\mathcal{B}_{1/2}$ acts as a building block in the forbidden pattern $\mathbb{B}_{3/2}$ that characterizes $\mathcal{B}_{3/2}$. Surprisingly, we find this observation confirmed if we compare the pattern $\mathbb{L}_{1/2}$ for $\mathcal{L}_{1/2}$ with the pattern $\mathbb{L}_{3/2}$ for $\mathcal{L}_{3/2}$ after an appropriate rewriting of the latter. Note from the definition above that we get in both cases with the same language operations from one level to the next. Together, this motivates the introduction of an iteration rule IT on patterns which continues the just observed formation procedure.

In general, starting with some initial pattern $\mathcal{I}$, our iterator generates for $n \geq 0$ classes of patterns $\mathbb{P}_n^{\mathcal{I}}$ which in turn define language classes $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ as usual by prohibiting the patterns $\mathbb{P}_n^{\mathcal{I}}$ in transition graphs. For a class $\mathcal{C}$ of languages let co$\mathcal{C}$ denote the class of their complements. We prove that

$$\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \cup \mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}) \cap \mathrm{co}\mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$$

and as the main technical result in this chapter that

$$\mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}).$$

With the latter we relate in a very general way Boolean operations and concatenation to the structural complexity of transition graphs. We show that the membership problems of the classes $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ for fixed $n \geq 0$ are efficiently decidable if that is true for $n = 0$.

Chapter Six

We consider in this chapter particular initial patterns $\mathcal{B}$ and $\mathcal{L}$ such that $\mathcal{FP}(\mathbb{P}_0^{\mathcal{B}}) = \mathcal{B}_{1/2}$ and $\mathcal{FP}(\mathbb{P}_0^{\mathcal{L}}) = \mathcal{L}_{1/2}$. It follows from our general results in Chapter 5 and from our characterization $\mathcal{B}_{3/2} = \mathcal{FP}(\mathbb{B}_{3/2})$ in Chapter 4 that

$$\begin{array}{ll} \mathcal{B}_{1/2} = \mathcal{FP}(\mathbb{P}_0^{\mathcal{B}}) & \qquad \mathcal{L}_{1/2} = \mathcal{FP}(\mathbb{P}_0^{\mathcal{L}}) \\ \mathcal{B}_{3/2} = \mathcal{FP}(\mathbb{P}_1^{\mathcal{B}}) & \qquad \mathcal{L}_{3/2} = \mathcal{FP}(\mathbb{P}_1^{\mathcal{L}}) \\ \mathcal{B}_{n+1/2} \subseteq \mathcal{FP}(\mathbb{P}_n^{\mathcal{B}}) & \qquad \mathcal{L}_{n+1/2} \subseteq \mathcal{FP}(\mathbb{P}_n^{\mathcal{L}}) \end{array}$$

Moreover, we see that all classes $\mathcal{FP}(\mathbb{P}_n^{\mathcal{B}})$ and $\mathcal{FP}(\mathbb{P}_n^{\mathcal{L}})$ have decidable membership problems and that they form hierarchies that exhaust the class of star–free languages. The inclusions above imply in particular a lower bound algorithm for the dot–depth of a given language L. One just has to determine the class $\mathcal{FP}(\mathbb{P}_n^{\mathcal{B}})$ for minimal n to which L belongs, and it follows that the dot–depth of L is strictly greater than $n - 1/2$ (another lower bound result for dot-depth n is known from [Wei93]). Then we provide some arguments that the forbidden pattern classes are not too large, e.g., if for $n \geq 2$ they were all equal to the class of star–free languages nothing would be won. For this end, we provide more structural similarities between the DDH and STH and the hierarchies of forbidden pattern classes: all hierarchies show the same inclusion structure and, interestingly, the typical languages that separate the levels of the DDH and STH also separate levelwise our forbidden pattern hierarchies. In particular, it holds that $\mathcal{FP}(\mathbb{P}_n^{\mathcal{L}})$ does not capture $\mathcal{B}_{n+1/2}$.

So here we introduce two strict and decidable hierarchies of star–free languages that are comparable (at least in one direction) to the DDH and STH. Up to now we have no evidence against the coincidence of the concatenation hierarchies and the respective forbidden pattern classes which we state as a conjecture (note that this would solve the dot–depth problem).

In the second part of the chapter we prove that even $\mathcal{L}_{5/2} = \mathcal{FP}(\mathbb{P}_2^{\mathcal{L}})$ holds if a two–letter alphabet is considered, i.e., in this special case we can show the reverse inclusion. The forbidden pattern characterization of $\mathcal{L}_{5/2}$ implies in particular its decidability in the two–letter case which extends the known results for the STH and has consequences in first–order logic. To obtain this result we show something more general: whenever it holds that $\mathcal{B}_{n+1/2} = \mathcal{FP}(\mathbb{P}_n^{\mathcal{B}})$ for some $n \geq 1$ and arbitrary alphabets then it follows that $\mathcal{L}_{n+3/2} = \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{L}})$ in case of a two–letter alphabet. Since the prerequisite of this implication holds for $n = 1$ by our previous work we obtain $\mathcal{L}_{5/2} = \mathcal{FP}(\mathbb{P}_2^{\mathcal{L}})$ for two–letter alphabets.

The forbidden pattern approach turns out to be a useful method in the context of concatenation hierarchies. It leads to new insights in the structure of the classes at the lower end and it allows to push the line of decidability in the DDH and in the STH a little higher than previously known. Moreover, it provides several promising starting points for further investigations and seems to have the capability to be successful in the general case.

Publications

The following papers contain results presented in this thesis.

[GS00c] C. Glaßer and H. Schmitz, *Decidable hierarchies of starfree languages.* Proceedings FST TCS 2000, 20th Conference on the Foundations of Software Technology and Theoretical Computer Science, LNCS 1974, pages 503–515, Springer Verlag, 2000.

[GS00a] C. Glaßer and H. Schmitz, *The Boolean structure of dot–depth one.* Preproceedings DCAGRS 2000, Second International Workshop on Descriptional Complexity of Automata, Grammars and Related Structures, London, Ontario, July 27-29, 2000.

[GS00d] C. Glaßer and H. Schmitz, *Languages of dot-depth* 3/2. Proceedings STACS 2000, 17th Symposium on Theoretical Aspects of Computer Science, LNCS 1770, pages 555–566, Springer Verlag, 2000.

[Sch00] H. Schmitz, *Restricted temporal logic and deterministic languages.* Journal of Automata, Languages and Combinatorics, 5(3): 325–341, 2000.

These and the technical reports [Sch99a, Sch99b, GS99, GS00b] relate to the single chapters of this thesis as follows.

– The content of Chapter 2 is from [Sch99a] and appeared together with [Gla99] as [GS00a].
– The results of Chapter 3 are from [Sch99b] which appeared as [Sch00].
– Chapter 4 is from [GS99] published as [GS00d].
– Chapters 5 and 6 contain the material from [GS00b], partially published as [GS00c].

1. Concatenation Hierarchies

We start this introductory chapter with some basic notations, more will be given as we continue. We fix some arbitrary finite alphabet A with $|A| \geq 2$. Elements of A are called letters and a word over A is a finite (possibly empty) concatenation of letters from A. The empty word is denoted by ε and the set of all (non–empty) words over A is denoted by A^* (A^+, respectively). For $w \in A^*$ we denote by $|w|$ its number of letters. A language L is a subset of A^+ and we call a set of languages also a class of languages . If L may contain ε we explicitely mention that L is a subset of A^*. Forthcoming definitions of languages will be made with respect to our fixed alphabet A and we will take care that it is clear from the context if we deal with a particular alphabet. If $w \in A^*$ and $L \subseteq A^*$ we define the left and right residuals of L as $w^{-1}L =_{\text{def}} \{ v \in A^* \mid wv \in L \}$ and $Lw^{-1} =_{\text{def}} \{ v \in A^* \mid vw \in L \}$. We write $\mathcal{P}(B)$ for the powerset of an arbitrary set B. Moreover, for a class $\mathcal{C}$ of languages we denote by $\text{co}\mathcal{C} =_{\text{def}} \{ A^+ \setminus L \mid L \in \mathcal{C} \}$ the set of complements with respect to A^+.

Regular languages over A are built up from the empty set and the singletons $\{a\}$ for $a \in A$ using Boolean operations (finite union, finite intersection and complementation), concatenation and iteration. The subclass of star–free languages SF over A is of particular interest for us. Here the iteration operation is not allowed, and—since we look at subsets of A^+—complements are taken with respect to A^+. For background on regular languages we refer to any standard textbook, e.g., [HU79].

1.1 Definition of Concatenation Hierarchies

A natural approach to further classify star–free languages is to look at the two different kinds of remaining operations: Boolean operations on one hand and concatenation on the other hand. If we emphasize the number of their alternating uses, this leads to the notion of concatenation hierarchies. The DDH and the STH are well–known concatenation hierarchies. To state their definition below, we specify closure operations on language classes. Denote for a class $\mathcal{C}$ of languages its closure under finite (possibly empty) union by $\text{FU}(\mathcal{C})$. Moreover, we set

$$\text{Pol}(\mathcal{C}) =_{\text{def}} \text{FU}(\{ L_0 L_1 \cdots L_n \mid n \geq 0 \text{ and } L_i \in \mathcal{C} \})$$

as the polynomial closure of $\mathcal{C}$. Note that $\text{Pol}(\mathcal{C})$ is exactly the closure of $\mathcal{C}$ under finite (possibly empty) union and finite (non–empty) concatenation. Furthermore, $\mathcal{C}$ is a subset of the polynomial closure of $\mathcal{C}$. For a second closure operation we consider Boolean operations. We denote the Boolean closure of a class $\mathcal{C}$ of languages of A^+ by $\text{BC}(\mathcal{C})$ (as before, taking complements with respect to A^+).

Definition 1.1 (DDH). The classes of the dot–depth hierarchy are defined as

$$\begin{array}{lll} \mathcal{B}_{1/2} & =_{\text{def}} & \mathrm{Pol}(\{\,\{w\} \mid w \in A^+\,\} \cup \{A^+\}), & \\ \mathcal{B}_{n+1} & =_{\text{def}} & \mathrm{BC}(\mathcal{B}_{n+1/2}) & \text{for } n \geq 0 \text{ and} \\ \mathcal{B}_{n+3/2} & =_{\text{def}} & \mathrm{Pol}(\mathcal{B}_{n+1}) & \text{for } n \geq 0. \end{array}$$

Definition 1.2 (STH). The classes of the Straubing–Thérien hierarchy are defined as

$$\begin{array}{llll} \mathcal{L}_{1/2} & =_{\text{def}} & \mathrm{Pol}(\{\, A^*aA^* \mid a \in A\,\}), & \\ \mathcal{L}_{n+1} & =_{\text{def}} & \mathrm{BC}(\mathcal{L}_{n+1/2}) & \text{for } n \geq 0 \text{ and} \\ \mathcal{L}_{n+3/2} & =_{\text{def}} & \mathrm{Pol}(\mathcal{L}_{n+1}) & \text{for } n \geq 0. \end{array}$$

Note that $\{w\}$ and A^+ as the complement of the empty set, and also A^*aA^* can be considered as simple languages in terms of alternation complexity with respect to Boolean operations and concatenation. We call the introduced classes also the levels of the DDH and STH where for integers n levels n are called full levels, and levels $n + 1/2$ are called half levels. The discussion in the forthcoming section, Section 1.2, relates these definitions to other definitions of concatenation hierarchies known from literature. The following inclusion relations in each hierarchy are easy to see from the definitions.

Proposition 1.3. *For $n \geq 0$ it holds that $\mathcal{B}_{n+1/2} \cup \mathrm{co}\mathcal{B}_{n+1/2} \subseteq \mathcal{B}_{n+1} \subseteq \mathcal{B}_{n+3/2} \cap \mathrm{co}\mathcal{B}_{n+3/2}$ and $\mathcal{L}_{n+1/2} \cup \mathrm{co}\mathcal{L}_{n+1/2} \subseteq \mathcal{L}_{n+1} \subseteq \mathcal{L}_{n+3/2} \cap \mathrm{co}\mathcal{L}_{n+3/2}$.*

We can also compare one hierarchy to the other by inclusion.

Proposition 1.4. *For $n \geq 1$ it holds that*

1. $\mathcal{L}_{n-1/2} \subseteq \mathcal{B}_{n-1/2} \subseteq \mathcal{L}_{n+1/2}$,
2. $\mathrm{co}\mathcal{L}_{n-1/2} \subseteq \mathrm{co}\mathcal{B}_{n-1/2} \subseteq \mathrm{co}\mathcal{L}_{n+1/2}$ *and*
3. $\mathcal{L}_n \subseteq \mathcal{B}_n \subseteq \mathcal{L}_{n+1}$.

Proof. Since for all $a \in A$ it holds that $A^*aA^* = \{a\} \cup aA^+ \cup A^+a \cup A^+aA^+ \in \mathcal{B}_{1/2}$ we obtain $\mathcal{L}_{1/2} \subseteq \mathcal{B}_{1/2}$. Moreover, it holds that $A^+ = \bigcup_{a \in A} A^*aA^* \in \mathcal{L}_{1/2}$, and for $w \in A^+$ with $w = a_1 \cdots a_n$ for letters $a_i \in A$ and $n \geq 1$ we obtain

$$\{w\} = \underbrace{A^*a_1A^* \cdots a_nA^*}_{\in \mathcal{L}_{1/2}} \cap \left(A^+ \setminus \underbrace{\bigcup_{b_1,\ldots,b_{n+1} \in A} A^*b_1A^* \cdots b_{n+1}A^*}_{\in \mathcal{L}_{1/2}} \right) \in \mathcal{L}_1.$$

In particular, $A^+ \in \mathcal{L}_{3/2}$ and $w \in \mathcal{L}_{3/2}$ for all $w \in A^+$ from which we get $\mathcal{B}_{1/2} \subseteq \mathcal{L}_{3/2}$. So we have seen $\mathcal{L}_{1/2} \subseteq \mathcal{B}_{1/2} \subseteq \mathcal{L}_{3/2}$ and the proposition follows from the monotony of $\mathrm{Pol}(\cdot)$ and $\mathrm{BC}(\cdot)$, and complementation. ❑

It is easy to see that the classes $\mathcal{B}_n$ for $n \geq 1$ coincide with the ones studied in [Eil76]. In particular, it is shown in [Eil76, Chapter IX.4] that $\bigcup_{n \geq 1} \mathcal{B}_n = \mathrm{SF}$. Together with Proposition 1.4 we see the following.

Proposition 1.5. *It holds that $\bigcup_{n \geq 1} \mathcal{L}_{n/2} = \bigcup_{n \geq 1} \mathcal{B}_{n/2} = \mathrm{SF}$.*

Figure 1.1 gives an overview.

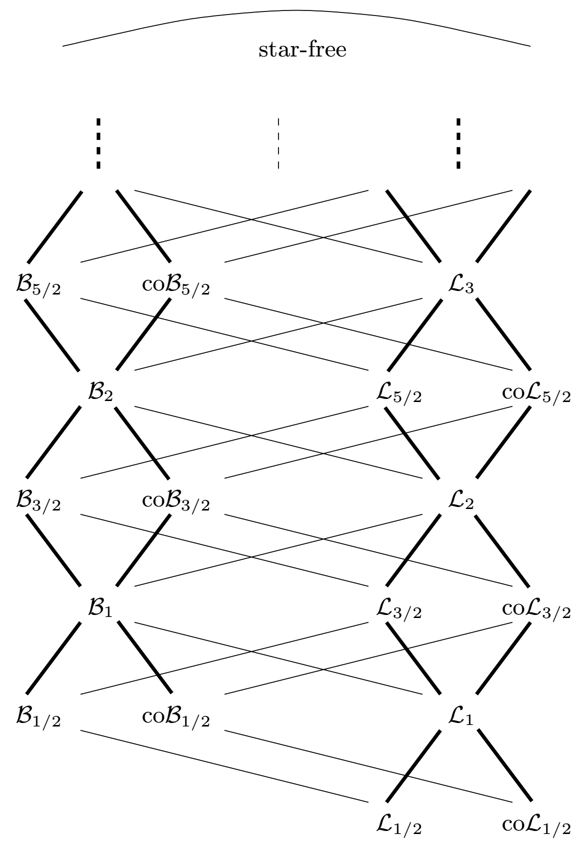


Fig. 1.1. Concatenation hierarchies of star–free languages. Inclusions hold from bottom to top.

1.2 Alternative Definitions and Normal Forms

The dot–depth hierarchy and the Straubing–Thérien hierarchy have gained much attention due to the still pending dot–depth problem (see Section 1.3 below). The purpose of this section here is to make our work comparable to other investigations from the literature and to take over known results to our notations. So we discuss alternative definitions and there are two points to look at. First, one finds other versions of the polynomial closure operation in the literature. Let $\mathcal{C}$ be a class of languages. Here are the definitions of the polynomial closure as chosen, e.g., with an algebraic approach recently in [PW97].

$$\begin{aligned}
\mathrm{Pol}^{\mathcal{L}}(\mathcal{C}) &=_{\mathrm{def}} \mathrm{FU}(\{\, L_0 a_1 L_1 \cdots a_n L_n \,|\, n \geq 0, L_i \in \mathcal{C} \text{ and } a_i \in A \,\}) \\
\mathrm{Pol}^{\mathcal{B}}(\mathcal{C}) &=_{\mathrm{def}} \mathrm{FU}(\{\, u_0 L_1 u_1 \cdots L_n u_n \,|\, n \geq 0, L_i \in \mathcal{C}, u_i \in A^* \text{ and if } n = 0 \text{ then } u_0 \neq \varepsilon \,\})
\end{aligned}$$

A second point is that languages may be defined in a way such that they contain the empty word. So we also want to see if that makes any difference. It is pointed out, e.g., in [Pin95] that this is a crucial point in the theory of varieties of finite semigroups. We denote the Boolean closure of a class $\mathcal{D}$ of languages of A^* by $\mathrm{BC}^*(\mathcal{D})$ (here taking complements with respect to A^*). Moreover, let $\mathrm{co}^*\mathcal{D} =_{\mathrm{def}} \{\, A^* \setminus L \,|\, L \in \mathcal{C} \,\}$ denote the set of complements with respect to A^*.

Definition 1.6 (DDH due to [Pin96]). Let $\mathcal{B}^+_{1/2}$ be the class of all languages of A^+ that can be written as finite unions of languages of the form $u_0 A^+ u_1 \cdots A^+ u_m$ where $m \geq 0$ and $u_i \in A^*$. For $n \geq 0$ let $\mathcal{B}^+_{n+1} =_{\mathrm{def}} \mathrm{BC}(\mathcal{B}^+_{n+1/2})$ and $\mathcal{B}^+_{n+3/2} =_{\mathrm{def}} \mathrm{Pol}^{\mathcal{B}}(\mathcal{B}^+_{n+1})$.

Definition 1.7 (STH due to [Str81, Thé81]). Let $\mathcal{L}^*_{1/2}$ be the class of all languages of A^* that can be written as finite unions of languages of the form $A^* a_1 A^* \cdots a_m A^*$ where $m \geq 0$ and $a_i \in A$. For $n \geq 0$ let $\mathcal{L}^*_{n+1} =_{\mathrm{def}} \mathrm{BC}^*(\mathcal{L}^*_{n+1/2})$ and $\mathcal{L}^*_{n+3/2} =_{\mathrm{def}} \mathrm{Pol}^{\mathcal{L}}(\mathcal{L}^*_{n+1})$.

These definitions are local to the remainder of Section 1.2. For the approach we follow in this thesis we find it suitable to have the inclusion relations from Proposition 1.4 at hand. We show in this section the following theorem.

Theorem 1.8. *It holds that*

1. $\mathcal{B}^+_{n/2} = \mathcal{B}_{n/2}$ *for* $n \geq 1$,
2. $\mathcal{L}^*_{1/2} = \mathcal{L}_{1/2} \cup \{A^*\}$ *and*
3. $\mathcal{L}^*_{n/2} = \mathcal{L}_{n/2} \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_{n/2} \,\}$ *for* $n \geq 2$.

So the classes $\mathcal{B}^+_{n/2}$ from Definition 1.6 and the classes $\mathcal{B}_{n/2}$ coincide, and the languages in $\mathcal{L}^*_{n/2}$ are up to the empty word the languages in $\mathcal{L}_{n/2}$. This enables us to carry over known normal forms and closure properties to our definitions in Subsection 1.2.3. The proof of Theorem 1.8 is given in Subsection 1.2.2. Let us recall known closure properties of the just defined classes.

Lemma 1.9 ([Arf91, PW97, Gla98]). *Let* $n \geq 1$ *and* $a \in A$.

1. *The classes* $\mathcal{B}^+_{n/2}, \mathrm{co}\mathcal{B}^+_{n/2}, \mathcal{L}^*_{n/2}$ *and* $\mathrm{co}^*\mathcal{L}^*_{n/2}$ *are closed under finite union and intersection.*
2. *Let* $\mathcal{C}$ *be one of the classes* $\mathcal{B}^+_{n/2}$ *or* $\mathrm{co}\mathcal{B}^+_{n/2}$. *Then* $a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \mathcal{C}$ *for* $L \in \mathcal{C}$.
3. *Let* $\mathcal{D}$ *be one of the classes* $\mathcal{L}^*_{n/2}$ *or* $\mathrm{co}^*\mathcal{L}^*_{n/2}$. *Then* $a^{-1}L, La^{-1} \in \mathcal{D}$ *for* $L \in \mathcal{D}$.

1.2.1 Polynomial Closure and Boolean Closure

We investigate the relationships between the different notions of polynomial closure and identify a condition for $\mathcal{C}$ under which these notions coincide.

Theorem 1.10. *It holds that* $\mathrm{Pol}(\mathcal{C}) = \mathrm{Pol}^{\mathcal{L}}(\mathcal{C}) = \mathrm{Pol}^{\mathcal{B}}(\mathcal{C})$ *for a class of languages* $\mathcal{C}$ *satisfying the conditions*

(a) $\{w\} \in \mathcal{C}$ *for* $w \in A^+$ *and*
(b) $a^{-1}L \cap A^+ \in \mathcal{C}$ *and* $La^{-1} \cap A^+ \in \mathcal{C}$ *for* $L \in \mathcal{C}$ *and* $a \in A$.

The theorem is immediate from the following lemma.

Lemma 1.11. *Let* $\mathcal{C}$ *be a class of languages.*

1. $\mathrm{Pol}^{\mathcal{L}}(\mathcal{C}) \subseteq \mathrm{Pol}^{\mathcal{B}}(\mathcal{C})$
2. $\mathrm{Pol}(\mathcal{C}) \subseteq \mathrm{Pol}^{\mathcal{B}}(\mathcal{C})$
3. *If* $\{w\} \in \mathcal{C}$ *for all* $w \in A^+$ *then* $\mathrm{Pol}^{\mathcal{B}}(\mathcal{C}) \subseteq \mathrm{Pol}(\mathcal{C})$.
4. *If* $\{w\}, a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \mathcal{C}$ *for* $w \in A^1 \cup A^2$, $a \in A$, $L \in \mathcal{C}$, *then* $\mathrm{Pol}(\mathcal{C}) \subseteq \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$.
5. *If* $\{a\} \in \mathcal{C}$ *for* $a \in A$ *then* $\mathrm{Pol}^{\mathcal{L}}(\mathcal{C}) \subseteq \mathrm{Pol}(\mathcal{C})$.

Proof. Statements 1, 2 and 5 can be easily verified, and we argue for statement 3. Let a language of the form $u_0L_1u_1\cdots L_nu_n$ be given with $n \geq 0$, $L_i \in \mathcal{C}$ and $u_i \in A^*$ such that if $n = 0$ then $u_0 \neq \varepsilon$. Then we can take out every $u_i = \varepsilon$ in this representation without changing the language. By assumption, $\{w\} \in \mathcal{C}$ for $w \in A^+$, so we may replace all remaining $u_i \in A^+$ by languages from $\mathcal{C}$. We obtain an equivalent expression of the form $L_0'L_1'\cdots L_{n'}'$ with $n' \geq 0$ and $L_i' \in \mathcal{C}$ (note that if $n = 0$ then $\{u_0\} \in \mathcal{C}$). This shows statement 3.

Let us turn to statement 4. Here we have the assumption that $\{w\} \in \mathcal{C}$ for $w \in A^+$ with length 1 or 2, and $a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \mathcal{C}$ for $a \in A$ and $L \in \mathcal{C}$. It suffices to show that $L_0L_1\cdots L_n \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$ for $n \geq 0$ and $L_i \in \mathcal{C}$. We prove this by induction on n. The induction base with $n = 0$ is trivial. Now we assume that we have proven $L_0L_1\cdots L_n \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$ for $n \geq 0$ and we want to show that $L_0L_1\cdots L_{n+1} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$. Since $L_0L_1\cdots L_n \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$ by induction hypothesis, it suffices to show for some $L' =_{\mathrm{def}} L_0'a_1L_1'\cdots a_lL_l'$ with $l \geq 0$, $L_i' \in \mathcal{C}$ and $a_i \in A$ that $L =_{\mathrm{def}} L' \cdot L_{n+1} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$. We obtain

$$L = \begin{cases} \Big(\bigcup_{a\in A} L'\cdot a\cdot \big(a^{-1}L_{n+1}\cap A^+\big)\Big) \cup \Big(\bigcup_{a\in L_{n+1}\cap A} L'\cdot a\Big) \cup L' & : \text{ if } \varepsilon \in L_{n+1} \\ \Big(\bigcup_{a\in A} L'\cdot a\cdot \big(a^{-1}L_{n+1}\cap A^+\big)\Big) \cup \Big(\bigcup_{a\in L_{n+1}\cap A} L'\cdot a\Big) & : \text{ otherwise} \end{cases}$$

By assumption we have $a^{-1}L_{n+1} \cap A^+ \in \mathcal{C}$. It follows that

$$\bigcup_{a\in A} L'\cdot a\cdot \big(a^{-1}L_{m+1}\cap A^+\big) \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C}).$$

Since also $L' \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$ it remains to show that $L' \cdot b \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$ for letters $b \in A$. If $l \geq 1$ then define $L'' =_{\mathrm{def}} L_0'a_1\cdots L_{l-1}'a_l$. Now consider the following case study of $L' \cdot b$.

$$L'\cdot b = \begin{cases} \Big(\bigcup_{a\in A} L''\cdot \big(L_l'a^{-1}\cap A^+\big)\cdot a\cdot b\Big) \cup \Big(\bigcup_{a\in L_l'\cap A} L''\cdot a\cdot b\Big) \cup L''\cdot b & : \text{ if } l\geq 1 \text{ and } \varepsilon\in L_l' \\ \Big(\bigcup_{a\in A} L''\cdot \big(L_l'a^{-1}\cap A^+\big)\cdot a\cdot b\Big) \cup \Big(\bigcup_{a\in L_l'\cap A} L''\cdot a\cdot b\Big) & : \text{ if } l\geq 1 \text{ and } \varepsilon\notin L_l' \\ \Big(\bigcup_{a\in A} \big(L_l'a^{-1}\cap A^+\big)\cdot a\cdot b\Big) \cup \Big(\bigcup_{a\in L_l'\cap A} \{ab\}\Big) \cup \{b\} & : \text{ if } l=0 \text{ and } \varepsilon\in L_l' \\ \Big(\bigcup_{a\in A} \big(L_l'a^{-1}\cap A^+\big)\cdot a\cdot b\Big) \cup \Big(\bigcup_{a\in L_l'\cap A} \{ab\}\Big) & : \text{ if } l=0 \text{ and } \varepsilon\notin L_l' \end{cases}$$

By assumption, $\{b\}, \{ab\} \in \mathcal{C}$ for letters a, b and $\tilde{L} =_{\mathrm{def}} L_l'a^{-1} \cap A^+ \in \mathcal{C}$. Hence for $a, b \in A$ it holds that

- if $l \geq 1$ then $L'' \cdot \big(L_l'a^{-1} \cap A^+\big) \cdot a \cdot b = L'' \cdot \tilde{L} \cdot a \cdot \{b\} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$,
- if $l \geq 1$ then $L'' \cdot a \cdot b = L'' \cdot \{ab\} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$,
- if $l \geq 1$ then $L'' \cdot b = L'' \cdot \{b\} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$, and
- $\big(L_l'a^{-1} \cap A^+\big) \cdot a \cdot b = \tilde{L} \cdot a \cdot \{b\} \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$.

Together with the case study this implies $L' \cdot b \in \mathrm{Pol}^{\mathcal{L}}(\mathcal{C})$, which completes the induction and the proof of statement 4. ❑

If the languages of two classes $\mathcal{C}$ and $\mathcal{D}$ differ only by the empty word, we show that this property is preserved by the polynomial closure operation. In other words, we obtain that also $\mathrm{Pol}(\mathcal{C})$ is equal to $\mathrm{Pol}(\mathcal{D})$ up to the empty word.

Lemma 1.12. *Let $\mathcal{C}$ be a class of languages of A^+ and $\mathcal{D}$ be a class of languages of A^*. If $\{\varepsilon\} \in \mathcal{D}$ and $\mathcal{D} = \mathcal{C} \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{C} \,\}$ then it holds that*

1. $\mathrm{Pol}(\mathcal{C})$ *is a class of languages of* A^+ *and* $\mathrm{Pol}(\mathcal{D})$ *is a class of languages of* A^*,
2. $\{\varepsilon\} \in \mathrm{Pol}(\mathcal{D})$ *and*
3. $\mathrm{Pol}(\mathcal{D}) = \mathrm{Pol}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{Pol}(\mathcal{C}) \,\}$.

Proof. Statements 1 and 2 follow immediately from the definition of $\mathrm{Pol}(\cdot)$. First, we show the inclusion $\mathrm{Pol}(\mathcal{D}) \subseteq \mathrm{Pol}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{Pol}(\mathcal{C}) \,\}$. Since the right hand side is closed under finite union, it suffices to prove the following claim.
Claim. It holds that $L_0 \cdots L_n \in \mathrm{Pol}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{Pol}(\mathcal{C}) \,\}$ for $n \geq 0$ and $L_i \in \mathcal{D}$.

We give a proof of the claim by induction on n. For the induction base let $n = 0$ and observe that the claim holds. Now assume that we have proven the claim for some $n \geq 0$ and we want to show it for $n + 1$. Let $L =_{\mathrm{def}} L_0 \cdots L_{n+1}$ with $L_i \in \mathcal{D}$. By induction hypothesis there exists some $L' \in \mathrm{Pol}(\mathcal{C})$, such that $L_0 \cdots L_n = L'$ or $L_0 \cdots L_n = L' \cup \{\varepsilon\}$. Since $L_{n+1} \in \mathcal{D}$ there exists an $L'' \in \mathcal{C} \subseteq \mathrm{Pol}(\mathcal{C})$ such that $L_{n+1} = L''$ or $L_{n+1} = L'' \cup \{\varepsilon\}$. This leads to the following four cases.

$$L = \begin{cases} L'L'' & : \text{ if } L_0 \cdots L_n = L' \text{ and } L_{n+1} = L'' \\ L'L'' \cup L' & : \text{ if } L_0 \cdots L_n = L' \text{ and } L_{n+1} = L'' \cup \{\varepsilon\} \\ L'L'' \cup L'' & : \text{ if } L_0 \cdots L_n = L' \cup \{\varepsilon\} \text{ and } L_{n+1} = L'' \\ L'L'' \cup L'' \cup L' \cup \{\varepsilon\} & : \text{ if } L_0 \cdots L_n = L' \cup \{\varepsilon\} \text{ and } L_{n+1} = L'' \cup \{\varepsilon\} \end{cases}$$

In any case there exists some $\tilde{L} \in \mathrm{Pol}(\mathcal{C})$ with $L = \tilde{L}$ or $L = \tilde{L} \cup \{\varepsilon\}$. This proves the claim.

Finally, we have to show the reverse inclusion. Since $\mathcal{C} \subseteq \mathcal{D}$ we have $\mathrm{Pol}(\mathcal{C}) \subseteq \mathrm{Pol}(\mathcal{D})$. So $L \cup \{\varepsilon\} \in \mathrm{Pol}(\mathcal{D})$ for $L \in \mathrm{Pol}(\mathcal{C})$ because $\{\varepsilon\} \in \mathrm{Pol}(\mathcal{D})$. This proves the lemma. ❑

We show the counterpart of Lemma 1.12 for Boolean closures.

Lemma 1.13. *Let $\mathcal{C}$ be a class of languages of A^+ and $\mathcal{D}$ be a class of languages of A^*. If $\{\varepsilon\} \in \mathcal{D}$ and $\mathcal{D} = \mathcal{C} \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{C} \,\}$ then it holds that*

1. $\mathrm{BC}(\mathcal{C})$ *is a class of languages of* A^+ *and* $\mathrm{BC}^*(\mathcal{D})$ *is a class of languages of* A^*,
2. $\{\varepsilon\} \in \mathrm{BC}^*(\mathcal{D})$ *and*
3. $\mathrm{BC}^*(\mathcal{D}) = \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$.

Proof. Statements 1 and 2 follow from the definition of $\mathrm{BC}(\cdot)$ and $\mathrm{BC}^*(\cdot)$. First, we show the inclusion $\mathrm{BC}^*(\mathcal{D}) \subseteq \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$. Note that languages from $\mathrm{BC}^*(\mathcal{D})$ can be written as finite unions of finite intersections of literals, which in turn are of the form L (positive literal) or $A^* \setminus L$ (negative literal) for some $L \in \mathcal{D}$. It is easy to see the right hand side of the equation in statement 3 is closed under finite union and intersection. So it suffices to show that all literals are elements of the right hand side. Since $\mathcal{D} \subseteq \mathcal{C} \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{C} \,\} \subseteq \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$ all positive literals are elements of the right hand side and it remains to show the following claim.
Claim. It holds that $A^* \setminus L \in \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$ for $L \in \mathcal{D}$.

Let $L \in \mathcal{D}$. Then there exists an $L' \in \mathcal{C}$ such that $L = L'$ or $L = L' \cup \{\varepsilon\}$. Since $L' \subseteq A^+$ we obtain $A^* \setminus L = (A^+ \setminus L') \cup \{\varepsilon\}$ in the first case and $A^* \setminus L = (A^+ \setminus L')$ in the second case. Because $A^+ \setminus L' \in \mathrm{BC}(\mathcal{C})$ we conclude that $A^* \setminus L \in \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$. This proves the claim and it follows that $\mathrm{BC}^*(\mathcal{D}) \subseteq \mathrm{BC}(\mathcal{C}) \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\}$.

Finally, we turn to the reverse inclusion. From $\mathcal{C} \subseteq \mathcal{D}$, $A^+ \setminus L = (A^* \setminus L) \cap (A^* \setminus \{\varepsilon\})$ for $L \subseteq A^+$ and $\{\varepsilon\} \in \mathrm{BC}^*(\mathcal{D})$ it follows that $\mathrm{BC}(\mathcal{C}) \subseteq \mathrm{BC}^*(\mathcal{D})$ and $\{\, L \cup \{\varepsilon\} \mid L \in \mathrm{BC}(\mathcal{C}) \,\} \subseteq \mathrm{BC}^*(\mathcal{D})$. This proves the lemma. ❑

1.2.2 Comparing Hierarchies: Proof of Theorem 1.8

Now we are ready to compare Definitions 1.1 and 1.2 with Definitions 1.6 and 1.7. First, we apply our Theorem 1.10.

Proposition 1.14.

1. *For $n \geq 1$ let $\mathcal{C}$ be one of the classes $\mathcal{B}^+_{n/2}$ or $\mathrm{co}\mathcal{B}^+_{n/2}$. Then* $\mathrm{Pol}(\mathcal{C}) = \mathrm{Pol}^{\mathcal{L}}(\mathcal{C}) = \mathrm{Pol}^{\mathcal{B}}(\mathcal{C})$.
2. *For $n \geq 2$ let $\mathcal{D}$ be one of the classes $\mathcal{L}^*_{n/2}$ or $\mathrm{co}^*\mathcal{L}^*_{n/2}$. Then* $\mathrm{Pol}(\mathcal{D}) = \mathrm{Pol}^{\mathcal{L}}(\mathcal{D}) = \mathrm{Pol}^{\mathcal{B}}(\mathcal{D})$.

Proof. To show the proposition we need to prove that the mentioned classes fulfill the two conditions from Theorem 1.10:

(a) $\{w\} \in \mathcal{C}$ for all $w \in A^+$ and
(b) $a^{-1}L \cap A^+ \in \mathcal{C}$ and $La^{-1} \cap A^+ \in \mathcal{C}$ for $L \in \mathcal{C}$ and $a \in A$.

In fact, it suffices to prove condition (a) for $\mathcal{B}^+_{1/2}$, $\mathrm{co}\mathcal{B}^+_{1/2}$ and $\mathcal{L}^*_1$ since these classes are included in the respective higher levels. So let $w \in A^+$. Then we have $\{w\} \in \mathcal{B}^+_{1/2}$ by definition, and we see that

$$\{w\} = A^+ \setminus \Bigg(\underbrace{\bigcup_{\substack{v \in A^+ \text{with} \\ |v|=|w|}} vA^+\varepsilon \cup \bigcup_{\substack{v \in A^+ \setminus \{w\} \\ \text{with } |v| \leq |w|}} v}_{\in \mathcal{B}^+_{1/2}} \Bigg) \in \mathrm{co}\mathcal{B}^+_{1/2}.$$

If $w = a_1 \cdots a_n$ for $n \geq 1$ and letters $a_i \in A$ we obtain (a) for $\mathcal{L}^*_1$ with

$$\{w\} = \underbrace{A^*a_1A^* \cdots a_nA^*}_{\in \mathcal{L}^*_{1/2}} \cap \Bigg(A^* \setminus \underbrace{\bigcup_{b_1,\ldots,b_{n+1} \in A} A^*b_1A^* \cdots b_{n+1}A^*}_{\in \mathcal{L}^*_{1/2}} \Bigg) \in \mathcal{L}^*_1.$$

It remains to verify condition (b) for both statements. This is clear for statement 1 by Lemma 1.9. We use the same lemma for statement 2 together with the closure under intersection. Note that $A^+ \in \mathcal{L}^*_{1/2} \subseteq \mathcal{L}^*_1$. ❑

The next two propositions will serve as the induction base for the proof of Theorem 1.8 below.

Proposition 1.15. *The following holds.*

1. $\mathcal{B}_{1/2} = \mathcal{B}^+_{1/2}$
2. *$\mathcal{B}_{1/2}$ is equal to the class of languages of A^+ that can be written as finite unions of languages of the form $u_0A^*u_1 \cdots A^*u_m$ where $m \geq 0$ and $u_i \in A^*$.*

Proof. Observe that languages from $\mathcal{B}_{1/2}$ are languages of A^+ that can be written as finite unions of languages of the form $u_0A^+u_1 \cdots A^+u_m$ where $m \geq 0$ and $u_i \in A^*$ (add some $u_i = \varepsilon$ if necessary). It follows that $\mathcal{B}_{1/2} \subseteq \mathcal{B}^+_{1/2}$. On the other hand, languages of A^+ having the form $u_0A^+u_1\cdots A^+u_m$ with $m \geq 0$ and $u_i \in A^*$, can be written as concatenations of A^+ and non–empty words (just drop all $u_i = \varepsilon$). This shows $\mathcal{B}_{1/2} \subseteq \mathcal{B}^+_{1/2}$ and statement 1 follows.

Now we exploit statement 1 for the proof of statement 2. Therefore, it suffices to show that for each language $L \subseteq A^+$ the following holds: L is a finite union of languages of the form $u_0A^*u_1\cdots A^*u_m$ with $m \geq 0$, $u_i \in A^*$ if and only if L is a finite union of languages of the form $u_0A^+u_1\cdots A^+u_m$ with $m \geq 0$, $u_i \in A^*$. This is easy to see since we can replace A^* and A^+ vice versa, due to $A^* = \{\varepsilon\} \cup A^+$ and $A^+ = \bigcup_{a\in A} aA^*$. This shows statement 2. ❑

Proposition 1.16. *The following holds.*

1. *$\mathcal{L}_{1/2}$ is equal to the class of languages of A^+ that can be written as finite unions of languages of the form $A^*a_1A^*\cdots a_mA^*$ where $m \geq 0$ and $a_i \in A$.*
2. *$\mathcal{L}_{1/2}$ is a class of languages of A^+ and $\mathcal{L}^*_{1/2} = \mathcal{L}_{1/2} \cup \{A^*\}$.*
3. *$\mathcal{L}_1$ is a class of languages of A^+ and $\mathcal{L}^*_1 = \mathcal{L}_1 \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$.*

Proof. Clearly, $\mathcal{L}_{1/2}$ and $\mathcal{L}_1$ are classes of languages of A^+. So to see statement 1 it suffices to mention that (i) the case $m = 0$ cannot occur since we speak about languages of A^+ and (ii) $A^* = A^*A^*$. Statement 2 follows immediately from statement 1 and the definition of $\mathcal{L}^*_{1/2}$. For statement 3 observe that $\mathcal{L}^*_1$ and $\mathcal{L}_1 \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$ are classes being closed under finite union and finite intersection. Furthermore, we have

$$\{\varepsilon\} = A^* \setminus \Big(\bigcup_{a\in A} A^*aA^* \Big) \quad \in \mathrm{co}^*\mathcal{L}^*_{1/2} \subseteq \mathcal{L}^*_1.$$

Thus it suffices to show that (i) $L, A^* \setminus L \in \mathcal{L}_1 \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$ for $L \in \mathcal{L}^*_{1/2}$ and (ii) $L', A^+ \setminus L' \in \mathcal{L}^*_1$ for $L' \in \mathcal{L}_{1/2}$. Since by the second statement of the lemma we have $\mathcal{L}^*_{1/2} = \mathcal{L}_{1/2} \cup \{A^*\}$ and because

$$A^* = \underbrace{\bigcup_{a\in A} A^*aA^*}_{\in\mathcal{L}_{1/2}} \cup \{\varepsilon\}$$

it follows that $\mathcal{L}^*_{1/2} \subseteq \mathcal{L}_1 \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$ and $\mathcal{L}_{1/2} \subseteq \mathcal{L}^*_1$. Hence it remains to show that

1. $A^* \setminus L \in \mathcal{L}_1 \cup \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$ for $L \in \mathcal{L}^*_{1/2}$ and
2. $A^+ \setminus L' \in \mathcal{L}^*_1$ for $L' \in \mathcal{L}_{1/2}$.

First let $L \in \mathcal{L}^*_{1/2}$. If $L \notin \mathcal{L}_{1/2}$ then $L = A^*$ and we obtain $A^* \setminus L \in \mathcal{L}_{1/2} \subseteq \mathcal{L}_1$. Otherwise we can write $A^* \setminus L = (A^+ \setminus L) \cup \{\varepsilon\} \in \{\, L \cup \{\varepsilon\} \mid L \in \mathcal{L}_1 \,\}$. Now let $L' \in \mathcal{L}_{1/2} \subseteq \mathcal{L}^*_{1/2}$. Since $A^+ \in \mathcal{L}^*_{1/2}$ and $A^+ \setminus L' = (A^* \setminus L') \cap A^+$ we obtain $A^+ \setminus L' \in \mathcal{L}^*_1$. This proves the proposition. ❑

We give a proof of Theorem 1.8. Recall with Proposition 1.16 that it remains to show that

1. $\mathcal{B}^+_{n/2} = \mathcal{B}_{n/2}$ for $n \geq 1$ and
2. $\mathcal{L}^*_{n/2} = \mathcal{L}_{n/2} \cup \big\{\, L \cup \{\varepsilon\} \,\big|\, L \in \mathcal{L}_{n/2} \,\big\}$ for $n \geq 2$.

Proof of Theorem 1.8. We show both statements by induction on n and start with statement 1. By Proposition 1.15 the assertion holds for $n = 1$ which is the induction base. We first assume that it holds for $n \geq 1$ with $n \equiv 1 \mod 2$ and we want to prove it for $n+1$. Then

$$\mathcal{B}_{(n+1)/2} = \mathrm{BC}(\mathcal{B}_{n/2}) = \mathrm{BC}(\mathcal{B}^+_{n/2}) = \mathcal{B}^+_{(n+1)/2}$$

with the induction hypothesis $\mathcal{B}_{n/2} = \mathcal{B}^+_{n/2}$. This shows in particular statement 1 for $n = 2$. So now we assume that it holds for $n \geq 2$ with $n \equiv 0 \mod 2$ and we want to prove it for $n+1$. Then we have by definition $\mathcal{B}_{(n+1)/2} = \mathrm{Pol}(\mathcal{B}_{n/2})$ and from Proposition 1.14 we get $\mathcal{B}^+_{(n+1)/2} = \mathrm{Pol}^{\mathcal{B}}(\mathcal{B}^+_{n/2}) = \mathrm{Pol}(\mathcal{B}^+_{n/2})$. The induction hypothesis provides $\mathrm{Pol}(\mathcal{B}_{n/2}) = \mathrm{Pol}(\mathcal{B}^+_{n/2})$.

We turn to statement 2. The induction base $n = 2$ is given in Proposition 1.16. Again, we first assume that the assertion holds for $n \geq 2$ with $n \equiv 0 \mod 2$ and we want to prove it for $n+1$. Then we have by definition $\mathcal{L}^*_{(n+1)/2} = \mathrm{Pol}^{\mathcal{L}}(\mathcal{L}^*_{n/2})$ and $\mathcal{L}_{(n+1)/2} = \mathrm{Pol}(\mathcal{L}_{n/2})$. It holds that $\mathcal{L}_{n/2}$ is a class of languages of A^+, $\{\varepsilon\} \in \mathcal{L}^*_1 \subseteq \mathcal{L}^*_{n/2}$, and from the induction hypothesis we obtain $\mathcal{L}^*_{n/2} = \mathcal{L}_{n/2} \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathcal{L}_{n/2} \right\}$. This shows that the classes $\mathcal{C} =_{\mathrm{def}} \mathcal{L}_{n/2}$ and $\mathcal{D} =_{\mathrm{def}} \mathcal{L}^*_{n/2}$ satisfy the assumptions of Lemma 1.12 and we obtain

$$\mathrm{Pol}(\mathcal{L}^*_{n/2}) = \mathrm{Pol}(\mathcal{L}_{n/2}) \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathrm{Pol}(\mathcal{L}_{n/2}) \right\}.$$

Because $n \geq 2$ we get from Proposition 1.14 that $\mathrm{Pol}(\mathcal{L}^*_{n/2}) = \mathrm{Pol}^{\mathcal{L}}(\mathcal{L}^*_{n/2}) = \mathcal{L}^*_{(n+1)/2}$.

This shows in particular statement 2 for $n = 3$. So now we assume that the assertion holds for $n \geq 3$ with $n \equiv 1 \mod 2$ and we want to prove it for $n+1$. Then we have by definition $\mathcal{L}^*_{(n+1)/2} = \mathrm{BC}^*(\mathcal{L}^*_{n/2})$ and $\mathcal{L}_{(n+1)/2} = \mathrm{BC}(\mathcal{L}_{n/2})$. It holds that $\mathcal{L}_{n/2}$ is a class of languages of A^+, $\{\varepsilon\} \in \mathcal{L}^*_1 \subseteq \mathcal{L}^*_{n/2}$, and from the induction hypothesis we obtain $\mathcal{L}^*_{n/2} = \mathcal{L}_{n/2} \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathcal{L}_{n/2} \right\}$. This shows that the classes $\mathcal{C} =_{\mathrm{def}} \mathcal{L}_{n/2}$ and $\mathcal{D} =_{\mathrm{def}} \mathcal{L}^*_{n/2}$ satisfy the assumptions of Lemma 1.13 and we obtain

$$\mathrm{BC}^*(\mathcal{L}^*_{n/2}) = \mathrm{BC}(\mathcal{L}_{n/2}) \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathrm{BC}(\mathcal{L}_{n/2}) \right\}.$$

So we get

$$\mathcal{L}^*_{(n+1)/2} = \mathcal{L}_{(n+1)/2} \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathcal{L}_{(n+1)/2} \right\}$$

which finishes the induction. **(End proof of Theorem 1.8.)**

Let us carry over Theorem 1.8 to the classes of complements.

Corollary 1.17. *It holds that*

1. $\mathrm{co}\mathcal{B}^+_{n+1/2} = \mathrm{co}\mathcal{B}_{n+1/2}$ *for* $n \geq 0$ *and*
2. $\mathrm{co}^*\mathcal{L}^*_{n+1/2} = \mathrm{co}\mathcal{L}_{n+1/2} \cup \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathrm{co}\mathcal{L}_{n+1/2} \right\}$ *for* $n \geq 1$.

Proof. Statement 1 is an immediate consequence of Theorem 1.8 from which we also get for $n \geq 1$ that

$$\begin{aligned} \mathrm{co}^*\mathcal{L}^*_{n+1/2} = \left\{ A^* \setminus L \,\middle|\, L \in \mathcal{L}^*_{n+1/2} \right\} &= \overbrace{\left\{ A^* \setminus L \,\middle|\, L \in \mathcal{L}_{n+1/2} \right\}}^{\text{languages with } \varepsilon} \cup \overbrace{\left\{ A^+ \setminus L \,\middle|\, L \in \mathcal{L}_{n+1/2} \right\}}^{\text{languages without } \varepsilon} \\ &= \left\{ L \cup \{\varepsilon\} \,\middle|\, L \in \mathrm{co}\mathcal{L}_{n+1/2} \right\} \cup \mathrm{co}\mathcal{L}_{n+1/2}. \end{aligned}$$

This shows the second statement. ❑

Note that $\mathcal{L}_{1/2}$ and $\mathcal{L}^*_{1/2}$ in Theorem 1.8 are some kind of exception since all classes $\mathcal{L}^*_{n/2}$ with $n \geq 2$ have the property $L \cup \{\varepsilon\} \in \mathcal{L}^*_{n/2} \iff L \setminus \{\varepsilon\} \in \mathcal{L}^*_{n/2}$ for all $L \subseteq A^*$. This does not hold for $\mathcal{L}^*_{1/2}$ because A^* is the only language in $\mathcal{L}^*_{1/2}$ which contains the empty word. However, we have the following uniform statement of this relation.

Corollary 1.18. *For $n \geq 1$ it holds that $\mathcal{L}_{n/2} = \mathcal{L}^*_{n/2} \cap \mathcal{P}(A^+)$.*

Proof. For $n \geq 2$ this follows from Theorem 1.8. By definition, $\mathcal{L}_{1/2}$ is a class of languages of A^+ and if we intersect both sides of $\mathcal{L}^*_{1/2} = \mathcal{L}_{1/2} \cup \{A^*\}$ with $\mathcal{P}(A^+)$ we get $\mathcal{L}_{1/2} = \mathcal{L}^*_{1/2} \cap \mathcal{P}(A^+)$. ❑

1.2.3 Normal Forms and Closure Properties

Finally, we give in this section some normal forms and closure properties for the hierarchy classes $\mathcal{L}_{n/2}$ and $\mathcal{B}_{n/2}$. They are adaptions of known results. We mention with the first statement in the following proposition the normal form for $\mathcal{L}_{3/2}$ known from [Arf87, Arf91].

Proposition 1.19. *The following holds.*

1. *$\mathcal{L}_{3/2}$ is equal to the class of languages of A^+ that can be written as finite unions of languages of the form $A_0^* a_1 A_0^* \cdots a_n A_n^*$ where $n \geq 0$, $a_i \in A$ and $A_i \subseteq A$.*
2. *$\mathcal{L}_{3/2}$ is equal to the class of languages of A^+ that can be written as finite unions of languages of the form $u_0 A_1^+ u_1 \cdots A_n^+ u_n$ where $n \geq 0$, $u_i \in A^*$ and $\emptyset \neq A_i \subseteq A$.*

Proof. In [Arf91] it is shown that $\mathcal{L}^*_{3/2}$ is equal to the class of languages of A^* that can be written as finite unions of languages of the form $A_0^* a_1 A_0^* \cdots a_n A_n^*$ where $n \geq 0$, $a_i \in A$ and $A_i \subseteq A$. By Corollary 1.18 we have

$$\mathcal{L}_{3/2} = \left\{ L \in \mathcal{L}^*_{3/2} \,\middle|\, L \subseteq A^+ \right\}$$

which shows statement 1. Observe that $\emptyset^* = \{\varepsilon\}$, and for $a \in A$ and $\emptyset \neq A' \subseteq A$ we have $A'^* = A'^+ \cup \{\varepsilon\}$ and $A'^+ = \bigcup_{a \in A'} aA'^*$. So it is easy to see by mutual substitution that the following statements are equivalent for every language $L \subseteq A^+$.

(1) L is a finite union of languages $A_0^* a_1 A_0^* \cdots a_n A_n^*$ with $n \geq 0$, $a_i \in A$ and $A_i \subseteq A$.
(2) L is a finite union of languages $u_0 A_1^* u_1 \cdots A_n^* u_n$ with $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$.
(3) L is a finite union of languages $u_0 A_1^+ u_1 \cdots A_n^+ u_n$ with $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$.

This shows statement 2. ❑

In [Gla98] normal forms for the levels $n + 1/2$ of the dot–depth hierarchy and the Straubing–Thérien hierarchy are given.

Lemma 1.20. *For $n \geq 1$ it holds that*

1. *$\mathcal{L}_{n+1/2} = \mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2})$ and*
2. *$\mathcal{B}_{n+1/2} = \mathrm{Pol}(\mathrm{co}\mathcal{B}_{n-1/2})$.*

Proof. By definition, $\mathrm{co}\mathcal{L}_{n-1/2} \subseteq \mathcal{L}_n$ and $\mathrm{co}\mathcal{B}_{n-1/2} \subseteq \mathcal{B}_n$ for $n \geq 1$. Thus we have $\mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2}) \subseteq \mathcal{L}_{n+1/2}$ and $\mathrm{Pol}(\mathrm{co}\mathcal{B}_{n-1/2}) \subseteq \mathcal{B}_{n+1/2}$ for $n \geq 1$. It remains to show the reverse inclusions. For this end, we recall the normal form result from [Gla98] which says for $n \geq 1$ that

$$\begin{aligned} \mathcal{L}^*_{n+1/2} &= \mathrm{Pol}^{\mathcal{L}}(\mathrm{co}^*\mathcal{L}^*_{n-1/2}) \text{ and} && (1.1)\\ \mathcal{B}^+_{n+1/2} &= \mathrm{Pol}^{\mathcal{B}}(\mathrm{co}\mathcal{B}^+_{n-1/2}). && (1.2) \end{aligned}$$

First we consider statement 1 for $n = 1$. By Proposition 1.19 languages from $\mathcal{L}_{3/2}$ can be written as finite unions of languages of the form $u_0 A_1^+ u_1 \cdots A_m^+ u_m$ where $m \geq 0$, $u_i \in A^*$ and $\emptyset \neq A_i \subseteq A$. Note that if $m = 0$ then $u_0 \neq \varepsilon$ since languages from $\mathcal{L}_{3/2}$ do not contain the empty word. Hence it suffices to show that $A'^+, \{a\} \in \mathrm{co}\mathcal{L}_{1/2}$ for $\emptyset \neq A' \subseteq A$ and $a \in A$ which can be seen as follows.

$$\begin{aligned} A'^+ &= A^+ \setminus \left(\bigcup_{a \in A \setminus A'} A^* a A^* \right) \in \mathrm{co}\mathcal{L}_{1/2} \\ \{a\} &= A^+ \setminus \left(\underbrace{\bigcup_{a' \in A \setminus \{a\}} A^* a' A^*}_{\substack{\text{words of length } \geq 1 \\ \text{containing a letter } a' \neq a}} \cup \underbrace{A^* a A^* a A^*}_{\text{words with } \geq 2\ a'\text{s}} \right) \in \mathrm{co}\mathcal{L}_{1/2} \end{aligned}$$

This shows $\mathcal{L}_{3/2} \subseteq \mathrm{Pol}(\mathrm{co}\mathcal{L}_{1/2})$. Now we consider statement 1 for some $n \geq 2$. Here we have $\mathrm{co}^*\mathcal{L}^*_{n-1/2} = \mathrm{co}\mathcal{L}_{n-1/2} \cup \{ L \cup \{\varepsilon\} \mid L \in \mathrm{co}\mathcal{L}_{n-1/2} \}$ by Corollary 1.17. Since $\{\varepsilon\} \in \mathrm{co}^*\mathcal{L}^*_{1/2} \subseteq \mathrm{co}^*\mathcal{L}^*_{n-1/2}$ we can apply Lemma 1.12 as before and we obtain

$$\mathrm{Pol}(\mathrm{co}^*\mathcal{L}^*_{n-1/2}) = \mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2}) \cup \{ L \cup \{\varepsilon\} \mid L \in \mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2}) \}. \tag{1.3}$$

From Proposition 1.14 we see that $\mathrm{Pol}(\mathrm{co}^*\mathcal{L}^*_{n-1/2}) = \mathrm{Pol}^{\mathcal{L}}(\mathrm{co}^*\mathcal{L}^*_{n-1/2})$. So together with (1.1) we can rewrite (1.3) as

$$\mathcal{L}^*_{n+1/2} = \underbrace{\mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2})}_{\text{languages without } \varepsilon} \cup \underbrace{\{ L \cup \{\varepsilon\} \mid L \in \mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2}) \}}_{\text{languages with } \varepsilon}. \tag{1.4}$$

We can compare this to Theorem 1.8 where we have

$$\mathcal{L}^*_{n+1/2} = \underbrace{\mathcal{L}_{n+1/2}}_{\text{languages without } \varepsilon} \cup \underbrace{\{ L \cup \{\varepsilon\} \mid L \in \mathcal{L}_{n+1/2} \}}_{\text{languages with } \varepsilon}. \tag{1.5}$$

Because the unions in (1.4) and (1.5) are disjoint we see that $\mathcal{L}_{n+1/2} = \mathrm{Pol}(\mathrm{co}\mathcal{L}_{n-1/2})$ which shows statement 1.

Let us consider statement 2 for $n \geq 1$. From (1.2) and Theorem 1.8 we obtain $\mathcal{B}_{n+1/2} = \mathrm{Pol}^{\mathcal{B}}(\mathrm{co}\mathcal{B}^+_{n-1/2})$. Together with Proposition 1.14 this yields $\mathcal{B}_{n+1/2} = \mathrm{Pol}(\mathrm{co}\mathcal{B}^+_{n-1/2})$. With Corollary 1.17 we get $\mathcal{B}_{n+1/2} = \mathrm{Pol}(\mathrm{co}\mathcal{B}_{n-1/2})$. ❑

Finally, we translate the closure properties from Lemma 1.9 to our definitions.

Lemma 1.21. *Let $n \geq 1$.*

1. *The classes $\mathcal{B}_{n/2}, \mathrm{co}\mathcal{B}_{n/2}, \mathcal{L}_{n/2}$ and $\mathrm{co}\mathcal{L}_{n/2}$ are closed under finite union and intersection.*
2. *Let $\mathcal{C}$ be one of the classes $\mathcal{B}_{n/2}$, $\mathrm{co}\mathcal{B}_{n/2}$, $\mathcal{L}_{n/2}$ or $\mathrm{co}\mathcal{L}_{n/2}$. Then $a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \mathcal{C}$ for $a \in A$ and $L \in \mathcal{C}$.*

Proof. For the classes $\mathcal{B}_{n/2}$ and $\mathrm{co}\mathcal{B}_{n/2}$ the lemma follows from Theorem 1.8 and Lemma 1.9. The closure of $\mathcal{L}_{n/2}$ under finite union and intersection for $n \geq 1$ is immediate from Lemma 1.9 and Corollary 1.18. This carries over to $\mathrm{co}\mathcal{L}_{n/2}$.

Now let $n \geq 1$, $a \in A$ and $L \in \mathcal{L}_{n/2}$. By Theorem 1.8 we have $\mathcal{L}_{n/2} \subseteq \mathcal{L}^*_{n/2}$. Thus $L \in \mathcal{L}^*_{n/2}$ and we obtain $La^{-1}, a^{-1}L \in \mathcal{L}^*_{n/2}$ by Lemma 1.9. Since $A^+ \in \mathcal{L}_{1/2} \subsetneq \mathcal{L}_{n/2}$ it follows from the closure under intersection and from Theorem 1.8 that $La^{-1} \cap A^+, a^{-1}L \cap A^+ \in \mathcal{L}_{n/2}$. Analogously this can be shown for $n \geq 1$, $a \in A$ and $L \in \mathrm{co}\mathcal{L}_{n+1/2}$ using Corollary 1.17 and Lemma 1.9.

Finally let $L' \in \mathrm{co}\mathcal{L}_{1/2}$ with $L' = A^+ \setminus L$ for some $L \in \mathcal{L}_{1/2}$. Then we have

$$
\begin{aligned}
L'a^{-1} \cap A^+ = \{\, v \in A^+ \mid va \in L' \,\} &= \{\, v \in A^+ \mid va \in A^+ \setminus L \,\} \\
&= \{\, v \in A^+ \mid va \notin L \,\} = A^+ \setminus \{\, v \in A^+ \mid va \in L \,\} \\
&= A^+ \setminus \underbrace{(La^{-1} \cap A^+)}_{\in \mathcal{L}_{1/2}} \in \mathrm{co}\mathcal{L}_{1/2}.
\end{aligned}
$$

Analogously one shows $a^{-1}L \cap A^+ \in \mathrm{co}\mathcal{L}_{1/2}$. ❑

1.3 The Dot–Depth Problem

The dot–depth problem is the question whether there exists an algorithms that outputs for a given language $L \subseteq A^+$ in the input the minimal $n \geq 1$ such that $L \in \mathcal{B}_{n/2}$. We also say that L has dot–depth $n/2$ if $L \in \mathcal{B}_{n/2}$ for a minimal n. As pointed out in the introduction, a reasonable way to approach the dot–depth problem is to consider the membership problems of fixed levels in concatenation hierarchies. We do this for the classes $\mathcal{B}_{n/2}$ and $\mathcal{L}_{n/2}$ and fixed $n \geq 1$. Recall with Figure 1.1 how these classes are comparable by inclusion, so we say with respect to these inclusions that one class has a higher concatenation complexity than another one, e.g., in these terms $\mathcal{B}_2$ is more complex than $\mathrm{co}\mathcal{B}_{3/2}$, and $\mathcal{B}_{3/2}$ is more complex than $\mathcal{L}_{3/2}$. Note that in light of Theorem 1.8 we may consider the classes $\mathcal{B}_{n/2}$ and $\mathcal{L}_{n/2}$ without loss of generality: it is easy to determine from a given DFA $\mathcal{M}$ accepting some language L whether it accepts the empty word, and we can construct some DFA $\mathcal{M}'$ accepting $L \setminus \{\varepsilon\}$.

We make some remarks concerning the strictness of the inclusions pictured in Figure 1.1. The strictness of the dot–depth hierarchy is shown in [BK78], a different proof by means of first–order logic can be found in [Tho84]. The proof given there even shows that $\mathcal{L}_n$ is strictly included in $\mathcal{B}_n$ for integers $n \geq 1$ from which we immediately get $\mathcal{L}_{n/2} \subsetneq \mathcal{B}_{n/2}$ for all $n \geq 1$. It is easy to see from this that also all inclusions $\mathcal{B}_n \subsetneq \mathcal{B}_{n+1/2}$ for $n \geq 1$ and $\mathcal{B}_{n+1/2} \subsetneq \mathcal{B}_{n+1}$ for $n \geq 0$ are strict. The same holds for the classes of complements, and $\mathcal{B}_{n+1/2} \neq \mathrm{co}\mathcal{B}_{n+1/2}$ for $n \geq 0$. So we derive from [Tho84] that in Figure 1.1 all inclusions between classes of the DDH are strict. No new argument is needed to do all this for the STH since the result $\mathcal{L}_n \subsetneq \mathcal{B}_n$ for $n \geq 1$ from [Tho84] also shows that $\mathcal{L}_n \subsetneq \mathcal{L}_{n+1}$ for $n \geq 1$.

1.3.1 Logical Characterizations

We recall a very natural connection between concatenation hierarchies and first–order logic. For an introduction to the field we refer to [Tho96].

Here formulas are considered using the binary relation symbol $<$, the constant symbols min and max, the function symbols S and P, and unary relation symbols π_a for each letter $a \in A$. They may also involve the equality symbol $=$, the Boolean connectives $\neg, \vee, \wedge$ and quantifiers $\exists, \forall$ bounding variables. Let Σ_n (Π_n) be the subclass of such formulas that have at most $n-1$ quantifier alternations, starting with an existential (universal, respectively) quantifier. We say that a language $L \subseteq A^+$ is definable by a formula of the logic $\mathrm{FO}[<, \min, \max, S, P]$ if there exists a sentence φ (i.e., a formula of the above type without free variables) such that all words $w \in L$ satisfy φ under the following interpretation: variables hold positions in w, $<$ is the usual $<$-relation on $\{1, \ldots, |w|\}$, $\min = 1$, $\max = |w|$, S (P) is the successor (predecessor) function on $\{1, \ldots, |w|\}$, and $\pi_a x$ means that the letter at position x is a. The following levelwise correspondence between the classes of concatenation hierarchies and the number of quantifier alternations are known.

Theorem 1.22 ([Tho82]). *Let $n \geq 1$ and let $L \subseteq A^+$.*

1. *$L \in \mathcal{B}_{n-1/2}$ if and only if L is definable by a Σ_n formula of* $\mathrm{FO}[<, \min, \max, S, P]$.
2. *$L \in \mathrm{co}\mathcal{B}_{n-1/2}$ if and only if L is definable by a Π_n formula of* $\mathrm{FO}[<, \min, \max, S, P]$.
3. *$L \in \mathcal{B}_n$ if and only if L is definable by a Boolean combination of Σ_n formulas of* $\mathrm{FO}[<, \min, \max, S, P]$.

Denote by $\mathrm{FO}[<]$ the restricted fragment, where the use of min, max, S and P in formulas is not allowed.

Theorem 1.23 ([Tho82, PP86]). *Let $n \geq 1$ and let $L \subseteq A^+$.*

1. *$L \in \mathcal{L}_{n-1/2}$ if and only if L is definable by a Σ_n formula of* $\mathrm{FO}[<]$.
2. *$L \in \mathrm{co}\mathcal{L}_{n-1/2}$ if and only if L is definable by a Π_n formula of* $\mathrm{FO}[<]$.
3. *$L \in \mathcal{L}_n$ if and only if L is definable by a Boolean combination of Σ_n formulas of* $\mathrm{FO}[<]$.

Due to these characterizations our results in Chapters 4 and 6 have consequences in first–order logic (cf. Corollaries 4.37 and 6.20).

1.3.2 Leaf Languages for Complexity Classes

There is also a close connection between concatenation hierarchies and complexity classes, both related via the so-called leaf language approach to define complexity classes. This approach was introduced in [BCS92, Ver93] and led to a number of interesting results giving new insights into the structure of complexity classes between P and PSPACE, e.g., [HLS+93, JMT94, KSV98, BV98, CHVW98]. We refer to these papers for a more comprehensive introduction, but briefly sketch the approach here. For undefined notions see [Pap94].

Let a nondeterministic polynomial–time Turing machine M output on every computation path a symbol from A and assume a fixed ordering on the set of all paths. This leads in a natural way to the notion of the *leafstring* of M on some input x when concatenating the output symbols at the leafs of the computation tree of M. Now a language $L \subseteq A^+$ gives rise to the class $\mathrm{Leaf}^{\mathrm{P}}(L)$ of all languages L' for which there exists a machine M of the above type such that for all x it holds that $x \in L'$ if and only if the leafstring of M on input x belongs to L. For some class $\mathcal{C}$ denote by $\mathrm{Leaf}^{\mathrm{P}}(\mathcal{C})$ the union of all classes $\mathrm{Leaf}^{\mathrm{P}}(L)$ with $L \in \mathcal{C}$. As an example, let us look at the class NP. By definition, a language $L' \in \mathrm{NP}$ is given by a nondeterministic polynomial–time machine M such that for all inputs x we have that x belongs to L' if and only if there is at least one accepting path in the computation tree of M on input x. Suppose that M outputs on accepting paths the symbol 1 and on rejecting paths the symbol 0. Hence NP is defined by the leaf language $L =_{\mathrm{def}} 0^*1\,\{0,1\}^*$. Note that $0^*1\,\{0,1\}^* = \{0,1\}^*\,1\,\{0,1\}^* \in \mathcal{L}_{1/2}$ and it can be easily seen from Proposition 1.15 that in fact $\mathrm{NP} = \mathrm{Leaf}^{\mathrm{P}}(\mathcal{B}_{1/2})$. Interestingly, this relation holds in general between the levels of the dot–depth hierarchy and the classes of the polynomial time hierarchy. Denote by Σ_n^{p} and Π_n^{p} for $n \geq 1$ the classes of the polynomial time hierarchy [Sto73].

Theorem 1.24 ([HLS⁺93, BV98, BKS98]). *Let* $n \geq 1$.

1. $\Sigma_n^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(\mathcal{B}_{n-1/2})$
2. $\Pi_n^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(\mathrm{co}\mathcal{B}_{n-1/2})$
3. $\mathrm{BC}(\Sigma_n^{\mathrm{p}}) = \mathrm{Leaf}^{\mathrm{P}}(\mathcal{B}_n)$
4. $\mathrm{NP}(n) = \mathrm{Leaf}^{\mathrm{P}}(\mathcal{B}_{1/2}(n))$

In the last statement $\mathrm{NP}(n)$ denotes the n-th level of the difference hierarchy over NP and $\mathcal{B}_{1/2}(n)$ denotes the n-th level of the difference hierarchy over $\mathcal{B}_{1/2}$ (for a formal definition see Definition 2.23). However, the above results are of the type that they deal with classes of leaf languages. An important questions in this context is what complexity classes are definable by a single leaf language. It is known that $\left\{ \mathrm{Leaf}^{\mathrm{P}}(L) \mid L \text{ non–trivial regular language} \right\}$ together with the inclusion relation forms an upper semilattice [Bor95]. The structure of this semilattice has been clarified in [Bor95, BKS98] for the classes at the lower end. Unfortunately, it seems to be a difficult task to do this for higher levels, i.e., to prove results supporting our intuition that more difficult regular languages lead to presumably broader complexity classes—opposed to the possibility that they may refine the upper semilattice of leaf language definable complexity classes. Such results cannot be achieved by union–style theorems like the ones above.

Here forbidden patterns may help since they make a positive assertion about the structure we find at least in the transiton graph of a DFA if the accepted language is not in some class any more. So the occurrence of a pattern can help to identify complexity classes that are at least included in the complexity class defined by the leaf language of the DFA having this pattern. In this way, the forbidden pattern result for $\mathcal{B}_{1/2}$ from [PW97] is exploited in [BKS98] to prove a gap theorem for the definability of complexity classes right above NP. To prove gap theorems along these lines for higher levels of the polynomial time hierarchy prerequistes forbidden pattern characterizations of the levels of the dot–depth hierarchy, clearly a difficult task. As pointed out earlier, forbidden pattern characterizations usually imply decidability of the respective membership problem.

We prove a result concerning the complexity class Δ_2^{p} and identify leaf language definable complexity classes in the upper semilattice around this class (cf. Theorem 3.31). As we will also see in Section 3.5, there is a close relation of Δ_2^{p} to regular languages definable in restricted temporal logic. In Section 4.5 we discuss possible consequences of the forbidden pattern characterization of $\mathcal{B}_{3/2}$ given in Chapter 4.

1.4 Connecting STH and DDH

In this section we introduce for $k \geq 0$ a family of hierarchies of classes $\mathcal{B}_{n/2,k}$ for $n \geq 1$ such that $\mathcal{L}_{n/2} = \mathcal{B}_{n/2,0}$ and $\mathcal{B}_{n/2}$ is just the union over all $\mathcal{B}_{n/2,k}$ for $k \geq 0$.

Recall from Proposition 1.16 that a language $L \subseteq A^+$ is in $\mathcal{L}_{1/2}$ if and only if it can be written as a finite union of languages of the form $A^*a_1A^* \cdots a_mA^*$ where $m \geq 0$ and $a_i \in A$. On the other hand, we have by Proposition 1.15 that L belongs to $\mathcal{B}_{1/2}$ if and only if it can be written as a finite union of languages of the form $u_0A^*u_1 \cdots A^*u_m$ where $m \geq 0$ and $u_i \in A^*$. So the difference between the two classes $\mathcal{L}_{1/2}$ and $\mathcal{B}_{1/2}$ is the possibility to specify a prefix and a suffix, and to fix two or more consecutive letters in the latter case. We have already noted that these two classes are distinct and it is also intuitively clear that finite unions cannot help to specify a longer block of consecutive letters within the resources of $\mathcal{L}_{1/2}$. A natural way to bridge these differences is to emphasize on the maximal block length. The parameter k with $k \geq 0$ will play this role in the forthcoming chapters.

The idea of looking at a parameterization in terms of block lengths is from [Sim72] and [Str85]. In the former, subhierarchies of $\mathcal{B}_1$ are studied where besides k also the parameter m, i.e., the number of specified blocks, is emphasized. Here each fixed pair (m,k) defines a subclass of $\mathcal{B}_1$, which is a Boolean algebra and which is characterized in [Sim72] in terms of certain equivalence relations on words (for an overview, see [Brz76]). A more general approach is chosen in [Str85]. As mentioned in the introduction, it is shown in this paper that the membership problems of $\mathcal{B}_n$ and $\mathcal{L}_n$ for integers n are equivalent with respect to decidability. This is achieved with an algebraic approach relating certain products of varieties of finite semigroups. We will take a careful look at the levels 1/2, 3/2 and intermediate classes, and give positive answers to several membership problems with an automata–theoretic approach in Chapters 2 to 4.

1.4.1 Block Decomposition of Words

Fix some $k \geq 0$. The set of all words from A^* of length k is denoted by A^k. Moreover, we denote by $A^{\leq k}$ ($A^{<k}$, $A^{\geq k}$, ...) the set of words from A^+ of length less or equal to k (less than k, greater or equal to k, ... respectively). Note that none of these sets contains the empty word, i.e., $A^{<0} = A^{\leq 0} = \emptyset$. For $x \in A^{\geq k}$ we denote by $p_k(x)$ the length-k prefix of x, and by $s_k(x)$ the length-k suffix of x. We call these the k-prefix and k-suffix of x. If $x \in A^{<k}$ we set $p_k(x) = s_k(x) =_{\text{def}} x$.

The k-decomposition of a word $x \in A^+$ is the sequence of each $k+1$ consecutive letters of x. In order to avoid confusion we denote elements from A^{k+1} as $\alpha, \beta, \gamma, \ldots$ and subsets of A^{k+1} as $\Sigma, \Gamma, \ldots$. Let $x = a_1a_2 \cdots a_{k+l} \in A^+$ for some $l \geq 1$. We call

$$\widehat{x} =_{\text{def}} (\alpha_1, \alpha_2, \ldots, \alpha_l)$$

the k-decomposition of x if $\alpha_i = a_i \cdots a_{i+k}$ for $1 \leq i \leq l$. If $x \in A^{\leq k}$ then we set $\widehat{x} =_{\text{def}} x$. The value of k will always be clear from the context when we use the notation $\widehat{x}$. Intuitively, k indicates by how many letters from A consecutive α_i overlap. For $x \in A^{\geq k+1}$ we set $\alpha(\widehat{x}) =_{\text{def}} \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ as the set of elements from A^{k+1} in the k-decomposition of x.

Next we want to define languages of words from A^+ that admit the same k-decomposition with respect to given elements and subsets of A^{k+1}.

Definition 1.25. Let $k \geq 0$. Let $\alpha_1, \ldots, \alpha_n \in A^{k+1}$ and $\Sigma_0, \Sigma_1, \ldots, \Sigma_n \subseteq A^{k+1}$ for some $n \geq 0$. For every $x \in A^+$ we say $x \in (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k$ if and only if $|x| \geq k+1$, $\widehat{x} = (\beta_1, \ldots, \beta_l)$ and there exist $0 = j_0 < j_1 < j_2 < \ldots < j_n < j_{n+1} = l+1$ such that

(a) $\beta_{j_i} = \alpha_i$ for $1 \leq i \leq n$ and
(b) $\beta_j \in \Sigma_i$ for $0 \leq i \leq n$ and $j_i < j < j_{i+1}$.

If we write an expression $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k$ we understand this as a syntactical object describing some language. While being aware of this, we do not distinguish between such an object and the language it stands for, unless stated otherwise. So the language $(\Sigma_0, \alpha_1, \Sigma_1, \alpha_2, \Sigma_2, \ldots, \alpha_n, \Sigma_n)_k$ consists of those words $x \in A^{\geq k+1}$, whose k-decomposition starts with a number (possibly zero) of elements from Σ_0, then α_1, followed by a number (possibly zero) of elements from Σ_1, then α_2 and so on. A subset of A^{k+1} in such an expression stands for possibly multiple occurrences of its elements. Note that the defined languages only contain words that admit a k-decomposition and that in case $k = 0$ we deal with the usual concatenation, e.g., $(A_0, a_1, A_1, a_2, A_2)_0 = A_0^* a_1 A_1^* a_2 A_2^*$ and $(A_0)_0 = A_0^+$. Without further definition we also use expressions of the form $(\alpha_1, \Sigma_1, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ and $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \Sigma_n, \alpha_{n+1})_k$ and thelike with the obvious meaning. We introduce other convenient notations.

Definition 1.26. Let $w, v \in A^*$, $\alpha_1, \ldots, \alpha_n \in A^{k+1}$ and $\Sigma_0, \ldots, \Sigma_n \subseteq A^{k+1}$ for some $n \geq 0$. Then we write $(w|\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n|v)_k$ instead of $\big(wA^* \cap A^*v \cap (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k\big)$.

Moreover, if all Σ_i are equal to A^{k+1} we do not want to mention them repeatedly in an expression. So we write $(\alpha_1, \alpha_2, \ldots, \alpha_m)_k$ instead of $(A^{k+1}, \alpha_1, A^{k+1}, \alpha_2, \ldots, A^{k+1}, \alpha_m, A^{k+1})_k$ for $m \geq 1$.

1.4.2 Connecting $\mathcal{L}_{1/2}$ and $\mathcal{B}_{1/2}$

We introduce the classes $\mathcal{B}_{1/2,k}$.

Definition 1.27. Let $k \geq 0$. The class $\mathcal{B}_{1/2,k}$ is the class of all languages $L \subseteq A^+$ that can be written as a finite union of languages L_i such that $L_i \subseteq A^{\leq k}$ or

$$L_i = (w|\alpha_1, \alpha_2, \ldots, \alpha_m|v)_k$$

where $m \geq 1, \alpha_1, \ldots, \alpha_m \in A^{k+1}$ and $w, v \in A^k$.

So here we are allowed to fix a prefix and a suffix of length k, and occurring blocks of length $k+1$ can be positively specified. Note that these blocks may overlap.

Proposition 1.28. *Let $k \geq 0$. It holds that*

1. $\mathcal{L}_{1/2} = \mathcal{B}_{1/2,0}$ *and*
2. $\mathcal{B}_{1/2,k} \subseteq \mathcal{B}_{1/2,k+1}$.

Proof. The first statement is obvious from Proposition 1.16. Just note that the case $m = 0$ implies $L = \emptyset$ since we deal with languages of A^+, and that $(\varepsilon|a_1, a_2, \ldots, a_m|\varepsilon)_0 = A^* a_1 A^* a_2 \cdots A^* a_m A^*$.

For the second statement let a language from $\mathcal{B}_{1/2,k}$ for some $k \geq 0$ be given. It suffices to show for some $L =_{\text{def}} (w|\alpha_1, \alpha_2, \ldots, \alpha_m|v)_k$ with $m \geq 1, \alpha_1, \ldots, \alpha_m \in A^{k+1}$ and $w, v \in A^k$ that $L \in \mathcal{B}_{1/2,k+1}$. We distinguish two cases.

Case 1. Suppose that $L \cap A^{k+m} = \emptyset$. We claim that

$$L \;=\; \bigcup \left(w'|\beta_1, \beta_2, \ldots, \beta_m|v'\right)_{k+1} \tag{1.6}$$

where the union ranges over all n, w', v' and $\beta_1, \ldots, \beta_m$ with $w' \in wA$, $v' \in Av$, $0 \leq n \leq m$, $\beta_l \in \alpha_l A$ for $1 \leq l \leq n$ and $\beta_l \in A\alpha_l$ for $n < l \leq m$. Clearly, this is a finite union.

We argue for the two inclusions. Let x be a word from the right hand side and fix some member of the union containing x. Suppose $x = c_1 \cdots c_{|x|}$ and $\widehat{x} = (\gamma_1, \ldots, \gamma_{|x|-k-1})$ for suitable $c_1, \ldots, c_{|x|} \in A$ and $\gamma_1, \ldots, \gamma_{|x|-k-1} \in A^{k+2}$. Note that we have fixed with $\widehat{x}$ a $(k+1)$-decomposition of x. By definition, there exist indices $1 \leq j_1 < j_2 < \cdots < j_m \leq |x| - k - 1$ such that $\beta_l = \gamma_{j_l}$ for $1 \leq l \leq m$. Let

$$i_l =_{\text{def}} \begin{cases} j_l & : \text{ if } l \leq n \\ j_l + 1 & : \text{ otherwise} \end{cases}$$

for $1 \leq l \leq m$. Then $1 \leq i_1 < i_2 < \cdots < i_m \leq |x| - k$. Because $\beta_l \in \alpha_l A$ for $1 \leq l \leq n$ and $\beta_l \in A\alpha_l$ for $n < l \leq m$ we obtain $\alpha_l = c_{i_l} \cdots c_{i_l+k}$ for $1 \leq l \leq m$. So $x \in (\alpha_1, \alpha_2, \ldots, \alpha_m)_k$ and since $w' \in wA$ and $v' \in Av$ we conclude $x \in L$.

Conversely, let $x \in L$. Choose suitable $c_1, \ldots, c_{|x|} \in A$ and $\gamma_1, \ldots, \gamma_{|x|-k} \in A^{k+1}$ such that $x = c_1 \cdots c_{|x|}$ and $\widehat{x} = (\gamma_1, \ldots, \gamma_{|x|-k})$. Here the latter is a k-decomposition. Again by definition, there are indices $1 \leq i_1 < i_2 < \cdots < i_m \leq |x| - k$ such that $\alpha_l = \gamma_{i_l}$ for $1 \leq l \leq m$. By assumption of this case we have $|x| \geq k + m + 1$, so there exists an index $1 \leq r \leq |x| - k$ such that $r \neq i_l$ for all $1 \leq l \leq m$. Therefore, we obtain $1 \leq j_1 < j_2 < \cdots < j_m \leq |x| - k - 1$ with the definition

$$j_l =_{\text{def}} \begin{cases} i_l & : \text{ if } i_l < r \\ i_l - 1 & : \text{ otherwise } (i_l > r) \end{cases}$$

for $1 \leq l \leq m$. Let $n =_{\text{def}} \max \{ 1 \leq l \leq m \,|\, j_l < r \} \cup \{0\}$ and $\beta_l =_{\text{def}} c_{j_l} \cdots c_{j_l+k+1}$ for $1 \leq l \leq m$. Then we obtain $\beta_l \in \alpha_l A$ for $1 \leq l \leq n$ and $\beta_l \in A\alpha_l$ for $n < l \leq m$. With $w' =_{\text{def}} p_{k+1}(x)$ and $v' =_{\text{def}} s_{k+1}(x)$ we conclude $x \in \left(w'|\beta_1, \beta_2, \ldots, \beta_m|v'\right)_{k+1}$. This shows (1.6) and completes the first case.

Case 2. Now assume $L \cap A^{k+m} \neq \emptyset$. Then this set has only one element x with $\widehat{x} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $w = p_k(\alpha_1)$ and $v = s_k(\alpha_m)$. We show how we can modify the first case by taking certain languages into the union on the right hand side in (1.6). If $m = 1$ then $x = \alpha_1$ and $|x| = k + 1$. So we can take $\{x\}$ to the union in (1.6) and still have a language from $\mathcal{B}_{1/2,k+1}$. Now assume that $m \geq 2$ and let $(\gamma_1, \gamma_2, \ldots, \gamma_{m-1})$ be the $(k+1)$-decomposition of x. Observe that for $1 \leq l \leq m - 1$ it holds that $p_{k+1}(\gamma_l) = \alpha_l$ and $s_{k+1}(\gamma_{m-1}) = \alpha_m$. We claim that it suffices to add

$$(\alpha_1|\gamma_1, \gamma_2, \ldots, \gamma_{m-1}|\alpha_m)_{k+1}$$

to the union in (1.6). Note that this is a language from $\mathcal{B}_{1/2,k+1}$. Clearly, x is in the set $(\alpha_1|\gamma_1, \ldots, \gamma_{m-1}|\alpha_m)_{k+1}$ since it has $(k+1)$-prefix α_1, $(k+1)$-suffix α_m and $(\gamma_1, \ldots, \gamma_{m-1})$ is just its $(k+1)$-decomposition. So it remains to show that $(\alpha_1|\gamma_1, \ldots, \gamma_{m-1}|\alpha_m)_{k+1} \subseteq L$. Therefore, let $u \in (\alpha_1|\gamma_1, \ldots, \gamma_{m-1}|\alpha_m)_{k+1}$ and let $1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq |u| - k - 1$ such that for the $(k+1)$-decomposition $\widehat{u} = (\zeta_1, \zeta_2, \ldots, \zeta_{|u|-k-1})$ it holds that $\gamma_l = \zeta_{i_l}$ for

$1 \le l \le m-1$. Now define $j_l =_{\text{def}} i_l$ for $1 \le l \le m-1$ and $j_m =_{\text{def}} i_{m-1}+1$. Then it holds that $1 \le j_1 < j_2 < \cdots < j_{m-1} < j_m \le |u| - k$ and $p_{k+1}(\zeta_{j_l}) = p_{k+1}(\gamma_l) = \alpha_l$ for $1 \le l \le m-1$. Recall that $s_{k+1}(\gamma_{m-1}) = \alpha_m$, so we find α_m starting at position j_m in u. Together, the indices j_l witness that $u \in (\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m)_k$. It remains to observe that $p_k(u) = p_k(\alpha_1) = w$ and $s_k(u) = s_k(\alpha_m) = v$. This finishes the second case and the proof of the proposition. ❑

Next we see that the union of all classes $\mathcal{B}_{1/2,k}$ amounts to $\mathcal{B}_{1/2}$.

Lemma 1.29. *It holds that* $\mathcal{B}_{1/2} = \bigcup_{k \ge 0} \mathcal{B}_{1/2,k}$.

Proof. We have to show two inclusions and need to swap between k-decompositions and the usual concatenation. We argue first for the inclusion from right to left. So let $k \ge 0$ and let $L \in \mathcal{B}_{1/2,k}$. Since $\mathcal{B}_{1/2}$ is closed under finite union and contains all finite sets, we may assume that

$$L = (w|\alpha_1, \alpha_2, \ldots, \alpha_m|v)_k$$

where $m \ge 1$, $\alpha_i \in A^{k+1}$ and $w, v \in A^k$. To show that $L \in \mathcal{B}_{1/2}$ we have to consider that the elements $\alpha_i \in A^{k+1}$ in the description of L may overlap, and we have to express this with usual concatenations.

