Cosmological Sectors in Loop Quantum Gravity

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Abstract

Loop Quantum Gravity is the most developed canonical quantization of General Relativity[1] based on Ashtekar's connection formulation of classical General Relativity and is as such a diffeomorphism-invariant $SU(2)$-gauge theory together with a set of scalar constraints that constrain the total Hamiltonian to vanish. The elementary degrees of freedom are Wilson lines of the Ashtekar connection and fluxes of the conjugated electric field through surfaces. The theory is constructed as a mathematically consistent generalization of lattice gauge theories that supports the diffeomorphisms as unitary transformations. The states of Loop Quantum Gravity are linear combinations of spin network functions, which describe microscopic gravity. It turns out that the geometry of the spin network states is highly distributional: quanta of area are carried on edges of the spin network function and quanta of volume on its vertices. It is conjectured that fine weaves of spin network functions could give rise to semiclassical geometries, but this picture is still under development.

This thesis is concerned with the description of macroscopic geometries through Loop Quantum Gravity, and there particularly with the description of cosmology within full Loop Quantum Gravity. For this purpose we depart from two distinct (classically virtually equivalent) ansätze: One is phase space reduction and the other is the restriction to particular states. It turns out that the quantum analogue of these two approaches are fundamentally different:

The quantum analogue of phase space reduction needs the reformulation in terms of the observable Poisson algebra, so it can be applied to the noncommutative quantum phase space: It rests on the observation that the observable Poisson algebra of classical canonical cosmology is induced by the embedding of the reduced cosmological phase space into the phase space of full General Relativity. Using techniques related to Rieffel-induction, we develop a construction for a noncommutative embedding that has a classical limit that is described by a Poisson embedding (chapter 4).

To be able to use this class of noncommutative embeddings for Loop Quantum Gravity, one needs a complete group of diffeomorphisms for the quantum theory, which is constructed (chapter 3). These two results are applied in chapter 5 to construct a quantum embedding of a cosmological sector into full Loop Quantum Gravity. The embedded cosmological sector turns out to be discrete, like standard Loop Quantum Cosmology[2] and can be interpreted as a superselection sector thereof; however due to pathologies of the dynamics of full Loop Quantum Gravity, one can not induce a meaningful dynamics for this cosmological sector.

[1] M-theory, as a theory of everything, is also a very far developed quantum theory, which does however not represent a canonical quantization of General Relativity, but a theory, which as a conjectured theory of everything has to describe classical gravity in a suitable limit.

[2] Standard Loop Quantum Cosmology is a loop quantization of classical cosmology that implements many features of full Loop Quantum Gravity.

The quantum analogue of restricting the space of states is achieved by explicitly constructing states for Loop Quantum Gravity with smooth geometry in chapter 6. These states do not exist within the Hilbert space of Loop Quantum Gravity, but as states on the observable algebra of Loop Quantum Gravity. This observable algebra is built from spin network functions, area operators and a restricted set of fluxes. For this algebra to be physically complete, we needed to construct a version of Loop Quantum Geometry based on a fundamental area operator. This version of Loop Quantum Geometry is constructed in chapter 8.

Since the smooth geometry states are not in the Hilbert space of standard Loop Quantum Gravity, we needed to calculate the Hilbert space representation that contains them using the GNS construction. This representation of the observable algebra can be illustrated as a classical condensate of geometry with quantum fluctuations thereon. Using these representations we construct a quantum-minisuperspace in chapter 7, which allows for an interpretation of standard Loop Quantum Cosmology in terms of these states and led us to conjecture a new approach for the implementation of dynamics for Loop Quantum Gravity.

Zusammenfassung

Die Schleifenquantengravitation ist die am weitesten entwickelte kanonische Quantisierung der Allgemeinen Relativitätstheorie[3], die auf der Ashtekar Zusammenhangsformulierung der klassischen Allgemeinen Relativitätstheorie basiert und ist als solche eine diffeomorphismusinvariante $SU(2)$-Eichtheorie mit einem Satz von skalaren Zwangsbedingungen, welche bewirken, dass der Gesamthamiltonian verschwindet. Die elementaren Freiheitsgrade sind Wilsonlinien des Ashtekar Zusammenhanges und Flüsse des konjugierten elektrischen Feldes durch Oberflächen. Die Theorie ist als eine mathematisch konsistente Verallgemeinerung von Gittereichtheorien konstruiert, welche die Diffeomorphismen als unitäre Transformationen trägt. Die Zustände der Schleifengravitation sind Linearkombinationen von Spinnetzwerkfunktionen, welche Mikrogravitation beschreiben. Es stellt sich heraus, dass die Geometrie der Spinnetzwerke hochgradig entartet ist: Flächenquanten werden auf Kanten des Spinnetzwerkes getragen, wogegen Volumensquanten an den Vertizes residieren. Es wird vermutet, das ein feines Gewebe von Spinnetzwerkfunktionen semiklassiche Geometrie erzeugen kann, aber dieses Bild ist noch unvollständig.

Die vorliegende Arbeit ist mit der Beschreibung makroskopischer Geometrien durch Schleifengravitation befasst und zwar insbesondere mit der Beschreibung von Kosmologie innerhalb der vollen Schleifengravitation. Für dieses Ziel verwenden wir zwei unterscheidliche (jedoch auf klassischem Level scheinbar äquivalente) Ansätze: Einerseits betrachten wir die Reduktion des Phasenraumes und andererseits die Beschränkung auf bestimmte Zustände. Es stellt sich jedoch heraus, dass sich die Quantenanaloga dieser beiden Zugänge fundamental unterscheiden:

Das Quantenanalogon der Phasenraumreduktion muss als Aussage über die Observablen-Poissonalgebra umformuliert werden bevor sie auf den nichtkommutativen Phasenraum von Quantentheorien angewendet werden kann: Die zugrundeliegende Beobachtung ist, dass die Observablen-Poissonalgebra von klassischer kanonischer Kosmologie durch die Einbettung des kosmologischen Phasenraumes in den Phasenraum der Allgemeinen Relativitätstheorie induziert wird. Damit können wir eine Technik, die von der Rieffelinduktion abgeschaut ist, anwenden um die Konstruktion einer nichtkommutativen Einbettung zu entwickeln, welche sich im klassischen Limes zu einer Poissoneinbettung reduziert (Kapitel 4).