For given $l \ge 1$ and $\beta_1, \ldots, \beta_l \in A^{k+1}$ we define the set $sh(\beta_1, \beta_2, \ldots, \beta_l)$ of shuffle words as follows. A word x belongs to $sh(\beta_1, \beta_2, \ldots, \beta_l)$ if and only if $\widehat{x} = (\gamma_1, \ldots, \gamma_n)$ for some $n \ge 1$ and there exist $1 \le j_1 < j_2 < \ldots < j_l \le n$ with $\gamma_{j_i} = \beta_i$ for $1 \le i \le l$ and $j_{i+1} - j_i \le k$ for $1 \le i \le l-1$. The latter condition ensures that all occurrences of β_i in x overlap, e.g., for $k = 2$ and $\beta_1 =_{\text{def}} abb$, $\beta_2 =_{\text{def}} bbc$ we have $sh(abb, bbc) = A^*abbcA^* \cup A^*abbbcA^*$. Moreover, we see that each $sh(\beta_1, \ldots, \beta_l)$ is a finite union of languages of the form A^*xA^* with $k + l \le |x| \le kl + 1$. Since $k + l \ge 1$ we get from Proposition 1.15 that $sh(\beta_1, \ldots, \beta_l) \in \mathcal{B}_{1/2}$.

Now we want to express L in terms of concatenations of sets of shuffle words and therefore rewrite L as $(wA^+ \cap A^+v) \cap (L' \cup sh(\alpha_1, \ldots, \alpha_m))$ where

$$L' =_{\text{def}} \bigcup_{\substack{1 \le n \le m-1 \\ 2 \le i_1 < i_2 < \ldots < i_n \le m}} sh(\alpha_1, \ldots, \alpha_{i_1-1}) sh(\alpha_{i_1}, \ldots, \alpha_{i_2-1}) \cdots sh(\alpha_{i_n}, \ldots, \alpha_m).$$

With the set L' we guess a number of n positions where the α_i from the description of L do not overlap. Since $\mathcal{B}_{1/2}$ is closed under concatenation, finite union and intersection (cf. Lemma 1.21) we obtain $L \in \mathcal{B}_{1/2}$.

We turn to the reverse inclusion, so let a language from $\mathcal{B}_{1/2}$ be given. In light of Proposition 1.28 it suffices to show that for each language $L =_{\text{def}} u_0A^+u_1 \cdots A^+u_n$ with $u_i \in A^+$ and $n \ge 0$ there is some $k \ge 0$ such that $L \in \mathcal{B}_{1/2,k}$. We show this by induction on n and prove the induction base for $n = 0$ and $n = 1$. In case $n = 0$ set $k_0 =_{\text{def}} |u_0|$ and we see that $\{u_0\} \in \mathcal{B}_{1/2,k_0}$. For the same reason we do not have to care about finite sets any more. Now let $n = 1$ and suppose $L = u_0A^+u_1$. Define $k_1 =_{\text{def}} |u_0| + |u_1|$. We claim that

$$L = \bigcup_{w,\alpha,v} (w|\ \alpha\ |v)_{k_1}$$

where the union ranges over all $w \in A^{k_1} \cap u_0A^*$, $v \in A^{k_1} \cap A^*u_1$ and all $\alpha \in A^{k_1+1}$. To see this we observe that L contains only words of length $\ge k_1 + 1$.

For the induction step suppose for $n \geq 1$ that $L = (D \cup L') \cdot A^+ u_{n+1}$ where $D \subseteq A^{\leq k_n}$ and $L' = (w|\alpha_1, \alpha_2, \ldots, \alpha_m|v)_{k_n}$ with $m \geq 1$, $w, v \in A^{k_n}$ and $\alpha_i \in A^{k_n+1}$. We assume here that L' is just a single such language again by Proposition 1.28. The case $D \cdot A^+ u_{n+1}$ was treated in the induction base. We can also suppose that $l =_{\text{def}} |u_{n+1}| \leq k_n$ since otherwise we get from Proposition 1.28 a representation of L' with a sufficiently large k_n. We claim that

$$L' \cdot A^+ u_{n+1} \quad = \quad \bigcup \ (w|\alpha_1, \alpha_2, \ldots, \alpha_m, va, \beta_1, \ldots, \beta_l|z)_{k_n}$$

where the union ranges over all $a \in A$, $z \in A^{k_n}$ with $s_l(z) = u_{n+1}$ and over all $\beta_j \in A^{k_n+1}$ for $1 \leq j \leq l$. For the inclusion from left to right observe that l is defined in a way that there are at least l occurrences of some $\beta_j \in A^{k_n+1}$ right of va. On the other hand, the l-suffix of z does not begin before va ends, which is due to the l occurrences of β_j. This completes the induction and the proof of the lemma. ❑

1.4.3 More Concatenation Hierarchies

The aim of this subsection is to carry over the relation between $\mathcal{L}_{1/2}$ and $\mathcal{B}_{1/2}$ to all other levels of the STH and DHH. Therefore we define for all $k \geq 0$ a hierarchy over $\mathcal{B}_{1/2,k}$.

Definition 1.30. Let $k \geq 0$. The classes of the concatenation hierarchy over $\mathcal{B}_{1/2,k}$ are defined as

$$\begin{aligned} \mathcal{B}_{n+1,k} &=_{\text{def}} \mathrm{BC}(\mathcal{B}_{n+1/2,k}) && \text{for } n \geq 0 \text{ and} \\ \mathcal{B}_{n+3/2,k} &=_{\text{def}} \mathrm{Pol}(\mathcal{B}_{n+1,k}) && \text{for } n \geq 0. \end{aligned}$$

The following proposition is the counterpart of Proposition 1.28.

Proposition 1.31. *Let $k \geq 0$ and $n \geq 1$. It holds that*

1. $\mathcal{L}_{n/2} = \mathcal{B}_{n/2,0}$ *and*
2. $\mathcal{B}_{n/2,k} \subseteq \mathcal{B}_{n/2,k+1}$.

Proof. The first statement is obvious from Proposition 1.28 because with $\mathcal{L}_{1/2} = \mathcal{B}_{1/2,0}$ we see that Definitions 1.2 and 1.30 coincide.

We show the second statement by induction on n. The induction base for $n = 1$ is given by Proposition 1.28. We first assume that the statement holds for $n \geq 1$ with $n \equiv 1 \mod 2$ and we want to prove it for $n + 1$. Then

$$\mathcal{B}_{(n+1)/2,k} = \mathrm{BC}(\mathcal{B}_{n/2,k}) \subseteq \mathrm{BC}(\mathcal{B}_{n/2,k+1}) = \mathcal{B}_{(n+1)/2,k+1}$$

by the induction hypothesis $\mathcal{B}_{n/2,k} \subseteq \mathcal{B}_{n/2,k+1}$ and the monotony of $\mathrm{BC}(\cdot)$. This shows in particular the second statement for $n = 2$. So now we assume that it holds for $n \geq 2$ with $n \equiv 0 \mod 2$ and we want to prove it for $n+1$. This is the same as before, just change $\mathrm{BC}(\cdot)$ to $\mathrm{Pol}(\cdot)$. ❑

There is also a general version of Lemma 1.29. To get this we first observe the following.

Proposition 1.32. *For $k \geq 0$ let $\mathcal{C}_k$ be a family of classes such that $\mathcal{C}_k \subseteq \mathcal{C}_{k+1}$. It holds that*

$$\mathrm{BC}(\textstyle\bigcup_{k\geq 0} \mathcal{C}_k) = \bigcup_{k\geq 0} \mathrm{BC}(\mathcal{C}_k) \quad \textit{and} \quad \mathrm{Pol}(\bigcup_{k\geq 0} \mathcal{C}_k) = \bigcup_{k\geq 0} \mathrm{Pol}(\mathcal{C}_k).$$

Proof. Suppose L is a Boolean combination of finitely many languages $L_i \in \mathcal{C}_{k_i}$ for $k_i \geq 0$. Then L is also a Boolean combination of languages $L_i \in \mathcal{C}_k$ with $k =_{\text{def}} \max k_i$ because $\mathcal{C}_{k_i} \subseteq \mathcal{C}_k$ by assumption. Conversely, if L is in $\text{BC}(\mathcal{C}_{k'})$ for some $k' \geq 0$ then it is also in $\text{BC}(\mathcal{C}_{k'}) \subseteq \text{BC}(\bigcup_{k \geq 0} \mathcal{C}_k)$. The second part of the proposition can be seen with the same arguments. ❑

Lemma 1.33. *It holds that* $\mathcal{B}_{n/2} = \bigcup_{k \geq 0} \mathcal{B}_{n/2,k}$ *for* $n \geq 1$.

Proof. We show the lemma by induction on n. The induction base for $n = 1$ is given by Lemma 1.29. We show the induction step for $n+1$ with $n \geq 1$ and $n \equiv 1 \mod 2$, for the other case just change $\text{BC}(\cdot)$ to $\text{Pol}(\cdot)$. It holds that

$$\mathcal{B}_{(n+1)/2} = \text{BC}(\mathcal{B}_{n/2}) = \text{BC}(\textstyle\bigcup_{k \geq 0} \mathcal{B}_{n/2,k}) = \bigcup_{k \geq 0} \text{BC}(\mathcal{B}_{n/2,k}) = \bigcup_{k \geq 0} \mathcal{B}_{(n+1)/2,k}$$

by the induction hypothesis $\mathcal{B}_{n/2} = \bigcup_{k \geq 0} \mathcal{B}_{n/2,k}$ and Proposition 1.32 together with Proposition 1.31. ❑

It is easy to see that all these relations hold also for the classes of complements and that for $n \geq 0$ it holds that $\mathcal{B}_{n+1/2,k} \cup \text{co}\mathcal{B}_{n+1/2,k} \subseteq \mathcal{B}_{n+1,k} \subseteq \mathcal{B}_{n+3/2,k} \cap \text{co}\mathcal{B}_{n+3/2,k}$. So we have the inclusions given in Figure 1.2. It pictures the landscape of classes that we study from Chapter 2 onwards.

1.5 Finite Automata and Forbidden Pattern Classes

A DFA $\mathcal{M}$ is given by $\mathcal{M} = (A, S, \delta, s_0, S')$, where A is the input alphabet, S is the set of states, $\delta : A \times S \to S$ is the total transition function, $s_0 \in S$ is the starting state and $S' \subseteq S$ is the set of accepting states. We denote by $L(\mathcal{M})$ the language accepted by $\mathcal{M}$ and by $|\mathcal{M}|$ the number of states of $\mathcal{M}$. We say a DFA is minimal if for all $s_1, s_2 \in S$ with $s_1 \neq s_2$ there exists some $z \in A^*$ such that $\delta(s_1, z) \in S' \Leftrightarrow \delta(s_2, z) \notin S'$. We can identify every $\mathcal{M}$ with its finite transition graph by taking S as the set of nodes, while edges are drawn and labelled with respect to the transition function. Since $\mathcal{M}$ is a deterministic automaton every $w \in A^*$ and $s \in S$ induce a unique path in the transition graph starting at s and labelled subsequently by the letters of w.

For notational convenience we extend the transition function to input words and, correspondingly, we look at the extended transition graph: the set of nodes is still S, but edges are drawn with respect to the extended transition function and have labels from A^*. This is an infinite directed graph with a finite number of nodes. We say that a state $s \in S$ has a loop $w \in A^*$ (has a w-loop, for short) if and only if $\delta(s, w) = s$. This is just a cycle in the extended transition graph at node s with label w. Every $w \in A^*$ induces a total mapping $\delta^w : S \to S$ with $\delta^w(s) =_{\text{def}} \delta(s, w)$ which also has an interpretation in the extended transition graph: we follow a path labelled w starting simultaneously at each node of the graph. Moreover, we say that a total mapping $\delta' : S \to S$ leads to a w-loop if and only if $\delta'(s)$ has a w-loop for all $s \in S$. We may also say for short that $v \in A^*$ leads to a w-loop if δ^v leads to a w-loop. We consider only automata where each state is reachable from the starting state and where the starting state is not accepting. Clearly, every DFA runs into a loop of w's if there are enough of them in the input.

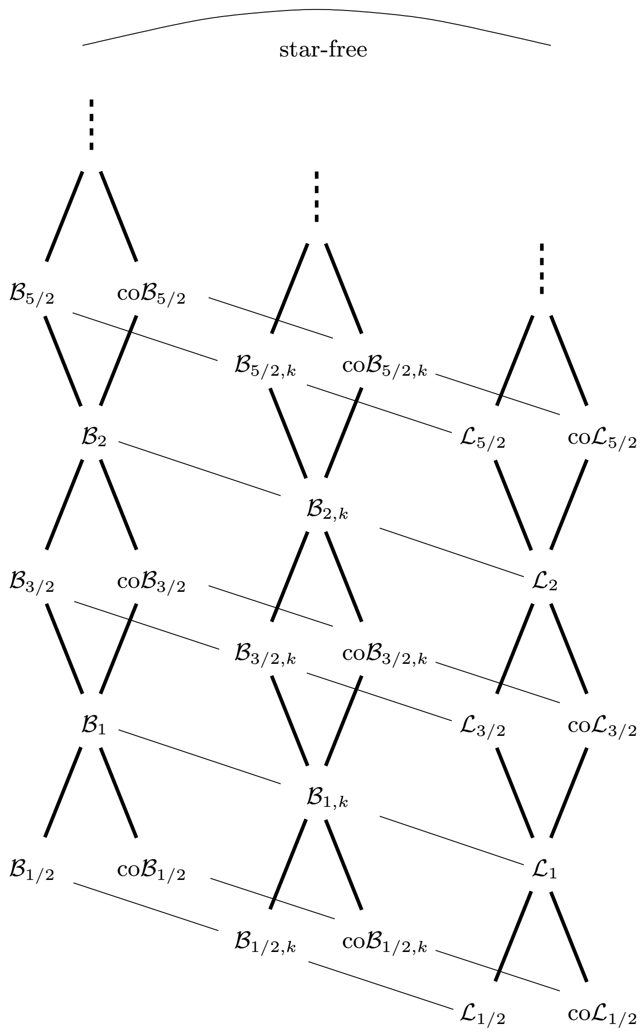


Fig. 1.2. Connecting STH and DDH via classes $\mathcal{B}_{n/2,k}$ for $k \geq 0$.

Proposition 1.34. *Let $\mathcal{M}$ be a* DFA *and $r \geq |\mathcal{M}|$. Then w^r leads to a $w^{r!}$-loop for all $w \in A^*$.*

Proof. Observe that w^r leads to a w^i-loop for some $1 \leq i \leq |\mathcal{M}|$. This is because $r \geq |\mathcal{M}|$, so there must be a state appearing twice after input $w^0, w^1, w^2, \ldots, w^r$. The proposition follows since every such w^i-loop can be considered as a $w^{r!}$-loop. ❑

It is also easy to find simultaneous loops between pairs of states if each two states of a pair are connected by the same word of sufficient length.

Proposition 1.35. *Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a* DFA*, let $l \geq 1$ and $v \in A^+$ with $|v| \geq |\mathcal{M}|^l$. Furthermore, let $(s_1, s_1'), \ldots, (s_l, s_l') \in S \times S$ such that $\delta(s_i, v) = s_i'$ for $1 \leq i \leq l$. Then there exist $\hat{s}_i \in S$, $x, y \in A^*$ and $v' \in A^+$ with $v = xv'y$ and $\delta(s_i, xv'y) = \delta(\hat{s}_i, v'y) = \delta(\hat{s}_i, y) = s_i'$ for all $1 \leq i \leq l$.*

Proof. Let v_j denote the prefix of v of length j with $0 \leq j \leq |v|$ and consider the sequence of l-tuples of states

$$(s_1^0, s_2^0, \ldots, s_l^0), (s_1^1, s_2^1, \ldots, s_l^1), \ldots, (s_1^{|v|}, s_2^{|v|}, \ldots, s_l^{|v|})$$

with $s_i^j =_{\text{def}} \delta(s_i, v_j)$ for $1 \leq i \leq l$ and $0 \leq j \leq |v|$. These are at least $(|\mathcal{M}|^l + 1)$ tuples, so there exist $0 \leq j_1 < j_1 \leq |v|$ with $(s_1^{j_1}, \ldots, s_l^{j_1}) = (s_1^{j_2}, \ldots, s_l^{j_2})$. Now rewrite v as $v = xv'y$ such that $x = v_{j_1}$ and $xv' = v_{j_2}$. ❑

A pattern is a subgraph of the extended transition graph with edges labelled by variables for words from A^*, denoted as $u, v, w, x, \ldots$. Sometimes patterns come with side conditions that must hold for the word variables. We define particular patterns by specifying the variables (eventually with a side condition) and by providing a figure of the subgraph (see Definition 2.15 and Figure 2.1 as an example). In such a figure some states are labelled by $+$ (accepting) or by $-$ (rejecting). If we want to express that one of two states is accepting if and only if the other one is rejecting we write $+/-$ and $-/+$. We say that a DFA $\mathcal{M}$ has a certain pattern if there is a subgraph in the extended transition graph of $\mathcal{M}$ as specified in the pattern definition, and if all side conditions hold when the labels are assigned to the word variables.

Definition 1.36. Let $\mathbb{P}$ denote a pattern. Then $\mathcal{FP}(\mathbb{P})$ denotes the class of languages $L \subseteq A^+$ such that there is some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ and $\mathcal{M}$ does not have pattern $\mathbb{P}$.

In all cases we consider in this thesis the classes $\mathcal{FP}(\mathbb{P})$ will be well–defined, i.e., for any two automata accepting L it holds that one has $\mathbb{P}$ if and only if the other one has $\mathbb{P}$. For a finite number of patterns $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n$ let $\mathcal{FP}(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$ denote the class of languages where all patterns $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n$ are simultaneously forbidden in an accepting DFA. A theorem stating $\mathcal{C} = \mathcal{FP}(\mathbb{P})$ or $\mathcal{C} = \mathcal{FP}(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ for a language class $\mathcal{C}$ is called a forbidden pattern characterization of $\mathcal{C}$. As mentioned earlier, such a result usually implies the decidability of the membership problem of $\mathcal{C}$ (even efficiently). To decide whether a given DFA has some pattern we have to verify the respective graph reachability conditions in its transition graph.

Let NL denote the class of languages that are decidable by a nondeterministic algorithm using space at most logarithmically in the input size (see, e.g., [Pap94] for details). It is known that this class is closed under complement [Sze87, Imm88]. So to show that the membership problem of some class $\mathcal{FP}(\mathbb{P})$ is in NL we may provide an algorithm that accepts if and only if the DFA in the input has pattern $\mathbb{P}$. We introduce some more notations to describe algorithms that look for patterns in transition graphs. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA. Then we define for $n \geq 1$, $s_i, s_i' \in S$ and $u \in A^*$ that

$$\begin{aligned}
(s_1, \ldots, s_n) \xrightarrow{u} (s_1', \ldots, s_n') &\iff_{\text{def}} \delta(s_i, u) = s_i' \text{ for all } 1 \leq i \leq n \\
(s_1, \ldots, s_n) \longrightarrow (s_1', \ldots, s_n') &\iff_{\text{def}} \text{there exists some } v \in A^* \text{ such that} \\
&\qquad\quad (s_1, \ldots, s_n) \xrightarrow{v} (s_1', \ldots, s_n') \\
(s_1, \ldots, s_n) \longrightarrow^+ (s_1', \ldots, s_n') &\iff_{\text{def}} \text{there exists some } w \in A^+ \text{ such that} \\
&\qquad\quad (s_1, \ldots, s_n) \xrightarrow{w} (s_1', \ldots, s_n')
\end{aligned}$$

If $n = 1$ we write $s_1 \xrightarrow{w} s_1'$, $s_1 \longrightarrow s_1'$ and $s_1 \longrightarrow^+ s_1'$, respectively. Assume that $n \geq 1$ is fixed. On input $(\mathcal{M}, W)$ with $W = \{\, (s_i, s_i') \in S \times S \mid 1 \leq i \leq n \,\}$ we can verify if

$$(s_1, \ldots, s_n) \longrightarrow (s_1', \ldots, s_n')$$

in nondeterministic logarithmic space NL. To see this we may guess $w \in A^*$ letter by letter and follow the paths which start in $s_1, \ldots, s_n$ and have label w with help of the transition function δ. After each guessed letter we store the new states on these paths in variables $(t_1, \ldots, t_n)$. Moreover, we guess in each step whether we have already reached the end of w, and if so, we check whether $t_i = s'_i$ for all $1 \le i \le n$. Because n is a constant to this algorithm the space needed is dominated by the space needed to store the tuple $(t_1, \ldots, t_n)$ which is logarithmic in the input size.

We recall the following theorem concerning star–free languages. It is the characterization of this class mentioned in the introduction.

Theorem 1.37 ([Sch65, MP71]). *Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a minimal* DFA*. Then $L(\mathcal{M})$ is star–free if and only if there is some $m \ge 0$ such that for all $w \in A^+$ and for all $s \in S$ it holds that $\delta(s, w^m) = \delta(s, w^{m+1})$.*

A minimal DFA $\mathcal{M}$ is called permutation–free if it has the above property. For later use we restate the previous theorem as follows.

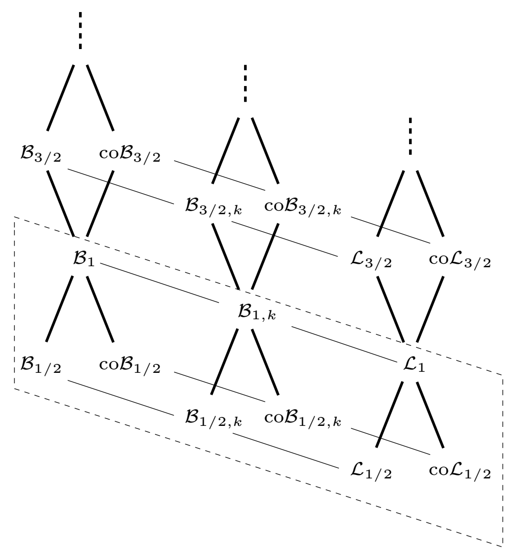
Proposition 1.38. *Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a minimal* DFA*. Then $L(\mathcal{M})$ is not star–free if and only if there exist $w \in A^+$, some $l \ge 2$ and pairwise distinct states $r_0, r_1, \ldots, r_{l-1} \in S$ such that $\delta(r_i, w) = r_{i+1}$ for $0 \le i \le l-1$ (with $r_l =_{\text{def}} r_0$).*

An obvious property of permutation–free automata is that they run into a w-loop after input of successive w's. Otherwise the automaton is not permutation–free as can be seen with Propositions 1.34 and 1.38.

Proposition 1.39. *Let $\mathcal{M}$ be a permutation–free* DFA *and $r \ge |\mathcal{M}|$. Then w^r leads to a w-loop for all $w \in A^*$.*

2. Dot–Depth One

We refer to the main results of this chapter. In Section 2.1 we recall generalizations of the subword relation introduced in [Ste85a] and prove that these relations $\preceq_k$ for $k \geq 0$ have a fundamental property: A^+ together with $\preceq_k$ is a well partial ordered set (cf. Theorem 2.10). This is exploited in Section 2.2 where we show that $\mathcal{B}_{1/2,k}$ is the class of all order ideals of $(A^+, \preceq_k)$ (cf. Theorem 2.12). We also give a forbidden pattern characterization of the classes $\mathcal{B}_{1/2,k}$ (cf. Theorem 2.18). In Section 2.3 we restate the main result from [Ste85a] which gives various characterizations of the classes $\mathcal{B}_{1,k}$ and which we refine in the following way.



First, we deal in Section 2.4 with the known characterization of $\mathcal{B}_{1,k}$ in terms of a finiteness condition on the number of alternations in $\preceq_k$-chains. We prove that the maximal number of such alternations with respect to a language L determines the location of L in the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ (cf. Theorem 2.30).

This has the mentioned finiteness condition as a corollary and we use our characterization to obtain strictness and decidability results for the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ (cf. Theorems 2.31 and 2.33). Such results are also known for the Boolean hierarchy over $\mathcal{B}_{1/2}$ [Gla99]. Taking them into account we identify in Section 2.5 a landscape that allows to study the question whether there exist trade–offs between the parameter k on one hand and Boolean operations on the other hand. We obtain a complete overview over the Boolean structure of $\mathcal{B}_1$ (see Figure 2.6). Finally, we show in Section 2.6 a forbidden pattern characterization of $\mathcal{B}_1$ (cf. Theorem 2.39).

2.1 Subword Relations

We start with a generalization of the well–known subword relation, here extended to k-decompositions of words.

Definition 2.1. Let $k \geq 0$ and let $u, v \in A^{\geq k+1}$. Suppose $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ and $\widehat{v} = (\beta_1, \ldots, \beta_n)$ for $m, n \geq 1$. We define

$$u \lhd_k v \iff_{\text{def}} \text{there exist } 1 \le j_1 < j_2 \ldots < j_m \le n \text{ such that } \beta_{j_i} = \alpha_i \text{ and}$$
$$u \preceq_k v \iff_{\text{def}} u \lhd_k v \text{ and } p_k(u) = p_k(v) \text{ and } s_k(u) = s_k(v).$$

Moreover, if $|u| \le k$ we write $u \lhd_k v$ or $u \preceq_k v$ if and only if $u = v$.

Since the k-decomposition of u has to be a subsequence of the k-decomposition of v, both relations are the usual subword relation in case $k = 0$. The relation $\preceq_k$ was introduced in [Ste85a] (see also the discussion in Section 2.7). Obviously, we have that $u \preceq_k v$ implies $u \lhd_k v$ and we see that both relations are reflexive and transitive. If $u \preceq_k v$ we also say that v is a k-extension of u.

A sequence of words $w_1, w_2, \ldots$ is called a $\lhd_k$-chain if $w_i \lhd_k w_{i+1}$ for all words in the sequence (analogously, we define $\preceq_k$-chains). We say that a chain has an alternation with respect to some given language L if $w_i \in L \Leftrightarrow w_{i+1} \notin L$ for some i.

2.1.1 Basic Properties and Elementary *k*-Extensions

We give basic properties of the defined relations. For some $w \in A^+$ denote by w^R its reverse and set $L^R =_{\text{def}} \{ w^R \mid w \in L \}$.

Proposition 2.2. *Let $k \ge 0$ and $u, v \in A^+$. Then $u \lhd_k v$ if and only if $u^R \lhd_k v^R$. The same holds for $\preceq_k$.*

Proof. The proposition is clear if $|u| \le k$. So suppose $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ and $\widehat{v} = (\beta_1, \ldots, \beta_n)$ for appropriate $m, n \ge 1$. Observe that for every $w \in A^{\ge k+1}$ with $\widehat{w} = (\gamma_1, \gamma_2, \ldots, \gamma_l)$ it holds that $\widehat{w^R} = (\gamma_l^R, \gamma_{l-1}^R, \ldots, \gamma_1^R)$. So if $1 \le j_1 < j_2 \ldots < j_m \le n$ witness that $u \lhd_k v$ we may take $j'_i =_{\text{def}} n - j_{m-i+1} + 1$ for $1 \le i \le m$ to see $u^R \lhd_k v^R$. Finally, it suffices to note that $p_k(w)^R = s_k(w^R)$ for all $w \in A^+$. ❑

With the next two propositions we isolate some arguments used in forthcoming proofs.

Proposition 2.3. *Let $k \ge 0$ and $x, y, z \in A^*$. It holds that*

1. *$u \lhd_k xuy$ for every $u \in A^{\ge k+1}$,*
2. *$uy \lhd_k uz \implies xuy \lhd_k xuz$ for every $u \in A^{\ge k}$ and*
3. *$xu \lhd_k yu \implies xuz \lhd_k yuz$ for every $u \in A^{\ge k}$.*

Proof. Statement 1 is obvious. To see statement 2 note that if $|uy| = k$ then $y = z = \varepsilon$. If $|uy| > k$ then suppose $1 \le j_1 < j_2 \ldots < j_m \le n$ for appropriate $m, n \ge 1$ witness that $uy \lhd_k uz$. Since uy and uz have the same k-prefix we may take $j'_i =_{\text{def}} i$ for $1 \le i \le |x|$ and $j'_i =_{\text{def}} j_{i-|x|} + |x|$ for $|x| + 1 \le i \le |x| + m$ to see that $xuy \lhd_k xuz$. Statement 3 follows from statement 2 using Proposition 2.2. ❑

Proposition 2.4. *Let $k \ge 0$ and $x, y, z \in A^*$. It holds that*

1. *$w_1 \preceq_k w_2 \implies xw_1y \preceq_k xw_2y$ for every $w_1, w_2 \in A^+$,*
2. *$xy \preceq_k xzy$ if $p_k(y) = p_k(zy) \in A^k$ and $xy \in A^{\ge k+1}$ and*
3. *$w_1x \preceq_k w_2x,\ xv_1 \preceq_k xv_2 \implies w_1xv_1 \preceq_k w_2xv_2$ for every $w_1, w_2, v_1, v_2 \in A^+$.*

Proof. Statement 1 is clear for $|w_1| \leq k$. For the other case note that from $w_1 \preceq_k w_2$ it follows that both words have the same k-prefix and k-suffix. Hence, we may apply Proposition 2.3.2 and then 2.3.3 to get $xw_1y \lhd_k xw_2y$. Since these words have again the same k-prefix and k-suffix we get $xw_1y \preceq_k xw_2y$. For statement 2 we may take $j_i =_{\text{def}} i$ for $1 \leq i \leq |x|$ and $j_i =_{\text{def}} i + |z|$ for $|x|+1 \leq i \leq |xy| - k$ as witnessing indices. Again, we see that both words have the same k-prefix and k-suffix. Moreover, the letter at position $j_{|x|}$ in xy and in xzy is followed by $p_k(y) = p_k(zy)$. If we apply statement 1 twice we get $w_1xv_1 \preceq_k w_2xv_1 \preceq_k w_2xv_2$ and obtain statement 3. ❑

One can understand $u \lhd_k v$ as a simultaneous insertion of factors at different positions in u to obtain v, while respecting certain context conditions depending on k. As a special case, there may only be one such insertion position. This gives rise to the notion of elementary k-extensions introduced next.

Definition 2.5. Let $k \geq 0$ and let $u, v \in A^{\geq k+1}$. Suppose $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ and $\widehat{v} = (\beta_1, \ldots, \beta_n)$ for $m, n \geq 1$. We define

$$\begin{array}{rcl} u \lhd_k^{\mathrm{e}} v & \Longleftrightarrow_{\text{def}} & \text{there exist } r \geq 0 \text{ and } 0 \leq l \leq m \text{ such that} \\ & & (\beta_1, \ldots, \beta_n) = (\alpha_1, \ldots, \alpha_l, \gamma_1, \ldots, \gamma_r, \alpha_{l+1}, \ldots, \alpha_m) \\ & & \text{for some } \gamma_1, \ldots, \gamma_r \in A^{k+1} \text{ and} \\ u \preceq_k^{\mathrm{e}} v & \Longleftrightarrow_{\text{def}} & u \lhd_k^{\mathrm{e}} v \text{ and } p_k(u) = p_k(v) \text{ and } s_k(u) = s_k(v). \end{array}$$

As before, if $|u| \leq k$ we write $u \lhd_k^{\mathrm{e}} v$ or $u \preceq_k^{\mathrm{e}} v$ if and only if $u = v$.

If $u \preceq_k^{\mathrm{e}} v$ we say that v is an elementary k-extension of u. Clearly, if $u \lhd_k^{\mathrm{e}} v$ then $u \lhd_k v$ and if $u \preceq_k^{\mathrm{e}} v$ then $u \preceq_k v$. Note also that if $u \lhd_k^{\mathrm{e}} v$ or $u \preceq_k^{\mathrm{e}} v$ then there are $x, y, z \in A^*$ with $u = xy$ and $v = xzy$. What we have described in Proposition 2.4.2 is in fact an elementary k-extension. We show with the following two propositions that we can decompose any k-extension into a finite number of elementary ones.

Proposition 2.6. *Let $k \geq 0$ and let $u, v \in A^+$. If $u \neq v$ and $u \preceq_k v$ then there exists some $w \in A^+$ with $w \neq u$ and $u \preceq_k^{\mathrm{e}} w \preceq_k v$.*

Proof. We may assume $u \preceq_k v$ with $u, v \in A^{\geq k+1}$. Let $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ and $\widehat{v} = (\beta_1, \ldots, \beta_n)$ for some $m, n \geq 1$. By definition it holds that $p_k(u) = p_k(v)$ and $s_k(u) = s_k(v)$ and there are $1 \leq j_1 < j_2 \ldots < j_m \leq n$ such that $\beta_{j_i} = \alpha_i$ for $1 \leq i \leq m$. Set $j_0 = 0$ and $j_{m+1} = n + 1$. Fix now a maximal l with $0 \leq l \leq m$ such that $j_l = l$ and set $r =_{\text{def}} j_{l+1} - j_l - 1$. Note that such an l exists since $u \neq v$, and that $r \geq 1$. We claim that there is a word $w \in A^{\geq k+1}$ such that

$$\widehat{w} = (\alpha_1, \ldots, \alpha_l, \beta_{l+1}, \ldots, \beta_{l+r}, \alpha_{l+1}, \ldots, \alpha_m).$$

Let a_i for $1 \leq i \leq m$ be the first letter of α_i, and let b_i for $1 \leq i \leq n$ be the first letter of β_i. Define $w =_{\text{def}} a_1 \cdots a_l b_{l+1} \cdots b_{l+r} a_{l+1} \cdots a_m s_k(\alpha_m)$. If we show that w has the above k-decomposition we immediately have $u \lhd_k^{\mathrm{e}} w$. In fact, we only need to show that α_l, β_{l+1} and $\beta_{l+r}, \alpha_{l+1}$ fit together in the sense that $s_k(\alpha_l) = p_k(\beta_{l+1})$ and $s_k(\beta_{l+r}) = p_k(\alpha_{l+1})$. To see that $s_k(\alpha_l) = p_k(\beta_{l+1})$ note that $l = j_l$ and thus $\alpha_l = \beta_l$. To see that $s_k(\beta_{l+r}) = p_k(\alpha_{l+1})$ observe that $l + r + 1 = j_{l+1}$ and thus $\alpha_{l+1} = \beta_{l+r+1}$.

If $0 < l < m$ then $p_k(w) = p_k(u)$ and $s_k(w) = s_k(u)$. If $l = 0$ then $p_k(w) = p_k(v)$ and hence $p_k(w) = p_k(u)$. The same holds for the k-suffix if $l = m$. So $u \preceq_k^{\mathrm{e}} w$.

We need to show $w \preceq_k v$. In fact, we only need to show $w \lhd_k v$. Due to the construction of w we may take as a witnessing sequence of indices $1, 2, \ldots, l, l+1, \ldots, l+r, j_{l+1}, j_{l+2}, \ldots, j_m$. Recall that $j_{l+1} = l + r + 1$. ❑

Proposition 2.7. *Let $k \geq 0$ and let $u, v \in A^+$ with $u \neq v$ and $u \preceq_k v$. Then there exist $l \geq 1$ and words $w_0, w_1, \ldots, w_l \in A^+$ with $w_0 = u, w_l = v$ such that for all $0 \leq i < l$ it holds that $w_i \preceq_k^{\mathrm{e}} w_{i+1}$. Moreover, all w_i are pairwise distinct.*

Proof. Define $w_0 =_{\mathrm{def}} u$. If v is already an elementary k-extension of u we are done with $w_1 =_{\mathrm{def}} v$. Otherwise we apply Proposition 2.6, set $w_1 =_{\mathrm{def}} w$ and obtain $w_0 \preceq_k^{\mathrm{e}} w_1$. Now we start over again with w_1 and v, apply Proposition 2.6 if $w_1 \neq v$, and so on. This procedure comes to an end since Proposition 2.6 provides a strict elementary k-extension and strict extensions are length increasing. ❑

Remark 2.8. In case $k = 0$ we can assume that $r = 1$ in an elementary 0-extension, simply insert one letter after another. This is because we do not have to respect any context conditions which is not true if $k \geq 1$. Consider over alphabet $A =_{\mathrm{def}} \{a, b, c\}$ for example $ab \preceq_1^{\mathrm{e}} abcb$ with $\widehat{ab} = (ab)$ and $\widehat{abcb} = (ab, bc, cb)$. We look for a word axb with $x \in A$ and $\widehat{axb} = (ax, xb)$ such that $ab \preceq_1^{\mathrm{e}} axb \preceq_1^{\mathrm{e}} abcb$. But $x = a$ ($x = b$) is not possible because there is no aa (bb, respectively) in $\widehat{abcb}$, and $x = c$ is not possible since then there is no ab in $\widehat{acb}$. This is a significant difference between $k = 0$ and arbitrary k.

2.1.2 Well Partial Ordered Sets

As it turns out, the word extensions we consider have a fundamental property: $(A^+, \preceq_k)$ for $k \geq 0$ is a well partial ordered set (wpos, for short). The proof we give below is based on an idea from [SS83] where A^+ and the usual subword relation are considered. A first proof of the latter was given in [Hig52].

For several equivalent properties, which may be used for defining well partial ordered sets, see [CK96, SS83]. We show here that in A^+ there exists neither an infinite strictly descending $\lhd_k$-chain, nor an infinite set of pairwise incomparable elements with respect to $\lhd_k$. This is equivalent to saying that for every non–empty subset of A^+ the set of minimal elements with respect to $\lhd_k$ in this subset is non–empty and finite [CK96]. In case $k = 0$ (i.e., the subword relation, also called division ordering) we encounter the fundamental theorem from Higman [Hig52].

Theorem 2.9. *Let $k \geq 0$. It holds that $(A^+, \lhd_k)$ is a wpos.*

Proof. First observe by a typical length argument (namely that $u \lhd_k v$ with $u \neq v$ implies $|u| < |v|$) that we only have to show that any set of pairwise incomparable elements is finite. Assume to the contrary that there is an infinite subset of A^+ such that all its elements are pairwise incomparable with respect to $\lhd_k$. In particular, there exist infinite sequences $\{f_i\}$ of words such that from $i < j$ it follows that $f_i \not\lhd_k f_j$. We will show that this is not true. For this consider any such sequence $\{f_i\}$ and note that all words in such a sequence must be different since $\lhd_k$ is reflexive. We choose from all sequences $\{f_i\}$ an 'earliest' sequence $\{u_i\}$ as follows (using the axiom of choice): let u_1 be a shortest word beginning some sequence $\{f_i\}$, then let u_2 be a shortest second word of any sequence $u_1, f_2, f_3 \ldots$, then let u_3 be a shortest third word of any sequence $u_1, u_2, f_3 \ldots$, and so on. Clearly, also for $\{u_i\}$ it holds that from $i < j$ it follows that $u_i \not\triangleleft_k u_j$. Since we have a finite alphabet there are words $u_{i_1} = awg_1, u_{i_2} = awg_2, \ldots$ with $i_1 < i_2 < \ldots$ for some $a \in A$ and some $w \in A^k$. Note that none of these g_j can be ε since $aw \triangleleft_k awv$ for arbitrary v.

Now we look at the sequence $u_1, u_2, \ldots, u_{i_1-1}, wg_1, wg_2, \ldots$ Denote this new sequence as $\{x_i\}$ which is 'earlier' than $\{u_i\}$ since $|wg_1| < |u_{i_1}|$. In order to obtain a contradiction to our construction we need to show that for all i, j with $i < j$ we can conclude $x_i \not\triangleleft_k x_j$. This is clear if $i, j \in \{1, \ldots, i_1 - 1\}$ by the same property for $\{u_i\}$. Now suppose $i \in \{1, \ldots, i_1 - 1\}$ and $j \geq i_1$ and assume $x_i \triangleleft_k x_j$ where $x_i = u_i$ and $x_j = wg_l$ for some $l \geq 1$. Since $wg_l \in A^{\geq k+1}$ we have by Proposition 2.3.1 that $wg_l \triangleleft_k awg_l = u_{i_l}$ and together $u_i \triangleleft_k u_{i_l}$, a contradiction. Finally let $i, j \geq i_1$ with $i < j$. Assume $x_i \triangleleft_k x_j$ with $x_i = wg_l$ and $x_j = wg_m$ for some $l < m$. By Proposition 2.3.2 we have that $awg_l \triangleleft_k awg_m$, so $u_{i_l} \triangleleft_k u_{i_m}$, again a contradiction. ❑

Theorem 2.10. *Let $k \geq 0$. It holds that $(A^+, \preceq_k)$ is a wpos.*

Proof. Suppose there exists an infinite subset of A^+ such that all its elements are pairwise incomparable with respect to $\preceq_k$. Then there is also an infinite subset L such that all words in L have the same k-prefix and k-suffix. So the words in L are pairwise incomparable with respect to $\triangleleft_k$ contradicting Theorem 2.9. ❑

Interestingly, it seems to be difficult to find a direct proof for $\preceq_k$. In fact, this is the reason why we introduced $\triangleleft_k$.

2.2 The Classes $\mathcal{B}_{1/2,k}$

There are close connections of k-extensions to the classes $\mathcal{B}_{1/2,k}$.

2.2.1 Order Ideals and Closure Properties

The closure of $u \in A^+$ under k-extensions is denoted as $\langle u \rangle_k =_{\mathrm{def}} \{ v \in A^+ \mid u \preceq_k v \}$. If $u \in A^{\leq k}$ then $\langle u \rangle_k = \{u\}$, so $\langle u \rangle_k \in \mathcal{B}_{1/2,k}$. If $u \in A^{\geq k+1}$ and $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ for some $m \geq 1$ then $\langle u \rangle_k = (p_k(u) | \alpha_1, \ldots, \alpha_m | s_k(u))_k$, so again $\langle u \rangle_k \in \mathcal{B}_{1/2,k}$.

Proposition 2.11. *Let $k \geq 0$ and $u \in A^+$. It holds that $\langle u \rangle_k \in \mathcal{B}_{1/2,k}$.*

Now we look at the closure of some $L \subseteq A^+$ under k-extensions, which we denote as $\langle L \rangle_k =_{\mathrm{def}} \bigcup_{u \in L} \langle u \rangle_k$. A language L is called an order ideal of $(A^+, \preceq_k)$ if and only if $L = \langle L \rangle_k$.

Theorem 2.12. *Let $k \geq 0$ and $L \subseteq A^+$. It holds that $L \in \mathcal{B}_{1/2,k}$ if and only if L is an order ideal of $(A^+, \preceq_k)$.*

Proof. Suppose $L = \langle L \rangle_k = \bigcup_{u \in L} \langle u \rangle_k$. We have seen before that $\langle u \rangle_k \in \mathcal{B}_{1/2,k}$. Since $(A^+, \preceq_k)$ is a wpos by Theorem 2.10 the set of distinct minimal elements $u \in L$ with respect to $\preceq_k$ is finite. Note that $u \preceq_k v$ if and only if $\langle u \rangle_k \supseteq \langle v \rangle_k$, so we may assume that the union $\bigcup_{u \in L} \langle u \rangle_k$ is finite. Hence, $L \in \mathcal{B}_{1/2,k}$.

Conversely, it suffices to show that $\langle L \rangle_k \subseteq L$. Suppose $u \in \langle L \rangle_k$. Then there is some $x \in L$ such that $x \preceq_k u$. If $x \in A^{\leq k}$ then $x = u$. If $x \in A^{\geq k+1}$ then x is in some set of the form $(w | \alpha_1, \ldots, \alpha_m | v)_k \subseteq L$ with $m \geq 1$, $w, v \in A^k$ and $\alpha_i \in A^{k+1}$. By definition of $\preceq_k$, the words x

and u have the same k-prefix and k-suffix, so $w = p_k(x) = p_k(u)$ and $v = s_k(x) = s_k(u)$. Moreover, all elements of the k-decomposition of x appear in this ordering in the k-decomposition of u, which holds in particular for $\alpha_1, \ldots, \alpha_m$. This shows $u \in (w|\alpha_1, \ldots, \alpha_m|v)_k \subseteq L$. ❑

The theorem also shows that the closure of an arbitrary language under k-extensions is in $\mathcal{B}_{1/2,k}$. In particular, we obtain a regular language independent of the language we start with.

Corollary 2.13. *Let $k \geq 0$ and $L \subseteq A^+$. It holds that $\langle L \rangle_k \in \mathcal{B}_{1/2,k}$.*

We continue with closure properties of the classes $\mathcal{B}_{1/2,k}$. The closure under right and left residuals is shown already for all $\mathcal{B}_{n/2,k}$ with $n \geq 1$.

Lemma 2.14. *Let $k \geq 0$ and $n \geq 1$.*

1. *The classes $\mathcal{B}_{1/2,k}$ and $\mathrm{co}\mathcal{B}_{1/2,k}$ are closed under finite union and intersection.*
2. *It holds that $a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \mathcal{B}_{n/2,k}$ for $a \in A$ and $L \in \mathcal{B}_{n/2,k}$.*

Proof. The closure of $\mathcal{B}_{1/2,k}$ under finite union is by definition. To prove the first statement it suffices to show the closure of $\mathcal{B}_{1/2,k}$ under finite intersection, since then both closure properties translate to $\mathrm{co}\mathcal{B}_{1/2,k}$ by DeMorgan's law. So suppose $L, L' \in \mathcal{B}_{1/2,k}$. We show that $L \cap L' = \langle L \cap L' \rangle_k$ from which $L \cap L' \in \mathcal{B}_{1/2,k}$ follows with Theorem 2.12. It suffices to argue for the inclusion $\langle L \cap L' \rangle_k \subseteq L \cap L'$. For every $u \in \langle L \cap L' \rangle_k$ there is some $x \in L \cap L'$ such that $x \preceq_k u$. Since by Theorem 2.12 we have $L = \langle L \rangle_k$ and $L' = \langle L' \rangle_k$ there are $y \in L$ and $y' \in L'$ such that $y \preceq_k x \preceq_k u$ and $y' \preceq_k x \preceq_k u$. So it holds that $u \in \langle L \rangle_k = L$ and $u \in \langle L' \rangle_k = L'$.

We turn to the second statement which we show by induction on n. It suffices to argue for the case of left residuals.

Induction base. Let $n = 1$. Since $\mathcal{B}_{1/2,0} = \mathcal{L}_{1/2}$ by Proposition 1.28 and together with Lemma 1.21 it remains to show the case when $k \geq 1$. So let a language $L \in \mathcal{B}_{1/2,k}$ be given. By definition, L is a finite union of a subset of $A^{\leq k}$ with languages of the form $(w|\alpha_1, \ldots, \alpha_m|v)_k \subseteq A^{\geq k+1}$ where $m \geq 1$, $w, v \in A^k$ and $\alpha_i \in A^{k+1}$. We can treat the members of this union separately since $a^{-1}(L_1 \cup L_2) = a^{-1}L_1 \cup a^{-1}L_2$ for arbitrary $L_1, L_2 \subseteq A^+$. Clearly, $a^{-1}D \cap A^+ \subseteq A^{\leq k}$ for $D \subseteq A^{\leq k}$. So fix some $L' =_{\mathrm{def}} (w|\alpha_1, \ldots, \alpha_m|v)_k$ with $m \geq 1$, $w, v \in A^k$ and $\alpha_i \in A^{k+1}$. Note that $k \geq 1$ and that $a^{-1}L' \subseteq A^{\geq k}$. So $a^{-1}L' \cap A^+ = a^{-1}L'$ and it remains to show that $a^{-1}L' \in \mathcal{B}_{1/2,k}$. Furthermore we may assume that $w = aw'$ for some $w' \in A^*$ since otherwise $a^{-1}L' = \emptyset \in \mathcal{B}_{1/2,k}$.

Case 1. Suppose $aw' \neq p_k(\alpha_1)$. Then $a^{-1}L' \subseteq A^{\geq k+1}$ and there are no words x in L' such that α_1 is the first element in the k-decomposition of x. We obtain

$$a^{-1}L' = \bigcup_{b \in A} (w'b|\alpha_1, \ldots, \alpha_m|v)_k \ \in \mathcal{B}_{1/2,k}.$$

Case 2. Suppose $aw' = p_k(\alpha_1)$ and $m \geq 2$. Then again $a^{-1}L' \subseteq A^{\geq k+1}$, but the k-decomposition of some $x \in L'$ may start with α_1. In this case we see that

$$a^{-1}L' = \bigcup_{b \in A} (w'b|\alpha_1, \ldots, \alpha_m|v)_k \ \cup \ (s_k(\alpha_1)|\alpha_2, \ldots, \alpha_m|v)_k \ \in \mathcal{B}_{1/2,k}.$$

Case 3. Suppose $aw' = p_k(\alpha_1)$ and $m = 1$. Then it holds that

$$a^{-1}L' = \bigcup_{b\in A}(w'b|\,\alpha_1\,|v)_k \;\cup\; \bigcup_{\beta\in A^{k+1}}(s_k(\alpha_1)|\,\beta\,|v)_k \;\cup\; \{\,v \mid s_k(\alpha_1)=v\,\} \;\in \mathcal{B}_{1/2,k}.$$

Induction step. We first assume that the assertion holds for $n \geq 1$ with $n \equiv 1 \mod 2$ and we want to prove it for $n+1$. So let $L \in \mathcal{B}_{(n+1)/2,k}$ and $a \in A$. By definition, L is a Boolean combination of languages from $\mathcal{B}_{n/2,k}$ that we can write as a finite union of intersections of sets E and $A^+ \setminus E$ with $E \in \mathcal{B}_{n/2,k}$. Again note that $a^{-1}(L_1 \cup L_2) = a^{-1}L_1 \,\cup\, a^{-1}L_2$ and also $a^{-1}(L_1 \cap L_2) = a^{-1}L_1 \,\cap\, a^{-1}L_2$ for all $L_1, L_2 \subseteq A^+$. So $a^{-1}L \cap A^+$ can be written as a finite union of intersections of sets $L_1 =_{\text{def}} a^{-1}E \,\cap\, A^+$ and $L_2 =_{\text{def}} a^{-1}(A^+ \setminus E) \,\cap\, A^+$. By hypothesis, we have $L_1 \in \mathcal{B}_{n/2,k} \subseteq \mathcal{B}_{(n+1)/2,k}$ and it remains to show that $L_2 \in \mathcal{B}_{(n+1)/2,k}$. Note that $E \subseteq A^+$ and denote for this proof by $\overline{L}$ the complement of L with respect to A^*. Then we can carry out the following calculation.

$$\begin{aligned}
L_2 &= a^{-1}(A^+ \setminus E) \,\cap\, A^+ \\
&= a^{-1}\left(A^+ \,\cap\, \overline{E}\right) \,\cap\, A^+ \\
&= a^{-1}A^+ \,\cap\, a^{-1}\overline{E} \,\cap\, A^+ \\
&= A^* \,\cap\, \overline{a^{-1}E} \,\cap\, A^+ \\
&= \left(\overline{a^{-1}E} \,\cup\, \overline{A^+}\right) \,\cap\, A^+ \\
&= A^+ \setminus \left(a^{-1}E \,\cap\, A^+\right)
\end{aligned}$$

By hypothesis, $L_2 \in \text{co}\mathcal{B}_{n/2,k} \subseteq \mathcal{B}_{(n+1)/2,k}$. This shows in particular the second statement for $n = 2$. So now we assume that it holds for $n \geq 2$ with $n \equiv 0 \mod 2$ and we want to prove it for $n+1$. Let $L \in \mathcal{B}_{(n+1)/2,k}$. By definition, L is a finite union of language $L_1L_2\cdots L_m$ with $m \geq 0$ and $L_i \in \mathcal{B}_{n/2,k}$. It suffices to consider each member of the union separately. In case $m = 0$ there is nothing to do and if $m = 1$ we can immediately apply the hypothesis. It remains to show that $a^{-1}\left(L_1L_2\cdots L_m\right) \,\cap\, A^+ = a^{-1}\left(L_1L_2\cdots L_m\right) \in \mathcal{B}_{(n+1)/2,k}$ for $m \geq 2$. Since $L_i \subseteq A^+$ we can write $a^{-1}\left(L_1L_2\cdots L_m\right) = \left((a^{-1}L_1)L_2\cdots L_m\right)$. Now we use that $\{b\}, L_1 \in \mathcal{B}_{n/2,k}$ which is a class closed under Boolean operations. So we can rewrite L_1 as $L_1 = L_1' \cup L_2''$ with $L_1' =_{\text{def}} L_1 \cap A$ and $L_1'' =_{\text{def}} L_1 \setminus L_1'$. Observe that again $L_1', L_1'' \in \mathcal{B}_{n/2,k}$. If we rewrite in

$$\left((a^{-1}L_1)L_2\cdots L_m\right) = \left((a^{-1}L_1' \cup a^{-1}L_1'')L_2\cdots L_m\right) = (a^{-1}L_1')L_2\cdots L_m \,\cup\, (a^{-1}L_1'')L_2\cdots L_m$$

the set L_1' as the finite union of its elements, we encounter either the empty set or $L_2\cdots L_m$ which are both in $\mathcal{B}_{(n+1)/2,k}$. In case of $a^{-1}L_1''$ observe that $L_1'' \subseteq A^{\geq 2}$ and $a^{-1}L_1'' = a^{-1}L_1'' \,\cap\, A^+ \in \mathcal{B}_{n/2,k}$ by hypothesis. So also $(a^{-1}L_1'')L_2\cdots L_m \in \mathcal{B}_{(n+1)/2,k}$ which completes the induction. ❑

2.2.2 Forbidden Pattern Characterization

We recall the forbidden patterns $\mathbb{L}_{1/2}$ and $\mathbb{B}_{1/2}$ characterizing $\mathcal{L}_{1/2}$ and $\mathcal{B}_{1/2}$, respectively.

Definition 2.15 ([PW97]).

1. Pattern $\mathbb{L}_{1/2}$ is defined as the subgraph given in Figure 2.1 with $x, w, z \in A^*$.
2. Pattern $\mathbb{B}_{1/2}$ is defined as the subgraph given in Figure 2.2 with $x, z \in A^*$ and $v, w \in A^+$.

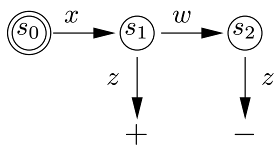


Fig. 2.1. Pattern $\mathbb{L}_{1/2}$.

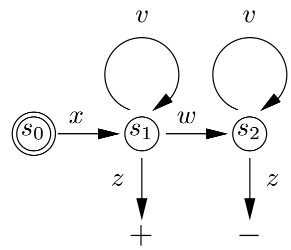


Fig. 2.2. Pattern $\mathbb{B}_{1/2}$.

It is easy to see that $\mathcal{FP}(\mathbb{L}_{1/2})$ is well–defined. The same holds for $\mathcal{FP}(\mathbb{B}_{1/2})$ which can be shown using the arguments from the second part of the following proof.

Theorem 2.16 ([PW97],[Arf91]). *It holds that*

1. $\mathcal{L}_{1/2} = \mathcal{FP}(\mathbb{L}_{1/2})$ *and*
2. $\mathcal{B}_{1/2} = \mathcal{FP}(\mathbb{B}_{1/2})$.

Proof. To show the first statement we recall [PW97, Theorem 8.5]. After rewriting their notations we obtain the following (together with Theorem 1.8). It is an automata–theoretic version of [Arf91, Theorem 3.3].

(a) Let $\mathcal{M}$ be a minimal DFA with $L(\mathcal{M}) \subseteq A^*$. Then $L(\mathcal{M}) \in \mathcal{L}_{1/2} \cup \{A^*\}$ if and only if $\mathcal{M}$ does not have a subgraph in its transition graph as depicted in Figure 2.1 with $x, w, z \in A^*$.

Suppose $L \in \mathcal{L}_{1/2}$ and let $\mathcal{M} = (A, S, \delta, s_0, S')$ be the minimal DFA with $L(\mathcal{M}) = L \subseteq A^+$. Assume that $\mathcal{M}$ has pattern $\mathbb{L}_{1/2}$ via $x, w, z \in A^*$. We apply (a) and see that $L = L(\mathcal{M}) \notin \mathcal{L}_{1/2} \cup \{A^*\}$, a contradiction. It follows that there exists an DFA accepting L that does not have pattern $\mathbb{L}_{1/2}$, so $L \in \mathcal{FP}(\mathbb{L}_{1/2})$ (recall Definition 1.36).

Conversely, let $L \in \mathcal{FP}(\mathbb{L}_{1/2})$. So there exists some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L \subseteq A^+$ such that $\mathcal{M}$ does not have pattern $\mathbb{L}_{1/2}$. We assume that $L \notin \mathcal{L}_{1/2}$ and show that this leads to a contradiction. Since $L \subseteq A^+$ we see that $L \notin \mathcal{L}_{1/2} \cup \{A^*\}$. So by (a), the minimal DFA $\mathcal{M}' = (A, S, \delta, s_0, S')$ accepting L has a subgraph in its transition graph as depicted in Figure 2.1 with $x, w, z \in A^*$. Then $xz \in L = L(\mathcal{M})$ and $xwz \notin L = L(\mathcal{M})$ with $x, w, z \in A^*$, which shows that $\mathcal{M}$ has pattern $\mathbb{L}_{1/2}$, a contradiction.

To see the second statement of the theorem we recall [PW97, Theorem 8.15]. After rewriting their notations we obtain the following (recall also by Theorem 1.8 that we talk about the same class of languages $\mathcal{B}_{1/2}$).

(b) Let $\mathcal{M}$ be a minimal DFA with $L(\mathcal{M}) \subseteq A^+$. Then $L(\mathcal{M}) \in \mathcal{B}_{1/2}$ if and only if $\mathcal{M}$ does not have a subgraph in its transition graph as depicted in Figure 2.2 with $x \in A^*$ and $v, w, z \in A^+$.

Suppose $L \in \mathcal{B}_{1/2}$ and let $\mathcal{M} = (A, S, \delta, s_0, S')$ be the minimal DFA with $L(\mathcal{M}) = L \subseteq A^+$. Assume that $\mathcal{M}$ has pattern $\mathbb{B}_{1/2}$ via $x, z \in A^*$ and $v, w \in A^+$. Then $\mathcal{M}$ also has pattern $\mathbb{B}_{1/2}$ via $x \in A^*$ and $v, w, z' \in A^+$ with $z' =_{\mathrm{def}} vz$. So we can apply (b) and see that $L = L(\mathcal{M}) \notin \mathcal{B}_{1/2}$, a contradiction. It follows that there exists an DFA accepting L that does not have pattern $\mathbb{B}_{1/2}$, so $L \in \mathcal{FP}(\mathbb{B}_{1/2})$.

Conversely, let $L \in \mathcal{FP}(\mathbb{B}_{1/2})$. So there exists some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L \subseteq A^+$ such that $\mathcal{M}$ does not have pattern $\mathbb{B}_{1/2}$. We assume that $L \notin \mathcal{B}_{1/2}$ and show that this leads to a contradiction. By (b), the minimal DFA $\mathcal{M}'$ accepting L has a subgraph in its transition graph as depicted in Figure 2.2 with $x \in A^*$ and $v, w, z \in A^+$.

Now let $r =_{\text{def}} |\mathcal{M}|$, and define $z' =_{\text{def}} z$, $x' =_{\text{def}} xv^r$, $w' =_{\text{def}} wv^r$ and $v' =_{\text{def}} v^{r!}$. Observe that $x', z' \in A^*$ and $v', w' \in A^+$. We obtain from Proposition 1.34 that x' and w' lead to a v'-loop. Moreover, we see from $\mathcal{M}'$ that $x'z' \in L = L(\mathcal{M})$ and $x'w'z' \notin L = L(\mathcal{M})$. So $\mathcal{M}$ has pattern $\mathbb{B}_{1/2}$, a contradiction. ❑

We show that the connection of $\mathcal{L}_{1/2}$ and $\mathcal{B}_{1/2}$ via the classes $\mathcal{B}_{1/2,k}$ has a natural counterpart on the pattern side.

Definition 2.17. Let $k \geq 0$. Pattern $\mathbb{B}_{1/2,k}$ is defined as the subgraph given in Figure 2.1 with $x, w, z \in A^*$ and the side conditions

– $|x| \geq k$, $|xz| \geq k + 1$ and
– $s_k(x) = s_k(xw)$.

Note that $\mathbb{L}_{1/2}$ is the same pattern as $\mathbb{B}_{1/2,0}$ since we look only at automata accepting languages from A^+. It is easy to see that $\mathcal{FP}(\mathbb{B}_{1/2,k})$ is well–defined. The following theorem gives in particular another proof of the first statement of Theorem 2.16.

Theorem 2.18. *Let $k \geq 0$. It holds that $\mathcal{B}_{1/2,k} = \mathcal{FP}(\mathbb{B}_{1/2,k})$.*

Proof. For the inclusion from left to right suppose $L \in \mathcal{B}_{1/2,k}$ and let $\mathcal{M}$ be a DFA accepting L. If $\mathcal{M}$ has pattern $\mathbb{B}_{1/2,k}$ with $x, w, z \in A^*$, then it is easy to see that $xz \preceq_k xwz$, e.g., apply Propositions 2.4.2 and 2.2. Since $xz \in L$ and $xwz \notin L$ we see that L is not an order ideal of $(A^+, \preceq_k)$. So by Theorem 2.12 we have $L \notin \mathcal{B}_{1/2,k}$, a contradiction.

For the reverse inclusion suppose $L \in \mathcal{FP}(\mathbb{B}_{1/2,k})$. Then there exists some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ such that $\mathcal{M}$ does not have pattern $\mathbb{B}_{1/2,k}$. Assume to the contrary that $L \notin \mathcal{B}_{1/2,k}$. Again by Theorem 2.12 there is some $u \in L$ and $v \notin L$ such that $u \preceq_k v$. Note that $|u| \geq k+1$ since otherwise $u = v$. We want to exploit this situation to find pattern $\mathbb{B}_{1/2,k}$ in $\mathcal{M}$. First, we apply the decomposition of k-extensions into elementary ones from Proposition 2.7. There must be at least one position i in the sequence of elementary k-extensions from u to v where $w_i \preceq_k^{\mathrm{e}} w_{i+1}$ with $w_i \in L$ and $w_{i+1} \notin L$. So we assume without loss of generality that $u \preceq_k^{\mathrm{e}} v$.

Let $\widehat{u} = (\alpha_1, \ldots, \alpha_m)$ for some $m \geq 1$ and $\widehat{v} = (\alpha_1, \ldots, \alpha_l, \gamma_1, \ldots, \gamma_r, \alpha_{l+1}, \ldots, \alpha_m)$ for some $0 \leq l \leq m$, $r \geq 1$ and $\gamma_i \in A^{k+1}$. Let $a_1, \ldots, a_l$ be the first letters of $\alpha_1, \ldots, \alpha_l$, and let $b_1, \ldots, b_r, c_{l+1}, \ldots, c_m$ be the last letters of $\gamma_1, \ldots, \gamma_r, \alpha_{l+1}, \ldots, \alpha_m$. Now define $x =_{\text{def}} a_1 \cdots a_l p_k(\gamma_1)$, $w =_{\text{def}} b_1 \cdots b_r$ and $z =_{\text{def}} c_{l+1} \cdots c_m$. Then $|x| \geq k$, $xz = u$, $|xz| \geq k+1$ and $xwz = v$. First assume $0 < l < m$. Then it holds that $s_k(x) = s_k(\alpha_l) = p_k(\alpha_{l+1}) = s_k(\gamma_r) = s_k(xw)$. If $l = 0$ then $x = p_k(\gamma_1) = p_k(\alpha_1) = s_k(xw)$ since $p_k(u) = p_k(v)$. Analogously, if $l = m$ then $s_k(x) = s_k(\alpha_l) = s_k(\gamma_r) = s_k(xw)$ since $s_k(u) = s_k(v)$. Together we see that $x, w, z \in A^*$ give rise to pattern $\mathbb{B}_{1/2,k}$ in $\mathcal{M}$, a contradiction. So $L \in \mathcal{B}_{1/2,k}$. ❑

One can understand pattern $\mathbb{B}_{1/2,k}$ as an elementary k-extensions in the transition graph of $\mathcal{M}$ leading from $L(\mathcal{M})$ to its complement, i.e., one alternation from + to −. It is clear that we encounter pattern $\mathbb{B}_{1/2}$ if k is large enough in comparison to $\mathcal{M}$.

Proposition 2.19. *Let $\mathcal{M}$ be a* DFA *and $k \geq |\mathcal{M}|^2$. If $\mathcal{M}$ has pattern $\mathbb{B}_{1/2,k}$ then $\mathcal{M}$ has pattern $\mathbb{B}_{1/2}$.*

Proof. Suppose $\mathcal{M} = (A, S, \delta, s_0, S')$ has pattern $\mathbb{B}_{1/2,k}$ via $x, w, z \in A^*$. Then there are two distinct pairs of states (s'_1, s_1) and (s'_2, s_2) such that for $v =_{\text{def}} s_k(x) = s_k(xw)$ we have $\delta(s'_i, v) = s_i$ for $i = 1, 2$. We apply Proposition 1.35 which provides $\hat{s}_i \in S$, $x', y' \in A^*$ and $v' \in A^+$ such that $v = x'v'y'$ and $\delta(s'_i, x'v'y') = \delta(\hat{s}_i, v'y') = \delta(\hat{s}_i, y') = s_i$ for $i = 1, 2$. It is easy to see that this yields pattern $\mathbb{B}_{1/2}$ in $\mathcal{M}$ since $\hat{s}_1$ and $\hat{s}_2$ have a non–empty v'-loop. ❑

So we see how pattern $\mathbb{L}_{1/2}$ turns in a natural way to pattern $\mathbb{B}_{1/2}$ as k increases. We describe now another proof of the second statement of Theorem 2.16. Let $L \in \mathcal{B}_{1/2}$ and let $\mathcal{M}$ be some DFA accepting L. Then L is in $\mathcal{B}_{1/2,k}$ for some $k \geq 0$ by Lemma 1.29 and hence $\mathcal{M}$ does not have pattern $\mathbb{B}_{1/2,k}$ by Theorem 2.18. But then $\mathcal{M}$ does also not have pattern $\mathbb{B}_{1/2}$ since $\mathbb{B}_{1/2}$ is a special case of $\mathbb{B}_{1/2,k}$ for all $k \geq 0$. On the other hand, if $L \notin \mathcal{B}_{1/2}$ then it is in none of the classes $\mathcal{B}_{1/2,k}$. So $\mathcal{M}$ has in particular pattern $\mathbb{B}_{1/2,k}$ for $k = |\mathcal{M}|^2$. From Proposition 2.19 we see that $\mathcal{M}$ has pattern $\mathbb{B}_{1/2}$.

Theorem 2.18 yields an NL–algorithm for the membership problem of $\mathcal{B}_{1/2,k}$ for fixed $k \geq 0$. We only have to look for pattern $\mathbb{B}_{1/2,k}$ in the transition graph of a given DFA. To do so we guess states s_1, s_2, s^+, s^- and check whether s^+ is accepting and s^- is rejecting. Then we verify $s_0 \longrightarrow s_1$ and $(s_1, s_2) \longrightarrow (s^+, s^-)$. We can also store the needed suffixes of size k because this is a contant to the algorithm.

There are similar NL–algorithm to find patterns $\mathbb{L}_{1/2}$ and $\mathbb{B}_{1/2}$ in transition graphs. Such algorithms were also pointed out in [PW97] for the membership problems of $\mathcal{L}_{1/2}$ and of $\mathcal{B}_{1/2}$. The bound on k from Proposition 2.19 allows the exact location of a language in the hierarchy of classes $\mathcal{B}_{1/2,k}$. Just repeat the algorithm for $\mathcal{B}_{1/2,k}$ with $k = 0, 1, \ldots, |\mathcal{M}|^2$. The latter is an algorithm that also decides the membership problem of $\mathcal{B}_{1/2}$.

2.3 Stern's Theorem

We turn to the classes $\mathcal{B}_{1,k}$ which are by definition the Boolean closure of $\mathcal{B}_{1/2,k}$. The forbidden patterns $\mathbb{B}_{1,k}$, $\hat{\mathbb{B}}_{1,k}$ and $\hat{\mathbb{B}}^{\text{rev}}_{1,k}$ characterize $\mathcal{B}_{1,k}$.

Definition 2.20 ([Ste85a]). Let $k \geq 0$.

1. Pattern $\mathbb{B}_{1,k}$ is defined as the subgraph given in Figure 2.3 with $x, y, y', u, v, w, w', z \in A^*$ and $|w| = |w'| = k$.
2. Pattern $\hat{\mathbb{B}}_{1,k}$ is defined as the subgraph given in Figure 2.4 with $x, u, v, w, z \in A^*$ and $|w| = k$.
3. Pattern $\hat{\mathbb{B}}^{\text{rev}}_{1,k}$ is defined as the subgraph given in Figure 2.5 with $x, u, v, w, z \in A^*$ and $|w| = k$.

The following is the main result from [Ste85a, Theorem 3.3] stated here in our notations. We have already defined the notion of $\preceq_k$-chains at the beginning of Section 2.1.

Theorem 2.21 ([Ste85a]). *Let $k \geq 0$, $L \subseteq A^+$ and let $\mathcal{M}$ be the minimal* DFA *accepting L. The following statements are equivalent.*

(1) $L \in \mathcal{B}_{1,k}$
(2) $L \in \mathcal{FP}(\mathbb{B}_{1,k}, \hat{\mathbb{B}}_{1,k}, \hat{\mathbb{B}}^{\text{rev}}_{1,k})$
(3) Any $\preceq_k$-chain has a finite number of alternations with respect to L.
(4) Any $\preceq_k$-chain has at most $2^{2|A|^{k+2}(k+1)^2|\mathcal{M}|}$ alternations with respect to L.

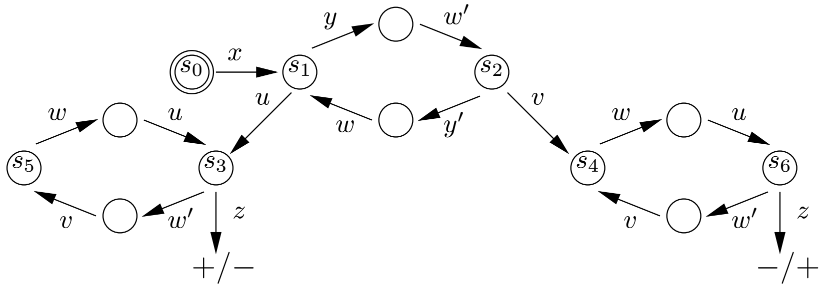


Fig. 2.3. Pattern $\mathbb{B}_{1,k}$ with $|w| = |w'| = k$.

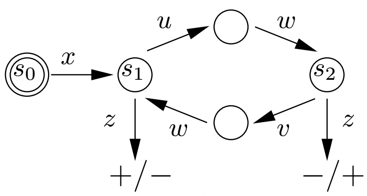


Fig. 2.4. Pattern $\hat{\mathbb{B}}_{1,k}$ with $|w| = k$.

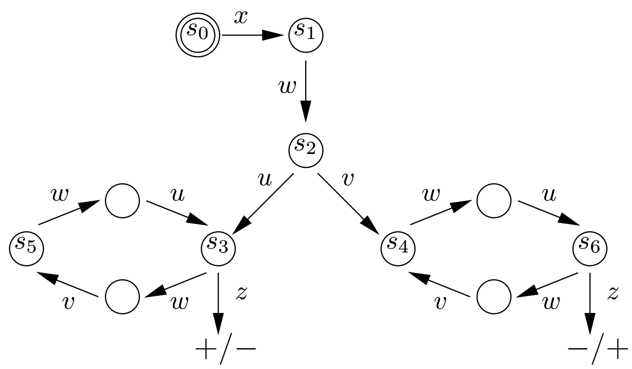


Fig. 2.5. Pattern $\hat{\mathbb{B}}^{\text{rev}}_{1,k}$ with $|w| = k$.

Let us mention that the patterns $\hat{\mathbb{B}}_{1,k}$ and $\hat{\mathbb{B}}^{\text{rev}}_{1,k}$ are connected via taking reverse languages. If $\mathcal{M}$ is some DFA then for any DFA $\hat{\mathcal{M}}$ with $L(\hat{\mathcal{M}}) = L(\mathcal{M})^R$ it holds that $\mathcal{M}$ has pattern $\hat{\mathbb{B}}_{1,k}$ if and only if $\hat{\mathcal{M}}$ has pattern $\hat{\mathbb{B}}^{\text{rev}}_{1,k}$. Moreover, $\mathcal{FP}(\mathbb{B}_{1,k}, \hat{\mathbb{B}}_{1,k}, \hat{\mathbb{B}}^{\text{rev}}_{1,k})$ is well–defined. We show both for very similar patterns in the forthcoming Propositions 3.6 and 3.7 and give proofs there. In case $k = 0$ we get a forbidden pattern characterization of $\mathcal{B}_{1,0} = \mathcal{L}_1$ which can be slightly simplified.

Proposition 2.22. *It holds that* $\mathcal{L}_1 = \mathcal{FP}(\hat{\mathbb{B}}_{1,0}, \hat{\mathbb{B}}^{\text{rev}}_{1,0})$.

Proof. It suffices to prove $\mathcal{FP}(\hat{\mathbb{B}}_{1,0}, \hat{\mathbb{B}}^{\text{rev}}_{1,0}) \subseteq \mathcal{FP}(\mathbb{B}_{1,0}, \hat{\mathbb{B}}_{1,0}, \hat{\mathbb{B}}^{\text{rev}}_{1,0})$. So let $L \in \mathcal{FP}(\hat{\mathbb{B}}_{1,0}, \hat{\mathbb{B}}^{\text{rev}}_{1,0})$. We need to show that there is some DFA accepting L that has none of the patterns $\mathbb{B}_{1,0}$, $\hat{\mathbb{B}}_{1,0}$ and $\hat{\mathbb{B}}^{\text{rev}}_{1,0}$. For this consider the minimal DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ and assume to the contrary that $\mathcal{M}$ has pattern $\mathbb{B}_{1,0}$. If $s_1 \neq s_2$ in this pattern then we found pattern $\hat{\mathbb{B}}_{1,0}$ because $\mathcal{M}$ is minimal. If $s_1 = s_2$ in this pattern then we found pattern $\hat{\mathbb{B}}^{\text{rev}}_{1,0}$. So $\mathcal{M}$ has patterns $\hat{\mathbb{B}}_{1,0}$ or $\hat{\mathbb{B}}^{\text{rev}}_{1,0}$, a contradiction. ❑

Note that this proof does not work for $k \geq 1$ since eventually $w \neq w'$ in pattern $\mathbb{B}_{1,k}$. This yields an NL–algorithm for the membership problem of $\mathcal{L}_1$. We guess in a straightforward way the involved states and verify the reachability conditions given by the patterns. Similar algorithms can be provided for the membership problem of $\mathcal{B}_{1,k}$ for fixed $k \geq 0$, investigated in [Ste85b, CH91]

2.4 The Boolean Hierarchy over $\mathcal{B}_{1/2,k}$

There is a natural connection between the class $\mathcal{B}_{1,k}$ and the number of alternations in $\preceq_k$-chains: the maximal number of such alternations determines the location of a language in the Boolean hierarchy over $\mathcal{B}_{1/2,k}$. Our Theorem 2.30 below has the equivalence of (1) and (3) in Theorem 2.21 as a corollary. We begin with the definition of Boolean hierarchies.

Definition 2.23 ([KSW87, CGH$^+$88]). Let $\mathcal{C}$ be a class of languages closed under union and intersection. The Boolean hierarchy over $\mathcal{C}$ is the family of classes $\mathcal{C}(l)$ and $\mathrm{co}\mathcal{C}(l)$ for $l \geq 1$ such that

1. $L \in \mathcal{C}(2l-1)$ if and only if there exist $L_1, L_2, \ldots, L_{2l-1} \in \mathcal{C}$ with $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{2l-1}$ and $L = \bigcup_{i=1}^{l-1}(L_{2i-1} \backslash L_{2i}) \cup L_{2l-1}$ and
2. $L \in \mathcal{C}(2l)$ if and only if there exist $L_1, L_2, \ldots, L_{2l} \in \mathcal{C}$ with $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{2l}$ and $L = \bigcup_{i=1}^{l}(L_{2i-1} \backslash L_{2i})$.

It is known from these papers that every class defined via a fixed but arbitrary Boolean combination of the languages from $\mathcal{C}$ coincides with one of the classes $\mathcal{C}(l)$ or $\mathrm{co}\mathcal{C}(l)$. We have taken this normal form result for the definition here. Moreover, the following inclusions are known.

Lemma 2.24 ([KSW87, CGH$^+$88]). *Let $\mathcal{C}$ be a class of languages closed under union and intersection. Then* $\mathrm{BC}(\mathcal{C}) = \bigcup_{l \geq 1} \mathcal{C}(l)$ *and* $\mathcal{C}(l) \cup \mathrm{co}\mathcal{C}(l) \subseteq \mathcal{C}(l+1) \cap \mathrm{co}\mathcal{C}(l+1)$ *for* $l \geq 1$.

The previous lemma can be applied in particular to the classes $\mathcal{B}_{1/2,k}$ due to Lemma 2.14.

2.4.1 A Membership Criterion

We fix some $k \geq 0$ for this subsection. The Boolean hierarchy over $\mathcal{B}_{1/2,k}$ is the family of classes $\mathcal{B}_{1/2,k}(l)$ and $\mathrm{co}\mathcal{B}_{1/2,k}(l)$ for $l \geq 1$. We introduce a notation for alternating $\preceq_k$-chains.

Definition 2.25. Let $L \subseteq A^+$, $m \geq 0$ and $w, v \in A^+$. We say that v is reachable from w by a $\preceq_k$-chain having m alternations with respect to L, in notation $w \overset{m,k}{\Longrightarrow}_L v$, if and only if there exist $w_0, \ldots, w_m \in A^+$ such that

1. $w = w_0 \preceq_k w_1 \preceq_k w_2 \preceq_k \ldots \preceq_k w_m \preceq_k v$ and
2. $w_i \in L$ if and only if $w_{i+1} \notin L$ for $0 \leq i \leq m-1$.

If $w_i \in L$ ($w_i \notin L$) we say that w_i has signature $+$ ($-$, respectively). Next we take a closer look at such chains and define the sets of words that can be reached from a word (not) in a given language L by m alternations.

Definition 2.26. For a language $L \subseteq A^+$ and $m \geq 0$ we define

1. $L_k^+(m) =_{\mathrm{def}} \left\{ v \in A^+ \,\middle|\, \exists w\ (w \in L \wedge w \overset{m,k}{\Longrightarrow}_L v) \right\}$ and
2. $L_k^-(m) =_{\mathrm{def}} \left\{ v \in A^+ \,\middle|\, \exists w\ (w \notin L \wedge w \overset{m,k}{\Longrightarrow}_L v) \right\}$.

Here are some properties of these sets.

Proposition 2.27. *Let $L \subseteq A^+$ and $m \geq 0$. It holds that*

1. $L_k^-(m) = (A^+ \setminus L)_k^+(m)$,
2. $L_k^+(m+1) \cup L_k^-(m+1) \subseteq L_k^+(m) \cap L_k^-(m)$ *and*
3. $L_k^+(m)$ *and* $L_k^-(m)$ *are languages in* $\mathcal{B}_{1/2,k}$.

Proof. To see statement 1 suppose $v \in L_k^-(m)$ for some $m \geq 0$. If we look at the witnessing $\preceq_k$-chain with m alternations, then going from L to $(A^+ \setminus L)$ just inverts its signature. Now this chain witnesses $v \in (A^+ \setminus L)_k^+(m)$. The second statement is due to the fact that a $\preceq_k$-chain with $m+1$ alternations is also a k-chain with m alternations since $\preceq_k$ is transitive. For statement 3 note from the definitions that $L_k^+(m)$ and $L_k^-(m)$ are order ideals of $(A^+, \preceq_k)$. ❑

Any language L can be expressed as a possibly infinite union of set differences of sets $L_k^+(m)$ and $L_k^-(m)$.

Proposition 2.28. *For $L \subseteq A^+$ the following holds.*

1. $L = \bigcup_{m\geq 0}^{\infty} \big(L_k^+(2m) \backslash L_k^+(2m+1)\big)$ *and*
   $(A^+ \setminus L) = \big(A^+ \backslash L_k^+(0)\big) \cup \bigcup_{m\geq 1}^{\infty} \big(L_k^+(2m-1) \backslash L_k^+(2m)\big)$.
2. $(A^+ \setminus L) = \bigcup_{m\geq 0}^{\infty} \big(L_k^-(2m) \backslash L_k^-(2m+1)\big)$ *and*
   $L = \big(A^+ \backslash L_k^-(0)\big) \cup \bigcup_{m\geq 1}^{\infty} \big(L_k^-(2m-1) \backslash L_k^-(2m)\big)$.

Proof. It suffices to prove statement 1 since the second statement follows from the first by Proposition 2.27. Let $m \geq 0$ and $v \in L_k^+(2m) \backslash L_k^+(2m+1)$. Because $v \in L_k^+(2m)$ there exists some $w \in L$ with $w \overset{2m,k}{\Longrightarrow}_L v$. Now observe that if $v \notin L$ then $w \overset{2m+1,k}{\Longrightarrow}_L v$ witnessed by the same $\preceq_k$-chain as before. But this is a contradiction to $v \notin L_k^+(2m+1)$. So $v \in L$. This shows the inclusion from right to left.

For the other inclusion let $v \in L$ and look at all sequences of words $w_0, w_1, \ldots, w_l \in A^+$ with $l \geq 0$ such that $w_0 \in L$, $w_l = v$, $w_0 \preceq_k w_1 \preceq_k \ldots \preceq_k w_l$, and $w_i \neq w_{i+1}$ for $0 \leq i \leq l-1$. Note that at least one such sequence exists. In fact, there is only a finite number of them since there are no infinite strictly descending $\preceq_k$-chains. We can associate with each such sequence its number of alternations with respect to L. Let $l_{\max}$ be the maximal such number over all considered sequences. First observe that $l_{\max} = 2m$ for some $m \geq 0$ because $w_0, w_{l_{\max}} \in L$. So we have $v \in L_k^+(2m)$ witnessed by the sequence with $l_{\max}$ alternations. It cannot hold that $v \in L_k^+(2m+1)$ due to the maximality of $l_{\max}$.

To see the statement for $(A^+ \setminus L)$ we can prove as before that $v \in L^+(2m-1) \backslash L^+(2m)$ implies $v \notin L$ for $m \geq 1$. Since for all $v \in A^+$ it holds that $v \overset{0,k}{\Longrightarrow}_L v$ we have $L \subseteq L_k^+(0)$. Hence $v \in A^+ \backslash L^+(0)$ implies $v \notin L$. This shows the inclusion from right to left. On the other hand, if $v \notin L$ then there is no $\preceq_k$-chain starting with some $w \in L$ and ending with v, or we may argue as before with the maximal number of alternations (which must be odd this time). ❑

In order to measure the number of inevitable alternations that occur with respect to a given language L we look for the maximal m such that the sets $L_k^+(m)$ and $L_k^-(m)$ are not empty.

Definition 2.29. For a language $L \subseteq A^+$ we define $m_k^+(L) =_{\text{def}} \sup \{\, m \mid L_k^+(m) \neq \emptyset \,\}$ and $m_k^-(L) =_{\text{def}} \sup \{\, m \mid L_k^-(m) \neq \emptyset \,\}$.

Since the measure m_k^+ gives the maximal number of alternations in $\preceq_k$-chains it is the same measure as used in Theorem 2.21. We relate the single classes of the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ to particular values of m_k^+ and m_k^-.

Theorem 2.30. *Let $L \subseteq A^+$ and $l \geq 1$. It holds that*

1. *$L \in \mathcal{B}_{1/2,k}(l)$ if and only if $m_k^+(L) < l$ and*
2. *$L \in \mathrm{co}\mathcal{B}_{1/2,k}(l)$ if and only if $m_k^-(L) < l$.*

Proof. We begin with statement 1 and we restrict ourselves to the case of even l, the other case being proved completely analogously.

Let $L \subseteq A^+$ with $m_k^+(L) < 2l$. Then $L_k^+(i) = \emptyset$ for all $i \geq 2l$. By Proposition 2.28 we can write L as

$$L = \bigcup_{i=0}^{l-1} \left(L_k^+(2i) \backslash L_k^+(2i+1) \right)$$

and from Proposition 2.27 we see with Definition 2.23 that $L \in \mathcal{B}_{1/2,k}(2l)$.

Now suppose $L \in \mathcal{B}_{1/2,k}(2l)$. By definition there exist $L_1, L_2, \dots, L_{2l} \in \mathcal{B}_{1/2,k}$ such that $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{2l}$ and $L = \bigcup_{i=1}^{l}(L_{2i-1} \backslash L_{2i})$. With $L_0 =_{\mathrm{def}} A^+$ and $L_{2l+1} =_{\mathrm{def}} \emptyset$ we obtain $(A^+ \setminus L) = \bigcup_{i=0}^{l}(L_{2i} \backslash L_{2i+1})$. So each word from A^+ is contained in some set $L_i \backslash L_{i+1}$ for some $i \in \{0, \dots, 2l\}$.

Assume to the contrary that $L_k^+(2l) \neq \emptyset$. Then by definition of $L_k^+(2l)$ there exist $w \in L$, some $v \in L_k^+(2l)$ and $w_0, w_1, \dots, w_{2l} \in A^+$ such that $w = w_0 \preceq_k w_1 \preceq_k \dots \preceq_k w_{2l} \preceq_k v$ with $w_{2i} \in L$ and $w_{2i-1} \notin L$. For any $i \in \{0, 1, \dots, 2l-1\}$ there must be two indices $j, j' \in \{0, \dots, 2l\}$ with $w_i \in L_j \backslash L_{j+1}$ and $w_{i+1} \in L_{j'} \backslash L_{j'+1}$. Since $w_i \in L \Leftrightarrow w_{i+1} \notin L$ these indices must be different. Note that $\langle L_j \rangle_k = L_j$ for all j. So from $w_i \preceq_k w_{i+1}$ we can conclude that $w_{i+1} \in L_j$ as well, which implies $j' > j$. Consequently, the words $w_0, w_1, \dots, w_{2l}$ are in $2l+1$ different sets $L_j \backslash L_{j+1}$ with $j \geq 1$ (since $w_0 \in L \subseteq L_1$). This is a contradiction since there are only $2l$ such sets. Hence $m_k^+(L) < 2l$.

Statement 2 follows from the first statement because $m_k^+(L) = m_k^-(A^+ \setminus L)$ which is immediate by Proposition 2.27. ❑

2.4.2 Strictness and Decidability Results

We give a strictness argument for the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ and show afterwards how the measures m_k^+ and m_k^- can be effectively computed.

Theorem 2.31. *For every $l \geq 1$ it holds that $\mathcal{B}_{1/2,k}(l) \not\subseteq \mathrm{co}\mathcal{B}_{1/2,k}(l)$.*

Proof. Since $|A| \geq 2$ there are two different letters $a, b \in A$. Let $\alpha =_{\mathrm{def}} a^{k+1}$. For a word $w \in A^{\geq k+1}$ we define $|w|_\alpha$ to be the number of occurrences of α in the k-decomposition $\widehat{w}$. For $r \geq 1$ define

1. $M_{2r-1} =_{\mathrm{def}} \left\{ w \in A^{\geq k+1} \,|\, |w|_\alpha \text{ is odd or } |w|_\alpha > 2r-1 \right\}$ and
2. $M_{2r} =_{\mathrm{def}} \left\{ w \in A^{\geq k+1} \,|\, |w|_\alpha \text{ is odd and } |w|_\alpha \leq 2r \right\}$.

We claim that for all $l \geq 1$ it holds that $m_k^-(M_l) = l$ and $m_k^+(M_l) = l-1$. Then we obtain from Theorem 2.30 that $M_l \in \mathcal{B}_{1/2,k}(l) \backslash \mathrm{co}\mathcal{B}_{1/2,k}(l)$. So it remains to prove this claim.

We first show that $m_k^-(M_l) \geq l$ and $m_k^+(M_l) \geq l-1$ for all $l \geq 1$. Therefore we define $y_i =_{\text{def}} ba^k a^i$ for all $i \geq 0$. It is easy to see that $|y_i|_\alpha = i$ and that $y_i \preceq_k^{\text{e}} y_{i+1}$ for all $i \geq 0$. As in the definition of the sets above we distinguish the cases of odd and even l. So let us assume first that $l = 2r-1$ for some $r \geq 1$. We see that $y_{2j} \notin M_{2r-1}$ and $y_{2j+1} \in M_{2r-1}$ for $0 \leq j \leq r-1$. So we may take $y_0 \preceq_k y_1 \preceq_k \ldots \preceq_k y_{2r-1}$ as a $\preceq_k$-chain having $2r-1$ alternations with respect to M_{2r-1} that witnesses $m_k^-(M_{2r-1}) \geq 2r-1$. The same $\preceq_k$-chain also witnesses that $m_k^+(M_{2r-1}) \geq 2r-2$. Now let $l = 2r$ for some $r \geq 1$. Again we observe that $y_{2j} \notin M_{2r}$ and $y_{2j+1} \in M_{2r}$ for $0 \leq j \leq r-1$. Since $y_{2r} \notin M_{2r}$ we may take $y_0 \preceq_k y_1 \preceq_k \ldots \preceq_k y_{2r}$ as a $\preceq_k$-chain having $2r$ alternations with respect to M_{2r} that witnesses $m_k^-(M_{2r}) \geq 2r$. The same $\preceq_k$-chain also witnesses that $m_k^+(M_{2r}) \geq 2r-1$.

We prove next that $m_k^-(M_l) \leq l$ and $m_k^+(M_l) \leq l-1$ for all $l \geq 1$. For a $\preceq_k$-chain involving words w_i note that from $w_i \preceq_k w_{i+1}$ it follows that $|w_i|_\alpha \leq |w_{i+1}|_\alpha$. Due to the definition of the sets M_l this inequality must be strict if there is an alternation with respect to these sets between w_i and w_{i+1}.

First assume again that $l = 2r-1$ for some $r \geq 1$. If $m_k^-(M_{2r-1}) > 2r-1$ then there are $w_0, w_1, \ldots, w_{2r} \in A^+$ such that $w_0 \preceq_k w_1 \preceq_k w_2 \preceq_k \ldots \preceq_k w_{2r}$ with $w_{2i} \notin M_{2r-1}$ and $w_{2i+1} \in M_{2r-1}$. It follows by the previous remark that $|w_{2r}|_\alpha > 2r-1$, a contradiction to $w_{2r} \notin M_{2r-1}$. If $m_k^+(M_{2r-1}) > 2r-2$ then there are $w_0, w_1, \ldots, w_{2r-1} \in A^+$ such that $w_0 \preceq_k w_1 \preceq_k w_2 \preceq_k \ldots \preceq_k w_{2r-1}$ with $w_{2i} \in M_{2r-1}$ and $w_{2i+1} \notin M_{2r-1}$. Since in particular $|w_0|_\alpha > 0$ it follows that $|w_{2r-1}|_\alpha > 2r-1$, a contradiction to $w_{2r-1} \notin M_{2r-1}$.

Now let $l = 2r$ for some $r \geq 1$. Suppose $m_k^-(M_{2r}) > 2r$. We may conclude as above that in a witnessing $\preceq_k$-chain having $(2r+1)$ alternations we have $|w_{2r+1}|_\alpha > 2r$, a contradiction to $w_{2r+1} \in M_{2r}$. If $m_k^+(M_{2r}) > 2r-1$ we obtain also as above for a witnessing $\preceq_k$-chain that $|w_{2r}|_\alpha > 2r$ since $|w_0|_\alpha > 0$. This contradicts that $w_{2r} \in M_{2r}$. ❑

It follows immediately with the complements of the witnessing languages that also $\mathrm{co}\mathcal{B}_{1/2,k}(l) \nsubseteq \mathcal{B}_{1/2,k}(l)$. So all classes of the Boolean hierarchy over $\mathcal{B}_{1/2,k}$ are distinct. We turn to the membership problems of these classes. Suppose some DFA $\mathcal{M}$ is given and fix some $l \geq 1$. We exploit the equivalence

$$L(\mathcal{M}) \in \mathcal{B}_{1/2,k}(l) \iff m_k^+(L(\mathcal{M})) < l \iff L(\mathcal{M})_k^+(l) = \emptyset$$

obtained from Theorem 2.30 and construct a nondeterministic finite automaton $\mathcal{M}_l$ from $\mathcal{M}$ that accepts $L(\mathcal{M})_k^+(l)$. An emptyness test will then provide the answer to the question whether $L(\mathcal{M}) \in \mathcal{B}_{1/2,k}(l)$. We carry out this construction in the following lemma where $\mathcal{M}_l$ realizes the idea of guessing a $\preceq_k$-chain having l alternations with respect to $L(\mathcal{M})$. We treat here only the measure m_k^+ because this translates in an obvious way to m_k^-, just consider the DFA for $A^+ \setminus L(\mathcal{M})$. For nondeterministic finite automata (NFA) we let $\delta : A \times S \to 2^S$.

Lemma 2.32. *Let $\mathcal{M}$ be a* DFA *and let $l \geq 1$. Then there exists some* NFA *$\mathcal{M}_l$ such that $|\mathcal{M}_l| \leq (2|\mathcal{M}||A|^{2k+2})^{l+3}$ and $L(\mathcal{M}_l) = L(\mathcal{M})_k^+(l)$.*

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ and set $L =_{\text{def}} L(\mathcal{M})$. We are interested in the set of words v that can be reached from some word $w \in L$ via a $\preceq_k$-chain having at least l alternations with respect to L. Note that from $l \geq 1$ it follows that $|v| > k+1$ and observe that $L \cap A^{\leq k+1} \subseteq L_k^+(0) \backslash L_k^+(1)$. The automaton $\mathcal{M}_l$ we have in mind guesses on input v a $\triangleleft_k$-chain of sufficient length and stores the k-prefix and the k-suffix of each word in the chain. It also remembers the states to which these words lead to in $\mathcal{M}$. Then $\mathcal{M}_l$ accepts v if and only if we find l alternations with respect to S' in this sequence of states and if the stored k-prefixes and k-suffixes match the ones of the input.

For each guessed word w_i of the $\lhd_k$-chain we store a quadruple $t_i = (c_i, d_i, f_i, r_i)$ where c_i (d_i) is the current k-prefix (k-suffix, resp.) of w_i, where f_i denotes a 0/1–valued variable, and r_i is the state $\mathcal{M}$ reaches after input w_i. The flag f_i tells us whether there is already a w_i in the guessed chain ($f_i = 1$) or whether the chain is still too short ($f_i = 0$). We also store in a variable c (d) the k-prefix (k-suffix, resp.) of the input. So let the set of states T be defined as the set of all tuples

$$[t_0, t_1, \ldots, t_l, c, d]$$

such that for $0 \le i \le l$ we have $t_i = (c_i, d_i, f_i, r_i)$ with $c_i, d_i \in A^{\le k} \cup \{\varepsilon\}$, $f_i \in \{0, 1\}$, $r_i \in S$ and $c, d \in A^{\le k} \cup \{\varepsilon\}$. Moreover, we set

$$s_0^l =_{\mathrm{def}} [\underbrace{(\varepsilon, \varepsilon, 0, s_0), (\varepsilon, \varepsilon, 0, s_0), \ldots, (\varepsilon, \varepsilon, 0, s_0)}_{(l+1)\text{–times}}, \varepsilon, \varepsilon]$$

as the starting state. Let $[t_0, t_1, \ldots, t_l, c, d], [t'_0, t'_1, \ldots, t'_l, c', d'] \in T$ be given with $t'_i = (c'_i, d'_i, f'_i, r'_i)$ for $0 \le i \le l$. We define that $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l([t'_0, t'_1, \ldots, t'_l, c', d'], a)$ if and only if $c = p_k(c'a)$, $d = s_k(d'a)$ and there exists some j with $0 \le j \le l + 1$ such that

1. for $0 \le i < j$ it holds that
   a) if $f'_i = 0$ then $t_i = (s_k(d'a), s_k(d'a), 0, \delta(s_0, s_k(d'a)))$,
   b) if $f'_i = 1$ then $t_i = t'_i$,
2. and for $j \le i \le l$ it holds that
   a) $d'_i = d'$ and $|d'_i| = k$, and
   b) $t_i = (c'_i, s_k(d'a), 1, \delta(r'_i, a))$.

Let us comment a little bit on this definition before we continue. Clearly, on the next letter a of the input we want to maintain the actual k-prefix and k-suffix of the input in c and d. When reading the first k letters of the input there is no nondeterministic choice possible for j because of 2.a, so it must be that $j = l + 1$. Since during this time all f'_i are 0, only the variables for the k-prefixes and k-suffixes are filled and the respective states of $\mathcal{M}$ are stored (see 1.a). If the input is longer than k and if we have already guessed a chain $w_0 \lhd_k w_1 \lhd_k \ldots \lhd_k w_l \lhd_k v$, then we may guess a smallest index j such that the new last element $d'a$ of the k-decomposition of the actual input appears in w_j first. We only obtain a $\lhd_k$-chain again if all words $w_j, w_{j+1}, \ldots, w_l, v$ have the same k-suffix, ensured by 2.a. With 2.b we keep all old k-prefixes and set the new k-suffixes and states. The quadruples for the words $w_0, \ldots, w_{j-1}$ are not affected (see 1.b). It can also be the case that we have guessed only a chain $w_m \lhd_k \ldots \lhd_k w_l \lhd_k v$ with $m > 0$ yet. If $j \ge m$ we keep the actual values with 1.a and 1.b. If $j < m$ then $w_j = w_{j+1} = \ldots = w_{m-1} = d'a$ are the new first words of the chain $w_j \lhd_k \ldots \lhd_k w_m \lhd_k \ldots \lhd_k w_l \lhd_k v$ and the flag variables $f_j, \ldots, f_{m-1}$ turn from 0 to 1 by 2.b.

We want to prove formally what we just described. Note that due to the definition of δ_l only states $[t_0, t_1, \ldots, t_l, c, d]$ can be reached such that $f_0 f_1 \cdots f_l \in 0^*1^*$. Moreover, it holds that $|T| \le (2|\mathcal{M}||A|^{2k+2})^{l+3}$.

Claim. Let $v \in A^{\geq k+1}$. Then $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, v)$ if and only if $c = p_k(v)$, $d = s_k(v)$ and there exist $w_m, w_{m+1}, \ldots, w_l \in A^{\geq k+1}$ for some m with $0 \leq m \leq l+1$ such that

1. $t_i = (d, d, 0, \delta(s_0, d))$ for $0 \leq i < m$,
2. $w_m \lhd_k w_{m+1} \lhd_k \ldots \lhd_k w_l \lhd_k v$ and
3. $t_i = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$ for $m \leq i \leq l$.

Proof of Claim. We prove the claim by induction on the length of v.
Induction base. Let $v = xa$ with $x \in A^k, a \in A$ and set $s' =_{\text{def}} \delta(s_0, x)$. It is easy to see that $\delta_l(s_0^l, x) = \{[(x, x, 0, s'), \ldots, (x, x, 0, s'), x, x]\}$. Now if $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, v)$ then there is some m with $0 \leq m \leq l+1$ such that $t_i = (c_i, d_i, 0, r_i)$ for $0 \leq i < m$ and $t_i = (c_i, d_i, 1, r_i)$ for $m \leq i \leq l$. Since $[t_0, t_1, \ldots, t_l, c, d]$ must emerge from $\delta_l(s_0^l, x)$ by choosing $j = m$ in the definition of δ_l, we have that $c = p_k(xa)$, $d = s_k(xa)$, $t_i = (s_k(xa), s_k(xa), 0, \delta(s_0, s_k(xa)))$ for $0 \leq i < m$ and $t_i = (x, s_k(xa), 1, \delta(s', a))$ for $m \leq i \leq l$. Since $xa = v$ we obtain $c = p_k(v)$, $d = s_k(v)$ and $t_i = (d, d, 0, \delta(s_0, d))$ for $0 \leq i < m$. If we define $w_i =_{\text{def}} v$ for $m \leq i \leq l$ then it holds that $w_m \lhd_k \ldots \lhd_k w_l \lhd_k v$. For $m \leq i \leq l$ we have $x = p_k(w_i)$, $s_k(xa) = s_k(w_i)$ and because $\delta(s', a) = \delta(\delta(s_0, x), a) = \delta(s_0, v) = \delta(s_0, w_i)$ we have shown the 'only-if'–part of the induction base.

To see the 'if'–part assume that $c = p_k(v)$, $d = s_k(v)$ and there are $w_m, \ldots, w_l \in A^{\geq k+1}$ for some m with $0 \leq m \leq l+1$ such that $w_m \lhd_k \ldots \lhd_k w_l \lhd_k v$. Then $c = p_k(xa)$, $d = s_k(xa)$ and we can conclude from $|v| = k+1$ that in fact $w_m = w_{m+1} = \ldots = w_l = v$. So the assumptions translate to $t_i = (s_k(xa), s_k(xa), 0, \delta(s_0, s_k(xa)))$ for $0 \leq i < m$ and $t_i = (p_k(xa), s_k(xa), 1, \delta(s_0, xa))$ for $m \leq i \leq l$. Now it can be seen that $[t_0, t_1, \ldots, t_l, c, d]$ is in $\delta_l(s_0^l, v)$ by choosing again $j = m$ in the definition of δ_l when going from $\delta_l(s_0^l, x)$ to $\delta_l(s_0^l, xa)$.

Induction step. Let va be given with $v \in A^{\geq k+1}$ and $a \in A$. We show the 'only-if'–part first. Suppose we know that $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, va)$ for some $[t_0, t_1, \ldots, t_l, c, d] \in T$. Then there is some m with $0 \leq m \leq l+1$ such that $t_i = (c_i, d_i, 0, r_i)$ for $0 \leq i < m$ and $t_i = (c_i, d_i, 1, r_i)$ for $m \leq i \leq l$. Moreover, there is some $[t'_0, t'_1, \ldots, t'_l, c', d'] \in \delta_l(s_0^l, v)$ with $t'_i = (c'_i, d'_i, f'_i, r'_i)$ for $0 \leq i \leq l$ such that $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l([t'_0, t'_1, \ldots, t'_l, c', d'], a)$. Let j with $0 \leq j \leq l+1$ denote the nondeterministic choice of this transition.

By hypothesis, we have $c' = p_k(v)$ and $d' = s_k(v)$, and by definition of δ_l it holds that $c = p_k(c'a)$ and $d = s_k(d'a)$. Hence $d = s_k(va)$ and because $|c'| = k$ we have $c = c' = p_k(v) = p_k(va)$. There also is some m' with $0 \leq m' \leq l+1$ such that $f'_i = 0$ for $0 \leq i < m'$ and $f'_i = 1$ for $m' \leq i \leq l$. Since $f_i = 0$ for $0 \leq i < m$ it must be that $m \leq m'$ and also $m \leq j$. So for $0 \leq i < m$ it holds by definition of δ_l that $t_i = (s_k(d'a), s_k(d'a), 0, \delta(s_0, s_k(d'a)))$. Hence $t_i = (d, d, 0, \delta(s_0, d))$ for $0 \leq i < m$ which shows statement 1 of the claim.

If $m < m'$ then it must be that $j = m$ because $f'_m = 0$ and $f_m = 1$. In this case the length of the guessed $\lhd_k$-chain strictly increases when going from v to va. If $m = m'$ then $m' \leq j \leq l$. We distinguish these two cases. By hypothesis there exist $w'_{m'}, w'_{m'+1}, \ldots, w'_l \in A^{\geq k+1}$ such that $w'_{m'} \lhd_k w'_{m'+1} \lhd_k \ldots \lhd_k w'_l \lhd_k v$ and $t'_i = (p_k(w'_i), s_k(w'_i), 1, \delta(s_0, w'_i))$ for $m' \leq i \leq l$.

Case 1. Suppose $m < m'$ and hence $j = m$. So by definition of δ_l it holds that $d'_i = d'$ and $|d'_i| = k$ for $m \leq i \leq l$, and since $f'_m = f'_{m+1} = \ldots = f'_{m'-1} = 0$ we have by hypothesis that $t'_i = (d', d', 0, \delta(s_0, d'))$ for $m \leq i < m'$. Define for $m \leq i < m'$ words $w_i =_{\text{def}} d'a$ and for $m' \leq i \leq l$ set $w_i =_{\text{def}} w'_i a$. For concise notations we set $w'_{l+1} =_{\text{def}} v$ and $w_{l+1} =_{\text{def}} va$. Then for $m \leq i < m'-1$ we have $d'a = w_i \lhd_k w_{i+1} = d'a$. Moreover, for $m' \leq i \leq l$ it holds that $s_k(w'_i) = d'_i = d' = s_k(v)$ and hence we have $w'_i a = w_i \lhd_k w_{i+1} = w'_{i+1}a$ for $m' \leq i \leq l$ by Proposition 2.3. We also see that $w_{m'-1} = d'a = s_k(w'_{m'})a \lhd_k w'_{m'}a = w_{m'}$. Together we have obtained $w_m \lhd_k \ldots \lhd_k w_l \lhd_k va$ which gives statement 2 of the claim.

For $m \leq i < m'$ we have by definition that $w_i = d'a$ and by hypothesis that $c'_i = d'$, $d'_i = d'$ and $r'_i = \delta(s_0, d')$. By definition of δ_l it holds that $t_i = (c'_i, s_k(d'a), 1, \delta(r'_i, a))$ and hence $t_i = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$. For $m' \leq i \leq l$ we have by definition $w_i = w'_i a$ and by hypothesis that $c'_i = p_k(w'_i)$, $d'_i = s_k(w'_i)$ and $r'_i = \delta(s_0, w'_i)$. By definition of δ_l we have $t_i = (c'_i, s_k(d'a), 1, \delta(r'_i, a))$, and since $d' = d'_i$ it follows that $t_i = (p_k(w'_i a), s_k(w'_i a), 1, \delta(s_0, w'_i a))$. Together we see that $t_i = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$ for $m \leq i \leq l$ which gives statement 3 of the claim.

Case 2. Now assume $m = m'$ and hence $m' \leq j \leq l$. Define for $m \leq i < j$ words $w_i =_{\text{def}} w'_i$ and for $j \leq i \leq l$ set $w_i =_{\text{def}} w'_i a$. Again, set for concise notations $w'_{l+1} =_{\text{def}} v$ and $w_{l+1} =_{\text{def}} va$. Statements 2 and 3 are clear for w_i with $m \leq i < j$ by hypothesis and since by definition of δ_l we have $t_i = t'_i$. From $w'_{j-1} \lhd_k w'_j$ we obtain $w_{j-1} = w'_{j-1} \lhd_k w'_j \lhd_k w'_j a = w_j$ by Proposition 2.3. By hypothesis we have $d'_i = s_k(w'_i)$ and from the definition of δ_l it follows that $s_k(w'_i) = d'$ for $j \leq i \leq l$. So we can conclude as in the first case that $w_i = w'_i a \lhd_k w'_{i+1} a = w_{i+1}$ for $j \leq i \leq l$. Together we have $w_m \lhd_k \ldots \lhd_k w_l \lhd_k va$.

Additionally to $d'_i = s_k(w'_i)$ it holds for $j \leq i \leq l$ by hypothesis that $c'_i = p_k(w'_i)$ and $r'_i = \delta(s_0, w'_i)$. By definition of δ_l we have as before for $j \leq i \leq l$ that $t_i = (c'_i, s_k(d'a), 1, \delta(r'_i, a))$ and it follows that $t_i = (p_k(w'_i a), s_k(w'_i a), 1, \delta(s_0, w'_i a)) = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$. This completes the proof for statements 2 and 3 for the second case.

Next we show the 'if'–part. Assume that $c = p_k(va)$, $d = s_k(va)$ and there exist $w_m, \ldots, w_l \in A^{\geq k+1}$ for some m with $0 \leq m \leq l+1$ such that $t_i = (d, d, 0, \delta(s_0, d))$ for $0 \leq i < m$ and $w_m \lhd_k \ldots \lhd_k w_l \lhd_k va$ and $t_i = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$ for $m \leq i \leq l$. We need to show $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, va)$ and start with the construction of a $\lhd_k$-chain for v from the given $\lhd_k$-chain for va in order to apply the hypothesis below.

Set $w_{l+1} =_{\text{def}} va$, $w'_{l+1} =_{\text{def}} v$, $c' =_{\text{def}} p_k(v)$ and $d' =_{\text{def}} s_k(v)$. We give an algorithm that computes a $\lhd_k$-chain for v stepwise for $n = l$ downto m. During this computation we will ensure that before each step n the following condition (C) holds.

(C) For $n+1 \leq i \leq l+1$ there are words w'_i such that $w_i = w'_i a$ and $s_k(w'_i) = d'$, and for $n+1 \leq i \leq l$ it holds that $w'_i \lhd_k w'_{i+1}$.

If we set initially $n = l$ then (C) holds. We proceed as follows.

Step n:
(1) If $n < m$ then STOP.
(2) If $w_n \lhd_k w'_{n+1}$ then STOP.
(3) If $|w_n a^{-1}| = k$ then STOP.
(4) Set $w'_n =_{\text{def}} w_n a^{-1}$ and continue with Step $n-1$.

Let us first argue why before Step $n-1$ again (C) holds. After line (1) we know that $n \geq m$. After line (2) we have $w_n \not\lhd_k w'_{n+1}$. It follows that $s_{k+1}(w_n) = d'a$ since otherwise we obtain from $w_n \lhd_k w'_{n+1}a$ and $s_k(w'_{n+1}) = d'$ that $w_n \lhd_k w'_{n+1}$. With the definition in line (4) we have $s_k(w'_n) = d'$ and since we know that $|w'_n| \geq k+1$ we obtain $w'_n \lhd_k w'_{n+1}$. The latter is because $w'_n a \lhd_k w'_{n+1} a$. It is clear that the proposed algorithm stops. We distinguish in cases where the algorithm stops.

Case 1. Suppose our algorithm stops in line (1) or line (2). In the former case we have by (C) that for $m \le i \le l+1$ there are words w_i' such that $w_i = w_i'a$ and $s_k(w_i') = d'$, and for $m \le i \le l$ it holds that $w_i' \lhd_k w_{i+1}'$. In the latter case we have $n \ge m$ and also by (C) that for $n+1 \le i \le l+1$ there are words w_i' such that $w_i = w_i'a$ and $s_k(w_i') = d'$, and for $n+1 \le i \le l$ it holds that $w_i' \lhd_k w_{i+1}'$. If we define for $m \le i \le n$ words $w_i' =_{\text{def}} w_i$ we obtain for $m \le i \le l$ that $w_i' \lhd_k w_{i+1}'$ since $w_n \lhd_k w_{n+1}'$.

Taking this together, we can state that for $m \le i \le l$ there are words w_i' such that $w_i' \lhd_k w_{i+1}'$. Moreover, for $m \le i \le n$ it holds that $w_i' = w_i$ and for $n+1 \le i \le l+1$ we have $s_k(w_i') = d'$ and $w_i = w_i'a$. Note that $m = n+1$ is possible. By hypothesis we know that $[t_0', t_1', \ldots, t_l', c', d'] \in \delta_l(s_0^l, v)$ with $t_i' = (d', d', 0, \delta(s_0, d'))$ for $0 \le i < m$ and $t_i' = (p_k(w_i'), s_k(w_i'), 1, \delta(s_0, w_i'))$ for $m \le i \le l$. Since for $n+1 \le i \le l+1$ we have $d_i' = s_k(w_i') = d'$ we can apply δ_l to $[t_0', t_1', \ldots, t_l', c', d']$ and input a with $j =_{\text{def}} n+1$. Recall that by definition of δ_l we obtain a state $[t_0'', t_1'', \ldots, t_l'', c'', d'']$ with $c'' = p_k(c'a)$, $d'' = s_k(d'a)$ and

– $t_i'' = (s_k(d'a), s_k(d'a), 0, \delta(s_0, s_k(d'a)))$ for $0 \le i < m$,
– $t_i'' = t_i'$ for $m \le i \le n$ and
– $t_i'' = (c_i', s_k(d'a), 1, \delta(r_i', a))$ for $n+1 \le i \le l$.

We have $c'' = c' = p_k(va) = c$ and $d'' = s_k(va) = d$. Moreover, it holds that $t_i'' = (d, d, 0, \delta(s_0, d))$ for $0 \le i < m$ and we have $t_i'' = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$ for $m \le i \le n$ because we have defined $w_i' = w_i$. Now let $n+1 \le i \le l$. Then $c_i' = p_k(w_i') = p_k(w_i'a)$ and $d' = d_i' = s_k(w_i')$ implies $s_k(d'a) = s_k(w_i'a)$. Finally, from $\delta(r_i', a) = \delta(\delta(s_0, w_i'), a) = \delta(s_0, w_i'a)$ it follows that $t_i'' = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$ for $n+1 \le i \le l$. Together this shows $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, va)$.

Case 2. Assume our algorithm stops in line (3). Then $n \ge m$. Since it did not stop in line (2) we have $s_{k+1}(w_n) = d'a$ and from $|w_n| = k+1$ it follows that $w_m = w_{m+1} = \ldots = w_n = d'a$. Moreover, by (C) we have for $n+1 \le i \le l+1$ that $w_i = w_i'a$ and $s_k(w_i') = d'$, and for $n+1 \le i \le l$ that $w_i' \lhd_k w_{i+1}'$. By hypothesis we know that $[t_0', t_1', \ldots, t_l', c', d'] \in \delta_l(s_0^l, v)$ with $t_i' = (d', d', 0, \delta(s_0, d'))$ for $0 \le i \le n$ and $t_i' = (p_k(w_i'), s_k(w_i'), 1, \delta(s_0, w_i'))$ for $n+1 \le i \le l$. Since we have for $m \le i \le n$ that $d_i' = d'$ and for $n+1 \le i \le l+1$ that $d_i' = s_k(w_i') = d'$ we can apply δ_l to $[t_0', t_1', \ldots, t_l', c', d']$ and input a with $j =_{\text{def}} m$. Similar to above, we obtain by definition of δ_l a state $[t_0'', t_1'', \ldots, t_l'', c'', d'']$ with $c'' = p_k(c'a)$, $d'' = s_k(d'a)$ and

– $t_i'' = (s_k(d'a), s_k(d'a), 0, \delta(s_0, s_k(d'a)))$ for $0 \le i < m$ and
– $t_i'' = (c_i', s_k(d'a), 1, \delta(r_i', a))$ for $m \le i \le l$.

We conclude as in the first case that $c'' = c$, $d'' = d$ and that for $0 \le i < m$ and $n+1 \le i \le l$ we have $t_i'' = t_i$. For $m \le i \le n$ we have $c_i'' = c_i' = d' = p_k(d'a)$ and $\delta(r_i', a) = \delta(\delta(s_0, d'), a) = \delta(s_0, d'a)$. Note for $m \le i \le n$ that $w_i = d'a$ and hence $t_i'' = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i))$. This shows $[t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, va)$. **(End proof of Claim.)**

We specify the set of acccepting states for $\mathcal{M}_l$. Let $[t_0, t_1, \ldots, t_l, c, d] \in T$ with $t_i = (c_i, d_i, f_i, r_i)$ for $0 \le i \le l$. We define $[t_0, t_1, \ldots, t_l, c, d] \in S_l'$ if and only if it holds for $0 \le i \le l$ that

– $f_i = 1$, $c_i = c$, $d_i = d$ and
– $r_i \in S' \leftrightarrow i \equiv 0 \mod 2$.

The first item ensures that we have guessed a $\preceq_k$-chain $w_0 \preceq_k w_1 \preceq_k \ldots \preceq_k w_l \preceq_k v$, while the second one takes alternations with respect to the given DFA $\mathcal{M}$ into consideration. We set $\mathcal{M}_l =_{\text{def}} (A, T, \delta_l, s_0^l, S_l')$ and claim that $L(\mathcal{M}_l) = L(\mathcal{M})_k^+(l)$. To see this we conclude

$$\begin{array}{rcl} v \in L(\mathcal{M}_l) & \iff & \delta_l(s_0^l, v) \cap S_l' \neq \emptyset \\ & \iff & \text{there exists some } [t_0, t_1, \ldots, t_l, c, d] \in \delta_l(s_0^l, v) \cap S_l' \\ & \iff & c = p_k(v), d = s_k(v) \text{ and there exist some } w_0, \ldots, w_l \text{ such that} \\ & & w_0 \lhd_k \ldots \lhd_k w_l \lhd_k v \text{ and for } 0 \leq i \leq l \text{ it holds that} \\ & & t_i = (p_k(w_i), s_k(w_i), 1, \delta(s_0, w_i)) \text{ with } p_k(w_i) = c, s_k(w_i) = d \text{ and} \\ & & (\delta(s_0, w_i) \in S' \leftrightarrow i \equiv 0 \mod 2) \\ & \iff & \text{there exist } w_0, \ldots, w_l \text{ such that } w_0 \preceq_k \ldots \preceq_k w_l \preceq_k v \text{ and} \\ & & (\delta(s_0, w_i) \in S' \leftrightarrow i \equiv 0 \mod 2) \text{ for } 0 \leq i \leq l \\ & \iff & v \in L(\mathcal{M})_k^+(l) \end{array}$$

where the third equivalence is due to our claim and the definition of S_l'. ❑

We use this lemma to show the decidability of the membership problem of $\mathcal{B}_{1/2,k}(l)$ for fixed $l \geq 1$ and $k \geq 0$. There is even an efficient algorithm for this which we sketch in the following proof.

Theorem 2.33. *For fixed $l \geq 1$ and $k \geq 0$ the membership problems for $\mathcal{B}_{1/2,k}(l)$ and co$\mathcal{B}_{1/2,k}(l)$ are decidable in nondeterministic logarithmic space* NL.

Proof. Suppose some DFA $\mathcal{M}$ is given. We consider the NFA $\mathcal{M}_l$ from the proof of Lemma 2.32 and we need to determine whether $L(\mathcal{M}_l) = \emptyset$. This is equivalent with the non–existence of a path in the transition graph of $\mathcal{M}_l$ between the starting state and one of its accepting states. Hence, we have to solve the graph non–accessibility problem for the transition graph of $\mathcal{M}_l = (A, T, \delta_l, s_0^l, S_l')$. To do this, we start with the initial state s_0^l and then continuously guess a next input letter and a nondeterministic choice j for the transition function δ_l. We then determine with help of δ_l the next state of $\mathcal{M}_l$, overwrite the actual state and check whether the new state is an accepting one. If so, the algorithm stops and accepts the input, otherwise we continue this procedure. The space needed to do this is dominated by the space needed to store a state of $\mathcal{M}_l$. Note that the needed counters to reconstruct δ_l remain small. So this can be done in space $\leq c \cdot (k \cdot l \cdot \log |A| + l \cdot \log |\mathcal{M}|)$ for some constant c. Since NL is closed under complement this shows the theorem. ❑

With help of Theorem 2.21 we can now bound l for fixed k, which yields an algorithm that determines for a regular language its exact location in the Boolean hierarchy over $\mathcal{B}_{1/2,k}$.

Theorem 2.34. *Let $L \subseteq A^+$ be a regular language. There is a recursive and monotone decreasing function f_L such that for $k \geq 0$ it holds that*

$$\begin{array}{ll} L \in \mathcal{B}_{1/2,k} & \textit{if } f_L(k) = 1, \\ L \in \mathcal{B}_{1/2,k}(l) \setminus \mathcal{B}_{1/2,k}(l-1) & \textit{if } f_L(k) = l \textit{ for } l \geq 2 \textit{ and} \\ L \notin \mathcal{B}_{1,k} & \textit{if } f_L(k) = \infty. \end{array}$$

Proof. Define $f_L(k) =_{\text{def}} m_k^+(L) + 1$ for $k \geq 0$. Observe that $u \preceq_{k+1} v$ implies $u \preceq_k v$ and hence every $\preceq_{k+1}$-chain is also a $\preceq_k$-chain. It follows that $m_{k+1}^+(L) \leq m_k^+(L)$ for all regular languages L and all $k \geq 0$. So f_L is a monotone decreasing function.

Let $\mathcal{M}$ be a DFA such that $L(\mathcal{M}) = L$ and let $k \geq 0$ be given. To compute $f_L(k)$ we need to determine $m_k^+(L) = m_k^+(L(\mathcal{M}))$. We may apply the algorithm from Theorem 2.33 for

$l = 1, 2, \ldots, 2^{2|A|^{k+2}(k+1)^2|\mathcal{M}|} + 1$ answering the questions $m_k^+(L(\mathcal{M})) < l$. If the first positive answer occurs then we output l as the value for $f_L(k)$. If all answers are negative then we output a special symbol for ∞. Note that this computation can be done by a binary search since the answer string is monotonic. The remaining parts of the theorem are a consequence of Theorems 2.30 and 2.21. ❑

For $k = 0$ the results of this subsection carry over to the classes $\mathcal{L}_{1/2}(l)$ and $\mathrm{co}\mathcal{L}_{1/2}(l)$ of the Boolean hierarchy over level 1/2 of the STH, i.e., it is a strict hierarchy of classes with decidable membership problems. Due to the logical characterization which we recalled in Theorem 1.23, this also holds for the Boolean hierarchy over the class of languages that are definable by a Σ_1 formula of FO[<].

2.5 The Boolean Structure of Dot–Depth One

We have already seen that $\mathcal{B}_{1/2} = \bigcup_{k \geq 0} \mathcal{B}_{1/2,k}$ and $\mathcal{B}_1 = \bigcup_{k \geq 0} \mathcal{B}_{1,k}$. This relation holds for every level of the Boolean hierarchy. Recall that we know from Lemma 1.21 that $\mathcal{B}_{1/2}$ is closed under union and intersection, so the classes $\mathcal{B}_{1/2}(l)$ and $\mathrm{co}\mathcal{B}_{1/2}(l)$ form the Boolean hierarchy over $\mathcal{B}_{1/2}$.

Proposition 2.35. *Let $l \geq 1$. It holds that*

1. $\mathcal{B}_{1/2}(l) = \bigcup_{k \geq 0} \mathcal{B}_{1/2,k}(l)$ *and*
2. $\mathcal{B}_{1/2,k}(l) \subsetneq \mathcal{B}_{1/2,k+1}(l)$ *for all* $k \geq 0$.

Proof. For the first statement let $L \in \mathcal{B}_{1/2}(l)$ via languages $L_1, L_2, \ldots, L_l \in \mathcal{B}_{1/2}$. By Lemma 1.29 we have $L_i \in \mathcal{B}_{1/2,k_i}$ for $1 \leq i \leq l$ and suitable $k_i \geq 0$. With Proposition 1.28 we see that $L_i \in \mathcal{B}_{1/2,k}$ for $k =_{\mathrm{def}} \max_{1 \leq i \leq l} k_i$ and hence $L \in \mathcal{B}_{1/2,k}(l)$. Conversely, let $L \in \mathcal{B}_{1/2,k}(l)$ for some $k \geq 0$. Since $\mathcal{B}_{1/2,k} \subseteq \mathcal{B}_{1/2}$ we immediately have $L \in \mathcal{B}_{1/2}(l)$.

To see the second statement we obtain $\mathcal{B}_{1/2,k}(l) \subseteq \mathcal{B}_{1/2,k+1}(l)$ from $\mathcal{B}_{1/2,k} \subseteq \mathcal{B}_{1/2,k+1}$ as before. That this inclusion is strict follows from $\mathcal{B}_{1/2,k+1} \not\subseteq \mathcal{B}_{1,k}$ which we show next.

Define for $k \geq 0$ languages $L_{k+1} =_{\mathrm{def}} a^{k+1}A^+$ and note that there are at least two different letters $a, b \in A$. Then $L_{k+1} \in \mathcal{B}_{1/2,k+1}$ because

$$L_{k+1} = \bigcup_{\alpha \in A^{k+2}} \bigcup_{w \in A^{k+1}} (a^{k+1}|\ \alpha\ |w)_{k+1}.$$

To see that $L_{k+1} \notin \mathcal{B}_{1,k}$ let $x_0 =_{\mathrm{def}} a^{k+2}$ and for $i \geq 0$ set $x_{2i+1} =_{\mathrm{def}} a^k b x_{2i}$ and $x_{2i+2} =_{\mathrm{def}} a x_{2i+1}$. Then for all $i \geq 0$ we have $x_i \preceq_k x_{i+1}$ and it holds that $x_{2i} \in L_{k+1}$ and $x_{2i+1} \notin L_{k+1}$. So $x_0 \preceq_k x_1 \preceq_k x_2 \preceq_k \ldots$ is a $\preceq_k$-chain having an infinite number of alternations with respect to L_{k+1}. Hence $L_{k+1} \notin \mathcal{B}_{1,k}$ by Theorem 2.21. ❑

These relations hold also for the classes $\mathrm{co}\mathcal{B}_{1/2,k}(l)$ which can be proved completely analogously. So the Boolean hierarchies over $\mathcal{B}_{1/2,k}$ amount to the Boolean hierarchy over $\mathcal{B}_{1/2}$. We have already noted that $m_{k+1}^+(L) \leq m_k^+(L)$ and $m_{k+1}^-(L) \leq m_k^-(L)$.

Definition 2.36. For a language $L \subseteq A^+$ we define $m^+(L) =_{\mathrm{def}} \min_{k \geq 0} m_k^+(L)$ and $m^-(L) =_{\mathrm{def}} \min_{k \geq 0} m_k^-(L)$.

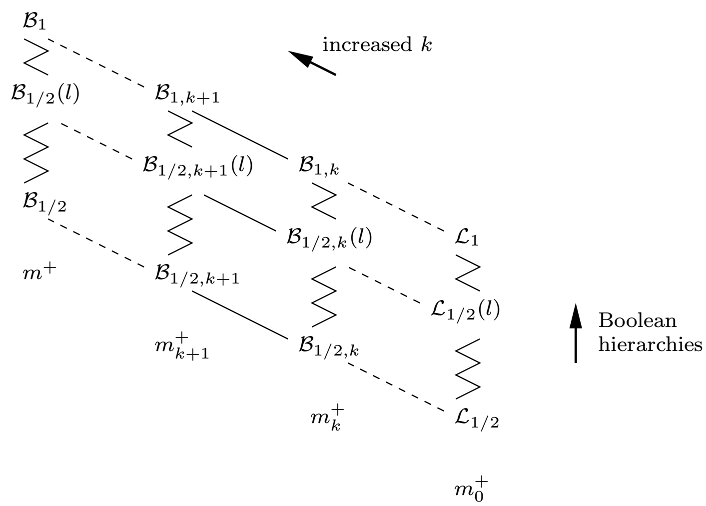


Fig. 2.6. The fine Boolean structure of $\mathcal{B}_1$

The measures m^+ and m^- relate to the single classes of the Boolean hierarchy over $\mathcal{B}_{1/2}$.

Proposition 2.37. *Let $L \subseteq A^+$ and let $l \geq 1$. It holds that*

1. *$L \in \mathcal{B}_{1/2}(l)$ if and only if $m^+(L) < l$ and*
2. *$L \in \mathrm{co}\mathcal{B}_{1/2}(l)$ if and only if $m^-(L) < l$.*

Proof. We only argue for the first statement. By Proposition 2.35 we have that $L \in \mathcal{B}_{1/2}(l)$ if and only there exists some $k \geq 0$ such that $L \in \mathcal{B}_{1/2,k}(l)$. By Theorem 2.30 the latter holds if and only if there exists some $k \geq 0$ such that $m_k^+(L) < l$ which in turn is equivalent to $m^+(L) < l$ by the definition of m^+. ❑

Unfortunately, it is not clear how to compute m^+ from m_k^+ although the latter is computable for fixed k. Here a result from [Gla99] can help where the classes of the Boolean hierarchy over $\mathcal{B}_{1/2}$ are characterized as in Proposition 2.37 but in terms of a measure $m_{\mathcal{M}}$ for a given DFA $\mathcal{M}$. This measure involves a relation on so–called structured words depending on $\mathcal{M}$. It follows that $m^+(L(\mathcal{M})) = m_{\mathcal{M}}$. It is shown in [Gla99] that the question if $m_{\mathcal{M}} < l$ for fixed l is decidable, which implies the decidability of the membership problem of $\mathcal{B}_{1/2}(l)$. Since the membership problem of $\mathcal{B}_1$ is decidable, one can compute the exact level of some language in the Boolean hierarchy over $\mathcal{B}_{1/2}$. A strictness argument for the Boolean hierarchy over $\mathcal{B}_{1/2}$ is also given in [Gla99]. In fact, for the latter the languages M_l for $k = 0$ from Theorem 2.31 can be used.

Figure 2.6 gives the structure of $\mathcal{B}_1$ in terms of Boolean combinations on one hand and in terms of the sequential parameter k on the other hand. This is a refinement of the figure at the beginning of this chapter. We show now how we can locate a given language L in this two–dimensional landscape and further investigate how the function f_L behaves (see Theorem 2.34). Observe that for a regular language L and for $l \geq 1$ it holds that $L \in \mathcal{B}_{1/2}(l)$ if and only if $\lim_{k\to\infty} f_L(k) \leq l$. So for $L \in \mathcal{B}_1$ the function f_L reaches the exact level of L in the Boolean hierarchy over $\mathcal{B}_{1/2}$ as k goes to infinity, and $f_L(k) = \infty$ for all $k \geq 0$ if $L \notin \mathcal{B}_1$. The following can be done for a given regular language $L \subseteq A^+$.

1. Determine if $L \in \mathcal{B}_1$ by one of the algorithms provided in [Kna83, Ste85b, CH91]. If $L \notin \mathcal{B}_1$ then $f_L(k) = \infty$ for all $k \geq 0$, otherwise continue.
2. Determine $l' =_{\mathrm{def}} m^+(L) + 1 = \lim_{k\to\infty} f_L(k)$ with help of the results in [Gla99].
3. Compute for $k = 0, 1, 2, \ldots$ the value of $f_L(k)$ (Theorem 2.34) until $f_L(k) = l'$ for some k. Then also $f_L(k') = l'$ for all $k' \geq k$.

All this can be carried out effectively. We may interpret the graph of f_L as follows. As long as $f_L(k)$ is infinite, it is not possible to describe L by combinations of blocks of length $k+1$. If L is in $\mathcal{B}_1$ then there is some minimal k_0 where this is possible. The minimal amount of Boolean complexity we need to spend for L in case of this particular k_0 is given by $l_0 =_{\mathrm{def}} f_L(k_0)$. If $l_0 = l'$ then k_0 is optimal in the sense that any larger k does not save Boolean combinations. If $l_0 > l'$ then trade–offs are possible and we can use the above algorithm to select the amount of descriptional complexity for L we like to spend — in terms of k versus Boolean combinations.

However, it remains to investigate if there are any trade–offs at all. For a partial answer we consider the following example language. Recall from the proof of Theorem 2.31 the language M_3 for $k = 1$ and define $L =_{\mathrm{def}} M_3 \cup \{bb\}$. Then we can compute via the above procedure the following table, in which we also provide witnessing $\preceq_k$-chains having the maximal number of alternations.

k	$f_L(k)$	witnessing $\preceq_k$-chain
0	∞	$aa \preceq_0 aba \preceq_0 aaba \preceq_0 ababa \preceq_0 aababa \preceq_0 \ldots$
1	5	$bb \preceq_1 bbabb \preceq_1 bbaabb \preceq_1 bbaabbaabb \preceq_1 bbaabbaabbaabb$
2	3	$b^3aab^3 \preceq_2 b^3aab^3aab^3 \preceq_2 b^3aab^3aab^3aab^3$
3	3	$b^4aab^4 \preceq_3 b^4aab^4aab^4 \preceq_3 b^4aab^4aab^4aab^4$
$\vdots$	$\vdots$	$\vdots$
∞	3	

Intuitively, there are no $\preceq_k$-chains having more alternations than our witnesses here, because $w \preceq_k u$ implies that $|w|_\alpha \leq |u|_\alpha$ for all $\alpha \in A^{k+1}$ (recall that $|w|_\alpha$ is the number of occurrences of α in the k-decomposition $\widehat{w}$). Note that L is a language of infinite cardinality having trade–offs in our setting. More precisely, the table says that L is not in $\mathcal{B}_{1,0}$, but it is in $\mathcal{B}_{1/2,1}(5)\backslash\mathcal{B}_{1/2,1}(4)$. We can do better if we choose $k = 2$, but no larger k has effects on the Boolean complexity. In particular, we have $L \in \mathcal{B}_{1/2}(3)\backslash\mathcal{B}_{1/2}(2)$. The reason for the jump between $k = 1$ and $k = 2$ here is that bb is too short to take part in $\preceq_k$-chains for $k \geq 2$. More complicated stepwise functions can be obtained by similar constructions.

What we observed here in the example is a more general phenomenon: whenever we can bound the length of at least one word in every $\preceq_k$-chain having the maximal number of alternations, then we can take for some $k' > k$ all these words into a finite set from $\mathcal{B}_{1/2,k'}$. This shortens all witnessing maximal chains and hence, reduces the level in the respective Boolean hierarchy. In particular, we have this case when there is only a finite number of $\preceq_k$-chains having a maximal number of alternations.

So the question remains whether we can still save Boolean combinations by increasing k if the condition we described does not hold. We feel the answer should be no, but could not give a proof yet. We think that this is an interesting point concerning dot–depth one languages and leave it as an open question here.

2.6 Forbidden Pattern Characterization of $\mathcal{B}_1$

We derive in this section a forbidden pattern characterization of $\mathcal{B}_1$ using Theorem 2.21.

Definition 2.38.

1. Pattern $\mathbb{B}_1$ is defined as the subgraph given in Figure 2.7 with $x, y, y', u, v, z \in A^*$ and $w, w' \in A^+$.
2. Pattern $\mathbb{D}$ is defined as the subgraph given in Figure 2.8 with $x, u, v, z \in A^*$ and $w \in A^+$.
3. Pattern $\mathbb{D}^{\mathrm{rev}}$ is defined as the subgraph given in Figure 2.9 with $x, u, v, z \in A^*$ and $w \in A^+$.

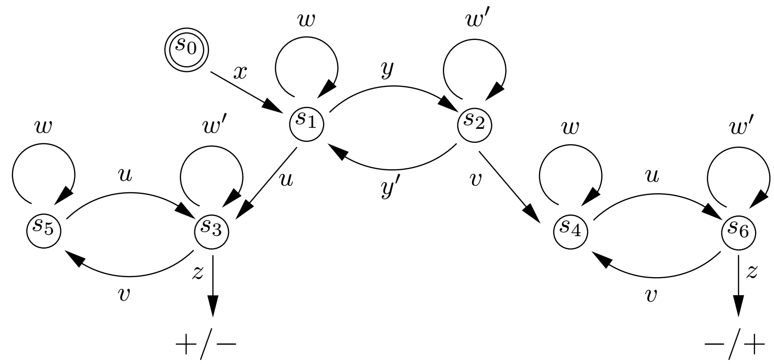


Fig. 2.7. Pattern $\mathbb{B}_1$ with $w, w' \in A^+$.

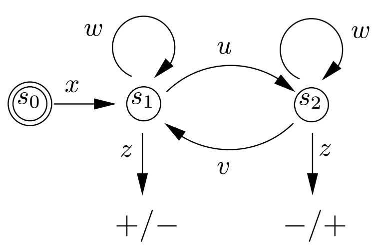


Fig. 2.8. Pattern $\mathbb{D}$ with $w \in A^+$.

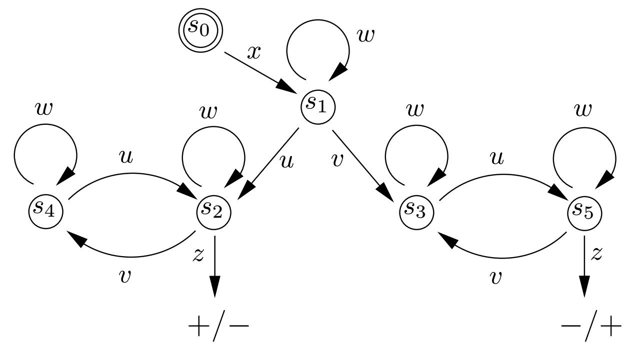


Fig. 2.9. Pattern $\mathbb{D}^{\mathrm{rev}}$ with $w \in A^+$.

To see that $\mathcal{FP}(\mathbb{B}_1)$ and $\mathcal{FP}(\mathbb{B}_1, \mathbb{D}, \mathbb{D}^{\mathrm{rev}})$ are well–defined we consider pattern $\mathbb{B}_1$ since no new argument is needed for the others. Suppose there is some DFA accepting some language L and which has pattern $\mathbb{B}_1$ via $x, y, y', u, v, z \in A^*$ and $w, w' \in A^+$. Let $\mathcal{M}$ be an arbitrary DFA with $L(\mathcal{M}) = L$ and set $r =_{\mathrm{def}} |\mathcal{M}|$. If we substitute

– $\hat{x} =_{\mathrm{def}} xw^r$, $\hat{y} =_{\mathrm{def}} y(w')^r$, $\hat{y}' =_{\mathrm{def}} y'w^r$,
– $\hat{u} =_{\mathrm{def}} u(w')^r$, $\hat{v} =_{\mathrm{def}} vw^r$ and
– $\hat{w} =_{\mathrm{def}} w^{r!}$, $\hat{w}' =_{\mathrm{def}} (w')^{r!}$ and $\hat{z} =_{\mathrm{def}} z$

we see with Proposition 1.34 that we find the required loops in $\mathcal{M}$ and one verifies that these words give rise to pattern $\mathbb{B}_1$ in $\mathcal{M}$.

Theorem 2.39. *It holds that* $\mathcal{B}_1 = \mathcal{FP}(\mathbb{B}_1) = \mathcal{FP}(\mathbb{B}_1, \mathbb{D}, \mathbb{D}^{\mathrm{rev}})$.

Proof. Suppose $L \in \mathcal{B}_1$ and let $\mathcal{M}$ be a DFA with $L(\mathcal{M}) = L$. Then there is some $k \geq 0$ such that $L \in \mathcal{B}_{1,k}$. By Theorem 2.21 we know that $\mathcal{M}$ has none of the patterns $\mathbb{B}_{1,k}$, $\hat{\mathbb{B}}_{1,k}$ and $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$. Assume to the contrary that $\mathcal{M}$ has pattern $\mathbb{B}_1$. Since $w, w' \in A^+$ we may take $\hat{w} =_{\text{def}} w^k$ and $\hat{w}' =_{\text{def}} (w')^k$ to see that $\mathcal{M}$ has pattern $\mathbb{B}_{1,k}$, a contradiction. So $\mathcal{M}$ is a DFA with $L(\mathcal{M}) = L$ which does not have pattern $\mathbb{B}_1$. This shows that $L \in \mathcal{FP}(\mathbb{B}_1)$.

Now let $L \in \mathcal{FP}(\mathbb{B}_1)$. Then there is some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ with $L(\mathcal{M}) = L$ and which does not have pattern $\mathbb{B}_1$. Assume to the contrary that $L \notin \mathcal{FP}(\mathbb{B}_1, \mathbb{D}, \mathbb{D}^{\text{rev}})$, so $\mathcal{M}$ has one of the patterns $\mathbb{B}_1$, $\mathbb{D}$ or $\mathbb{D}^{\text{rev}}$. It suffices to show that $\mathbb{D}$ and $\mathbb{D}^{\text{rev}}$ are special cases of $\mathbb{B}_1$. First suppose $\mathcal{M}$ has pattern $\mathbb{D}$ via $x, u, v, z \in A^*$ and $w \in A^+$ involving states s_1 and s_2. Then we set

- $\hat{x} =_{\text{def}} x$, $\hat{y} =_{\text{def}} u$, $\hat{y}' =_{\text{def}} v$,
- $\hat{w} =_{\text{def}} w$, $\hat{w}' =_{\text{def}} w$, $\hat{z} =_{\text{def}} z$ and
- $\hat{u} =_{\text{def}} \varepsilon$, $\hat{v} =_{\text{def}} \varepsilon$.

These words give rise to pattern $\mathbb{B}_1$ in $\mathcal{M}$ involving the states $\hat{s}_1 = \hat{s}_3 = \hat{s}_5 = s_1$ and $\hat{s}_2 = \hat{s}_4 = \hat{s}_6 = s_2$. As the second case, assume that $\mathcal{M}$ has pattern $\mathbb{D}^{\text{rev}}$ via $x, u, v, z \in A^*$ and $w \in A^+$ involving states s_1 to s_5. This time we just set $y =_{\text{def}} \varepsilon$, $y' =_{\text{def}} \varepsilon$ and $w' =_{\text{def}} w$ to see together with x, u, v, w, z that $\mathcal{M}$ has pattern $\mathbb{B}_1$.

Finally, let $L \in \mathcal{FP}(\mathbb{B}_1, \mathbb{D}, \mathbb{D}^{\text{rev}})$. Then there is some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ with $L(\mathcal{M}) = L$ and which does not have any of the patterns $\mathbb{B}_1$, $\mathbb{D}$ or $\mathbb{D}^{\text{rev}}$. Assume to the contrary that $L \notin \mathcal{B}_1$, so for all $k \geq 0$ it holds that $L \notin \mathcal{B}_{1,k}$. It follows from Theorem 2.21 and because $\mathcal{FP}(\mathbb{B}_1, \mathbb{D}, \mathbb{D}^{\text{rev}})$ is well–defined that for all $k \geq 0$ the DFA $\mathcal{M}$ has one of the patterns $\mathbb{B}_{1,k}$, $\hat{\mathbb{B}}_{1,k}$ or $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$. We look at the case $k = |\mathcal{M}|^5$ and show that pattern $\mathbb{B}_{1,k}$, $\hat{\mathbb{B}}_{1,k}$ or $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ in $\mathcal{M}$ implies pattern $\mathbb{B}_1$, $\mathbb{D}$ or $\mathbb{D}^{\text{rev}}$ in $\mathcal{M}$, respectively, which is a contradiction.

Case 1. Assume that $\mathcal{M}$ has pattern $\mathbb{B}_{1,k}$ via $x, y, y', v, u, w, w', z \in A^*$ with $|w| = |w'| = k$ (see Figure 2.3). We apply Proposition 1.35 to w' and the three pairs of states

- $(\delta(s_0, xy), \delta(s_0, xyw'))$, $(\delta(s_0, xu), \delta(s_0, xuw'))$ and $(\delta(s_0, xyw'vwu), \delta(s_0, xyw'vwuw'))$.

This provides states $\hat{s}_2$, $\hat{s}_3$ and $\hat{s}_6$, respectively, such that $w' = w_1'\hat{w}'w_2'$ and $\hat{s}_i$ has a $\hat{w}'$-loop for $i = 2, 3, 6$. We do the same thing for w and

- $(\delta(s_0, xyw'y'), \delta(s_0, x))$, $(\delta(s_0, xuw'v), \delta(s_0, xuw'vw))$ and $(\delta(s_0, xyw'v), \delta(s_0, xyw'vw))$

to get states $\hat{s}_1$, $\hat{s}_5$ and $\hat{s}_4$ having a $\hat{w}$-loop where $w = w_1\hat{w}w_2$. Now we find pattern $\mathbb{B}_1$ involving states $\hat{s}_1$ to $\hat{s}_6$ when considering

- $\hat{x} =_{\text{def}} xyw'y'w_1$, $\hat{y} =_{\text{def}} w_2yw_1'$, $\hat{y}' =_{\text{def}} w_2'y'w_1$,
- $\hat{u} =_{\text{def}} w_2uw_1'$, $\hat{v} =_{\text{def}} w_2'vw_1$, $\hat{z} =_{\text{def}} w_2'vwuz$ and
- $\hat{w}, \hat{w}' \in A^+$.

Case 2. Assume that $\mathcal{M}$ has pattern $\hat{\mathbb{B}}_{1,k}$ via $x, u, v, w, z \in A^*$ with $|w| = k$ (see Figure 2.4). We apply Proposition 1.35 to w and

- $(\delta(s_0, xuwv), \delta(s_0, x))$ and $(\delta(s_0, xu), \delta(s_0, xuw))$

to get states $\hat{s}_1$ and $\hat{s}_2$, respectively, having a $\hat{w}$-loop where $w = w_1\hat{w}w_2$. We find pattern $\mathbb{D}$ involving states $\hat{s}_1$ and $\hat{s}_2$ when considering

- $\hat{x} =_{\text{def}} xuwvw_1$, $\hat{u} =_{\text{def}} w_2uw_1$, $\hat{v} =_{\text{def}} w_2vw_1$,
- $\hat{z} =_{\text{def}} w_2z$ and $\hat{w} \in A^+$.

Case 3. Assume that $\mathcal{M}$ has pattern $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ via $x, u, v, w, z \in A^*$ with $|w| = k$ (see Figure 2.5). We apply Proposition 1.35 to w and

– $(\delta(s_0, x), \delta(s_0, xw))$, $(\delta(s_0, xwu), \delta(s_0, xwuw))$, $(\delta(s_0, xwv), \delta(s_0, xwvw))$,
– $(\delta(s_0, xwuwv), \delta(s_0, xwuwvw))$ and $(\delta(s_0, xwvwu), \delta(s_0, xwvwuw))$

to get states $\hat{s}_i$ for $1 \leq i \leq 5$, respectively, having a $\hat{w}$-loop where $w = w_1\hat{w}w_2$. We find pattern $\mathbb{D}^{\text{rev}}$ involving states $\hat{s}_1$ to $\hat{s}_5$ when considering

– $\hat{x} =_{\text{def}} xw_1$, $\hat{u} =_{\text{def}} w_2uw_1$, $\hat{v} =_{\text{def}} w_2vw_1$,
– $\hat{z} =_{\text{def}} w_2vwuz$ and $\hat{w} \in A^+$.

❑

The previous theorem yields an NL–algorithm for the membership problem of $\mathcal{B}_1$ as follows. Let a DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ be given. We guess states $s_1, \dots s_6, s^+, s^- \in S$ and check whether s^+ is accepting and s^- is rejecting. Then we verify $s_0 \longrightarrow s_1$ and $(s_3, s_6) \longrightarrow (s^+, s^-)$. If the latter fails we check $(s_3, s_6) \longrightarrow (s^-, s^+)$. Now it remains to verify that $(s_1, s_4, s_5) \longrightarrow^+ (s_1, s_4, s_5)$, $(s_2, s_3, s_6) \longrightarrow^+ (s_2, s_3, s_6)$, $s_1 \longrightarrow s_2$, $s_2 \longrightarrow s_1$, $(s_5, s_4) \longrightarrow (s_3, s_6)$ and $(s_3, s_6) \longrightarrow (s_5, s_4)$. Similar algorithms have been investigated in [Ste85b, CH91].

2.7 Discussion and Bibliographic Notes

We want to make a few remarks concerning the major characterizations of $\mathcal{B}_{1,k}$ from [Ste85a, Theorem 3.3]. With the way we have cited this theorem as Theorem 2.21 we have translated the notations from [Ste85a] as follows. Pattern $\hat{\mathbb{B}}_{1,k}$ is given as the property of 'k-stableness', i.e., a minimal DFA is k-stable if and only if it does not have pattern $\hat{\mathbb{B}}_{1,k}$. The patterns $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ and $\mathbb{B}_{1,k}$ are called 'forks of type I' and 'forks of type II', respectively. For systematic reasons we chose here a uniform treatment in terms of patterns. However, there is a little difference in the pattern definitions. It is additionally required in [Ste85a] that the states s_3 and s_6 in pattern $\mathbb{B}_{1,k}$ and also the states s_3 and s_6 in pattern $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ must be in distinct strongly connected components. It is easy to see that we can drop this condition if all three patterns are forbidden: if s_3 and s_6 are in the same strongly connected component then we find pattern $\hat{\mathbb{B}}_{1,k}$ (in both cases).

Moreover, in [Ste85a] the classes $\mathcal{B}_{1,k}$ are defined differently, i.e., based on certain equivalence relations from [Sim72]. To show that this definition coincides with our definition, one has to show that $L \in \mathcal{B}_{1,k}$ if and only if there exists some $m \geq 1$ such that L is in the Boolean algebra generated by languages L_i with $L_i \subseteq A^{<k+m}$ or $L_i = (w|\alpha_1, \dots, \alpha_m|v)_k$ where $\alpha_i \in A^{k+1}$ and $w, v \in A^k$. For the 'if'–part we only need to observe that languages $L_i \subseteq A^{<k+m}$ are in the Boolean closure of $\mathcal{B}_{1/2,k}$. This is due to the fact that we can express words x with lengths $\geq k+1$ as

$$\{x\} = \langle x\rangle_k \setminus \bigcup_{\substack{x \preceq_k y \\ x \neq y}} \langle y\rangle_k \quad \in \text{BC}(\mathcal{B}_{1/2,k})$$

since the latter union is finite by Theorem 2.12. For the reverse implication one can show that the mentioned Boolean algebras are included in each other for increasing m. This is very similar to Proposition 1.28.

The relation $\preceq_k$ is introduced as so–called 'k-embeddings' in [Ste85a, Section 1.2]. Unfortunately, the definition given there is misleading since it requires that u and v have the same prefix of length $2k$ if there is a k-embedding from u to v (this is inconsistent with [Ste85a, Theorem 3.3]). A look at the proofs in [Ste85a] shows that in fact k-embeddings are used in the way we have defined $\preceq_k$ here. We have not yet mentioned [Ste85a, Proposition 4.1] which says that a language L is in $\mathcal{B}_1$ if and only if it is in $\mathcal{B}_{1,k}$ with $k \leq |\mathcal{M}|^3$ and where $\mathcal{M}$ is the minimal DFA accepting L. We want to remark that another proof of the remaining statements of Theorem 2.21 can be concluded from [Sch99c].

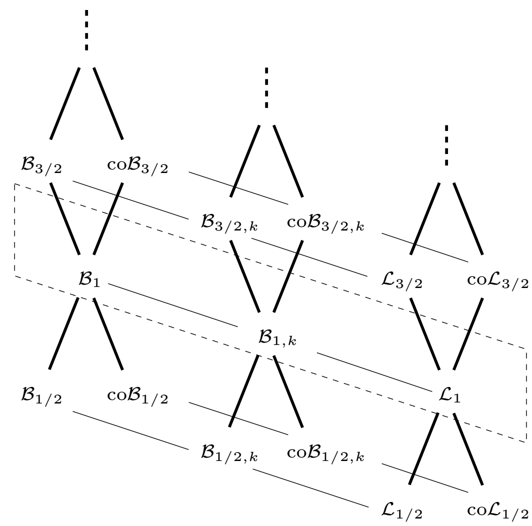
The characterization of $\mathcal{B}_1$ in terms of $\mathbb{B}_1$ can be compared to the algebraic condition from [Kna83] where the decidability of the membership problem of $\mathcal{B}_1$ was first shown. The algebraic condition is reflected in a straightforward way in the structure of the subgraph defined by $\mathbb{B}_1$.

The patterns $\mathbb{D}$ and $\mathbb{D}^{\text{rev}}$ itself characterize the classes of languages that have locally $\mathcal{R}$-trivial and locally $\mathcal{L}$-trivial semigroups, respectively, which is a result from [CPP93]. It looks like the authors have rediscovered these patterns since they do not mention [Ste85a]. We will further investigate these classes in the following chapter.

Finally, we want to mention that for the case $k = 0$ the results of Subsection 2.4 can be found in [SW98], where the usual subword relation and the Boolean hierarchy over level 1/2 of the STH are studied. The hint to look at [Hig52] for a connection of the subword relation to order ideals is from Dietrich Kuske, Dresden.

3. Deterministic Languages and Restricted Temporal Logic

We refer to the main results of this chapter. We define the classes of k-deterministic languages in Section 3.1 and isolate their main property (cf. Lemma 3.3). For fixed $k \geq 0$ we prove in Section 3.2 a forbidden pattern characterization of these classes (cf. Theorem 3.5). In Section 3.3 we turn to restricted temporal logic, recall the needed definitions and introduce fragments of this logic in terms of the nesting depth $\leq k$ of the next operator. These fragments give rise to the so–called next hierarchy of classes of languages that are definable by formulas restricted in this way. Then we show that the languages of level k of the next hierarchy are just the k-deterministic languages (cf. Theorem 3.17). Our characterization in terms of forbidden patterns allows to give concise proofs of decidability and strictness results for the next hierarchy (cf. Theorems 3.21 and 3.22).



In Section 3.4 we investigate the relation of k-deterministic languages to the DDH and the STH, and we see how these classes fit into this landscape (cf. Figure 3.3). Finally, in Section 3.5 we show that there are close connections between the complexity class Δ_2^{p} and languages definable in restricted temporal logic (cf. Theorem 3.31).

3.1 Generalized Deterministic Languages

Recall with Definition 1.25 that a language $(\Sigma_1, \alpha_1, \ldots, \Sigma_{n+1}, \alpha_{n+1})_k$ with $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ consists of those words $x \in A^{\geq k+1}$ whose k-decomposition starts with a number (possibly zero) of elements from Σ_1, then α_1, followed by a number (possibly zero) of elements from Σ_2, then α_2 and so on, and ends with α_{n+1}. Of special interest in this chapter are languages that admit a unique such decomposition.

Definition 3.1. Let $k \geq 0$ and $L \subseteq A^+$.

1. L is left k-deterministic if and only if there exist $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ with $L = (\Sigma_1, \alpha_1, \ldots, \Sigma_n, \alpha_n, \Sigma_{n+1}, \alpha_{n+1})_k$ and for $1 \leq i \leq n$ it holds that $\alpha_i \notin \Sigma_i$.
2. L is right k-deterministic if and only if there exist $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ with $L = (\alpha_1, \Sigma_1, \alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ and for $2 \leq i \leq n+1$ it holds that $\alpha_i \notin \Sigma_i$.

Note that the requirement $\alpha_i \notin \Sigma_i$ does not range over the last (first) index. We refer to these languages as k-deterministic languages. Deterministic languages for $k = 0$ and sets $A_0^* a_1 A_1^* \cdots a_n A_n^*$ were studied in [Eil76, Pin86].

Definition 3.2. Let $k \geq 0$. Then $\mathcal{D}_k^{\text{left}}$ is the class of languages that can be written as finite unions of left k-deterministic languages. Moreover, $\mathcal{D}_k^{\text{right}}$ is the class of languages that can be written as finite unions of right k-deterministic languages. We may eventually take a finite set $D \subseteq A^{\leq k}$ to each of these languages.

It is easy to see that a language belongs to $\mathcal{D}_k^{\text{left}}$ if and only if its reverse belongs to $\mathcal{D}_k^{\text{right}}$ due to the symmetric definitions. The following lemma demonstrates the effect of the property $\alpha_i \notin \Sigma_i$ in the representation of a language L. It basically says that the partitioning of a word x that witnesses $x \in L$ must be the same for all $xy \in L$.

Lemma 3.3. *Let $k, s \geq 0$. Let $\alpha_1, \ldots, \alpha_s \in A^{k+1}$ and $\Sigma_1, \ldots, \Sigma_{s+1} \subseteq A^{k+1}$ with $\alpha_i \notin \Sigma_i$ for $1 \leq i \leq s$. If x and xy belong to $(\Sigma_1, \alpha_1, \ldots, \Sigma_s, \alpha_s, \Sigma_{s+1})_k$ with $\widehat{x} = (\beta_1, \ldots, \beta_l)$ and $\widehat{xy} = (\beta_1, \ldots, \beta_l, \beta_{l+1}, \ldots, \beta_{l+m})$, then $\beta_{l+i} \in \Sigma_{s+1}$ for $1 \leq i \leq m$.*

Proof. Suppose all prerequisites of the lemma are given. Since $x \in L$ there are $1 \leq j_1 < j_2 < \ldots < j_s \leq l$ with $\beta_{j_i} = \alpha_i$ and $\beta_j \in \Sigma_i$ with $j_i < j < j_{i+1}$ for all $0 \leq i \leq s$ (set $j_0 =_{\text{def}} 0$ and $j_{s+1} =_{\text{def}} l+1$). The same holds for $xy \in L$, so there are $1 \leq j'_1 < j'_2 < \ldots < j'_s \leq l+m$ with $\beta_{j'_i} = \alpha_i$ and $\beta_j \in \Sigma_i$ with $j'_i < j < j'_{i+1}$ for all $0 \leq i \leq s$ (set $j'_0 =_{\text{def}} 0$ and $j'_{s+1} =_{\text{def}} l+m+1$).

We show $j_i = j'_i$ for all $0 \leq i \leq s$ by induction on i. The induction base is clear by definition, so assume that $j_{i+1} \neq j'_{i+1}$ for some i with $0 \leq i < s$. Without loss of generality we may suppose that $j'_{i+1} > j_{i+1}$. Moreover, we know that $\beta_{j'_i+1}, \ldots, \beta_{j'_{i+1}-1} \in \Sigma_{i+1}$. Since by hypothesis $j_i = j'_i$ this sequence in fact starts with elements $\beta_{j_i+1}, \ldots, \beta_{j_{i+1}-1}, \beta_{j_{i+1}}$ and we get $\beta_{j_{i+1}} = \alpha_{i+1} \in \Sigma_{i+1}$, a contradiction.

Especially, we have $j_s = j'_s \leq l < l+i$ for $1 \leq i \leq m$. So from $\beta_j \in \Sigma_{s+1}$ for $j'_s < j \leq l+m$ we conclude $\beta_{l+i} \in \Sigma_{s+1}$ for $1 \leq i \leq m$. ❑

As a special case we have that if $\Sigma_{s+1} = \emptyset$ and x belongs to a language L as above, then there is no $y \in A^+$ such that xy is in L. A dual lemma holds for right k-deterministic languages.

3.2 Forbidden Pattern Characterization of $\mathcal{D}_k^{\text{left}}$ and $\mathcal{D}_k^{\text{right}}$

We define the following patterns.

Definition 3.4. Let $k \geq 0$.

1. Pattern $\mathbb{D}_k$ is defined as the subgraph given in Figure 3.1 with $x, u, v, w, z \in A^*$, $a \in A$ and $|w| = k$.
2. Pattern $\mathbb{D}_k^{\text{rev}}$ is defined as the subgraph given in Figure 3.2 with $x, u, v, w, z \in A^*$, $a \in A$ and $|w| = k$.

This definition for $k = 0$ can already be found in [EW96]. Note that the letter $a \in A$ makes the difference to the definition of the patterns $\hat{\mathbb{B}}_{1,k}$ and $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ (see Definition 2.20). We come back to this point in Section 3.4. In this section, we prove the following forbidden pattern characterization of the classes $\mathcal{D}_k^{\text{left}}$ and $\mathcal{D}_k^{\text{right}}$.

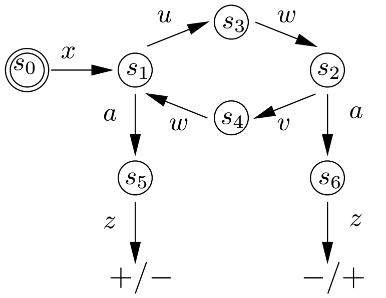


Fig. 3.1. Pattern $\mathbb{D}_k$ with $|w| = k$ and $a \in A$.

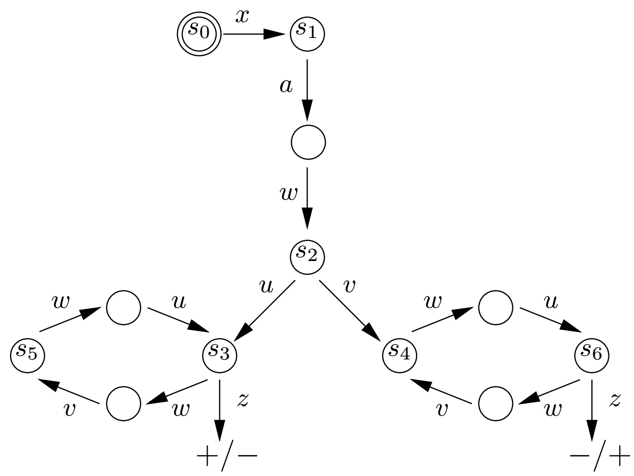


Fig. 3.2. Pattern $\mathbb{D}_k^{\text{rev}}$ with $|w| = k$ and $a \in A$.

Theorem 3.5. *Let $k \geq 0$. It holds that*

1. $\mathcal{D}_k^{\text{left}} = \mathcal{FP}(\mathbb{D}_k)$ *and*
2. $\mathcal{D}_k^{\text{right}} = \mathcal{FP}(\mathbb{D}_k^{\text{rev}})$.

Before we give a proof in Subsection 3.2.2 we observe some properties of the just defined patterns. It follows from Proposition 3.7 below that $\mathcal{FP}(\mathbb{D}_k)$ and $\mathcal{FP}(\mathbb{D}_k^{\text{rev}})$ are well–defined.

3.2.1 Basic Properties

We show with the next two propositions that the patterns $\mathbb{D}_k$ and $\mathbb{D}_k^{\text{rev}}$ are related via the reverse of the accepted languages, and that if some DFA has one of these patterns, then any DFA accepting the same language also has this pattern.

Proposition 3.6. *Let $k \geq 0$. Let $\mathcal{M}$ and $\hat{\mathcal{M}}$ be two* DFA*'s such that $L(\hat{\mathcal{M}}) = L(\mathcal{M})^R$. Then $\mathcal{M}$ has pattern $\mathbb{D}_k$ if and only if $\hat{\mathcal{M}}$ has pattern $\mathbb{D}_k^{\text{rev}}$.*

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ and let $\hat{\mathcal{M}} = (A, \hat{S}, \hat{\delta}, \hat{s}_0, \hat{S}')$ with $L(\hat{\mathcal{M}}) = L(\mathcal{M})^R$. Set $L =_{\text{def}} L(\mathcal{M})$.

First suppose $\mathcal{M}$ has pattern $\mathbb{D}_k$ for some $k \geq 0$ witnessed by $x, u, v, w, z \in A^*$ and $a \in A$ such that $|w| = k$. We assume without loss of generality that $\delta(s_0, xaz) \in S'$ and $\delta(s_0, xuwaz) \notin S'$. Then for all $l \geq 0$ we have $x(uwvw)^l az \in L$ and $x(uwvw)^l uwaz \notin L$. So for all $l \geq 0$ we know that $z^R a(w^R v^R w^R u^R)^l x^R \in L^R$ and $z^R a w^R u^R (w^R v^R w^R u^R)^l x^R = z^R a(w^R u^R w^R v^R)^l w^R u^R x^R \notin L^R$.

Due to the finiteness of $\hat{S}$ there are $r_1, l_1, r_2, l_2 \geq 1$ such that $\hat{\delta}(\hat{s}_0, z^R a(w^R v^R w^R u^R)^{r_1}) = \hat{\delta}(\hat{s}_0, z^R a(w^R v^R w^R u^R)^{r_1+l_1})$ and $\hat{\delta}(\hat{s}_0, z^R a(w^R u^R w^R v^R)^{r_2}) = \hat{\delta}(\hat{s}_0, z^R a(w^R u^R w^R v^R)^{r_2+l_2})$. Without loss of generality we may assume that $l_1 \geq r_1$ and $l_2 \geq r_2$ (otherwise consider appropriate multiples of l_1 and l_2). Let t be the smallest common multiple of l_1 and l_2. Then it holds that

$$\hat{\delta}(\hat{s}_0, z^R a(w^R v^R w^R u^R)^t) = \hat{\delta}(\hat{s}_0, z^R a(w^R v^R w^R u^R)^{rt})$$

and

$$\hat{\delta}(\hat{s}_0, z^R a(w^R u^R w^R v^R)^t) = \hat{\delta}(\hat{s}_0, z^R a(w^R u^R w^R v^R)^{rt})$$

for all $r \geq 1$, because $t \geq l_1 \geq r_1$ and $t \geq l_2 \geq r_2$. We define

$$\begin{aligned}
\hat{x} &=_{\text{def}} z^R, \\
\hat{u} &=_{\text{def}} (u^R w^R v^R w^R)^{2t-1} u^R, \\
\hat{v} &=_{\text{def}} (v^R w^R u^R w^R)^t v^R, \\
\hat{w} &=_{\text{def}} w^R \text{ and} \\
\hat{z} &=_{\text{def}} x^R.
\end{aligned}$$

Then the state

$$\hat{\delta}(\hat{s}_0, \hat{x}a\hat{w}\hat{u}) = \hat{\delta}(\hat{s}_0, z^R a w^R (u^R w^R v^R w^R)^{2t-1} u^R) = \hat{\delta}(\hat{s}_0, z^R a (w^R u^R w^R v^R)^{2t-1} w^R u^R)$$

has a loop with label

$$(w^R v^R w^R u^R)^{3t} = w^R \cdot (v^R w^R u^R w^R)^t v^R \cdot w^R \cdot (u^R w^R v^R w^R)^{2t-1} u^R = \hat{w}\hat{v}\hat{w}\hat{u}.$$

On the other hand, the state

$$\hat{\delta}(\hat{s}_0, \hat{x}a\hat{w}\hat{v}) = \hat{\delta}(\hat{s}_0, z^R a w^R (v^R w^R u^R w^R)^t v^R) = \hat{\delta}(\hat{s}_0, z^R a (w^R v^R w^R u^R)^t w^R v^R)$$

has a loop with label

$$(w^R u^R w^R v^R)^{3t} = w^R \cdot (u^R w^R v^R w^R)^{2t-1} u^R \cdot w^R \cdot (v^R w^R u^R w^R)^t v^R = \hat{w}\hat{u}\hat{w}\hat{v}.$$

One verifies that $\hat{x}, \hat{u}, \hat{v}, \hat{w}, \hat{z} \in A^*$ and $a \in A$ with $|\hat{w}| = k$ witness that $\hat{\mathcal{M}}$ has pattern $\mathbb{D}_k^{\text{rev}}$.

Conversely, suppose that $\hat{\mathcal{M}}$ has pattern $\mathbb{D}_k^{\text{rev}}$ witnessed by $x, u, v, w, z \in A^*$ and $a \in A$ with $|w| = k$. We assume without loss of generality $\hat{\delta}(\hat{s}_0, xawuz) \in \hat{S}'$ and $\hat{\delta}(\hat{s}_0, xawvwuz) \notin \hat{S}'$. Then for all $l \geq 0$ we have $xawu(wvwu)^l z \in L^R$ and $xa(wvwu)^{l+1} z \notin L^R$. So for all $l \geq 1$ we know that $z^R(u^R w^R v^R w^R)^l u^R w^R a x^R \in L$ and $z^R(u^R w^R v^R w^R)^l a x^R \notin L$.

Due to the finiteness of S there are $r, t \geq 1$ such that $\delta(s_0, z^R(u^R w^R v^R w^R)^r) = \delta(s_0, z^R(u^R w^R v^R w^R)^{r+t})$. We define

$$\begin{aligned}
\hat{x} &=_{\text{def}} z^R(u^R w^R v^R w^R)^r, \\
\hat{u} &=_{\text{def}} u^R, \\
\hat{v} &=_{\text{def}} (v^R w^R u^R w^R)^{t-1} v^R, \\
\hat{w} &=_{\text{def}} w^R \text{ and} \\
\hat{z} &=_{\text{def}} x^R.
\end{aligned}$$

Then the state $\delta(s_0, \hat{x}) = \delta(s_0, z^R(u^R w^R v^R w^R)^r)$ has a loop with label

$$(u^R w^R v^R w^R)^t = u^R \cdot w^R \cdot (v^R w^R u^R w^R)^{t-1} v^R \cdot w^R = \hat{u}\hat{w}\hat{v}\hat{w}.$$

It is easy to verify that $\hat{x}, \hat{u}, \hat{v}, \hat{w}, \hat{z} \in A^*$ and $a \in A$ with $|\hat{w}| = k$ witness that $\mathcal{M}$ has the pattern $\mathbb{D}_k$. ❑

Proposition 3.7. *Let $k \geq 0$. Let $\mathcal{M}$ and $\hat{\mathcal{M}}$ be two* DFA*'s such that $L(\mathcal{M}) = L(\hat{\mathcal{M}})$. Then $\mathcal{M}$ has pattern $\mathbb{D}_k$ if and only if $\hat{\mathcal{M}}$ has pattern $\mathbb{D}_k$. The same holds for $\mathbb{D}_k^{\text{rev}}$.*

Proof. It suffices to show one implication. So suppose $\mathcal{M}$ has pattern $\mathbb{D}_k$ and let $\mathcal{M}'$ be some DFA with $L(\mathcal{M}') = L(\mathcal{M})^R$. By Proposition 3.6 we see that $\mathcal{M}'$ has pattern $\mathbb{D}_k^{\text{rev}}$. Now observe that $L(\mathcal{M}') = L(\hat{\mathcal{M}})^R$ so again by Proposition 3.6 we obtain that $\hat{\mathcal{M}}$ has pattern $\mathbb{D}_k$. This can also be carried out for pattern $\mathbb{D}_k^{\text{rev}}$. ❑

3.2.2 Forbidden Pattern Characterization: Proof of Theorem 3.5

We show the two inclusions of the first statement of Theorem 3.5 in Lemma 3.8 and Lemma 3.11. The second statement of Theorem 3.5 is an easy consequence of the first statement and Proposition 3.6. Recall that the duality via reversion of languages holds also between $\mathcal{D}_k^{\text{left}}$ and $\mathcal{D}_k^{\text{right}}$.

Lemma 3.8. *Let $k \geq 0$. It holds that $\mathcal{D}_k^{\text{left}} \subseteq \mathcal{FP}(\mathbb{D}_k)$.*

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA with $L(\mathcal{M}) \in \mathcal{D}_k^{\text{left}}$. We assume to the contrary that $\mathcal{M}$ has pattern $\mathbb{D}_k$ and show that this leads to a contradiction.

Suppose $\mathcal{M}$ has pattern $\mathbb{D}_k$ via $x, u, v, w, z \in A^*$ and $a \in A$ such that $|w| = k$. Without loss of generality we may assume that $\delta(s_0, xaz) \in S'$ and $\delta(s_0, xuwaz) \notin S'$. Then for all $l \geq 0$ it holds that $x(uwvw)^l az \in L(\mathcal{M})$ and $x(uwvw)^l uwaz \notin L(\mathcal{M})$. Since $L(\mathcal{M})$ is a finite union of left k-deterministic languages, there is a one such language L such that $x(uwvw)^l az \in L$ for infinitely many l. By definition, there exist $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ such that $L = (\Sigma_1, \alpha_1, \ldots, \Sigma_n, \alpha_n, \Sigma_{n+1}, \alpha_{n+1})_k$ and for $1 \leq i \leq n$ it holds that $\alpha_i \notin \Sigma_i$. We consider the k-decompositions of the selected words $x(uwvw)^l az \in L$ and look in particular at two parts of it, namely at the k-decomposition of $x(uwvw)^l$ and at the one of waz. Note that if we put them together we get exactly the k-decomposition of $x(uwvw)^l az$ because $|w| = k$. We want to determine the position of the last element of the k-decomposition of $x(uwvw)^l$ in the left k-deterministic representation of L. At first glance, there are two possibilities: it must be that there is some i with $0 \leq i \leq n$ such that for infinitely many l we have

$$x(uwvw)^l \in (\Sigma_1, \ldots, \alpha_i, \Sigma_{i+1})_k \text{ and } waz \in (\Sigma_{i+1}, \alpha_{i+1}, \ldots, \Sigma_{n+1}, \alpha_{n+1})_k \tag{3.1}$$

or there is some i with $1 \leq i \leq n$ ($i \neq n+1$ since $az \neq \varepsilon$) such that for infinitely many l

$$x(uwvw)^l \in (\Sigma_1, \ldots, \Sigma_i, \alpha_i)_k \text{ and } waz \in (\Sigma_{i+1}, \alpha_{i+1}, \ldots, \Sigma_{n+1}, \alpha_{n+1})_k. \tag{3.2}$$

But Lemma 3.3 tells us that case (3.2) is not possible: it cannot be for two distinct values of l that $x(uwvw)^l \in (\Sigma_1, \ldots, \Sigma_i, \alpha_i)_k$ since $\alpha_j \notin \Sigma_j$ for $1 \leq j \leq i$. So let $1 < l_1 < l_2$ such that $x(uwvw)^{l_1}, x(uwvw)^{l_2} \in (\Sigma_1, \ldots, \alpha_i, \Sigma_{i+1})_k$ as stated in (3.1), and define $x' =_{\text{def}} x(uwvw)^{l_1}$ and $y' =_{\text{def}} (uwvw)^{l_2 - l_1}$. We can apply Lemma 3.3 to x' and $x'y'$ and obtain that $\alpha(\widehat{wy'}) \in \Sigma_{i+1}$. Since $l_2 - l_1 \geq 1$ we have in particular that $\alpha(\widehat{wuw}) \in \Sigma_{i+1}$ from which $x(uwvw)^{l_1} uw \in (\Sigma_1, \ldots, \alpha_i, \Sigma_{i+1})_k$ follows. Now we recall from (3.1) that it holds that $waz \in (\Sigma_{i+1}, \alpha_{i+1}, \ldots, \Sigma_{n+1}, \alpha_{n+1})_k$. If we put these pieces together we finally get $x(uwvw)^{l_1} uwaz \in (\Sigma_1, \ldots, \alpha_i, \Sigma_{i+1}, \ldots, \Sigma_{n+1}, \alpha_{n+1})_k \subseteq L(\mathcal{M})$ which is a contradiction. ❑

Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA. We start working our way towards the reverse implication and give a finite decomposition of $L(\mathcal{M})$ into subsets that remain in a strongly connected component (SCC). To do so, we unfold the acyclic graph of SCC's to a tree and emphasize in each SCC on the entry state, the leaving state, and a prefix and suffix of constant length. The absence of pattern $\mathbb{D}_k$ intuitively says that within an SCC it cannot be distinguished between different occurrences of a word of length $k+1$. This will allow a description of the languages in each SCC over A^{k+1}.