Um diese Konstruktion der Einbettung auf die Schleifenquantengravitation anwenden zu können benötigt man eine vollständige Diffeomorphismengruppe für die Quantentheorie, welche in Kapitel 3 erarbeitet wird. Diese beiden Ergebnisse werden in Kapitel 5 angewendet um die Quanteneinbettung eines kosmolo-

[3]Die M-Theorie, als eine Theorie von Allem, ist eine ebenfalls sehr weit entwickelte Quantentheorie, welche aber keine kanonische Quantisierung der Allgemeinen Relativitätstheorie darstellt, sondern eine Theorie, die als vermutliche Theory of Everything auch klassische Gravitation in einem geeigneten Limes beschreiben muss.

gischen Sektors in die volle Schleifengravitation zu konstruieren. Dieser ist, wie die standard Schleifenkosmologie[4] diskret und kann als Auswahlsektor derselben interpretiert werden; aufgrund von Pathologien in der Dynamik der vollen Schleifengravitation lässt sich aus dieser jedoch keine sinnvolle Dynamik für den kosmologischen Sektor induzieren.

Das Quantenanalogon der Beschränkung des Raumes der Zustände basiert auf der expliziten Konstruktion von Zuständen, die eine glatte räumliche Geometrie beschreiben (Kapitel 6). Diese Zustände existieren zwar nicht im Hilbertraum der Schleifenquantengravitation, aber als Zustände auf der Observablenalgebra der Schleifenquantengravitation. Diese Observablenalgebra wird aus den Spinnetzwerken, den Flächenoperatoren und einer eingeschränkten Menge der Flüsse konstruiert. Um zu zeigen, dass diese Observablenalgebra physikalisch vollständig ist benötigen wir eine Schleifenquantengeometrie, die auf einem fundamentalen Flächenoperator aufbaut. Diese Schleifenquantengeometrie wird in Kapitel 8 konstruiert.

Nachdem die Zustände mit glatter Geometrie nicht im Hilbertraum der standard Schliefengravitation liegen, müssen wir aus diesen Zuständen Hilbertraumdarstellungen der Observablenalgebra durch die GNS-Konstruktion erschaffen. Diese Darstellung kann mit dem Bild eines klassichen Kondensats von Geometrie, um welches Quantenfluktuationen existieren, illustriert werden. Ausgehend von diesen Darstellungen konstruieren wir in Kapitel 7 einen Quanten-Minisuperraum, welcher eine Interpretation der standard Schleifenkosmologie durch diese Zustände erlaubt. Dieser Zugang gab uns ausserdem den Hinweis auf eine mögliche Konstruktion einer Dynamik für die volle Schleifenquantengravitation.

[4]Die standard Schleifenkosmologie ist eine Schleifenquantiserung der klassichen Kosmologie, die viele Eigenschaften der vollen Schleifengravitation besitzt.

Contents

Chapter 1

Introduction

There are two very well tested fundamental theories that are used to describe modern physics: On one side there is General Relativity (GR) describing gravity through curvature of space-time, whose source is the energy momentum density, and on the other side there are Quantum Field Theories (QFTs) describing elementary particles and their interactions as the dynamics of excitations of quantum fields. These two theories have rather separated domains of validity in everyday life: GR describes gravity from sub-millimeter scale up to the size of the universe, QFTs on the other hand describe the dynamics of sub-atomic particles. The real world does of course not distinguish between these domains of validity. The very early universe for example had to go through an era where the excitations of quantum fields reached very high energy momentum densities, so the interaction with gravity must have been important at this stage. From a philosophical perspective one would say there is only one physics and both GR and QFTs should come out of a theory that describes this physics. This theory of Quantum Gravity (QG) is however not yet discovered, partly because GR and QFTs are formulated in two different branches of mathematics:

GR is formulated using differential geometry, particularly pseudo-Riemannian manifolds and fiber bundles thereon. The inherent diffeomorphism symmetry of GR is encoded in the coordinate independence of differential geometry. Although there are fundamental difficulties in constructing interacting Lorentz invariant QFTs in more than 2+1 dimensions, one has the standard model of particle physics formulated as a perturbation series for interacting QFTs around free QFTs which is tested with remarkable precision, so there is good evidence for the validity of QFTs. The mathematical description of QFTs is however formulated in terms of operator algebras on Hilbert spaces and unitary covariant actions of the symmetry groups on this Hilbert space. The progress in Noncommutative Geometry (see e.g. [1]) made a description of differential geometry on Riemannian spin manifolds in terms of operator algebras available, and allows for the formulation of a classical field for gravity in terms of operator algebras.

There is however evidence form quantum information theory that suggests that a quantum system can not consistently interact with a classical system[1], so QG is expected to be a QFT as well. Loop Quantum Gravity (LQG) is an approach to construct a canonical QFT using the $SU(2)$ connection formulation [2] of gravity (For further information on LQG see [3, 4, 5]). It is formulated as a Hilbert space representation of an operator algebra built from holonomies of the $SU(2)$-gauge field and the fluxes of the conjugate $su(2)$-electric field. The ground state is a diffeomorphism invariant generalization of the ground state of a lattice gauge field theory, which turns out to correspond to a "no geometry" state. The guiding principle for the construction of the kinematics of LQG is the consistent implementation of the group of kinematic gauge transformations of GR, which consists of the usual $SU(2)$-gauge transformations and the spatial diffeomorphisms. There is another set of gauge transformations generated by scalar constraints, which constrain the total Hamiltonian to vanish. A phenomenologically acceptable implementation of this set of "dynamical" constraints is ongoing research despite recent advances [6, 7].

Unlike super-string theory, LQG does not attempt to provide a unification of all matter and force fields coming from one fundamental object, but LQG can be adapted to carry all standard matter. There are however ideas that standard model matter and forces may already be present but disguised in pure LQG [8, 9, 10, 11]. The highly distributional character of geometry in LQG suggests that a smooth geometry as we experience it in everyday life arises under some coarse graining that corresponds to the experimental resolution of geometry; small scale fluctuations that can not be resolved experimentally can have properties of elementary particles. Before this idea can be put on a firm basis, one needs to understand smooth geometries in LQG. It is the purpose of this thesis to contribute to this understanding and providing a general construction of smooth sectors of LQG and to apply it in particular to extract a cosmological sector from LQG.

Loop Quantum Cosmology (LQC) has been pioneered by Bojowald [12, 13, 14] and is in analogy to Bianchi cosmology, which is a symmetry reduction of classical GR, a symmetry reduced model of LQG. It is often presented as a toy model, but the elementary operators of LQC are constructed by symmetrizing the respective LQG operators [14, 15]. The particular aim of this thesis is to provide an embedding of LQC into LQG. This is a nontrivial issue, because the kinematic quantum configuration space[2] of LQC can not be embedded into the kinematic quantum configuration space of LQG [17]. This thesis is concerned with two approaches for this embedding: The first is concerned with embedding the observable algebra, which amounts to the direct construction of a reduced observable algebra; the second is concerned with constructing a Hilbert space with state vectors that have the desired symmetry properties.