A strongly connected component of the transition graph of $\mathcal{M}$ is a maximal set of states $C \subseteq S$ such that there is a path between any two states in C. We allow this path to be empty, so S is a finite union of disjoint SCC's. Let $p \in S$ and set $L(p) =_{\text{def}} \{\, x \in A^+ \mid \delta(s_0, x) = p \,\}$. We immediately obtain

$$L(\mathcal{M}) = \bigcup_{s \in S'} L(s). \tag{3.3}$$

If C is an SCC and $p, q \in C$ we define $L(p, C, q) =_{\text{def}} \{\, x \in A^* \mid \delta(p, x) = q \,\}$ as the set of words that remain in C between two particular states. For any $p \in S$ we have

$$L(p) = \bigcup L(p_1, C_1, q_1) a_1 L(p_2, C_2, q_2) \cdots a_n L(p_{n+1}, C_{n+1}, q_{n+1}) \tag{3.4}$$

where the union ranges over any sequence of distinct SCC's $C_1, \ldots, C_{n+1}$ with $n \geq 0$ such that $p_1 = s_0$, $q_{n+1} = p$, $p_i, q_i \in C_i$ for $1 \leq i \leq n+1$ and $\delta(q_i, a_i) = p_{i+1}$ for $1 \leq i \leq n$. Observe that for any $x \in L(p)$ we can find distinct witnesses $C_1, \ldots, C_{n+1}$ on the path from s_0 to p given by x, and that the above union is finite. For $k \geq 0$ we define

$$L^{<k}(p, C, q) =_{\text{def}} L(p, C, q) \ \cap \ \Big(\bigcup_{i=0}^{k-1} A^i\Big) \quad \text{and} \quad L^{\geq k}(p, C, q) =_{\text{def}} L(p, C, q) \ \cap \ \big(A^k \cup A^{>k}\big).$$

So we split the set $L(p, C, q)$ in the set of words with lengths $< k$ and in the set of words with lengths $\geq k$. Note that also the empty word is considered. It holds that

$$L(p, C, q) = L^{<k}(p, C, q) \cup L^{\geq k}(p, C, q). \tag{3.5}$$

Let $u, v \in A^*$ and define $L^{\geq k}(p, u, C, v, q) =_{\text{def}} L^{\geq k}(p, C, q) \cap (uA^* \cap A^* v)$ as the set of all words in $L^{\geq k}(p, C, q)$ with prefix u and suffix v. Then it holds that

$$L^{\geq k}(p, C, q) = \bigcup_{u, v \in A^k} L^{\geq k}(p, u, C, v, q). \tag{3.6}$$

Now we are ready to give the decomposition of $L(\mathcal{M})$

Lemma 3.9. *Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a* DFA *and let $k \geq 0$. Then $L(\mathcal{M})$ can be written as a finite union of languages $w_0 L_1 a_1 w_1 L_2 a_2 w_2 \cdots L_n a_n w_n L_{n+1}$ with $n \geq 0$, $w_i \in A^*$ for $0 \leq i \leq n$ and there exist distinct SCC's $C_1, \ldots, C_{n+1}$ in the transition graph of $\mathcal{M}$, states $p_i, q_i \in C_i$ for $1 \leq i \leq n+1$, words $u_i, v_i \in A^k$ for $1 \leq i \leq n$ and words $u_{n+1} \in A^k$ and $v_{n+1} \in A^{k+1}$ such that*

1. $\delta(s_0, w_0) = p_1$ *and* $q_{n+1} \in S'$,
2. $\delta(q_i, a_i w_i) = p_{i+1}$ *and* $\delta(q_i, a_i) \notin C_i$ *for* $1 \leq i \leq n$,
3. $L_i = L^{\geq k}(p_i, u_i, C_i, v_i, q_i)$ *for* $1 \leq i \leq n$ *and*
4. $L_{n+1} = L^{<k+1}(p_{n+1}, C_{n+1}, q_{n+1}) \cup L^{\geq k+1}(p_{n+1}, u_{n+1}, C_{n+1}, v_{n+1}, q_{n+1})$.

Proof. To obtain the required representation, we start with (3.3), substitute the occurring languages using (3.4) to (3.6), rewrite all finite sets as the finite unions of their elements and apply the identity $L(L' \cup L'') = LL' \cup LL''$. Note that we fix in C_{n+1} the k-prefix u_{n+1} and the $(k+1)$-suffix v_{n+1}, which we can do with a suitable adaption of (3.6). Note also that we may ignore empty sets in the stepwise decomposition just described. So we obtain for each language L_i for $1 \leq i \leq n+1$ a witnessing SCC C_i with the properties stated in the lemma. ❑

The following proposition isolates the pattern arguments used in the proof of Lemma 3.11. For an SCC C set $S(C) =_{\text{def}} \{\, \alpha \in A^{k+1} \mid \text{there exists some } q \in C \text{ such that } \delta(q,\alpha) \in C \,\}$.

Proposition 3.10. *Let $\mathcal{M}$ be a minimal* DFA *and fix a decomposition as given by Lemma 3.9. Then $\mathcal{M}$ has pattern $\mathbb{D}_k$ in all of the following cases.*

1. *There exist $1 \leq i \leq n$ and $x \in A^* v_i$ with $\delta(p_i, x) \in C_i$ and $\delta(p_i, xa_i) \neq \delta(q_i, a_i)$.*
2. *There exists some $x \in A^* v_{n+1}$ with $\delta(p_{n+1}, x) \in C_{n+1}$ and $\delta(p_{n+1}, x) \neq q_{n+1}$.*
3. *There exist $1 \leq i \leq n+1$ and $x \in u_i A^+$ with $\alpha(\widehat{x}) \subseteq S(C_i)$ and $\delta(p_i, x) \notin C_i$.*

Proof. We additionally fix for $1 \leq i \leq n+1$ some state $q_i' \in C_i$ with $\delta(q_i', v_i) = q_i$ (such q_i' exist since $L_i \neq \emptyset$). To see that $\mathcal{M}$ has $\mathbb{D}_k$, we specify in each case some $w \in A^k$ and witnessing states $s_1, \ldots, s_6$ (see Figure 3.1) in the SCC C_i. Note that s_3 is reachable from s_1, and s_4 is reachable from s_2 since they will be in the same SCC, and we only need to argue that s_5 and s_6 are distinct, because $\mathcal{M}$ is minimal.

1. Let $x = x'v_i$ and set $w =_{\text{def}} v_i$, $s_1 =_{\text{def}} \delta(p_i, x)$, $s_2 =_{\text{def}} q_i$, $s_3 =_{\text{def}} q_i'$, $s_4 =_{\text{def}} \delta(p_i, x')$, $s_5 =_{\text{def}} \delta(p_i, xa_i)$, $s_6 =_{\text{def}} \delta(q_i, a_i)$. By assumption, $s_5 \neq s_6$.

2. Let $x = x'v_{n+1}$, and $v_{n+1} = wa$ for $w \in A^k$ and $a \in A$. Then set $s_1 =_{\text{def}} \delta(p_{n+1}, x'w)$, $s_2 =_{\text{def}} \delta(q_{n+1}', w)$, $s_3 =_{\text{def}} q_{n+1}'$, $s_4 =_{\text{def}} \delta(p_{n+1}, x')$, $s_5 =_{\text{def}} \delta(p_{n+1}, x)$, $s_6 =_{\text{def}} q_{n+1}$. By assumption, $s_5 \neq s_6$.

3. Let wa for $w \in A^k$ and $a \in A$ be the leftmost factor of x such that there is a state $s \in C_i$ with $\delta(s, w) \in C_i$ and $\delta(s, wa) \notin C_i$. Such a factor exists because $|x| \geq k+1$ and because x has k-prefix u_i with $\delta(p_i, u_i) \in C_i$. Since $wa \in \alpha(\widehat{x}) \subseteq S(C_i)$ there are also states $s', s'' \in C_i$ such that $\delta(s', wa) = s''$ and $s' \neq s$. Set $s_1 =_{\text{def}} \delta(s', w)$, $s_2 =_{\text{def}} \delta(s, w)$, $s_3 =_{\text{def}} s$, $s_4 =_{\text{def}} s'$, $s_5 =_{\text{def}} s''$, $s_6 =_{\text{def}} \delta(s, wa)$. Then $s_5 \neq s_6$ because $s_5 \in C_i$ and $s_6 \notin C_i$. ❑

Lemma 3.11. *Let $k \geq 0$. It holds that $\mathcal{FP}(\mathbb{D}_k) \subseteq \mathcal{D}_k^{\text{left}}$.*

Proof. Let a language from $\mathcal{FP}(\mathbb{D}_k)$ be given. We can consider the minimal DFA $\mathcal{M}$ accepting this language because $\mathcal{FP}(\mathbb{D}_k)$ is well–defined. We show that any language $L =_{\text{def}} w_0 L_1 a_1 w_1 L_2 a_2 w_2 \cdots L_n a_n w_n L_{n+1}$ from Lemma 3.9 is in $\mathcal{D}_k^{\text{left}}$ if $\mathcal{M}$ does not have pattern $\mathbb{D}_k$.

Step 1. There is no need to fix q_i in the sets L_i for $1 \leq i \leq n$ since any path labeled v_i in C_i and followed by a_i must end in the same state, or we find pattern $\mathbb{D}_k$. Therefore we define for $1 \leq i \leq n$ sets

$$V_i =_{\text{def}} \left\{ x \in A^{\geq k} \,\middle|\, \delta(p_i, x) \in C_i \right\} \cap (u_i A^* \cap A^* v_i)$$

and do the same thing for C_{n+1} regarding just the $(k+1)$-suffix. Let

$$V_{n+1} =_{\text{def}} \left\{ x \in A^{\geq k+1} \,\middle|\, \delta(p_{n+1}, x) \in C_{n+1} \right\} \cap (u_{n+1} A^* \cap A^* v_{n+1}) \,.$$

Clearly, $L_i \subseteq V_i$. With $B =_{\text{def}} L^{<k+1}(p_{n+1}, C_{n+1}, q_{n+1})$ we obtain that

$$L = w_0 V_1 a_1 w_1 V_2 a_2 w_2 \cdots V_n a_n w_n (B \cup V_{n+1}).$$

The inclusion from right to left is an easy consequence of the first and second statement of Proposition 3.10.

Step 2. We are interested in a description of the sets V_i over A^{k+1}. Given some x with $\widehat{x} = (\alpha_1, \ldots, \alpha_l)$ we write for short $(\ldots, \widehat{x}, \ldots)_k$ instead of $(\ldots, \alpha_1, \emptyset, \alpha_2, \emptyset, \ldots, \emptyset, \alpha_l, \ldots)_k$ in this proof. Let $y_0 =_{\text{def}} w_0 u_1$, $y_i =_{\text{def}} v_i a_i w_i u_{i+1}$ for $1 \leq i \leq n$ and $B' =_{\text{def}} v_n a_n w_n B$.

As the first case, assume that $w_0 \neq \varepsilon$ and that $n > 0$. We claim that

$$\begin{aligned} L \;=\; & \bigcup_{y_{n+1} \in B'} (\widehat{y_0}, S(C_1), \widehat{y_1}, S(C_2), \widehat{y_2}, \ldots, S(C_n), \widehat{y_{n+1}})_k \;\cup \\ & (\widehat{y_0}, S(C_1), \widehat{y_1}, S(C_2), \widehat{y_2}, \ldots, S(C_n), \widehat{y_n}, S(C_{n+1}), v_{n+1})_k \,. \end{aligned} \tag{3.7}$$

For the inclusion from left to right in (3.7) it suffices to observe where we find the k-decomposition of an arbitrary $x \in V_i$ for $1 \leq i \leq n+1$ on the right hand side. If $|x| = k$ then $x = u_i = v_i$ and no element from $S(C_i)$ occurs. If $|x| \geq k+1$ then $\delta(p_i, x) \in C_i$ by definition, and hence $\alpha(\widehat{x}) \subseteq S(C_i)$. The prefix u_i and the suffix v_i of x allow the connection to $\widehat{y_{i-1}}$ and $\widehat{y_i}$ before and after each $S(C_i)$ in the description.

For the inclusion from right to left in (3.7) we show for any x with k-prefix u_i, with k-suffix v_i and with $\alpha(\widehat{x}) \subseteq S(C_i)$ (if $|x| \geq k+1$) that $\delta(p_i, x) \in C_i$, and hence $x \in V_i$. In case $1 \leq i \leq n$ and $|x| = k$ we have $\delta(p_i, x) \in C_i$, and if $1 \leq i \leq n$ and $|w| \geq k+1$ we can apply the third statement of Proposition 3.10. Now we look at $i = n+1$. So let x be given with $\widehat{x} = (\alpha_1, \ldots, \alpha_{l-1}, v_{n+1})$ such that $\alpha_j \in S(C_{n+1})$ for $1 \leq j \leq l-1$. We can apply the third statement of Proposition 3.10 again because $v_{n+1} \in S(C_{n+1})$.

The remaining cases deal with little modifications due to lengths of words. No new argument is needed, so we just state the respective representation of L. For the second case let $w_0 = \varepsilon$ and $n > 0$. If V_1 contains only words of length $\geq k+1$ we can state

$$\begin{aligned} L \;=\; \bigcup_{u_1 a \in S(C_1)} \Big[& \bigcup_{y_{n+1} \in B'} (u_1 a, S(C_1), \widehat{y_1}, S(C_2), \ldots, S(C_n), \widehat{y_{n+1}})_k \;\cup \\ & (u_1 a, S(C_1), \widehat{y_1}, S(C_2), \widehat{y_2}, \ldots, S(C_n), \widehat{y_n}, S(C_{n+1}), v_{n+1})_k \Big]. \end{aligned} \tag{3.8}$$

If V_1 contains a word of length k then this word is $u_1 = v_1$ and we add to the union above each set again, this time starting with $(\widehat{y_1}, S(C_2), \widehat{y_2}, \ldots)_k$.

The third case is $n = 0$ and $w_0 \neq \varepsilon$. Then we have $L = w_0 B \cup w_0 V_1$ and see that

$$L \;=\; \big(w_0 B \cap A^{\leq k}\big) \;\cup \bigcup_{\substack{x \in w_0 B \\ |x| > k}} (\widehat{x})_k \;\cup \big(\widehat{w_0 u_1}, S(C_1), v_1\big)_k \,. \tag{3.9}$$

Finally, for the last case suppose $n = 0$ and $w_0 = \varepsilon$. Then $L = B \cup V_1$ and we have

$$L \;=\; B \;\cup \bigcup_{\alpha \in V_1 \cap A^{k+1}} (\emptyset, \alpha)_k \;\cup \bigcup_{u_1 a \in S(C_1)} (u_1 a, S(C_1), v_1)_k \,. \tag{3.10}$$

Step 3. In any of the sets on the right hand side in (3.7) to (3.10) and for any $1 \leq i \leq n$ it holds that $v_i a_i \notin S(C_i)$. Otherwise we can apply the first statement of Proposition 3.10 to find pattern $\mathbb{D}_k$ in C_i. To see that L is left k-deterministic observe that we can insert in (3.7) to (3.10) the empty set at any position in the description over A^{k+1}. Note that in case of $(\ldots, \alpha, \emptyset, \beta, \ldots)_k$ we have fulfilled $\beta \notin \emptyset$. Note that the last index is not subject to the requirement $\alpha_i \notin \Sigma_i$. In our construction we have indeed $v_{n+1} \in S(C_{n+1})$. This completes the proof of the lemma. ❑

With help of the pattern characterization of $\mathcal{D}_k^{\text{left}}$ it is easy to see that these classes are in fact Boolean algebras. Just note that the part of the transition graph of a DFA where a pattern can appear, does not change when inverting acceptance.

Proposition 3.12. *Let $k \geq 0$. It holds that $\mathcal{D}_k^{\text{left}}$ is closed under finite union, finite intersection and complementation. The same holds for $\mathcal{D}_k^{\text{right}}$.*

3.3 Restricted Temporal Logic

We turn to temporal logic, and contribute to the study of the expressive power of its fragments, which are obtained by omitting one or the other of the usual temporal operators *next* ($\mathbf{X}$), *eventually* ($\mathbf{F}$) and *until* ($\mathbf{U}$). With little modifications we use notations from [Wil99]. Formulas of temporal logic TL over an alphabet A are built up from the elements of A as atomic formulas, using the Boolean connectives $\wedge, \vee, \neg$, the unary operators $\mathbf{X}$ and $\mathbf{F}$ and the binary operator $\mathbf{U}$. We interpret formulas over finite words. A fragment TL[.] of TL is a subset of TL where only the use of the temporal operators specified in brackets is allowed, e.g., $\mathrm{TL}[] \subseteq \mathrm{TL}[\mathbf{F}] \subseteq \mathrm{TL}[\mathbf{X},\mathbf{F}] \subseteq \mathrm{TL}[\mathbf{X},\mathbf{F},\mathbf{U}] = \mathrm{TL}$.

In Subsection 3.3.1, Theorem 3.14, we recall known forbidden pattern characterizations of the fragments $\mathrm{TL}[\mathbf{F}]$ and $\mathrm{TL}[\mathbf{X},\mathbf{F}]$ from [CPP93, EW96]. The latter fragment is also known as restricted temporal logic (RTL). Characterizations of this type allow to decide whether or not a given language is definable when only the restricted formalism of the respective fragment can be used. Note that in case of $\mathrm{TL}[\mathbf{F}]$ we are not allowed to specify the next event, while in case of $\mathrm{TL}[\mathbf{X},\mathbf{F}]$ unrestricted use of $\mathbf{X}$ is possible. We fill the room between these two positions and give a comprehensive answer to the question of how many nested uses of $\mathbf{X}$ are needed to express a certain property in restricted temporal logic. As it turns out in Subsection 3.3.2, if we put the bound k on the nesting depth of $\mathbf{X}$ we encounter exactly the right k-deterministic languages (see Theorem 3.17). We subsume our characterizations and discuss consequences in Subsection 3.3.3.

3.3.1 Definitions and Known Results

For a word $x = a_1a_2\cdots a_n \in A^+$ and some $i \in \{1,\ldots,n\}$ we define what it means that an RTL formula φ is true in x at position i, in notation $(x,i) \models \varphi$.

1. $(x,i) \models a$ if and only if $a_i = a$ (for all $a \in A$).
2. $(x,i) \models \varphi \vee \psi$ if and only if $(x,i) \models \varphi$ or $(x,i) \models \psi$ (analogously for $\wedge, \neg$).
3. $(x,i) \models \mathbf{X}\varphi$ if and only if $i < n$ and $(x,i+1) \models \varphi$.
4. $(x,i) \models \mathbf{F}\varphi$ if and only if there exists some j with $i < j \leq n$ and $(x,j) \models \varphi$.

Given an RTL formula φ and a word x we say that x is a model of φ, in notation $x \models \varphi$, if and only if $(x,1) \models \varphi$. We write $L(\varphi) =_{\text{def}} \{\, x \in A^+ \,|\, x \models \varphi \,\}$ for the set of all models of φ and for a class of formulas $\Phi \subseteq \mathrm{RTL}$ we denote by $L(\Phi) =_{\text{def}} \{\, L(\varphi) \,|\, \varphi \in \Phi \,\}$ the class of Φ-definable languages. E.g., for the RTL formula $\varphi = \mathbf{F}(a \wedge \mathbf{X}b)$ we have $L(\varphi) = AA^*abA^*$. Note that the *eventually* operator $\mathbf{F}$ is defined here in a way such that the quantified position is strictly greater than the actual position.

So far we have defined the future version of restricted temporal logic. One can also think of a past version simply by reversing the ordering. We will treat both cases in parallel and use in the proofs the formalism we find more suitable. An easy duality argument then allows to carry over the results from one version to another. If we substitute the temporal operators $\mathbf{X}$ by $\mathbf{Y}$ (*previously*) and $\mathbf{F}$ by $\mathbf{P}$ (*eventually in the past*) we obtain formulas of restricted past temporal logic (RPTL). Again, we define for a given word $w = a_1 a_2 \cdots a_n \in A^+$ and some $i \in \{1, \ldots, n\}$ what it means that such a formula φ is true in w at position i (with 1. and 2. as above).

3' $(w, i) \models \mathbf{Y}\varphi$ if and only if $i > 1$ and $(w, i-1) \models \varphi$.
4' $(w, i) \models \mathbf{P}\varphi$ if and only if there exists some j with $1 \leq j < i$ and $(w, j) \models \varphi$.

Given an RPTL formula φ and a word w we say that w is a model of φ, in notation $w \models \varphi$, if and only if $(w, |w|) \models \varphi$. Furthermore we carry over all definitions introduced for RTL formulas in an obvious way to RPTL formulas, e.g., the definition of fragments and definable languages. We will ensure that it is always clear from the context with what kind of formula we deal. Note that for the RTL formula a we have $L(a) = aA^*$ and for the RPTL formula a it holds that $L(a) = A^*a$. However, there is an easy way to turn an RTL formula into an RPTL formula and vice versa, such that the defined languages are just the reverse of one another, and such that the syntactic structure of the formula is maintained. The dual of an RTL formula (RPTL formula) φ is the RPTL formula (RTL formula) $\overline{\varphi}$ where each occurrence of $\mathbf{X}$ is substituted by $\mathbf{Y}$ ($\mathbf{Y}$ by $\mathbf{X}$) and each occurrence of $\mathbf{F}$ by $\mathbf{P}$ ($\mathbf{P}$ by $\mathbf{F}$, respectively).

Proposition 3.13. *Let $w \in A^+$.*

1. *For any $\varphi \in \mathrm{RTL}$ it holds that $w \models \varphi$ if and only if $w^R \models \overline{\varphi}$.*
2. *For any $\varphi \in \mathrm{RPTL}$ it holds that $w \models \varphi$ if and only if $w^R \models \overline{\varphi}$.*

Proof. We only argue for the first statement, since $\varphi = \overline{\overline{\varphi}}$. Let $\varphi \in \mathrm{RTL}$ and fix some $w = a_1 \cdots a_n \in A^+$. We show by induction on the structure of φ that for all $1 \leq i \leq n$ it holds that $(w, i) \models \varphi$ if and only if $(w^R, n-i+1) \models \overline{\varphi}$. Note that the i-th letter of w is the $(n-i+1)$-th letter of w^R, so if $\varphi = a$ for some $a \in A$ then $(w, i) \models a \iff a_i = a \iff (w^R, n-i+1) \models a$. If φ is a Boolean combination of formulas for which the induction hypothesis holds, there is nothing to prove. Now assume $\varphi = \mathbf{X}\psi$. Then

$$\begin{aligned} (w, i) \models \mathbf{X}\psi &\iff i < n \text{ and } (w, i+1) \models \psi \\ &\overset{\text{hyp}}{\iff} n - i > 0 \text{ and } (w^R, n-i) \models \overline{\psi} \\ &\iff (w^R, n-i+1) \models \mathbf{Y}\overline{\psi} = \overline{\varphi}. \end{aligned}$$

Finally, suppose $\varphi = \mathbf{F}\psi$. Then we can conclude

$$\begin{aligned} (w, i) \models \mathbf{F}\psi &\iff \text{there exists some } i < j \leq n \text{ with } (w, j) \models \psi \\ &\overset{\text{hyp}}{\iff} \text{there exists some } 1 \leq n-j+1 < n-i+1 \text{ with } (w^R, n-j+1) \models \overline{\psi} \\ &\iff (w^R, n-i+1) \models \mathbf{P}\overline{\psi} = \overline{\varphi}. \end{aligned}$$

In particular, this shows $(w, 1) \models \varphi$ if and only if $(w^R, n) \models \overline{\varphi}$. ❑

Now we recall the following.

Theorem 3.14 ([CPP93, EW96]). *Let $k \geq 0$. It holds that*

1. $L(\mathrm{TL}[\mathbf{Y}, \mathbf{P}]) = \mathcal{FP}(\mathbb{D})$,
2. $L(\mathrm{TL}[\mathbf{P}]) = \mathcal{FP}(\mathbb{D}_0)$,
3. $L(\mathrm{TL}[\mathbf{X}, \mathbf{F}]) = \mathcal{FP}(\mathbb{D}^{\mathrm{rev}})$ *and*
4. $L(\mathrm{TL}[\mathbf{F}]) = \mathcal{FP}(\mathbb{D}_0^{\mathrm{rev}})$.

Note that $L(\mathrm{TL}[\mathbf{Y}, \mathbf{P}]) = L(\mathrm{RPTL})$ and $L(\mathrm{TL}[\mathbf{X}, \mathbf{F}]) = L(\mathrm{RTL})$. To take a closer look at the fine structure of the class of RTL-definable languages we define the notion of the next depth of an RTL formula, i.e., we count the number of nested uses of the $\mathbf{X}$ operator. If φ is an RTL formula then $nd(\varphi)$ denotes its next depth. More precisely, we define inductively $nd(a) =_{\mathrm{def}} 0$ for all $a \in A$, $nd(\varphi \vee \psi) = nd(\varphi \wedge \psi) =_{\mathrm{def}} max\{nd(\varphi), nd(\psi)\}$ and $nd(\mathbf{X}\varphi) =_{\mathrm{def}} nd(\varphi) + 1$ ($\neg$ and $\mathbf{F}$ have no effect). Let $k \geq 0$ and denote with $\mathrm{TL}[\mathbf{X}(k), \mathbf{F}] =_{\mathrm{def}} \{\, \varphi \in \mathrm{TL}[\mathbf{X}, \mathbf{F}] \mid nd(\varphi) \leq k \,\}$ the set of all RTL formulas φ having next depth at most k. One easily relates these definitions to Theorem 3.14 due to $\mathrm{TL}[\mathbf{F}] = \mathrm{TL}[\mathbf{X}(0), \mathbf{F}]$ and $\mathrm{RTL} = \mathrm{TL}[\mathbf{X}, \mathbf{F}] = \bigcup_{k \geq 0} \mathrm{TL}[\mathbf{X}(k), \mathbf{F}]$. Within the next hierarchy $\{\, L(\mathrm{TL}[\mathbf{X}(k), \mathbf{F}]) \mid k \geq 0 \,\}$ a language has next depth k if it is in one of these classes for minimal k. Analogously, one defines the previously depth of an RPTL formula and the previously hierarchy. Note that a language L has next depth k if and only if L^R has previously depth k, which is an easy consequence of Proposition 3.13.

Fortunately the next operator $\mathbf{X}$ shows a nice property: it commutes with $\wedge, \vee$ and $\mathbf{F}$ (even with $\mathbf{U}$), and also in case of negation we can do something similar. The following switching rules [Eme90] allow us to bring all $\mathbf{X}$ operators next to atomic formulas.

Proposition 3.15. *Let $\varphi, \psi \in \mathrm{RTL}$ and let $w = a_1 \cdots a_n \in A^+$. For all $1 \leq i \leq n$ it holds that*

1. $(w, i) \models \mathbf{XF}\varphi \iff (w, i) \models \mathbf{FX}\varphi$,
2. $(w, i) \models \mathbf{X}(\varphi \vee \psi) \iff (w, i) \models (\mathbf{X}\varphi \vee \mathbf{X}\psi)$,
3. $(w, i) \models \mathbf{X}(\varphi \wedge \psi) \iff (w, i) \models (\mathbf{X}\varphi \wedge \mathbf{X}\psi)$ *and*
4. $(w, i) \models \mathbf{X}\neg\varphi \iff (w, i) \models ((\neg\mathbf{X}\varphi) \wedge (\mathbf{X} \bigvee_{a \in A} a))$.

Proof. We only show the first and the last statement since the others are easily seen. Let $\varphi \in \mathrm{RTL}$ and $w = a_1 \cdots a_n \in A^+$ and suppose first that $(w, i) \models \mathbf{XF}\varphi$ for some i with $1 \leq i \leq n$. So $i < n$ and there exists some j with $i + 1 < j \leq n$ such that $(w, j) \models \varphi$. Then $j' =_{\mathrm{def}} j - 1$ witnesses that $(w, i) \models \mathbf{FX}\varphi$. Conversely, assume $(w, i) \models \mathbf{FX}\varphi$. Then there is some j with $i < j \leq n$ such that $j < n$ and $(w, j + 1) \models \varphi$. For $j' =_{\mathrm{def}} j + 1$ it holds that $i + 1 < j' \leq n$ and $(w, j') \models \varphi$, so $(w, i) \models \mathbf{XF}\varphi$.

To see the last statement, suppose $(w, i) \models \mathbf{X}\neg\varphi$. Then $i < n$ and it is not the case that $(w, i+1) \models \varphi$. Additionally, since $i < n$ it holds that $(w, i) \models (\mathbf{X} \bigvee_{a \in A} a)$. Conversely, assume $(w, i) \models (\neg\mathbf{X}\varphi)$ and $(w, i) \models (\mathbf{X} \bigvee_{a \in A} a)$. From the first part we known that $i < n$ is not true or it is not the case that $(w, i + 1) \models \varphi$. From the second part we conclude that $i < n$ so together we obtain $i < n$ and it is not the case that $(w, i + 1) \models \varphi$, so $(w, i) \models \mathbf{X}\neg\varphi$. ❑

Observe that in statement 4 we have to ensure that there is at least one more letter right to the actual position in order to establish the implication from right to left. It is important to see that these rules preserve the next depth of the formula in question. This gives rise to the definition of the set $X_k =_{\mathrm{def}} \{\underbrace{\mathbf{X} \cdots \mathbf{X}}_{l \text{ times}} a \mid a \in A, 0 \leq l \leq k\}$ of all atomic formulas a

with a prefix of at most k next operators. Denote by $\langle X_k \rangle$ the set of all RTL formulas built up from elements of X_k using $\vee, \wedge, \neg$ and $\mathbf{F}$. The following proposition is a consequence of Proposition 3.15.

Proposition 3.16. *Let $k \geq 0$. It holds that $L(\mathrm{TL}[\mathbf{X}(k), \mathbf{F}]) = L(\langle X_k \rangle)$.*

3.3.2 From Logic to Languages and Back

We prove in this subsection the following theorem.

Theorem 3.17. *Let $k \geq 0$. It holds that*

1. $L(\mathrm{TL}[\mathbf{Y}(k), \mathbf{P}]) = \mathcal{D}_k^{\mathrm{left}}$ *and*
2. $L(\mathrm{TL}[\mathbf{X}(k), \mathbf{F}]) = \mathcal{D}_k^{\mathrm{right}}$.

Again due to the duality argument we may restrict ourselves to one statement. This time we show the second one in Lemma 3.18 and Lemma 3.19 (together with Proposition 3.16).

Lemma 3.18. *Let $k \geq 0$. It holds that $L(\langle X_k \rangle) \subseteq \mathcal{D}_k^{\mathrm{right}}$.*

Proof. Let $\varphi \in \langle X_k \rangle$. We show $L(\varphi) \in \mathcal{D}_k^{\mathrm{right}}$ by induction on the structure of φ. For the induction base let $\varphi \in X_k$, so $\varphi = \underbrace{\mathbf{X} \cdots \mathbf{X}}_{l \text{ times}} a$ for some $a \in A$ and $0 \leq l \leq k$. Then $L(\varphi) = A^l a A^*$. For $i, j \geq 0$ we define $B_{i,j} =_{\mathrm{def}} A^i a A^j$. One verifies that

$$L(\varphi) \quad = \quad \bigcup_{0 \leq j < k-l} B_{l,j} \quad \cup \quad \bigcup_{\alpha \in B_{l,k-l}} (\alpha, A^{k+1})_k$$

where we have a finite union of right k-deterministic languages on the right hand side.

By Proposition 3.12 it suffices to show the induction step for $\varphi = \mathbf{F}\psi$ with $L(\psi) \in \mathcal{D}_k^{\mathrm{right}}$. It holds that $L(\mathbf{F}\psi) = AA^*L(\psi) = \bigcup_{a \in A} aA^*L(\psi)$. Suppose $L(\psi)$ is a finite union of right k-deterministic languages with some finite set $D \subseteq A^{\leq k}$. We are done if we prove for any member L of this union that $aA^*L \in \mathcal{D}_k^{\mathrm{right}}$.

For $L = D$ and $D' =_{\mathrm{def}} \{\, xz \in A^{k+1} \mid z \in D \,\}$ one verifies that

$$aA^*D \quad = \quad (aA^*D \cap A^{\leq k}) \quad \cup \quad \bigcup_{\alpha \in aA^*D \,\cap\, A^{k+1}} (\alpha, \emptyset)_k \quad \cup \quad \bigcup_{\beta \in aA^k} \; \bigcup_{\gamma \in D'} (\beta, A^{k+1}, \gamma, \emptyset)_k$$

where again we have a finite union of right k-deterministic languages.

Now let $L = (\alpha_1, \Sigma_1, \alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ with $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ such that for $2 \leq i \leq n+1$ it holds that $\alpha_i \notin \Sigma_i$. Obviously, aA^*L is equal to

$$L' \;=_{\mathrm{def}} \; \bigcup_{\alpha \in aA^k} \; (\alpha, A^{k+1}, \alpha_1, \Sigma_1, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k.$$

We may not necessarily have a right k-deterministic representation of L' on the right hand side since possibly $\alpha_1 \in \Sigma_1$. But we can choose in the k-decomposition of a word from L' the rightmost occurrence of α_1 in the sequence of elements from Σ_1. So with $\Sigma_1' =_{\mathrm{def}} \Sigma_1 \backslash \{\alpha_1\}$ we have

$$L' \quad = \quad \bigcup_{\alpha \in aA^k} \; (\alpha, A^{k+1}, \alpha_1, \Sigma_1', \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$$

with $\alpha_i \notin \Sigma_i$ for $2 \leq i \leq n+1$ by hypothesis and $\alpha_1 \notin \Sigma_1'$. ❑

Lemma 3.19. *Let $k \geq 0$. It holds that $\mathcal{D}_k^{\text{right}} \subseteq L(\langle X_k \rangle)$.*

Proof. To define a finite set D with words of length at most k by a formula of next depth $\leq k$, it suffices to show this for some $w = a_1 \cdots a_n$ with $a_i \in A$ and $n \leq k$. We obtain $L(\varphi) = \{w\}$ with

$$\varphi =_{\text{def}} \Big(\bigwedge_{0 \leq i \leq n-1} \underbrace{\mathbf{X} \cdots \mathbf{X}}_{i \text{ times}} a_{i+1} \Big) \;\wedge\; \neg \Big(\bigvee_{a \in A} \underbrace{\mathbf{X} \cdots \mathbf{X}}_{n \text{ times}} a \Big).$$

Note that we can define the empty set with the second conjunct of φ setting $n = 0$.

It remains to prove for a non–empty right k-deterministic language L that there is an RTL formula $\varphi \in \langle X_k \rangle$ with $L(\varphi) = L$. Let us look at some $L = (\alpha_1, \Sigma_1, \alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ with $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ such that for $2 \leq i \leq n+1$ it holds that $\alpha_i \notin \Sigma_i$. For a language L' denote by $\mathit{suf}(L') =_{\text{def}} \{\, y \in A^{\geq k+1} \mid \text{there exists some } x \in A^* \text{ with } xy \in L' \,\}$ the set of all suffixes of length $\geq k+1$ from words in L'. We make the following observation, which ensures that we always find for certain suffixes y some x such that $xy \in L$.

Suppose $y \in (\alpha_i, \Sigma_i, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ for some $1 \leq i \leq n+1$. Then there is some $x \in A^*$ such that $xy \in L$ since otherwise $L = \emptyset$. Moreover, if $y \in (\beta, \Sigma_i, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ for some $1 \leq i \leq n+1$ and $\beta \in \Sigma_i$ then we can also assume that there is always some $x \in A^*$ such that $xy \in L$. To see this assume for the moment that there is some y for which there is no $x \in A^*$ such that $xy \in L$. But then also for all $y' \in (\beta, \Sigma_i, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ it holds that there is no $x \in A^*$ such that $xy \in L$ because k-decompositions overlap by at most k letters. Hence, we may take $\Sigma_i' =_{\text{def}} \Sigma_i \backslash \{\beta\}$ instead of Σ_i without changing the language. This procedure comes to an end.

For notational convenience we write $\mathbf{G}\phi$ instead of $\neg\mathbf{F}\neg\phi$ for all $\phi \in \text{RTL}$. We define for all $\alpha = a_1 \cdots a_{k+1} \in A^{k+1}$ formulas

$$\psi(\alpha) =_{\text{def}} \bigwedge_{0 \leq l \leq k} \underbrace{\mathbf{X} \cdots \mathbf{X}}_{l \text{ times}} a_{l+1} \quad \text{with} \quad L(\psi(\alpha)) = \alpha A^*$$

and for each $\Sigma \subseteq A^{k+1}$ formulas

$$\psi(\Sigma) =_{\text{def}} \bigvee_{\alpha \in \Sigma} \psi(\alpha) \quad \text{with} \quad L(\psi(\Sigma)) = \Sigma A^*.$$

Moreover, set $\chi =_{\text{def}} \neg\psi(A^{k+1})$ with $L(\chi) = A^{\leq k}$. Then $\psi(\alpha), \psi(\Sigma), \chi \in \langle X_k \rangle$.

The proof is by induction on n. Additionally, we make available in each step a formula φ' defining $\mathit{suf}(L(\varphi))$. For the induction base let $n = 0$. So $L = (\alpha_1, \Sigma_1)_k$ and we set

$$\varphi =_{\text{def}} \psi(\alpha_1) \wedge \mathbf{G}(\psi(\Sigma_1) \vee \chi) \quad \text{and} \quad \varphi' =_{\text{def}} (\psi(\alpha_1) \vee \psi(\Sigma_1)) \wedge \mathbf{G}(\psi(\Sigma_1) \vee \chi).$$

Then $\varphi, \varphi' \in \langle X_k \rangle$ and it holds that $L(\varphi) = L$ and $L(\varphi') = \mathit{suf}(L(\varphi))$.

Now let $n \geq 1$ and $L = (\alpha_1, \Sigma_1, \alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$. By hypothesis, there exist $\varphi_1, \varphi_1' \in \langle X_k \rangle$ such that $L(\varphi_1) = (\alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$ and $L(\varphi_1') = \mathit{suf}(L(\varphi_1))$. We define

$$\begin{aligned} \varphi &=_{\text{def}} \big(\psi(\alpha_1) \wedge \mathbf{F}\varphi_1\big) \wedge \mathbf{G}(\psi(\Sigma_1) \vee \varphi_1' \vee \chi) \qquad \text{and} \\ \varphi' &=_{\text{def}} \Big[\big((\psi(\alpha_1) \vee \psi(\Sigma_1)) \wedge \mathbf{F}\varphi_1\big) \wedge \mathbf{G}(\psi(\Sigma_1) \vee \varphi_1' \vee \chi)\Big] \vee \varphi_1'. \end{aligned}$$

Then $\varphi, \varphi' \in \langle X_k \rangle$ since $\varphi_1, \varphi_1' \in \langle X_k \rangle$. Observe that $L \subseteq L(\varphi)$ is easy to verify: every word w in L starts with α_1 and there is a strict suffix of w from $L(\varphi_1)$. Moreover, for all positions strictly greater than the first, it holds that there begins

– some element from Σ_1 (between α_1 and α_2), or
– a suffix of length $\geq k+1$ of $L(\varphi_1)$, or
– some word of length $\leq k$.

Similarly, we see $suf(L) \subseteq L(\varphi')$. It remains to argue for the reverse inclusions.

Let $w \in L(\varphi)$ with $\widehat{w} = (\beta_1, \ldots, \beta_l)$ for some $l \geq 1$. Then $\beta_1 = \alpha_1$ and there is some strict suffix $z \in L(\varphi_1)$ of w with $\widehat{z} = (\beta_i, \ldots, \beta_l)$ for some i with $2 \leq i \leq l$. If $i = 2$ then no element from Σ_1 occurs, $s_k(\alpha_1) = p_k(\alpha_2)$ and $w \in L$. For $i \geq 3$ we want to show that the definition of φ ensures $\{\beta_2, \ldots, \beta_{i-1}\} \in \Sigma_1$. Suppose that this is not the case. By the formula $\mathbf{G}(\ldots)$ in the definition of φ there is some j with $2 \leq j \leq i-1$ and some $y \in A^+$ such that $\widehat{yz} = (\beta_j, \ldots, \beta_i, \ldots, \beta_l)$ and $yz \in L(\varphi_1')$. Observe that $y \in A^+$ because $j \leq i-1$. Since $L(\varphi_1') = suf(L(\varphi_1))$ there is some $x \in A^*$ with $xyz \in L(\varphi_1)$. Note with the observation from the beginning of the proof that such an x always exists. So we obtain that z and xyz are in $L(\varphi_1) = (\alpha_2, \Sigma_2, \ldots, \alpha_{n+1}, \Sigma_{n+1})_k$. Recall that for $2 \leq i \leq n+1$ we have $\alpha_i \notin \Sigma_i$ and observe that $|xyz| > |z|$. This is a contradiction to the dual version of Lemma 3.3 for right deterministic languages.

The same arguments prove $L(\varphi') \subseteq suf(L)$. ❑

3.3.3 The Next Hierarchy

Taking together Theorems 3.5 and 3.17 we obtain the following characterization of the previously hierarchy and the next hierarchy.

Theorem 3.20. *Let $k \geq 0$. It holds that*

1. $L(\mathrm{TL}[\mathbf{Y}(k), \mathbf{P}]) = \mathcal{D}_k^{\mathrm{left}} = \mathcal{FP}(\mathbb{D}_k)$ *and*
2. $L(\mathrm{TL}[\mathbf{X}(k), \mathbf{F}]) = \mathcal{D}_k^{\mathrm{right}} = \mathcal{FP}(\mathbb{D}_k^{\mathrm{rev}})$.

We immediately obtain $L(\mathrm{RPTL}) = \bigcup_{k \geq 0} \mathcal{D}_k^{\mathrm{left}}$ and $L(\mathrm{RTL}) = \bigcup_{k \geq 0} \mathcal{D}_k^{\mathrm{right}}$. Note also that for $k = 0$ we have given another proof of the second and fourth statement of Theorem 3.14. Using Theorem 3.20 we obtain an NL–algorithm for the membership problem of $\mathcal{D}_k^{\mathrm{left}}$ for fixed k as in case of the forbidden pattern characterizations before. We additionally guess some letter a and a word w of length k (hence count to k, which is a constant to the algorithm), and verify the required reachability conditions. There is a similar algorithm for the classes $\mathcal{D}_k^{\mathrm{right}}$ so the levels of the previously hierarchy and of the next hierarchy have membership problems decidable in NL.

Theorem 3.21. *For fixed $k \geq 0$ the membership problem of $\mathcal{D}_k^{\mathrm{left}}$ is decidable in nondeterministic logarithmic space* NL*. The same holds for $\mathcal{D}_k^{\mathrm{right}}$.*

Recall Proposition 2.19. The same proof shows that if $k \geq |\mathcal{M}|^2$ and $\mathcal{M}$ has pattern $\mathbb{D}_k$ the $\mathcal{M}$ has pattern $\mathbb{D}$. As before, we see how pattern $\mathbb{D}_k$ turns in a natural way to pattern $\mathbb{D}$ as k increases. The same holds for the patterns $\mathbb{D}_k^{\mathrm{rev}}$ and $\mathbb{D}^{\mathrm{rev}}$. Moreover, we can determine the exact level of a language in the previously hierarchy and in the next hierarchy. We simply apply the algorithm for the membership problems for $k = 0, \ldots, |\mathcal{M}|^2$, which is an algorithm that also decides the membership problems of $L(\mathrm{RTL})$ and $L(\mathrm{RPTL})$. In particular, this yields another proof of the remaining statements of Theorem 3.14, in the same way as in case of the patterns $\mathbb{B}_{1/2,k}$ and $\mathbb{B}_{1/2}$. Also a strictness result can be easily achieved with help of our forbidden pattern characterization.

Theorem 3.22. *For all $k \geq 0$ it holds that $\mathcal{D}_k^{\text{left}} \subsetneq \mathcal{D}_{k+1}^{\text{left}}$ and $\mathcal{D}_k^{\text{right}} \subsetneq \mathcal{D}_{k+1}^{\text{right}}$.*

Proof. Suppose a DFA $\mathcal{M}$ has $\mathbb{D}_k$ for some $k \geq 0$. Then it also has $\mathbb{D}_l$ for $0 \leq l \leq k$. So $\mathcal{D}_k^{\text{left}} \subseteq \mathcal{D}_{k+1}^{\text{left}}$. We may assume that there are different letters $a, b \in A$ and define witnessing languages as $L_{k+1} =_{\text{def}} A^*b^{k+1} \backslash A^*b^{k+2}A^*$. If we look at the minimal DFA accepting L_{k+1} we observe that it has pattern $\mathbb{D}_k$ but it does not have pattern $\mathbb{D}_{k+1}$. Thus, by Theorem 3.20 we have $L_{k+1} \in \mathcal{D}_{k+1}^{\text{left}} \backslash \mathcal{D}_k^{\text{left}}$. We may take L_{k+1}^R to see $\mathcal{D}_k^{\text{right}} \neq \mathcal{D}_{k+1}^{\text{right}}$. ❑

3.4 Relations to Concatenation Hierarchies

We have already observed that the patterns $\hat{\mathbb{B}}_{1,k}$ and $\mathbb{D}_k$ are very similar: they differ only in the occurrence of the letter a (compare Figures 2.4 and 3.1). We discuss and clarify the inclusion structure between the occurring classes. First, we give a formal language representation of $\mathcal{FP}(\hat{\mathbb{B}}_{1,k})$. Instead of fixing the last (first) element in the definition of k-deterministic languages, let us fix only the k-suffix (k-prefix) of some last (first) element. We define that L is weak left k-deterministic if and only if there exist $v \in A^k$, $\alpha_1, \ldots, \alpha_n \in A^{k+1}$ and $\Sigma_1, \ldots, \Sigma_{n+1} \subseteq A^{k+1}$ for some $n \geq 0$ such that $L = ((\Sigma_1, \alpha_1, \ldots, \Sigma_n, \alpha_n, \Sigma_{n+1})_k \cap A^*v)$ and for $1 \leq i \leq n$ it holds that $\alpha_i \notin \Sigma_i$. There is an analog definition for weak right k-deterministic languages. We denote the classes of finite unions of such languages as $\hat{\mathcal{D}}_k^{\text{left}}$ and $\hat{\mathcal{D}}_k^{\text{right}}$ (again, we may take some finite set $D \subseteq A^{\leq k}$ to each language). Slight modifications in the proof of Theorem 3.5 allow to establish the following.

Theorem 3.23. *Let $k \geq 0$. It holds that*

1. $\hat{\mathcal{D}}_k^{\text{left}} = \mathcal{FP}(\hat{\mathbb{B}}_{1,k})$ *and*
2. $\hat{\mathcal{D}}_k^{\text{right}} = \mathcal{FP}(\hat{\mathbb{B}}_{1,k}^{\text{rev}})$.

Next we see that the difference in the pattern definitions really causes a difference in the languages classes.

Proposition 3.24. *Let $k \geq 0$. It holds that*

1. $\hat{\mathcal{D}}_k^{\text{left}} \subsetneq \mathcal{D}_k^{\text{left}} \subsetneq \hat{\mathcal{D}}_{k+1}^{\text{left}}$ *and*
2. $\hat{\mathcal{D}}_k^{\text{right}} \subsetneq \mathcal{D}_k^{\text{right}} \subsetneq \hat{\mathcal{D}}_{k+1}^{\text{right}}$.

Proof. We only show the first statement. It is an easy obseration that if a DFA has pattern $\hat{\mathbb{B}}_{k+1}$ then it also has pattern $\mathbb{D}_k$, and if it has $\mathbb{D}_k$ then it also has pattern $\hat{\mathbb{B}}_k$. To see that these inclusions are strict we look again at the witnessing language from Proposition 3.22 and observe that the minimal automaton accepting L_{k+1} has pattern $\mathbb{D}_k$ but it does not even have pattern $\hat{\mathbb{B}}_{k+1}$. On the other hand, the minimal automaton accepting $L'_k =_{\text{def}} A^*b^{k+1}$ has $\hat{\mathbb{B}}_k$ but it does not have $\mathbb{D}_k$. Analog arguments hold for the reverse patterns and languages. ❑

Figure 3.3 gives a summary of the structural results and refines the figure from the beginning of this chapter. We have proved (or provided alternative proofs of) forbidden pattern characterizations of all pictured classes (except for $\mathcal{B}_{3/2} \cap \text{co}\mathcal{B}_{3/2}$ and $\mathcal{L}_{3/2} \cap \text{co}\mathcal{L}_{3/2}$ which we treat in the next chapter). From this, efficient algorithms for their membership problems are easily derived. Since we have for a given DFA $\mathcal{M}$ a bound on k we can exactly locate any $L(\mathcal{M})$ in this landscape. Note that if for some language classes $\mathcal{C}_1$ and $\mathcal{C}_2$ we have $\mathcal{C}_1 = \mathcal{FP}(\mathbb{P}_1)$ and $\mathcal{C}_2 = \mathcal{FP}(\mathbb{P}_2)$ then $\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{FP}(\mathbb{P}_1, \mathbb{P}_2)$.

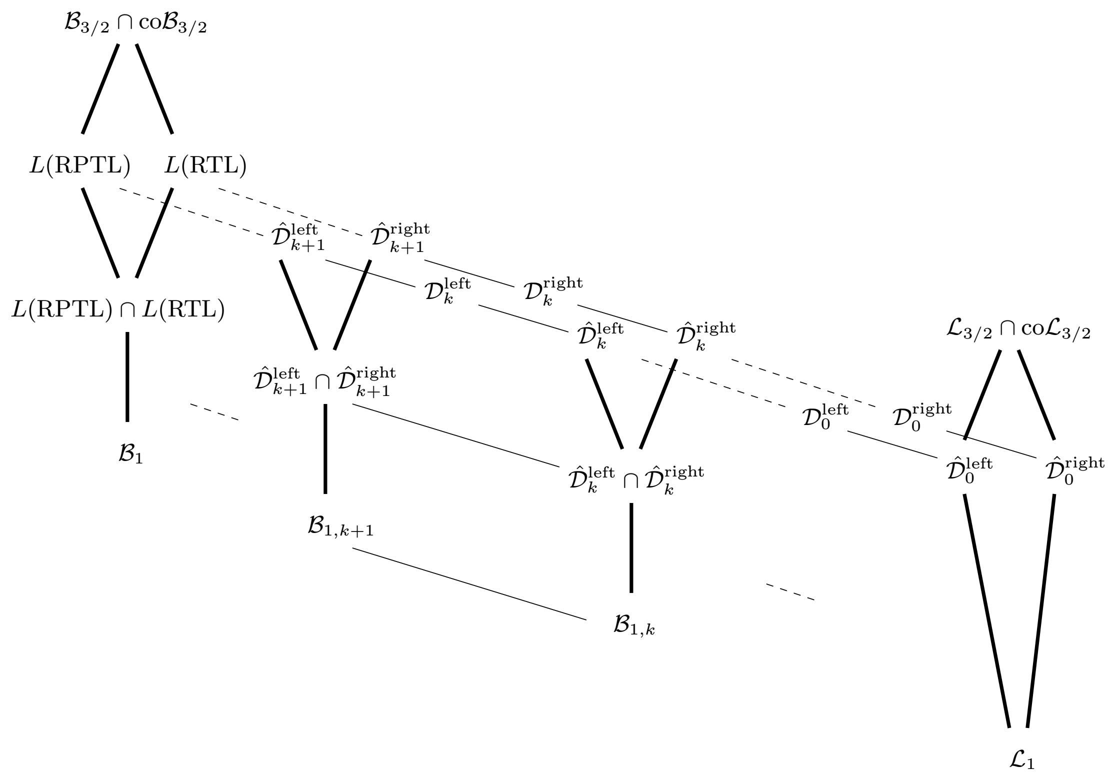


Fig. 3.3. Classes of generalized deterministic languages. All inclusions are strict.

We turn to strictness issues. As pointed out in [EW96] it holds that $L(\mathrm{RTL}) \subseteq \mathcal{B}_{3/2}$. Since the former is a Boolean algebra and the latter is closed under reversion of languages it follows that $L(\mathrm{RTL}) \cup L(\mathrm{RPTL}) \subseteq \mathcal{B}_{3/2} \cap \mathrm{co}\mathcal{B}_{3/2}$. This inclusion is strict because the languages from [EW96] that separate the levels of the until–hierarchy are in $\mathcal{B}_{3/2}$ but not in $L(\mathrm{RTL})$ (the until–hierarchy is defined via the nesting depth of $\mathbf{U}$ in formulas from TL, see [EW96, TW96]).

The case $k = 0$ in Figure 3.3 is somewhat special. Recall from Proposition 2.22 together with Theorem 3.23 that $\mathcal{L}_1 = \hat{\mathcal{D}}_0^{\text{left}} \cap \hat{\mathcal{D}}_0^{\text{right}}$. This equation is known as the fact that a finite monoid is $\mathcal{J}$-trivial if and only if it is $\mathcal{R}$-trivial and $\mathcal{L}$-trivial (see, e.g., [Pin86]), here expressed in terms of forbidden patterns, and $\hat{\mathcal{D}}_0^{\text{left}}$ and $\hat{\mathcal{D}}_0^{\text{right}}$ are the deterministic languages studied in [Eil76].

In contrast, the inclusions $\mathcal{B}_{1,k} \subsetneq \hat{\mathcal{D}}_k^{\text{right}} \cap \hat{\mathcal{D}}_k^{\text{right}}$ are strict for $k \geq 1$ which is due to the following example. Let $\mathcal{M} =_{\text{def}} (A, S, \delta, s_0, S')$ with $A = \{0, 1, a, b\}$, $S =_{\text{def}} \{s_0, \ldots, s_3\}$ and $S' =_{\text{def}} \{s_2\}$. The transition function δ is given in Figure 3.4. Observe that $\mathcal{M}$ is minimal. We claim that $L(\mathcal{M}) \in \hat{\mathcal{D}}_1^{\text{left}} \cap \hat{\mathcal{D}}_1^{\text{right}} \setminus \mathcal{B}_{1,1}$. If we set

- $\hat{x} =_{\text{def}} \varepsilon$, $\hat{y} =_{\text{def}} \varepsilon$, $\hat{y}' =_{\text{def}} \varepsilon$,
- $\hat{u} =_{\text{def}} b$, $\hat{v} =_{\text{def}} a$, $\hat{z} =_{\text{def}} \varepsilon$,
- $\hat{w} =_{\text{def}} 0$ and $\hat{w}' =_{\text{def}} 1$

then $\hat{x}, \hat{y}, \hat{y}', \hat{u}, \hat{v}, \hat{w}, \hat{w}', \hat{z} \in A^*$ and $|\hat{w}| = |\hat{w}'| = 1$ witness hat $\mathcal{M}$ has pattern $\mathbb{B}_{1,1}$. So $L(\mathcal{M}) \notin \mathcal{B}_{1,1}$ by Theorem 2.21. Observe that $\mathcal{M}$ even has pattern $\mathbb{B}_{1,k}$ for all $k \geq 1$, so

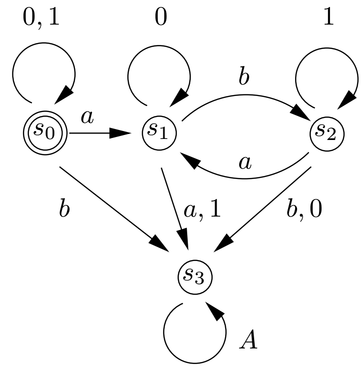


Fig. 3.4. Automaton $\mathcal{M}$ with $L(\mathcal{M}) \in \hat{\mathcal{D}}_1^{\text{left}} \cap \hat{\mathcal{D}}_1^{\text{right}} \setminus \mathcal{B}_{1,1}$.

$L(\mathcal{M}) \notin \mathcal{B}_1$. It remains to show that $\mathcal{M}$ has neither pattern $\hat{\mathbb{B}}_{1,1}$ nor pattern $\hat{\mathbb{B}}_{1,1}^{\text{rev}}$. The former is easily seen, because there is no strongly connected component C in the transition graph of $\mathcal{M}$ such that there is some letter $c \in A$ and two distinct states $s, s' \in C$ for which $\delta(s, c), \delta(s', c) \in C$. The same observation can be made when looking at the minimal DFA accepting $L(\mathcal{M})^R$, so with Theorem 3.23 and Proposition 3.6 we obtain $L(\mathcal{M}) \in \hat{\mathcal{D}}_1^{\text{left}} \cap \hat{\mathcal{D}}_1^{\text{right}}$.

It follows in particular that $\mathcal{B}_1 \subsetneq L(\text{RPTL}) \cap L(\text{RTL})$ and as demonstrated, this is not hard to see once forbidden patterns are known. However, before a first proof of strictness was given (by algebraic methods), it was conjectured for a while that equality might hold [Pin00]. We give an informal interpretation why we encounter strictness for $k \geq 1$ but not for $k = 0$. Let $k \geq 0$ and recall from Theorem 2.21 that $\mathcal{B}_{1,k}$ is characterized by the finiteness condition on the number of alternations in $\preceq_k$-chains. Pattern $\hat{\mathbb{B}}_{1,k}$ induces for $i \geq 1$ a $\preceq_k$-chain

$$x(uwvw)^i z \preceq_k^{\text{e}} x(uwvw)^i uwz \preceq_k^{\text{e}} x(uwvw)^i uwvwz = x(uwvw)^{i+1} z$$

having an infinite number of alternations, where the next alternation happens right to the position of the previous one, i.e., every inserted factor remains untouched. It is sufficient here to have one word w of length k which ensures the needed context condition. The same can be observed in case of pattern $\hat{\mathbb{B}}_{1,k}^{\text{rev}}$ where insertion happens always left to the previous alternation. These two types of $\preceq_k$-chains are forbidden for languages in $\hat{\mathcal{D}}_k^{\text{left}} \cap \hat{\mathcal{D}}_k^{\text{right}}$. But there still exist languages in this class which lead to an infinite number of alternations, because a third type of $\preceq_k$-chain is possible: also factors inserted just into a previously inserted factor may cause alternation. To see this, we look at pattern $\mathbb{B}_{1,k}$ which makes the difference between $\mathcal{B}_{1,k}$ and $\hat{\mathcal{D}}_k^{\text{left}} \cap \hat{\mathcal{D}}_k^{\text{right}}$. Here we obtain for $i \geq 1$ a $\preceq_k$-chain

$$\begin{array}{ll} & x(yw'y'w)^i u(w'vwu)^i z \\ \preceq_k^{\text{e}} & x(yw'y'w)^i yw'vwu(w'vwu)^i z \\ \preceq_k^{\text{e}} & x(yw'y'w)^i yw'y'wuw'vwu(w'vwu)^i z \\ = & x(yw'y'w)^{i+1} u(w'vwu)^{i+1} z \end{array}$$

having an infinite number of alternations. Note that $yw'vw$ is inserted into $y'w \cdot uw'$ and $y'wuw'$ is inserted into $yw' \cdot vw$ and both insertions lead to an alternation. Moreover, w and w' ensure the needed context conditions. As we have pointed out in Remark 2.8, this third type of $\preceq_k$-chain does not appear if $k = 0$ but it cannot be avoided if $k \geq 1$. If $k = 0$ we can insert one letter after another, since no context conditions have to be considered.

3.5 RTL–definable Languages and their Relation to Δ_2^{p}

We turn to the connection of regular languages to complexity classes defined via leaf languages as mentioned in Subsection 1.3.2. First, we show a result of the type of Theorem 1.24, i.e., Δ_2^{p} is just the class of all languages that can be accepted by leaf languages from the Boolean closure of $L(\mathrm{RTL})$ and $L(\mathrm{RPTL})$. Then we make some progress on the question of what single complexity classes are definable by regular leaf languages. If we consider the complexity class defined by a leaf language that is neither in $L(\mathrm{RTL})$ nor in $L(\mathrm{RPTL})$, then this complexity class contains at least Δ_2^{p} or a class from a short list of other classes (cf. Theorem 3.31). With the list we provide hereby, we identify more complexity classes in the upper semilattice of leaf language definable classes. However, since the upper bound Δ_2^{p} for leaf languages from $L(\mathrm{RTL}) \cap L(\mathrm{RPTL})$ meets our lower bound for languages not in $L(\mathrm{RTL}) \cap L(\mathrm{RPTL})$, this does not show a gap in terms of leaf language definability (as, e.g., in [Bor95, BKS98]). But it draws a line such that leaf languages with higher concatenation complexity do not refine the upper semilattice of leaf language definable complexity classes below this line. For background on standard complexity classes we refer to [Pap94].

Lemma 3.25. *It holds that* $\Delta_2^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(L(\mathrm{RTL}))$.

Proof. To see $\Delta_2^{\mathrm{p}} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L(\mathrm{RTL}))$ one may consider a characterization of Δ_2^{p} from [Wag90, Theorem 6.5]. With help of this characterization it is easy to see that $\Delta_2^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(L_1)$ with $L_1 =_{\mathrm{def}} A^*10^*$ and $A =_{\mathrm{def}} \{0,1,2\}$. Since the minimal automaton accepting L_1 does not have pattern $\mathbb{D}^{\mathrm{rev}}$ we have by Theorem 3.14 that $L_1 \in L(\mathrm{RTL})$. Interestingly, this automaton has pattern $\mathbb{D}$ so $L_1 \in L(\mathrm{RTL}) \setminus L(\mathrm{RPTL})$. The language L_1 is in fact a weak right 0-deterministic language. There is also a weak right 1-deterministic language over a two–letter alphabet which can be used instead.

Now let $L \in L(\mathrm{RTL})$. By our previous results, there is some $k \geq 0$ such that $L \in \mathcal{D}_k^{\mathrm{right}}$. First suppose $k = 0$. Since Δ_2^{p} is closed under union it is sufficient to show that $\mathrm{Leaf}^{\mathrm{P}}(L) \subseteq \Delta_2^{\mathrm{p}}$ for some right 0-deterministic language L. So let $a_1, \ldots, a_{n+1} \in A$ and $A_1, \ldots, A_{n+1} \subseteq A$ for some $n \geq 0$ such that for $2 \leq i \leq n+1$ it holds that $a_i \notin A_i$ and $L = a_1 A_1^* a_2 A_2^* \cdots a_{n+1} A_{n+1}^*$. Any language accepted by some nondeterministic polynomial time Turing machine M via leaf language L can also be accepted by the following polynomial time algorithm using an NP–oracle.

1. Find the first computation path p_{n+1} from the right with output a_{n+1} using a binary search and suitable oracle queries. The oracle answers questions of the type "does there exist a path p right of p_{n+1} with output a_{n+1}?" One can check with one more oracle query if all paths right of p_{n+1} produce an output from A_{n+1}.
2. Repeat this procedure for $i = n, n-1, \ldots, 2$, i.e., find the first path p_i left from p_{i+1} with output a_i by a binary search as above. Again, one can check with one oracle query if all paths between p_i and p_{i+1} produce an output from A_i.
3. Check whether all paths between the first path and p_2 have a result from A_1 and whether the first path has the result a_1 with one more oracle query.

Now suppose $k \geq 1$. We adapt the above algorithm and do not only check a single path, but blocks of $k+1$ adjacent paths each time. Since k is a constant to this algorithm we only have a polynomial time increase. ❑

Since $\mathrm{Leaf}^{\mathrm{P}}(L) = \mathrm{Leaf}^{\mathrm{P}}(L^R)$ for all L and because Δ_2^{p} is closed under Boolean operations, we immediately have the following corollary.

Corollary 3.26. *It holds that* $\Delta_2^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(\mathrm{BC}(\mathcal{L}(\mathrm{RTL}) \cup \mathcal{L}(\mathrm{RPTL})))$.

This result can be strengthened to $\Delta_2^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(\mathcal{B}_{3/2} \cap \mathrm{co}\mathcal{B}_{3/2})$ [BSS99]. Now we implement a catalogue of patterns in automata that are typical for certain complexity classes. We define patterns of type 1 through type 4 by drawing their graph in Figures 3.5 to 3.8, respectively, just as in case of forbidden patterns. Additionally, we require the side conditions

– $z \in A^*$, $d, e \in A^+$ and
– there is some $w \in A^+$ such that $\delta(s, w) = s$ for each pictured state s (except $+/-, \ldots$).

Note that from the existence of a pattern of type 1 to 4 it follows that $d \neq e$.

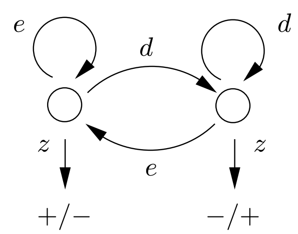


Fig. 3.5. Pattern of type 1.

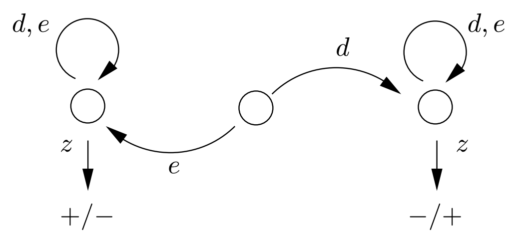


Fig. 3.6. Pattern of type 2.

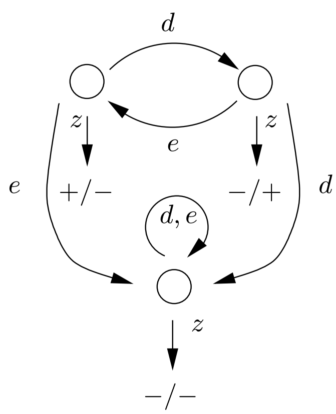


Fig. 3.7. Pattern of type 3.

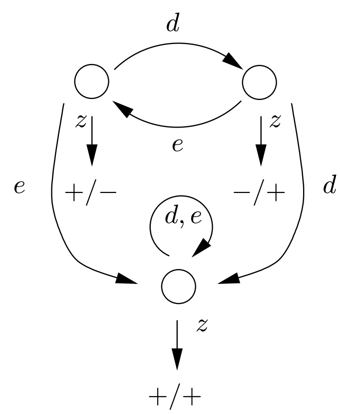


Fig. 3.8. Pattern of type 4.

We introduce the complexity class ACP which is closely related to the just defined patterns.

Definition 3.27. Set $L_2 =_{\mathrm{def}} (0^*10^*2)^*0^*$. We define $\mathrm{ACP} =_{\mathrm{def}} \mathrm{Leaf}^{\mathrm{P}}(L_2)$.

Proposition 3.28. *Let* $L \subseteq A^+$ *and let* $\mathcal{M}$ *be a* DFA *with* $L(\mathcal{M}) = L$.

1. *If* $\mathcal{M}$ *has a pattern of type* 1 *then* $\Delta_2^{\mathrm{p}} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L)$.
2. *If* $\mathcal{M}$ *has a pattern of type* 2 *then* $\Delta_2^{\mathrm{p}} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L)$.
3. *If* $\mathcal{M}$ *has a pattern of type* 3 *then* $\mathrm{ACP} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L)$.
4. *If* $\mathcal{M}$ *has a pattern of type* 4 *then* $\mathrm{coACP} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L)$.

Proof. All statements can be proved similarly. As an example, we show the first statement. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a DFA which has a pattern of type 1, and let $L = L(\mathcal{M})$. Moreover, let $p, q \in S$, $z \in A^*$ and $d, e, w \in A^+$ witness that there is a pattern of type 1 in the transition graph of $\mathcal{M}$ and suppose $\delta(s_0, x) = p$ for some $x \in A^*$. We assume without loss of generality that $\delta(p, z) \notin S'$ and $\delta(q, z) \in S'$. Recall now that $\Delta_2^{\mathrm{p}} = \mathrm{Leaf}^{\mathrm{P}}(B^*10^*)$ with $B =_{\mathrm{def}} \{0, 1, 2\}$ and let $L' \in \mathrm{Leaf}^{\mathrm{P}}(B^*10^*)$ for some language $L' \in \Delta_2^{\mathrm{p}}$ be witnessed by a nondeterministic polynomial time Turing machine M. To see that $L' \in \mathrm{Leaf}^{\mathrm{P}}(L)$ we reconstruct M as follows. We add a leftmost path spanning a computation tree having x as its leafstring, we add a rightmost path spanning a computation tree having z as its leafstring, and for every path of M, if M outputs 0 (1, 2) we append a computation tree with leafstring w (d, e, respectively). ❑

Note that we may consider the patterns of type 1 to 4 as DFA's itself, which shows that the complexity classes from Proposition 3.28 are leaf language definable. Now we prove that we encounter one of the patterns of type 1 to 4 if some DFA has pattern $\mathbb{D}$.

Lemma 3.29. *Let $L \subseteq A^+$ and let $\mathcal{M}$ be the minimal* DFA *with $L(\mathcal{M}) = L$. If $\mathcal{M}$ has pattern $\mathbb{D}$ then $L \notin$* SF *or $\mathcal{M}$ has a pattern of type* 1, 2, 3 *or* 4.

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be the minimal DFA accepting L. We assume that $\mathcal{M}$ has pattern $\mathbb{D}$ and that $\mathcal{M}$ is permutation–free, since otherwise $L \notin$ SF. By Theorem 1.37 there is some $c \geq 1$ such that for all $s \in S$ and all $w \in A^*$ it holds that $\delta(s, w^c) = \delta(s, w^{c+1})$. Under these assumptions we show below by case distinction that we always find in $\mathcal{M}$ one of the patterns of type 1 to 4.

In particular, let $p, q \in S$ and $x, w, u, v, z \in A^*$ with $|w| \geq 1$ witness that $\mathcal{M}$ has pattern $\mathbb{D}$. So $\delta(s_0, x) = p = \delta(p, w) = \delta(q, v)$, $\delta(p, u) = q = \delta(q, w)$, $\delta(p, z) \in S'$ and $\delta(p, z) \notin S'$ (or vice versa). Since $\mathcal{M}$ is permutation–free it holds that $u \neq v$. As a first step we set $u' =_{\mathrm{def}} uw^c$ and $v' =_{\mathrm{def}} vw^c$. Then still $\delta(p, u') = q$ and $\delta(q, v') = p$, but now there is a w-loop at any state in $\mathcal{M}$ that can be reached with u' or v'. We write u and v instead of u' and v' for short.

Case 1. Suppose $\delta(p, v) = p$ or $\delta(q, u) = q$. Without loss of generality we assume $\delta(p, v) = p$. Then we found a pattern of type 1 with $d =_{\mathrm{def}} vu$ and $e =_{\mathrm{def}} v$.

Case 2. Suppose $\delta(p, v) \neq p$ and $\delta(q, u) \neq q$. We define $d =_{\mathrm{def}} (uv)^c u$ and $e =_{\mathrm{def}} (vu)^c v$. Our strategy will be as follows. Let s, r be two distinct states such that $\delta(s, d) = r$ and $\delta(r, e) = s$, and note that this holds in particular for p, q. We define two operations $\longrightarrow_R$ ('right') and $\longrightarrow_L$ ('left') on such pairs (s, r) resulting in one of the desired patterns, in a new pair (s', r') fulfilling certain helpful properties, or in some case we have treated before.

We start with the operation $\longrightarrow_R$ on (s, r) having the above property, i.e., s and r are distinct and $\delta(s, d) = r$ and $\delta(r, e) = s$. Let $p_0 =_{\mathrm{def}} s$ and $q_0 =_{\mathrm{def}} r$. For $1 \leq i \leq c$ set $q_i =_{\mathrm{def}} \delta(q_{i-1}, d)$ and $p_i =_{\mathrm{def}} \delta(q_i, e)$. Then it holds for $0 \leq i \leq c$ that $\delta(q_i, e) = p_i$ and $\delta(p_i, d) = q_i$. Additionally we have $\delta(q_c, d) = q_c$. We distinguish two cases. Assume that $p_c \neq q_c$. Then, since F is minimal, we found a situation as in Case 1. Otherwise we have $p_c = q_c$ and we take the minimal j with $0 \leq j < c$ such that $p_j \neq q_j$. Then $\delta(p_j, d) = q_j$, $\delta(q_j, e) = p_j$ and additionally $\delta(q_j, d) = q_{j+1} = \delta(q_{j+1}, e) = \delta(q_{j+1}, d)$. We keep the pair (p_j, q_j) with $p_j \neq q_j$ as a result of operation $\longrightarrow_R$ on (s, r). Moreover, we see that there is some $0 \leq l < c$ such that $\delta(r, d^l) = q_j$. We denote this operation by $(s, r) \xrightarrow{l}_R (p_j, q_j)$.

The operation $\longrightarrow_L$ on a pair (s, r) is a dual version of $\longrightarrow_R$. We investigate what happens on input e at state s. Let $p_0 =_{\mathrm{def}} s$ and $q_0 =_{\mathrm{def}} r$. Similar as above, for $1 \leq i \leq c$ we set

$p_i =_{\mathrm{def}} \delta(p_{i-1}, e)$ and $q_i =_{\mathrm{def}} \delta(p_i, d)$. Then it holds for $0 \leq i \leq c$ that $\delta(p_i, d) = q_i$ and $\delta(q_i, e) = p_i$. Additionally we have $\delta(p_c, e) = p_c$. If $p_c \neq q_c$ we also find a situation as in Case 1, and otherwise we keep for a minimal j such that $p_j \neq q_j$ the pair (p_j, q_j), where additionally $\delta(p_j, e) = p_{j+1} = \delta(p_{j+1}, e) = \delta(p_{j+1}, d)$. There is some $0 \leq l < c$ such that $\delta(s, e^l) = p_j$. We denote this operation by $(s, r) \xrightarrow{l}_L (p_j, q_j)$.

Now look at the sequence of pairs that results from alternating applications of the operations $\longrightarrow_R$ and $\longrightarrow_L$ to p and q, starting with $\longrightarrow_R$. Set $p_0 =_{\mathrm{def}} p$ and $q_0 =_{\mathrm{def}} q$. Every application produces a pair of distinct states having the described properties, or we are done. So we may assume that for all $i \geq 0$ there are pairs (p_{2i}, q_{2i}) and (p_{2i+1}, q_{2i+1}), and also $l_{2i}, l_{2i+1} \geq 0$ such that $(p_{2i}, q_{2i}) \xrightarrow{l_{2i}}_R (p_{2i+1}, q_{2i+1}) \xrightarrow{l_{2i+1}}_L (p_{2(i+1)}, q_{2(i+1)})$ with $p_{2i} \neq q_{2i}$ and $p_{2i+1} \neq q_{2i+1}$.

Case 2.a. Suppose there is a pair (s, r) that appears in the sequence after an $\longrightarrow_R$ operation and also after an $\longrightarrow_L$ operation. Then there are states s' and r' such that $\delta(s, e) = s' = \delta(s', d) = \delta(s', e)$ and $\delta(r, d) = r' = \delta(r', d) = \delta(r', e)$. If $s' \neq r'$ then we found a pattern of type 2, witnessed by the states s', s, r' and words e and dd. Now assume that $s' = r'$ and let z witness that s and r are distinct. If $\delta(s', z) = \delta(r', z) \notin S'$ then we found a pattern of type 3 witnessed by s, r and s'. If $\delta(s', z) = \delta(r', z) \in S'$ then this is a pattern of type 4.

Case 2.b. Any pair (s, r) of the sequence appears only after $\longrightarrow_R$ operations or it appears only after $\longrightarrow_L$ operations. Then there must be i, j with $1 \leq i < j$ such that $(p_{2i}, q_{2i}) = (p_{2j}, q_{2j})$. By construction, there are states p'_{2i} and q'_{2i+1} such that $\delta(p_{2i}, e) = p'_{2i} = \delta(p'_{2i}, d) = \delta(p'_{2i}, e)$ and $\delta(q_{2i+1}, d) = q'_{2i+1} = \delta(q'_{2i+1}, d) = \delta(q'_{2i+1}, e)$. Moreover, there is some $l_{2i} \geq 0$ with $\delta(q_{2i}, d^{l_{2i}}) = q_{2i+1}$ and in fact $l_{2i} \geq 1$ because otherwise we are in Case 2.a. There is also a word ef with $f \in \{d, e\}^*$ such that $\delta(q_{2i+1}, ef) = p_{2j} = p_{2i}$. Furthermore, it must hold that $p_{2i} \neq q_{2i+1}$ since otherwise $p_{2i} = q_{2i}$ follows. Now we have a situation as in Case 2.a witnessed by the states $p_{2i}, q_{2i+1}, p'_{2i}, q'_{2i+1}$ and words $d' =_{\mathrm{def}} dd^{l_{2i}}$ and $e' =_{\mathrm{def}} ef$. ❑

We can show the same fact in case $\mathcal{M}$ has pattern $\mathbb{D}^{\mathrm{rev}}$.

Lemma 3.30. *Let $L \subseteq A^+$ and let $\mathcal{M}$ be the minimal* DFA *with $L(\mathcal{M}) = L$. If $\mathcal{M}$ has pattern $\mathbb{D}^{\mathrm{rev}}$ then $L \notin \mathrm{SF}$ or $\mathcal{M}$ has a pattern of type* 1, 2, 3 *or* 4.

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be the minimal DFA accepting L. Let $s_1, \ldots, s_5 \in S$ and $x, w, u, v, z \in A^*$ with $|w| \geq 1$ witness that $\mathcal{M}$ has pattern $\mathbb{D}^{\mathrm{rev}}$. Suppose $s_2 \neq s_4$ or $s_3 \neq s_5$. Then we found pattern $\mathbb{D}$ and we can apply Lemma 3.29. On the other hand, if $s_2 = s_4$ and $s_3 = s_5$ then $u \neq v$ and we found a pattern of type 2. ❑

Finally, we obtain the following theorem.

Theorem 3.31. *Let $L \subseteq A^+$ be a regular language.*

1. *If $L \in \mathcal{L}(\mathrm{RTL}) \cap \mathcal{L}(\mathrm{RPTL})$ then $\mathrm{Leaf}^{\mathrm{P}}(L) \subseteq \Delta_2^{\mathrm{p}}$.*
2. *If $L \notin \mathcal{L}(\mathrm{RTL}) \cap \mathcal{L}(\mathrm{RPTL})$ then $\mathrm{Leaf}^{\mathrm{P}}(L)$ contains one of the classes Δ_2^{p}, ACP, coACP or $\mathrm{MOD}_p\mathrm{P}$ for some prime p.*

Proof. Let $L = L(\mathcal{M})$ for the minimal DFA $\mathcal{M}$. The first statement is a consequence of Corollary 3.26. If $L \notin \mathcal{L}(\mathrm{RTL}) \cap \mathcal{L}(\mathrm{RPTL})$ then $\mathcal{M}$ has one of the patterns $\mathbb{D}$ or $\mathbb{D}^{\mathrm{rev}}$ by Theorem 3.14. We may apply Lemma 3.29 or Lemma 3.30 to obtain that $\mathcal{M}$ is not permutation–free or we find a pattern of type 1, 2, 3 or 4 in the transition graph of $\mathcal{M}$. For the former case it is known that $\mathrm{MOD}_p\mathrm{P} \subseteq \mathrm{Leaf}^{\mathrm{P}}(L)$ for some prime p [Bor95]. In the latter case we apply Proposition 3.28. ❑

The classes $\mathrm{MOD}_p\mathrm{P}$ do not come into place, if we consider only star–free languages. Since there is evidence that $\mathrm{MOD}_p\mathrm{P}$ is not a subclass of some level of the polynomial time hierarchy (otherwise the latter collapses, see [Tod91]), we may take a closer look at the case of star–free leaf languages. The relation of ACP to the classes of the polynomial time hierarchy is of particular interest. One can show that $\mathrm{ACP} \subseteq \Pi_2^{\mathrm{p}}$ and that $\Delta_2^{\mathrm{p}} \subseteq \mathrm{ACP}$ if and only if $\mathrm{ACP} = \Pi_2^{\mathrm{p}}$. Certainly, there is more to investigate. In case of the first statement of Theorem 3.31 we have no evidence that this is optimal in the sense that there exists some $L \in L(\mathrm{RTL}) \cap L(\mathrm{RPTL})$ such that $\mathrm{Leaf}^{\mathrm{P}}(L) = \Delta_2^{\mathrm{p}}$. In fact, the typical languages with this property are in $L(\mathrm{RTL}) \setminus L(\mathrm{RPTL})$ or $L(\mathrm{RPTL}) \setminus L(\mathrm{RTL})$ (see the proof of Lemma 3.25). So also here it remains to investigate if there is a presumably smaller class than Δ_2^{p} for which the first statement holds. We have pointed out at the end of the previous section what properties are typical for languages in $L(\mathrm{RTL}) \cap L(\mathrm{RPTL})$.

3.6 Discussion and Bibliographic Notes

We make some more remarks concerning the results of this chapter. Let us consider the generalized deterministic languages first, for which another characterization can be added, this time derived from the theory of finite semigroups. We look only at one of the two dual cases. Denote by $\mathbf{R}$ the variety of finite $\mathcal{R}$-trivial monoids and by $\mathbf{LR}$ the variety of finite locally $\mathcal{R}$-trivial semigroups. It is shown in [Eil76] that $\hat{\mathcal{D}}_0^{\mathrm{left}}$ forms the language variety corresponding to $\mathbf{R}$, and we see from [CPP93] that $L(\mathrm{RPTL})$ forms the language variety corresponding to $\mathbf{LR}$. Moreover, it is known from [Eil76, Chapter V.12] that $\mathbf{LR} = \mathbf{R} * \mathbf{D}$, the variety resulting from the right unitary, semidirect products of members from $\mathbf{R}$ and $\mathbf{D}$ (the variety of definite semigroups). So the general results from [Str85] for varieties of this form can be applied here: it holds that $\mathbf{R} * \mathbf{D}$ is the union of all $\mathbf{R} * \mathbf{D}_{\boldsymbol{k}}$ (the latter is the variety of k-definite semigroups) and each $\mathbf{R} * \mathbf{D}_{\boldsymbol{k}}$ is characterized using certain generalizations of the congruences corresponding to $\mathbf{R}$. Now the following can be done. Starting with the congruences for $\mathbf{R}$ known from [BF80], one can show that languages from $\hat{\mathcal{D}}_k^{\mathrm{left}}$ can be described by the generalized congruences used in [Str85]. It follows that $\hat{\mathcal{D}}_k^{\mathrm{left}}$ is the language variety corresponding to $\mathbf{R} * \mathbf{D}_{\boldsymbol{k}}$. So in this case we have additionally to Theorem 3.23 a levelwise algebraic characterization. This sketches also a way to show that the unions of weak left k-deterministic languages can be made disjoint.

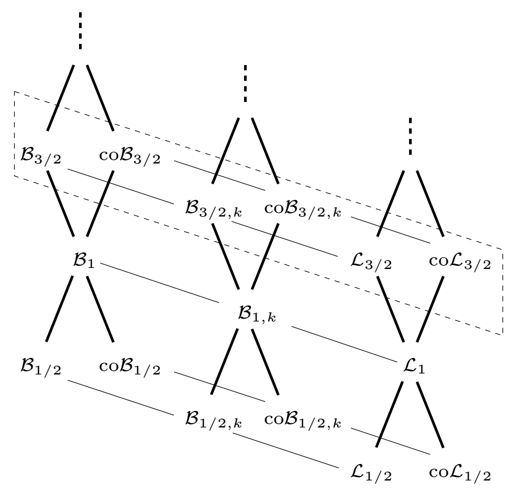
Further investigations may involve, e.g., to find a logical characterization of $\hat{\mathcal{D}}_k^{\mathrm{left}}$, or to look for an algebraic characterization of $\mathcal{D}_k^{\mathrm{left}}$ (using the forbidden patterns, for instance). One can also look at a non–strict version of $\mathbf{F}$: this makes a difference if $\mathbf{F}$ is the only temporal operator, while when arbitrary use of $\mathbf{X}$ is added, both versions have the same expressive power [Wil98]. There are several characterizations known for the classes $\mathcal{L}_{3/2} \cap \mathrm{co}\mathcal{L}_{3/2}$ and $\mathcal{B}_{3/2} \cap \mathrm{co}\mathcal{B}_{3/2}$ in terms of first–order logic restricted to two variables, in terms of restricted temporal logic where future and past opertors are allowed to use at the same time, and in terms of unambiguous languages [EVW97, PW97, TW98].

Recall with Theorem 2.39 the forbidden pattern characterization of $\mathcal{B}_1$. Together with Theorem 1.24 this could be used to look for a theorem on leaf language definability similar to Theorem 3.31. However, to find pattern types that appear in a DFA having pattern $\mathbb{B}_1$ seems to be difficult, because a lot more combinations are possible than in case of pattern $\mathbb{D}$.

The work done in this chapter was initiated by a proof of Theorem 3.17 for $k = 0$, provided by Klaus W. Wagner.

4. Dot–Depth 3/2

In Chapter 2 we have generalized the forbidden pattern $\mathbb{L}_{1/2}$ characterizing $\mathcal{L}_{1/2}$ to show a forbidden pattern characterization of $\mathcal{B}_{1/2,k}$ via pattern $\mathbb{B}_{1/2,k}$. It was fairly easy to establish a bound on k if a DFA is given, which in turn yields the forbidden pattern characterization of $\mathcal{B}_{1/2}$ (see Theorem 2.18 and Proposition 2.19 and the discussion following there). The same observation can be made when going from $\mathcal{D}_0^{\text{left}}$ via $\mathcal{D}_k^{\text{left}}$ to $L(\text{RPTL})$ (see the discussion following Theorem 3.21). In these two cases, the forbidden pattern characterizations of $\mathcal{B}_{1/2}$ and $L(\text{RPTL})$ were previously known, and our refinements lead to other proofs of these results. In this chapter, we exploit this approach one more time: we restate the known forbidden pattern for $\mathbb{L}_{3/2}$ characterizing $\mathcal{L}_{3/2}$ from [PW97] (cf. Theorem 4.2), define a generalized pattern $\mathcal{B}_{3/2,k}$ and provide a forbidden pattern characterization of $\mathcal{B}_{3/2,k}$ (cf. Theorem 4.22).



This yields the decidability of the membership problem of $\mathcal{B}_{3/2,k}$ (cf. Theorem 4.30) and enables us to prove the strictness of the hierarchy of classes $\mathcal{B}_{3/2,k}$ for $k \geq 0$ (cf. Theorem 4.29). However, no forbidden pattern characterization of $\mathcal{B}_{3/2}$ (or any other effective characterization) was known before and the situation is more involved than in the previous chapters. We prove a bound on k from which we derive a forbidden pattern characterization of $\mathcal{B}_{3/2}$ (cf. Theorem 4.32). This implies the decidability of the membership problem of $\mathcal{B}_{3/2}$ (cf. Theorem 4.35) and has consequences in first–order logic (cf. Corollary 4.37). In Section 4.5 we see that the forbidden pattern characterization of $\mathcal{B}_{3/2}$ has also an algebraic interpretation (cf. Theorem 4.38) and we sketch consequences for complexity theory (cf. Theorem 4.40).

First, we develop in Section 4.1 a combinatorial tool that (for a given DFA $\mathcal{M}$) allows to partition words w of arbitrary length into factors w_i of bounded length, such that every second factor w_{2j} is idempotent in $\mathcal{M}$, i.e., w_{2j} leads to a w_{2j}-loop (cf. Theorem 4.3). Moreover, we provide in Section 4.2 a normal form for languages in $\mathcal{B}_{3/2,k}$ (cf. Theorem 4.9) which generalizes (and gives in case $k = 0$ another proof) of the normal form result from [Arf91] stated in Proposition 1.19. Let us first recall the following.

Definition 4.1 ([PW97]). Pattern $\mathbb{L}_{3/2}$ is defined as the subgraph given in Figure 4.1 with $x, z \in A^*$, $v, w \in A^+$ and $\alpha(vwv) \subseteq \alpha(vv)$.

Note that the subgraph in Figure 4.1 is the same as in case of pattern $\mathbb{B}_{1/2}$, but has the additional side condition $\alpha(vwv) \subseteq \alpha(vv)$ (recall also that $\alpha(x)$ denotes the set of letters occurring in x for any $x \in A^*$). The latter is equivalent to $\alpha(w) \subseteq \alpha(v)$ but in order to have a uniform treatment with our generalizations below we state it this way. It holds that $\mathcal{FP}(\mathbb{L}_{3/2})$ is well–defined as can be seen with the arguments used in the following proof.

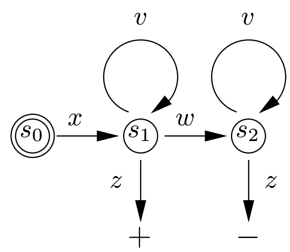


Fig. 4.1. Pattern $\mathbb{L}_{3/2}$ with $\alpha(vwv) \subseteq \alpha(vv)$.

Theorem 4.2 ([PW97]). *It holds that* $\mathcal{L}_{3/2} = \mathcal{FP}(\mathbb{L}_{3/2})$.

Proof. To show the theorem we recall [PW97, Theorem 8.9]. After rewriting their notations we obtain the following (together with Theorem 1.8).