To explain the idea behind the construction of the reduced observable algebra, it is useful to consider the symmetry reduction of a classical Hamilto-

[1] A quantum system can be coupled to a classical external source, but the back-reaction of the quantum field onto the source is problematic.

[2] The quantum configuration space is the spectrum of the C^*-completion of the algebra generated by the configuration observables.

nian field theory: Let us denote the fields on the Cauchy surface Σ by ϕ_i and the canonically conjugated momentum densities π^j, so the pairs of field- and momentum density-configurations (ϕ_i, π^j) furnish canonical phase space coordinates. Assuming a Lie-algebra $\mathfrak{L}$ of vector fields v^k on Σ, one can search for solutions to $\mathcal{L}_v(\phi_i, \pi^j) = 0$, which defines the symmetric subspace Γ_{sym} of the full phase space Γ. Moreover, let χ^a be a set of constraint functions on the phase space, so the constraint surface $\chi^a = 0$ defines the physical phase space Γ_{phys}. The intersection $\Gamma_{sym} \cap \Gamma_{phys}$ of the symmetric phase space Γ_{sym} and the physical phase space Γ_{phys} is the desired physical symmetric phase space Γ_o of the reduced theory.

A quantum theory is defined on a noncommutative phase space. To be more precise, we call a Hilbert space representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathfrak{A}$ of quantum observables together with a unitary covariant representation U of the group of gauge transformations and symmetries on $\mathcal{H}$ a quantum theory. The Gel'fand-Naimark theorem states that any commutative C^*-algebra is isomorphic to an algebra of continuous complex valued functions on a locally compact Hausdorff space, which is isomorphic to the spectrum of the C^*-algebra when endowed with the Gel'fand topology. The quantum observable algebra $\mathfrak{A}$ is a completion of the classical observable algebra $\mathfrak{A}^\infty = C^\infty(\Gamma)$ consisting of smooth functions on phase space. The rays in the Hilbert space $\mathcal{H}$ are the sates that the quantum system can attain, which correspond classically to probability measures on the classical phase space. The restriction to an embedded subspace $i : \Gamma_o \to \Gamma$ implies a pull-back of the observable algebra $\mathfrak{A}_o^\infty := i^*\mathfrak{A}^\infty$ to an observable algebra on the reduced phase space Γ_o. The induced Poisson structure on $\mathfrak{A}_o^\infty$ implies that i is a Poisson embedding. The desired quantum embedding is thus a generalization of Poisson embeddings for noncommutative algebras.

The direct construction of such an embedding does not exist for noncommutative algebras, but one can take a slight detor that can be applied to the noncommutative case as well: Consider a vector bundle (E, π, Γ) over Γ and a second vector bundle (E_o, π_o, Γ_o) over Γ_o and consider a vector bundle morphism η, such that the projection π of $\eta(E_o)$ to Γ is a Poisson embedding i, so i is encoded in η. The Serre-Swan theorem states the equivalence of categories between vector bundles and finitely generated projective modules over the commutative C^*-algebra of continuous complex valued functions on the base space. Going to the completions reveals that noncommutative vector bundles are nothing else than Hilbert-C^*-modules over the respective noncommutative C^*-algebra that serves as the noncommutative base space. We thus have to construct embeddings of Hilbert-C^*-modules that preserve the algebraic structure for a dense set of operators, because the commutators of the quantum theory reflect the classical Poisson bracket.

It turns out that Hilbert-C^*-modules for a large class of physically interesting C^*-algebras are generated by commutative C^*-algebras $C(\mathbb{X})$. This allows for the construction of noncommutative vector bundle morphisms through embeddings of the spectra $i : \mathbb{X}_o \to \mathbb{X}$ in these special cases. It turns out that these ebeddings preserve the algebraic structures and that a generalization of Rieffel induction allows for calculating the induced representation for the embedded algebra. We thus define a quantum Poisson embedding using this embedding of Hilbert-C^*-modules. Moreover, it turns out that the continuous complex valued functions on the quantum configuration space furnish such a module for many interesting applications.

However, due to the nonembeddability of the kinematic quantum configuration space of LQC into the kinematic configuration space of LQG it is impossible to construct an embedding of the noncommutative vector bundles in this fashion. The reduced phase space Γ_o however is the intersection of the constraint surface Γ_{phys} with the symmetric sector of the phase space Γ_{sym}. The idea is thus to apply our construction to the diffeomorphism constraint surface. Thus, after defining a suitable gauge for the diffeomorphisms, we are able to construct a quantum Poisson embedding for a cosmological sector into full LQG.

The application of this quantum Poisson embedding provides a setting for a systematic study of the interplay between diffeomorphism invariance and symmetry reduction. The non-triviality of this relation is shown by the result that the extracted cosmological system has configurations variables that are very similar to the ones of a super-selection sector of standard Loop Quantum Cosmology, but its full operator algebra turns out to be different from standard Loop Quantum Cosmology. The homogeneous isotropic sector of pure gravity turns out to be quantum mechanics on a circle and a simple matter model turns out to be quantum mechanics on a torus. The dynamics of our system seems pathological at first sight, and we give both mathematical and physical reasons for this behavior and we explain a strategy to cure these pathologies.

The idea behind the construction of states that have expectation values for geometric operators that satisfy certain symmetry properties is to directly construct the respective positive linear functionals on the full observable algebra of Loop Quantum Gravity and to perform the GNS-construction to obtain a Hilbert space representation thereof. It turns out that a slight modification of the observable algebra of Loop Quantum Gravity, that still contains all connection operators and all geometric operators of the full theory, allows for the construction of states with vacuum expectation values for the geometric operators that match the classical expectation values for these geometric observables in a given classical geometry. The defining equation for these state functionals is a slight adaption of the state equation for harmonic oscillator coherent states $\omega_\alpha(a) := \langle \alpha, \pi_o(a)\alpha \rangle$ for Weyl-operators to the adapted observable algebra that underlies Loop Quantum Gravity. It turns out that the GNS-ground states Ω_{E_o} are eigenstates of the geometric observables with eigenvalues described by a classical 3-geometry E_o.