(a) Let $\mathcal{M}$ be a minimal DFA with $L(\mathcal{M}) \subseteq A^*$. Then $L(\mathcal{M}) \in \mathcal{L}_{3/2} \cup \{\, L \cup \{\varepsilon\} \,|\, L \in \mathcal{L}_{3/2} \,\}$ if and only if $\mathcal{M}$ does not have a subgraph in its transition graph as depicted in Figure 4.1 with $x, v, w, z \in A^*$ and $\alpha(w) = \alpha(v)$.

Suppose $L \in \mathcal{L}_{3/2}$ and let $\mathcal{M} = (A, S, \delta, s_0, S')$ be the minimal DFA with $L(\mathcal{M}) = L \subseteq A^+$. Assume that $\mathcal{M}$ has pattern $\mathbb{L}_{3/2}$ via $x \in A^*$, $v, w \in A^+$ and $\alpha(vwv) \subseteq \alpha(vv)$. Now let $w' =_{\mathrm{def}} vwv$ and observe that $\mathcal{M}$ still has pattern $\mathbb{L}_{3/2}$ via $x, z \in A^*$, $v, w' \in A^+$ and $\alpha(w') = \alpha(v)$. We apply (a) and see that $L = L(\mathcal{M}) \notin \mathcal{L}_{3/2} \cup \{\, L \cup \{\varepsilon\} \,|\, L \in \mathcal{L}_{3/2} \,\}$, a contradiction. It follows that there exists some DFA accepting L which does not have pattern $\mathbb{L}_{3/2}$. Hence $L \in \mathcal{FP}(\mathbb{L}_{3/2})$.

Conversely, let $L \in \mathcal{FP}(\mathbb{L}_{3/2})$. So there exists some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L \subseteq A^+$ such that $\mathcal{M}$ does not have pattern $\mathbb{L}_{3/2}$. We assume that $L \notin \mathcal{L}_{3/2}$ and show that this leads to a contradiction. By (a), the minimal DFA $\mathcal{M}'$ accepting L has a subgraph in its transition graph as depicted in Figure 4.1 with $x, v, w, z \in A^*$ and $\alpha(w) = \alpha(v)$. Note that $w \in A^+$ because the states s_1 and s_2 in the pattern are distinct. It follows that also $v \in A^+$.

We argue as in the case of Theorem 2.16 for $\mathbb{B}_{1/2}$. Let $r =_{\mathrm{def}} |\mathcal{M}|$ and define $z' =_{\mathrm{def}} z$, $x' =_{\mathrm{def}} xv^r$, $w' =_{\mathrm{def}} wv^r$ and $v' =_{\mathrm{def}} v^{r!}$. Observe that $x', z' \in A^*$, $v', w' \in A^+$ and that

$$\alpha(v'w'v') = \alpha(v^{r!}wv^r v^{r!}) = \alpha(vw) = \alpha(v) = \alpha(v^{r!}v^{r!}) = \alpha(v'v').$$

We obtain from Proposition 1.34 that x' and w' lead to a v'-loop in $\mathcal{M}$. Moreover, we see from $\mathcal{M}'$ that $x'z' \in L = L(\mathcal{M})$ and $x'w'z' \notin L = L(\mathcal{M})$. So $\mathcal{M}$ has pattern $\mathbb{L}_{3/2}$, a contradiction. ❑

This forbidden pattern characterization implies the decidability of the membership problem of $\mathcal{L}_{3/2}$.

4.1 How to find Automata Loops in Words

A useful tool in further proofs is the fact that we can find factors in a word that lead to loops in a given DFA. It is important here to analyse the length needed to find such a factor, depending on the size of the DFA in question. For this end, we define a bounding function $\mathcal{K}(n)$ as

$$\mathcal{K}(n) =_{\text{def}} (n+1)^{(n+1)^{(n+1)}}$$

and prove in this section the following rather technical theorem. Let δ denote the transition function of the given DFA $\mathcal{M}$.

Theorem 4.3. *For every* DFA $\mathcal{M}$ *and for all* $v_0, \ldots, v_n \in A^+$ *there exist an* $m \geq 0$ *and indices* $0 = i_0 < i_1 < \cdots < i_{2m+1} = n+1$ *such that*

1. $i_{j+1} - i_j \leq \mathcal{K}(|\mathcal{M}|)$ *for* $0 \leq j \leq 2m$ *and*
2. $\delta^{uu} = \delta^u$ *for all* $u = v_{i_j} v_{i_j+1} \cdots v_{i_{j+1}-1}$ *with* $1 \leq j < 2m$ *and* $j \equiv 1 \mod 2$*.*

The proof is given in Subsection 4.1.2. To give some intuition we state what this means for factors of length one, i.e., letters.

Corollary 4.4. *For every* DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ *and every* $w \in A^+$ *there exist words* $w_0, \ldots, w_m, u_1, \ldots, u_m \in A^{\leq \mathcal{K}(|F|)}$ *such that* $w = w_0 u_1 w_1 \cdots u_m w_m$ *and* $\delta^{u_i} = \delta^{u_i u_i}$ *for* $1 \leq i \leq m$*.*

We use Theorem 4.3 for arbitrary factors v_i in the proof of Lemma 4.27 below. This is the main lemma from which we derive the forbidden pattern characterization of $\mathcal{B}_{3/2,k}$.

4.1.1 How to find One Loop

We first show with a rather rough estimation, that $\mathcal{K}(n)$ does not become too small if we repeatedly divide it by n^n. This will make the proof of Lemma 4.7 below better readable.

Proposition 4.5. *Let* $n \geq 1$*,* $m_1 =_{\text{def}} \lfloor \mathcal{K}(n)/2 \rfloor$ *and* $m_{i+1} =_{\text{def}} \lfloor m_i/n^n \rfloor - 1$ *for* $i \geq 1$*. For* $1 \leq i \leq n^n + 1$ *it holds that*

$$m_i \geq (2n^n)^{(n^n+3-i)}.$$

Proof. We will prove the lemma by induction on i with $1 \leq i \leq n^n + 1$. For the induction base let $i = 1$. We distinguish two cases, first suppose $n = 1$. By definition of $\mathcal{K}(n)$, we have $m_1 = 8 = (2n^n)^{(n^n+3-1)}$. Now let $n \geq 2$. By the binomial theorem we have in this case $(n+1)^{n+1} \geq n^{n+1} + (n+1)\,n^n + (n+1)\,n + 1 \geq n^{n+1} + 2n + 2$ and $n^n + n \cdot n^{n-1} \leq (n+1)^n$. So the following estimation shows the induction base.

$$\begin{aligned}(2n^n)^{(n^n+3-1)} = \left(n^n + n\cdot n^{n-1}\right)^{(n^n+3-1)} &\leq (n+1)^{n(n^n+2)}\\ &\leq (n+1)^{\left((n+1)^{(n+1)}-2\right)}\\ &\leq (n+1)^{\left((n+1)^{(n+1)}-1\right)}-1\\ &\leq \frac{(n+1)^{\left((n+1)^{(n+1)}\right)}}{2}-1\\ &\leq \lfloor\mathcal{K}(n)/2\rfloor = m_1\end{aligned}$$

For the induction step, suppose that we have already shown $m_k \geq (2n^n)^{(n^n+3-k)}$ with $1 \leq k < n^n+1$. By definition, $m_{k+1} = \lfloor m_k/n^n\rfloor - 1$. From the induction hypothesis we obtain

$$m_{k+1} \geq \frac{(2n^n)^{(n^n+3-k)}}{n^n} - 2.$$

Since $n \geq 1$ and $k < n^n+1$ we have $(2n^n)^{(n^n+3-k)}/n^n \geq 8$. It follows that

$$m_{k+1} \geq \frac{(2n^n)^{(n^n+3-k)}}{n^n} - 2 = 2\cdot(2n^n)^{(n^n+3-(k+1))} - 2 \geq (2n^n)^{(n^n+3-(k+1))}.$$

❑

The key argument for Lemma 4.7 below is the iterated use of the fact that there is only a finite number of mappings $\delta' : S \to S$ when a finite set S is given. We isolate the iteration step in the following lemma. Let a word v be given with a factorization $v = v_1v_2\cdots v_l$ for sufficiently large l. Among the mappings $\delta^{v_1\cdots v_j}$ some coincide if l is large enough. Suppose for instance, there are x, y, z, v' such that $v = xyzv'$ and $\delta^x = \delta^{xy} = \delta^{xyz}$. Then δ^x leads to a y-loop and also to a z-loop. We repeat this selection procedure on the now coarser factorization $xyzv' = v = v_1v_2\cdots v_l$, and collect the hereby encountered mappings in the set Δ.

In order to make this precise, let $v_0, v_1, \ldots, v_l \in A^+$ and define $v[i,j] =_{\mathrm{def}} v_iv_{i+1}\cdots v_{j-1}$ for all $0 \leq i < j \leq l+1$ as the concatenation of the respective words. We work with indices $i_0, \ldots, i_m$ in order to allow iterated applications.

Lemma 4.6. *Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a* DFA *and let $v_0, v_1, \ldots, v_l \in A^+$ be given. Furthermore, let $0 \leq i_0 < i_1 < \cdots < i_m \leq l$ and suppose that Δ is a set of total mappings $\delta' : S \to S$ such that every $\delta' \in \Delta$ leads to a $v[i_j, i_{j+1}]$-loop for all $0 \leq j < m$. Then there exist indices $i'_0 < i'_1 < \cdots < i'_n$ with $n =_{\mathrm{def}} \lfloor m/(|S|^{|S|})\rfloor$ such that*

1. *$\{i'_0, i'_1, \ldots, i'_n\} \subseteq \{i_0, i_1, \ldots, i_m\}$,*
2. *every $\delta' \in \Delta$ leads to a $v[i'_0, i'_1]$-loop (for $n \geq 1$) and*
3. *every $\delta' \in \Delta \cup \left\{\delta^{v[i'_0,i'_1]}\right\}$ leads to a $v[i'_j, i'_{j+1}]$-loop for all $1 \leq j < n$.*

Proof. First, set $i'_0 =_{\mathrm{def}} i_0$. This shows in particular the lemma for $n = 0$. If $n = 1$ we set $i'_1 =_{\mathrm{def}} i_1$ and we are done. Suppose $n \geq 2$ and set $\delta_{i_j} =_{\mathrm{def}} \delta^{v[i_0,i_j]}$ for $1 \leq j \leq m$. Since there are at most $|S|^{|S|}$ total mappings $S \to S$, there exist mappings appearing several times in the list $\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_m}$. From these mappings we choose a mapping $\overline{\delta}$ that appears most frequently, say $\overline{\delta}$ appears n' times. So $n' \geq \lfloor m/(|S|^{|S|})\rfloor = n$. Let $i'_1, i'_2, \ldots, i'_n \in \{i_1, i_2, \ldots, i_m\}$ such that $i'_1 < i'_2 < \cdots < i'_n$ and $\overline{\delta} = \delta_{i'_j}$ for $1 \leq j \leq n$.

Since $\{i'_1, i'_2, \ldots, i'_n\} \subseteq \{i_1, i_2, \ldots, i_m\}$ we see the first statement. By assumption, every $\delta' \in \Delta$ leads to a $v[i_j, i_{j+1}]$-loop for all $0 \leq j < m$. It follows that every $\delta' \in \Delta$ leads also to a u-loop, where u is an arbitrary concatenation of words $v[i_j, i_{j+1}]$ with $0 \leq j < m$. Particularly, every $\delta' \in \Delta$ leads to a $v[i'_j, i'_{j+1}]$-loop for all $0 \leq j < n$, thus the second statement follows. The same argument shows also the third statement for $\delta' \in \Delta$.

It remains to show that $\delta^{v[i'_0, i'_1]}$ leads to a $v[i'_j, i'_{j+1}]$-loop for all $1 \leq j < n$. By the choice of $\overline{\delta}$ we have that $\delta^{v[i'_0, i'_j]} = \overline{\delta} = \delta^{v[i'_0, i'_{j+1}]}$ for all $1 \leq j < n$ and we see that $\overline{\delta}$ leads to an $v[i'_j, i'_{j+1}]$-loop for all $1 \leq j < n$. Since $\delta^{v[i'_0, i'_1]} = \overline{\delta}$ the third statement follows. ❑

Note that the second statement in the previous lemma and also third statement for $\delta' \in \Delta$ follow immediately from the first statement. We explicitely state them here to focus on what is important in the following proof. We use the same finiteness argument as before: the mapping we add to Δ in Lemma 4.6 cannot always be a new mapping. So if the number of factors we start with is large enough to allow many applications of Lemma 4.6, then we find a mapping δ^u that has already been added to δ before, say δ'. But this means by the second statement of Lemma 4.6 that δ' leads to a u-loop, and hence $\delta' = \delta^u = \delta^{uu}$.

Lemma 4.7. *For every* DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ *and for all* $v_0, v_1, \ldots, v_l \in A^+$ *with* $l =_{\text{def}} \lfloor \mathcal{K}(|\mathcal{M}|)/2 \rfloor$ *there exist* $0 \leq g < h \leq l$ *such that* $\delta^{uu} = \delta^u$ *with* $u =_{\text{def}} v_g v_{g+1} \cdots v_{h-1}$.

Proof. Let $n =_{\text{def}} |\mathcal{M}|$. Initially, let $m^{(1)} =_{\text{def}} l$, $\Delta^{(1)} =_{\text{def}} \emptyset$ and $i_r^{(1)} =_{\text{def}} r$ for $0 \leq r \leq l$. We apply Lemma 4.6 the first time and obtain for $n^{(1)} =_{\text{def}} \lfloor m^{(1)}/n^n \rfloor$ indices ${i'}_r^{(1)}$ with $0 \leq r \leq n^{(1)}$ such that

1. $\left\{ {i'}_r^{(1)} \,\middle|\, 0 \leq r \leq n^{(1)} \right\} \subseteq \left\{ i_r^{(1)} \,\middle|\, 0 \leq r \leq m^{(1)} \right\}$ and
2. $\delta^{v[{i'}_0^{(1)}, {i'}_1^{(1)}]}$ leads to a $v[{i'}_r^{(1)}, {i'}_{r+1}^{(1)}]$-loop for all $1 \leq r < n^{(1)}$.

Now we want to start over after position ${i'}_1^{(1)}$ and set $m^{(2)} =_{\text{def}} n^{(1)} - 1$, $\Delta^{(2)} =_{\text{def}} \Delta^{(1)} \cup \left\{ \delta^{v[{i'}_0^{(1)}, {i'}_1^{(1)}]} \right\}$ and $i_r^{(2)} =_{\text{def}} {i'}_{r+1}^{(1)}$ for $0 \leq r < n^{(1)}$. We apply Lemma 4.6 again.

In general, after the j-th application of Lemma 4.6, we obtain for $n^{(j)} =_{\text{def}} \lfloor m^{(j)}/n^n \rfloor$ the indices ${i'}_r^{(j)}$ with $0 \leq r \leq n^{(j)}$ such that

1. $\left\{ {i'}_r^{(j)} \,\middle|\, 0 \leq r \leq n^{(j)} \right\} \subseteq \left\{ i_r^{(j)} \,\middle|\, 0 \leq r \leq m^{(j)} \right\}$,
2. every $\delta' \in \Delta^{(j)}$ leads to a $v[{i'}_0^{(j)}, {i'}_1^{(j)}]$-loop (for $n^{(j)} > 0$) and
3. every $\delta' \in \Delta^{(j)} \cup \left\{ \delta^{v[{i'}_0^{(j)}, {i'}_1^{(j)}]} \right\}$ leads to a $v[{i'}_r^{(j)}, {i'}_{r+1}^{(j)}]$-loop for all $1 \leq r < n^{(j)}$.

Moreover, with $m^{(j+1)} =_{\text{def}} n^{(j)} - 1$, $\Delta^{(j+1)} =_{\text{def}} \Delta^{(j)} \cup \left\{ \delta^{v[{i'}_0^{(j)}, {i'}_1^{(j)}]} \right\}$ and $i_r^{(j+1)} =_{\text{def}} {i'}_{r+1}^{(j)}$ for $0 \leq r < n^{(j)}$ we can carry out the $(j+1)$-st application of Lemma 4.6.

We chose l at the beginning large enough such that we can apply Lemma 4.6 sufficiently often to face the same mapping twice. This can be seen as follows. By Proposition 4.5 we have that $m^{(j)} \geq (2n^n)^{(n^n+3-j)}$ for $1 \leq j \leq n^n + 1$. It follows that $n^{(j)} = \lfloor m^{(j)}/n^n \rfloor \geq (2n^n)^{(n^n+2-j)} - 1 \geq 1$ for $1 \leq j \leq n^n + 1$. Particularly, the indices ${i'}_0^{(j)}$ and ${i'}_1^{(j)}$ exist for $1 \leq j \leq n^n + 1$.

On one hand, at the end of each step j we take $\delta^{v[{i'}_0^{(j)}, {i'}_1^{(j)}]}$ to $\Delta^{(j)}$ and obtain $\Delta^{(j+1)}$. On the other hand, there are at most n^n total mappings $S \to S$. Therefore, there exists a step

t with $1 \le t \le n^n + 1$ such that $\overline{\delta} =_{\mathrm{def}} \delta^{v[i'^{(t)}_0, i'^{(t)}_1]}$ is already an element of $\Delta^{(t)}$. From the second statement of Lemma 4.6 it follows that $\overline{\delta}$ leads to a $v[i'^{(t)}_0, i'^{(t)}_1]$-loop. With $g =_{\mathrm{def}} i'^{(t)}_0$, $h =_{\mathrm{def}} i'^{(t)}_1$ and $u =_{\mathrm{def}} v_g v_{g+1} \cdots v_{h-1}$ we have $u = v[i'^{(t)}_0, i'^{(t)}_1]$. Thus $\overline{\delta} = \delta^u$ leads to a u-loop and hence $\delta^{uu} = \delta^u$. ❑

4.1.2 Proof of Theorem 4.3

Now we give a proof of Theorem 4.3. If we do not have the particular number l of words, but factors $v_0, v_1, \ldots, v_n$ for arbitrary n, we can partition them in a number of factors such that in each factor there are only $\mathcal{K}(|\mathcal{M}|)$ words v_i, and every second factor u has in fact the property $\delta^{uu} = \delta^u$. Note that we can understand $v_0, v_1, \ldots, v_n$ as some word $v = v_0 v_1 \cdots v_n$ with n markers attached to it. So we obtain for words of arbitrary length a factorization with the described properties.

Proof of Theorem 4.3. Let $l =_{\mathrm{def}} \lfloor \mathcal{K}(|\mathcal{M}|)/2 \rfloor$. If $n < \mathcal{K}(|\mathcal{M}|)$ then we set $m =_{\mathrm{def}} 0$, $i_0 =_{\mathrm{def}} 0$, $i_1 =_{\mathrm{def}} n+1$ and we are done. Otherwise we partition the list $v_1, \ldots, v_n$ from left to right into factors such that every factor contains $l+1$ words v_j. We obtain $m \ge 1$ such factors $B_1, \ldots, B_m$ and $r \le l$ remaining words $v_{n-r+1}, \ldots, v_n$. For every factor $B_t = (v_j, v_{j+1}, \ldots, v_{j+l})$ with $j = (t-1)(l+1)+1$ and $1 \le t \le m$ we apply Lemma 4.7 and we obtain indices $j \le g_t < h_t \le j+l$ such that $\delta^{uu} = \delta^u$ with $u =_{\mathrm{def}} v_{g_t} v_{g_t+1} \cdots v_{h_t-1}$. Now let $i_0 =_{\mathrm{def}} 0$, $i_{2m+1} =_{\mathrm{def}} n+1$ and $i_{2t-1} =_{\mathrm{def}} g_t$, $i_{2t} =_{\mathrm{def}} h_t$ for $1 \le t \le m$. Since $0 = i_0 < i_1 < \cdots < i_{2m+1} = n+1$ we already have the second statement of Theorem 4.3.

It remains to show the first statement. For $1 \le t \le m$ it holds that $i_{2t} - i_{2t-1} = h_t - g_t \le l < \mathcal{K}(|\mathcal{M}|)$. For $1 \le t < m$ we have $B_t = (v_j, v_{j+1}, \ldots, v_{j+l})$ and $B_{t+1} = (v_{j+l+1}, v_{j+l+2}, \ldots, v_{j+2l+1})$ with $j = (t-1)(l+1)+1$. Since

$$j \le g_t < h_t \le j+l \;<\; j+l+1 \le g_{t+1} < h_{t+1} \le j+2l+1$$

it follows that $g_{t+1} - h_t \le (j+2l) - (j+1) = 2l-1 < \mathcal{K}(|\mathcal{M}|)$. Moreover $i_1 - i_0 = g_1 \le l < \mathcal{K}(|\mathcal{M}|)$, so we have shown $i_{j+1} - i_j \le \mathcal{K}(|\mathcal{M}|)$ for $0 \le j \le 2m-1$.

We are left with $i_{2m+1} - i_{2m}$. Observe that $B_m = (v_{n-r-l}, v_{n-r-l+1}, \ldots, v_{n-r})$ and that $i_{2m} = h_m > n-r-l$. So

$$i_{2m+1} - i_{2m} = n+1-i_{2m} \;<\; n+1-n+r+l = r+l+1 \le 2l+1 \le \mathcal{K}(|\mathcal{M}|)+1$$

and hence $i_{2m+1} - i_{2m} \le \mathcal{K}(|\mathcal{M}|)$. **(End proof of Theorem 4.3.)**

4.2 A Normal Form for $\mathcal{B}_{3/2,k}$

By definition, languages in $\mathcal{B}_{3/2,k}$ are finite unions of concatenations of languages, that are in turn Boolean combinations of languages from $\mathcal{B}_{1/2,k}$. In case $k=0$ we have $\mathcal{B}_{3/2,0} = \mathcal{L}_{3/2}$ for which the following normal form is known [Arf91]. Every language from $\mathcal{L}_{3/2}$ can be written as a finite union of languages of the form $A_0^* a_1 A_0^* \cdots a_n A_n^*$ where $n \ge 0$, $a_i \in A$ and $A_i \subseteq A$ (see Proposition 1.19). The natural way to carry this over to arbitrary k is to look at expressions of the form $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ (recall Definition 1.25).

Definition 4.8. Let $k \geq 0$. The class $\widetilde{\mathcal{B}}_{3/2,k}$ is the class of all languages $L \subseteq A^+$ that can be written as a finite union of languages L_i such that $L_i \subseteq A^{\leq k}$ or

$$L_i = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$$

where $m \geq 1$, $\alpha_1, \ldots, \alpha_m \in A^{k+1}$ and $\Sigma_0, \Sigma_1, \ldots, \Sigma_m \subseteq A^{k+1}$.

Note that we require here $m \geq 1$ which is no restriction since $(\Sigma_0)_k$ contains only words of length $\geq k+1$ and hence $(\Sigma_0)_k = \bigcup_{\beta \in \Sigma_0} (\Sigma_0, \beta, \Sigma_0)_k$ (the latter is also true if $\Sigma_0 = \emptyset$). We show in this section that in fact $\mathcal{B}_{3/2,k} = \widetilde{\mathcal{B}}_{3/2,k}$ for all $k \geq 0$, which gives for the special case $k = 0$ another proof of the result from [Arf91].

Theorem 4.9. *Let $k \geq 0$. It holds that $\mathcal{B}_{3/2,k} = \widetilde{\mathcal{B}}_{3/2,k}$.*

The proof of this theorem is given in Subsection 4.2.3. While preparing this proof, we show an even stronger result in the following subsection.

4.2.1 A Normal Form for $\widetilde{\mathcal{B}}_{3/2,k}$

We show with the following theorem that we may assume in the expressions of the form $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ that every $\beta \in \Sigma_i$ appears first as some α_j with $j \leq i$.

Theorem 4.10. *Let $k \geq 0$. Every $L \in \widetilde{\mathcal{B}}_{3/2,k}$ can be written as a finite union of languages L_i such that $L_i \subseteq A^{\leq k}$ or $L_i = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ where $m \geq 1$, $\alpha_1, \ldots, \alpha_m \in A^{k+1}$, $\Sigma_0, \ldots, \Sigma_m \subseteq A^{k+1}$ such that for $0 \leq i \leq m$ it holds that*

$$\Sigma_i \subseteq \{\alpha_1, \ldots, \alpha_i\}.$$

The proof of this theorem is immediate from the next lemma. Some more definitions are needed. In order to distinguish between languages and their formal representation we use now the term *expression* and mean the syntactical object $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k$ that describes a language. Let us formally describe what we mean with the notion of first occurrence.

Definition 4.11. For every $\beta \in A^{k+1}$ and every expression $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k$ with $n \geq 0$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ we define the position of the first occurrence of β as

$$\begin{aligned} \beta_{\min}(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k &=_{\text{def}} \min(\{1 \leq i \leq n \mid \beta = \alpha_i\} \cup \{n+1\}) \text{ and} \\ \beta_{\text{MIN}}(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k &=_{\text{def}} \min(\{0 \leq i \leq n \mid \beta \in \Sigma_i\} \cup \{n+1\}). \end{aligned}$$

Here $\beta_{\min}$ is the leftmost position of β as some α_j in an expression, and β_{MIN} is the index of the leftmost set Σ_i in which β is contained. Next we look at two cardinalities. The first measures the size of an expression and the other one gives the number of words from A^{k+1} for which the property of first occurrence is violated, i.e., it measures the number of different $\beta \in A^{k+1}$ that occur in an expression at first in some set Σ_i. We call this the number of transpositions. These two cardinalities will be used in the proof of the following lemma by two nested inductions.

Definition 4.12. We define the size of an expression and its number of transpositions as

$$\mathrm{size}(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k =_{\mathrm{def}} n + \sum_{0 \le i \le n} |\Sigma_i| \text{ and}$$

$$\mathrm{Tr}\,(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_n, \Sigma_n)_k =_{\mathrm{def}} \left| \left\{ \beta \in A^{k+1} \mid \beta_{\mathrm{MIN}}\,(\Sigma_0, \ldots, \Sigma_n)_k < \beta_{\min}\,(\Sigma_0, \ldots, \Sigma_n)_k \right\} \right|.$$

The following lemma says that we can always find equivalent expressions, i.e., expressions that describe the same language, with zero transpositions. It applies in particular to the languages from $\widetilde{\mathcal{B}}_{3/2,k}$ and if the number of transpositions is zero, then it holds that $\Sigma_i \subseteq \{\alpha_1, \ldots, \alpha_i\}$. So Lemma 4.13 proves Theorem 4.10.

Lemma 4.13. *Let $k \ge 0$. Every language given by an expression $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $m \ge 0$ can be written as a finite union of languages, each of which represented by expressions of the form $(\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$ such that $\mathrm{Tr}\,(\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k = 0$ and $n \ge m$.*

Proof. We show the lemma by induction on $\mathrm{Tr}\,(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$. The induction base is trivial. Now assume that we have shown the lemma for all languages given by expressions $(\Theta_0, \theta_1, \Theta_1, \ldots, \theta_p, \Theta_p)_k$ with $\mathrm{Tr}\,(\Theta_0, \theta_1, \Theta_1, \ldots, \theta_p, \Theta_p)_k \le l$ and we have to show the following claim.

> **Claim.** Every language given by an expression $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $m \ge 0$ and $\mathrm{Tr}\,(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k = l+1$ can be written as a finite union of languages represented by expressions of the form $(\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$ with $\mathrm{Tr}\,(\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k = 0$ and $n \ge m$.

We will prove this claim by a second induction on $\mathrm{size}(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$. If it holds that $\mathrm{size}(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k = 0$, then it must be that $m = 0$ and $\Sigma_0 = \emptyset$. Thus $\mathrm{Tr}\,(\Sigma_0)_k = 0$, which proves the induction base of the second induction. Now assume that claim has been shown for all languages given by expressions $(\Theta_0, \theta_1, \Theta_1, \ldots, \theta_p, \Theta_p)_k$ with $\mathrm{Tr}\,(\Theta_0, \theta_1, \Theta_1, \ldots, \theta_p, \Theta_p)_k = l+1$ and $\mathrm{size}(\Theta_0, \theta_1, \Theta_1, \ldots, \theta_p, \Theta_p)_k \le r$.

Let $E =_{\mathrm{def}} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ be an expression with $\mathrm{Tr}(E) = l+1$ and $\mathrm{size}(E) = r+1$. Thus there exists some $\beta \in A^{k+1}$ and indices $0 \le i_1 < i_2 \le m+1$ with $i_1 = \beta_{\mathrm{MIN}}(E)$ and $i_2 = \beta_{\min}(E)$. We define the expressions

$$\begin{aligned} E_1 &=_{\mathrm{def}} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_{i_1}, \Sigma_{i_1} \setminus \{\beta\}, \qquad\quad \alpha_{i_1+1}, \Sigma_{i_1+1}, \ldots, \alpha_m, \Sigma_m)_k \text{ and} \\ E_2 &=_{\mathrm{def}} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_{i_1}, \Sigma_{i_1} \setminus \{\beta\}, \beta, \Sigma_{i_1}, \alpha_{i_1+1}, \Sigma_{i_1+1}, \ldots, \alpha_m, \Sigma_m)_k. \end{aligned}$$

Since $\beta \in \Sigma_{i_1}$ it is easy to see that the language given by E is just the union of the languages given by E_1 and E_2. Moreover, the length (i.e., number of components) of E_1 and E_2 is greater or equal to the length of E and $\mathrm{size}(E_1) = r$. The expression E_1 was obtained from E by removing β from Σ_{i_1}, thus we have $\beta'_{\mathrm{MIN}}(E) \le \beta'_{\mathrm{MIN}}(E_1)$ and $\beta'_{\min}(E) = \beta'_{\min}(E_1)$ for all $\beta' \in A^{k+1}$. It follows that $\mathrm{Tr}(E_1) \le \mathrm{Tr}(E) = l+1$. If $\mathrm{Tr}(E_1) < l+1$ then the language given by E_1 can be written as a finite union of languages represented by expressions E' with $\mathrm{Tr}(E') = 0$ by the first induction hypothesis. If $\mathrm{Tr}(E_1) = l+1$ then the language given by E_1 can be written as a finite union of languages represented by expressions E' with $\mathrm{Tr}(E') = 0$ by the second induction hypothesis (because $\mathrm{size}(E_1) = r$).

Now we want to show that we reduced the number of transpositions with the construction of E_2 and hence $\mathrm{Tr}(E_2) < l+1$. Let $\beta' \in A^{k+1} \setminus \{\beta\}$. It follows from the definition of E_2 that

${\beta'}_{\mathrm{MIN}}(E_2) \neq i_1 + 1$ because from $\beta' \in \Sigma_{i_1}$ also $\beta' \in \Sigma_{i_1} \setminus \{\beta\}$ would follow. Furthermore, we have

$$
\begin{aligned}
{\beta'}_{\mathrm{MIN}}(E) &= \begin{cases} {\beta'}_{\mathrm{MIN}}(E_2) & : \text{ if } {\beta'}_{\mathrm{MIN}}(E_2) \leq i_1 \\ {\beta'}_{\mathrm{MIN}}(E_2) - 1 & : \text{ otherwise, i.e., if } {\beta'}_{\mathrm{MIN}}(E_2) \geq i_1 + 2, \text{ and} \end{cases} \\
{\beta'}_{\min}(E) &= \begin{cases} {\beta'}_{\min}(E_2) & : \text{ if } {\beta'}_{\min}(E_2) \leq i_1 \\ {\beta'}_{\min}(E_2) - 1 & : \text{ otherwise, i.e., if } {\beta'}_{\min}(E_2) \geq i_1 + 2. \end{cases}
\end{aligned}
$$

Therefore, for every $\beta' \in A^{k+1} \setminus \{\beta\}$ it holds that

$$
{\beta'}_{\mathrm{MIN}}(E_2) < {\beta'}_{\min}(E_2) \quad \Longrightarrow \quad {\beta'}_{\mathrm{MIN}}(E) < {\beta'}_{\min}(E). \tag{4.1}
$$

Since $i_1 = \beta_{\mathrm{MIN}}(E)$, we obtain

$$
\beta_{\mathrm{MIN}}(E_2) \geq \beta_{\min}(E_2). \tag{4.2}
$$

From (4.1), (4.2) and from $\beta_{\mathrm{MIN}}(E) < \beta_{\min}(E)$ it follows that $\mathrm{Tr}(E_2) < \mathrm{Tr}(E) = l + 1$. Hence E_2 can be represented as a finite union of expressions E' with $\mathrm{Tr}(E') = 0$ by the first induction hypothesis. This completes our second induction and proves the claim. So the induction step of the first induction is completed. ❑

4.2.2 Basic Properties of $\widetilde{\mathcal{B}}_{3/2,k}$

We provide some auxiliary results concerning $\widetilde{\mathcal{B}}_{3/2,k}$ and start with closure properties.

Proposition 4.14. *Let $k \geq 0$. It holds that $a^{-1}L \cap A^+, La^{-1} \cap A^+ \in \widetilde{\mathcal{B}}_{3/2,k}$ for $a \in A$ and $L \in \widetilde{\mathcal{B}}_{3/2,k}$.*

Proof. We show that $a^{-1}L \cap A^+ \in \widetilde{\mathcal{B}}_{3/2,k}$ for $L =_{\mathrm{def}} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $m \geq 1$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$. The other case follows from the closure of $\widetilde{\mathcal{B}}_{3/2,k}$ under reversion, which is easy to see from the definition. Note also that $a^{-1}D \cap A^+ \subseteq A^{\leq k}$ for $D \subseteq A^{\leq k}$. By Theorem 4.10 we may assume without loss of generality that $\Sigma_0 = \emptyset$ and that $\Sigma_1 \in \{\{\alpha_1\}, \emptyset\}$. Furthermore, we may also assume that $L \neq \emptyset$. So $\alpha_1 = aw$ for some $a \in A$ and $w \in A^k$ since otherwise $a^{-1}L = \emptyset$. We distinguish two cases.

Case 1. Assume that $m = 1$. If $\Sigma_1 = \emptyset$ then $L = (\emptyset, aw, \emptyset)_k$ and $a^{-1}L \cap A^+ = \{w\} \cap A^+$. If $k = 0$ then the latter set is empty and belongs to $\widetilde{\mathcal{B}}_{3/2,k}$, otherwise $\{w\} \subseteq A^{\leq k}$ and belongs also to $\widetilde{\mathcal{B}}_{3/2,k}$. Now suppose $\Sigma_1 = \{aw\}$. If $p_k(aw) \neq w$ then we may set $\Sigma_1 = \emptyset$ without changing the language. We have treated this before, so suppose $p_k(aw) = w$. Then

$$
a^{-1}L \cap A^+ = a^{-1} (\emptyset, aw, \{aw\})_k \cap A^+ = \left(\{w\} \cup (\{aw\})_k\right) \cap A^+.
$$

Note that $(\{aw\})_k$ contains only words of length $\geq k + 1$ and we argue as before that we have obtained a set in $\widetilde{\mathcal{B}}_{3/2,k}$.

Case 2. Now let $m \geq 2$. It must be that $p_k(\alpha_2) = w$ since otherwise $L = \emptyset$. If $\Sigma_1 = \emptyset$ then

$$
\begin{aligned}
a^{-1}L \cap A^+ &= a^{-1} (\emptyset, aw, \emptyset, \alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \cap A^+ \\
&= (\alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \cap A^+ \\
&= (\emptyset, \alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \qquad \in \widetilde{\mathcal{B}}_{3/2,k}.
\end{aligned}
$$

Finally, suppose $\Sigma_1 = \{aw\}$. Again, if $p_k(aw) \neq w$ then we may set $\Sigma_1 = \emptyset$ without changing the language. We have treated this before, so suppose $p_k(aw) = w$ and recall that $p_k(\alpha_2) = w$. Then

$$\begin{aligned} a^{-1}L \cap A^+ &= a^{-1}(\emptyset, aw, \{aw\}, \alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \cap A^+ \\ &= (\{aw\}, \alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \cap A^+ \\ &= (\{aw\}, \alpha_2, \Sigma_2, \ldots, \alpha_m, \Sigma_m)_k \qquad \in \widetilde{\mathcal{B}}_{3/2,k}. \end{aligned}$$

❑

We show next that $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under polynomial closure. Recall that the polynomial closure is exactly the closure under finite union and concatenation since concatenation distributes over finite unions.

Proposition 4.15. *Let $k \geq 0$. It holds that $\widetilde{\mathcal{B}}_{3/2,k} = \mathrm{Pol}(\widetilde{\mathcal{B}}_{3/2,k})$.*

Proof. We need to argue for the inclusion from right to left. It suffices show for any two languages $L_1, L_2 \in \widetilde{\mathcal{B}}_{3/2,k}$ that $L_1 \cdot L_2 \in \widetilde{\mathcal{B}}_{3/2,k}$ since $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under finite union by definition. For the same reason it remains to consider the following cases.

First, suppose $L_2 \subseteq A^{\leq k}$. If also $L_1 \subseteq A^{\leq k}$ we consider $(L_1 \cdot L_2) \cap A^{\leq k}$ and $(L_1 \cdot L_2) \cap A^{\geq k+1}$ separately. The former language is in $\widetilde{\mathcal{B}}_{3/2,k}$ and to see this for the latter, note that for any word $w \in A^{\geq k+1}$ we have $\{w\} = (\emptyset, \alpha_1, \emptyset, \alpha_2, \ldots \emptyset, \alpha_m, \emptyset)_k$ if $\widehat{w} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

If $L_1 = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $k \geq 0$, $m \geq 1$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ we see that $L_1 \cdot a \in \widetilde{\mathcal{B}}_{3/2,k}$ for some $a \in A$ by

$$L_1 \cdot a = \bigcup_{\beta \in A^k \cdot a} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m, \beta, \emptyset)_k\,.$$

This covers also the case that $L_1 \subseteq A^{\leq k}$ and L_2 has the form $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$.

Finally, suppose $L_1 = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ and $L_2 = (\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$ with $k \geq 0$, $m, n \geq 1$, $\alpha_i, \gamma_i \in A^{k+1}$ and $\Sigma_i, \Gamma_i \subseteq A^{k+1}$, and we want to show $L_1 \cdot L_2 \in \widetilde{\mathcal{B}}_{3/2,k}$. All we have to do is to ensure that there are exactly k elements in the k-decomposition of any word in $L_1 \cdot L_2$ between the rightmost element from Σ_m (or α_m) and the leftmost element from Γ_0 (or γ_1). It holds that

$$L_1 \cdot L_2 = \bigcup_{\beta_1, \ldots, \beta_k \in A^{k+1}} (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m, \beta_1, \emptyset, \beta_2, \ldots, \emptyset, \beta_k, \Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$$

where we have a language from $\widetilde{\mathcal{B}}_{3/2,k}$ on the right hand side. ❑

We can also isolate each single α_i in $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$.

Lemma 4.16. *Let $k \geq 0$ and $L = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ for $m \geq 1$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$. For all $1 \leq h \leq m$ it holds that*

$$L = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_{h-1}, \Sigma_{h-1})_k\, p_k(\alpha_h)^{-1} \cdot \alpha_h \cdot s_k(\alpha_h)^{-1} (\Sigma_h, \alpha_{h+1}, \Sigma_{h+1}, \ldots, \alpha_m, \Sigma_m)_k.$$

Proof. For some fixed h with $1 \le h \le m$ we set $L_1 =_{\text{def}} (\Sigma_0, \alpha_1, \Sigma_1, \dots, \alpha_{h-1}, \Sigma_{h-1})_k$ and $L_2 =_{\text{def}} (\Sigma_h, \alpha_{h+1}, \Sigma_{h+1}, \dots, \alpha_m, \Sigma_m)_k$. We argue first for the inclusion from left to right, so let $w \in L$ with $w = a_1 \cdots a_{l+k}$ and $\widehat{w} = (\beta_1, \dots, \beta_l)$ for $l \ge 1$, $a_i \in A$ and $\beta_i \in A^{k+1}$. By definition, there exist $0 = j_0 < j_1 < j_2 < \dots < j_m < j_{m+1} = l+1$ such that

(a) $\beta_{j_i} = \alpha_i$ for $1 \le i \le m$ and
(b) $\beta_j \in \Sigma_i$ for $0 \le i \le m$ and $j_i < j < j_{i+1}$.

With $w_1 =_{\text{def}} a_1 a_2 \cdots a_{j_h-1}$, $w_2 =_{\text{def}} a_{j_h+k+1} a_{j_h+k+2} \cdots a_{l+k}$ and $w_1' =_{\text{def}} w_1 \cdot p_k(\alpha_h)$, $w_2' =_{\text{def}} s_k(\alpha_h) \cdot w_2$ we obtain

$$\begin{aligned} w_1' &= a_1 a_2 \cdots a_{j_h-1} \cdot a_{j_h} a_{j_h+1} \cdots a_{j_h+k-1} \quad \text{and} \\ w_2' &= a_{j_h+1} a_{j_h+2} \cdots a_{j_h+k} \cdot a_{j_h+k+1} a_{j_h+k+2} \cdots a_{l+k}. \end{aligned}$$

Hence $w = w_1 \alpha_h w_2$, $\widehat{w_1'} = (\beta_1, \dots, \beta_{j_h-1})$ and $\widehat{w_2'} = (\beta_{j_h+1}, \dots, \beta_l)$. It follows that $w_1' \in L_1$ and $w_2' \in L_2$. Thus we obtain $w_1 \in L_1 p_k(\alpha_h)^{-1}$ and $w_2 \in s_k(\alpha_h)^{-1} L_2$. Therefore, we have $w \in L_1 p_k(\alpha_h)^{-1} \cdot \alpha_h \cdot s_k(\alpha_h)^{-1} L_2$.

Conversely, let $w \in L_1 p_k(\alpha_h)^{-1} \cdot \alpha_h \cdot s_k(\alpha_h)^{-1} L_2$, i.e., there exist words $w_1 \in L_1 p_k(\alpha_h)^{-1}$, $w_2 \in s_k(\alpha_h)^{-1} L_2$ such that $w = w_1 \alpha_h w_2$. Observe that $w_1, w_2 \in A^+$ since $L_1, L_2 \subseteq A^{\ge k+1}$. For suitable $a_1, \dots, a_{l+k} \in A$ and $\beta_i \in A^{k+1}$ we have $w = a_1 \cdots a_{l+k}$, $\widehat{w} = (\beta_1, \dots, \beta_l)$ and $\beta_{i_h} = \alpha_h$ with $i_h =_{\text{def}} |w_1| + 1$. Define $w_1', w_2' \in A^+$ such that $\widehat{w_1'} = (\beta_1, \dots, \beta_{i_h-1})$ and $\widehat{w_2'} = (\beta_{i_h+1}, \dots, \beta_l)$ and notice that such w_1', w_2' exist. It follows that $w_1' = a_1 \cdots a_{i_h+k-1}$ and $w_2' = a_{i_h+1} \cdots a_{l+k}$, hence $w_1' = w_1 p_k(\alpha_h)$ and $w_2' = s_k(\alpha_h) w_2$. We obtain that $w_1' \in L_1 = (\Sigma_0, \alpha_1, \Sigma_1, \dots, \alpha_{h-1}, \Sigma_{h-1})_k$ and $w_2' \in L_2 = (\Sigma_h, \alpha_{h+1}, \Sigma_{h+1}, \dots, \alpha_m, \Sigma_m)_k$. Since $\widehat{w} = (\beta_1, \dots, \beta_l)$, $\widehat{w_1'} = (\beta_1, \dots, \beta_{i_h-1})$ and $\widehat{w_2'} = (\beta_{i_h+1}, \dots, \beta_l)$, we conclude $w \in L = (\Sigma_0, \alpha_1, \Sigma_1, \dots, \alpha_m, \Sigma_m)_k$. ❑

In general, we must not distribute concatenation over intersections. The situation changes if the concatenation is at a certain position, namely where some factor β occurs the first time.

Lemma 4.17. *Let $k \ge 0$. Let $C_1, C_2, D_1, D_2 \subseteq A^*$ and $\beta \in A^{k+1}$. Furthermore, set $v =_{\text{def}} p_k(\beta)$ and $w =_{\text{def}} s_k(\beta)$. If $(C_1 \cup D_1) \subseteq (A^* \setminus A^*\beta A^*)$ then it holds that*

$$(C_1 \cap D_1)\, v^{-1} \cdot \beta \cdot w^{-1} (C_2 \cap D_2) = \left(C_1 v^{-1} \cdot \beta \cdot w^{-1} C_2\right) \cap \left(D_1 v^{-1} \cdot \beta \cdot w^{-1} D_2\right).$$

Proof. It suffices to show that

$$\left(C_1 v^{-1} \cap D_1 v^{-1}\right) \cdot \beta \cdot \left(w^{-1} C_2 \cap w^{-1} D_2\right) = \left(C_1 v^{-1} \cdot \beta \cdot w^{-1} C_2\right) \cap \left(D_1 v^{-1} \cdot \beta \cdot w^{-1} D_2\right).$$

We argue first for the inclusion from right to left. So let u be an element of the right hand side. Then there exist words $c_1 \in C_1 v^{-1}$, $c_2 \in w^{-1} C_2$, $d_1 \in D_1 v^{-1}$ and $d_2 \in w^{-1} D_2$ such that $u = c_1 \cdot \beta \cdot c_2 = d_1 \cdot \beta \cdot d_2$. We want to show $c_1 = d_1$. Assume $|c_1 \cdot \beta| < |d_1 \cdot \beta|$. Because $c_1 \cdot \beta$ is a proper prefix of $d_1 \cdot \beta$ there exists an $l \ge 1$ and letters $b_1, \dots, b_l \in A$ such that $d_1 \cdot \beta = c_1 \cdot \beta \cdot b_1 \cdots b_l$. Since $d_1 \in D_1 v^{-1}$ we have $d_1 v = d_1 p_k(\beta) = c_1 \cdot \beta \cdot b_1 \cdots b_{l-1} \in D_1$. This is a contradiction, because $D_1 \subseteq (A^* \setminus A^*\beta A^*)$ by assumption. We obtain $|c_1 \cdot \beta| \ge |d_1 \cdot \beta|$, and the same argument shows $|c_1 \cdot \beta| \le |d_1 \cdot \beta|$. Therefore, $c_1 = d_1$ and $c_2 = d_2$. It follows that $u \in \left(C_1 v^{-1} \cap D_1 v^{-1}\right) \cdot \beta \cdot \left(w^{-1} C_2 \cap w^{-1} D_2\right)$.

Conversely, let $u \in \left(C_1 v^{-1} \cap D_1 v^{-1}\right) \cdot \beta \cdot \left(w^{-1} C_2 \cap w^{-1} D_2\right)$. Then there exits some $u_1 \in \left(C_1 v^{-1} \cap D_1 v^{-1}\right)$ and $u_2 \in \left(w^{-1} C_2 \cap w^{-1} D_2\right)$ such that $u = u_1 \cdot \beta \cdot u_2$. It follows that $u_1 \cdot \beta \cdot u_2$ is an element of $\left(C_1 v^{-1} \cdot \beta \cdot w^{-1} C_2\right)$ and also of $\left(D_1 v^{-1} \cdot \beta \cdot w^{-1} D_2\right)$. ❑

4.2.3 Proof of Theorem 4.9

The crucial part in the proof of Theorem 4.9 is to show that $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under intersection with languages from co$\mathcal{B}_{1/2}$. This is stated in Lemma 4.20 below and we prepare the proof of this lemma with the following two propositions. First, we turn certain languages from co$\mathcal{B}_{1/2}$ into finite unions of particular languages from $\widetilde{\mathcal{B}}_{3/2,k}$. Note that here no k-prefix and k-suffix is specified.

Proposition 4.18. *Let $k \geq 0$. Let $L = \big(A^+ \setminus (\alpha_1, \ldots, \alpha_m)_k\big)$ with $m \geq 1$ and $\alpha_i \in A^{k+1}$. It holds that*

$$L = \bigcup_{1\leq i\leq m} \Big(A^{k+1}\backslash\{\alpha_1\}, \alpha_1,\ A^{k+1}\backslash\{\alpha_2\}, \alpha_2,\ \ldots,\ A^{k+1}\backslash\{\alpha_{i-1}\}, \alpha_{i-1},\ A^{k+1}\backslash\{\alpha_i\}\Big)_k \cup A^{\leq k}.$$

Proof. Let us first define for $1 \leq i \leq m$ the sets

$$L_i =_{\text{def}} \Big(A^{k+1}\backslash\{\alpha_1\}, \alpha_1,\ A^{k+1}\backslash\{\alpha_2\}, \alpha_2,\ \ldots,\ A^{k+1}\backslash\{\alpha_{i-1}\}, \alpha_{i-1},\ A^{k+1}\backslash\{\alpha_i\}\Big)_k.$$

The inclusion from right to left is easy to see. Just note that $A^{\leq k} \subseteq L$ and that $L_i \subseteq \big(A^+ \setminus (\alpha_1, \ldots, \alpha_i)_k\big)$ for $1 \leq i \leq m$. Hence $L_i \subseteq L$ for $1 \leq i \leq m$.

We turn to the inclusion from left to right. Let $w \in L$. If $1 \leq |w| \leq k$ then we are done. Otherwise $\widehat{w} = (\beta_1, \ldots, \beta_l)$ for suitable $l \geq 1$ and $\beta_i \in A^{k+1}$. Let $n \geq 1$ be minimal such that there do not exist indices $1 \leq i_1 < i_2 < \cdots < i_n \leq l$ with $\beta_{i_j} = \alpha_j$ for $1 \leq j \leq n$. Notice that $n \leq m$ because otherwise $w \in (\alpha_1, \ldots, \alpha_m)_k$, a contradiction. For $i_0 =_{\text{def}} 0$ and $j = 1, \ldots, n-1$ let $i_j =_{\text{def}} \min\{\, h > i_{j-1} \,|\, \beta_h = \alpha_j \,\}$ and observe that those minima exist. Thus for $1 \leq j < n$ we have $\beta_{i_j} = \alpha_j$ and $\beta_h \in A^{k+1} \setminus \{\alpha_j\}$ for $i_{j-1} < h < i_j$. It follows from the definition of n that $\beta_h \in A^{k+1} \setminus \{\alpha_n\}$ for all $i_{n-1} < h \leq l$. Therefore, $w \in L_n$. ❑

With this representation at hand we show the following.

Proposition 4.19. *Let $k \geq 0$. It holds that $L \cap L' \in \widetilde{\mathcal{B}}_{3/2,k}$ for languages $L \in \widetilde{\mathcal{B}}_{3/2,k}$ and $L' = \big(A^+ \setminus (\alpha_1, \ldots, \alpha_m)_k\big)$ with $m \geq 1$ and $\alpha_i \in A^{k+1}$.*

Proof. Note that $\big(A^+ \setminus \big(\alpha_1, \ldots, \alpha_m\big)_k\big) \cap D = D$ for every $D \subseteq A^{\leq k}$. Thus by Proposition 4.18 and distributive laws it suffices to prove the following claim.

> **Claim.** Let $C = (\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$ with $n \geq 1$, $\gamma_i \in A^{k+1}$ and $\Gamma_i \subseteq A^{k+1}$. Let $D = \big(A^{k+1}\backslash\{\alpha_1\}, \alpha_1, A^{k+1}\backslash\{\alpha_2\}, \alpha_2, \ldots, A^{k+1}\backslash\{\alpha_{m-1}\}, \alpha_{m-1},\ A^{k+1}\backslash\{\alpha_m\}\big)_k$ with $m \geq 1$ and $\alpha_i \in A^{k+1}$. Then $C \cap D \in \widetilde{\mathcal{B}}_{3/2,k}$.

The proof is by induction on m.

Induction base. For $m = 1$ we have $D = \big(A^{k+1} \setminus \{\alpha_1\}\big)_k$. By Theorem 4.10 and distributive laws, we may assume without loss of generality that $\Gamma_i \subseteq \{\gamma_1, \ldots, \gamma_i\}$ for $0 \leq i \leq n$. So if $\alpha_1 \notin \{\gamma_1, \ldots, \gamma_n\}$ then $C \cap D = C \in \widetilde{\mathcal{B}}_{3/2,k}$ and otherwise $C \cap D = \emptyset \in \widetilde{\mathcal{B}}_{3/2,k}$.

Induction step. Suppose the claim holds for some $m \geq 1$ and we want to show it for $m+1$. Again by Theorem 4.10 and distributive laws, we may assume $\Gamma_i \subseteq \{\gamma_1, \ldots, \gamma_i\}$ for $0 \leq i \leq n$. If $\alpha_1 \notin \{\gamma_1, \ldots, \gamma_n\}$ we have $C \subseteq \big(A^{k+1} \setminus \{\alpha_1\}\big)_k$ and $C \cap D = \emptyset$. Note that α_1 appears in the k-decomposition of every word in D. Otherwise, let $j =_{\text{def}} \min\{\, i \,|\, \gamma_i = \alpha_1 \,\}$. By Lemma 4.16 we obtain with $\beta =_{\text{def}} \alpha_1$, $v =_{\text{def}} p_k(\alpha_1)$ and $w =_{\text{def}} s_k(\alpha_1)$ that

$$
\begin{array}{rcl}
C & = & \overbrace{(\Gamma_0, \gamma_1, \Gamma_1, \dots, \gamma_{j-1}, \Gamma_{j-1})_k}^{C_1 =_{\mathrm{def}}} v^{-1} \cdot \beta \cdot w^{-1} \overbrace{(\Gamma_j, \gamma_{j+1}, \Gamma_{j+1}, \dots, \gamma_n, \Gamma_n)_k}^{C_2 =_{\mathrm{def}}} \quad \text{and} \\
D & = & \underbrace{\left(A^{k+1} \setminus \{\alpha_1\}\right)_k}_{D_1 =_{\mathrm{def}}} v^{-1} \cdot \beta \cdot w^{-1} \underbrace{\left(A^{k+1} \setminus \{\alpha_2\}, \alpha_2, \dots, A^{k+1} \setminus \{\alpha_m\}, \alpha_m, A^{k+1} \setminus \{\alpha_{m+1}\}\right)_k}_{D_2 =_{\mathrm{def}}}.
\end{array}
$$

Since $\Gamma_i \subseteq \{\gamma_1, \dots, \gamma_i\}$ for $0 \le i \le n$ it follows from the definition of j that $C_1 \subseteq (A^* \setminus A^* \beta A^*)$ and it also holds that $D_1 \subseteq (A^* \setminus A^* \beta A^*)$. Thus we can apply Lemma 4.17 and obtain

$$C \cap D = (C_1 \cap D_1)\, v^{-1} \cdot \beta \cdot w^{-1} \, (C_2 \cap D_2)\,.$$

Observe that $C_1 \cap D_1 = C_1 \in \widetilde{\mathcal{B}}_{3/2,k}$ (if $j = 1$ then $C_1 = (\emptyset)_k = \emptyset \in \widetilde{\mathcal{B}}_{3/2,k}$). We see with the hypothesis that $C_2 \cap D_2 \in \widetilde{\mathcal{B}}_{3/2,k}$ (if $j = n$ then $C_2 = \bigcup_{\gamma \in \Gamma_n} (\Gamma_n, \gamma, \Gamma_n)_k$). Because $C_1, C_2, D_1, D_2 \subseteq A^{\ge k+1}$ and $|v| = |w| = k$ it follows from Proposition 4.14 that $(C_1 \cap D_1)\, v^{-1} \in \widetilde{\mathcal{B}}_{3/2,k}$ and $w^{-1} (C_2 \cap D_2) \in \widetilde{\mathcal{B}}_{3/2,k}$. Note with Proposition 4.15 that $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under concatenation and that $\{\beta\} = (\emptyset, \beta, \emptyset)_k \in \widetilde{\mathcal{B}}_{3/2,k}$. We conclude that $C \cap D \in \widetilde{\mathcal{B}}_{3/2,k}$ which proves the claim. ❑

Lemma 4.20. *Let $k \ge 0$. It holds that $L \cap L' \in \widetilde{\mathcal{B}}_{3/2,k}$ for $L \in \widetilde{\mathcal{B}}_{3/2,k}$ and $L' \in \mathrm{co}\mathcal{B}_{1/2,k}$.*

Proof. By definition, languages from $\mathrm{co}\mathcal{B}_{1/2,k}$ are finite intersections of languages L_i such that $L_i = A^+ \setminus D$ for some $D \subseteq A^{\le k}$ or $L_i = \left(A^+ \setminus (w|\alpha_1, \dots, \alpha_m|v)_k\right)$ where $m \ge 1$, $\alpha_i \in A^{k+1}$ and $w, v \in A^k$. It is easy to see that $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under intersection with languages of the form $A^+ \setminus D$, so it remains to treat the other case. Let $L' =_{\mathrm{def}} \left(A^+ \setminus (w|\alpha_1, \dots, \alpha_m|v)_k\right)$ be a language as above. By definition we have $(w|\alpha_1, \dots, \alpha_m|v)_k = (\alpha_1, \dots, \alpha_m)_k \cap wA^* \cap A^*v$ and it holds that $A^{\le k} \subseteq \left(A^+ \setminus (\alpha_1, \dots, \alpha_m)_k\right)$. So we obtain

$$L' = \left(A^+ \setminus (\alpha_1, \dots, \alpha_m)_k\right) \;\cup\; \left((A^+ \setminus wA^*) \cap A^{\ge k+1}\right) \;\cup\; \left((A^+ \setminus A^*v) \cap A^{\ge k+1}\right).$$

By Proposition 4.19 it remains to show that $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under intersection with languages

$$
\begin{array}{rcl}
\left((A^+ \setminus wA^*) \cap A^{\ge k+1}\right) & = & \displaystyle\bigcup_{\beta \in A^{k+1} \setminus wA} \left(\emptyset, \beta, A^{k+1}\right)_k \quad \text{and} \\
\left((A^+ \setminus A^*v) \cap A^{\ge k+1}\right) & = & \displaystyle\bigcup_{\beta \in A^{k+1} \setminus Av} \left(A^{k+1}, \beta, \emptyset\right)_k.
\end{array}
$$

It suffices to show for $C =_{\mathrm{def}} (\Gamma_0, \gamma_1, \Gamma_1, \dots, \gamma_n, \Gamma_n)_k$ with $n \ge 1$, $\gamma_i \in A^{k+1}$ and $\Gamma_i \subseteq A^{k+1}$, and $D =_{\mathrm{def}} (\emptyset, \beta, A^{k+1})_k$ with $\beta \in A^{k+1}$ that $C \cap D \in \widetilde{\mathcal{B}}_{3/2,k}$. The same arguments can also be applied to languages $(A^{k+1}, \beta, \emptyset)_k$. We obtain

$$
C \cap D = \begin{cases}
(\emptyset, \beta, \Gamma_0, \gamma_1, \Gamma_1, \dots, \gamma_n, \Gamma_n)_k \cup (\emptyset, \gamma_1, \Gamma_1, \dots, \gamma_n, \Gamma_n)_k & : \text{ if } \beta \in \Gamma_0 \wedge \gamma_1 = \beta \\
(\emptyset, \beta, \Gamma_0, \gamma_1, \Gamma_1, \dots, \gamma_n, \Gamma_n)_k & : \text{ if } \beta \in \Gamma_0 \wedge \gamma_1 \ne \beta \\
(\emptyset, \gamma_1, \Gamma_1, \dots, \gamma_n, \Gamma_n)_k & : \text{ if } \beta \notin \Gamma_0 \wedge \gamma_1 = \beta \\
\emptyset & : \text{ if } \beta \notin \Gamma_0 \wedge \gamma_1 \ne \beta.
\end{cases}
$$

This shows $C \cap D \in \widetilde{\mathcal{B}}_{3/2,k}$. ❑

Proof of Theorem 4.9. We want to show now that $\mathcal{B}_{3/2,k} = \widetilde{\mathcal{B}}_{3/2,k}$ for $k \geq 0$. For the inclusion from right to left note that $\mathcal{B}_{3/2,k}$ is closed under finite union and that any $D \subseteq A^{\leq k}$ is in $\mathcal{B}_{1/2,k} \subseteq \mathcal{B}_{3/2,k}$. So it remains to show for languages $(\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $m \geq 1$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$ that they belong to $\mathcal{B}_{3/2,k}$. We do this a little more general for $L =_{\text{def}} (\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_n, \Gamma_n)_k$ with $\gamma_i \in A^{k+1}$ and $\Gamma_i \subseteq A^{k+1}$, where we allow $n \geq 0$. The proof is by induction on n.

For $n = 0$ we have

$$(\Gamma_0)_k = \bigcap_{\substack{\alpha \in A^{k+1} \setminus \Gamma_0 \\ w,v \in A^k}} \left(A^+ \setminus (w|\ \alpha\ |v)_k\right) \ \cap \ \left(A^+ \setminus A^{\leq k}\right)$$

which shows that $(\Gamma_0)_k \in \text{co}\mathcal{B}_{1/2} \subseteq \mathcal{B}_{3/2,k}$. For the induction step we suppose that $L = (\Gamma_0, \gamma_1, \Gamma_1, \ldots, \gamma_{n+1}, \Gamma_{n+1})_k$ which we can write by Lemma 4.16 as

$$L = \underbrace{(\Gamma_0)_k}_{L_1 =_{\text{def}}} p_k(\gamma_1)^{-1} \ \cdot \ \gamma_1 \ \cdot \ s_k(\gamma_1)^{-1} \underbrace{(\Gamma_1, \gamma_2, \Gamma_2, \ldots, \gamma_{n+1}, \Gamma_{n+1})_k}_{L_2 =_{\text{def}}}.$$

By hypothesis, we have $L_1, L_2 \in \mathcal{B}_{3/2,k}$. Since $L_1, L_2 \subseteq A^{\geq k+1}$ we obtain from Lemma 2.14 that also $L_1 p_k(\gamma_1)^{-1}, s_k(\gamma_1)^{-1} L_2 \in \mathcal{B}_{3/2,k}$. With the observation that $\{\gamma_1\} \in \mathcal{B}_{1,k}$ we finally get $L \in \mathcal{B}_{3/2,k}$.

We turn to the more difficult inclusion $\mathcal{B}_{3/2,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$ and argue first that $\mathcal{B}_{1/2,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$. It suffices to show that $L \in \widetilde{\mathcal{B}}_{3/2,k}$ for $L =_{\text{def}} (w|\alpha_1, \ldots, \alpha_n|v)_k$ with $n \geq 1$, $w, v \in A^k$ and $\alpha_i \in A^{k+1}$. By definition of $\widetilde{\mathcal{B}}_{3/2,k}$ we know that $L' =_{\text{def}} \left(A^{k+1}, \alpha_1, A^{k+1}, \ldots, \alpha_n, A^{k+1}\right)_k \in \widetilde{\mathcal{B}}_{3/2,k}$. Let $L_1' =_{\text{def}} w\left(w^{-1}L'\right)$ and observe that $L_1' = L' \cap wA^*$. Since $|w| = k$ and $L' \subseteq A^{\geq k+1}$ we can apply Proposition 4.14 and obtain $w^{-1}L' \in \widetilde{\mathcal{B}}_{3/2,k}$. Because $\{w\} \in \widetilde{\mathcal{B}}_{3/2,k}$ and $\widetilde{\mathcal{B}}_{3/2,k}$ is closed under concatenation by Proposition 4.15 we have that $L_1' \in \widetilde{\mathcal{B}}_{3/2,k}$. It follows that $L_1' \subseteq A^{\geq k+1}$ and we can do the same thing for the k-suffix v. Therefore, let $L_2' =_{\text{def}} \left(L_1' v^{-1}\right) v$ for which the same arguments show that $L_2' = L_1' \cap A^* v$ and $L_2' \in \widetilde{\mathcal{B}}_{3/2,k}$. We conclude that $L = wA^* \cap A^*v \cap L' = L_2' \in \widetilde{\mathcal{B}}_{3/2,k}$ which shows $\mathcal{B}_{1/2,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$.

Now we show that $\mathcal{B}_{1,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$. Recall from Lemma 2.14 that $\mathcal{B}_{1/2,k}$ and $\text{co}\mathcal{B}_{1/2,k}$ are closed under intersection. So any language from $\mathcal{B}_{1,k}$ can be written as a finite union of languages C, D or $C \cap D$ with $C \in \mathcal{B}_{1/2,k}$ and $D \in \text{co}\mathcal{B}_{1/2,k}$. It follows from Lemma 4.20 that in particular $\text{co}\mathcal{B}_{1/2,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$, so with the same lemma we see that C, D and $C \cap D$ belong to $\widetilde{\mathcal{B}}_{3/2,k}$. Hence, $\mathcal{B}_{1,k} \subseteq \widetilde{\mathcal{B}}_{3/2,k}$. Together with Proposition 4.15 we finally obtain $\mathcal{B}_{3/2,k} = \text{Pol}(\mathcal{B}_{1,k}) \subseteq \text{Pol}(\widetilde{\mathcal{B}}_{3/2,k}) = \widetilde{\mathcal{B}}_{3/2,k}$. **(End proof of Theorem 4.9.)**

4.3 Forbidden Pattern Characterization of $\mathcal{B}_{3/2,k}$

Let us look again at the definition of the pattern $\mathbb{L}_{3/2}$ characterizing $\mathcal{L}_{3/2}$ (see Definition 4.1). It is defined as the subgraph in Figure 4.2 with the condition that $\alpha(vwv) \subseteq \alpha(vv)$. We generalize this condition to k-decompositions and make the following definition.

Definition 4.21. Let $k \geq 0$. Pattern $\mathbb{B}_{3/2,k}$ is defined as the subgraph given in Figure 4.2 with $x, z \in A^*$, $w \in A^+$, $v \in A^{\geq k+1}$ and $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$.

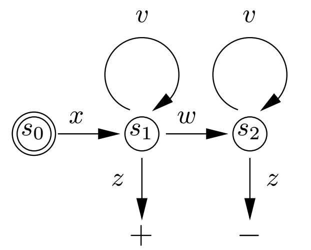


Fig. 4.2. Pattern $\mathbb{B}_{3/2,k}$ with $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$.

Recall that $\alpha(\widehat{x})$ is the set of factors of length $k+1$ in the k-decomposition of $x \in A^{\geq k+1}$. So in case $k = 0$ we encounter pattern $\mathbb{L}_{3/2}$ and no new argument is needed to see that also $\mathcal{FP}(\mathbb{B}_{3/2,k})$ is well–defined. We prove in this section the following theorem, which gives in the special case $k = 0$ another proof of Theorem 4.2.

Theorem 4.22. *Let $k \geq 0$. It holds that $\mathcal{B}_{3/2,k} = \mathcal{FP}(\mathbb{B}_{3/2,k})$.*

The two inclusions are given in Lemma 4.24 in Subsection 4.3.1 and Lemma 4.28 in Subsection 4.3.2. We discuss consequences of Theorem 4.22 in Subsection 4.3.3.

4.3.1 The Easy Inclusion

If a language $L = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ is given, then there is some n such that we can insert w with $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$ into $xv^n z \in L$ and still have a word in L. This can be seen by the following argument. If n is large enough then there must be some Σ_i such that $\alpha(\widehat{vv}) \subseteq \Sigma_i$. It follows from $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$ that we do not leave Σ_i if w is inserted.

Lemma 4.23. *Let $k \geq 0$ and let $L = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ with $m \geq 1$, $\alpha_i \in A^{k+1}$ and $\Sigma_i \subseteq A^{k+1}$. Moreover, let $n \geq 4m + 3$. Then for all $x, z \in A^*$, $w \in A^+$ and $v \in A^{\geq k+1}$ with $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$ there exists some p with $1 \leq p < n$ such that*

$$xv^n z \in L \implies xv^p wv^{n-p} z \in L.$$

Proof. Let $xv^n z \in L$ and choose suitable $l \geq 1$, $a_1, \ldots, a_{l+k} \in A$ and $\beta_i \in A^{k+1}$ such that $xv^n z = a_1 \cdots a_{l+k}$ and $\widehat{xv^n z} = (\beta_1, \ldots, \beta_l)$. By definition of L there exist $0 = j_0 < j_1 < j_2 < \ldots < j_m < j_{m+1} = l + 1$ such that

(a) $\beta_{j_i} = \alpha_i$ for $1 \leq i \leq m$ and
(b) $\beta_j \in \Sigma_i$ for $0 \leq i \leq m$ and $j_i < j < j_{i+1}$.

We denote the position in $xv^n z$ where the i-th v starts by q_i, i.e., for $1 \leq i \leq n$ we set $q_i =_{\text{def}} 1 + |x| + (i-1)|v|$. Since $|v| \geq k+1$ we obtain n different positions such that $1 \leq q_1 < q_2 < \cdots < q_n \leq l$ and $q_i - q_{i-1} = |v|$ for $2 \leq i \leq n$. By assumption we have that $n \geq 4m + 3$, so at least $3(m+1)$ of the positions q_i are different from $j_1, \ldots, j_m$. By the pigeon hole principle there exist h, p with $0 \leq h \leq m$ and $1 \leq p \leq n-2$ such that $j_h < q_p < q_{p+1} < q_{p+2} < j_{h+1}$. We fix these positions and set $\lambda =_{\text{def}} q_m$, $\mu =_{\text{def}} q_{m+1}$ and $\nu =_{\text{def}} q_{m+2}$. Since $a_\lambda a_{\lambda+1} \cdots a_{\mu-1} = a_\mu a_{\mu+1} \cdots a_{\nu-1} = v$ and $|v| \geq k+1$ we have

$$\alpha(\widehat{vv}) \subseteq \Sigma_h. \tag{4.3}$$

Now we look at $u =_{\text{def}} xv^pwv^{n-p}z$ and choose suitable $b_1, \ldots, b_{l+k+|w|} \in A$, $\gamma_i \in A^{k+1}$ such that $u = b_1 \cdots b_{l+k+|w|}$ and $\widehat{u} = \big(\gamma_1, \ldots, \gamma_{l+|w|}\big)$. Observe that

$$a_i = \begin{cases} b_i & : \text{ if } 1 \leq i \leq \mu - 1, \\ b_{i+|w|} & : \text{ if } \mu \leq i \leq l+k \end{cases}$$

and

$$\beta_j = \begin{cases} \gamma_j & : \text{ if } 1 \leq j \leq \mu - k - 1, \\ \gamma_{j+|w|} & : \text{ if } \mu \leq j \leq l. \end{cases} \tag{4.4}$$

Let $l_i =_{\text{def}} j_i$ for $0 \leq i \leq h$ and $l_i =_{\text{def}} j_i + |w|$ for $h < i \leq m+1$. Since $j_h < \lambda$ and $j_{h+1} > \mu$, we have $l_h \leq \mu - k - 1$ and $l_{h+1} \geq \mu + |w|$. From (4.4) it follows that $\gamma_j = \beta_j$ for $1 \leq j \leq l_h$ and $\gamma_j = \beta_{j-|w|}$ for $l_{h+1} \leq j \leq l + |w|$. Thus we have $\gamma_{l_i} = \gamma_{j_i} = \beta_{j_i}$ for $1 \leq i \leq h$ and $\gamma_{l_i} = \gamma_{j_i+|w|} = \beta_{j_i}$ for $h < i \leq m$. Therefore, we obtain

$$\begin{aligned} &\gamma_{l_i} = \alpha_i \text{ for } i \text{ with } 1 \leq i \leq m, \\ &\gamma_j = \beta_j \in \Sigma_i \text{ for } i,j \text{ with } 0 \leq i < h \text{ and } l_i = j_i < j < j_{i+1} = l_{i+1} \text{ and} \\ &\gamma_j = \beta_{j-|w|} \in \Sigma_i \text{ for } i,j \text{ with } h < i \leq m \text{ and } l_i = j_i + |w| < j < j_{i+1} + |w| = l_{i+1}. \end{aligned} \tag{4.5}$$

To see that $u \in L = (\Sigma_0, \alpha_1, \Sigma_1, \ldots, \alpha_m, \Sigma_m)_k$ it remains to show $\gamma_j \in \Sigma_h$ for $l_h < j < l_{h+1}$. This is clear for $l_h < j < \mu - k$ and for all $\mu + |w| \leq j < l_{h+1}$ due to (4.4). So we have to show $\gamma_j \in \Sigma_h$ for $\mu - k \leq j < \mu + |w|$. Observe that

$$b_{\mu-k}b_{\mu-k+1}b_{\mu-k+2} \cdots b_{\mu+|w|+k-1} = s_k(v) \cdot w \cdot p_k(v).$$

So for $\mu - k \leq j < \mu + |w|$ we have $\gamma_j \in \alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv}) \subseteq \Sigma_h$ by (4.3). This shows

$$\gamma_j \in \Sigma_h \text{ for } l_h < j < l_{h+1}. \tag{4.6}$$

We summarize (4.5) and (4.6) as

(a) $\gamma_{l_i} = \alpha_i$ for $1 \leq i \leq m$ and
(b) $\gamma_j \in \Sigma_i$ for $0 \leq i \leq m$ and $l_i < j < l_{i+1}$

which shows $u \in L$. ❑

We immediately see that the insertion stated in Lemma 4.23 contradicts the occurrence of pattern $\mathbb{B}_{3/2,k}$.

Lemma 4.24. *Let $k \geq 0$. It holds that $\mathcal{B}_{3/2,k} \subseteq \mathcal{FP}(\mathbb{B}_{3/2,k})$.*

Proof. Let $L \in \mathcal{B}_{3/2,k}$ and let $\mathcal{M}$ be some DFA accepting L. By Theorem 4.9 we can write L as

$$L = \bigcup_{i=1}^{m} (\Sigma_{i,0}, \alpha_{i,1}, \Sigma_{i,1}, \ldots, \alpha_{i,m_i}, \Sigma_{i,m_i})_k \ \cup \ D$$

for some $m \geq 0$, $m_i \geq 1$, $\alpha_{i,j} \in A^{k+1}$, $\Sigma_{i,j} \subseteq A^{k+1}$ and $D \subseteq A^{\leq k}$. Assume to the contrary that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$ witnessed by $x, z \in A^*$, $w \in A^+$ and $v \in A^{\geq k+1}$ such that $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$. Then $xv^iz \in L$ and $xv^iwv^jz \notin L$ for all $i, j \geq 0$. In particular, $xv^nz \in L$ with $n =_{\text{def}} \max\{\, 4m_i + 3 \mid 1 \leq i \leq m \,\} \cup \{k+1\}$. Hence $|xv^nz| \geq k+1$ and there exists an l with $xv^nz \in (\Sigma_{l,0}, \alpha_{l,1}, \Sigma_{l,1}, \ldots, \alpha_{l,m_l}, \Sigma_{l,m_l})_k$. Since $n \geq 4m_l + 3$ we can apply Lemma 4.23 and we obtain $xv^pwv^{n-p}z \in (\Sigma_{l,0}, \alpha_{l,1}, \Sigma_{l,1}, \ldots, \alpha_{l,m_l}, \Sigma_{l,m_l})_k \subseteq L$ for a suitable p with $1 \leq p < n$. This is a contradiction to the occurrence of pattern $\mathbb{B}_{3/2,k}$ in $\mathcal{M}$ since the state we reach after input $xv^pwv^{n-p}z$ is rejecting. We conclude that $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2,k}$. ❑

4.3.2 The More Complicated Inclusion

We turn to the inclusion $\mathcal{FP}(\mathbb{B}_{3/2,k}) \subseteq \mathcal{B}_{3/2,k}$ which is more difficult to handle than the reverse inclusion. The reason is that we can only use the fact that some DFA $\mathcal{M}$ does *not* have a certain structure. We apply this argument to all words $x \in L(\mathcal{M})$ and derive for each such x a subset of $L(\mathcal{M})$ that can be described by expressions of bounded size. Due to this bound we then conclude that $L(\mathcal{M})$ itself can be described by a finite number of expressions. We consider for $k \geq 0$ expressions E of the form

$$w_0 \cdot \left(v_1|\Sigma_1|v_1'\right)_k \cdot w_1 \cdots \left(v_n|\Sigma_n|v_n'\right)_k \cdot w_n$$

where $n \geq 0$, $w_i \in A^+$, $v_j, v_j' \in A^k$ and $\Sigma_j \subseteq A^{k+1}$ for $0 \leq i \leq n$ and $1 \leq j \leq n$. In order not to overload notations, we identify each such expression E with the language described by E (recall Definitions 1.25 and 1.26). So at the same time we will make statements like '$x \in E$' and also talk about the size of E as a syntactical object. For fixed $k \geq 0$ we denote the set of all such expressions by $\mathcal{E}_k$. Observe also that if we start with an expression $E \in \mathcal{E}_k$ and we replace a factor of some w_i by an expression $E' \in \mathcal{E}_k$, then the resulting expression is again in $\mathcal{E}_k$ (even if we replace w_i completely by E').

Definition 4.25. Let $k \geq 0$. For $E \in \mathcal{E}_k$ with $E = w_0 \cdot (v_1|\Sigma_1|v_1')_k \cdot w_1 \cdots (v_n|\Sigma_n|v_n')_k \cdot w_n$ we define the size of E as

$$\|E\| =_{\mathrm{def}} |w_0 w_1 \cdots w_n|.$$

Since the w_i are non–empty words, there exist for fixed $k \geq 0$ only a finite number of expressions in $\mathcal{E}_k$ having the same size. We define a function that helps to analyze the size of expressions in the following lemma. The variables a, m and n will be associated with the size of the alphabet A, the size of the automaton $\mathcal{M}$ and the cardinality of $\alpha(\widehat{w})$ for a given word w, respectively.

Definition 4.26.

$$\mathcal{S}(k,a,m,n) =_{\mathrm{def}} \begin{cases} k & : \text{ if } n = 0 \\ 2m^m + k + 1 & : \text{ if } n = 1 \\ 3\mathcal{K}(m) \cdot (5m^m a^k + 1) \cdot \mathcal{S}(k,a,m,n-1) & : \text{ otherwise} \end{cases}$$

Recall that $\mathcal{K}(m) = (m+1)^{(m+1)^{(m+1)}}$ is defined at the beginning of Section 4.1 where it was used to find automata loops in words. The following main lemma states, under the assumption that a DFA $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2,k}$, that for any word x we can find an expression E_x of bounded size such that $x \in E_x$ and if $x \in L(F)$ then $E_x \subseteq L(\mathcal{M})$. We consider also prefixes x' and suffixes x'' in order to allow an inductive proof.

Lemma 4.27. *Let $k \geq 0$. Let $\mathcal{M}$ be a* DFA *which does not have pattern $\mathbb{B}_{3/2,k}$. For every $x \in A^+$ there exists an expression $E_x \in \mathcal{E}_k$ with $x \in E_x$ and $\|E_x\| \leq \mathcal{S}(k, |A|, |\mathcal{M}|, |\alpha(\widehat{x})|)$ such that for all $x', x'' \in A^*$ it holds that*

$$x' x x'' \in L(\mathcal{M}) \implies x' E_x x'' \subseteq L(\mathcal{M}).$$

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a DFA which does not have pattern $\mathbb{B}_{3/2,k}$. For this proof we extend the definition of $\alpha(\widehat{x})$ to words $x \in A^{\leq k}$ and set $\alpha(\widehat{x}) =_{\mathrm{def}} \emptyset$ for such x. The proof is by induction on $N =_{\mathrm{def}} |\alpha(\widehat{x})|$ with $0 \leq N \leq |A|^{k+1}$. For the induction base we consider the cases $N = 0$ and $N = 1$.

Induction base. For $N = 0$ we set $E_x =_{\mathrm{def}} x$ and we are done. Now let $x \in A^+$ with $N = |\alpha(\widehat{x})| = 1$. If $|x| \leq \mathcal{S}(k, |A|, |\mathcal{M}|, 1)$ we set again $E_x =_{\mathrm{def}} x$ and we are done. Otherwise, we have $|x| > 2|\mathcal{M}|^{|\mathcal{M}|} + k + 1$. Since $|x| \geq k+2$ we obtain by comparing letters that $x = a^{|x|}$ for some $a \in A$. We consider the mappings δ^{a^i} induced by prefixes a^i of x. There exist i, j with $1 \leq i < j \leq |\mathcal{M}|^{|\mathcal{M}|} + 1$ such that $\delta^{a^i} = \delta^{a^j}$. So for all $m \geq 0$ it holds that

$$\delta^{a^i} = \delta^{a^{i+m(j-i)}}. \tag{4.7}$$

We choose some l with $1 \leq l \leq j - i$ such that $|x| - i - l \equiv 0 \mod (j - i)$. From (4.7) we obtain

$$\delta^{a^{i+l}} = \delta^x. \tag{4.8}$$

Now define

$$E_x =_{\mathrm{def}} a^i \cdot \left(a^k \middle| \left\{a^{k+1}\right\} \middle| a^k\right)_k \cdot a^l$$

and observe that $1 \leq i, l \leq |\mathcal{M}|^{|\mathcal{M}|}$. Because $|x| > 2|\mathcal{M}|^{|\mathcal{M}|} + k + 1$ we have $|x| - i - l \geq k + 1$. So it follows that $x \in E_x$. Moreover, it holds that $E_x \in \mathcal{E}_k$ and $\|E_x\| = i + l \leq 2|\mathcal{M}|^{|\mathcal{M}|} \leq \mathcal{S}(k, |A|, |\mathcal{M}|, 1)$.

Let $x', x'' \in A^*$ be given such that $x'xx'' \in L(\mathcal{M})$. Furthermore, set $s_1 =_{\mathrm{def}} \delta(s_0, x'a^i)$. Then by (4.7) we have $\delta(s_1, a^{m(j-i)}) = s_1$ for all $m \geq 0$. Assume that there exists some $h \geq k + 1$ such that $\delta(s_1, a^h a^l x'') \notin L(\mathcal{M})$. Let $s_2 =_{\mathrm{def}} \delta(s_1, a^h)$ and define $\tilde{x} =_{\mathrm{def}} x'a^i$, $\tilde{v} =_{\mathrm{def}} a^{(k+1)(j-i)}$, $\tilde{w} =_{\mathrm{def}} a^h$ and $\tilde{z} =_{\mathrm{def}} a^l x''$.

Then $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{\tilde{v}\tilde{v}})$ since only the letter a occurs, and $|\tilde{v}| = (j - i)(k + 1) \geq k + 1$. Because s_1 has a $\tilde{v}$-loop and $\tilde{v}$ is a sequence of a's, we do not leave this loop with the word a^h, and we obtain that s_2 also has a $\tilde{v}$-loop. Furthermore, we have $\delta(s_0, \tilde{x}) = s_1$, $\delta(s_1, \tilde{z}) \in S'$ by (4.8) and $\delta(s_2, \tilde{z}) \notin S'$ by assumption. This shows that we found pattern $\mathbb{B}_{3/2,k}$ in $\mathcal{M}$, witnessed by $\tilde{x}, \tilde{z} \in A^*$, $\tilde{w} \in A^+$ and $\tilde{v} \in A^{\geq k+1}$ with $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{\tilde{v}\tilde{v}})$, a contradiction. So we have for all $h \geq k + 1$ that $\delta(s_0, x'a^i a^h a^l x'') = \delta(s_1, a^h a^l x'') \in L(\mathcal{M})$. In particular, $x'E_x x'' \subseteq L(\mathcal{M})$ which shows the induction base.

Induction step. We state the induction hypothesis.

> For all $x \in A^+$ with $0 \leq |\alpha(\widehat{x})| \leq N < |A|^{k+1}$ there exists some $E_x \in \mathcal{E}_k$ with $x \in E_x$ and $\|E_x\| \leq \mathcal{S}(k, |A|, |\mathcal{M}|, |\alpha(\widehat{x})|)$ such that for all $x', x'' \in A^*$ it holds that
>
> $$x'xx'' \in L(\mathcal{M}) \implies x'E_x x'' \subseteq L(\mathcal{M}).$$

Let $x \in A^+$ be given with $\alpha(\widehat{x}) = N + 1$ and $N \geq 1$. We start with a decomposition of x into so–called 'sectors', i.e., we decompose x into factors s_i such that $|\alpha(\widehat{s_i})| \leq N$ (actually we will have $|\alpha(\widehat{s_i})| = N$ for all sectors except for the last one). This is done in the following way: we start with x and determine the longest prefix s_1 of x such that $|\alpha(\widehat{s_1})| \leq N$. Now we start over with $s_1^{-1}x$, determine the longest prefix s_2 of $s_1^{-1}x$ such that $|\alpha(\widehat{s_2})| \leq N$ and proceed with $(s_1 s_2)^{-1}x$. If we continue this procedure, we obtain a factorization of x into sectors $s_1, s_2, \ldots, s_l$ for some $l \geq 2$ such that

1. $x = s_1 s_2 \cdots s_l$ with $|s_i| \geq k+1$ for $1 \leq i < l$ (since $|\alpha(\widehat{s_i})| = N \geq 1$ for $1 \leq i < l$),
2. $\alpha(\widehat{s_i}) \subsetneq \alpha(\widehat{x})$ for $1 \leq i \leq l$ and
3. $\alpha(\widehat{s_i s_{i+1}}) = \alpha(\widehat{x})$ for $1 \leq i < l$ (since we have chosen maximal prefixes s_i).

Before we continue, we give an outline of the further argumentation. Sectors do not have to be factors of x for which we know how to bound their length, and we can also not state a particular bound on the number of sectors. The main task of the induction step is to replace the unbounded number of consecutive sectors by a bounded number of terms of the form $(v|\Sigma|v')_k$ in a way such that (i) we do not leave $L(\mathcal{M})$ if we started with $x \in L(\mathcal{M})$ and (ii) we obtain an expression where only a bounded number of sectors and of terms of the form $(v|\Sigma|v')_k$ are left. The induction hypothesis then provides expressions of bounded size for the remaining sectors. Note that the closure under concatenation of the class in question is necessary for this approach. Together, we obtain an expression of bounded size containing x and being a subset of $L(\mathcal{M})$ which will prove the lemma. We distinguish the two cases when the number l of sectors is already reasonable small, and when it is not.

Case 1. Assume for the number of sectors l that $l \leq 3\mathcal{K}(|\mathcal{M}|) \cdot (5|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1)$. Then there is not much to do since l is reasonable small and $|\alpha(\widehat{s_i})| \leq N$ for $1 \leq i \leq l$. Thus, by induction hypothesis, we find for $1 \leq i \leq l$ expressions $E_{s_i} \in \mathcal{E}_k$ such that $s_i \in E_{s_i}$, $\|E_{s_i}\| \leq \mathcal{S}(k, |A|, |F|, N)$ and

$$s_i' E_{s_i} s_i'' \subseteq L(\mathcal{M}) \text{ for all } s_i', s_i'' \in A^* \text{ with } s_i' s_i s_i'' \in L(\mathcal{M}). \tag{4.9}$$

We define $E_x =_{\text{def}} E_{s_1} \cdot E_{s_2} \cdots E_{s_l}$. It follows that $x \in E_x$ and $\|E_x\| \leq \mathcal{S}(k, |A|, |\mathcal{M}|, N+1)$.