The GNS-construction from these states yields spin network functions that are embedded into classical geometric backgrounds. To obtain a unitary representation of the gauge- and diffeomorphism transformations, one needs to consider the direct sum of these GNS representations over all densitized inverse triads $\phi(\Lambda^{-1}E_o\Lambda)$ that describe the same geometry as E_o. The gauge- and diffeomorphism invariant Hilbert space can be obtained using the group averaging procedure and turns out to contain gauge-invariantly coupled gauge invariant spin network functions that are embedded into a background geometry modulus isometries of the background geometry. Moreover, it turns out that the classical E_o-geometry can be recovered from each normalized vector of the GNS Hilbert space through quantum measurements, which is the reason for calling it the "essential geometry".

Using these states, one can consider a classical minisuperspace and take all occurring 3-geometries E_o and build a quantum minisuperspace by taking the direct sum over the GNS-ground state vectors Ω_{E_o}. It turns out that these states are in a one-one correspondence to Bojowalds μ_o-states for Loop Quantum Cosmology. One is thus able to impose the Loop Quantum Cosmology dynamics on this minisuperspace. The new feature in this case is however that one has a representation of the full observable algebra of Loop Quantum Gravity available, so one can consider genuine "Loop" fluctuations around these states without having to model them beforehand. A different approach to the dynamics of these states can be obtained as follows [20]:

The gauge-variant embedded spin network functions are eigenstates of the geometric operators, so there is a distributional geometry E_o that describes their geometry. The dynamics of Quantum Gravity is solved by the construction of the mutual kernel of the constraints. The existence of classical solutions with particular geometries $E_o^{sol.}$ suggests to construct the mutual kernel as the closure of the span of the Ω_{E_o} states for which there is a point (A_o, E_o) on the classical constraint surface. To determine which embedded spin network states lie in the kernel, one can use the existence of the essential geometry and define it to lie in the kernel if the distributional geometry describing the embedded spin network state is attained as the limit of a sequence $(A_o, E_o)_n$ of points on the constraint surface as $n \to \infty$.

This thesis is organized as follows:

- In section 1 of chapter 2, we will consider classical Hamiltonian Bianchi I cosmology as the example of choice for symmetry reduction in this thesis and make the case for reducing a quantum theory using quantum embeddings. This chapter also serves as an introduction into the cosmological background that is needed for the rest of the text. The rest of chapter 2 is devoted to explain the problem of symmetry reducing a quantum theory.

- Chapter 3 is concerned with constructing the complete group of gauge transformations. The main result is that Loop Quantum Gravity should be invariant under an extension of the group of analytic diffeomorphisms, that is large enough to map any graph onto any other graph in the same iso-knot class. This result is necessary for the later construction of the quantum Poisson embedding.

- We describe the physical ideas behind and the actual implementation of our construction of quantum Poisson embeddings in chapter 4. A detailed description of this procedure can be found in [21].

- We construct a quantum Poisson embedding for cosmological sectors in LQG in chapter 5. The main result of this construction is that there is an embedding of cosmological sectors into LQG and that the super-selection sectors in standard LQC can be interpreted as such embedded sectors. An expanded discussion is contained in [22].

- The construction of the states with smooth geometries is performed in chapter 6, which contains a discussion of the kinematics of this representation of Loop Quantum Gravity as well.
- These states are used to construct minisuper-spaces in chapter 7 and the implementation of the dynamics of standard Loop Quantum Cosmology as well as an idea for the construction of a fundamental dynamics for these states are discussed.
- We supplement the construction of smooth geometry states with the construction of a version of quantum geometry that is based on a fundamental area operator. This version of geometry is essential for the construction of the adapted observable algebra used to construct the smooth geometry states in chapter 6 and justifies the adaption of the observable algebra from a physical perspective.
- These results are summarized in chapter 9.
- We supplement the thesis with appendices that provide the necessary background on C^*-algebras, Hilbert-C^*-modules and Rieffel induction (These mathematical issues are covered in appendix A).
- The necessary background about GR in Ashtekar variables, a useful definition of quantum field theories and standard Loop Quantum Gravity as such a quantum field theory as well as standard Loop Quantum Cosmology is provided in appendix B.
- Geometric operators are essential for a theory of quantum gravity. We thus devote appendix C to length-, area- and volume operators.

Chapters 3 to 8 contain the individual results of this work and are written with the intention to present these results as self contained as it is possible in a thesis.

Chapter 2

Explanation of the Problem

To expose the problem considered in this thesis, we give an overview of classical Hamiltonian cosmology (sections 2.1.1, 2.1.2). Applied to considering observables, this raises the question of how to impose spatial symmetry on the observables in General Relativity, which we consider classically in section 2.1.3. We use the effects of different operator orderings (section 2.2) and the fact that only a few operator orderings are admissible in quantum field theories to make the case for applying a symmetry reduction directly to the quantum theory (section 2.3).

2.1 Classical Hamiltonian Cosmology

Cosmology is usually discussed in the Lagrangian setup, however in order to gain insight into cosmological sectors of LQG we look at canonical GR and its symmetry reduction.

2.1.1 Hamiltonian General Relativity

Given a globally hyperbolic four-dimensional pseudo-Riemannian manifold $\mathbb{X}^4$, it is topologically equivalent to $\mathbb{R} \times \Sigma$, where Σ is a Riemannian manifold. A foliation $X : \mathbb{R} \times \Sigma \to \mathbb{X}^4$ allows to split the metric g on $\mathbb{X}^4$ into a spatial metric q together with a lapse function N and a shift vector field N^a on Σ, such that the metric g can be expressed (in chart $(t, x^a) := X(t_o, \sigma^a)$) as

$$g = (q_{ab}N^a N^b - N^2) dt \otimes dt + q_{ab}N^b dt \vee dx^a + q_{ab} dx^a \otimes dx^b. \tag{2.1}$$

Using Σ as a Cauchy surface, we use q, N, N^a as configuration variables and denote the conjugate momenta by P^{ab}, Π, Π_a, such that the Einstein Hilbert action $S = \frac{1}{\kappa}\int_{\mathbb{X}^4} \sqrt{-|g|} d^4x R[g]$ can be rewritten in these canonical variables as $S = \frac{1}{\kappa}\int_{\mathbb{R}\times\Sigma} dt d^3\sigma \left(\dot{q}_{ab}P^{ab} + \dot{N}\Pi + \dot{N}^a\Pi_a - (N^a V_a + |N|C) + \lambda\Pi + \lambda_a\Pi^a\right)$, where λ, λ^a are Lagrange multipliers constraining Π and Π^a, such that the equations of motion for N and N^a are just integrals of these Lagrange multipliers, so we can turn them into Lagrange multipliers themselves yielding the ADM action

$$S = \frac{1}{\kappa} \int_{\mathbb{R}\times\Sigma} \left(\dot{q}_{ab} P^{ab} - (N^a V_a + |N| C) \right), \tag{2.2}$$

where N and N^a are now considered as Lagrange multipliers and V^a and C denote the vector- and scalar- constraint respectively, whose expression is:

$$\begin{array}{rcl} V_a & = & q_{ac} D_b P^{bc} \\ C & = & \frac{1}{\sqrt{|q|}} q_{ac} q_{bd} P^{ab} P^{cd} - \frac{1}{2\sqrt{|q|}} (q_{ab} P^{ab})^2 - \sqrt{|q|} R[q], \end{array} \tag{2.3}$$

where R denotes the curvature and D the covariant derivative w.r.t. the spatial metric q. The first summand in equation 2.2 is a symplectic potential implying the Poisson bracket $\{F(q), P(f)\} = -\kappa F(f)$ and the second summand denotes the total Hamiltonian that is constrained to vanish, where $F(g) = \int_\Sigma d^3\sigma F^{ab}(\sigma) q_{ab}(\sigma)$ and $P(f) := \int_\Sigma d^3\sigma P^{ab}(\sigma) f_{ab}(\sigma)$ for a symmetric co-tensor valued density F and a symmetric tensor valued scalar f. A classical trajectory $g(t)$ is evolved using the Hamiltonian $H = \int_\Sigma d^3\sigma \left(N^a V_a + |N| C\right)$ using equation 2.1 and treating N, N^a as Lagrange multipliers.

2.1.2 Bianchi Cosmology

Many aspects of cosmology are due to spatial homogeneity of the universe. These geometries are classified in Bianchi types. The notion of homogeneity is introduced by a group $\mathbb{G}$ of isometries $\phi : \Sigma \to \Sigma$, i.e. we assume $\phi^* q = q$. The 3-dimensional symmetry group shall act freely and transitively on Σ, such that the spatial manifold equals the group manifold ($\Sigma = \mathbb{G}$). The invariance condition can be written as:$\mathcal{L}_X g = 0$, where X denotes the Killing vector-field, whose action generates the flow ϕ_t, then it is obvious, that the Killing vector-fields X generate the Lie-algebra $\mathfrak{G}$ of $\mathbb{G}$, such that the Lie-algebra $\mathfrak{G}$ of $\mathbb{G}$ is given by the Lie-bracket of the vector fields X:

$$[X_i, X_j] = C^k_{ij} X_k, \tag{2.4}$$

for $i, j, k = 1, 2, 3$. For any globally hyperbolic manifold and three-dimensional Lie-group, acting freely and transitively on the spacelike hypersurfaces, it turns out that the unit normal vector n of the spacelike hypersurfaces (now considered as group orbits) yield a natural choice for a timelike vector-field, which is constant under orbits of $\mathbb{G}$:

$$\mathcal{L}_X n = 0 \forall X \in \mathfrak{G}. \tag{2.5}$$

Let us then use triads e^a, that commute with the Killing-vectorfields $[e^a, X_i] = 0$. Bundling these three vector-fields with n, we obtain four linearly independent vector fields e_μ, such that we can use their integrals as coordinates and we obtain a line-element $ds^2 = dt^2 - (g_{ij} dx^i dx^j)$. It turns out, that all the commutation relations $[e_\alpha, e_\beta] = \gamma^\mu_{\alpha\beta} e_\mu$ are purely functions of time (which is the coordinate obtained as the integral of n). This implies that we can classify all the Bianchi-cosmologies by the commutation relations of the frames e_μ, since the the e_i furnish an equivalent Lie-algebra in each spatial hypersurface. The standard classification of Bianchi types is then usually stated in terms of the decomposition:

$$\gamma^i_{jk} = \epsilon_{jkl} n^{il} + a_j \delta^i_k - a_k \delta^i_j, \tag{2.6}$$

where n, a are constants in the group-invariant frame satisfying $n^{ij} a_j = 0$. Type A cosmologies arise for $a_i = 0$ (no shift) and type B stands for nonvanishing a_i. Using the signs of the eigenvalues n_1, n_2, n_3, we can classify the Bianchi-type A cosmologies by:

Bianchi type A	n_1	n_2	n_3
I	0	0	0
II	+1	0	0
VI_o	0	+1	−1
VII_o	0	+1	+1
$VIII$	−1	+1	+1
IX	+1	+1	+1.

(2.7)

It turns out, that the type B Bianchi cosmologies are classified on terms of a the sign of h defined by $a^2 = hn_2n_3$, where $a = (1, 0, 0)$, such that:

Bianchi type B	n_1	n_2	n_3
V	0	0	0
IV	0	0	+1
VI_h	0	+1	−1
VII_h	0	+1	+1,

(2.8)

where h stands short for the sign of h.
We can further simplify these models by assuming local rotational symmetry (LRS) (i.e. rotational symmetry w.r.t one fixed axis) or by assuming complete isotropy (i.e. rotational symmetry w.r.t. all rotations). This is done by the introduction of symmetry transformations corresponding to the respective rotations, which amounts to the introduction of one or three additional Killing-vector fields.
Let us now consider the simplest case: Let $\Sigma = \mathbb{R}^3$ and let the group $\mathbb{R}^3$ act on Σ by translation. Moreover consider the action of $SO(3)$ by rotations on Σ. Taking a global chart (x_i), we can consider these two group actions to be generated by the vector fields:

$$\begin{aligned} X_i &= \partial_i, \\ Y_i &= \epsilon_{ijk} x_j \partial_k, \end{aligned} \tag{2.9}$$

where $i = 1, 2, 3$. The combined group is the Euclidean group $E(3)$ and the vector fields (X_i, Y_i) are a particular representation of their Lie-algebra acting on the homogeneous space $\mathbb{R}^3$. The $E(3)$-invariant line-element is then given by:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2, \tag{2.10}$$

where (x_i) denote cartesian coordinates on $\mathbb{R}^3$.
Taking the three homogeneous spaces: the flat $\mathbb{R}^3$, the open three hyperboloid $\mathbb{H}^3$ and the closed sphere $\mathbb{S}^3$, which we obtain as the Bianchi types I, IX and V respectively, which can each be considered in the homogeneous, locally rotationally symmetric and isotropic case. The isotropic line-element is given by:

$$ds^2 = dt^2 - a(t)((1 - kr^2)dr^2 + r^2(d\theta^2 + \sin(\theta)d\phi^2)), \tag{2.11}$$

where $k = 0$ represents isotropic type I, $k = 1$ isotropic type V and $k = -1$ isotropic type IX and (r, θ, ϕ) denote global coordinates. Throughout this thesis we are mostly concerned with type I cosmologies to avoid technical difficulties.