Now let $x', x'' \in A^*$ such that $x'xx'' \in L(\mathcal{M})$. We need to show that $x'E_x x'' \subseteq L(\mathcal{M})$. Let $y \in E_x$. Then for $1 \leq i \leq l$ there exist $y_i \in E_{s_i}$ such that $y = y_1 y_2 \cdots y_l$. Starting with $x'xx'' = x's_1 s_2 \cdots s_l x'' \in L(\mathcal{M})$ we will step by step replace the sectors s_i by y_i without leaving $L(\mathcal{M})$. For the first step we define $s_l' =_{\text{def}} x's_1 \cdots s_{l-1}$ and $s_l'' =_{\text{def}} x''$. So we have $x'xx'' = s_l' s_l s_l'' \in L(\mathcal{M})$. From (4.9) it follows that $s_l' E_{s_l} s_l'' \subseteq L(\mathcal{M})$ and $s_l' y_l s_l'' \in L(\mathcal{M})$. This shows $x's_1 \cdots s_{l-1} y_l x'' \in L(\mathcal{M})$. For the second step let $s_{l-1}' =_{\text{def}} x's_1 \cdots s_{l-2}$ and $s_{l-1}'' =_{\text{def}} y_l x''$. So we have $s_{l-1}' s_{l-1} s_{l-1}'' \in L(\mathcal{M})$ by the previous step, and again from (4.9) it follows that $s_{l-1}' y_{l-1} s_{l-1}'' \in L(\mathcal{M})$. This shows $x's_1 \cdots s_{l-2} y_{l-1} y_l x'' \in L(\mathcal{M})$. If we continue this procedure we finally obtain $x'y_1 \cdots y_l x'' = x'yx'' \in L(\mathcal{M})$. Hence $x'E_x x'' \subseteq L(\mathcal{M})$.

Case 2. Suppose for the number of sectors l that $l > 3\mathcal{K}(|\mathcal{M}|) \cdot (5|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1)$. Define $v_i =_{\text{def}} s_{2i-1} s_{2i}$ for $1 \leq i \leq \lfloor l/2 \rfloor - 1$ and $v_{\lfloor l/2 \rfloor} =_{\text{def}} s_{2\lfloor l/2 \rfloor - 1} \cdots s_l$. Note that if l is odd, then $v_{\lfloor l/2 \rfloor}$ contains three sectors. So $x = v_1 v_2 \cdots v_{\lfloor l/2 \rfloor}$ and every v_i contains at least two and at most three sectors with the effect that $\alpha(\widehat{v_i}) = \alpha(\widehat{x})$. Now we apply Theorem 4.3 to the list of words $v_1, v_2, \ldots, v_{\lfloor l/2 \rfloor}$ and obtain a new list of words $x_0', x_1, x_1', x_2, x_2', \ldots, x_m, x_m'$ with $m \geq 1$ (since $l > 3\mathcal{K}(|\mathcal{M}|)$) such that $x = x_0' x_1 x_1' x_2 x_2' \cdots x_m x_m'$ and the following fact holds.

Fact 1.

1. Every x_i (and also every x_i') is a concatenation of at least one and at most $\mathcal{K}(|\mathcal{M}|)$ words v_j. Thus it contains at least two and at most $3\mathcal{K}(|\mathcal{M}|)$ sectors.
2. For every x_i it holds that $\delta^{x_i x_i} = \delta^{x_i}$.

Next we assign for $1 \leq i \leq m$ to each x_i a tag representing the mapping $\delta_i =_{\text{def}} \delta^{x_0' x_1 x_1' \cdots x_i}$ and also $s_k(x_i)$, the k-suffix of x_i. Note that there are at most $|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k$ different tags. Now the task is to find maximal factors between some x_i and x_j having the same tags. We call such a maximal factor a 'region' and we argue below, how the number of regions can be bounded.

Let us start with an algorithm that describes how to determine regions. First we choose some j_1, j_1' with $1 \leq j_1 < j_1' \leq m$ such that $x_{j_1}, x_{j_1'}$ have the same tag and $j_1' - j_1$ is maximal. With (j_1, j_1') we have found the first region and we mark for $j_1 \leq i \leq j_1'$ all x_i as already used, i.e., we mark all x_i within this region. We call x_{j_1} $(x_{j_1'})$ the left border (right border, respectively) and the word $x'_{j_1} x_{j_1+1} x'_{j_1+1} \cdots x_{j_1'-1} x'_{j_1'-1}$ the content of the region (j_1, j_1'). In a second step, we choose j_2, j_2' with $1 \leq j_2 < j_2' \leq m$ such that none of the $x_{j_2}, x_{j_2+1}, \ldots, x_{j_2'}$ is marked, x_{j_2} and $x_{j_2'}$ have the same tag and $j_2' - j_2$ is maximal. With (j_2, j_2') we have found the second region (with left border x_{j_2}, right border $x_{j_2'}$ and content $x'_{j_2} x_{j_2+1} x'_{j_2+1} \cdots x_{j_2'-1} x'_{j_2'-1}$). Again we mark all x_i for $j_2 \leq i \leq j_2'$ as already used. Analogously, we proceed in the following steps, until no more pairs of indices can be found that fulfill the selection condition. We obtain non–intersecting regions $(j_1, j_1'), (j_2, j_2'), \ldots, (j_n, j_n')$.

If we denote the left (right) borders of a region by r_i (r_i', respectively), the content of a region by b_i and the gaps between consecutive regions by b_i', we can write x as

$$x = b_0' \ r_1 b_1 r_1' \ b_1' \ r_2 b_2 r_2' \ b_2' \ \cdots \ r_n b_n r_n' \ b_n'.$$

We treat the content b_i of a region below and show that we can give a short representation. Before we do so, we want to establish a bound on the number of sectors that do *not* belong to the content of some region (see statement 5 of the following fact). In order to keep the argumentation transparent we also give some auxiliary statements.

Fact 2.

1. There do not exist two different regions (i, i') and (j, j') that both have the same tags attached to their borders.
2. There do not exist $i < j$ such that x_i, x_j are not marked and both have the same tags attached.
3. The number of the x_i that are not marked and the number n of regions is bounded by $|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k$.
4. The total number of sectors which lie in some gap b_i' is bounded by $3\mathcal{K}(|\mathcal{M}|) \cdot (3|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1)$.
5. The total number of sectors which lie in some r_i, r_i' or b_i' is bounded by $3\mathcal{K}(|\mathcal{M}|) \cdot (5|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1)$.
6. For every r_i it holds that $\delta^{r_i r_i} = \delta^{r_i}$. The same holds for every r_i'.

Proof of Fact 2. We begin to argue for the first statement of Fact 2. Suppose there exist regions (i, i') and (j, j') such that $i < i' < j < j'$ and $x_i, x_{i'}, x_j, x_{j'}$ have the same tags. If there is no other region between (i, i') and (j, j'), we should have chosen (i, j') instead of (i, i') and (j, j') in order to maximize the size of the region. On the other hand, if there exist a region $(\tilde{i}, \tilde{i}')$ between (i, i') and (j, j'), we should have chosen (i, j') instead of $(\tilde{i}, \tilde{i}')$, again in order to maximize the size of the region. So in both cases we obtain a contradiction. This shows the first statement, and the second can be seen analogously. The third statement is an easy consequence of the first and second statement.

We turn to statement 4. Each b_i' has the form $x_j' x_{j+1} x_{j+1}' \cdots x_{j'} x_{j'}'$ for $j \leq j'$ by the construction of regions. The number of last factors $x_{j'}'$ can be bounded by the total number of gaps b_i' which is by statement 3 less then or equal to $|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1$. Also by statement 3 there are at most $|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k$ factors x_i in x that are not marked. This bounds also the number of factors x_i' before some unmarked x_{i+1} in x. Together, it follows that the total number of all x_i and x_i' that lie in some gap is bounded by

$$2 \cdot \left(|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k\right) + \left(|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1\right) = 3 \cdot |\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1.$$

By statement 1 of Fact 1, every x_i (and x_i') contains at most $3\mathcal{K}(|\mathcal{M}|)$ sectors, which in turn gives statement 4.

By the definition of regions, every r_i (and r_i') consists of exactly one x_i. So from statement 1 of Fact 1 and statement 3 of Fact 2 we obtain that the number of sectors that lie in some r_i or r_i' is bounded by

$$3\mathcal{K}(|\mathcal{M}|) \cdot 2 \cdot |\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k.$$

If we add this number to the quantity given in statement 4 we obtain what is required for statement 5. From statement 2 of Fact 1 we obtain statement 6. **(End proof of Fact 2.)**

As an intermediate step we define an expression E_x' for words x that satisfy the above fact, where we replace each b_i by $(p_k(b_i)|\alpha(\widehat{x})|s_k(b_i))_k$.

$$\begin{aligned} E_x' \;=_{\text{def}}\; b_0'\; & r_1 \cdot \;\big(p_k(b_1)|\alpha(\widehat{x})|s_k(b_1)\big)_k \;\cdot r_1'\; b_1' \,\cdot \\ & r_2 \cdot \;\big(p_k(b_2)|\alpha(\widehat{x})|s_k(b_2)\big)_k \;\cdot r_2'\; b_2' \,\cdot \\ & \qquad\qquad\vdots \\ & r_n \cdot \;\big(p_k(b_n)|\alpha(\widehat{x})|s_k(b_n)\big)_k \;\cdot r_n'\; b_n' \end{aligned}$$

Although the number of remaining sectors in the b_i', r_i and r_i' is bounded (see statement 5 of Fact 2), we cannot bound the size of E_x' yet. This is because the size of each remaining sector is still unbounded. We will treat these sectors below, using the induction hypothesis, and then finally obtain the needed expression E_x for x. Let us first prove that E_x' has the properties stated in the lemma, except for the size requirement.

Since for $1 \leq i \leq n$ each b_i is a factor of x and consists of at least one x_j' which in turn consists of at least two sectors, we have $\alpha(\widehat{b_i}) = \alpha(\widehat{x})$ and hence $x \in E_x'$. Now consider arbitrary $x', x'' \in A^*$ and suppose $x'xx'' \in L(\mathcal{M})$. We will show $x'E_x'x'' \subseteq L(\mathcal{M})$ using the argument that $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2,k}$. So let $y \in E_x'$. Then there are for $1 \leq i \leq n$ words $y_i \in (p_k(b_i)|\alpha(\widehat{x})|s_k(b_i))_k$ such that

$$y = b_0' \cdot r_1 y_1 r_1' \cdot b_1' \cdot r_2 y_2 r_2' \cdot b_2' \cdots r_n y_n r_n' \cdot b_n'.$$

Similar to Case 1 above, we turn $x'xx''$ into $x'yx''$ by showing that we can replace each b_i in x by y_i for $i = n$ downto 1, and always obtain a word in $L(\mathcal{M})$. We demonstrate the first step of this procedure and argue how it can be repeated. Define

$$y_n' =_{\text{def}} b_0' r_1 b_1 r_1' b_1' r_2 b_2 r_2' b_2' \cdots b_{n-1}' \qquad \text{and} \qquad y_n'' =_{\text{def}} b_n'.$$

Then $x'xx'' = x'y_n'r_n \cdot b_n \cdot r_n'y_n''x'' \in L(\mathcal{M})$ and we assume to the contrary that it holds that $x'y_n'r_n \cdot y_n \cdot r_n'y_n''x'' \notin L(\mathcal{M})$. Define

$$\begin{aligned}
\tilde{x} &=_{\mathrm{def}} x'y'_n r_n, \\
\tilde{v} &=_{\mathrm{def}} r'_n, \\
\tilde{w} &=_{\mathrm{def}} y_n r'_n, \\
\tilde{z} &=_{\mathrm{def}} y''_n x'', \\
s_1 &=_{\mathrm{def}} \delta(s_0, \tilde{x}) \quad \text{and} \\
s_2 &=_{\mathrm{def}} \delta(s_0, \tilde{x}\tilde{w}).
\end{aligned}$$

We claim that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$ witnessed by $\tilde{x}, \tilde{z} \in A^*$, $\tilde{w} \in A^+$, $\tilde{v} \in A^{\geq k+1}$ and states s_1 and s_2. Observe that $\tilde{v} = r'_n$ which contains at least two sectors and hence $\tilde{v} \in A^{\geq k+1}$ (only the last sector of x may be shorter than $k+1$). Note also that by assumption $\delta(s_2, \tilde{z}) \notin S'$. By the choice of regions, we know that r_n and r'_n have the same tags, hence $y'_n r_n$ and $y'_n r_n b_n r'_n$ induce the same mappings on the states of $\mathcal{M}$. It follows that $\delta^{\tilde{x}} = \delta^{\tilde{x} b_n r'_n}$. In particular, it holds that $\delta(s_0, \tilde{x} b_n r'_n) = s_1$ and since $x'xx'' = \tilde{x} b_n r'_n \tilde{z} \in L(\mathcal{M})$ we conclude $\delta(s_1, \tilde{z}) \in S'$. Moreover, statement 6 of Fact 2 holds for r'_n and hence both states s_1 and s_2 have a $\tilde{v}$-loop (note that $s_1 = \delta(s_0, \tilde{x} b_n r'_n)$). So we found a subgraph in the transition graph of $\mathcal{M}$ which is a candidate for pattern $\mathbb{B}_{3/2,k}$.

It remains to verify the condition for the respective k-decompositions, more precisely, we need to show $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{\tilde{v}\tilde{v}})$ which is the same as $\alpha(\widehat{r'_n y_n r'_n r'_n}) \subseteq \alpha(\widehat{r'_n r'_n})$. By the choice of r'_i each such word contains at least two successive sectors (see statement 1 of Fact 1). So by the choice of sectors we have $\alpha(\widehat{x}) = \alpha(\widehat{r'_n}) \subseteq \alpha(\widehat{r'_n r'_n})$. We show that $\alpha(\widehat{r'_n y_n r'_n}) \subseteq \alpha(\widehat{x})$.

Since r'_n is a factor of x the inclusion is clear for every element in the k-decomposition of $r'_n y_n r'_n$ which is a factor of r'_n. Also $\alpha(\widehat{y_n}) \subseteq \alpha(\widehat{x})$ is clear by definition of y_n. So we are left with the factors of length $k+1$ in $r'_n y_n r'_n$ that contain letters from r'_n and also from y_n. It holds that

$$\alpha(\widehat{s_k(r'_n) p_k(y_n)}) = \alpha(\widehat{s_k(r_n) p_k(b_n)}) \subseteq \alpha(\widehat{x})$$

since r'_n and r_n have the same k-suffix, $y_n \in (p_k(b_n)|\alpha(\widehat{x})|s_k(b_n))_k$ and $s_k(r_n)p_k(b_n)$ is a factor of x. We can also state

$$\alpha(\widehat{s_k(y_n) p_k(r'_n)}) = \alpha(\widehat{s_k(b_n) p_k(r'_n)}) \subseteq \alpha(\widehat{x})$$

with the same arguments. Together, we have shown $\alpha(\widehat{r'_n y_n r'_n}) \subseteq \alpha(\widehat{x}) \subseteq \alpha(\widehat{r'_n r'_n})$ and it follows that $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{\tilde{v}\tilde{v}})$ because $|\tilde{v}| \geq k+1$. Hence we found pattern $\mathbb{B}_{3/2,k}$, a contradiction. We conclude that $x'y'_n r_n y_n r'_n y''_n x'' \in L(\mathcal{M})$.

The previous argumentation was independent of the particular prefix $x'y'_n$ and the suffix $y''_n x''$, so we can repeat this step for y_{n-1}. It is crucial here that we proceed from right to left, so all tags left to the actual substitution position remain valid, i.e., the tags still stand for the mapping induced by the respective prefix. If we define $y'_{n-1} =_{\mathrm{def}} b'_0 r_1 b_1 r'_1 b'_1 r_2 b_2 r'_2 b'_2 \cdots b'_{n-2}$ and $y''_{n-1} =_{\mathrm{def}} b'_{n-1} r_n y_n r'_n b'_n$ then $x'y'_{n-1} r_{n-1} b_{n-1} r'_{n-1} y''_{n-1} x'' \in L(\mathcal{M})$ by the previous step. Now we want to substitute b_{n-1} by y_{n-1} and we observe that we have the same starting position as in the previous step. In the same way we get $x'y'_{n-1} r_{n-1} y_{n-1} r'_{n-1} y''_{n-1} x'' \in L(\mathcal{M})$. If we repeat this procedure n-times, we obtain $x'yx'' \in L(\mathcal{M})$, which shows $x'E'_x x'' \subseteq L(\mathcal{M})$.

The remaining task is to construct the expression E_x with bounded size. We start with expression E'_x and apply the induction hypothesis to each sector s_i in E'_x. Note that we have

$|\alpha(\widehat{s_i})| \leq N$ by the construction of the sectors, so we may replace each remaining s_i by E_{s_i} and obtain the expression E_x.

Finally, let us verify what is required for E_x. Since $x \in E'_x$ and $s_i \in E_{s_i}$ for every sector s_i in E'_x, we obtain $x \in E'_x \subseteq E_x$. Now let $x', x'' \in A^*$ with $x'xx'' \in L(\mathcal{M})$. We already know that $x'E'_xx'' \subseteq L(\mathcal{M})$. Exactly as in Case 1 we can show that we do not leave $L(\mathcal{M})$ if we replace in $x'E'_xx''$ each sector s_i by E_{s_i}. This shows $x'E_xx'' \subseteq L(\mathcal{M})$. Now we consider the size of E_x. By definition, E'_x is a concatenation of sectors s_i and terms of the form $(p_k(b_i)|\alpha(\widehat{x})|s_k(b_i))_k$. Since the latter terms do not influence the size of expressions, we obtain

$$\|E_x\| = \sum_{\substack{\text{sectors } s_i \\ \text{in } E'_x}} \|E_{s_i}\|.$$

By induction hypothesis we have $\|E_{s_i}\| \leq \mathcal{S}(k, |A|, |\mathcal{M}|, N)$ for all sectors s_i of E'_x. By statement 5 of Fact 2 we know that the number of sectors in E'_x is less than or equal to $3\mathcal{K}(|\mathcal{M}|) \cdot (5|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1)$. It follows that

$$\|E_x\| \leq 3\mathcal{K}(|\mathcal{M}|) \cdot (5|\mathcal{M}|^{|\mathcal{M}|} \cdot |A|^k + 1) \cdot \mathcal{S}(k, |A|, |\mathcal{M}|, N) = \mathcal{S}(k, |A|, |\mathcal{M}|, N + 1).$$

This completes the induction. ❑

We show the remaining inclusion.

Lemma 4.28. *Let $k \geq 0$. It holds that $\mathcal{FP}(\mathbb{B}_{3/2,k}) \subseteq \mathcal{B}_{3/2,k}$.*

Proof. If $L \in \mathcal{FP}(\mathbb{B}_{3/2,k})$ then there exists some DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ which does not have pattern $\mathbb{B}_{3/2,k}$. We apply Lemma 4.27 to every $x \in L(\mathcal{M})$ and obtain corresponding expressions E_x with $x \in E_x$. Since $x = \varepsilon x \varepsilon \in L(\mathcal{M})$ we have $E_x = \varepsilon E_x \varepsilon \subseteq L(\mathcal{M})$. So it holds that

$$L(\mathcal{M}) = \bigcup_{x \in L(\mathcal{M})} E_x$$

Also by Lemma 4.27, the size of E_x is bounded by $\mathcal{S}(k, |A|, |\mathcal{M}|, |\alpha(\widehat{x})|)$, so in particular $\mathcal{S}(k, |A|, |\mathcal{M}|, |A|^{k+1})$ bounds the size of each E_x. Because there is only a finite number of expressions in $\mathcal{E}_k$ having the same size, the above union is finite.

It remains to show that languages described by expressions from $\mathcal{E}_k$ are in $\mathcal{B}_{3/2,k} = \widetilde{\mathcal{B}}_{3/2,k}$. Languages of the form $\{u\}$ with $u \in A^+$ are in $\widetilde{\mathcal{B}}_{3/2,k}$, because they can be written either as $\{u\} \subseteq A^{\leq k}$ or as $(\emptyset, \alpha_1, \emptyset, \alpha_2, \ldots, \emptyset, \alpha_n, \emptyset)_k$ if $\widehat{u} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let us consider languages of the form $(w|\Sigma|v)_k$ with $w, v \in A^k$ and $\Sigma \subseteq A^{k+1}$. For $k = 0$ they can be written as $(\Sigma)_k \in \widetilde{\mathcal{B}}_{3/2,k}$. If $k \geq 1$ we have $(w|\Sigma|v)_k = ((w(w^{-1}(\Sigma)_k))v^{-1})v$. Since $|w| = |v| = k$ and $(\Sigma)_k \subseteq A^{\geq k+1}$ we can apply Proposition 4.14. Together with Proposition 4.15 we obtain $(w|\Sigma|v)_k \in \widetilde{\mathcal{B}}_{3/2,k}$. Thus we have shown that languages of the form $\{u\}$ and $(w|\Sigma|v)_k$ are in $\widetilde{\mathcal{B}}_{3/2,k}$. Again by Proposition 4.15 we conclude that languages described by expressions from $\mathcal{E}_k$ are in $\widetilde{\mathcal{B}}_{3/2,k}$. Together we see that $L(\mathcal{M})$ is a finite union of languages from $\widetilde{\mathcal{B}}_{3/2,k}$. Hence $L(\mathcal{M}) \in \widetilde{\mathcal{B}}_{3/2,k} = \mathcal{B}_{3/2,k}$ by Theorem 4.9. ❑

4.3.3 Strictness and Decidability Results

As in the case of the forbidden pattern characterizations we have obtained earlier, Theorem 4.22 has the decidability of $\mathcal{B}_{3/2,k}$ for fixed k as a consequence. Moreover, it follows that the hierarchy of classes $\mathcal{B}_{3/2,k}$ is strict, which we show first. We could provide a witnessing language L and analyse that a DFA accepting L has pattern $\mathbb{B}_{3/2,k}$ but not pattern $\mathbb{B}_{3/2,k+1}$. However, we use instead a easy counting argument and the normal form result $\mathcal{B}_{3/2,k} = \widetilde{\mathcal{B}}_{3/2,k}$ from Theorem 4.9 in the following proof.

Theorem 4.29. *For all $k \geq 0$ it holds that $\mathcal{B}_{3/2,k} \subsetneq \mathcal{B}_{3/2,k+1}$.*

Proof. It holds that $\mathcal{B}_{3/2,k} \subseteq \mathcal{B}_{3/2,k+1}$ for all $k \geq 0$ by Proposition 1.31. Let $a, b \in A$ be different letters, let $w_i =_{\mathrm{def}} \left(a^{k+1}b\right)^i$ for $i \geq 1$ and define

$$L =_{\mathrm{def}} \left(a^{k+1}b, \left\{ a^i b a^{k+1-i} \middle| \; 0 \leq i \leq k+1 \right\}, a^{k+1}b\right)_{k+1}.$$

So $L \in \widetilde{\mathcal{B}}_{3/2,k+1}$ and it is easy to see that $L = \{ w_i \,|\, i \geq 2 \}$. We assume that also $L \in \widetilde{\mathcal{B}}_{3/2,k}$ and show that this is not true. Since $L \subseteq A^{\geq k+2}$, we have

$$L = \bigcup_{i=1}^{m} (\Sigma_{i,0}, \alpha_{i,1}, \Sigma_{i,1}, \ldots, \alpha_{i,m_i}, \Sigma_{i,m_i})_k$$

for some $m \geq 1$, $m_i \geq 1$, $\alpha_{i,j} \in A^{k+1}$ and $\Sigma_{i,j} \subseteq A^{k+1}$. Let n be the maximum over all m_i. There exists some l with $1 \leq l \leq m$ such that $w_{2n+2} \in (\Sigma_{l,0}, \alpha_{l,1}, \Sigma_{l,1}, \ldots, \alpha_{l,m_l}, \Sigma_{l,m_l})_k$. The k-decomposition of w_{2n+2} consists of $(2n+2)(k+2) - k \geq n + (n+1)(k+2)$ elements of A^{k+1}. By the pigeon hole principle there exists some $1 \leq h \leq m_l$ such that at least $k+2$ consecutive elements of this k-decomposition are assigned to $\Sigma_{l,h}$. By definition of w_{2n+2} the element a^{k+1} must appear in this sequence of words of length $k+1$. Therefore, we can pump up the word w_{2n+2} with letters a without leaving L. This is a contradiction, because in every word from L the number of consecutive letters a is bounded by $k+1$. So $L \in \widetilde{\mathcal{B}}_{3/2,k+1} \backslash \widetilde{\mathcal{B}}_{3/2,k}$. ❑

Next we turn to decidability issues and give in the following proof an efficient algorithm for the membership problem of $\mathcal{B}_{3/2,k}$ for fixed k.

Theorem 4.30. *For fixed $k \geq 0$ the membership problem of $\mathcal{B}_{3/2,k}$ is decidable in nondeterministic logarithmic space* NL.

Proof. Let some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ be given. Note that $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2,k}$ if and only if $L(\mathcal{M}) \in \mathcal{B}_{3/2,k}$ by Theorem 4.22 and since $\mathcal{FP}(\mathbb{B}_{3/2,k})$ is well–defined. So it suffices provide an algorithm that decides whether $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$. If $x, z \in A^*$, $w \in A^+$ and $v \in A^{\geq k+1}$ witness the occurrence of pattern $\mathbb{B}_{3/2,k}$ then we may assume that w has length $\geq k+1$, otherwise we take vwv instead of w. Note that this does not affect the side condition.

1. Guess states $s_1, s_2, s^+, s^- \in S$ and store them. Check whether $s^+ \in S'$, $s^- \notin S'$, $s_0 \longrightarrow s_1$ and $(s_1, s_2) \longrightarrow (s^+, s^-)$. Reject if any of this fails.
2. Check $(s_1, s_2) \longrightarrow^+ (s_1, s_2)$ and while doing this, perform the following. Let $v \in A^+$ be the sequence of continuously guessed letters. Start guessing the end of v not before $|v| \geq k+1$ and store $p_k(v)$. While guessing v store also the actual value of $s_k(v)$ and the set $M' =_{\mathrm{def}} \alpha(\widehat{v})$. If $(s_1, s_2) \xrightarrow{v} (s_1, s_2)$ then determine $M =_{\mathrm{def}} \alpha(\widehat{vv})$ with help of M', $p_k(v)$ and $s_k(v)$, and store the sets M', M.

3. Check $s_1 \longrightarrow^+ s_2$ and while doing this, perform the following. Let $w \in A^+$ be the sequence of continuously guessed letters. As before, start guessing the end of w not before $|w| \geq k+1$, store $p_k(w)$, the actual value of $s_k(w)$ and also the set $N' =_{\text{def}} \alpha(\widehat{w})$. If $s_1 \xrightarrow{w} s_2$ then determine the set $N =_{\text{def}} \alpha(\widehat{vwv})$ from N' with help of $s_k(v), p_k(v), s_k(w), p_k(w)$ and the set M'. Store the set N.
4. Accept if $N \subseteq M$, otherwise reject.

It is easy to see that this algorithm accepts if and only if $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$. The space needed to store the states s_1, s_2, s^+, s^- and to check the reachability conditions is $O(\log |\mathcal{M}|)$. For the sets M', M, N', N the algorithm uses space at most $O(k) \cdot |A|^{k+1}$ which is a constant to the algorithm. ❑

For the case $k = 0$ a similar algorithm is provided in [PW97].

4.4 Forbidden Pattern Characterization of $\mathcal{B}_{3/2}$

Now we have the following interesting situation. Suppose some DFA $\mathcal{M}$ is given. If $L(\mathcal{M})$ is in $\mathcal{B}_{3/2}$ then it is in $\mathcal{B}_{3/2,k}$ for some $k \geq 0$ and we already know how to decide this for $k = 0, 1, 2, \ldots$. On the other hand, if $L(\mathcal{M}) \notin \mathcal{B}_{3/2}$ then it has for all $k \geq 0$ the patterns $\mathbb{B}_{3/2,k}$. But due to the finiteness of $\mathcal{M}$ these patterns should not be all different, i.e., we can do something similar as in case of $\mathcal{B}_{1/2}$ and $\mathcal{D}_k^{\text{left}}$: if k is sufficiently large in comparison to $\mathcal{M}$ then we identify a single pattern $\mathbb{B}_{3/2}$ in $\mathcal{M}$ which characterizes $\mathcal{B}_{3/2}$. As usual, the decidability of the membership problem of $\mathcal{B}_{3/2}$ follows from this forbidden pattern characterization.

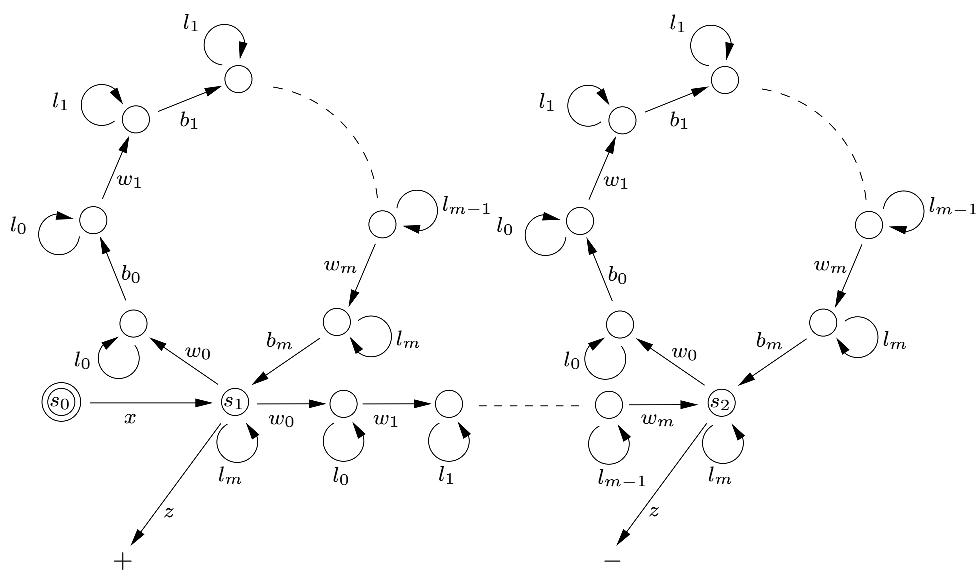


Fig. 4.3. Pattern $\mathbb{B}_{3/2}$.

Definition 4.31. Pattern $\mathbb{B}_{3/2}$ is defined as the subgraph given in Figure 4.3 with $m \geq 0$, $x, z \in A^*$ and $w_i, l_i, b_i \in A^+$.

That $\mathcal{FP}(\mathbb{B}_{3/2})$ is well–defined follows from Proposition 5.14 together with Theorem 6.4 where this is shown for generalized patterns of which $\mathbb{B}_{3/2}$ is a special case. We prove in this section the following theorem.

Theorem 4.32. *It holds that* $\mathcal{B}_{3/2} = \mathcal{FP}(\mathbb{B}_{3/2})$.

The proof is given in Subsection 4.4.1 and consequences are discussed in Subsection 4.4.2.

4.4.1 Proof of Theorem 4.32

We prepare the proof with the following two lemmas.

Lemma 4.33. *Suppose some* DFA $\mathcal{M}$ *has pattern* $\mathbb{B}_{3/2,k}$ *with* $k > \mathcal{K}(|\mathcal{M}|)$ *witnessed by* $x, z \in A^*$, $w \in A^+$ *and* $v \in A^{\geq k+1}$. *Then we may assume that* v *and* w *are of the form* $v = v'u$ *and* $w = w'u$ *such that* $1 \leq |u| \leq \mathcal{K}(|\mathcal{M}|)$ *and* $\delta^{uu} = \delta^u$.

Proof. We define below words $\tilde{x}, \tilde{v}, \tilde{w}, \tilde{z}$ which show that $\mathcal{M}$ has another instance of pattern $\mathbb{B}_{3/2,k}$ having the properties required in the lemma. By Corollary 4.4 we can write v as $v = v_0 u_1 v_1 \cdots u_m v_m$ such that $u_i, v_i \in A^{\leq \mathcal{K}(|\mathcal{M}|)}$ and $\delta^{u_i u_i} = \delta^{u_i}$. Since $|v| \geq k+1 > \mathcal{K}(|\mathcal{M}|)$ it must be that $m \geq 1$. Therefore, with $\dot{v} =_{\mathrm{def}} v_0 u_1 v_1 \cdots u_{m-1} v_{m-1}$, $\dot{u} =_{\mathrm{def}} u_m$ and $\ddot{v} =_{\mathrm{def}} v_m$ we can rewrite v as $v = \dot{v}\dot{u}\ddot{v}$ with $\dot{u}, \ddot{v} \in A^{\leq \mathcal{K}(|\mathcal{M}|)}$ and $\delta^{\dot{u}\dot{u}} = \delta^{\dot{u}}$. Now define

$$\begin{aligned} \tilde{x} &=_{\mathrm{def}} x\dot{v}\dot{u}, \\ \tilde{v} &=_{\mathrm{def}} \ddot{v}\dot{v}\dot{u}, \\ \tilde{w} &=_{\mathrm{def}} \ddot{v}w\dot{v}\dot{u} \text{ and} \\ \tilde{z} &=_{\mathrm{def}} \ddot{v}z. \end{aligned}$$

With $v' =_{\mathrm{def}} \ddot{v}\dot{v}$, $w' =_{\mathrm{def}} \ddot{v}w\dot{v}$ we can write $\tilde{v}$ and $\tilde{w}$ as $\tilde{v} = v'\dot{u}$ and $\tilde{w} = w'\dot{u}$ such that $1 \leq |\dot{u}| \leq \mathcal{K}(|\mathcal{M}|)$ and $\delta^{\dot{u}\dot{u}} = \delta^{\dot{u}}$. To see that $\tilde{x}, \tilde{v}, \tilde{w}$ and $\tilde{z}$ give rise to pattern $\mathbb{B}_{3/2,k}$ in $\mathcal{M}$ note that $\tilde{w} \in A^+$ and $|\tilde{v}| = |v| \geq k+1$. It remains to verify $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{\tilde{v}\tilde{v}})$. Since $\tilde{v}\tilde{w}\tilde{v}$ is a factor of $vvwvv$ and $|v| \geq k+1$ we obtain $\alpha(\widehat{\tilde{v}\tilde{w}\tilde{v}}) \subseteq \alpha(\widehat{vvwvv}) \subseteq \alpha(\widehat{vv})$. With the argument that $|v| = |\tilde{v}| \geq k+1$ one verifies that $\alpha(\widehat{vv}) = \alpha(\widehat{\tilde{v}\tilde{v}})$. ❑

Lemma 4.34. *Let* $\mathcal{M}$ *be a* DFA *and* $k \geq 3\mathcal{K}(|\mathcal{M}|)$. *If* $\mathcal{M}$ *has pattern* $\mathbb{B}_{3/2,k}$ *then* $\mathcal{M}$ *has pattern* $\mathbb{B}_{3/2}$.

Proof. First observe for $y_1, y_2 \in A^{\geq k'+1}$ and $k' \geq k$ that if $\alpha(\widehat{y_1}) \subseteq \alpha(\widehat{y_2})$ with respect to k'-decomposition then also $\alpha(\widehat{y_1}) \subseteq \alpha(\widehat{y_2})$ with respect to k-decomposition. To see this note that any factor of length $k+1$ is a factor of some factor of length $k'+1$.

So we may suppose that the DFA $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$ with $k = 3\mathcal{K}(|\mathcal{M}|)$ witnessed by $x, z \in A^*$, $w \in A^+$ and $v \in A^{\geq k+1}$. With Lemma 4.33 we assume that v and w are of the form $v = v'u$ and $w = w'u$ such that $1 \leq |u| \leq \mathcal{K}(|\mathcal{M}|)$ and $\delta^{uu} = \delta^u$. It follows that the states s_1 and s_2 in pattern $\mathbb{B}_{3/2,k}$ both have a u-loop. Next we obtain with help of Corollary 4.4 a factorization of w' with $w' = w'_0 u_1 w'_1 \cdots u_m w'_m$ such that $w'_i, u_i \in A^{\leq \mathcal{K}(|\mathcal{M}|)}$ and $\delta^{u_i u_i} = \delta^{u_i}$. Let $u_0 =_{\mathrm{def}} u$ and $u_{m+1} =_{\mathrm{def}} u$. Then we have $u_0, w'_0, u_1, \ldots, w'_m, u_{m+1} \in A^{\leq \mathcal{K}(|\mathcal{M}|)}$ and $\delta^{u_i u_i} = \delta^{u_i}$ for $0 \leq i \leq m+1$.

We want to see how the factors $u_i w'_i u_{i+1}$ appear as factors in some loop at s_1 and at s_2. Observe that $|u_i w'_i u_{i+1}| \leq 3\mathcal{K}(|\mathcal{M}|) < k+1$ for $0 \leq i \leq m$ and that $u_0 w'_0 u_1 \cdots w'_m u_{m+1} =$

$uw'u = uw$ is a factor of vw. So each $u_i w'_i u_{i+1}$ for $0 \le i \le m$ appears in some element of the k-decomposition of vwv. From the condition $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$ it follows that for $0 \le i \le m$ each $u_i w'_i u_{i+1}$ is a factor of vv. Hence for $0 \le i \le m$ there exist v'_i, v''_i such that $vv = v'_i u_i w'_i u_{i+1} v''_i$. We make the following definitions in order to show that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2}$. Set $m' =_{\text{def}} m+1$ and $l_i =_{\text{def}} u_i$ for $0 \le i \le m'$. Furthermore, we define

$$\begin{aligned}
b_0 &=_{\text{def}} v'_0 u_0, \\
b_i &=_{\text{def}} v''_{i-1} v'_i u_i \quad \text{for } 1 \le i \le m'-1, \\
b_{m'} &=_{\text{def}} v''_m u_{m'}, \\
w_0 &=_{\text{def}} u_0 \quad \text{and} \\
w_i &=_{\text{def}} w'_{i-1} u_i \quad \text{for } 1 \le i \le m'.
\end{aligned}$$

Observe for $0 \le i \le m'$ that $w_i, l_i, b_i \in A^+$ and that there is a l_i-loop in $\mathcal{M}$ after each $w_i, b_i \in A^+$ since they have suffix $u_i = l_i$ and $\delta^{u_i u_i} = \delta^{u_i}$. It remains to show that s_1 and s_2 have a loop with label $w_0 b_0 w_1 b_1 \cdots w_{m'} b_{m'}$ and that we get from s_1 to s_2 with $w_0 w_1 \cdots w_{m'}$. To see this we look at the factorizations

$$\begin{aligned}
w_0 b_0 w_1 b_1 \cdots b_m w_{m'} b_{m'} &= \overbrace{u_0}^{w_0} \overbrace{v'_0 u_0}^{b_0} \overbrace{w'_0 u_1}^{w_1} \overbrace{v''_0 v'_1 u_1}^{b_1} \overbrace{w'_1 u_2}^{w_2} \overbrace{v''_1 v'_2 u_2}^{b_2} \cdots \overbrace{v''_{m-1} v'_m u_m}^{b_m} \overbrace{w'_m u_{m'}}^{w_{m'}} \overbrace{v''_m \, u_{m'}}^{b_{m'}} \\
&= \underbrace{u_0}_{u} \underbrace{v'_0 u_0 w'_0 u_1 v''_0}_{vv} \underbrace{v'_1 u_1 w'_1 u_2 v''_1}_{vv} v'_2 u_2 \cdots v''_{m-1} \underbrace{v'_m u_m w'_m u_{m+1} v''_m}_{vv} \underbrace{u_{m'}}_{u} \\
&= u v^{2m'} u
\end{aligned}$$

and

$$\begin{aligned}
w_0 w_1 w_2 \cdots w_m w_{m'} &= \overbrace{u_0}^{w_0} \cdot \overbrace{w'_0 u_1}^{w_1} \overbrace{w'_1 u_2}^{w_2} \cdots \overbrace{w'_{m-1} u_m}^{w_m} \overbrace{w'_m u_{m'}}^{w_{m'}} \\
&= \underbrace{u_0}_{u} \underbrace{w'_0 u_1 w'_1 u_2 \cdots w'_{m-1} u_m w'_m}_{w'} \underbrace{u_{m'}}_{u} \\
&= u w' u.
\end{aligned}$$

Recall that s_1 and s_2 have a u-loop and that $w = w'u$. ❑

Proof of Theorem 4.32. We have to show $\mathcal{B}_{3/2} = \mathcal{FP}(\mathbb{B}_{3/2})$. For the inclusion from left to right, let $L \in \mathcal{B}_{3/2}$ and let $\mathcal{M}$ be some DFA with $L(\mathcal{M}) = L$. We assume that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2}$ and show that this leads to a contradiction. Suppose $m \ge 0$, $x, z \in A^*$ and $w_i, l_i, b_i \in A^+$ witness that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2}$ and let $k \ge 0$. We define

$$\begin{aligned}
v &=_{\text{def}} w_0 \cdot l_0^k b_0 l_0^k \cdot w_1 \cdot l_1^k b_1 l_1^k \cdots w_m \cdot l_m^k b_m l_m^k \quad \text{and} \\
w &=_{\text{def}} w_0 \cdot l_0^k \cdot w_1 \cdot l_1^k \cdots w_{m-1} \cdot l_{m-1}^k \cdot w_m \cdot l_m^k.
\end{aligned}$$

Note that $\delta(s_0, x) = \delta(s_0, xv)$, $\delta(s_0, xw) = \delta(s_0, xwv)$, $\delta(s_0, xz) \in S'$ and that $\delta(s_0, xwz) \notin S'$. Moreover, it holds that $w \in A^+$ because $m \ge 0$ and $w_0 \in A^+$, and $v \in A^{\ge k+1}$ because $m \ge 0$ and $w_0 l_0^k \in A^{\ge k+1}$. So with $x, z \in A^*$, $w \in A^+$ and $v \in A^{\ge k+1}$ we have found a subgraph in $\mathcal{M}$ that is of the form as required for pattern $\mathbb{B}_{3/2,k}$. It remains to show $\alpha(\widehat{vwv}) \subseteq \alpha(\widehat{vv})$ with respect to k-decomposition. For $\beta \in \alpha(\widehat{vwv})$ we distinguish three cases.

Case 1: Suppose β is a factor of v. Then $\beta \in \alpha(\widehat{vv})$.

Case 2: Suppose β is a factor of w. Since $|l_i| \geq 1$, we have $|l_i^k| \geq k$ for $0 \leq i \leq m$. Because $|\beta| = k+1$ each occurrence of β in w overlaps at most with one of the w_i. In particular, β is a factor of $w_0 l_0^k$ or it is a factor of $l_i^k w_{i+1} l_{i+1}^k$ for some $0 \leq i < m$. From the definition of v we see that $w_0 l_0^k$ is a factor of v and that $l_i^k w_{i+1} l_{i+1}^k$ for all $0 \leq i < m$ is a factor of v, so $\beta \in \alpha(\widehat{vv})$.

Case 3: Suppose β is a factor of $s_k(v)p_k(w)$ or $s_k(w)p_k(v)$. In both cases is β a factor of $l_m^k w_0 l_0^k$. It follows that β is also a factor of vv, so $\beta \in \alpha(\widehat{vv})$.

Together, we have shown that an arbitrary DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ has pattern $\mathbb{B}_{3/2,k}$ for arbitrary $k \geq 0$, if it has pattern $\mathbb{B}_{3/2}$. By Theorem 4.22 we have $L \notin \bigcup_{k\geq 0} \mathcal{B}_{3/2,k} = \mathcal{B}_{3/2}$, a contradiction. It follows that $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2}$, so $L \in \mathcal{FP}(\mathbb{B}_{3/2})$.

Conversely, let $L \in \mathcal{FP}(\mathbb{B}_{3/2})$ and let $\mathcal{M}$ be some DFA with $L(\mathcal{M}) = L$ that does not have pattern $\mathbb{B}_{3/2}$. Assume to the contrary that $L \notin \mathcal{B}_{3/2} = \bigcup_{k\geq 0} \mathcal{B}_{3/2,k}$. So for all $k \geq 0$ it holds that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2,k}$. In particular, it has pattern $\mathbb{B}_{3/2,k}$ for $k = 3\mathcal{K}(|\mathcal{M}|)$. We apply Lemma 4.34 and obtain that $\mathcal{M}$ has pattern $\mathbb{B}_{3/2}$, a contradiction. It follows that $L \in \mathcal{B}_{3/2}$.

(End proof of Theorem 4.32.)

4.4.2 Decidability Results

Let some regular language $L \subseteq A^+$ be given via a DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$. Due to Theorem 4.32 we can decide whether $L \in \mathcal{B}_{3/2}$ by looking at the transition graph of $\mathcal{M}$ as follows. If $\mathcal{M}$ does not have pattern $\mathbb{B}_{3/2}$ then $L \in \mathcal{B}_{3/2}$ by Theorem 4.32, and if $\mathcal{M}$ has pattern $\mathbb{B}_{3/2}$ then $L \notin \mathcal{B}_{3/2}$ because $\mathcal{FP}(\mathbb{B}_{3/2})$ is well–defined.

Theorem 4.35. *The membership problem of $\mathcal{B}_{3/2}$ is decidable in nondeterministic logarithmic space* NL.

We postpone the proof of this theorem until Chapter 5. There we show for classes defined via certain generalized forbidden patterns that their membership problems can be decided in NL. Since $\mathcal{FP}(\mathbb{B}_{3/2})$ is a special case of these classes we refer here to the forthcoming proof of Theorem 6.15. We give an informal description how we can find pattern $\mathbb{B}_{3/2}$ in a given transition graph.

As in case of the patterns treated earlier, we first guess states s_1, s_2, s^+ and s^- and see if $s_0 \longrightarrow s_1$, $(s_1, s_2) \longrightarrow (s^+, s^-)$ and if s^+ is accepting and s^- is rejecting. Now we start a loop in the algorithm, and guess states r_1, r_2, r_3 and verify $(s_1, s_1, s_2) \longrightarrow^+ (r_1, r_2, r_3)$, i.e., the latter are reachable by w_0. Then we guess two more states q_1, q_2 and check $(r_1, r_2, r_3, q_1, q_2) \longrightarrow^+ (r_1, r_2, r_3, q_1, q_2)$, i.e., they all have an l_0-loop, and see if $(r_2, r_3) \longrightarrow^+ (q_1, q_2)$, i.e., there is some b_0 between r_2 and q_1, and also between r_3 and q_2. If $r_1 = s_2$, $q_1 = s_1$ and $q_2 = s_2$ we accept, otherwise we continue this procedure and start over again, this time with r_1, q_1, q_2 instead of s_1, s_1, s_2. Note that we only need to store a constant number of states at a time.

Now we know that $\mathcal{B}_{3/2}$ and also all classes $\mathcal{B}_{3/2,k}$ have decidable membership problems. The following is an immediate consequence.

Theorem 4.36. *There exists a recursive function which outputs for a given regular language $L \subseteq A^+$ the minimal $k \geq 0$ such that $L \in \mathcal{B}_{3/2,k}$ (or some special symbol if $L \notin \mathcal{B}_{3/2}$).*

Finally, we draw the connection to first–order logic. Theorem 1.22 provides the following corollary of Theorem 4.35.

Corollary 4.37. *Given a regular language L it is decidable whether L is definable by a Σ_2 formula of the logic* $\mathrm{FO}[<, \min, \max, S, P]$.

4.5 Discussion and Further Consequences

In this section, we look at consequences of Theorem 4.35 for finite semigroups and leaf languages. We leave Theorem 4.38 and 4.40 below without proof, and understand them as directions to further research.

First, we look at the algebraic approach to regular languages, which we do not follow in this thesis. As mentioned in the introduction, many results — among them the forbidden pattern characterizations of $\mathcal{L}_{1/2}$, $\mathcal{B}_{1/2}$ and $\mathcal{L}_{3/2}$ from [PW97] — have been obtained in this theory. For an introduction to the field see [Pin96]. We have given other proofs of these characterizations, and we sketch in what follows an algebraic interpretation of Theorem 4.35.

Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some minimal DFA and denote by $T_{\mathcal{M}} =_{\mathrm{def}} \{\, \delta^w \mid w \in A^+ \,\}$ the transition semigroup of $\mathcal{M}$. In general, we can associate with a semigroup T an order relation $\leq$ (reflexive, transitive, antisymmetric) on T which respects the multiplication of T, i.e., for every $\mu, \nu, \gamma \in T$ with $\mu \leq \nu$ we have $\mu\gamma \leq \nu\gamma$ and $\gamma\mu \leq \gamma\nu$. This is called the ordered semigroup $(T, \leq)$. In particular, we define in case of $T_{\mathcal{M}}$ that $\mu \leq \nu$ if and only if $\alpha\nu\beta(s_0) \in S'$ implies $\alpha\mu\beta(s_0) \in S'$ for all $\alpha, \beta \in T_{\mathcal{M}} \cup \{\mathrm{id}\}$ (where id denotes the identity mapping on T). It is easy to see that $(T_{\mathcal{M}}, \leq)$ is an ordered semigroup, which we call the ordered transition semigroup of L. We need one more notion from semigroup theory. For every element μ of a semigroup T we define the ω-power of μ as $\mu^\omega =_{\mathrm{def}} \mu^i$ where $i = \inf \left\{\, j \geq 1 \,\middle|\, \mu^j = \mu^{2j} \,\right\}$. Note that for a finite semigroup T it holds that μ^ω is always an element of T. Using Theorem 4.35 one can now show the following.

Theorem 4.38. *Let $L \subseteq A^+$ be a regular language and $(T_{\mathcal{M}}, \leq)$ be the ordered transition semigroup of L. Then $L \in \mathcal{B}_{3/2}$ if and only if $(T_{\mathcal{M}}, \leq)$ satisfies all inequalities $\{\, E_m \mid m \geq 0 \,\}$ for any choice of τ_i, β_i and γ_i from $T_{\mathcal{M}}$ where*

$$\begin{aligned} E_m &=_{\mathrm{def}} \beta^\omega \gamma \beta^\omega \leq \beta^\omega \ \textit{with} \\ \gamma &=_{\mathrm{def}} \gamma_0 \tau_0^\omega \gamma_1 \tau_1^\omega \gamma_2 \tau_2^\omega \cdots \gamma_m \tau_m^\omega \ \textit{and} \\ \beta &=_{\mathrm{def}} \gamma_0 \tau_0^\omega \beta_0 \tau_0^\omega \gamma_1 \tau_1^\omega \beta_1 \tau_1^\omega \gamma_2 \tau_2^\omega \beta_2 \tau_2^\omega \cdots \gamma_m \tau_m^\omega \beta_m \tau_m^\omega . \end{aligned}$$

These inequalities reflect in a straightforward way the subgraph given by pattern $\mathbb{B}_{3/2}$. Moreover, they characterize the variety of finite ordered semigroups which corresponds to the positive $+$–variety of languages as which $\mathcal{B}_{3/2}$ can be understood (see [Arf91, PW97]). Another characterization of this variety of semigroups can be found in [PW97]. However, the decidability of the membership problem of $\mathcal{B}_{3/2}$ could not be derived in [PW97] and was left as an open question, which we answer with the work done in this chapter.

Let us look at leaf languages now and see what we can conclude from Theorem 4.35. We already know that $\mathrm{Leaf}^{\mathrm{P}}(L) \subseteq \Sigma_2^{\mathrm{p}}$ if $L \in \mathcal{B}_{3/2}$ by Theorem 1.24. If $L \notin \mathcal{B}_{3/2}$ then a DFA $\mathcal{M}$ with $L(\mathcal{M}) = L$ has pattern $\mathbb{B}_{3/2}$ by Theorem 4.35, and we can exploit this to show what complexity classes we encounter at least in $\mathrm{Leaf}^{\mathrm{P}}(L)$. In order to state Theorem 4.40 below we need to recall some more notations from complexity theory. For the remainder of this section, let x, y denote strings over $\{0,1\}$. Moreover, for a set M let $\|M\|$ denote its cardinality. For a language class $\mathcal{C}$ we define

$$
\begin{array}{lll}
L \in \exists^{\mathrm{u}}\cdot\mathcal{C} & \Longleftrightarrow_{\mathrm{def}} & \text{there exists } D \in \mathcal{C} \text{ and polynomial } p \text{ such that} \\
& & (a)\ \ \forall x\ \big[\ \|\{\,(x,y) \mid |y| = p(|x|) \wedge (x,y) \in D\,\}\| \leq 1\ \big] \text{ and} \\
& & (b)\ \ x \in L \Leftrightarrow \exists y\ \big[\ |y| = p(|x|) \wedge (x,y) \in D\ \big] \\
L \in \forall^{\mathrm{u}}\cdot\mathcal{C} & \Longleftrightarrow_{\mathrm{def}} & \text{there exists } D \in \mathcal{C} \text{ and polynomial } p \text{ such that} \\
& & (a)\ \ \forall x\ \big[\ \|\{\,(x,y) \mid |y| = p(|x|) \wedge (x,y) \in D\,\}\| \leq 1\ \big] \text{ and} \\
& & (b)\ \ x \in L \Leftrightarrow \forall y\ \big[\ |y| = p(|x|) \implies (x,y) \in D\ \big] \\
L \in \exists!\cdot\mathcal{C} & \Longleftrightarrow_{\mathrm{def}} & \text{there exists } D \in \mathcal{C} \text{ and polynomial } p \text{ such that} \\
& & x \in L \Leftrightarrow \text{there exists exactly one } y \text{ with } \big[\ |y| = p(|x|) \wedge (x,y) \in D\ \big]
\end{array}
$$

For more background on these quantifiers we refer to [NR98]. If we drop condition (a) above we obtain the quantifiers $\exists\cdot$ and $\forall\cdot$ that build up the polynomial time hierarchy when applied to P in an alternating way. For comparison, we state the main theorem from [BKS98].

Theorem 4.39 ([BKS98]). *Let $L \subseteq A^+$ be a regular language.*

1. *If $L \in \mathcal{B}_{1/2}$ then* $\mathrm{Leaf}^{\mathrm{P}}(L) \subseteq \mathrm{NP}$.
2. *If $L \notin \mathcal{B}_{1/2}$ then* $\mathrm{Leaf}^{\mathrm{P}}(L)$ *contains one of the classes* $\forall\cdot\mathrm{P}$, $\mathrm{co}\exists!\cdot\mathrm{P}$ *or* $\mathrm{MOD}_p\mathrm{P}$ *for some prime p.*

The first statement is known from [HLS+93]. For the second statement, note that it is quite easy to see from pattern $\mathbb{B}_{1/2}$ that if $L \notin \mathcal{B}_{1/2}$ but star–free, then $\mathrm{Leaf}^{\mathrm{P}}(L)$ contains $\forall^{\mathrm{u}}\cdot\mathrm{P}$. The main task carried out in [BKS98] is resolving the promise condition (a) in the quantor $\forall^{\mathrm{u}}\cdot$ which can be achieved by looking at the possible continuations of pattern $\mathbb{B}_{1/2}$ in a DFA.

Now pattern $\mathbb{B}_{3/2}$ for $\mathcal{B}_{3/2}$ is known, and we can show the following.

Theorem 4.40. *Let $L \subseteq A^+$ be a regular language.*

1. *If $L \in \mathcal{B}_{3/2}$ then* $\mathrm{Leaf}^{\mathrm{P}}(L) \subseteq \Sigma_2^{\mathrm{P}}$.
2. *If $L \notin \mathcal{B}_{3/2}$ then* $\mathrm{Leaf}^{\mathrm{P}}(L)$ *contains one of the classes* $\forall\cdot\exists^{\mathrm{u}}\cdot\mathrm{P}$, $\mathrm{co}\exists!\cdot\exists^{\mathrm{u}}\cdot\mathrm{P}$ *or* $\mathrm{MOD}_p\mathrm{P}$ *for some prime p.*

Here also the first statement is known from [HLS+93]. For the second statement note that it is quite easy to see from pattern $\mathbb{B}_{3/2}$ that if $L \notin \mathcal{B}_{3/2}$ but star–free, then $\mathrm{Leaf}^{\mathrm{P}}(L)$ contains $\forall^{u}\exists^{\mathrm{u}}\cdot\mathrm{P}$. Using the techniques from [BKS98] together with some additional constructions we can resolve the promise condition of the outer quantor $\forall^{\mathrm{u}}\cdot$. Certainly, there is more to investigate in this direction.

Finally, we want to remark that the decidability of the membership problem of $\mathcal{B}_{3/2}$ follows also from very recently provided results in [PW00] (although not explicitly mentioned there). The authors extend the result from [Str85] to levels $n + 1/2$: for all $n \geq 0$ the membership problem of $\mathcal{B}_{n+1/2}$ is decidable if and only if the membership problem of $\mathcal{L}_{n+1/2}$ is decidable.

5. A Theory of Forbidden Patterns

We refer to the main results of this chapter. First, we consider the patterns $\mathbb{B}_{1/2}$ and $\mathbb{B}_{3/2}$ again and observe how $\mathbb{B}_{1/2}$ acts as a building block in $\mathbb{B}_{3/2}$. Surprisingly, we find this confirmed if we compare the patterns $\mathbb{L}_{1/2}$ and $\mathbb{L}_{3/2}$. However, to reveal this relation in the latter case we need to rewrite $\mathbb{L}_{3/2}$ in an appropriate way. If we continue the just observed formation procedure this leads in a natural way to an iteration rule IT (cf. Definition 5.3). We consider this rule for arbitrary initial patterns $\mathcal{I}$ fulfilling some reasonable weak assumptions. In Section 5.1 we show how IT can be used to define classes of patterns $\mathbb{P}_n^{\mathcal{I}}$ for $n \geq 0$ (cf. Definitions 5.1 to 5.5). After some technical results that allow to handle these patterns we give the main result of this chapter: we prove that a complementation followed by a polynomial closure operation on the language side is captured by our iteration rule on the forbidden pattern side (cf. Theorem 5.13). Moreover, we investigate the inclusion structure between the forbidden pattern classes (cf. Theorem 5.19) and we treat their decidability (cf. Theorem 5.25).

We recall pattern $\mathbb{B}_{1/2}$ from Figure 2.2 and give the significant part of it in Figure 5.1 again. The *loop–structure* of the pattern is just the v-loop at s_1 and at s_2, and we call $p = (v, w) \in A^+ \times A^+$ the *bridge–structure* since it forms the subgraph that bridges from s_1 to s_2.

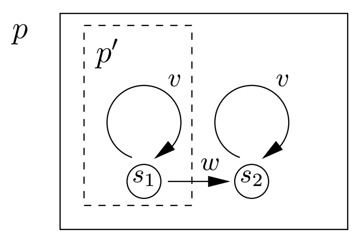


Fig. 5.1. Forbidden pattern for $\mathcal{B}_{1/2}$ with $p = (v, w) \in A^+ \times A^+$ and loop–structure p'.

Now we look at pattern $\mathbb{B}_{3/2}$ from Figure 4.3 and give its significant part in Figure 5.2 again. Here the loop–structure p' is more complex: it is the sequence of words $w_0, w_1, \ldots, w_m$ for $m \geq 0$ such that between each w_i, w_{i+1} there is the bridge–structure p_i from some pattern $\mathbb{B}_{1/2}$. Moreover, we get from s_1 with $w_0 w_1 \cdots w_m$ to s_2 and after each prefix $w_0 w_1 \cdots w_i$ we reach a state with the loop–structure p_i' (corresponding to the bridge–structure p_i between w_i and w_{i+1}).

So how may a next iteration step look like? There should be two states s_1 and s_2 both having the same loop–structure as follows. There are words $w_0, w_1, \ldots, w_m$ such that between each w_i, w_{i+1} there is the bridge–structure p' now from some pattern $\mathbb{B}_{3/2}$. Furthermore, we

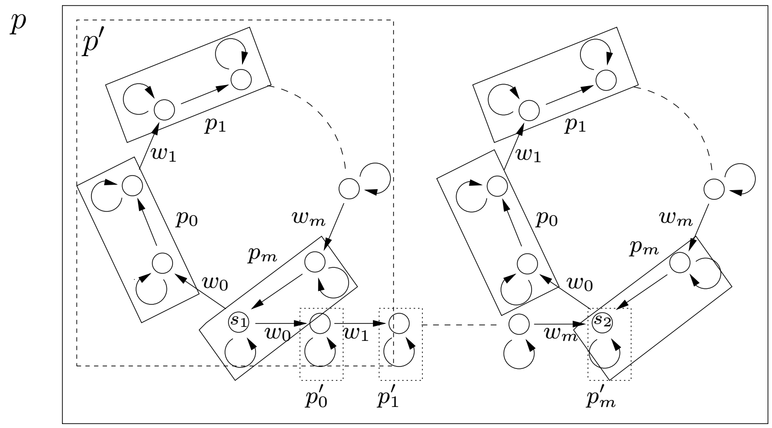


Fig. 5.2. Forbidden pattern for $\mathcal{B}_{3/2}$ with $p = (w_0, p_0, w_1, p_1, \ldots, w_m, p_m)$ and loop–structure p'. Note that $p_i \in A^+ \times A^+$ are patterns of type $\mathbb{B}_{1/2}$ with loop–structure p'_i.

should find after every prefix $w_0 w_1 \cdots w_i$ a state with the loop–structure p' from the respective pattern $\mathbb{B}_{3/2}$ that appeared between w_i and w_{i+1}. This formation procedure is made precise in the next section.

However, if this should make any sense in connection with the DDH and STH in general, we must look first at $\mathcal{L}_{1/2}$ and $\mathcal{L}_{3/2}$. Recall from Lemma 1.20 that $\mathcal{B}_{3/2} = \mathrm{Pol}(\mathrm{co}\mathcal{B}_{1/2})$ and $\mathcal{L}_{3/2} = \mathrm{Pol}(\mathrm{co}\mathcal{L}_{1/2})$. We give pattern $\mathbb{L}_{1/2}$ from Figure 2.1 again in Figure 5.3. Here the loop–structure is just an ε-loop at s_1 and at s_2, and the bridge–structure is $p = (\varepsilon, w) \in \{\varepsilon\} \times A^*$.

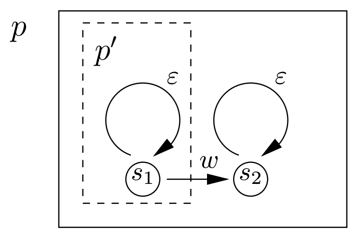


Fig. 5.3. Forbidden pattern for $\mathcal{L}_{1/2}$ with $p = (\varepsilon, w) \in \{\varepsilon\} \times A^*$ and loop–structure p'.

Now we look at pattern $\mathbb{L}_{3/2}$ from Figure 4.1 and do some rewriting before we state its significant part in Figure 5.4. In fact, this figure looks just like Figure 5.2. The only difference is that the loop–structures p'_i from some pattern $\mathbb{L}_{1/2}$ are ε-loops. So the pattern given in Figure 5.4 is equivalent to saying that for some $m \geq 0$ there are words $w_0, w_1, \ldots, w_m$ and $b_0, b_1, \ldots, b_m$ such that for $\hat{v} =_{\mathrm{def}} w_0 b_0 w_1 b_1 \cdots w_{m-1} b_m w_m$ and $\hat{w} =_{\mathrm{def}} w_0 w_1 \cdots w_m$ we have

$$s_1 \xrightarrow{\hat{v}} s_1 \text{ and } s_2 \xrightarrow{\hat{v}} s_2 \text{ and } s_1 \xrightarrow{\hat{w}} s_2.$$

We will prove formally in Theorem 6.4 that this is equivalent to pattern $\mathbb{L}_{3/2}$ but note for now that $\alpha(\hat{w}) \subseteq \alpha(\hat{v})$.

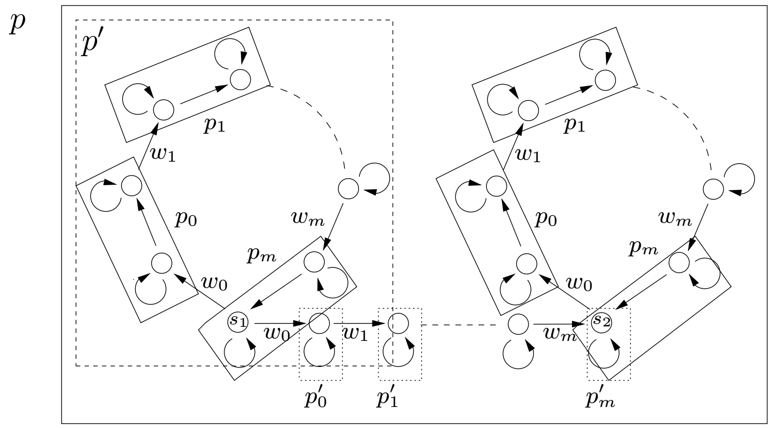


Fig. 5.4. Forbidden pattern for $\mathcal{L}_{3/2}$ with $p = (w_0, p_0, w_1, p_1, \ldots, w_m, p_m)$ and loop–structure p'. Note that $p_i \in \{\varepsilon\} \times A^*$ are patterns of type $\mathbb{L}_{1/2}$ with loop–structure $p'_i = \varepsilon$.

5.1 Pattern Iteration

We make precise what we just observed, define the iteration rule IT and provide some useful constructions that let us handle the iterated patterns. Everything in this chapter is valid for an arbitrary initial pattern $\mathcal{I}$ fulfilling some reasonable weak assumptions.

5.1.1 How to Define Iterated Patterns

Let us first say what an initial pattern is.

Definition 5.1. We define an initial pattern $\mathcal{I}$ to be a subset of $A^* \times A^*$ such that for all $r \geq 1$ and $v, w \in A^*$ it holds that $(v, w) \in \mathcal{I} \implies (v, v), (v^r, w \cdot v^r) \in \mathcal{I}$.

Note that this requirement is just what is needed to show that $\mathcal{FP}(\mathbb{B}_{1/2})$ is well–defined. In order to cope with the inductive nature of the iteration rule we refine what we understand under the notion "some DFA $\mathcal{M}$ has pattern $\mathbb{P}$". So far we had to find the particular subgraph from the definition of $\mathbb{P}$ in the transition graph of $\mathcal{M}$. This will still be the case but we consider $\mathbb{P}$ now to be a set of tuples of words, where each tuple is an instance of the pattern. We say in Definition 5.5 below that a DFA $\mathcal{M}$ has pattern $\mathbb{P}$ if there exists some tuple $p \in \mathbb{P}$ that witnesses certain reachability condition, namely that a certain subgraph appears at some state (the loop–structure) and that two states are connected by a certain subgraph (the bridge–structure). This is consistent with our prior definitions of patterns since the witnessing word variables are all existentially quantified. As the first step of an inductive definition we consider an initial pattern.

Definition 5.2. For $p = (v, w) \in \mathcal{I}$ and given states s, s_1, s_2 of some DFA $\mathcal{M}$ we say

– p appears at $s \iff_{\text{def}} s$ has a v-loop and
– s_1, s_2 are connected via p (in notation $s_1 \overset{p}{\rightsquigarrow} s_2$) $\iff_{\text{def}}$ p appears at s_1 and s_2, and $s_1 \xrightarrow{w} s_2$.

It is easy to compare this definition to the patterns $\mathbb{B}_{1/2}$ and $\mathbb{L}_{1/2}$ with $\mathcal{I} = A^+ \times A^+$ and $\mathcal{I} = \{\varepsilon\} \times A^*$, respectively. Now we formalize the iteration rule.

Definition 5.3. For every set $\mathbb{P}$ we define

$$\mathrm{IT}(\mathbb{P}) =_{\mathrm{def}} \left\{ (w_0, p_0, \ldots, w_m, p_m) \,\middle|\, m \geq 0, p_i \in \mathbb{P}, w_i \in A^+ \right\}.$$

Moreover, for an initial pattern $\mathcal{I}$ we set $\mathbb{P}_0^{\mathcal{I}} =_{\mathrm{def}} \mathcal{I}$ and $\mathbb{P}_{n+1}^{\mathcal{I}} =_{\mathrm{def}} \mathrm{IT}(\mathbb{P}_n^{\mathcal{I}})$ for $n \geq 0$ to obtain iterated patterns starting with $\mathcal{I}$. We have to say what it means that we find an iterated pattern in the transition graph of some DFA.

Definition 5.4. For some $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathrm{IT}(\mathbb{P}_{n+1}^{\mathcal{I}})$ with $n \geq 0$ and given states s, s_1, s_2 of some DFA $\mathcal{M}$ we say

– p appears at $s \Longleftrightarrow_{\mathrm{def}}$ there exist states $q_0, r_0, \ldots, q_m, r_m$ of $\mathcal{M}$ such that

$$s \xrightarrow{w_0} q_0 \overset{p_0}{\rightsquigarrow} r_0 \xrightarrow{w_1} q_1 \overset{p_1}{\rightsquigarrow} r_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m \overset{p_m}{\rightsquigarrow} r_m = s$$

– s_1, s_2 are connected via p (in notation $s_1 \overset{p}{\rightsquigarrow} s_2$) $\Longleftrightarrow_{\mathrm{def}}$ p appears at s_1 and s_2, and there exist states $q_0, \ldots, q_m$ of $\mathcal{M}$ such that p_i appears at state q_i for $0 \leq i \leq m$ and

$$s_1 \xrightarrow{w_0} q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m = s_2.$$

If we compare the definition of $\mathbb{P}_n^{\mathcal{I}}$ for $\mathcal{I} = A^+ \times A^+$ and $\mathcal{I} = \{\varepsilon\} \times A^*$ with the patterns $\mathbb{B}_{3/2}$ and $\mathbb{L}_{3/2}$ as given in Figures 5.2 and 5.4, respectively, we see how our previous observations are reflected. Finally, we define what it means that some DFA has pattern $\mathbb{P}_n^{\mathcal{I}}$.

Definition 5.5. For a DFA $\mathcal{M} = (A, S, \delta, s_0, S')$, an initial pattern $\mathcal{I}$ and $n \geq 0$ we say that $\mathcal{M}$ has pattern $\mathbb{P}_n^{\mathcal{I}}$ if and only if there exist $s_1, s_2 \in S$, $x, z \in A^*$ and $p \in \mathbb{P}_n^{\mathcal{I}}$ such that $\delta(s_0, x) = s_1$, $\delta(s_1, z) \in S'$, $\delta(s_2, z) \notin S'$ and $s_1 \overset{p}{\rightsquigarrow} s_2$.

5.1.2 Some Technical Results

We give two useful constructions to obtain from a given pattern $p \in \mathbb{P}_n^{\mathcal{I}}$ a new pattern from $\mathbb{P}_n^{\mathcal{I}}$ having certain nice properties. Before this, we fix a word $\overline{p}^{\circ}$ obtained from the loop–structure of p (call this the *loop–word*), and a word $\overline{p}$ derived from the bridge–structure of p (the *bridge–word*).

Definition 5.6. Let $\mathcal{I}$ be an initial pattern. For $p = (v, w) \in \mathbb{P}_0^{\mathcal{I}}$ we define $\overline{p} =_{\mathrm{def}} w$ and $\overline{p}^{\circ} =_{\mathrm{def}} v$. For $n \geq 0$ and $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_{n+1}^{\mathcal{I}}$ we define $\overline{p} =_{\mathrm{def}} w_0 \cdots w_m$ and $\overline{p}^{\circ} =_{\mathrm{def}} w_0 \overline{p_0} \cdots w_m \overline{p_m}$.

We describe two constructions. First, for $p \in \mathbb{P}_n^{\mathcal{I}}$ some $\lambda(p) \in \mathbb{P}_n^{\mathcal{I}}$ can be defined such that if p appears at some state s then s, s are connected via $\lambda(p)$ (cf. Definition 5.8 and Lemma 5.9). Secondly, in Definition 5.10 we pump up the loop–structure of p to construct for given $r \geq 3$ some $\pi(p, r) \in \mathbb{P}_n^{\mathcal{I}}$ such that

– if two states are connected via p, then they are also connected via $\pi(p, r)$ (cf. Lemma 5.11) and
– in every DFA $\mathcal{M}$ with $|\mathcal{M}| \leq r$ the words $\overline{\pi(p, r)}$ and $\overline{\pi(p, r)}^{\circ}$ lead to states where $\pi(p, r)$ appears (cf. Lemma 5.12).
– in every DFA $\mathcal{M}$ with $|\mathcal{M}| \leq r$ the words $\overline{\pi(p, r)}^{\circ}$ and $\overline{\pi(p, r)}^{\circ}\,\overline{\pi(p, r)}$ lead to states that are connected via $\pi(p, r)$ (cf. Lemma 5.12).

Let us begin with some easy to see statements. In particular the second and third statement show why we call $\overline{p}$ the bridge–word and $\overline{p}^{\circ}$ the loop–word of p.

Proposition 5.7. *Let $\mathcal{I}$ be an initial pattern, $n \geq 0$, $p \in \mathbb{P}_n^{\mathcal{I}}$ and let s, s_1, s_2 be states of some* DFA*.*

1. *If $n \geq 1$ then $\overline{p}, \overline{p}^{\circ} \in A^+$.*
2. *If $s_1 \stackrel{p}{\rightsquigarrow} s_2$ then $s_1 \xrightarrow{\overline{p}} s_2$ and p appears at s_1 and at s_2.*
3. *If p appears at state s then $s \xrightarrow{\overline{p}^{\circ}} s$.*
4. *If p appears at state s and if $p = (w_0, p_0, \ldots, w_m, p_m)$ with $p_i \in \mathbb{P}_{n-1}^{\mathcal{I}}$ for $n \geq 1$, then also p_m appears at state s.*

All statements are immediate from the definitions (for the third use the second statement). We give the construction of $\lambda(p)$.

Definition 5.8. Let $\mathcal{I}$ be an initial pattern. For $p = (v, w) \in \mathbb{P}_0^{\mathcal{I}}$ we define $\lambda(p) =_{\mathrm{def}} (v, v)$. For $n \geq 1$ and $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_n^{\mathcal{I}}$ we define $\lambda(p) =_{\mathrm{def}} (\overline{p}^{\circ}, \lambda(p_m))$.

The following lemma states the announced property of $\lambda(p)$.

Lemma 5.9. *For every initial pattern $\mathcal{I}$, $n \geq 0$ and $p \in \mathbb{P}_n^{\mathcal{I}}$ we have $\lambda(p) \in \mathbb{P}_n^{\mathcal{I}}$. Moreover, if p appears at state s of some* DFA *then s, s are connected via $\lambda(p)$, i.e., $s \stackrel{\lambda(p)}{\rightsquigarrow} s$.*

Proof. We prove the lemma by induction on n. For $n = 0$ we have $p = (v, w) \in \mathbb{P}_0^{\mathcal{I}} = \mathcal{I}$. By Definition 5.1 it holds that $\lambda(p) = (v, v) \in \mathcal{I} = \mathbb{P}_0^{\mathcal{I}}$. If p appears at some state s we have $\delta(s, v) = s$ by definition. Therefore, the states s, s are connected via $\lambda(p) = (v, v)$ by Definition 5.2.

Assume the lemma holds for some $n \geq 0$ and we want to prove it for $n+1$. Let $p \in \mathbb{P}_{n+1}^{\mathcal{I}}$ such that for some $m \geq 0$, $p_i \in \mathbb{P}_n^{\mathcal{I}}$ and $w_i \in A^+$ we have $p = (w_0, p_0, \ldots, w_m, p_m)$. By Proposition 5.7 we have $\overline{p}^{\circ} \in A^+$ and from induction hypothesis we know that $\lambda(p_m) \in \mathbb{P}_n^{\mathcal{I}}$. So with Definition 5.3 we see that $\lambda(p) = (\overline{p}^{\circ}, \lambda(p_m)) \in \mathrm{IT}(\mathbb{P}_n^{\mathcal{I}}) = \mathbb{P}_{n+1}^{\mathcal{I}}$.

It remains to show that the states s, s are connected via $\lambda(p) = (\overline{p}^{\circ}, \lambda(p_m))$ in some DFA if p appears at state s. By Proposition 5.7 we know that p_m appears at state s so we get from the induction hypothesis that s, s are connected via $\lambda(p_m)$. Since $\delta(s, \overline{p}^{\circ}) = s$ by Proposition 5.7 we obtain that $\lambda(p)$ appears at state s. Now let $s_1 =_{\mathrm{def}} s$, $s_2 =_{\mathrm{def}} s$ and $q_0 =_{\mathrm{def}} s$. Then $q_0 = s_2$ and since p appears at state s it follows from Proposition 5.7 that $\delta(s_1, \overline{p}^{\circ}) = q_0$. We have already seen that s, s are connected via $\lambda(p_m)$, particularly $\lambda(p_m)$ appears at state $s = q_0$ by Proposition 5.7. This shows that s, s are connected via $\lambda(p)$. ❑

The second construction, i.e., the construction of $\pi(p, r)$ is more involved.

Definition 5.10. Let $\mathcal{I}$ be an initial pattern and $r \geq 3$. For $p = (v, w) \in \mathbb{P}_0^{\mathcal{I}}$ let $\pi(p, r) =_{\mathrm{def}} (v^{r!}, w \cdot v^{r!})$. For $n \geq 1$ and $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_n^{\mathcal{I}}$ we define the following:

$$\begin{aligned}
p_i' &=_{\mathrm{def}} \pi(p_i, r) \\
w &=_{\mathrm{def}} w_0 \cdot \overline{p_0'}^{\circ} \cdot \overline{p_0'} \cdot w_1 \cdot \overline{p_1'}^{\circ} \cdot \overline{p_1'} \cdots w_m \cdot \overline{p_m'}^{\circ} \cdot \overline{p_m'} \\
\pi(p, r) &=_{\mathrm{def}} (w_0 \cdot \overline{p_0'}^{\circ}, p_0', w_1 \cdot \overline{p_1'}^{\circ}, p_1', \ldots, w_m \cdot \overline{p_m'}^{\circ}, p_m', \underbrace{w, \lambda(p_m'), \ldots, w, \lambda(p_m')}_{(r!-1) \text{ times } \text{“}w, \lambda(p_m')\text{”}})
\end{aligned}$$

With the next lemma we show that pattern $\pi(p,r)$ is equivalent to p in the sense that it appears at a state and connects states in some DFA if p does.

Lemma 5.11. *Let $\mathcal{I}$ be an initial pattern, $r \geq 3$, $n \geq 0$, $p \in \mathbb{P}_n^{\mathcal{I}}$ and let s, s_1, s_2 be states of some* DFA.

1. *It holds that $\pi(p,r) \in \mathbb{P}_n^{\mathcal{I}}$.*
2. *If p appears at some state s then also $\pi(p,r)$ appears at s.*
3. *If $s_1 \overset{p}{\rightsquigarrow} s_2$ then s_1, s_2 are connected also via $\pi(p,r)$, i.e., $s_1 \overset{\pi(p,r)}{\rightsquigarrow} s_2$.*

Proof. We show the three statements simultaneously by induction on n.
Induction base. For $n = 0$ we have $p = (v, w) \in \mathbb{P}_0^{\mathcal{I}} = \mathcal{I}$ and $p' =_{\text{def}} \pi(p,r) = (v^{r!}, w \cdot v^{r!})$. By Definition 5.1 we get $p' \in \mathcal{I} = \mathbb{P}_0^{\mathcal{I}}$. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a DFA and $s, s_1, s_2 \in S$. If p appears at state s then $\delta(s, v) = s$. Hence $\delta(s, v^{r!}) = s$ and so p' appears at state s. If s_1, s_2 are connected via p, then $s_1 = \delta(s_1, v)$ and $s_2 = \delta(s_1, w) = \delta(s_2, v)$. It follows that $s_1 = \delta(s_1, v^{r!})$ and $s_2 = \delta(s_1, w \cdot v^{r!}) = \delta(s_2, v^{r!})$. Thus s_1, s_2 are also connected via p'.
Induction step. Suppose we have shown the lemma for some $n \geq 0$ and we want to show it for $n+1$. Let $p \in \mathbb{P}_{n+1}^{\mathcal{I}}$, choose suitable $m \geq 0$, $p_i \in \mathbb{P}_n^{\mathcal{I}}$ and $w_i \in A^+$ such that $p = (w_0, p_0, \ldots, w_m, p_m)$ and set $p' =_{\text{def}} \pi(p,r)$. As in Definition 5.10 let $p_i' =_{\text{def}} \pi(p_i, r)$ and $w =_{\text{def}} w_0 \cdot \overline{p_0'}^{\circ} \cdot \overline{p_0'} \cdots w_m \cdot \overline{p_m'}^{\circ} \cdot \overline{p_m'}$. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA. First we show the following claim.

Claim. Let $s' \in S$ such that p appears at s'. Then it holds that $\delta(s', w) = s'$.

Proof of Claim. If p appears at s' then there exist states $q_0, r_0, \ldots, q_m, r_m$ of $\mathcal{M}$ such that

$$s' \xrightarrow{w_0} q_0 \overset{p_0}{\rightsquigarrow} r_0 \xrightarrow{w_1} q_1 \overset{p_1}{\rightsquigarrow} r_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m \overset{p_m}{\rightsquigarrow} r_m = s'.$$

By induction hypothesis we know that q_i, r_i are also connected via p_i' for $0 \leq i \leq m$. From Proposition 5.7 we see that $\delta(q_i, \overline{p_i'}) = r_i$ for $0 \leq i \leq m$. Again by Proposition 5.7 the state q_i has a $\overline{p_i'}^{\circ}$-loop for $0 \leq i \leq m$. It follows that $\delta(s', w_0 \cdot \overline{p_0'}^{\circ} \cdot \overline{p_0'} \cdots w_i \cdot \overline{p_i'}^{\circ} \cdot \overline{p_i'}) = r_i$ for $0 \leq i \leq m$. This shows $\delta(s', w) = r_m = s'$. **(End proof of Claim.)**

We use similar arguments to show the induction step. For the first statement observe that by hypothesis we have $p_0', \ldots, p_m' \in \mathbb{P}_n^{\mathcal{I}}$. By Lemma 5.9 it holds that $\lambda(p_m') \in \mathbb{P}_n^{\mathcal{I}}$. Since $w \in A^+$ and also $w_0\overline{p_0'}^{\circ}, \ldots, w_m\overline{p_m'}^{\circ} \in A^+$ it follows that $\pi(p,r) \in \mathrm{IT}(\mathbb{P}_n^{\mathcal{I}}) = \mathbb{P}_{n+1}^{\mathcal{I}}$.

Let $s, s_1, s_2 \in S$. For the second statement assume that p appears at state s. By definition, there exist states $q_0, r_0, \ldots, q_m, r_m$ of $\mathcal{M}$ such that

$$s \xrightarrow{w_0} q_0 \overset{p_0}{\rightsquigarrow} r_0 \xrightarrow{w_1} q_1 \overset{p_1}{\rightsquigarrow} r_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m \overset{p_m}{\rightsquigarrow} r_m = s.$$

Using the additional states $q_j =_{\text{def}} s$ and $r_j =_{\text{def}} s$ for $m+1 \leq j \leq m'$ with $m' =_{\text{def}} m + r! - 1$ we show that also p' appears at state s. Therefore we have to show that

$$s \xrightarrow{w_0\overline{p_0'}^{\circ}} q_0 \overset{p_0'}{\rightsquigarrow} r_0 \xrightarrow{w_1\overline{p_1'}^{\circ}} q_1 \overset{p_1'}{\rightsquigarrow} r_1 \xrightarrow{w_2\overline{p_2'}^{\circ}} \cdots \xrightarrow{w_m\overline{p_m'}^{\circ}} q_m \overset{p_m'}{\rightsquigarrow} r_m \xrightarrow{w} q_{m+1} \overset{\lambda(p_m')}{\rightsquigarrow} r_{m+1} \cdots \xrightarrow{w} q_{m'} \overset{\lambda(p_m')}{\rightsquigarrow} r_{m'}.$$

Note that $r_{m'} = s$ and let $1 \leq j \leq m$. Since $q_j \overset{p_j}{\rightsquigarrow} r_j$ we have by Proposition 5.7 that p_j appears at q_j. We obtain from the hypothesis that p_j' appears at q_j and from Proposition 5.7 that $\delta(q_j, \overline{p_j'}^{\circ}) = q_j$. So for $1 \leq j \leq m$ we see that

$$s \xrightarrow{w_0\overline{p'_0}^{\circ}} q_0 \quad \text{and} \quad r_{j-1} \xrightarrow{w_j\overline{p'_j}^{\circ}} q_j$$

while the former is shown with the same arguments. By hypothesis we have that q_i, r_i are connected via p'_i for $0 \leq i \leq m$. It remains to show that $\delta(s,w) = s$ and that s,s are connected via $\lambda(p'_m)$. For the former we apply our claim from above, for the latter note with Proposition 5.7 that p_m appears at $r_m = s$. So by hypothesis we have that p'_m appears at s and Lemma 5.9 says that s,s are connected via $\lambda(p'_m)$.

We turn to the third statement and assume that s_1, s_2 are connected via p. By definition, p appears at s_1 and s_2, and there exist states $q_0, \dots, q_m$ of $\mathcal{M}$ such that p_i appears at state q_i for $0 \leq i \leq m$ and

$$s_1 \xrightarrow{w_0} q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m = s_2.$$

We already know from the second statement that p' appears at s_1 and s_2. Using the additional states $q_j =_{\mathrm{def}} s_2$ for $m+1 \leq j \leq m'$ with $m' =_{\mathrm{def}} m + r! - 1$ we first show that

$$s_1 \xrightarrow{w_0\overline{p'_0}^{\circ}} q_0 \xrightarrow{w_1\overline{p'_1}^{\circ}} q_1 \xrightarrow{w_2\overline{p'_2}^{\circ}} \cdots \xrightarrow{w_m\overline{p'_m}^{\circ}} q_m \xrightarrow{w} q_{m+1} \cdots \xrightarrow{w} q_{m'}.$$

Note that $q_{m'} = s_2$ and let $0 \leq i \leq m$. By assumption we know that p_i appears at q_i and from the hypothesis we get that also p'_i appears at q_i. So we have by Proposition 5.7 that $\delta(q_i, \overline{p'_i}^{\circ}) = q_i$ and together with the assumption we see for $0 \leq i < m$ that

$$s \xrightarrow{w_0\overline{p'_0}^{\circ}} q_0 \quad \text{and} \quad q_i \xrightarrow{w_i\overline{p'_i}^{\circ}} q_{i+1}.$$

Since p appears at s_2 our claim shows for $m \leq j < m'$ that $\delta(q_j, w) = \delta(s_2, w) = s_2 = q_{j+1}$.

We know that p_i appears at q_i for $0 \leq i \leq m$ and by hypothesis we obtain that p'_i appears at q_i. In particular p'_m appears at $q_m = s_2$. So from Lemma 5.9 together with Proposition 5.7 we obtain that $\lambda(p'_m)$ appears at q_j for $m+1 \leq j \leq m'$. ❑

With the construction of $\pi(p,r)$ we have obtained a possibility to find patterns from $\mathbb{P}^{\mathcal{I}}_n$ in automata for which we only require that their size is less or equal to r. Note in particular that we do not require in the following lemma that $p \in \mathbb{P}^{\mathcal{I}}_n$ appears somewhere in $\mathcal{M}$ or that it connects some states, we just have the size restriction.

Lemma 5.12. *Let $\mathcal{I}$ be an initial pattern, $r \geq 3$, $n \geq 0$, $p \in \mathbb{P}^{\mathcal{I}}_n$ and let $\mathcal{M}$ be a* DFA *with $|\mathcal{M}| \leq r$. It holds that*

1. *$\overline{\pi(p,r)}^{\circ}$ leads to states in $\mathcal{M}$ where $\pi(p,r)$ appears,*
2. *$\overline{\pi(p,r)}$ leads to states in $\mathcal{M}$ where $\pi(p,r)$ appears and*
3. *$\overline{\pi(p,r)}^{\circ}$ and $\overline{\pi(p,r)}^{\circ}\overline{\pi(p,r)}$ lead to states in $\mathcal{M}$ that are connected via $\pi(p,r)$.*

Proof. We prove the lemma by induction on n.
Induction base. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA with $|\mathcal{M}| \leq r$. If $n = 0$ then we have $p = (v, w) \in \mathbb{P}^{\mathcal{I}}_0$ and $\pi(p,r) = (v^{r!}, w \cdot v^{r!})$. Since v^r leads to $v^{r!}$-loops in $\mathcal{M}$ by Proposition 1.34 we obtain that $\overline{\pi(p,r)}^{\circ} = v^{r!-r} \cdot v^r$ and $\overline{\pi(p,r)} = w \cdot v^{r!-r} \cdot v^r$ lead to states where $\pi(p,r)$ appears. Hence $\overline{\pi(p,r)}^{\circ}$ and $\overline{\pi(p,r)}^{\circ} \cdot \overline{\pi(p,r)}$ lead to states which are connected via $\pi(p,r)$.
Induction step. Suppose we have shown the lemma for some $n \geq 0$ and we want to show it for $n+1$. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA with $|\mathcal{M}| \leq r$. Furthermore, let $p = (w_0, p_0, \dots, w_m, p_m) \in \mathbb{P}^{\mathcal{I}}_{n+1}$ and assume that w, p'_i are as in Definition 5.10. First we show the following claim.