2.1.3 Imposing Symmetry on the Observables

The symmetry reduced physical phase space is the intersection of the symmetric part of the phase space and the constraint surface. In order to be able to transfer the symmetry reduction of the classical system to the quantum theory, we will consider the symmetry reduction of the observable algebra, whose noncommutative quantum analogue replaces the classical phase space. We will therefore proceed as follows: We calculate the symmetric phase space (we denote its elements (q_{sym}, P_{sym})) and evaluate the dependence of a given observable on the symmetric phase space. This generates equivalence classes of full observables that depend in the same way on the reduced phase space. This is the quotient of the full observable algebra by the ideal of phase space functions that vanish on the symmetric phase space. A Dirac observable depends on the physical phase space only, so the equivalence classes of functions on the symmetry reduced physical phase space are labeled by equivalence classes of Dirac observables, that depend equivalently on the symmetry reduced phase space.

This procedure is usually evaded by constructing a reduced set of constraints that define the reduced physical phase space as a constraint surface of the reduced phase space. This is a classically equivalent procedure, but the quantum analogue of this procedure is not unambiguous.

Let us consider isotropic Bianchi I cosmology, so the expression for the metric in the symmetric chart is $q_{sym}(t) = a(t)(dx_1^2 + dx_2^2 + dx_3^2)$ and the expression for the isotropic momentum is in this chart $P^{ab}_{sym}(t) = p(t)\delta^{ab}$. A general observable is a function $O(q, P)$. Let us introduce (local) canonical phase space coordinates $(a, q_\perp, p, P_\perp)$ such that (a, p) furnish coordinates for the symmetric phase space and $(q_\perp, P_\perp)$ for the complement of the symmetric phase space, so $\{a, p\} = -\frac{V}{\kappa}$ and $\{a, c_\perp\} = 0 = \{p, c_\perp\}$ for all coordinates $c_\perp$ of the complement. A symmetric observable is a function $O(q, P) = f(a, p)$ and since the symmetric phase space satisfies $q_\perp = 0 = P_\perp$, one has equivalence classes of observables $O(q, P) = \sum_i f_i(a, p)g_i(q_\perp, P_\perp)$ which coincide on the symmetric phase space, i.e. $\sum_i f_i^{(1)}(a, p)g_i^{(1)}(0, 0) = \sum_j f_j^{(2)}(a, p)g_j^{(2)}(0, 0)$.

Finding Dirac observables in the reduced phase space simplifies significantly due to the fact that (a, p) Poisson commute with the $(q_\perp, P_\perp)$ and the gauge transformation equations turn into the usual Friedman equations after choosing a suitable matter extension of the phase space, introducing symmetric coordinates thereon and using a suitable matter Hamiltonian. The symmetry reduced observable algebra is given by phase space functions of the form $O(a, q_\perp, p, P_\perp) = f(a, p)$, but this is the same as the quotient of the full observable algebra by the ideal of functions on phase space $F(a, q_\perp, p, P_\perp) - F(a, 0, p, 0)$, that vanish at the symmetric phase space. The same argument holds for the constraint surface.

Using an expansion $F = \sum_{ijKL} f_{ijKL} a^i p^j q_\perp^K P_\perp^L$ and $f := F(a, q_\perp, p, P_\perp) - F(a, 0, p, 0) = \sum_{ij} f_{ij00} a^i p^j$ for the associated elements of the quotient, where K, L are multi indices, we can calculate the restriction of the Poisson bracket of F_1, F_2 to the symmetric phase space as $\{F_1, F_2\}|_{\Gamma_{sym}} = \{f_1, f_2\}$, where we used that $q_\perp = 0 = P_\perp$ on Γ_{sym}. The symmetry reduction is thus a Poisson embedding, because the Poisson bracket for the elements f of the quotient algebra is independent of the particular representative F in the full observable algebra. We can thus induce the Poisson structure for the symmetry reduced observable algebra by defining it as the full Poisson bracket of the equivalence classes of observables that depend equivalently on the symmetry reduced phase space.

The time evolution of the Dirac observables is trivial, all Dirac observables are constants of motion, since GR is a constrained Hamiltonian theory, so we have solved the symmetry reduced model once we have calculated the Dirac observables of the reduced system. The quantum analogue of this idea is the first approach used in this thesis.

2.2 The Case for Quantum Symmetry Reduction

The main point of this section is that symmetry reduction of a quantum theory generally differs from the quantization of a classically reduced system, even if the same quantization map is used. This is due to the factor ordering ambiguities that are fixed as long as one works purely at the quantum level, but that arise as soon as one leaves the quantum regime. We will use a very simplified picture in this section, that still makes the essential point:

Suppose there is a full quantum theory $(\mathcal{H}, \pi, \mathfrak{A}, H)$ consisting of a Hilbert space representation $(\mathcal{H}, \pi)$ of the C^*-algebra $\mathfrak{A}$ of quantum observables and a Hermitian operator H that serves as the Hamiltonian of the system. Furthermore, suppose that there is a projection P on $\mathcal{H}$ that projects down to the symmetric part of the theory. Then we can induce a reduced observable algebra $\mathfrak{A}_{sym} := \{PaP : a \in \mathfrak{A}\}$ and an induced representation of $\mathfrak{A}_{sym}$ on $P(\mathcal{H})$ by $\pi_{sym}(PaP)Pv := P\pi(a)v$. The induced dynamics of this theory is then governed by the Hamiltonian $H_{sym} = PHP$. This Hamiltonian still has the factor ordering that is dictated by the full theory. Since finding the right factor ordering is essential for the existence of a quantum theory with an infinite number of degrees of freedom, we can not overestimate the importance of keeping it fixed. However, if one performs a symmetry reduction at the classical level and then quantizes (even when using the same quantization map) the particular induced factor ordering of the full quantum theory is lost. We will therefore consider the consequences of choosing different factor orderings in the quantization of a classical system.

Factor ordering ambiguities arise whenever one quantizes a classical system. Let us consider quantum mechanics in one dimension and adopt a quantization map $\hat{.} : f(p,q) \mapsto f(\hat{p},\hat{q})$, where f is analytical, and such that $[\hat{p},\hat{q}] = i$. Let us then consider two one dimensional distinct classical systems, whose dynamics is governed by $H_1 = p^2 + V_1(q)$ and $H_2 = p^2 + V_2(q)$ respectively, where we assume that $\{p,q\} = 1$ in both cases and without loss of physical generality assume that both potentials are C^1 and positive, but finite for any finite q. Let us rewrite the Hamilton function H_2 as

$$H_2 = \frac{1}{f(q)} p f^2(q) p \frac{1}{f(q)} + V_2(q) + c,$$

which is an equivalent classical Hamiltonian, if f vanishes nowhere and c is a constant.

Let us now apply the quantization map $\hat{.}$ to both Hamiltonians:

$$\begin{array}{rcl} \hat{H}_1 & = & \hat{p}^2 + V_1(\hat{q}) \\ \hat{H}_2 & = & \frac{1}{f(\hat{q})}\hat{p}f^2(\hat{q})\hat{p}\frac{1}{f(\hat{q})} + V_2(\hat{q}) + c \\ & = & \left(p + \frac{1}{f(\hat{q})}[\hat{p}, f(\hat{q})]\right)\left(\hat{p} + f(\hat{q})[\hat{p}, \frac{1}{f(\hat{q})}]\right) + V_2(\hat{q}) + c \\ & = & \hat{p}^2 - i\left[\hat{p}, \frac{f'(\hat{q})}{f(\hat{q})}\right] + \left(\frac{f'(\hat{q})}{f(\hat{q})}\right)^2 + V_2(\hat{q}) + c \\ & = & \hat{p}^2 + \frac{f''(\hat{q})}{f(\hat{q})} + V_2(q) + c. \end{array} \quad (2.12)$$

Hence, if the solution f of the ODE

$$f''(q) + (V_2(q) - V_1(q) + c)f(q) = 0$$

vanishes nowhere, then because of the assumed properties of the potential, we obtain an analytic function $f(q)$ that is finite for every q. Thus, if $V_2 - V_1$ is bounded from below on $\mathbb{R}$ then we can choose a c such that $V_2 - V_1 + c \geq 0$, which implies that the solution $f(q)$ to the ODE with the initial condition $f(0) = 1, f'(0) = 0$ satisfies $\infty > f(q) \geq 1$ for all values of q.

Thus, if $V_2 - V_1$ is bounded from below, then we can use the factor ordering ambiguity to obtain a quantization of H_2 that equals H_1:

$$\hat{H}_2 = \hat{p}^2 + \frac{f''(\hat{q})}{f(\hat{q})} + V_2(q) + c = \hat{p}^2 + V_1(\hat{q}) = \hat{H}_1. \quad (2.13)$$

This disturbing feature of factor ordering ambiguities obviously carries through to quantum mechanics on $\mathbb{R}^n$ for arbitrary n. We are thus not able to recover the "correct quantum potential" from the classical theory, thus whenever we reduce a theory classically and quantize it again, even when we are able to use the same quantization map, we are not able to predict the reduced quantum theory unambiguously. This is the reason, why we consider a quantum version of symmetry reduction in this thesis.

2.3 Symmetry Reduction of a Quantum Theory

We reduced the observables of a classical system in section 2.1.3 and saw that the reduced observable algebra coincides with the quotient of the full observable algebra over the ideal of observables that vanish at the symmetric phase space. The symmetry reduced phase space Γ_{sym} is naturally embedded into the full phase space Γ by an identity map $i : \Gamma_{sym} \to \Gamma$ and we saw that the embedding i $(\Gamma_{sym}, \{.,.\}_{sym})$ on the symmetry reduced phase space satisfies the condition $\{f \circ i, g \circ i\} = \{f, g\}_{sym} \circ i$, so i is a Poisson embedding. The pull-back under i furnishes the symmetry reduction of the Poisson algebras, so a quantum symmetry reduction is precisely a non-commutative Poisson embedding, i.e. the reduction of the observable algebra $\mathfrak{A}$ yields the symmetry reduced observable algebra $\mathfrak{A}_{red} = \{i^* f : f \in \mathfrak{A}\}$.

Before we explain the strategy for the construction of non-commutative Poisson embeddings, let us consider the physical interpretation of reducing an observable algebra as a reduced sensitivity of the measurements in a classical system: An element f of the observable algebra corresponds to the operation of a measurement on the system, i.e. given the state of the system which is given by a distribution $\mathfrak{d}$ on phase space, then the expectation value $\mathfrak{d}(f) = \int_\Gamma \mu_\mathfrak{d} f$ represents the expectation value of the outcome of our measurement. The pull-back under a Poisson embedding is then the restriction of the sensitivity of our measurement.

Mathematically, we say that our reduced observables are those that are insensitive to precisely those observables, that vanish at the embedded phase space. Using this point of view lets us generalize the reduction problem to quantum systems. The general strategy to "quantize a statement" is to formulate the statement for a Poisson system, i.e. stating it as a problem on the algebra of observables and to reformulate this statement in such a way that the commutativity of the observable algebra becomes irrelevant.

With our classical preparations, we have already stated the problem in terms of the observable algebra, and we want to construct the quantum analogue of the pull-back under a Poisson embedding. The problem with this statement is however that the "phase space of quantum mechanics is noncommutative", i.e. it is not a topological space but rather a noncommutative algebra of observables, which is thought of as the algebra of continuous functions on a noncommutative phase space.

If we naively apply the Gel'fand-Naimark theorem we proceed as follows: A commutative C^*-algebra is isomorphic to the algebra of continuous functions on its spectrum, which is a locally compact Hausdorff space in the Gel'fand topology. Thus we could conclude that we are looking for the pullback under an embedding of spectra. This procedure is however flawed as we can see by considering a rather simple example: Take the C^*-algebra of ordinary quantum mechanics on any locally compact group, then by the Stone-von Neumann theorem we know, that there is only one unitary equivalence class of regular irreducible representations of this algebra, thus the spectrum consists of only one point. This means that this quantum system is embeddable into any other