Claim. It holds that $w^{r!-1}$ leads to states in $\mathcal{M}$ where $\pi(p,r)$ appears.

Proof of Claim. Observe that $w^{r!-1} = w^{r!-1-r}w^r$ leads to a $w^{r!}$-loop in $\mathcal{M}$ by Proposition 1.34 since $r \geq 3$ and hence $r! \geq r+1$. So let s be a state of $\mathcal{M}$ which has a $w^{r!}$-loop and we will show that $\pi(p,r)$ appears at s. Define the following witnessing states.

$$\begin{array}{lll}
q_0 =_{\mathrm{def}} \delta(s, w_0 \cdot \overline{p'_0}^{\circ}) & r_0 =_{\mathrm{def}} \delta(q_0, \overline{p'_0}) & \\
q_i =_{\mathrm{def}} \delta(r_{i-1}, w_i \cdot \overline{p'_i}^{\circ}) & r_i =_{\mathrm{def}} \delta(q_i, \overline{p'_i}) & \text{for } 1 \leq i \leq m \\
q_{m+j} =_{\mathrm{def}} \delta(r_m, w^j) & r_{m+j} =_{\mathrm{def}} q_{m+j} & \text{for } 1 \leq j \leq r!-1
\end{array}$$

So we have the following situation where $m' =_{\mathrm{def}} m + r! - 1$.

$$\begin{array}{l}
s \xrightarrow{w_0 \cdot \overline{p'_0}^{\circ}} q_0 \xrightarrow{\overline{p'_0}} r_0 \xrightarrow{w_1 \cdot \overline{p'_1}^{\circ}} q_1 \xrightarrow{\overline{p'_1}} r_1 \xrightarrow{w_2 \cdot \overline{p'_2}^{\circ}} \cdots \xrightarrow{w_m \cdot \overline{p'_m}^{\circ}} q_m \xrightarrow{\overline{p'_m}} r_m \quad \text{and} \\
r_m \xrightarrow{w} q_{m+1} = r_{m+1} \xrightarrow{w} q_{m+2} = r_{m+2} \xrightarrow{w} \cdots \xrightarrow{w} q_{m'} = r_{m'}
\end{array}$$

It follows from induction hypothesis that q_i, r_i are connected via p'_i for $0 \leq i \leq m$. Moreover, the hypothesis also shows that p'_m appears at q_j for $m+1 \leq j \leq m'$ since $\overline{p'_m}$ is a suffix of w. From Lemma 5.9 we get that q_j, r_j are connected via $\lambda(p'_m)$. Finally, by the definition of w we have $r_m = \delta(s,w)$ and $r_{m'} = \delta(r_m, w^{r!-1}) = \delta(s, w^{r!}) = s$. Hence we have shown that

$$\begin{array}{l}
s \xrightarrow{w_0 \cdot \overline{p'_0}^{\circ}} q_0 \overset{p'_0}{\rightsquigarrow} r_0 \xrightarrow{w_1 \cdot \overline{p'_1}^{\circ}} q_1 \overset{p'_1}{\rightsquigarrow} r_1 \xrightarrow{w_2 \cdot \overline{p'_2}^{\circ}} \cdots \xrightarrow{w_m \cdot \overline{p'_m}^{\circ}} q_m \overset{p'_m}{\rightsquigarrow} r_m \quad \text{and} \\
r_m \xrightarrow{w} q_{m+1} \overset{\lambda(p'_m)}{\rightsquigarrow} r_{m+1} \xrightarrow{w} q_{m+2} \overset{\lambda(p'_m)}{\rightsquigarrow} r_{m+2} \xrightarrow{w} \cdots \xrightarrow{w} q_{m'} \overset{\lambda(p'_m)}{\rightsquigarrow} r_{m'} = s.
\end{array}$$

So $\pi(p,r)$ appears at s. **(End proof of Claim.)**

Since $\overline{p'_m}$ is a suffix of w it follows from the induction hypothesis that w leads to states where p'_m appears. From Lemma 5.9 and Proposition 5.7 we obtain that w leads to a $\overline{\lambda(p'_m)}$-loop in $\mathcal{M}$. Hence our claim holds also for $\left(w \cdot \overline{\lambda(p'_m)}\right)^{r!-1}$. Now observe that

$$\begin{array}{rcl}
\overline{\pi(p,r)} & = & w_0 \cdot \overline{p'_0}^{\circ} \cdots w_m \cdot \overline{p'_m}^{\circ} \cdot w^{r!-1} \quad \text{and} \\
\overline{\pi(p,r)}^{\circ} & = & w_0 \cdot \overline{p'_0}^{\circ} \cdot \overline{p'_0} \cdots w_m \cdot \overline{p'_m}^{\circ} \cdot \overline{p'_m} \cdot \left(w \cdot \overline{\lambda(p'_m)}\right)^{r!-1}.
\end{array}$$

So the claim says that $\overline{\pi(p,r)}$ and $\overline{\pi(p,r)}^{\circ}$ lead to states in $\mathcal{M}$ where $\pi(p,r)$ appears. This shows statements 1 and 2 of the lemma.

We turn to statement 3 and choose an arbitrary state s of $\mathcal{M}$. For $s_1 =_{\mathrm{def}} \delta(s, \overline{\pi(p,r)}^{\circ})$ and $s_2 =_{\mathrm{def}} \delta(s, \overline{\pi(p,r)}^{\circ} \cdot \overline{\pi(p,r)})$ we show that s_1, s_2 are connected via $\pi(p,r)$. Let $m' =_{\mathrm{def}} m + r! - 1$ and define the following witnessing states.

$$\begin{array}{ll}
q_0 =_{\mathrm{def}} \delta(s_1, w_0 \cdot \overline{p'_0}^{\circ}) & \\
q_{i+1} =_{\mathrm{def}} \delta(q_i, w_{i+1} \cdot \overline{p'_{i+1}}^{\circ}) & \text{for } 0 \leq i < m \\
q_{j+1} =_{\mathrm{def}} \delta(q_j, w) & \text{for } m \leq j < m'
\end{array}$$

We have already seen that $\pi(p,r)$ appears at s_1 and at s_2. Observe that $q_{m'} = \delta(s_1, \overline{\pi(p,r)}) = s_2$. So it remains to show that p'_i appears at q_i for $0 \leq i \leq m$ and that $\lambda(p'_m)$ appears at q_j for $m+1 \leq j \leq m'$.

By induction hypothesis we have that $\overline{p'_i}^{\circ}$ leads to states in $\mathcal{M}$ where p'_i appears. Hence p'_i appears at state q_i for $0 \leq i \leq m$. Since $\overline{p'_m}$ is a suffix of w the induction hypothesis shows that p'_m appears at q_j for all j with $m+1 \leq j \leq m'$. With Lemma 5.9 we see that q_j, q_j are connected via $\lambda(p'_m)$, so in particular $\lambda(p'_m)$ appears at state q_j. ❑

5.2 Pattern Iterator versus Polynomial Closure

We relate in this section in a general way Boolean operations and concatenation to the structural complexity of transition graphs, i.e., we show with the following theorem that a complementation followed by a polynomial closure operation on the language side is captured by our iteration rule on the forbidden pattern side.

Theorem 5.13. *Let $\mathcal{I}$ be an initial pattern and let $n \geq 0$. It holds that*

$$\mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}).$$

The proof is given in the next subsection, while we show in Subsection 5.2.2 what inclusion relations hold between the forbidden pattern classes. Subsection 5.2.3 investigates under what circumstances the classes $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ have decidable membership problems. Let us begin with the fact that for all $n \geq 0$ the classes $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ are well–defined.

Proposition 5.14. *Let $\mathcal{I}$ be an initial pattern, $n \geq 0$ and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two* DFA*'s such that $L(\mathcal{M}_1) = L(\mathcal{M}_2)$. Then it holds that $\mathcal{M}_1$ has pattern $\mathbb{P}_n^{\mathcal{I}}$ if and only $\mathcal{M}_2$ has pattern $\mathbb{P}_n^{\mathcal{I}}$.*

Proof. It suffices to show one implication, so suppose $\mathcal{M}_1$ has pattern $\mathbb{P}_n^{\mathcal{I}}$. Then there are states s_1, s_2 in $\mathcal{M}_1$ that are connected via some $p \in \mathbb{P}_n^{\mathcal{I}}$ such that $\delta(s_0, x) = s_1$, $\delta(s_1, z) \in S'$, $\delta(s_2, z) \notin S'$ for suitable $x, z \in A^*$ and if s_0 is the starting state of $\mathcal{M}_1$ and if S' is its set of accepting states. Define $r =_{\mathrm{def}} |\mathcal{M}_2|$ and $p' =_{\mathrm{def}} \pi(p, r)$. We obtain from Lemma 5.11 that s_1, s_2 in $\mathcal{M}_1$ are also connected via $p' \in \mathbb{P}_n^{\mathcal{I}}$. So by Proposition 5.7 we have $x\overline{p'}^{\circ}z \in L(\mathcal{M}_1) = L(\mathcal{M}_2)$ and $x\overline{p'}^{\circ}\overline{p'}z \notin L(\mathcal{M}_1) = L(\mathcal{M}_2)$. Now define s_1' and s_2' to be the states in $\mathcal{M}_2$ that can be reached from its starting state on input $x\overline{p'}^{\circ}$ and $x\overline{p'}^{\circ}\overline{p'}$, respectively. By Lemma 5.12 we get that s_1' and s_2' are connected via p' in $\mathcal{M}_2$. Since we reach from s_1' (s_2') with z an accepting state of $\mathcal{M}_2$ (rejecting state, respectively) this shows that $\mathcal{M}_2$ has pattern $\mathbb{P}_n^{\mathcal{I}}$. ❑

5.2.1 Proof of Theorem 5.13

We isolate the main argument of the proof in the following lemma. It says that under certain assumptions we can replace bridge–words by their respective loop–words without leaving the language of some DFA.

Lemma 5.15. *Let $\mathcal{I}$ be an initial pattern, $r \geq 3$, $n \geq 0$ and $p \in \mathbb{P}_{n+1}^{\mathcal{I}}$. Let $\mathcal{M}$ be a* DFA *with $|\mathcal{M}| \leq r$ which does not have pattern $\mathbb{P}_n^{\mathcal{I}}$. Then for all $x, z \in A^*$ it holds that*

$$x \cdot \overline{\pi(p,r)} \cdot z \in L(\mathcal{M}) \implies x \cdot \overline{\pi(p,r)}^{\circ} \cdot z \in L(\mathcal{M}).$$

Proof. We choose suitable $m \geq 0$, $w_i \in A^+$ and $p_i \in \mathbb{P}_n^{\mathcal{I}}$ such that $p = (w_0, p_0, \ldots, w_m, p_m)$. For $0 \leq i \leq m$ let as before $p_i' =_{\mathrm{def}} \pi(p_i, r)$ and $w =_{\mathrm{def}} w_0 \cdot \overline{p_0'}^{\circ} \cdot \overline{p_0'} \cdots w_m \cdot \overline{p_m'}^{\circ} \cdot \overline{p_m'}$. By Definitions 5.10 and 5.6 we have that

$$\begin{aligned}
p' =_{\mathrm{def}} \pi(p,r) &= (w_0 \cdot \overline{p_0'}^{\circ}, p_0', w_1 \cdot \overline{p_1'}^{\circ} \ldots, w_m \cdot \overline{p_m'}^{\circ}, p_m', w, \lambda(p_m'), \ldots, w, \lambda(p_m')) \\
\overline{p'}^{\circ} &= w_0 \cdot \overline{p_0'}^{\circ} \cdot \overline{p_0'} \cdot w_1 \cdot \overline{p_1'}^{\circ} \cdots w_m \cdot \overline{p_m'}^{\circ} \cdot \overline{p_m'} \cdot \left(w \cdot \overline{\lambda(p_m')}\right)^{r!-1} \quad \text{and} \\
\overline{p'} &= w_0 \cdot \overline{p_0'}^{\circ} \cdot \qquad w_1 \cdot \overline{p_1'}^{\circ} \cdots w_m \cdot \overline{p_m'}^{\circ} \cdot \qquad \left(w \qquad\quad \right)^{r!-1}
\end{aligned}$$

where the term "$w, \lambda(p'_m)$" in $\pi(p,r)$ is repeated $(r!-1)$ times. Now let $x, z \in A^*$ such that $x\overline{p'}z \in L(\mathcal{M})$. First, we show that we can successively insert the $\overline{p'_i}$ in $\overline{p'}$. By assumption we have

$$\underbrace{xw_0}_{x'=_{\mathrm{def}}}\ \overline{p'_0}^{\circ}\ \underbrace{w_1\overline{p'_1}^{\circ} w_2\overline{p'_2}^{\circ} \cdots w_m\overline{p'_m}^{\circ} w^{r!-1}z}_{z'=_{\mathrm{def}}} \in L(\mathcal{M}).$$

We show that $x'\overline{p'_0}^{\circ}\,\overline{p'_0}z' \in L(\mathcal{M})$. From Lemma 5.12 we see that the states $s_1 =_{\mathrm{def}} \delta(s_0, x'\overline{p'_0}^{\circ})$ and $s_2 =_{\mathrm{def}} \delta(s_0, x'\overline{p'_0}^{\circ}\,\overline{p'_0})$ are connected via p'_0. Note that $p'_0 \in \mathbb{P}^{\mathcal{I}}_n$ by Lemma 5.11. If $x'\overline{p'_0}^{\circ}\,\overline{p'_0}z' \notin L(\mathcal{M})$ then we have $\delta(s_0, x'\overline{p'_0}^{\circ}) = s_1$, $\delta(s_1, z') \in S'$, $\delta(s_2, z') \notin S'$ and the states s_1, s_2 are connected via $p'_0 \in \mathbb{P}^{\mathcal{I}}_n$. It follows that $\mathcal{M}$ has pattern $\mathbb{P}^{\mathcal{I}}_n$ which is a contradiction to the assumption. Thus starting from

$$xw_0\overline{p'_0}^{\circ} \quad w_1\overline{p'_1}^{\circ} w_2\overline{p'_2}^{\circ} \cdots w_m\overline{p'_m}^{\circ} w^{r!-1}z \in L(\mathcal{M})$$

we have shown

$$xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ} w_2\overline{p'_2}^{\circ} \cdots w_m\overline{p'_m}^{\circ} w^{r!-1}z \in L(\mathcal{M}).$$

With the same procedure we obtain:

$$\begin{array}{llllllll}
xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ} & w_2\overline{p'_2}^{\circ} & w_3\overline{p'_3}^{\circ} & w_4\overline{p'_4}^{\circ} & \cdots & w_m\overline{p'_m}^{\circ} & w^{r!-1}z \in L(\mathcal{M}) \\
xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ} & & w_3\overline{p'_3}^{\circ} & w_4\overline{p'_4}^{\circ} & \cdots & w_m\overline{p'_m}^{\circ} & w^{r!-1}z \in L(\mathcal{M}) \\
xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ}\,\overline{p'_2}w_3\overline{p'_3}^{\circ} & & & w_4\overline{p'_4}^{\circ} & \cdots & w_m\overline{p'_m}^{\circ} & w^{r!-1}z \in L(\mathcal{M}) \\
xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ}\,\overline{p'_2}w_3\overline{p'_3}^{\circ}\,\overline{p'_3}w_4\overline{p'_4}^{\circ} & & & & \cdots & w_m\overline{p'_m}^{\circ} & w^{r!-1}z \in L(\mathcal{M}) \\
 & & & \vdots & & & \\
xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ}\,\overline{p'_2}w_3\overline{p'_3}^{\circ}\,\overline{p'_3}w_4\overline{p'_4}^{\circ} & & & & \cdots & w_m\overline{p'_m}^{\circ}\,\overline{p'_m}w^{r!-1}z \in L(\mathcal{M}) &
\end{array} \tag{5.1}$$

Now we have to deal with the $\overline{\lambda(p'_m)}$. Note that by definition $\overline{p'_m}$ is a suffix of w. From Lemma 5.12 it follows that w leads to states in $\mathcal{M}$ where p'_m appears. By Lemma 5.9 we get for all $s' \in S$ and $s =_{\mathrm{def}} \delta(s', w)$ that s, s are connected via $\lambda(p'_m)$. Now Proposition 5.7 says that w leads to states in $\mathcal{M}$ which have a $\overline{\lambda(p'_m)}$-loop. So starting with (5.1) we concluded

$$xw_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ}\,\overline{p'_2}w_3\overline{p'_3}^{\circ}\,\overline{p'_3}w_4\overline{p'_4}^{\circ} \cdots w_m\overline{p'_m}^{\circ}\,\overline{p'_m}\ w\overline{\lambda(p'_m)}\ w^{r!-2}z \in L(\mathcal{M}).$$

We can repeat this argument for the remaining $(r!-2)$ occurrences of w to see that

$$x\,\underbrace{w_0\overline{p'_0}^{\circ}\,\overline{p'_0}w_1\overline{p'_1}^{\circ}\,\overline{p'_1}w_2\overline{p'_2}^{\circ}\,\overline{p'_2}w_3\overline{p'_3}^{\circ}\,\overline{p'_3}w_4\overline{p'_4}^{\circ} \cdots w_m\overline{p'_m}^{\circ}\,\overline{p'_m}w\overline{\lambda(p'_m)}\ldots w\overline{\lambda(p'_m)}}_{=\overline{p'}^{\circ}}\,z \in L(\mathcal{M})$$

where the term "$w, \overline{\lambda(p'_m)}$" is repeated $(r!-1)$ times. This shows that $x\overline{p'}^{\circ}z \in L(\mathcal{M})$. ❑

Proof of Theorem 5.13. We assume that there exists an $L \in \mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n))$ which is no element of $\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_{n+1})$ and show that this leads to a contradiction. From $L \in \mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n))$ it follows that

$$L = \bigcup_{i=1}^{k} L_{i,0}L_{i,1}\cdots L_{i,k_i}$$

for some $k \geq 0$, $k_i \geq 0$ and $L_{i,j} \in \mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a DFA with $L(\mathcal{M}) = L$. For $1 \leq i \leq k$ and $0 \leq j \leq k_i$ let $\mathcal{M}_{i,j}$ be a DFA with $L(\mathcal{M}_{i,j}) = L_{i,j}$ and let $\mathcal{M}'_{i,j}$ be a DFA with $L(\mathcal{M}'_{i,j}) = A^+ \setminus L_{i,j}$. Furthermore, we define

$$r =_{\mathrm{def}} \max\left(\{\, |\mathcal{M}_{i,j}|, |\mathcal{M}'_{i,j}| \mid 1 \leq i \leq k \wedge 0 \leq j \leq k_i \,\} \cup \{|\mathcal{M}|, 3\} \cup \{\, k_i + 1 \mid 1 \leq i \leq k \,\}\right).$$

The DFA $\mathcal{M}$ has pattern $\mathbb{P}_{n+1}^{\mathcal{I}}$ since $L \notin \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$ by assumption. So there exist $s_1, s_2 \in S$, $x, z \in A^*$, $p \in \mathbb{P}_{n+1}^{\mathcal{I}}$ such that $\delta(s_0, x) = s_1$, $\delta(s_1, z) \in S'$, $\delta(s_2, z) \notin S'$ and the states s_1, s_2 are connected via p. It follows that $L \neq \emptyset$ and $k \geq 1$. By Lemma 5.11 the states s_1, s_2 are also connected via pattern $p' =_{\mathrm{def}} \pi(p, r)$. From Proposition 5.7 it follows that $x\left(\overline{p'}^{\circ}\right)^i z \in L$ for all $i \geq 0$. Thus there exists an i' with $1 \leq i' \leq k$ such that

$$x\left(\overline{p'}^{\circ}\right)^r z \in L_{i',0} L_{i',1} \cdots L_{i',k_{i'}}.$$

Since $r \geq k_{i'} + 1$ it follows with a pigeon hole argument that there exist $0 \leq j' \leq k_{i'}$ and words $x', x'', z', z'' \in A^*$ such that

1. $x\left(\overline{p'}^{\circ}\right)^r z = x''x'\overline{p'}^{\circ} z'z''$,
2. $x''x' = x\left(\overline{p'}^{\circ}\right)^i$ and $z'z'' = \left(\overline{p'}^{\circ}\right)^j z$ for some $i, j \geq 0$ and
3. $x'' \in L_{i',0} L_{i',1} \cdots L_{i',j'-1}$, $\quad x'\overline{p'}^{\circ} z' \in L_{i',j'}$ $\quad$ and $\quad z'' \in L_{i',j'+1} L_{i',j'+2} \cdots L_{i',k_{i'}}$.

Since $|\mathcal{M}_{i',j'}| \leq r$ the word $\overline{p'}^{\circ}$ leads to states in $\mathcal{M}_{i',j'}$ where p' appears by Lemma 5.12. In particular, such a state has a $\overline{p'}^{\circ}$-loop by Proposition 5.7. From $x'\overline{p'}^{\circ} z' \in L_{i',j'}$ it follows that for all $i \geq 1$ we have

$$x'\left(\overline{p'}^{\circ}\right)^i z' \in L_{i',j'}. \tag{5.2}$$

Because s_1, s_2 are connected via p' in $\mathcal{M}$ we have by Proposition 5.7 that $\delta(s_0, x''x') = s_1$, $\delta(s_2, z'z'') = \delta(s_2, z)$ and $\delta(s_1, \overline{p'}) = s_2$. Assume to the contrary that $x'\overline{p'}^{\circ}\overline{p'}z' \in L_{i',j'}$. Then we obtain $x''x'\overline{p'}^{\circ}\overline{p'}z'z'' \in L$. It follows that

$$\delta(s_2, z) = \delta(s_2, z'z'') = \delta(s_1, \overline{p'}z'z'') = \delta(s_1, \overline{p'}^{\circ}\overline{p'}z'z'') = \delta(s_0, x''x'\overline{p'}^{\circ}\overline{p'}z'z'') \in S'.$$

This is a contradiction since $\delta(s_2, z) \notin S'$. So we have seen that

$$x'\overline{p'}^{\circ}\overline{p'}z' \notin L_{i',j'}.$$

In other terms, it holds that

$$x'\overline{p'}^{\circ}\overline{p'}z' \in A^+ \setminus L_{i',j'} = L(\mathcal{M}'_{i',j'}) \tag{5.3}$$

because $|x'\overline{p'}^{\circ}\overline{p'}z'| \geq |x'\overline{p'}^{\circ} z'| \geq 1$. Recall that $L(\mathcal{M}'_{i',j'}) \in \mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ and hence the DFA $\mathcal{M}'_{i',j'}$ does not have pattern $\mathbb{P}_n^{\mathcal{I}}$. Since $|\mathcal{M}'_{i',j'}| \leq r$ we can apply Lemma 5.15 and together with (5.3) we obtain $x'\overline{p'}^{\circ}\overline{p'}^{\circ} z' \in L(\mathcal{M}'_{i',j'})$. It follows that $x'\overline{p'}^{\circ}\overline{p'}^{\circ} z' \notin A^+ \setminus L(\mathcal{M}'_{i',j'}) = L_{i',j'}$. This is a contradiction to (5.2). So $\mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$.

(End proof of Theorem 5.13.)

5.2.2 Inclusion Relations

In this subsection we show that if some initial pattern $\mathcal{I}$ satisfies a certain weak property then the inclusion $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \cup \text{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}) \cap \text{co}\mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$ holds for all $n \geq 0$ (cf. Theorem 5.19).

Definition 5.16. Let $\mathcal{I}_1, \mathcal{I}_2$ be initial patterns and let $n_1, n_2 \geq 0$. We define that $\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2}$ if and only if for every $p_2 \in \mathbb{P}_{n_2}^{\mathcal{I}_2}$ there exists a $p_1 \in \mathbb{P}_{n_1}^{\mathcal{I}_1}$ such that for every DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ and all states $s, s_1, s_2 \in S$ the following holds:

1. If p_2 appears at s then p_1 appears at s.
2. If $s_1 \overset{p_2}{\rightsquigarrow} s_2$ then $s_1 \overset{p_1}{\rightsquigarrow} s_2$.

If the relation $\preceq$ holds on one level, then it also holds on the next.

Proposition 5.17. *Let $\mathcal{I}_1, \mathcal{I}_2$ be initial patterns and let $n_1, n_2 \geq 0$. It holds that*

$$\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2} \implies \mathbb{P}_{n_1+1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2+1}^{\mathcal{I}_2}.$$

Proof. Suppose $\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2}$. So for given $p_2 = (w_{2,0}, p_{2,0}, \ldots, w_{2,m}, p_{2,m}) \in \mathbb{P}_{n_2+1}^{\mathcal{I}_2}$ with $w_{2,i} \in A^+$ and $p_{2,i} \in \mathbb{P}_{n_2}^{\mathcal{I}_1}$ there exist $p_{1,0}, \ldots, p_{1,m} \in \mathbb{P}_{n_1}^{\mathcal{I}_1}$ such that for every DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ and all states $s, s_1, s_2 \in S$ the following holds:

a. If $p_{2,i}$ appears at s then $p_{1,i}$ appears at s.
b. If $s_1 \overset{p_{2,i}}{\rightsquigarrow} s_2$ then $s_1 \overset{p_{1,i}}{\rightsquigarrow} s_2$.

Define $p_1 =_{\text{def}} (w_{2,0}, p_{1,0}, \ldots, w_{2,m}, p_{1,m})$ and observe that $p_1 \in \mathbb{P}_{n_1+1}^{\mathcal{I}_1}$. Now let $\mathcal{M} = (A, S, \delta, s_0, S')$ be some DFA and $s, s_1, s_2 \in S$. We have to show the following:

(i) If p_2 appears at s then p_1 appears at s.
(ii) If $s_1 \overset{p_2}{\rightsquigarrow} s_2$ then $s_1 \overset{p_1}{\rightsquigarrow} s_2$.

Suppose that p_2 appears at s in $\mathcal{M}$. Then there exist states $q_0, r_0, \ldots, q_m, r_m$ of $\mathcal{M}$ such that

$$s \xrightarrow{w_{2,0}} q_0 \overset{p_{2,0}}{\rightsquigarrow} r_0 \xrightarrow{w_{2,1}} q_1 \overset{p_{2,1}}{\rightsquigarrow} r_1 \xrightarrow{w_{2,2}} \cdots \xrightarrow{w_{2,m}} q_m \overset{p_{2,m}}{\rightsquigarrow} r_m = s.$$

Using b. from above this implies that

$$s \xrightarrow{w_{2,0}} q_0 \overset{p_{1,0}}{\rightsquigarrow} r_0 \xrightarrow{w_{2,1}} q_1 \overset{p_{1,1}}{\rightsquigarrow} r_1 \xrightarrow{w_{2,2}} \cdots \xrightarrow{w_{2,m}} q_m \overset{p_{1,m}}{\rightsquigarrow} r_m = s$$

which shows that p_1 appears at s.

Now assume that s_1, s_2 are connected via p_2. Then p_2 appears at s_1 and s_2, and there exist states $q_0, \ldots, q_m$ of $\mathcal{M}$ such that $p_{2,i}$ appears at state q_i for $0 \leq i \leq m$ and

$$s_1 \xrightarrow{w_{2,0}} q_0 \xrightarrow{w_{2,1}} q_1 \xrightarrow{w_{2,2}} \cdots \xrightarrow{w_{2,m}} q_m = s_2.$$

From (i) we obtain that p_1 appears at s_1 and s_2, and if we apply a. we get that $p_{1,i}$ appears at state q_i for $0 \leq i \leq m$. This shows that s_1, s_2 are connected via p_1. So $\mathbb{P}_{n_1+1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2+1}^{\mathcal{I}_2}$. ❑

Proposition 5.18. *Let $\mathcal{I}_1, \mathcal{I}_2$ be initial patterns and let $n_1, n_2 \geq 0$. It holds that*

$$\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2} \implies \mathcal{FP}(\mathbb{P}_{n_1}^{\mathcal{I}_1}) \subseteq \mathcal{FP}(\mathbb{P}_{n_2}^{\mathcal{I}_2}).$$

Proof. Suppose $\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2}$. For any language $L \notin \mathcal{FP}(\mathbb{P}_{n_2}^{\mathcal{I}_2})$ we show that $L \notin \mathcal{FP}(\mathbb{P}_{n_1}^{\mathcal{I}_1})$. Let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a DFA with $L(\mathcal{M}) = L$. Observe that $\mathcal{M}$ has pattern $\mathbb{P}_{n_2}^{\mathcal{I}_2}$ by assumption. So there exist $s_1, s_2 \in S$, $x, z \in A^*$ and $p_2 \in \mathbb{P}_{n_2}^{\mathcal{I}_2}$ such that $\delta(s_0, x) = s_1$, $\delta(s_1, z) \in S'$, $\delta(s_2, z) \notin S'$ and the states s_1, s_2 are connected via p_2. Because $\mathbb{P}_{n_1}^{\mathcal{I}_1} \preceq \mathbb{P}_{n_2}^{\mathcal{I}_2}$ there exists some $p_1 \in \mathbb{P}_{n_1}^{\mathcal{I}_1}$ such that the states s_1, s_2 are connected via p_1. It follows that $\mathcal{M}$ has also pattern $\mathbb{P}_{n_1}^{\mathcal{I}_1}$. This shows $L \notin \mathcal{FP}(\mathbb{P}_{n_1}^{\mathcal{I}_1})$ and hence $\mathcal{FP}(\mathbb{P}_{n_1}^{\mathcal{I}_1}) \subseteq \mathcal{FP}(\mathbb{P}_{n_2}^{\mathcal{I}_2})$. ❑

The proof of the following theorem is an immediate consequence of these propositions.

Theorem 5.19. *Let $\mathcal{I}$ be an initial pattern with $\mathbb{P}_0^{\mathcal{I}} \preceq \mathbb{P}_1^{\mathcal{I}}$. For $n \geq 0$ it holds that*

$$\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \cup \mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}) \cap \mathrm{co}\mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}}).$$

Proof. From Proposition 5.17 we obtain with the assumption $\mathbb{P}_0^{\mathcal{I}} \preceq \mathbb{P}_1^{\mathcal{I}}$ that $\mathbb{P}_n^{\mathcal{I}} \preceq \mathbb{P}_{n+1}^{\mathcal{I}}$ for all $n \geq 0$. By Proposition 5.18 this implies $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$ for all $n \geq 0$. We conclude that also $\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathrm{co}\mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$ for all $n \geq 0$. Observe from Theorem 5.13 that $\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) \subseteq \mathrm{Pol}(\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})) \subseteq \mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$ which also implies $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}}) = \mathrm{co}(\mathrm{co}\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})) \subseteq \mathrm{co}\mathcal{FP}(\mathbb{P}_{n+1}^{\mathcal{I}})$. ❑

5.2.3 Decidability of Pattern Classes

We show that the membership problem of the class $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ for fixed $n \geq 0$ is decidable in nondeterministic logarithmic space NL whenever the following can be done for the initial pattern $\mathcal{I}$ in these space bounds: decide for a given DFA $\mathcal{M}$ and a constant number of states whether $\mathcal{I}$ appears at these states and whether they are connected via $\mathcal{I}$. Note that the decidability of $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$ has to depend on the initial pattern, since an undecidable set $\mathcal{I}$ (which can be easily constructed) may lead to undecidable pattern classes.

We define two problems addressing the question of the existence of paths and patterns that appear simultaneously in a DFA. The first problem $\textsc{Reach}_k$ has already been considered in Section 1.5, however, we have not fixed a notation yet.

Definition 5.20. Let $k \geq 1$. We define $\textsc{Reach}_k$ to be the set of pairs $(\mathcal{M}, W)$ such that

1. $\mathcal{M} = (A, S, \delta, s_0, S')$ is a DFA,
2. $W \subseteq S \times S$ with $|W| \leq k$ and
3. $(s_1, \ldots, s_{|W|}) \longrightarrow^+ (s'_1, \ldots, s'_{|W|})$ for $(s_i, s'_i) \in W$.

We have argued in Section 1.5 that for fixed $k \geq 0$ it holds that $\textsc{Reach}_k$ is in NL: start at $(s_1, \ldots, s_{|W|})$, guess a non–empty path and continuously compare the actual tuple of states with $(s'_1, \ldots, s'_{|W|})$.

Proposition 5.21. *Let $k \geq 1$. It holds that* $\textsc{Reach}_k \in \mathrm{NL}$.

The second problem has two parameters, additionally to k bounding the number of states as before, there is a parameter n refering to $\mathcal{FP}(\mathbb{P}_n^{\mathcal{I}})$.

Definition 5.22. Let $\mathcal{I}$ be an initial pattern, $n \geq 0$ and $k \geq 1$. We define $\textsc{Pattern}^{\mathcal{I}}_{n,k}$ to be the set of all triples $(\mathcal{M}, T_1, T_2)$ such that

1. $\mathcal{M} = (A, S, \delta, s_0, S')$ is a DFA,
2. $T_1 \subseteq S$ with $|T_1| \leq k$,
3. $T_2 \subseteq S \times S$ with $|T_2| \leq k$ and
4. there exists some $p \in \mathbb{P}^{\mathcal{I}}_n$ such that for all $s \in T_1$ and all $(s_1, s_2) \in T_2$ it holds that
   a) p appears at s and
   b) $s_1 \overset{p}{\rightsquigarrow} s_2$.

Before we continue, we need to make a remark concerning our computation model. Our algorithm in the forthcoming proof works on nondeterministic Turing machines having a read–only input tape and a read–write working tape. We prerequisite some standard enconding of a DFA and of the respective set of states. Moreover, a Turing machine represents a single state of a DFA on its working tape by the number of the state in binary. Note that the space needed to do this for a contant number of states is bounded logarithmically in the input size.

Furthermore, the type of Turing machine we have in mind have access to an oracle via an additional write–only query tape. The restrictions for the query tape are as follows. From the moment where the machine writes the first letter on the query tape, it is not allowed to make nondeterministic choices until the oracle is asked. It receives the answer whether the string on the query tape belongs to the oracle set by changing to a respective state. After doing this the query tape is empty. When we determine the amount of space the machine uses then only the working tape is considered. For details about this computation model we refer to [RST82], an introduction to oracle computation can be found in [Pap94].

The next lemma says that the problem $\textsc{Pattern}^{\mathcal{I}}_{n,k}$ is decidable in nondeterministic logarithmic space when the machine has access to $\textsc{Pattern}^{\mathcal{I}}_{(n-1),3k}$ as an oracle. Note that the first index is reduced to $n-1$.

Lemma 5.23. *Let $\mathcal{I}$ be an initial pattern, $n \geq 1$ and $k \geq 1$. It holds that*

$$\textsc{Pattern}^{\mathcal{I}}_{n,k} \in \mathrm{NL}^{\textsc{Pattern}^{\mathcal{I}}_{(n-1),3k}}.$$

Proof. We give an algorithm for $\textsc{Pattern}^{\mathcal{I}}_{n,k}$ in Table 5.1. This algorithm has access to a $\textsc{Reach}_{4k}$ oracle and to a $\textsc{Pattern}^{\mathcal{I}}_{(n-1),3k}$ oracle. The notations in the table are adopted from Figures 5.5 and 5.6 where we give an example of the case that a pattern appears at some state and of the case that two states are connected via a pattern, respectively. We show that this algorithm works in logarithmic space and decides $\textsc{Pattern}^{\mathcal{I}}_{n,k}$. By Proposition 5.21 we have $\textsc{Reach}_k \in \mathrm{NL}$. Since the access to an oracle from NL does not rise the power of an NL machine, i.e., $\mathrm{NL}^{\mathrm{NL}} = \mathrm{NL}$ [Sze87, Imm88], we can do the same computation without the $\textsc{Reach}_{4k}$ oracle and obtain the required algorithm.

First we observe that the algorithm accesses the oracle in the way as described above. For this we only have to consider step 4. Since on one hand we have already computed the sets W, T'_1 and T'_2 (and stored on the working–tape) and on the other hand $\mathcal{M}$ is stored on the input–tape, we can actually write down the queries $(\mathcal{M}, W)$ and $(\mathcal{M}, T'_1, T'_2)$ on the query–tape without making any nondeterministic choices.

Let us analyse the space on the working–tape which is needed on input $(\mathcal{M}, T_1, T_2)$. Note that our algorithm uses only a constant number of variables which is bounded by $O(k)$.

Moreover, all variables except T_1', T_2', W contain numbers of states of $\mathcal{M}$, which can be stored in logarithmic space. Each of the variables T_1', T_2', W contains a set consisting of at most $4k$ (pairs of) numbers of states. This shows that our algorithm works in logarithmic space.

In the remaining part of the proof we argue that our algorithm really decides $\text{PATTERN}^{\mathcal{I}}_{n,k}$. First we want to see that the computation has an accepting path if $(\mathcal{M}, T_1, T_2) \in \text{PATTERN}^{\mathcal{I}}_{n,k}$. For this let $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}^{\mathcal{I}}_n$ be a witnessing pattern. We denote the involved states in the occurrence of p at s_i as in Figure 5.5, and we denote the involved states of the connection of q_j, r_j via p as in Figure 5.6. Now consider that particular path of the computation, where we carry out exactly $m+1$ passes of the loop and where we guess the states $\phi_{i,l}, \psi_{i,l}, \alpha_{j,l}, \beta_{j,l}, \gamma_{j,l}, \delta_{j,l}, \lambda_{j,l}$ at the beginning of the l-th pass of the loop (starting with pass 0). It can be easily verified that this is an accepting path.

Now suppose that the computation on input $(\mathcal{M}, T_1, T_2)$ has an accepting path, and fix one of these paths. Choose m such that on this path the loop is passed $m+1$ times. Note that in each pass of the loop we receive positive answers to the queries $(\mathcal{M}, W) \in \text{REACH}_{4k}$ and $(\mathcal{M}, T_1', T_2') \in \text{PATTERN}^{\mathcal{I}}_{(n-1),3k}$ because otherwise the fixed path would be rejecting. It follows that for each pass l there exists a word $w_l \in A^+$ witnessing $(\mathcal{M}, W) \in \text{REACH}_{4k}$ and there exists a pattern $p_l \in \mathbb{P}^{\mathcal{I}}_{n-1}$ witnessing $(\mathcal{M}, T_1', T_2') \in \text{PATTERN}^{\mathcal{I}}_{(n-1),3k}$. Now define $p =_{\text{def}} (w_0, p_0, \ldots, w_m, p_m)$. Using the states $\phi_{i,l}, \psi_{i,l}, \alpha_{j,l}, \beta_{j,l}, \gamma_{j,l}, \delta_{j,l}, \lambda_{j,l}$ which were guessed at the beginning of the l-th pass of the loop, we can verify that a) p appears at all $s_i \in T_1$ and b) all q_i, r_i with $(q_i, r_i) \in T_2$ are connected via p. ❑

The following corollary is immediate, just note again that $\text{NL}^{\text{NL}} = \text{NL}$ [Sze87, Imm88].

Corollary 5.24. *Let $\mathcal{I}$ be an initial pattern such that $\text{PATTERN}^{\mathcal{I}}_{0,k} \in \text{NL}$ for all $k \geq 1$. Then $\text{PATTERN}^{\mathcal{I}}_{n,k} \in \text{NL}$ for all $n \geq 0$ and $k \geq 1$.*

Finally, we obtain the efficient decidability of the membership problem of $\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n)$ for fixed $n \geq 0$ under the assumptions that the appearance (and connection) of the initial pattern can be efficiently verified at a constant number of states.

Theorem 5.25. *Let $\mathcal{I}$ be an initial pattern with $\text{PATTERN}^{\mathcal{I}}_{0,k} \in \text{NL}$ for $k \geq 1$. For fixed $n \geq 0$ the membership problem of $\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n)$ is decidable in nondeterministic logarithmic space* NL.

Proof. Let a DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ be given. We first guess states $s_1, s_2, s^+, s^- \in S$ and check whether s^+ (s^-) is accepting (rejecting, respectively), and if $s_0 \longrightarrow s_1$, $(s_1, s_2) \longrightarrow (s^+, s^-)$, i.e., we test if $(\mathcal{M}, \{(s_0, s_1)\}) \in \text{REACH}_1$ and $(\mathcal{M}, \{(s_1, s^+), (s_2, s^-)\}) \in \text{REACH}_2$. This is possible in nondeterministic logarithmic space by Proposition 5.21. It remains to check whether $(\mathcal{M}, \emptyset, \{(s_1, s_2)\}) \in \text{PATTERN}^{\mathcal{I}}_{n,1}$ which is also possible in NL by Corollary 5.24. ❑

5.3 Discussion

We want to mention that our result $\text{Pol}(\text{co}\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n)) \subseteq \mathcal{FP}(\mathbb{P}^{\mathcal{I}}_{n+1})$ generalizes the usually easier to prove inclusion in forbidden pattern characterizations. Of course we are also interested in the reverse inclusion. Further investigations may involve to look for particular initial patterns $\mathcal{I}$ for which the reverse inclusion holds — or does not hold: it is possible that the iteration rule is too strong in the sense that $\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_{n+1})$ is a much broader class than $\text{Pol}(\text{co}\mathcal{FP}(\mathbb{P}^{\mathcal{I}}_n))$. However, the applications in the next chapter give some evidence that this is not true in case of the concatenation hierarchies we are interested in.

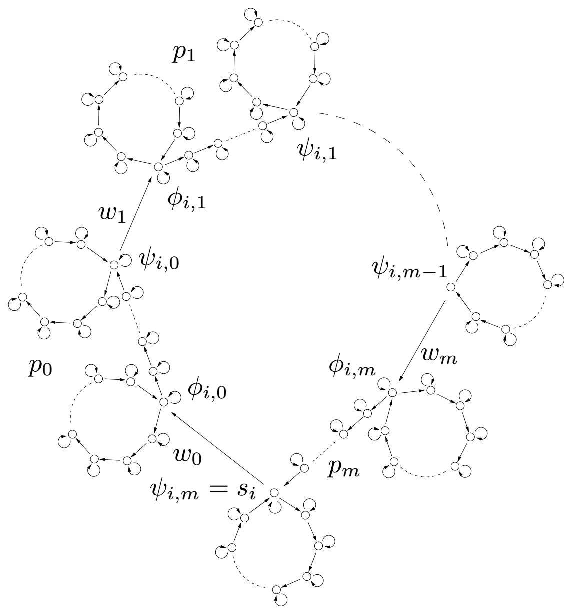


Fig. 5.5. Example of $p = (w_0, p_0, w_1, p_1, \ldots, w_m, p_m)$ that appears at state s_i.

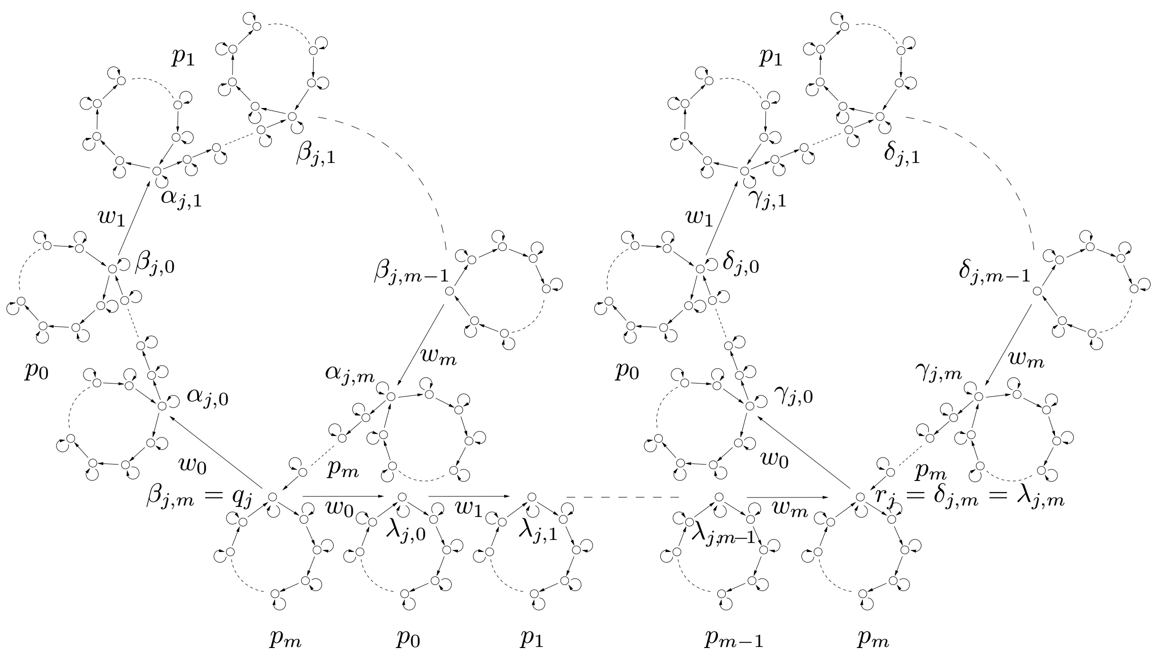


Fig. 5.6. Example of states q_j, r_j that are connected via $p = (w_0, p_0, w_1, p_1, \ldots, w_m, p_m)$.

Step, Label	Command	Remark
1.	Let $t_1 := \|T_1\|$, $t_2 := \|T_2\|$ and let s_i, q_j, r_j such that $T_1 = \{s_1, \ldots, s_{t_1}\}$ and $T_2 = \{(q_1, r_1), \ldots, (q_{t_2}, r_{t_2})\}$.	Note that t_1, t_2 are bounded by the constant k. We have to decide whether there is a $p \in \mathbb{P}_n^{\mathcal{I}}$ such that a) p appears at all s_i (Figure 5.5) and b) all q_j, r_j are connected via p (Figure 5.6).
2.	For $1 \leq i \leq t_1$ and $1 \leq j \leq t_2$ let $\psi_i^{\text{start}} := s_i \qquad \beta_j^{\text{start}} := q_j$ $\delta_j^{\text{start}} := r_j \qquad \lambda_j^{\text{start}} := q_j$	Variables marked with 'start' contain the starting point from where we have to guess and check the next fragment of the pattern.
3. `loop:`	Guess states ϕ_i, ψ_i for $1 \leq i \leq t_1$, states $\alpha_j, \beta_j, \gamma_j, \delta_j, \lambda_j$ for $1 \leq j \leq t_2$ and let $T_1' := \{ \lambda_j \mid 1 \leq j \leq t_2 \}$ $T_2' := \{ (\phi_i, \psi_i) \mid 1 \leq i \leq t_1 \} \cup \{ (\alpha_j, \beta_j) \mid 1 \leq j \leq t_2 \} \cup \{ (\gamma_j, \delta_j,) \mid 1 \leq j \leq t_2 \}$ $W := \{ (\psi_i^{\text{start}}, \phi_i) \mid 1 \leq i \leq t_1 \} \cup \{ (\beta_j^{\text{start}}, \alpha_j) \mid 1 \leq j \leq t_2 \} \cup \{ (\delta_j^{\text{start}}, \gamma_j) \mid 1 \leq j \leq t_2 \} \cup \{ (\lambda_j^{\text{start}}, \lambda_j) \mid 1 \leq j \leq t_2 \}$	The guessed states correspond to Figures 5.5 and 5.6. In the l-th pass of this loop (starting with pass 0) the variables $\phi_i, \psi_i, \alpha_j, \beta_j, \gamma_j, \delta_j, \lambda_j$ used in the algorithm correspond to $\phi_{i,l}, \psi_{i,l}, \alpha_{j,l}, \beta_{j,l}, \gamma_{j,l}, \delta_{j,l}, \lambda_{j,l}$, respectively. Moreover, at the beginning of the l-th pass we have a correspondence between $\psi_i^{\text{start}}, \beta_j^{\text{start}}, \delta_j^{\text{start}}, \lambda_j^{\text{start}}$ and $\psi_{i,l-1}, \beta_{j,l-1}, \delta_{j,l-1}, \lambda_{j,l-1}$, respectively. Using T_1' and T_2' we ask the oracle whether there is a pattern p_l that connects (and appears at) the guessed states. With W we test the existence of a word w_l.
4.	Ask the following queries and reject when a negative answer is given. $(\mathcal{M}, W) \in \text{REACH}_{4k}$ $(\mathcal{M}, T_1', T_2') \in \text{PATTERN}_{(n-1),3k}^{\mathcal{I}}$	If at least one negative answer is given then the states guessed in the previous step do not witness that there is a pattern from $\mathbb{P}_n^{\mathcal{I}}$.
5.	For $1 \leq i \leq t_1$ and $1 \leq j \leq t_2$ let $\psi_i^{\text{start}} := \psi_i \qquad \beta_j^{\text{start}} := \beta_j$ $\delta_j^{\text{start}} := \delta_j \qquad \lambda_j^{\text{start}} := \lambda_j$	Here we set the next starting points after a successfull check of the previous fragment of the pattern.
6.	Jump nondeterministically to `loop` or to `exit`.	Guess whether we have already checked the right number of fragments of the pattern, i.e., whether the number of passes equals m.
7. `exit:`	Accept if and only if the following conditions hold for all $1 \leq i \leq t_1$ and $1 \leq j \leq t_2$: $\psi_i = s_i \qquad \beta_j = q_j$ $\delta_j = r_j \qquad \lambda_j = r_j$	It remains to check whether the guessed loops have reached their starting points, and whether the path which was guessed via λ_j leads from q_j to r_j.

Table 5.1. An algorithm which decides $(\mathcal{M}, T_1, T_2) \in \text{PATTERN}_{n,k}^{\mathcal{I}}$ on input of a DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ and sets $T_1 \subseteq S$ and $T_2 \subseteq S \times S$ with $|T_1|, |T_2| \leq k$.

6. Lower Bounds and a Decidability Result for the STH

We consider in this chapter two special initial patterns $\mathcal{L}$ and $\mathcal{B}$ corresponding to the concatenation hierarchies we are interested in. For notational convenience we write $\mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_n^{\mathcal{B}}$ instead of $\mathcal{FP}(\mathbb{P}_n^{\mathcal{L}})$ and $\mathcal{FP}(\mathbb{P}_n^{\mathcal{B}})$, respectively.

The main results of this chapter are as follows. In Section 6.1 we see what consequences of our previous results can be derived for the DDH and STH. We provide the inclusion relations between forbidden pattern classes (cf. Theorem 6.3) and the inclusion relations between the STH and DDH and forbidden pattern classes (cf. Theorem 6.4). All classes $\mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_n^{\mathcal{B}}$ contain only star–free languages (cf. Theorem 6.7) and can be separated using languages that also separate the DDH and STH (cf. Corollary 6.11). Then we show that all pattern classes have decidable membership problems (cf. Theorem 6.15) and derive from this a lower bound algorithm for the dot–depth of a given regular language. In Section 6.2 we show that even $\mathcal{L}_{5/2} = \mathcal{FP}_2^{\mathcal{L}}$ holds if a two–letter alphabet is considered (cf. Corollary 6.18). This implies in particular the decidability of $\mathcal{L}_{5/2}$ in the two–letter case (cf. Corollary 6.19) and has consequences in first–order logic (cf. Corollary 6.20). In fact, we show more general that whenever $\mathcal{B}_{n+1/2} = \mathcal{FP}_n^{\mathcal{B}}$ for some $n \geq 1$ and arbitrary alphabets then $\mathcal{L}_{n+3/2} = \mathcal{FP}_{n+1}^{\mathcal{L}}$ in case of a two–letter alphabet (cf. Theorem 6.17).

6.1 Consequences for Concatenation Hierarchies

In case of the pattern $\mathbb{L}_{1/2}$ for $\mathcal{L}_{1/2}$ it was required that there is some $w \in A^*$ such that for two states s_1, s_2 it holds that $s_1 \xrightarrow{w} s_2$, while in case of pattern $\mathbb{B}_{1/2}$ for $\mathcal{B}_{1/2}$ there must be $v, w \in A^+$ such that for two states s_1, s_2 we have $s_1 \xrightarrow{w} s_2$, and s_1 and s_2 both have a v-loop.

Definition 6.1. We define the following initial patterns.

$$\begin{aligned} \mathcal{L} &=_{\text{def}} \{\varepsilon\} \times A^* \\ \mathcal{B} &=_{\text{def}} A^+ \times A^+ \end{aligned}$$

It is easy to see that $\mathcal{L}$ and $\mathcal{B}$ are indeed initial patterns. Figure 6.1 summarizes the results of Section 6.1.

6.1.1 DDH and STH versus Pattern Classes

Let us consider the inclusion relations between classes of concatenation hierarchies and forbidden pattern classes.

Proposition 6.2. *It holds that* $\mathbb{P}_0^{\mathcal{L}} \preceq \mathbb{P}_1^{\mathcal{L}}$, $\mathbb{P}_0^{\mathcal{B}} \preceq \mathbb{P}_1^{\mathcal{B}}$, $\mathbb{P}_0^{\mathcal{L}} \preceq \mathbb{P}_0^{\mathcal{B}}$ *and* $\mathbb{P}_0^{\mathcal{B}} \preceq \mathbb{P}_1^{\mathcal{L}}$.

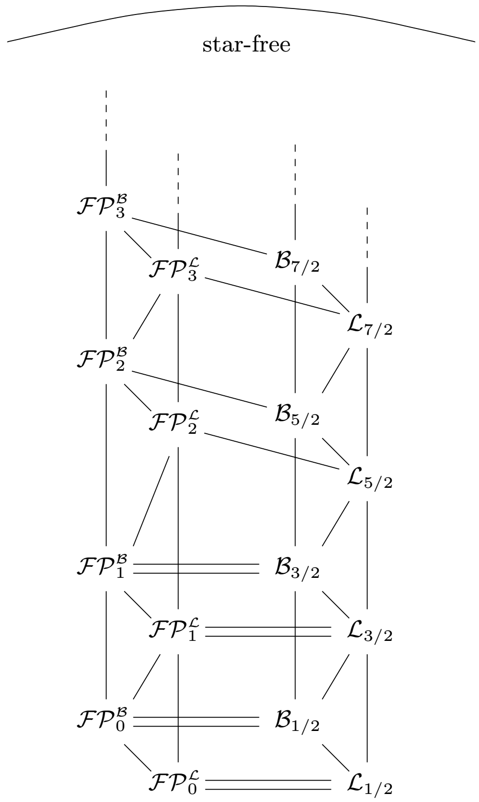


Fig. 6.1. Concatenation hierarchies and forbidden pattern classes. Doubled lines stand for equality.

Proof. We show $\mathbb{P}_0^{\mathcal{B}} \preceq \mathbb{P}_1^{\mathcal{L}}$ first. Let $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_1^{\mathcal{L}}$ with $m \geq 0$, $w_i \in A^+$ and $p_i \in \mathbb{P}_0^{\mathcal{L}} = \mathcal{L} = \{\varepsilon\} \times A^*$ for $0 \leq i \leq m$. Define $\tilde{p} =_{\text{def}} (\overline{p}^{\circ}, \overline{p})$. Note that $\tilde{p} \in \mathbb{P}_0^{\mathcal{B}} = \mathcal{B} = A^+ \times A^+$ because $\overline{p}^{\circ}, \overline{p} \in A^+$ by Proposition 5.7. Now let s, s_1, s_2 be states of some DFA. If p appears at s then s has a $\overline{p}^{\circ}$-loop by Proposition 5.7, so $\tilde{p}$ appears at s. If s_1, s_2 are connected via p then p appears at s_1 and at s_2 and so does $\tilde{p}$. Moreover, $s_1 \xrightarrow{\overline{p}} s_2$ by Proposition 5.7 and hence s_1, s_2 are connected via $\tilde{p}$.

We give the constructions that witness the remaining relations. In case of $\mathbb{P}_0^{\mathcal{L}} \preceq \mathbb{P}_0^{\mathcal{B}}$ let $p = (v, w) \in \mathbb{P}_0^{\mathcal{B}} = A^+ \times A^+$ and define $\tilde{p} =_{\text{def}} (\varepsilon, w)$. For $\mathbb{P}_0^{\mathcal{B}} \preceq \mathbb{P}_1^{\mathcal{B}}$ let $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_1^{\mathcal{B}}$ and set $\tilde{p} =_{\text{def}} (\overline{p}^{\circ}, \overline{p})$. Finally, to see $\mathbb{P}_0^{\mathcal{L}} \preceq \mathbb{P}_1^{\mathcal{L}}$ let $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_1^{\mathcal{L}}$ and define $\tilde{p} =_{\text{def}} (\varepsilon, \overline{p})$. ❑

So the following theorem is an immediate consequence of Theorem 5.19 and Propositions 5.17 and 5.18.

Theorem 6.3. *For $n \geq 0$ the following inclusions hold.*

1. $\mathcal{FP}_n^{\mathcal{L}} \cup \mathrm{co}\mathcal{FP}_n^{\mathcal{L}} \subsetneq \mathcal{FP}_{n+1}^{\mathcal{L}} \cap \mathrm{co}\mathcal{FP}_{n+1}^{\mathcal{L}}$
2. $\mathcal{FP}_n^{\mathcal{B}} \cup \mathrm{co}\mathcal{FP}_n^{\mathcal{B}} \subsetneq \mathcal{FP}_{n+1}^{\mathcal{B}} \cap \mathrm{co}\mathcal{FP}_{n+1}^{\mathcal{B}}$
3. $\mathcal{FP}_n^{\mathcal{L}} \subsetneq \mathcal{FP}_n^{\mathcal{B}}$
4. $\mathcal{FP}_n^{\mathcal{B}} \subsetneq \mathcal{FP}_{n+1}^{\mathcal{L}}$

If we compare these relations to Proposition 1.3 and 1.4 we see the same inclusion structure as in case of the STH and DDH. However, the connections between pattern classes and classes of concatenation hierarchies are even closer.

Theorem 6.4. *For $n \geq 0$ it holds that*

1. $\mathcal{L}_{1/2} = \mathcal{FP}_0^{\mathcal{L}}$, $\mathcal{L}_{3/2} = \mathcal{FP}_1^{\mathcal{L}}$ *and* $\mathcal{L}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{L}}$ *and*
2. $\mathcal{B}_{1/2} = \mathcal{FP}_0^{\mathcal{B}}$, $\mathcal{B}_{3/2} = \mathcal{FP}_1^{\mathcal{B}}$ *and* $\mathcal{B}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{B}}$.

We prepare the proof with the following two propositions. Recall from Proposition 5.7 that if some $p \in \mathbb{P}_n^{\mathcal{I}}$ appears at some state s then s has a $\overline{p}^{\circ}$-loop. In case of $\mathbb{P}_1^{\mathcal{L}}$ also the reverse implication holds.

Proposition 6.5. *Let $p \in \mathbb{P}_1^{\mathcal{L}}$. For every state s of some* DFA *it holds that p appears at s if and only if s has a $\overline{p}^{\circ}$-loop.*

Proof. We need to show the 'if'–part. Let $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_1^{\mathcal{L}}$ with $m \geq 0$, $w_i \in A^+$ and $p_i = (\varepsilon, b_i) \in \mathbb{P}_0^{\mathcal{L}} = \{\varepsilon\} \times A^*$ for $0 \leq i \leq m$. It holds that $\overline{p}^{\circ} = w_0\overline{p_0}w_1\overline{p_1} \cdots w_m\overline{p_m}$ and $\overline{p_i} = b_i$. So for any state s of some DFA that has a $\overline{p}^{\circ}$-loop there are states $q_0, r_0, \ldots, q_m, r_m$ such that

$$s \xrightarrow{w_0} q_0 \xrightarrow{b_0} r_0 \xrightarrow{w_1} q_1 \xrightarrow{b_1} r_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m \xrightarrow{b_m} r_m = s.$$

Since $p_i = (\varepsilon, b_i)$ for $0 \leq i \leq m$ we have in fact

$$s \xrightarrow{w_0} q_0 \overset{p_0}{\rightsquigarrow} r_0 \xrightarrow{w_1} q_1 \overset{p_1}{\rightsquigarrow} r_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m \overset{p_m}{\rightsquigarrow} r_m = s$$

which shows that p appears at s. ❑

Next we see that a pattern $p \in \mathbb{P}_0^{\mathcal{B}}$ with an additional alphabet condition is in fact a pattern from $\mathbb{P}_1^{\mathcal{L}}$.

Proposition 6.6. *Let $p = (l, b) \in \mathbb{P}_0^{\mathcal{B}}$ such that $\alpha(b) \subseteq \alpha(l)$ and l and b have the same first letter. Then there is some $p' \in \mathbb{P}_1^{\mathcal{L}}$ with $\overline{p'}^{\circ} = l^n$ for $n =_{\mathrm{def}} |b|$ such that for every* DFA *$\mathcal{M} = (A, S, \delta, s_0, S')$ and $s_1, s_2 \in S$ it holds that $s_1 \overset{p}{\rightsquigarrow} s_2$ implies $s_1 \overset{p'}{\rightsquigarrow} s_2$.*

Proof. Recall that $p = (l, b) \in A^+ \times A^+$ and let $b = a_0a_1 \cdots a_m$ for $m \geq 0$ and $a_i \in A$. Because $\{a_0, \ldots, a_m\} \subseteq \alpha(l)$ we can rewrite l for all $0 \leq i \leq m$ as $l = l_ia_il_i'$. Since l and b start with the same first letter by assumption we may assume that $l_0 = \varepsilon$. Now define $p' =_{\mathrm{def}} (w_0, p_0, \ldots, w_m, p_m)$ with $w_i =_{\mathrm{def}} a_i$ for $0 \leq i \leq m$, $p_i =_{\mathrm{def}} (\varepsilon, l_j'l_{j+1})$ for $0 \leq j < m$ and $p_m =_{\mathrm{def}} (\varepsilon, l_m')$. Then $p_i \in \mathbb{P}_0^{\mathcal{L}}$, $w_i \in A^+$ and hence $p' \in \mathbb{P}_1^{\mathcal{L}}$. Observe also that

$$\overline{p'}^{\circ} = w_0\overline{p_0}w_1\overline{p_1} \cdots w_m\overline{p_m} = a_0 \cdot l_0'l_1 \cdot a_1 \cdot l_1'l_2 \cdots a_m \cdot l_m' = l_0a_0l_0' \cdot l_1a_1l_1' \cdots l_ma_ml_m' = l^{m+1}$$

and $m + 1 = |b|$.

Let us first show that p' appears at s_1 and at s_2. Since p appears at s_1 and at s_2 we know that these states have an l-loop. So they also have a $\overline{p'}^{\circ}$-loop and hence p' appears at s_1 and at s_2 by Proposition 6.5. Because $\overline{p_i}^{\circ} = \varepsilon$ for $0 \leq i \leq m$ and $\overline{p'} = w_0w_1 \cdots w_m = b$ we see from $s_1 \xrightarrow{b} s_2$ that s_1, s_2 are connected via p'. ❑

Proof of Theorem 6.4. It is easy to see that $\mathcal{FP}_0^{\mathcal{L}} = \mathcal{FP}(\mathbb{L}_{1/2})$ and $\mathcal{FP}_0^{\mathcal{B}} = \mathcal{FP}(\mathbb{B}_{1/2})$ just by comparing definitions. For $\mathbb{L}_{1/2}$ and $\mathbb{B}_{1/2}$ see Definition 2.15, for $\mathbb{P}_0^{\mathcal{L}}$ and $\mathbb{P}_0^{\mathcal{B}}$ consider $\mathcal{L}$ and $\mathcal{B}$ and Definitions 5.2 and 5.5. Actually, we have set–up the definition of $\mathbb{P}_n^{\mathcal{I}}$ in order to obtain this. For the same reason we see that also $\mathcal{FP}_1^{\mathcal{B}} = \mathcal{FP}(\mathbb{B}_{3/2})$. Here we look at Definition 4.31 and consider the first iteration step $\mathrm{IT}(\mathbb{P}_0^{\mathcal{B}}) = \mathbb{P}_1^{\mathcal{B}}$ in Definition 5.4. So by Theorem 2.16 we have $\mathcal{L}_{1/2} = \mathcal{FP}(\mathbb{L}_{1/2}) = \mathcal{FP}_0^{\mathcal{L}}$ and $\mathcal{B}_{1/2} = \mathcal{FP}(\mathbb{B}_{1/2}) = \mathcal{FP}_0^{\mathcal{B}}$, and Theorem 4.32 yields $\mathcal{B}_{3/2} = \mathcal{FP}(\mathbb{B}_{3/2}) = \mathcal{FP}_1^{\mathcal{B}}$. The inclusions $\mathcal{L}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{B}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{B}}$ for $n \geq 0$ follow from Theorem 5.13 and Lemma 1.20 together with $\mathcal{L}_{1/2} = \mathcal{FP}_0^{\mathcal{L}}$, $\mathcal{B}_{1/2} = \mathcal{FP}_0^{\mathcal{B}}$ and the monotony of $\mathrm{Pol}(\cdot)$ and $\mathrm{BC}(\cdot)$. It remains to argue that $\mathcal{FP}_1^{\mathcal{L}} \subseteq \mathcal{L}_{3/2}$.

We have by Theorem 4.2 that $\mathcal{L}_{3/2} = \mathcal{FP}(\mathbb{L}_{3/2})$ so it suffices to show that if some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ has pattern $\mathbb{L}_{3/2}$ then it has pattern $\mathbb{P}_1^{\mathcal{L}}$. So assume that $\mathcal{M}$ has pattern $\mathbb{L}_{3/2}$ witnessed by $x, z \in A^*$, $s_1, s_2 \in S$ and $v, w \in A^+$ with $\alpha(vwv) \subseteq \alpha(vv)$. Define $l =_{\mathrm{def}} vv$ and $b =_{\mathrm{def}} vwv$ and observe that $p =_{\mathrm{def}} (l, b) \in \mathbb{P}_0^{\mathcal{B}}$. By assumption we have $\alpha(b) \subseteq \alpha(l)$ and by definition it holds that l and b start with the same first letter. Note that s_1, s_2 are connected via p since s_1 and s_2 have an l-loop and $\delta(s_1, b) = s_2$. Hence we may apply Proposition 6.6 and obtain that s_1, s_2 are connected via some $p' \in \mathbb{P}_1^{\mathcal{L}}$. It follows that $\mathcal{M}$ has pattern $\mathbb{P}_1^{\mathcal{L}}$. **(End proof of Theorem 6.4.)**

6.1.2 Pattern Classes are Starfree

We show that all classes $\mathcal{FP}_n^{\mathcal{B}}$ and $\mathcal{FP}_n^{\mathcal{L}}$ contain only star–free languages. To do this, we prove that if some minimal DFA $\mathcal{M}$ is not permutation–free, then $\mathcal{M}$ has all patterns $\mathbb{P}_n^{\mathcal{B}}$ for arbitrary $n \geq 0$. Recall that SF denotes the class of star–free languages.

Theorem 6.7. *It holds that* $\bigcup_{n\geq 0} \mathcal{FP}_n^{\mathcal{L}} = \bigcup_{n\geq 0} \mathcal{FP}_n^{\mathcal{B}} = \mathrm{SF}$.

Note that we already have $\mathrm{SF} \subseteq \bigcup_{n\geq 0} \mathcal{L}_{n+1/2}$ by Proposition 1.5, that $\mathcal{L}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{L}}$ by Theorem 6.4 and that $\mathcal{FP}_n^{\mathcal{L}} \subseteq \mathcal{FP}_n^{\mathcal{B}}$ by Theorem 6.3. So the proof of the previous theorem is immediate from Lemma 6.9 below. We show the first the following auxiliary lemma.

Lemma 6.8. *Let $n \geq 0$ and let $\mathcal{M} = (A, S, \delta, s_0, S')$ be a* DFA *such that there exist $w \in A^+$, $l \geq 2$ and $r_0, r_1, \ldots, r_{l-1} \in S$ with $\delta(r_i, w) = r_{i+1}$ for $0 \leq i \leq l-1$ (with $r_l =_{\mathrm{def}} r_0$). Then for all m with $1 \leq m \leq l-1$ there exists some $p_m \in \mathbb{P}_n^{\mathcal{B}}$ such that for all j, j' with $0 \leq j, j' \leq l-1$ and $r_j \xrightarrow{w^m} r_{j'}$ it holds that $r_j \overset{p_m}{\rightsquigarrow} r_{j'}$.*

Proof. Let $\mathcal{M}$ be a DFA with the properties states in the lemma. The proof is by induction on n. For the induction base let $n = 0$ and let some m with $1 \leq m \leq l-1$ be given. Define $p_m =_{\mathrm{def}} (w^l, w^m)$ and observe that $p_m \in \mathbb{P}_0^{\mathcal{B}}$ because $l, m \geq 1$. Since for $0 \leq i \leq l-1$ all states r_i have a w^l-loop, we see that p_m appears at r_j and also at $r_{j'}$. Due to $\delta(r_j, w^m) = r_{j'}$ we have that r_j and $r_{j'}$ are connected via p_m.

Suppose the lemma holds for some $n \geq 0$ and we want to show it for $n+1$. Again, let some m with $1 \leq m \leq l-1$ be given. Set $m' =_{\mathrm{def}} l - m$. Then $1 \leq m' \leq l-1$ and the induction hypothesis provides some $\hat{p}_{m'} \in \mathbb{P}_n^{\mathcal{B}}$. Now define $p_m =_{\mathrm{def}} (w^m, \hat{p}_{m'})$. Then $p_m \in \mathbb{P}_{n+1}^{\mathcal{B}}$ and we have to show for given j, j' with $0 \leq j, j' \leq l-1$ and $\delta(r_j, w^m) = r_{j'}$ that r_j and $r_{j'}$ are connected via p_m. Since $r_j \xrightarrow{w^m} r_{j'} \xrightarrow{w^{m'}} r_j$ we obtain by hypothesis that

$$r_j \xrightarrow{w^m} r_{j'} \overset{\hat{p}_{m'}}{\rightsquigarrow} r_j$$

and hence p_m appears at r_j. Because $r_{j'} \xrightarrow{w^m} r_{(j'+m \bmod l)} \xrightarrow{w^{m'}} r_{j'}$ we get by hypothesis that

$$r_{j'} \xrightarrow{w^m} r_{(j'+m \bmod l)} \overset{\hat{p}_{m'}}{\rightsquigarrow} r_{j'}$$

and hence p_m appears at $r_{j'}$. Finally, note that $\delta(r_j, w^m) = r_{j'}$ and that $\hat{p}_{m'}$ appears at $r_{j'}$. So r_j and $r_{j'}$ are connected via p_m. ❑

Lemma 6.9. *For all $n \geq 0$ it holds that $\mathcal{FP}_n^{\mathcal{B}} \subseteq \mathrm{SF}$.*

Proof. Let $L \in \mathcal{FP}_n^{\mathcal{B}}$ for some $n \geq 0$. Then the minimal DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ with $L(\mathcal{M}) = L$ does not have pattern $\mathbb{P}_n^{\mathcal{B}}$ since $\mathcal{FP}_n^{\mathcal{B}}$ is well–defined. Assume to the contrary that $L \notin \mathrm{SF}$. By Proposition 1.38 there exist $w \in A^+$, some $l \geq 2$ and pairwise distinct states $r_0, r_1, \ldots, r_{l-1} \in S$ such that $\delta(r_i, w) = r_{i+1}$ for $0 \leq i \leq l-1$ (with $r_l =_{\mathrm{def}} r_0$). Since we deal with distinct states in a minimal DFA there exists some $z \in A^*$ and $0 \leq j < k \leq l-1$ such that $\delta(r_j, z) \in S' \Leftrightarrow \delta(r_k, z) \notin S'$. By renaming states we can assume that $j = 0$ and $1 \leq k \leq l-1$. Moreover, we may suppose that $\delta(r_0, z) \in S'$ and $\delta(r_k, z) \notin S'$ because if it is the other way round, then we rename again and take r_k as r_0, r_{k+1} as r_1, …, r_{k-1} as r_{l-1}. Since $\delta(r_0, w^k) = r_k$ we can apply Lemma 6.8 for $m = k$ and obtain that r_0, r_k are connected via some $p \in \mathbb{P}_n^{\mathcal{B}}$. Taking some $x \in A^*$ with $\delta(s_0, x) = r_0$ into account shows that $\mathcal{M}$ has pattern $\mathbb{P}_n^{\mathcal{B}}$ which is a contradiction. So $L \in \mathrm{SF}$. ❑

6.1.3 Strictness of Pattern Hierarchies

We want to show the strictness of the forbidden pattern hierarchies in a certain way, namely we take witnessing languages from [Tho84] that were used there to separate the classes of the dot–depth hierarchy. As remarked in [Tho84], these languages can also be used to show that the Straubing–Thérien hierarchy is strict. A first proof of strictness of the DDH was given in [BK78] using similar languages. We could also do our separation here with these languages, but to facilitate the exposition we stick to [Tho84]. We assume in this subsection that $A = \{a, b\}$ and we separate our hierarchies when defined over A. This can also be done for larger alphabets, see Remark 6.14 below. We show the following theorem.

Theorem 6.10. *For all $n \geq 0$ it holds that $\mathcal{FP}_n^{\mathcal{L}} \subsetneq \mathcal{FP}_n^{\mathcal{B}}$.*

The proof is given at the end of this subsection where it remains to argue for the strictness due to Theorem 6.3. We immediatly have the following corollary. Observe with Theorem 6.10 and Theorem 6.3 that for $n \geq 1$ we have $\mathcal{FP}_{n-1}^{\mathcal{L}} \subsetneq \mathcal{FP}_{n-1}^{\mathcal{B}} \subseteq \mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_{n-1}^{\mathcal{B}} \subseteq \mathcal{FP}_n^{\mathcal{L}} \subsetneq \mathcal{FP}_n^{\mathcal{B}}$.

Corollary 6.11. *For all $n \geq 1$ it holds that $\mathcal{FP}_{n-1}^{\mathcal{L}} \subsetneq \mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_{n-1}^{\mathcal{B}} \subsetneq \mathcal{FP}_n^{\mathcal{B}}$.*

Inspired by [Tho84] we define a family of patterns $\mathbb{W}_n$ for $n \geq 1$.

Definition 6.12. Let $n \geq 1$. Pattern $\mathbb{W}_n$ is defined as the subgraph given in Figure 6.2.

Lemma 6.13. *Let $n \geq 2$. There exist $p, p' \in \mathbb{P}_{n-1}^{\mathcal{L}}$ such that for every* DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ *and for every occurrence $r_0, r_1, \ldots, r_n \in S$ of $\mathbb{W}_n$ in $\mathcal{M}$ it holds that*

$$r_0 \overset{p}{\rightsquigarrow} r_1 \quad \text{and} \quad r_1 \overset{p'}{\rightsquigarrow} r_0.$$

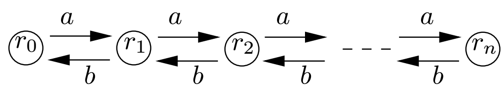


Fig. 6.2. Pattern $\mathbb{W}_n$ with $n \geq 1$.

Proof. The proof is by induction on n. For the induction base let $n = 2$ and define $\hat{p} =_{\text{def}} (ab, a)$ and $\hat{p}' =_{\text{def}} (ab, abb)$. Because all involved words are in A^+ it holds that $\hat{p}, \hat{p}' \in \mathbb{P}_0^{\mathcal{B}}$. We may apply Proposition 6.6 to see that there are $p, p' \in \mathbb{P}_1^{\mathcal{L}}$ such that if two states are connected via $\hat{p}$ or $\hat{p}'$ in a DFA $\mathcal{M}$ then they are connected via p or p', respectively. The definition of p and p' does not depend on $\mathcal{M}$ so it suffices to show the induction base for $\hat{p}$ and $\hat{p}'$.

Let the states $r_0, r_1, r_2 \in S$ witness that some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ has pattern $\mathbb{W}_2$. Then r_0 and r_1 have an ab-loop and hence $\hat{p}$ appears at r_0 and r_1, and $\hat{p}'$ appears at r_0 and r_1. Since $\delta(r_0, a) = r_1$ and $\delta(r_1, abb) = r_0$ we see that r_0, r_1 are connected via $\hat{p}$, and that r_1, r_0 are connected via $\hat{p}'$.

Suppose the lemma holds for some $n \geq 2$ and we want to show it for $n + 1$. By hypothesis there are $\hat{p}, \hat{p}' \in \mathbb{P}_{n-1}^{\mathcal{L}}$ for some $n \geq 2$ having the properties stated in the lemma. We define

$$p =_{\text{def}} (a, \hat{p}') \quad \text{and} \quad p' =_{\text{def}} (ab, \hat{p}, b, \lambda(\hat{p})).$$

Then it holds that $p, p' \in \mathbb{P}_n^{\mathcal{L}}$ since also $\lambda(\hat{p}) \in \mathbb{P}_{n-1}^{\mathcal{L}}$ by Lemma 5.9. Now let some DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ be given such that $r_0, r_1, \ldots, r_n, r_{n+1} \in S$ witness an occurrence of pattern $\mathbb{W}_{n+1}$ in $\mathcal{M}$. Since $r_0, \ldots, r_n$ and $r_1, \ldots, r_{n+1}$ witness two ocurrences of pattern $\mathbb{W}_n$ in $\mathcal{M}$ we can apply the induction hypothesis and obtain that

a. $r_0 \overset{\hat{p}}{\rightsquigarrow} r_1 \overset{\hat{p}}{\rightsquigarrow} r_2$ and
b. $r_2 \overset{\hat{p}'}{\rightsquigarrow} r_1 \overset{\hat{p}'}{\rightsquigarrow} r_0$.

It follows that $\hat{p}$ appears at r_0 and at r_1. So Lemma 5.9 shows that

c. $r_0 \overset{\lambda(\hat{p})}{\rightsquigarrow} r_0$ and $r_1 \overset{\lambda(\hat{p})}{\rightsquigarrow} r_1$.

Let us verify that r_0, r_1 are connected via p. We obtain with b. that p appears at r_0 and also at r_1 because

$$r_0 \xrightarrow{a} r_1 \overset{\hat{p}'}{\rightsquigarrow} r_0 \quad \text{and} \quad r_1 \xrightarrow{a} r_2 \overset{\hat{p}'}{\rightsquigarrow} r_1.$$

It follows that $\hat{p}'$ appears at r_1, so $\delta(r_0, a) = r_1$ implies that r_0, r_1 are connected via p. Now we want so see that r_1, r_0 are connected via p'. We obtain from a. and c. that p' appears at r_1 and at r_0 because

$$r_1 \xrightarrow{ab} r_1 \overset{\hat{p}}{\rightsquigarrow} r_2 \xrightarrow{b} r_1 \overset{\lambda(\hat{p})}{\rightsquigarrow} r_1 \quad \text{and} \quad r_0 \xrightarrow{ab} r_0 \overset{\hat{p}}{\rightsquigarrow} r_1 \xrightarrow{b} r_0 \overset{\lambda(\hat{p})}{\rightsquigarrow} r_0.$$

It follows that $\hat{p}$ appears at r_1 and that $\lambda(\hat{p})$ appears at r_0. So

$$r_1 \xrightarrow{ab} r_1 \xrightarrow{b} r_0$$

shows that r_1, r_0 are connected via p'. ❑

Proof of Theorem 6.10. We need to show $\mathcal{FP}_n^{\mathcal{L}} \subsetneq \mathcal{FP}_n^{\mathcal{B}}$ for all $n \geq 0$. The case $n = 0$ is easily seen since $\mathcal{FP}_0^{\mathcal{L}} = \mathcal{L}_{1/2} \subsetneq \mathcal{B}_{1/2} = \mathcal{FP}_0^{\mathcal{B}}$ by Theorem 6.4 and Proposition 2.35. Also the case $n = 1$ is already settled since $\mathcal{FP}_1^{\mathcal{L}} = \mathcal{L}_{3/2} \subsetneq \mathcal{B}_{3/2} = \mathcal{FP}_1^{\mathcal{B}}$ by Theorem 6.4 and Theorem 4.29. So we may assume $n \geq 2$. We consider witnessing languages L_n such that $L_n \in \mathcal{FP}_n^{\mathcal{B}} \setminus \mathcal{FP}_n^{\mathcal{L}}$. Therefore, we recall the definition of a particular family of languages from [Tho84]. For $n \geq 2$ let L_n be the set of words $w \in A^+$ such that

– $|w|_a - |w|_b = n$ and
– for every prefix v of w it holds that $0 \leq (|v|_a - |v|_b) \leq n$.

Recall that $|w|_a$ for a letter $a \in A$ denotes the number of occurrences of a in w. It is shown in [Tho84] that $L_n \in \mathcal{B}_n$ (these languages are denoted as L_n^+ there). With Theorem 6.4 we have $L_n \in \mathcal{B}_n \subseteq \mathcal{B}_{n+1/2} \subseteq \mathcal{FP}_n^{\mathcal{B}}$ for all $n \geq 1$ and it remains to prove that $L_n \notin \mathcal{FP}_n^{\mathcal{L}}$. For this we define a DFA $\mathcal{M}_n$ with $L(\mathcal{M}_n) = L_n$ as follows. Let $\mathcal{M}_n =_{\text{def}} (A, S, \delta, r_0, \{r_n\})$ with $S =_{\text{def}} \{r_0, r_1, \ldots r_n, r^-\}$ and

– $\delta(r_i, a) =_{\text{def}} r_{i+1}$ and $\delta(r_{i+1}, b) =_{\text{def}} r_i$ for $0 \leq i \leq n-1$,
– $\delta(r_n, a) =_{\text{def}} r^-$ and $\delta(r_0, b) =_{\text{def}} r^-$ and
– $\delta(r^-, a) = \delta(r^-, b) =_{\text{def}} r^-$.

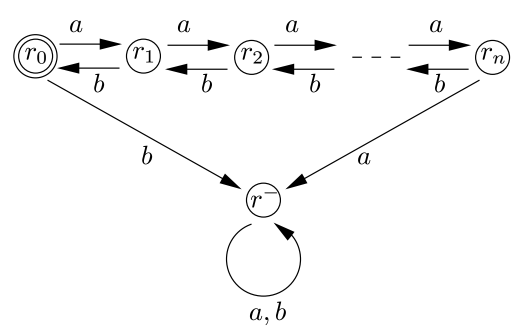


Fig. 6.3. Automaton $\mathcal{M}_n$ with $L(\mathcal{M}_n) = L_n$.

The DFA $\mathcal{M}_n$ is given in Figure 6.3 and it is easy to see that $L(\mathcal{M}_n) = L_n$. We show for all $n \geq 2$ that $\mathcal{M}_n$ has pattern $\mathbb{P}_n^{\mathcal{L}}$. It follows from this that $L(\mathcal{M}_n) = L_n \notin \mathcal{FP}_n^{\mathcal{L}}$.

Observe that $\mathcal{M}_n$ has pattern $\mathbb{W}_n$ witnessed by $r_0, r_1, \ldots, r_n$. So from Lemma 6.13 we obtain that there exists some $\hat{p} \in \mathbb{P}_{n-1}^{\mathcal{L}}$ such that r_0, r_1 are connected via $\hat{p}$. Now define $p =_{\text{def}} (ab, \hat{p}, b, \lambda(\hat{p}))$ as in the induction step in the proof of Lemma 6.13. Then $p \in \mathbb{P}_n^{\mathcal{L}}$ and we show that r_0, r^- are connected via p.

We obtain that p appears at r_0 because

$$r_0 \xrightarrow{ab} r_0 \overset{\hat{p}}{\rightsquigarrow} r_1 \xrightarrow{b} r_0 \overset{\lambda(\hat{p})}{\rightsquigarrow} r_0.$$

Here we use that r_0, r_1 are connected via $\hat{p}$ from Lemma 6.13 and that r_0, r_0 are connected via $\lambda(\hat{p})$ by Lemma 5.9 since $\hat{p}$ appears at r_0. It also holds that p appears at r^- because this is a sink state and one can show with a trivial induction that for all $n \geq 0$ and all $\tilde{p} \in \mathbb{P}_n^{\mathcal{L}}$

it holds that $\tilde{p}$ appears at a sink. For the same reason also $\lambda(\hat{p})$ appears at r^- and we have already noticed that $\hat{p}$ appears at r_0. Together with

$$r_0 \xrightarrow{ab} r_0 \xrightarrow{b} r^-$$

this shows that r_0, r^- are connected via p. Finally we define $x =_{\text{def}} \varepsilon$ and $z =_{\text{def}} a^n$ to see that $\mathcal{M}_n$ has pattern $\mathbb{P}_n^{\mathcal{L}}$. **(End proof of Theorem 6.10.)**

Remark 6.14. Suppose we deal with some alphabet A such that $|A| \geq 3$, e.g., $A = \{a, b, c_1, \cdots, c_l\}$ for some $l \geq 1$. If we define $\mathcal{M}_n$ such that $\delta(s, c_i) = r^-$ for $1 \leq i \leq l$ and for all $s \in S$, we still find the required patterns. This means on the language side that we intersect the expressions for L_n with $\{a,b\}^+ = A^+ \setminus \bigcup_{1 \leq i \leq l} A^* c_i A^* \in \mathrm{co}\mathcal{B}_{1/2}$. The latter does not increase the dot–depth since $L_n \in \mathcal{B}_n$ which is a Boolean algebra that includes $\mathrm{co}\mathcal{B}_{1/2}$. Together this allows to prove Theorem 6.10 also in case of a larger alphabet.

6.1.4 Decidability, Lower Bounds and a Conjecture for the Dot–Depth Problem

We carry over our general decidability result for pattern classes.

Theorem 6.15. *For fixed $n \geq 0$ the membership problems of $\mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_n^{\mathcal{B}}$ are decidable in nondeterministic logarithmic space* NL.

Proof. It holds that $\textsc{Pattern}_{0,k}^{\mathcal{L}}, \textsc{Pattern}_{0,k}^{\mathcal{B}} \in \mathrm{NL}$ for each $k \geq 1$. To see this observe that due to the definition of the initial patterns the problems $\textsc{Pattern}_{0,k}^{\mathcal{L}}$ and $\textsc{Pattern}_{0,k}^{\mathcal{B}}$ are just reachability problems very similar to $\textsc{Reach}_k$ which can be solved in NL. Now the theorem follows from Theorem 5.25. ❑

Since membership to SF is decidable, Theorem 6.15 yields an algorithm to determine the minimal n such that a given regular language is in $\mathcal{FP}_n^{\mathcal{B}}$. This is in fact a lower bound algorithm for the dot–depth of a given language: if $L \in \mathcal{FP}_n^{\mathcal{B}} \setminus \mathcal{FP}_{n-1}^{\mathcal{B}}$ then the dot–depth of L is strictly greater than $n - 1/2$ by Theorem 6.4.

We have seen many structural similarities between the STH and DDH on one hand, and the hierarchies of the forbidden pattern classes $\mathcal{FP}_n^{\mathcal{L}}$ and $\mathcal{FP}_n^{\mathcal{B}}$ on the other hand. Among them we have shown strictness and the upper bound SF. At this point we have no evidence against the following conjecture.

Conjecture 6.16. *For all $n \geq 0$ it holds that $\mathcal{B}_{n+1/2} = \mathcal{FP}_n^{\mathcal{B}}$ and $\mathcal{L}_{n+1/2} = \mathcal{FP}_n^{\mathcal{L}}$.*

Note that this conjectures an effective characterization of all levels $n + 1/2$ of the DDH and STH. As stated in Theorem 6.4 we already now that the conjecture is true for $n \in \{0, 1\}$. If we look at the witnessing language L_n from Theorem 6.10 again we see that $L_n \in \mathcal{B}_{n+1/2}$ but $L_n \notin \mathcal{FP}_n^{\mathcal{L}}$. So $\mathcal{B}_{n+1/2} \nsubseteq \mathcal{FP}_n^{\mathcal{L}}$ which shows that $\mathcal{FP}_n^{\mathcal{L}}$ captures $\mathcal{L}_{n+1/2}$ but not $\mathcal{B}_{n+1/2}$. We may conclude that our pattern classes are not too ‘broad’.

6.2 $\mathcal{L}_{5/2}$ is Decidable for Two–Letter Alphabets

We prove in this section the following theorem.

Theorem 6.17. *Let $n \geq 1$. If $\mathcal{B}_{n+1/2} = \mathcal{FP}_n^{\mathcal{B}}$ for arbitrary alphabets, then $\mathcal{L}_{n+3/2} = \mathcal{FP}_{n+1}^{\mathcal{L}}$ in case of a two–letter alphabet.*

With the latter we mean that we consider the STH defined over some alphabet A with $|A| = 2$, and correspondingly we consider for $\mathcal{FP}_{n+1}^{\mathcal{L}}$ only automata with input alphabet A. Note that the inclusion $\mathcal{L}_{n+3/2} \subseteq \mathcal{FP}_{n+1}^{\mathcal{L}}$ is from Theorem 6.4 and holds unconditionally. The proof of Theorem 6.17 is given at the end of Subsection 6.2.3. From Theorem 6.4 we know that $\mathcal{B}_{3/2} = \mathcal{FP}_1^{\mathcal{B}}$. Here we had no restrictions on the size of the alphabet over which $\mathcal{B}_{3/2}$ is defined.

Corollary 6.18. *It holds that $\mathcal{L}_{5/2} = \mathcal{FP}_2^{\mathcal{L}}$ in case of a two–letter alphabet.*

We give $\mathbb{P}_2^{\mathcal{L}}$ in Figure 6.4. The following corollary is an immediate consequence of Theorem 6.15.

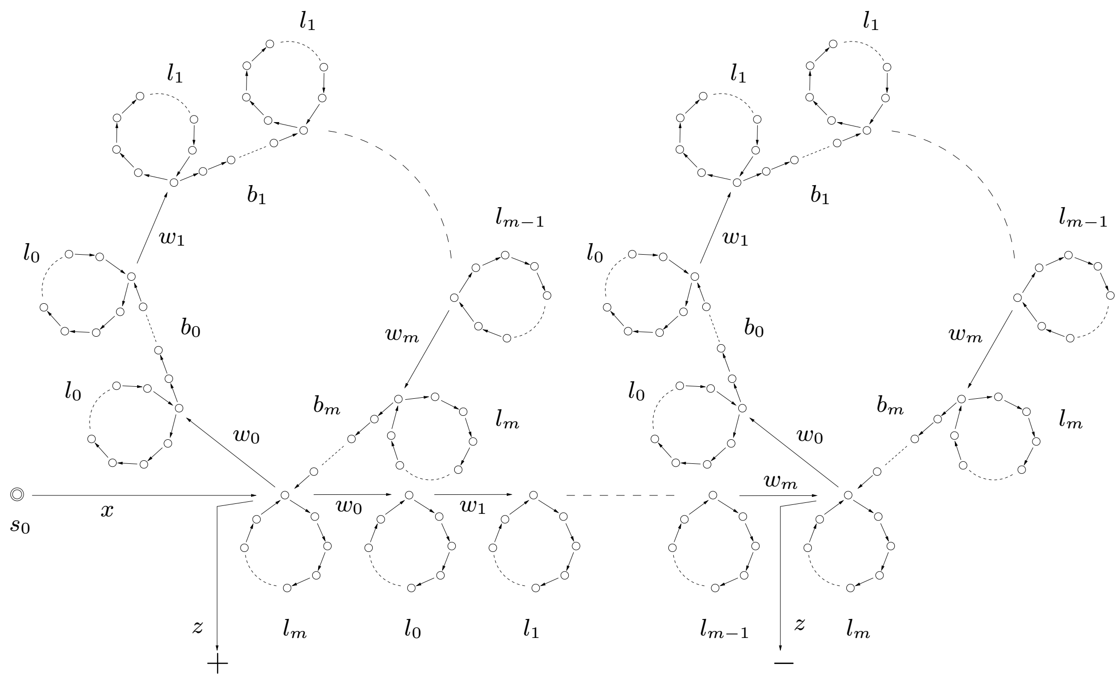


Fig. 6.4. Forbidden pattern for $\mathcal{FP}_2^{\mathcal{L}}$ with $p = (w_0, p_0, \ldots, w_m, p_m) \in \mathbb{P}_2^{\mathcal{L}}$ where $w_i \in A^+$, $p_i \in \mathbb{P}_1^{\mathcal{L}}$ and $\overline{p_i} = b_i$ and $\overline{p_i}^{\circ} = l_i$. It holds that $\mathcal{L}_{5/2} = \mathcal{FP}_2^{\mathcal{L}}$ if $|A| = 2$.

Corollary 6.19. *The membership problem of $\mathcal{L}_{5/2}$ defined over a two–letter alphabet is decidable in nondeterministic logarithmic space* NL.

Finally, we draw the connection to first–order logic with help of Theorem 1.23.

Corollary 6.20. *Given a language $L \subseteq \{0,1\}^+$ it is decidable whether L is definable by a Σ_3 formula of the logic* FO$[<]$.

6.2.1 Changing the Alphabet

Let $A =_{\mathrm{def}} \{a, b\}$ for the remainder of this section. We want to show for $n \geq 1$ the inclusion $\mathcal{FP}^{\mathcal{L}}_{n+1} \subseteq \mathcal{L}_{n+3/2}$ when both classes are defined with respect to A. To do so, we reduce this case to the assumption $\mathcal{FP}^{\mathcal{B}}_{n} = \mathcal{B}_{n+1/2}$ from Theorem 6.17 where $\mathcal{FP}^{\mathcal{B}}_{n}$ and $\mathcal{B}_{n+1/2}$ are defined with respect to some arbitrary large but fixed alphabet. The reduction is as follows. Note with Theorem 6.7 and Theorem 1.37 that there exists for every $L \in \mathcal{FP}^{\mathcal{L}}_{n+1}$ some permutation–free DFA accepting L. So let us fix some arbitrary permutation–free DFA $\mathcal{M} = (A, S, \delta, s_0, S')$ and set $r =_{\mathrm{def}} |\mathcal{M}|$. The idea in this section is straightforward: nothing new happens in $\mathcal{M}$ if the same letter appears $\geq r$ times consecutively in the input (see Proposition 1.39). We encode the behaviour of $\mathcal{M}$ in these finitely many cases over some larger alphabet. Therefore, we define $A_{\mathcal{M}} =_{\mathrm{def}} A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}$ with

$$A^a_{\mathcal{M}} =_{\mathrm{def}} \{a_1, \ldots, a_r\} \text{ and } A^b_{\mathcal{M}} =_{\mathrm{def}} \{b_1, \ldots, b_r\}$$

and some function $f_{\mathcal{M}} : A^+ \to (A_{\mathcal{M}})^+$ that achieves the encoding. The function $f_{\mathcal{M}}$ will map every block from $\{a\}^+ \cup \{b\}^+$ of maximal length to a single letter from $A_{\mathcal{M}}$, where the index of the letter corresponds to the block size up to threshold r. First, we write every $w \in A^+$ as $w = w_1 w_2 \cdots w_k$ for some $k \geq 1$ and factors w_i of maximal length such that $w_i \in \{a\}^+ \cup \{b\}^+$. Call this the A-factorization of w and observe that this factorization is unique due to the maximality condition.

Definition 6.21. Let $w \in A^+$ and let $w = w_1 w_2 \cdots w_k$ for some $k \geq 1$ be the A-factorization of w. Then $f_{\mathcal{M}}(w) =_{\mathrm{def}} c_1 c_2 \cdots c_k \in (A_{\mathcal{M}})^+$ with

$$c_i =_{\mathrm{def}} \begin{cases} a_{\min\{l,r\}} & : \quad w_i = a^l \\ b_{\min\{l,r\}} & : \quad w_i = b^l \end{cases}$$

for $1 \leq i \leq k$ and $l \geq 1$.

We say that $c \in A_{\mathcal{M}}$ has type a if $c \in A^a_{\mathcal{M}}$ and it has type b if $c \in A^b_{\mathcal{M}}$. Observe that $w_i \in \{a\}^+$ if and only if c_i has type a and $w_i \in \{b\}^+$ if and only if c_i has type b. Note also that $|f_{\mathcal{M}}(w)| = k$ if and only if the A-factorization of w has k factors w_i. Moreover, we define for $L \subseteq A^+$ and for a class of languages $\mathcal{C}$ that

$$f_{\mathcal{M}}(L) =_{\mathrm{def}} \bigcup_{w \in L} \{f_{\mathcal{M}}(w)\} \quad \text{and} \quad f_{\mathcal{M}}(\mathcal{C}) =_{\mathrm{def}} \{\, f_{\mathcal{M}}(L) \mid L \in \mathcal{C} \,\}.$$

Since the definition of $f_{\mathcal{M}}$ is based on the single factors of an A-factorization we can also concatenate the $f_{\mathcal{M}}(w_i)$ and obtain $f_{\mathcal{M}}(w)$.

Proposition 6.22. *Let $w \in A^+$ and let $w = w_1 w_2 \cdots w_k$ for some $k \geq 1$ be the A-factorization of w. Then $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(w_1) f_{\mathcal{M}}(w_2) \cdots f_{\mathcal{M}}(w_k)$ with $f_{\mathcal{M}}(w_i) \in A_{\mathcal{M}}$.*

Proof. By definition, $f_{\mathcal{M}}(w) = c_1 c_2 \cdots c_k$ such that for $1 \leq i \leq k$ we have $c_i = a_{\min\{l_i,r\}}$ if $w_i = a^{l_i}$ or $c_i = b_{\min\{l_i,r\}}$ if $w_i = b^{l_i}$ for some $l_i \geq 1$. Fix some i with $1 \leq i \leq k$ and assume without loss of generality that $w_i \in \{a\}^+$. Then the A-factorization of w_i is just $w_i = a^{l_i}$ and by definition we have $f_{\mathcal{M}}(w_i) = a_{\min\{l_i,r\}}$. So $c_i = a_{\min\{l_i,r\}} = f_{\mathcal{M}}(w_i)$. It follows that $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(w_1) f_{\mathcal{M}}(w_2) \cdots f_{\mathcal{M}}(w_k)$. ❑

We will use the previous proposition without further reference. Let us also note that we can take factors of w that respect its A-factorization and obtain the respective factor of $f_{\mathcal{M}}(w)$.

Proposition 6.23. *Let $w \in A^+$ and let $w = w_1 w_2 \cdots w_k$ be the A-factorization of w for some $k \geq 1$. If $f_{\mathcal{M}}(w) = c_1 c_2 \cdots c_k$ for $c_i \in A_{\mathcal{M}}$ then for all j, j' with $1 \leq j \leq j' \leq k$ it holds that $f_{\mathcal{M}}(w_j w_{j+1} \cdots w_{j'}) = c_j c_{j+1} \cdots c_{j'}$.*

Proof. It holds that $f_{\mathcal{M}}(w_i) = c_i$ for $1 \leq i \leq k$. Since $w_j w_{j+1} \cdots w_{j'}$ is an A-factorization we have

$$c_j c_{j+1} \cdots c_{j'} = f_{\mathcal{M}}(w_j) f_{\mathcal{M}}(w_{j+1}) \cdots f_{\mathcal{M}}(w_{j'}) = f_{\mathcal{M}}(w_j w_{j+1} \cdots w_{j'}).$$

❑

Next we see that $f_{\mathcal{M}}$ does what we intended, namely that $\mathcal{M}$ cannot distinguish words having the same encoding.

Proposition 6.24. *Let $w, v \in A^+$ with $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(v)$. Then $\delta(s, w) = \delta(s, v)$ for all $s \in S$. In particular, it holds that $w \in L(\mathcal{M}) \Leftrightarrow v \in L(\mathcal{M})$.*

Proof. By assumption, $f_{\mathcal{M}}(w) = c_1 c_2 \cdots c_k = f_{\mathcal{M}}(v)$ for some $k \geq 1$ and $c_i \in A_{\mathcal{M}}$. Let $w = w_1 w_2 \cdots w_k$ and let $v = v_1 v_2 \cdots v_k$ be the A-factorizations of w and v, respectively. Fix some $s \in S$ and some i with $1 \leq i \leq k$. We argue that $\delta(s, w_i) = \delta(s, v_i)$. Assume without loss of generality that c_i is of type a and hence $w_i, v_i \in \{a\}^+$. If $c_i = a_j$ with $1 \leq j \leq r - 1$ then $w_i = a^j = v_i$ and $\delta(s, w_i) = \delta(s, v_i)$. If $c_i = a_r$ then $w_i = a^l$ and $v_i = a^{l'}$ with $l, l' \geq r$. By Proposition 1.39 we have that $\delta(s, a^l) = \delta(s, a^r) = \delta(s, a^{l'})$. It follows inductively that $\delta(s, w_1 w_2 \cdots w_i) = \delta(s, v_1 v_2 \cdots v_i)$ for all $s \in S$ and $1 \leq i \leq k$. Hence, $\delta(s, w) = \delta(s, v)$ and in particular $\delta(s_0, w) = \delta(s_0, v)$. ❑

However, not all words from $(A_{\mathcal{M}})^+$ can appear in the range of $f_{\mathcal{M}}$. The maximality condition in A-factorizations ensures that the types of letters in $f_{\mathcal{M}}(w)$ alternate between a and b. We call these words well–formed.

Definition 6.25. Define $WF_{\mathcal{M}} =_{\text{def}} f_{\mathcal{M}}(A^+)$ as the set of well–formed words of $(A_{\mathcal{M}})^+$. For $\mu \in WF_{\mathcal{M}}$ let $L_\mu =_{\text{def}} \{\, v \in A^+ \mid f_{\mathcal{M}}(v) = \mu \,\}$.

By definition, none of the sets L_μ is empty. The alternation condition of letter types is characteristic for $WF_{\mathcal{M}}$.

Proposition 6.26. *It holds that $\mu \in WF_{\mathcal{M}}$ if and only if the letters in μ alternate between type a and b.*

Proof. Let $\mu = c_1 c_2 \cdots c_k \in (A_{\mathcal{M}})^+$ for some $k \geq 1$. If $\mu = f_{\mathcal{M}}(w)$ for some $w \in A^+$ then the letters c_i alternate between $A^a_{\mathcal{M}}$ and $A^b_{\mathcal{M}}$ due to the maximality condition in the A-factorization of w. Conversely, we may consider $w =_{\text{def}} w_1 w_2 \cdots w_k$ with $w_i =_{\text{def}} a^j$ if $c_i = a_j \in A^a_{\mathcal{M}}$ and $w_i =_{\text{def}} b^j$ if $c_i = b_j \in A^b_{\mathcal{M}}$. Then $w = w_1 w_2 \cdots w_k$ is the A-factorization of w with $f_{\mathcal{M}}(w_i) = c_i$. So $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(w_1) f_{\mathcal{M}}(w_2) \cdots f_{\mathcal{M}}(w_k) = c_1 c_2 \cdots c_k$ and hence $f_{\mathcal{M}}(w) = \mu \in WF_{\mathcal{M}}$. ❑

Since this condition holds also for factors of well–formed words, these are again well–formed.

Proposition 6.27. *Every non–empty factor of a well–formed word is well–formed.*

Finally, we see that well–formed words behave nicely with $f_{\mathcal{M}}$.

Proposition 6.28. *Let $w, v \in A^+$. It holds that*

$$f_{\mathcal{M}}(w)f_{\mathcal{M}}(v) \in WF_{\mathcal{M}} \Longleftrightarrow f_{\mathcal{M}}(w)f_{\mathcal{M}}(v) = f_{\mathcal{M}}(wv).$$

Proof. It suffices to argue for the 'only-if'–part. Suppose $w = w_1 \cdots w_k$ and $v = w_{k+1} \cdots w_{k+l}$ for $k, l \geq 1$ are the A-factorizations of w and v, respectively. Since $f_{\mathcal{M}}(w)f_{\mathcal{M}}(v) \in WF_{\mathcal{M}}$ we have by Proposition 6.26 that $w_k \in \{a\}^+ \Leftrightarrow w_{k+1} \in \{b\}^+$. So the A-factorization of wv is $wv = w_1 \cdots w_k w_{k+1} \cdots w_{k+l}$. Hence, $f_{\mathcal{M}}(wv) = f_{\mathcal{M}}(w_1) \cdots f_{\mathcal{M}}(w_k) f_{\mathcal{M}}(w_{k+1}) \cdots f_{\mathcal{M}}(w_{k+l}) = f_{\mathcal{M}}(w)f_{\mathcal{M}}(v)$. ❑

6.2.2 Transformation of Patterns

The goal of this subsection is to prove Lemma 6.29 below. It says that if $L(\mathcal{M}) \in \mathcal{FP}^{\mathcal{L}}_{n+1}$ for $n \geq 1$ and for some permutation–free DFA $\mathcal{M}$ with a two–letter input alphabet, then $f_{\mathcal{M}}(L(\mathcal{M})) \in \mathcal{FP}^{\mathcal{B}}_n$. In the rather technical proof we do the following. Starting with the given DFA $\mathcal{M}$ we define another DFA $\tilde{\mathcal{M}}$ with $L(\tilde{\mathcal{M}}) = f_{\mathcal{M}}(L(\mathcal{M}))$. Then we show that if $\tilde{\mathcal{M}}$ had pattern $\mathbb{P}^{\mathcal{B}}_n$ then $\mathcal{M}$ would have even pattern $\mathbb{P}^{\mathcal{L}}_{n+1}$. The fact that $|A| = 2$ and that only well–formed words appear in pattern $\mathbb{P}^{\mathcal{B}}_n$ in the transition graph of $\tilde{\mathcal{M}}$ are the main arguments to see that the loops in the innermost bridge–structures of pattern $\mathbb{P}^{\mathcal{B}}_n$ in $\tilde{\mathcal{M}}$ have letters of type a and of type b. It follows from this that the innermost bridge–structures of the pattern in $\mathcal{M}$ satisfy a certain alphabet condition, hence $\mathcal{M}$ has in fact pattern $\mathbb{P}^{\mathcal{L}}_{n+1}$ (similar to $\mathbb{B}_{1/2}$ versus $\mathbb{L}_{3/2}$).

Lemma 6.29. *Let $n \geq 1$ and let $\mathcal{M}$ be a permutation–free* DFA *with input alphabet $A = \{a, b\}$. If $L(\mathcal{M}) \in \mathcal{FP}^{\mathcal{L}}_{n+1}$ then $f_{\mathcal{M}}(L(\mathcal{M})) \in \mathcal{FP}^{\mathcal{B}}_n$.*

Proof. Let $\mathcal{M} = (A, S, \delta, s_0, S')$, set $r =_{\mathrm{def}} |\mathcal{M}|$ and $L =_{\mathrm{def}} L(\mathcal{M})$. We use the notations from the previous subsection. The proof has three steps.

Step 1: Construction of $\tilde{\mathcal{M}}$. The automaton $\tilde{\mathcal{M}}$ we have in mind has input alphabet $A_{\mathcal{M}}$ and simulates $\mathcal{M}$ as follows. If $\tilde{\mathcal{M}}$ reads for instance a_j for $1 \leq j \leq r$ from the input then it behaves like $\mathcal{M}$ on input a^j. Since we want that $\tilde{\mathcal{M}}$ rejects whenever the input is not well–formed, we have to store the type of the previous input letter in a second component in order to ensure the alternation of letter types. We introduce a rejecting sink state $\perp$ for all non well–formed inputs to $\tilde{\mathcal{M}}$. As we will see, this construction preserves the structure of the transition graph of $\mathcal{M}$ in a way such that we can conclude from a pattern in $\tilde{\mathcal{M}}$ to a pattern in $\mathcal{M}$. Now define $\tilde{\mathcal{M}} =_{\mathrm{def}} (A_{\mathcal{M}}, \tilde{S}, \tilde{\delta}, (s_0, \varepsilon), \tilde{S}')$ with

a. $\tilde{S} =_{\mathrm{def}} \{S \times \{a, b\}\} \cup \{(s_0, \varepsilon), \perp\}$,
b. $\tilde{S}' =_{\mathrm{def}} S' \times \{a, b\}$,
c. $\tilde{\delta}(\perp, c) =_{\mathrm{def}} \perp$ for all $c \in A_{\mathcal{M}}$,
d. $\tilde{\delta}((s_0, \varepsilon), a_j) =_{\mathrm{def}} (\delta(s_0, a^j), a)$ for all $a_j \in A^a_{\mathcal{M}}$,
e. $\tilde{\delta}((s_0, \varepsilon), b_j) =_{\mathrm{def}} (\delta(s_0, b^j), b)$ for all $b_j \in A^b_{\mathcal{M}}$,
f. $\tilde{\delta}((s, a), a_j) =_{\mathrm{def}} \perp$ for all $a_j \in A^a_{\mathcal{M}}$ and $s \in S$,
g. $\tilde{\delta}((s, a), b_j) =_{\mathrm{def}} (\delta(s, b^j), b)$ for all $b_j \in A^b_{\mathcal{M}}$ and $s \in S$,
h. $\tilde{\delta}((s, b), a_j) =_{\mathrm{def}} (\delta(s, a^j), a)$ for all $a_j \in A^a_{\mathcal{M}}$ and $s \in S$ and
i. $\tilde{\delta}((s, b), b_j) =_{\mathrm{def}} \perp$ for all $b_j \in A^b_{\mathcal{M}}$ and $s \in S$.

We set $\tilde{L} =_{\text{def}} L(\tilde{\mathcal{M}})$. With the following two claims we make precise the relation between $\mathcal{M}$ and $\tilde{\mathcal{M}}$, i.e., we show how we find a path of the transition graph of $\mathcal{M}$ in the transition graph of $\tilde{\mathcal{M}}$ and vice versa.

> **Claim 1.** Let $w \in A^+$ and $s \xrightarrow{w} s'$ in $\mathcal{M}$ for some $s, s' \in S$. Let $t_f \in A$ ($t_l \in A$) be the first letter (last letter, respectively) of w. For $t' \in A \cup \{\varepsilon\}$ with $t' \neq t_f$ and $(s, t') \in \tilde{S}$ it holds that $(s, t') \xrightarrow{f_{\mathcal{M}}(w)} (s', t_l)$ in $\tilde{\mathcal{M}}$.

Proof of Claim 1. Let $w = w_1 w_2 \cdots w_k$ with $k \geq 1$ be the A-factorization of w. Moreover, let $f_{\mathcal{M}}(w) = c_1 c_2 \cdots c_k$ and define $t_i \in A$ to be the type of c_i. Hence, $t_1 t_2 \cdots t_k$ consists of alternating a's and b's, and in particular $t_1 = t_f$ and $t_k = t_l$. For $1 \leq i \leq k$ define $v_i =_{\text{def}} a^j$ if $c_i = a_j$ and $v_i =_{\text{def}} b^j$ if $c_i = b_j$. Since $f_{\mathcal{M}}(w_i) = c_i = f_{\mathcal{M}}(v_i)$ we have $\delta(p, w_i) = \delta(p, v_i)$ for all $p \in S$ by Proposition 6.24. Now assume that $t' \in A \cup \{\varepsilon\}$ with $t' \neq t_f$ and $(s, t') \in \tilde{S}$. We show inductively that $\tilde{\delta}((s, t'), c_1 \cdots c_i) = (\delta(s, w_1 \cdots w_i), t_i)$ for $1 \leq i \leq k$.

Induction base. Let $i = 1$ and assume without loss of generality that $t_1 = t_f = a$. Hence, $c_1 = a_j$ and $v_1 = a^j$ for some $1 \leq j \leq r$, and $t' \in \{b, \varepsilon\}$ by assumption. First suppose $t' = \varepsilon$. Since we require that $(s, t') \in \tilde{S}$ we only make an assertion in case $s = s_0$. We conclude from $\delta(s_0, w_1) = \delta(s_0, v_1)$ and from d. in the definition of $\tilde{\mathcal{M}}$ that

$$\tilde{\delta}((s, t'), c_1) = \tilde{\delta}((s_0, \varepsilon), a_j) = (\delta(s_0, a^j), a) = (\delta(s_0, v_1), a) = (\delta(s_0, w_1), t_1).$$

Now assume that $t' = b$. Then we see with $\delta(s, w_1) = \delta(s, v_1)$ and h. in the definition of $\tilde{\mathcal{M}}$ that

$$\tilde{\delta}((s, t'), c_1) = \tilde{\delta}((s, b), a_j) = (\delta(s, a^j), a) = (\delta(s, v_1), a) = (\delta(s, w_1), t_1).$$

The case $t_1 = t_f = b$ is completely analogously involving e. and g. from the definition of $\tilde{\mathcal{M}}$.

Induction step. Suppose we have $\tilde{\delta}((s, t'), c_1 \cdots c_i) = (\delta(s, w_1 \cdots w_i), t_i)$ for some i with $1 \leq i < k$ and we want to show this for $i + 1$. Let us assume without loss of generality that $t_i = a$. Hence, $c_i = a_j$, $v_i = a^j$ and $t_{i+1} = b$, $c_{i+1} = b_{j'}$, $v_{i+1} = b^{j'}$ for some $1 \leq j, j' \leq r$.

We conclude from the hypothesis, from $\delta(r, w_{i+1}) = \delta(r, v_{i+1})$ for arbitrary $r \in S$ and from g. in the definition of $\tilde{\mathcal{M}}$ that

$$\begin{aligned}
\tilde{\delta}((s, t'), c_1 \cdots c_i c_{i+1}) &= \tilde{\delta}(\tilde{\delta}((s, t'), c_1 \cdots c_i), c_{i+1}) \\
&= \tilde{\delta}((\delta(s, w_1 \cdots w_i), t_i), c_{i+1}) \\
&= \tilde{\delta}((\delta(s, w_1 \cdots w_i), a), b_{j'}) \\
&= (\delta(\delta(s_0, w_1 \cdots w_i), b^{j'}), b) \\
&= (\delta(\delta(s_0, w_1 \cdots w_i), v_{i+1}), b) \\
&= (\delta(\delta(s_0, w_1 \cdots w_i), w_{i+1}), t_{i+1}) \\
&= (\delta(s_0, w_1 \cdots w_i w_{i+1}), t_{i+1}).
\end{aligned}$$

The case of $t_i = b$ is completely analogously involving h. from the definition of $\tilde{\mathcal{M}}$. This completes the induction and it follows that

$$\tilde{\delta}((s, t'), f_{\mathcal{M}}(w)) = \tilde{\delta}((s, t'), c_1 \cdots c_k) = (\delta(s, w_1 \cdots w_k), t_k) = (\delta(s, w), t_l) = (s', t_l).$$

(End proof of Claim 1.)

Claim 2. Suppose $(s,t) \xrightarrow{\mu} (s',t')$ in $\tilde{\mathcal{M}}$ for some $\mu \in (A_{\mathcal{M}})^+$ and $(s,t),(s',t') \in \tilde{S}$ with $s,s' \in S$, $t \in A \cup \{\varepsilon\}$ and $t' \in A$. Then $\mu \in WF_{\mathcal{M}}$ and $s \xrightarrow{w} s'$ in $\mathcal{M}$ for all $w \in L_\mu$.

Proof of Claim 2. Let $\mu = c_1c_2\cdots c_k$ for some $k \geq 1$ and define t_i as before to be the type of c_i. Note that if $t_1 = t$ then $\tilde{\delta}((s,t),c_1) = \bot$ due to f. and i. in the definition of $\tilde{\mathcal{M}}$, and $\tilde{\delta}((s,t),\mu) = \bot$ due to c. So $t_1 \neq t$.

Now assume to the contrary that $\mu \notin WF_{\mathcal{M}}$. Then we get from Proposition 6.26 that there is some minimal i with $1 \leq i < k$ such that $c_i, c_{i+1} \in A^a_{\mathcal{M}}$ or $c_i, c_{i+1} \in A^b_{\mathcal{M}}$. So $t_i = t_{i+1}$ and $c_1c_2\cdots c_i \in WF_{\mathcal{M}}$. Hence, there is some $v \in A^+$ with first letter t_1 and last letter t_i such that $f_{\mathcal{M}}(v) = c_1\cdots c_i$ and there is some $s'' \in S$ with $\delta(s,v) = s''$. Because $t_1 \neq t$ we have all prerequisites of Claim 1 and obtain from it that $\tilde{\delta}((s,t),c_1c_2\cdots c_i) = (s'',t_i)$. From f. and i. in the definition of $\tilde{\mathcal{M}}$ we see that $\tilde{\delta}((s'',t_i),c_{i+1}) = \bot$ since $t_i = t_{i+1}$, and from c. we get $\tilde{\delta}((s,t),\mu) = \bot$, a contradiction to our assumption in Claim 2 that $\tilde{\delta}((s,t),\mu) = (s',t')$. This shows that $\mu \in WF_{\mathcal{M}}$.

Now let $w \in L_\mu$. Then $f_{\mathcal{M}}(w) = \mu$ and w has first letter t_1 and last letter t_k. Moreover, there is some $s'' \in S$ such that $\delta(s,w) = s''$. Since we know that $t_1 \neq t$ we can apply Claim 1 again and obtain

$$(s',t') = \tilde{\delta}((s,t),\mu) = \tilde{\delta}((s,t),f_{\mathcal{M}}(w)) = (s'',t_k).$$

So $\delta(s,w) = s'' = s'$. **(End proof of Claim 2.)**

We immediately obtain from these claims that $\tilde{L} = L(\tilde{\mathcal{M}}) = f_{\mathcal{M}}(L(\mathcal{M})) = f_{\mathcal{M}}(L)$. To see this suppose first $\mu \in \tilde{L}$. Then $\tilde{\delta}((s_0,\varepsilon),\mu) = (s,d) \in \tilde{S}'$ for some $s \in S'$ and $d \in A$ by b. in the definition of $\tilde{\mathcal{M}}$. We can apply Claim 2 to see $\mu \in WF_{\mathcal{M}}$. So there exists some $w \in A^+$ with $\mu = f_{\mathcal{M}}(w)$ and again by Claim 2 we obtain $w \in L$ from $\delta(s_0,w) = s$. Conversely, suppose $w \in L$. Then $\delta(s_0,w) = s$ for some $s \in S'$ and by Claim 1 we see that $\tilde{\delta}((s_0,\varepsilon),f_{\mathcal{M}}(w)) = (s,t) \in \tilde{S}'$ for some $t \in A$. Hence $f_{\mathcal{M}}(w) \in \tilde{L}$.

So we have seen in this first step that $\tilde{L} = f_{\mathcal{M}}(L) \subseteq WF_{\mathcal{M}}$.

Step 2: Pattern transformation. We show in this second step that if two states $(s,t),(s',t')$ are connected in $\tilde{\mathcal{M}}$ via some $\tilde{p} \in \mathbb{P}^{\mathcal{B}}_n$ for $n \geq 0$ then there exists some $p \in \mathbb{P}^{\mathcal{L}}_{n+1}$ such that s,s' are connected in $\mathcal{M}$ via p. We require here as an assumption that the two states that are connected via $\tilde{p}$ in $\tilde{\mathcal{M}}$ are not $\bot$. It is shown in Step 3 that this is no restriction.

Claim 3. Let $n \geq 0$ and $\tilde{p} \in \mathbb{P}^{\mathcal{B}}_n$ such that $(s,t) \overset{\tilde{p}}{\rightsquigarrow} (s',t')$ in $\tilde{\mathcal{M}}$ for some $(s,t),(s',t') \in \tilde{S}$ with $s,s' \in S$ and $t,t' \in A \cup \{\varepsilon\}$. Then there exists some $p \in \mathbb{P}^{\mathcal{L}}_{n+1}$ such that for all $(s_1,t_1),(s_2,t_2),(s_3,t_3) \in \tilde{S}$ with $s_1,s_2,s_3 \in S$ and $t_1,t_2,t_3 \in A \cup \{\varepsilon\}$ the following holds.

1. If $\tilde{p}$ appears at (s_1,t_1) in $\tilde{\mathcal{M}}$, then p appears at s_1 in $\mathcal{M}$.
2. If $(s_2,t_2) \overset{\tilde{p}}{\rightsquigarrow} (s_3,t_3)$ in $\tilde{\mathcal{M}}$, then $s_2 \overset{p}{\rightsquigarrow} s_3$ in $\mathcal{M}$.

Proof of Claim 3. The proof is by induction on n.

Induction base. Let $n = 0$. Then $\tilde{p} = (\nu,\mu) \in \mathbb{P}^{\mathcal{B}}_0$ for some $\nu,\mu \in (A_{\mathcal{M}})^+$. Since $(s,t),(s',t') \in \tilde{S}$ are connected via $\tilde{p}$ it follows that $\tilde{\delta}((s,t),\nu^i) = (s,t)$ for all $i \geq 1$ and $\tilde{\delta}((s,t),\mu) = (s',t')$. We obtain from Claim 2 that $\nu^i, \mu \in WF_{\mathcal{M}}$ for $i \geq 1$. So we can consider L_ν and L_μ, and there are $v \in L_\nu$ and $w \in L_\mu$ with $f_{\mathcal{M}}(v) = \nu$ and $f_{\mathcal{M}}(w) = \mu$. Note that

$v, w \in A^+$ and define $p' =_{\text{def}} (vv, vwv)$. Then $p' \in \mathbb{P}_0^{\mathcal{B}}$ (with respect to A), and vv and vwv start with the same letter. Moreover, it holds that $\alpha(vv) = A$. To see this suppose $\alpha(vv) \neq A$. Then all letters of ν have the same type $\alpha(v)$ and $\nu^2 \notin WF_{\mathcal{M}}$, a contradiction. It follows that $\alpha(vwv) \subseteq A = \alpha(vv)$. We apply Proposition 6.6 with $l =_{\text{def}} vv$ and $b =_{\text{def}} vwv$ and obtain some $p \in \mathbb{P}_1^{\mathcal{L}}$ such that $\overline{p}^\circ = v^{|vwv|}$ and if any two states in $\mathcal{M}$ are connected via p' then they are also connected via p.

To show the first statement suppose $\tilde{p}$ appears at (s_1, t_1) in $\tilde{\mathcal{M}}$. Then $\tilde{\delta}((s_1, t_1), \nu) = (s_1, t_1)$ and we obtain from Claim 2 that $\delta(s_1, v) = s_1$. So $\delta(s_1, \overline{p}^\circ) = s_1$ and Proposition 6.5 shows that p appears at s_1 in $\mathcal{M}$. For the second statement assume that $(s_2, t_2), (s_3, t_3)$ are connected via $\tilde{p}$ in $\tilde{\mathcal{M}}$. By definition, $\tilde{\delta}((s_2, t_2), \nu) = (s_2, t_2)$, $\tilde{\delta}((s_3, t_3), \nu) = (s_3, t_3)$ and $\tilde{\delta}((s_2, t_2), \mu) = (s_3, t_3)$. We get from Claim 2 that $\delta(s_2, v) = s_2 = \delta(s_2, vv)$, $\delta(s_3, v) = s_3 = \delta(s_3, vv)$ and $\delta(s_2, vwv) = \delta(s_2, wv) = \delta(s_3, v) = s_3$. So s_2, s_3 are connected via p' in $\mathcal{M}$ and we have already noted from Proposition 6.6 that they are also connected via p.

Induction step. Suppose the lemma holds for some $n \geq 0$ and we want to show it for $n+1$. So let $\tilde{p} = (\mu_0, \tilde{p}_0, \ldots, \mu_m, \tilde{p}_m) \in \mathbb{P}_{n+1}^{\mathcal{B}}$ with $\mu_i \in (A_{\mathcal{M}})^+$ and $\tilde{p}_i \in \mathbb{P}_n^{\mathcal{B}}$ for $0 \leq i \leq m$. By assumption, there are $(s, t), (s', t') \in \tilde{S}$ that are connected via $\tilde{p}$, and in particular $\tilde{p}$ appears at (s, t). So there exist states $\tilde{q}_0, \tilde{r}_0, \ldots, \tilde{q}_m, \tilde{r}_m$ of $\tilde{\mathcal{M}}$ such that

$$(s, t) \xrightarrow{\mu_0} \tilde{q}_0 \overset{\tilde{p}_0}{\rightsquigarrow} \tilde{r}_0 \xrightarrow{\mu_1} \tilde{q}_1 \overset{\tilde{p}_1}{\rightsquigarrow} \tilde{r}_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} \tilde{q}_m \overset{\tilde{p}_m}{\rightsquigarrow} \tilde{r}_m = (s, t)$$

Note that $\tilde{q}_i \xrightarrow{\overline{\tilde{p}_i}} \tilde{r}_i$ in $\tilde{\mathcal{M}}$ for $0 \leq i \leq m$. Since $(s, t) \neq \perp$ and because of c. in the definition of $\tilde{\mathcal{M}}$ it follows that the states $\tilde{q}_i, \tilde{r}_i$ for $0 \leq i \leq m$ are not $\perp$. So we can rewrite these states as $\tilde{q}_i = (q_i, t_{q_i})$ and $\tilde{r}_i = (r_i, t_{r_i})$ for $0 \leq i \leq m$, for suitable $q_i, r_i \in S$ and $t_{q_i}, t_{r_i} \in A$. Due to the construction of $\tilde{\mathcal{M}}$ the latter are not ε and we have

$$(s, t) \xrightarrow{\mu_0} (q_0, t_{q_0}) \overset{\tilde{p}_0}{\rightsquigarrow} (r_0, t_{r_0}) \xrightarrow{\mu_1} (q_1, t_{q_1}) \overset{\tilde{p}_1}{\rightsquigarrow} (r_1, t_{r_1}) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} (q_m, t_{q_m}) \overset{\tilde{p}_m}{\rightsquigarrow} (r_m, t_{r_m}) = (s, t).$$

We see that each $\tilde{p}_i$ connects some states in $\mathcal{M}$ that are not $\perp$. So the induction hypothesis provides for each $\tilde{p}_i$ some $p_i \in \mathbb{P}_{n+1}^{\mathcal{L}}$ such that for all $(s'_1, t'_1), (s'_2, t'_2), (s'_3, t'_3) \in \tilde{S}$ with $s'_1, s'_2, s'_3 \in S$ and $t'_1, t'_2, t'_3 \in A \cup \{\varepsilon\}$ the following holds.

(H1) If $\tilde{p}_i$ appears at (s'_1, t'_1) in $\tilde{\mathcal{M}}$, then p_i appears at s'_1 in $\mathcal{M}$.

(H2) If $(s'_2, t'_2) \overset{\tilde{p}_i}{\rightsquigarrow} (s'_3, t'_3)$ in $\tilde{\mathcal{M}}$, then $s'_2 \overset{p_i}{\rightsquigarrow} s'_3$ in $\mathcal{M}$.

Moreover, we get from Claim 2 that $\overline{\tilde{p}}^\circ \in WF_{\mathcal{M}}$. So with Proposition 6.27 we have $\mu_i \in WF_{\mathcal{M}}$ for $0 \leq i \leq m$. In particular, there are $w_i \in L_{\mu_i}$ such that $f_{\mathcal{M}}(w_i) = \mu_i$ for $0 \leq i \leq m$. Define now $p =_{\text{def}} (w_0, p_0, \ldots, w_m, p_m)$ and observe that $p \in \mathbb{P}_{n+2}^{\mathcal{L}}$.

We turn to the first statement. Suppose $\tilde{p}$ appears at (s_1, t_1) in $\tilde{\mathcal{M}}$. Then there exist states $\tilde{q}'_0, \tilde{r}'_0, \ldots, \tilde{q}'_m, \tilde{r}'_m$ of $\tilde{\mathcal{M}}$ such that

$$(s_1, t_1) \xrightarrow{\mu_0} \tilde{q}'_0 \overset{\tilde{p}_0}{\rightsquigarrow} \tilde{r}'_0 \xrightarrow{\mu_1} \tilde{q}'_1 \overset{\tilde{p}_1}{\rightsquigarrow} \tilde{r}'_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} \tilde{q}'_m \overset{\tilde{p}_m}{\rightsquigarrow} \tilde{r}'_m = (s_1, t_1)$$

With the same argument as above, we can rewrite these states as $\tilde{q}'_i = (q'_i, t_{q'_i})$ and $\tilde{r}'_i = (r'_i, t_{r'_i})$ for $0 \leq i \leq m$, for suitable $q'_i, r'_i \in S$ and $t_{q'_i}, t_{r'_i} \in A$. So we have

$$(s_1, t_1) \xrightarrow{\mu_0} (q'_0, t_{q'_0}) \overset{\tilde{p}_0}{\rightsquigarrow} (r'_0, t_{r'_0}) \xrightarrow{\mu_1} (q'_1, t_{q'_1}) \overset{\tilde{p}_1}{\rightsquigarrow} (r'_1, t_{r'_1}) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} (q'_m, t_{q'_m}) \overset{\tilde{p}_m}{\rightsquigarrow} (r'_m, t_{r'_m}) = (s_1, t_1).$$

From Claim 2 and the induction hypothesis (H2) we obtain that

$$s_1 \xrightarrow{w_0} q_0' \overset{p_0}{\rightsquigarrow} r_0' \xrightarrow{w_1} q_1' \overset{p_1}{\rightsquigarrow} r_1' \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m' \overset{p_m}{\rightsquigarrow} r_m' = s_1$$

which shows that p appears at s_1 in $\mathcal{M}$.

Next we want to prove the second statement. Therefore assume that $(s_2, t_2), (s_3, t_3)$ are connected via $\tilde{p}$ in $\tilde{\mathcal{M}}$. Then $\tilde{p}$ appears at (s_2, t_2) and also at (s_3, t_3), and there are states $\tilde{q}_i \in \tilde{S}$ with $0 \le i \le m$ such that $\tilde{p}_i$ appears at state $\tilde{q}_i$ for $0 \le i \le m$ and

$$(s_2, t_2) \xrightarrow{\mu_0} \tilde{q}_0 \xrightarrow{\mu_1} \tilde{q}_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} \tilde{q}_m = (s_3, t_3).$$

As before, none of the states $\tilde{q}_i$ for $0 \le i \le m$ can be $\perp$, so we can rewrite them as $\tilde{q}_i = (q_i, t_{q_i})$ for $0 \le i \le m$, for suitable $q_i \in S$ and $t_{q_i} \in A$. Then

$$(s_2, t_2) \xrightarrow{\mu_0} (q_0, t_{q_0}) \xrightarrow{\mu_1} (q_1, t_{q_1}) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} (q_m, t_{q_m}) = (s_3, t_3).$$

From Claim 2 we obtain that

$$s_2 \xrightarrow{w_0} q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} q_m = s_3.$$

By the induction hypothesis (H1) we know that p_i appears at state q_i for $0 \le i \le m$ and for the first statement we obtain that p appears at s_2 and at s_3. This shows that s_2, s_3 are connected via p. **(End proof of Claim 3.)**

Note that Claim 3 provides in particular for $\tilde{p} \in \mathbb{P}_n^{\mathcal{B}}$ with $(s, t) \overset{\tilde{p}}{\rightsquigarrow} (s', t')$ in $\tilde{\mathcal{M}}$ some $p \in \mathbb{P}_{n+1}^{\mathcal{L}}$ such that $s \overset{p}{\rightsquigarrow} s'$ in $\mathcal{M}$.

Step 3: From $\tilde{\mathcal{M}}$ to $\mathcal{M}$. We prove in this final step the following claim.

Claim 4. Let $n \ge 1$. If $\tilde{\mathcal{M}}$ has pattern $\mathbb{P}_n^{\mathcal{B}}$ then $\mathcal{M}$ has pattern $\mathbb{P}_{n+1}^{\mathcal{L}}$.

Proof of Claim 4. Suppose that $\tilde{\mathcal{M}}$ has pattern $\mathbb{P}_n^{\mathcal{B}}$. Then there exist $\tilde{s}_1, \tilde{s}_2 \in \tilde{S}$, $\eta, \zeta \in (A_{\mathcal{M}})^*$, $\tilde{p} \in \mathbb{P}_n^{\mathcal{B}}$, $\tilde{s}^+ \in \tilde{S}'$ and $\tilde{s}^- \notin \tilde{S}'$ such that

$$(s_0, \varepsilon) \xrightarrow{\eta} \tilde{s}_1 \overset{\tilde{p}}{\rightsquigarrow} \tilde{s}_2 \quad \text{and} \quad \tilde{s}_1 \xrightarrow{\zeta} \tilde{s}^+ \quad \text{and} \quad \tilde{s}_2 \xrightarrow{\zeta} \tilde{s}^-. \tag{6.1}$$

We may assume without loss of generality that $\eta, \zeta \in (A_{\mathcal{M}})^+$. To see this note that $\tilde{s}_1$ and $\tilde{s}_2$ have a $\overline{\tilde{p}}^{\circ}$-loop by Proposition 5.7 (but (s_0, ε) has no loop by construction). Let $\tilde{p} = (\mu_0, \tilde{p}_0, \ldots, \mu_m, \tilde{p}_m)$ for some $m \ge 0$ with $\mu_i \in (A_{\mathcal{M}})^+$ and $\tilde{p}_i \in \mathbb{P}_{n-1}^{\mathcal{B}}$. We already now that $\eta \cdot \zeta \in WF_{\mathcal{M}}$ because $\eta \cdot \zeta \in L(\tilde{\mathcal{M}}) = f_{\mathcal{M}}(L(\mathcal{M})) \subseteq WF_{\mathcal{M}}$. Now we want to show this for $\eta \cdot \overline{\tilde{p}} \cdot \zeta$. Since $\tilde{p}$ appears at $\tilde{s}_1$ there exist states $\tilde{q}_0, \tilde{r}_0, \ldots, \tilde{q}_m, \tilde{r}_m$ of $\tilde{\mathcal{M}}$ such that

$$(s_0, \varepsilon) \xrightarrow{\eta} \tilde{s}_1 \xrightarrow{\mu_0} \tilde{q}_0 \overset{\tilde{p}_0}{\rightsquigarrow} \tilde{r}_0 \xrightarrow{\mu_1} \tilde{q}_1 \overset{\tilde{p}_1}{\rightsquigarrow} \tilde{r}_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_m} \tilde{q}_m \overset{\tilde{p}_m}{\rightsquigarrow} \tilde{r}_m = \tilde{s}_1 \xrightarrow{\zeta} \tilde{s}^+.$$

Note that $\tilde{q}_i \xrightarrow{\overline{\tilde{p}_i}} \tilde{r}_i$ and that $\tilde{q}_i$ and $\tilde{r}_i$ have a non–empty $\overline{\tilde{p}_i}^{\circ}$-loop for $0 \le i \le m$ because $n \ge 1$. With the same argument as before we see that

$$\begin{array}{llllllllllll} \eta \cdot \mu_0 \cdot & & \overline{\tilde{p}_0} & & \cdot\, \mu_1 \cdot & & \overline{\tilde{p}_1} & & \cdots\ \mu_m \cdot & & \overline{\tilde{p}_m} & & \cdot\ \zeta \subseteq WF_{\mathcal{M}} \text{ and} \\ \eta \cdot \mu_0 \cdot & \overline{\tilde{p}_0}^{\circ} \cdot & \overline{\tilde{p}_0} & \cdot\, \overline{\tilde{p}_0}^{\circ} & \cdot\, \mu_1 \cdot & \overline{\tilde{p}_1}^{\circ} \cdot & \overline{\tilde{p}_1} & \cdot\, \overline{\tilde{p}_1}^{\circ} & \cdots\ \mu_m \cdot & \overline{\tilde{p}_m}^{\circ} \cdot & \overline{\tilde{p}_m} & \cdot\, \overline{\tilde{p}_m}^{\circ} & \cdot\ \zeta \subseteq WF_{\mathcal{M}}. \end{array}$$

To argue that

$$\eta \cdot \overline{\tilde{p}} \cdot \zeta \quad = \quad \eta \cdot \mu_0 \cdot \mu_1 \cdots \mu_m \cdot \zeta \quad \in WF_{\mathcal{M}}$$

we need to observe that the types of every two consecutive letters alternate. We see from above that it suffices to show for $0 \leq i \leq m$ that the type of the last letter of μ_i is the same as the type of the last letter of $\overline{\tilde{p}_i}$. But this is clear because the type of the last letter of μ_i is different from the type of the first letter of $\overline{\tilde{p}_i}^{\circ}$ which in turn is different from the type of the last letter of $\overline{\tilde{p}_i}$. Note that there are only two types a and b.

Because $\eta \cdot \zeta \in WF_{\mathcal{M}}$ and $\eta \cdot \overline{\tilde{p}} \cdot \zeta \in WF_{\mathcal{M}}$ none of the states $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{s}^+$ and $\tilde{s}^-$ is the sink state $\perp$. So we can rewrite them as

$$\tilde{s}_1 = (s_1, t_{s_1}) \text{ and } \tilde{s}_2 = (s_2, t_{s_2}) \text{ and } \tilde{s}^+ = (s^+, t_{s^+}) \text{ and } \tilde{s}^- = (s^-, t_{s^-})$$

for suitable $s_1, s_2, s^+, s^- \in S$ with $s^+ \in S'$, $s^- \notin S'$ and $t_{s_1}, t_{s_2}, t_{s^+}, t_{s^-} \in A$. It follows that we can rewrite (6.1) as

$$(s_0, \varepsilon) \xrightarrow{\eta} (s_1, t_{s_1}) \overset{\tilde{p}}{\rightsquigarrow} (s_2, t_{s_2}) \quad \text{and} \quad (s_1, t_{s_1}) \xrightarrow{\zeta} (s^+, t_{s^+}) \quad \text{and} \quad (s_2, t_{s_2}) \xrightarrow{\zeta} (s^-, t_{s^-}).$$

Furthermore, we have $\eta, \zeta \in WF_{\mathcal{M}}$. So let $x, z \in A^+$ such that $f_{\mathcal{M}}(x) = \eta$ and $f_{\mathcal{M}}(z) = \zeta$. From Claim 2 and Claim 3 we obtain that

$$s_0 \xrightarrow{x} s_1 \overset{p}{\rightsquigarrow} s_2 \quad \text{and} \quad s_1 \xrightarrow{z} s^+ \quad \text{and} \quad s_2 \xrightarrow{z} s^-$$

for some $p \in \mathbb{P}^{\mathcal{L}}_{n+1}$. This shows that $\mathcal{M}$ has pattern $\mathbb{P}^{\mathcal{L}}_{n+1}$. **(End proof of Claim 4.)**

Now the proof of Lemma 6.29 is as follows. Assume $f_{\mathcal{M}}(L(\mathcal{M})) \notin \mathcal{FP}^{\mathcal{B}}_n$. Since $L(\tilde{\mathcal{M}}) = f_{\mathcal{M}}(L(\mathcal{M}))$ by Step 1 this implies that $\tilde{\mathcal{M}}$ has pattern $\mathbb{P}^{\mathcal{B}}_n$. From Claim 4 we obtain that $\mathcal{M}$ has pattern $\mathbb{P}^{\mathcal{L}}_{n+1}$ and hence $L(\mathcal{M}) \notin \mathcal{FP}^{\mathcal{L}}_{n+1}$. This shows that $L(\mathcal{M}) \in \mathcal{FP}^{\mathcal{L}}_{n+1}$ implies $f_{\mathcal{M}}(L(\mathcal{M})) \in \mathcal{FP}^{\mathcal{B}}_n$ as stated in the lemma. ❑

6.2.3 Transformation of Expressions

The main result of this subsection is Lemma 6.34 that allows together with Lemma 6.29 a proof of Theorem 6.17. For some permutation–free DFA $\mathcal{M}$ with input alphabet $A = \{a, b\}$ denote for $R \subseteq (A_{\mathcal{M}})^+$ by $f^{-1}_{\mathcal{M}}(R)$ the set of all words w from A^+ such that $f_{\mathcal{M}}(w) \in R$. We show in Lemma 6.34 that for $n \geq 1$ and for every $R \subseteq WF_{\mathcal{M}}$ with $R \in \mathcal{B}_{n+1/2}$ we can find a language $T(R) \subseteq A^+$ such that $T(R) = f^{-1}_{\mathcal{M}}(R)$. Moreover, $T(R)$ does not have a much higher concatenation complexity than R, i.e., $T(R) \in \mathcal{L}_{n+3/2}$. To prepare the proof of this we fix some permutation–free DFA $\mathcal{M}$ with input alphabet $A = \{a, b\}$. First, we observe the following two propositions concerning L_μ and $WF_{\mathcal{M}}$.

Proposition 6.30. *Let $\mu \in WF_{\mathcal{M}}$. It holds that $L_\mu \in \mathcal{L}_{3/2}$ and $L_\mu = f^{-1}_{\mathcal{M}}(\{\mu\})$.*

Proof. Let $\mu = c_1 c_2 \cdots c_k \in WF_{\mathcal{M}}$ for some $k \geq 1$. Recall that by Definition 6.25 it holds that $L_\mu = \{\, v \in A^+ \mid f_{\mathcal{M}}(v) = \mu \,\}$. So we immediately have $L_\mu = f^{-1}_{\mathcal{M}}(\{\mu\})$ and it remains to show that $L_\mu \in \mathcal{L}_{3/2}$. For $c \in A_{\mathcal{M}}$ define

$$T(c) \quad =_{\text{def}} \quad \begin{cases} a^j & : \quad c = a_j \text{ and } 1 \leq j \leq r-1 \\ b^j & : \quad c = b_j \text{ and } 1 \leq j \leq r-1 \\ a^r \cdot \{a\}^* & : \quad c = a_r \\ b^r \cdot \{b\}^* & : \quad c = b_r \end{cases}$$

and set $T(\mu) =_{\text{def}} T(c_1) \cdot T(c_2) \cdots T(c_k)$. Note with Proposition 1.19 that $T(\mu) \in \mathcal{L}_{3/2}$. We show that $L_\mu = T(\mu)$.

Let $w \in A^+$ be given and assume $w \in L_\mu$. Then $f_{\mathcal{M}}(w) = \mu = c_1c_2 \cdots c_k$ and w has the A-factorization $w = w_1w_2 \cdots w_k$ with $f_{\mathcal{M}}(w_i) = c_i$ for $1 \leq i \leq k$. Now fix some i with $1 \leq i \leq k$ and suppose without loss of generality that $c_i = a_j$ for some $1 \leq j \leq r$. Since c_i has type a we have $w_i = a^l$ for some $l \geq 1$ with $j = \min\{r, l\}$. If $1 \leq j \leq r-1$ then $j = l$ and $w_i = a^l = a^j = T(c_i)$. If $j = r$ then $l \geq r$ and $w_i = a^l \in a^r \cdot \{a\}^* = T(c_i)$. We put all factors together and see that $w \in T(\mu)$.

Conversely, suppose $w \in T(\mu)$. We can write w as $w = v_1v_2 \cdots v_k$ with $v_i \in T(c_i)$ ($v_i = T(c_i)$, respectively) for $1 \leq i \leq k$. Since $\mu \in WF_{\mathcal{M}}$ the types of the c_i alternate between a and b. It follows that $v_i \in \{a\}^+ \Leftrightarrow v_{i+1} \in \{b\}^+$ for $1 \leq i < k$. So $w = v_1v_2 \cdots v_k$ is the A-factorization of w and $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(v_1v_2 \cdots v_k) = f_{\mathcal{M}}(v_1)f_{\mathcal{M}}(v_2) \cdots f_{\mathcal{M}}(v_k)$. The definition of each $T(c_i)$ is such that $v_i \in T(c_i)$ ($v_i = T(c_i)$, respectively) implies $f_{\mathcal{M}}(v_i) = c_i$ for $1 \leq i \leq k$. Hence $f_{\mathcal{M}}(w) = f_{\mathcal{M}}(v_1)f_{\mathcal{M}}(v_2) \cdots f_{\mathcal{M}}(v_k) = c_1c_2 \cdots c_n = \mu$ which shows $w \in L_\mu$. ❑

Proposition 6.31. *It holds that $WF_{\mathcal{M}} \in \text{co}\mathcal{B}_{1/2}$.*

Proof. Consider the following definition.

$$T =_{\text{def}} (A_{\mathcal{M}})^+ \setminus \Big(\bigcup_{\nu \in A^a_{\mathcal{M}} A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}} A^b_{\mathcal{M}}} (A_{\mathcal{M}})^* \cdot \nu \cdot (A_{\mathcal{M}})^* \Big)$$

Note with Proposition 1.15 that $T \in \text{co}\mathcal{B}_{1/2}$ and we want to show $T = WF_{\mathcal{M}}$. Let $\mu = c_1c_2 \cdots c_k \in (A_{\mathcal{M}})^+$ for some $k \geq 1$ be given and recall from Proposition 6.26 that $\mu \in WF_{\mathcal{M}}$ if and only if $c_i \in A^a_{\mathcal{M}} \Leftrightarrow c_{i+1} \in A^b_{\mathcal{M}}$ for $1 \leq i < k$. So if every two consecutive letters in μ have different type then μ is in none of the sets subtracted from $(A_{\mathcal{M}})^+$ in the definition of T. If two consecutive letters c_ic_{i+1} in μ have the same type then $c_ic_{i+1} \in A^a_{\mathcal{M}}A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}A^b_{\mathcal{M}}$ and $\mu \notin T$. ❑

The following two lemmas will help to give the proof of Lemma 6.34. In particular, Lemma 6.33 serves as a part of the induction base there. We treat in Lemma 6.32 languages R that may contain words that are not well–formed.

Lemma 6.32. *Let $\mu_0, \mu_1, \ldots, \mu_m \in (A_{\mathcal{M}})^+$ for $m \geq 0$ and let $l_1, r_1, l_2, r_2, \ldots, l_m, r_m \in A$ such that $\mu_0 \cdot A^{l_1}_{\mathcal{M}}$, $A^{r_i}_{\mathcal{M}} \cdot \mu_i \cdot A^{l_{i+1}}_{\mathcal{M}}$, $A^{r_m}_{\mathcal{M}} \cdot \mu_m \subseteq WF_{\mathcal{M}}$ for $1 \leq i < m$. For the language*

$$R = \mu_0 \ \cdot \ A^{l_1}_{\mathcal{M}}(A_{\mathcal{M}})^* A^{r_1}_{\mathcal{M}} \ \cdot \ \mu_1 \ \cdot \ A^{l_2}_{\mathcal{M}}(A_{\mathcal{M}})^* A^{r_2}_{\mathcal{M}} \ \cdot \ \mu_2 \cdots A^{l_m}_{\mathcal{M}}(A_{\mathcal{M}})^* A^{r_m}_{\mathcal{M}} \ \cdot \ \mu_m$$

there exists some $T(R) \subseteq A^+$ with $T(R) \in \mathcal{L}_{3/2}$ such that $T(R) = f^{-1}_{\mathcal{M}}(R)$.

Proof. We define the transformation of R as

$$T(R) =_{\text{def}} L_{\mu_0} \ \cdot \ l_1B_1r_1 \ \cdot \ L_{\mu_1} \ \cdot \ l_2B_2r_2 \ \cdot \ L_{\mu_2} \cdots l_mB_mr_m \ \cdot \ L_{\mu_m}$$

with

$$B_i \quad =_{\text{def}} \quad \begin{cases} A^* & : \ l_i \neq r_i \\ A^*bA^* & : \ l_i = r_i = a \\ A^*aA^* & : \ l_i = r_i = b \end{cases}$$

for $1 \leq i \leq m$. Proposition 6.30 shows that $L_{\mu_i} \in \mathcal{L}_{3/2}$ for $0 \leq i \leq m$ and from Proposition 1.19 we see that also $l_i B_i r_i \in \mathcal{L}_{3/2}$ for $1 \leq i \leq m$. Since $\mathcal{L}_{3/2}$ is closed under concatenation we obtain $T(R) \in \mathcal{L}_{3/2}$.

Now we want to show that $T(R) = f_{\mathcal{M}}^{-1}(R)$. First let $w \in f_{\mathcal{M}}^{-1}(R)$ and we have to show that $w \in T(R)$. Let $w = w_1 w_2 \cdots w_k \in A^+$ for some $k \geq 1$ be the A-factorization of w. Then $f_{\mathcal{M}}(w) = c_1 c_2 \cdots c_k \in WF_{\mathcal{M}}$ for some $c_i \in A_{\mathcal{M}}$ and because $f_{\mathcal{M}}(w) \in R$ we can write it as

$$f_{\mathcal{M}}(w) = \mu_0 \cdot d_1 \nu_1 e_1 \cdot \mu_1 \cdot d_2 \nu_2 e_2 \cdot \mu_2 \cdots d_m \nu_m e_m \cdot \mu_m$$

with $d_i \in A_{\mathcal{M}}^{l_i}$, $e_i \in A_{\mathcal{M}}^{r_i}$ and $\nu_i \in (A_{\mathcal{M}})^*$ for $1 \leq i \leq m$. Let $1 < j_1 < j_1' < j_2 < j_2' \cdots < j_m < j_m' < k$ such that $d_i = c_{j_i}$, $\nu_i = c_{j_i+1} \cdots c_{j_i'-1}$ and $e_i = c_{j_i'}$ for $1 \leq i \leq m$, and $\mu_i = c_{j_i'+1} \cdots c_{j_{i+1}-1}$ for $0 \leq i \leq m$ (set $j_0' =_{\mathrm{def}} 0$ and $j_{m+1} =_{\mathrm{def}} k+1$). We apply Proposition 6.23 and obtain $f_{\mathcal{M}}(w_{j_i'+1} \cdots w_{j_{i+1}-1}) = c_{j_i'+1} \cdots c_{j_{i+1}-1} = \mu_i$ for $0 \leq i \leq m$. By Proposition 6.30 this implies that $w_{j_i'+1} \cdots w_{j_{i+1}-1} \in L_{\mu_i}$ for $0 \leq i \leq m$.

Also by Proposition 6.23 we see that $f_{\mathcal{M}}(w_{j_i} \cdots w_{j_i'}) = c_{j_i} \cdots c_{j_i'} = d_i \nu_i e_i$ with $w_{j_i} \in \{l_i\}^+$ and $w_{j_i'} \in \{r_i\}^+$ for $1 \leq i \leq m$. Now fix some i with $1 \leq i \leq m$. If $l_i \neq r_i$ then $w_{j_i} \cdots w_{j_i'} \in l_i A^* r_i = l_i B_i r_i$. If $l_i = r_i$ we may assume without loss of generality that $l_i = r_i = a$ and hence $w_{j_i}, w_{j_i'} \in \{a\}^+$. But since the latter are factors of maximal length and because $j_i < j_i'$ there must be some j with $j_i < j < j_i'$ and $w_j \in \{b\}^+$. So $w_{j_i} \cdots w_j \cdots w_{j_i'} \in aA^*bA^*a = l_i B_i r_i$. We put all factors together and see that $w \in T(R)$.

Conversely, assume that $w \in T(R)$ and we have to show that $f_{\mathcal{M}}(w) \in R$. Since $w \in T(R)$ we can write w as

$$w = u_0 \cdot l_1 v_1 r_1 \cdot u_1 \cdot l_2 v_2 r_2 \cdot u_2 \cdots l_m v_m r_m \cdot u_m$$

with $v_i \in B_i$ for $1 \leq i \leq m$ and $u_i \in L_{\mu_i}$ for $0 \leq i \leq m$. We obtain from Proposition 6.30 that $f_{\mathcal{M}}(u_i) = \mu_i$ for $0 \leq i \leq m$.

Now fix some i with $1 \leq i \leq m$ and let $f_{\mathcal{M}}(l_i v_i r_i) = f_1 f_2 \cdots f_{n'}$ for some $n' \geq 1$ and $f_j \in A_{\mathcal{M}}$. First suppose $l_i \neq r_i$. Then f_1 has type l_i and $f_{n'}$ has different type r_i. So $n' \geq 2$ and $f_1 f_2 \cdots f_{n'} \in A_{\mathcal{M}}^{l_i} (A_{\mathcal{M}})^* A_{\mathcal{M}}^{r_i}$. If $l_i = r_i$ we may assume without loss of generality that $l_i = r_i = a$. Then f_1 and $f_{n'}$ have both type a. By definition of $T(R)$ we see $v_i \in B_i = A^* b A^*$. It follows that there is some factor from $\{b\}^+$ in $a v_i a$. Hence there must be some j with $1 < j < n'$ such that f_j has type b. So $n' \geq 3$ and $f_1 f_2 \cdots f_{n'} \in A_{\mathcal{M}}^{a} (A_{\mathcal{M}})^* A_{\mathcal{M}}^{a} = A_{\mathcal{M}}^{l_i} (A_{\mathcal{M}})^* A_{\mathcal{M}}^{r_i}$.

Define $\varphi_i =_{\mathrm{def}} f_{\mathcal{M}}(l_i v_i r_i)$ for $1 \leq i \leq m$. We have just shown that $\varphi_i \in A_{\mathcal{M}}^{l_i} (A_{\mathcal{M}})^* A_{\mathcal{M}}^{r_i}$ for $1 \leq i \leq m$ and obtain with $f_{\mathcal{M}}(u_i) = \mu_i$ for $0 \leq i \leq m$ from above that

$$\mu \;=_{\mathrm{def}}\; f_{\mathcal{M}}(u_0) \cdot f_{\mathcal{M}}(l_1 v_1 r_1) \cdot f_{\mathcal{M}}(u_1) \cdots f_{\mathcal{M}}(l_m v_m r_m) \cdot f_{\mathcal{M}}(u_m) \;=\; \mu_0 \cdot \varphi_1 \cdot \mu_1 \cdots \varphi_m \cdot \mu_m \;\in\; R.$$

We argue that μ is well–formed with a few observations. Let $t(c)$ for $c \in A_{\mathcal{M}}$ denote the type of c and recall that $p_1(x)$ ($s_1(x)$) is just the first letter (last letter, respectively) of a word x. It holds that

– $t(s_1(\mu_0)) \neq l_1 = t(p_1(\varphi_1))$ because $\mu_0 \cdot A_{\mathcal{M}}^{l_1} \subseteq WF_{\mathcal{M}}$,
– $t(s_1(\varphi_i)) = r_i \neq t(p_1(\mu_i))$ and $t(s_1(\mu_i)) \neq l_{i+1} = t(p_1(\varphi_{i+1}))$ because $A_{\mathcal{M}}^{r_i} \cdot \mu_i \cdot A_{\mathcal{M}}^{l_{i+1}} \subseteq WF_{\mathcal{M}}$ for $1 \leq i < m$ and
– $t(s_1(\varphi_m)) = r_m \neq t(p_1(\mu_m))$ because $A_{\mathcal{M}}^{r_m} \cdot \mu_m \subseteq WF_{\mathcal{M}}$.

We apply repeatedly Proposition 6.27 and Proposition 6.28 to $\mu \in WF_{\mathcal{M}}$ and obtain

$$f_{\mathcal{M}}(u_0) \cdot f_{\mathcal{M}}(l_1 v_1 r_1) \cdot f_{\mathcal{M}}(u_1) \cdots f_{\mathcal{M}}(l_m v_m r_m) \cdot f_{\mathcal{M}}(u_m) \;=\; f_{\mathcal{M}}(u_0 \cdot l_1 v_1 r_1 \cdot u_1 \cdots l_m v_m r_m \cdot u_m).$$

So $f_{\mathcal{M}}(w) = \mu \in R$. ❑

Let $B \subseteq A^+$, $C \subseteq (A_{\mathcal{M}})^+$. We write for short $\overline{B}$ instead of $A^+ \backslash B$ and $\overline{C}$ instead of $(A_{\mathcal{M}})^+ \backslash C$.

Lemma 6.33. *Let $R \subseteq WF_{\mathcal{M}}$ with $R \in \mathrm{co}\mathcal{B}_{1/2}$. There exists some $T(R) \subseteq A^+$ such that $T(R) \in \mathrm{co}\mathcal{L}_{3/2}$ and $T(R) = f_{\mathcal{M}}^{-1}(R)$.*

Proof. Since $\overline{R} \in \mathcal{B}_{1/2}$ we can write $\overline{R}$ by Proposition 1.15 as a finite union of languages R' of the form

$$R' = \mu_0(A_{\mathcal{M}})^+\mu_1(A_{\mathcal{M}})^+ \cdots \mu_{m-1}(A_{\mathcal{M}})^+\mu_m$$

with $m \geq 0$ and $\mu_i \in (A_{\mathcal{M}})^*$ for $0 \leq i \leq m$. Let U denote the finite union of all letters in $A_{\mathcal{M}}$. We can assume without loss of generality that for $0 \leq i \leq m$ we have $\mu_i \neq \varepsilon$. To see this first assume that $\mu_0 = \varepsilon$. Then it must be that $m \geq 1$ and we rewrite the leftmost occurrence of $(A_{\mathcal{M}})^+$ as $U \cup U(A_{\mathcal{M}})^+$ and distribute the concatenations over the occurring unions. If $\mu_m = \varepsilon$ we can do a similar thing. Finally, if $\mu_i = \varepsilon$ for $0 < i < m$ we rewrite $(A_{\mathcal{M}})^+(A_{\mathcal{M}})^+$ as $U(A_{\mathcal{M}})^+$ and continue as before.

Moreover, we can rewrite each $(A_{\mathcal{M}})^+$ as $U \cup A_{\mathcal{M}}(A_{\mathcal{M}})^*A_{\mathcal{M}}$ and distribute again the concatenations over the occurring unions. So we obtain as a first step that $\overline{R}$ is a finite union of languages R' of the form

$$R' = \mu_0 \cdot (A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}) \cdot (A_{\mathcal{M}})^* \cdot (A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}) \cdot \mu_1 \cdots (A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}) \cdot (A_{\mathcal{M}})^* \cdot (A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}}) \cdot \mu_m$$

with $m \geq 0$ and $\mu_i \in (A_{\mathcal{M}})^+$ for $0 \leq i \leq m$. Now we distribute the concatenations over the remaining unions $(A^a_{\mathcal{M}} \cup A^b_{\mathcal{M}})$ and find that $\overline{R}$ is a finite union of languages R' of the form

$$R' = \mu_0 \cdot A^{l_1}_{\mathcal{M}} \cdot (A_{\mathcal{M}})^* \cdot A^{r_1}_{\mathcal{M}} \cdot \mu_1 \cdot A^{l_2}_{\mathcal{M}} \cdot (A_{\mathcal{M}})^* \cdot A^{r_2}_{\mathcal{M}} \cdot \mu_2 \cdots A^{l_m}_{\mathcal{M}} \cdot (A_{\mathcal{M}})^* \cdot A^{r_m}_{\mathcal{M}} \cdot \mu_m \tag{6.2}$$

with $m \geq 0$, letters $l_i, r_i \in A$ and $\mu_i \in (A_{\mathcal{M}})^+$ for $0 \leq i \leq m$.

Observe that if for some R' the condition

$$\left(\mu_0 A^{l_1}_{\mathcal{M}} \quad \cup \quad \bigcup_{1 \leq i < m} A^{r_i}_{\mathcal{M}} \mu_i A^{l_{i+1}}_{\mathcal{M}} \quad \cup \quad A^{r_m}_{\mathcal{M}} \mu_m \right) \quad \cap \quad \overline{WF_{\mathcal{M}}} \quad \neq \quad \emptyset \tag{6.3}$$

is true then $R' \subseteq \overline{WF_{\mathcal{M}}}$ which can be seen as follows. Suppose ν witnesses that (6.3) holds and let us further assume without loss of generality that $\nu \in A^{r_i}_{\mathcal{M}} \mu_i A^{l_{i+1}}_{\mathcal{M}}$ for some fixed i with $1 \leq i < m$ (the cases $\nu \in \mu_0 A^{l_1}_{\mathcal{M}}$ and $\nu \in A^{r_m}_{\mathcal{M}} \mu_m$ can be seen analogously). Then also any other $\nu' \in A^{r_i}_{\mathcal{M}} \mu_i A^{l_{i+1}}_{\mathcal{M}}$ is in $\overline{WF_{\mathcal{M}}}$ since ν and ν' have the same sequence of types of letters from $A_{\mathcal{M}}$. If there is some $\beta \in R' \cap WF_{\mathcal{M}}$ then we have by definition of R' that β has some factor $\nu' \in A^{r_i}_{\mathcal{M}} \mu_i A^{l_{i+1}}_{\mathcal{M}}$ with $\nu' \in \overline{WF_{\mathcal{M}}}$. Because $\beta \in WF_{\mathcal{M}}$ and non–empty factors are again well–formed by Proposition 6.27 this is a contradiction.

Suppose that for $k \geq 0$ the sets $R'_1, R'_2, \ldots, R'_k$ of the form as stated in (6.2) occur in the finite union describing $\overline{R}$. We turn to the description of R now. Let $I \subseteq \{1, \ldots, k\}$ be the set of indices such that R'_i does not satisfy (6.3) for all $i \in I$ and set $\overline{I} =_{\mathrm{def}} \{1, \ldots, k\} \backslash I$. As pointed out before, we have $R'_i \subseteq \overline{WF_{\mathcal{M}}}$ and hence $WF_{\mathcal{M}} \subseteq \overline{R'_i}$ for all $i \in \overline{I}$. Recall that $R \subseteq WF_{\mathcal{M}}$ is assumed in the lemma. So we have

$$\begin{array}{rcll} R & = & R \ \cap \ WF_{\mathcal{M}} & \text{since } R \subseteq WF_{\mathcal{M}} \\ & = & \bigcap_{1\le i\le k} \overline{R'_i} \ \cap \ WF_{\mathcal{M}} & \\ & = & \bigcap_{i\in I} \overline{R'_i} \ \cap \ \bigcap_{i\in \overline{I}} \overline{R'_i} \ \cap \ WF_{\mathcal{M}} & \\ & = & \bigcap_{i\in I} \overline{R'_i} \ \cap \ WF_{\mathcal{M}} & \text{because } WF_{\mathcal{M}} \ \subseteq \ \bigcap_{i\in \overline{I}} \overline{R'_i}. \end{array}$$

Since for all $i \in I$ the sets R'_i do not fulfil (6.3) it follows that we can apply Lemma 6.32 to each of them. Denote for all $i \in I$ by $T(R'_i)$ the languages of A^+ with $T(R'_i) \in \mathcal{L}_{3/2}$ and $T(R'_i) = f_{\mathcal{M}}^{-1}(R'_i)$ provided by Lemma 6.32 and define

$$T(R) \quad =_{\text{def}} \quad \bigcap_{i\in I} \overline{T(R'_i)} \ .$$

The class $\text{co}\mathcal{L}_{3/2}$ is closed under intersection by definition, so $T(R) \in \text{co}\mathcal{L}_{3/2}$. To show that $T(R) = f_{\mathcal{M}}^{-1}(R)$ let $w \in A^+$ be given. We observe that

$$\begin{array}{rcll} w \in f_{\mathcal{M}}^{-1}(R) & \iff & f_{\mathcal{M}}(w) \in R & \\ & \iff & f_{\mathcal{M}}(w) \in \overline{R'_i} \text{ for all } i \in I \text{ and } f_{\mathcal{M}}(w) \in WF_{\mathcal{M}} & \\ & \iff & f_{\mathcal{M}}(w) \in \overline{R'_i} \text{ for all } i \in I & \text{since } f_{\mathcal{M}}(w) \in WF_{\mathcal{M}} \\ & \iff & w \in \overline{T(R'_i)} \text{ for all } i \in I & \text{since } T(R'_i) = f_{\mathcal{M}}^{-1}(R'_i) \\ & \iff & w \in T(R). & \end{array}$$

❑

Now we transfer this relation inductively to all higher levels of the DDH and STH.

Lemma 6.34. *Let $n \geq 1$ and let $\mathcal{M}$ be a permutation–free* DFA *which has input alphabet $A = \{a, b\}$.*

1. *For every $R \subseteq WF_{\mathcal{M}}$ with $R \in \text{co}\mathcal{B}_{n-1/2}$ there exists some $T(R) \subseteq A^+$ such that $T(R) \in \text{co}\mathcal{L}_{n+1/2}$ and $T(R) = f_{\mathcal{M}}^{-1}(R)$.*
2. *For every $R \subseteq WF_{\mathcal{M}}$ with $R \in \mathcal{B}_{n+1/2}$ there exists some $T(R) \subseteq A^+$ such that $T(R) \in \mathcal{L}_{n+3/2}$ and $T(R) = f_{\mathcal{M}}^{-1}(R)$.*

Proof. We prove both statements simultaneously by induction on n.
Induction base. Let $n = 1$. The first statement holds by Lemma 6.33 and we have to show the second statement. Recall from Lemma 1.20 that $\mathcal{B}_{3/2} = \text{Pol}(\text{co}\mathcal{B}_{1/2})$. So R can be written as a finite union of languages R' for which in turn there are languages $L_0, L_1, \ldots, L_m \subseteq (A_{\mathcal{M}})^+$ for some $m \geq 0$ such that

$$R' = L_0 L_1 \cdots L_m$$

with $L_i \in \text{co}\mathcal{B}_{1/2}$ for $0 \leq i \leq m$. From $R \subseteq WF_{\mathcal{M}}$ we see $R' \subseteq WF_{\mathcal{M}}$ for each member R' of the union. Moreover, for each R' it holds that $L_i \subseteq WF_{\mathcal{M}}$ for all $0 \leq i \leq m$ since otherwise there is a word in R' (having a factor) that is not well–formed. Let us define the transformation $T(L_i) \subseteq A^+$ of L_i to be the set from $\text{co}\mathcal{L}_{3/2}$ with $T(L_i) = f_{\mathcal{M}}^{-1}(L_i)$ provided by the first statement. Now set

$$T(R') =_{\text{def}} T(L_0)T(L_1)\cdots T(L_m)$$

and define $T(R)$ to be the union of all $T(R')$ ranging over all R'. Because $T(L_i) \in \text{co}\mathcal{L}_{3/2}$ for $0 \le i \le m$ we have $T(R') \in \text{Pol}(\text{co}\mathcal{L}_{3/2})$ and also $T(R) \in \text{Pol}(\text{co}\mathcal{L}_{3/2})$. So $T(R) \in \mathcal{L}_{5/2}$ by Lemma 1.20.

It remains to show that $T(R) = f_{\mathcal{M}}^{-1}(R)$ which we do for every member of the union $T(R')$ separately. First let $w \in f_{\mathcal{M}}^{-1}(R')$ and let $w = w_1w_2\cdots w_k \in A^+$ for some $k \ge 1$ be the A-factorization of w. Then $f_{\mathcal{M}}(w) \in R'$ and $f_{\mathcal{M}}(w) = c_1c_2\cdots c_k \in WF_{\mathcal{M}}$ with $c_i \in A_{\mathcal{M}}$. We can write $f_{\mathcal{M}}(w)$ as $f_{\mathcal{M}}(w) = \mu_0\mu_1\cdots\mu_m$ with $\mu_i \in L_i$ for $0 \le i \le m$. Let $1 = j_0 < j_1 < \cdots < j_m < k$ such that $\mu_i = c_{j_i}\cdots c_{j_{i+1}-1}$ for $0 \le i \le m$ (set $j_{m+1} =_{\text{def}} k+1$). We apply Proposition 6.23 and obtain $f_{\mathcal{M}}(w_{j_i}\cdots w_{j_{i+1}-1}) = \mu_i \in L_i$ for $0 \le i \le m$. Since $T(L_i) = f_{\mathcal{M}}^{-1}(L_i)$ we obtain $w_{j_i}\cdots w_{j_{i+1}-1} \in T(L_i)$ for $0 \le i \le m$. If we put these factors together we get $w \in T(R')$.

Conversely, let $w \in T(R')$. We can write w as $w = u_0u_1\cdots u_m$ with $u_i \in T(L_i)$ for $0 \le i \le m$. Because $T(L_i) = f_{\mathcal{M}}^{-1}(L_i)$ we have $f_{\mathcal{M}}(u_i) \in L_i$ for $0 \le i \le m$. So $f_{\mathcal{M}}(u_0)f_{\mathcal{M}}(u_1)\cdots f_{\mathcal{M}}(u_m) \in R' \subseteq WF_{\mathcal{M}}$. We apply repeatedly Proposition 6.27 and Proposition 6.28 and obtain $f_{\mathcal{M}}(u_0)f_{\mathcal{M}}(u_1)\cdots f_{\mathcal{M}}(u_m) = f_{\mathcal{M}}(u_0u_1\cdots u_m) = f_{\mathcal{M}}(w) \in R'$. Hence, $w \in f_{\mathcal{M}}^{-1}(R')$

Induction step. Assume the lemma holds for some $n \ge 1$ and we want to show it for $n+1$. We begin with the first statement and suppose that $R \in \text{co}\mathcal{B}_{n+1/2}$. Define $R' =_{\text{def}} \overline{R} \cap WF_{\mathcal{M}}$. It holds that $R' \in \mathcal{B}_{n+1/2}$ because $WF_{\mathcal{M}} \in \text{co}\mathcal{B}_{1/2} \subseteq \mathcal{B}_{n+1/2}$ for $n \ge 1$ by Proposition 6.31 and since $\mathcal{B}_{n+1/2}$ is closed under intersection by Lemma 1.21. Moreover, we have

$$\overline{R'} \cap WF_{\mathcal{M}} \;=\; \overline{\overline{R} \cap WF_{\mathcal{M}}} \cap WF_{\mathcal{M}} \;=\; \left(R \cup \overline{WF_{\mathcal{M}}}\right) \cap WF_{\mathcal{M}} \;=\; R \cap WF_{\mathcal{M}}. \tag{6.4}$$

So we see that

$$\begin{aligned} R &= R \cap WF_{\mathcal{M}} && \text{since } R \subseteq WF_{\mathcal{M}} \\ &= \overline{R'} \cap WF_{\mathcal{M}} && \text{by (6.4).} \end{aligned}$$

Since $R' \in \mathcal{B}_{n+1/2}$ is a subset of $WF_{\mathcal{M}}$ we can apply the induction hypothesis of the second statement. Denote by $T(R')$ the language of A^+ with $T(R') \in \mathcal{L}_{n+3/2}$ and $T(R') = f_{\mathcal{M}}^{-1}(R')$ provided by the hypothesis and define

$$T(R) \quad =_{\text{def}} \quad \overline{T(R')}\ .$$

Note that $T(R) \in \text{co}\mathcal{L}_{n+3/2}$ and let some $w \in A^+$ be given in order to show $T(R) = f_{\mathcal{M}}^{-1}(R)$. We observe that

$$\begin{aligned} w \in f_{\mathcal{M}}^{-1}(R) &\iff f_{\mathcal{M}}(w) \in R \\ &\iff f_{\mathcal{M}}(w) \in \overline{R'} \text{ and } f_{\mathcal{M}}(w) \in WF_{\mathcal{M}} \\ &\iff f_{\mathcal{M}}(w) \in \overline{R'} && \text{since } f_{\mathcal{M}}(w) \in WF_{\mathcal{M}} \\ &\iff w \in \overline{T(R')} && \text{since } T(R') = f_{\mathcal{M}}^{-1}(R') \\ &\iff w \in T(R). \end{aligned}$$

This completes the induction step for the first statement and we turn to the second statement.

Let $R \in \mathcal{B}_{n+3/2}$ with $R \subseteq WF_{\mathcal{M}}$ be given and recall from Lemma 1.20 that $\mathcal{B}_{n+3/2} = \mathrm{Pol}(\mathrm{co}\mathcal{B}_{n+1/2})$. Now we can proceed exactly as in the proof of the induction base for the second statement. Just apply what we have shown for $\mathrm{co}\mathcal{B}_{n+1/2}$ in the induction step for the first statement. ❑

Finally, we give the proof of Theorem 6.17.

Proof of Theorem 6.17. It suffices to show for $n \geq 1$ the inclusion $\mathcal{FP}^{\mathcal{L}}_{n+1} \subseteq \mathcal{L}_{n+3/2}$ (defined over $A = \{a, b\}$) under the assumption that $\mathcal{B}_{n+1/2} = \mathcal{FP}^{\mathcal{B}}_{n}$ for arbitrary alphabets. So let $L \in \mathcal{FP}^{\mathcal{L}}_{n+1}$. By Theorem 6.7 and Theorem 1.37 there is some permutation–free DFA $\mathcal{M}$ with $L = L(\mathcal{M}) \subseteq A^+$. It follows from Lemma 6.29 and our assumption that $f_{\mathcal{M}}(L(\mathcal{M})) \in \mathcal{FP}^{\mathcal{B}}_{n} = \mathcal{B}_{n+1/2}$. Since $f_{\mathcal{M}}(L(\mathcal{M})) \subseteq WF_{\mathcal{M}}$ we obtain from Lemma 6.34 some $T(f_{\mathcal{M}}(L(\mathcal{M}))) \subseteq A^+$ with $T(f_{\mathcal{M}}(L(\mathcal{M}))) \in \mathcal{L}_{n+3/2}$ such that $T(f_{\mathcal{M}}(L(\mathcal{M}))) = f^{-1}_{\mathcal{M}}(f_{\mathcal{M}}(L(\mathcal{M})))$. It holds that $L(\mathcal{M}) \subseteq f^{-1}_{\mathcal{M}}(f_{\mathcal{M}}(L(\mathcal{M})))$ and we want to argue that also the reverse inclusion holds. So let $w \in f^{-1}_{\mathcal{M}}(f_{\mathcal{M}}(L(\mathcal{M})))$ and hence $f_{\mathcal{M}}(w) \in f_{\mathcal{M}}(L(\mathcal{M}))$. It follows that there is some $v \in L(\mathcal{M})$ with $f_{\mathcal{M}}(v) = f_{\mathcal{M}}(w)$ and Proposition 6.24 shows that also $w \in L(\mathcal{M})$. Together we see that

$$L = L(\mathcal{M}) = f^{-1}_{\mathcal{M}}(f_{\mathcal{M}}(L(\mathcal{M}))) = T(f_{\mathcal{M}}(L(\mathcal{M}))) \in \mathcal{L}_{n+3/2}.$$

(End proof of Theorem 6.17.)

6.3 Discussion

We want to make a few more remarks concerning Conjecture 6.16. It may turn out that the conjecture does not hold, which we think is certainly possible. However, we did not find evidence against it and we even may interpret the results of Section 6.2 as another argument supporting it. If an effective characterization of the DDH and STH is possible at all, and if this can be done in terms of forbidden patterns, we believe that the work done in the last two chapters is a step in this direction. Our approach is dynamic in the sense that we can eventually learn from counter–examples and adjust the iteration rule respectively.

Independently from the validity of Conjecture 6.16 we have obtained with the forbidden pattern classes two strict and decidable hierarchies of star–free languages that are comparable (at least in one direction) to the DDH and STH. We think that it is an interesting task to further investigate these hierarchies, i.e., to look for characterizations in terms of formal languages, logic or finite semigroups.

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