C^*-algebra, particularly into $\mathbb{C}$, which we view as a degenerate quantum system, whose phase space consists of one point only. Moreover $\mathbb{D}_2$, the algebra of diagonal 2×2-matrices, is "larger" than any Heisenberg system. This is in clear contradiction to the idea that embeddability defines a partial ordering corresponding to the size of the associated classical phase space.
We have a much better understanding using the reduced sensitivity point of view: We are able to define an ideal of observables that our measurements are insensitive to such that the observable algebra of the reduced quantum system arises as the quotient of the full observable algebra by this ideal. This would allow to define the observable algebra, however other than in a classical theory, where all pure states are evaluations at points in phase space, we need a Hilbert-space representation for the observable algebra in addition to the algebra itself, whose elements represent the pure states of the system. Furthermore, we know that there are in general many inequivalent representations of a given C^*-algebra. This shows that we can not separate the reduction of the observable algebra from the reduction of its Hilbert space representation.
A strategy that we could follow to construct a reduced quantum algebra and its Hilbert-space representation is to "try to read the rules of quantization off", then to reduce the classical theory using the pull-back under a Poisson embedding and then quantizing this system "using the extracted rules for quantization". But, we will not be able to find the induced operator ordering for the reduced system, hence using the argument of section 2.2 there are many inequivalent ways of constructing a C^*-algebra and its Hilbert-space representation from a given classical system, and it is in no way clear that using two sets of rules that yield the same full quantum system yield equivalent quantum systems for a reduced system.
Consequently, we do not want to "reduce and quantize again", but we want to reduce a quantum system, i.e. given a triple $(\mathfrak{A}, \pi, \mathcal{H})$, where $\mathfrak{A}$ is a C^*-algebra representing the quantum observables of a system and π is a representation of this algebra as a subset of the bounded operators on a Hilbert-space $\mathcal{H}$, we seek the construction $\mathcal{E}$ of the reduced system $(\mathfrak{A}_o, \pi_o, \mathcal{H}_o)$ directly from $(\mathfrak{A}, \pi, \mathcal{H})$.
Let us now formalize the requirements of the quantum Poisson map $\mathcal{E}_i$, corresponding to a classical Poisson embedding i:
First, we want that the construction reproduces the right classical limit, i.e. we want for the observable algebras that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & \mathfrak{A} & \longrightarrow & \mathfrak{A}_o & \\ \hbar \to 0 & \downarrow & i^* & \downarrow & \hbar \to 0 \\ & C^\infty_c(\Gamma) & \longrightarrow & C^\infty_c(\Gamma_o). & \end{array} \tag{2.14}$$

Here Γ, Γ_o are the full and reduced phase space respectively, i denotes the embedding of the reduced into the full phase space and $\mathfrak{A}, \mathfrak{A}_o$ are the C^*-algebras that represent the quantum observables of the associated quantum systems. This diagram needs some explanation: We need to specify the way in which we take the classical limit indicated by $\hbar \to 0$. Our notion of classical limit will be fixed when we restrict ourselves to transformation group systems, i.e. classical

systems in which we give a polarization, such that we are able to talk about a configuration space and second by considering the group generated by the exponentiated Poisson action of the momenta. There is a simple correspondence between the classical Poisson systems and the associated C^*-algebras, such that taking the classical limit is an easy procedure.
Second, we want to construct the "right Hilbert space representation": Physically we want that the expectation values of our observables are matched by corresponding expectation values in the full theory. We can reduce the number of assumptions by noticing that any representation of a C^*-algebra arises as a direct sum of cyclic representations out of vacuum states Ω_i, where the summands are of the kind $\langle\psi_a, \pi(b)\psi_c\rangle_i = \Omega_i(a^*bc)$. This allows us to restrict our attention to vacuum expectation values, thus we demand that there is a dense subset $\mathcal{D}$ in the reduced C^*-algebra and that there is a vacuum state ω_i, corresponding to Ω_i, on the reduced quantum algebra such that a map $\mathcal{E} : \mathcal{D} \subset \mathfrak{A}_o \to \mathfrak{A}$ matches vacuum expectation values:

$$\omega_i(a) = \Omega_i(\mathcal{E}(a)) \;\; \forall a \in \mathcal{D}. \tag{2.15}$$

This condition ensures that expectation values of the reduced Hilbert space representation is a subset of the expectation values in the full theory, as we would expect it from being a subsystem.
Third, one would like to constrain the dynamics to coincide with the dynamics of the full theory. Let us consider the corresponding classical situation: Given a Poisson submanifold embedded into a larger Poisson manifold, it is generally not the case that the Hamilton vector field of the full Hamiltonian will be tangential to the submanifold. The situation for quantum theories is analogous: Consider the von-Neumann equation for a density operator ρ, which reads using the correspondence map $\mathcal{E}$:

$$i\partial_t \mathcal{E}(\rho) = [H, \mathcal{E}(\rho)], \tag{2.16}$$

which implies that if H contains "mixing matrix elements" one obtains a dynamics that moves away from the image of $\mathcal{E}$. This forces us to use the reduced sensitivity interpretation, which tells us that our measurements are insensitive to an ideal of observables, and that our dynamics has to be corrected by building the quotient. Since the ultimate goal of this work is the extraction of a sector from Quantum Gravity, which is a theory with constrained dynamics, we will not discuss details about dynamics but rather focus on the imposition of constraints.
Our strategy to construct a quantum Poisson embedding will be as follows: We will restrict our attention to Lie-transformation group systems and use groupoids as the linking structure between classical and quantum systems. Lie-transformation groupoids are very useful for this purpose, because they are on the one hand classical spaces, which we are able to treat with methods of topology, but on the other hand, one can define a noncommutative algebra of functions on a groupoid using the convolution product, which is precisely the product of the corresponding quantum algebra. Another feature of groupoids

is that they act on spaces in a way very similar to the representation of a C^*-algebra on a Hilbert-space. This allows one to use Morita theory, i.e. the theory of categories of isomorphism classes of representations of groupoids. It turns out that Morita theory for groupoids with Haar measures induces the Morita theory for the corresponding groupoid C^*-algebras. Thus we use the structure of a groupoid as a commutative space on the one hand, which allows us to construct embeddings, and constructions similar to Morita theory on the other hand to construct an equivalent notion of embedding, such that it can be applied to C^*-algebras. The resulting procedure will be the noncommutative version of constructing an embedding by embedding a vector bundle over the reduced phase space into a larger vector bundle over the unreduced phase space and recovering the phase space embedding using the bundle projection of the embedded subbundle in the full vector bundle. This procedure is developed in chapter 4 and applied to extract a cosmological sector from LQG in chapter 5.

Chapter 3

Physical Diffeomorphisms in Loop Quantum Gravity

General Relativity is a theory of the dynamics of geometry and as such invariant under diffeomorphisms which are physically nothing else than gauge transformations in the sense of Dirac. The quantum theory should thus implement the diffeomorphisms as unitary transformations. The configuration space of the quantum theory (i.e. the spectrum of the configuration observable algebra) is however a distributional extension of the classical configuration space, so one needs to carefully extend the action of the diffeomorphisms to the quantum theory. Dirac's procedure of postponing the imposition of gauge invariance until after quantization is expected to yield a quantization of the gauge orbits of the classical theory. We show in this section however that one has to use an extension of the diffeomorphism group for this expectation to hold. This result is important for the course of this thesis, because the precise "size" of the gauge group is very important for the cosntruction of the quantum reduction.

3.1 Explanation of the Problem

Quantization of a classical Hamiltonian system requires a system of real "elementary" variables that separate points in classical phase space, which is closed under taking Poisson brackets, so the elementary variables form a Lie-algebra under the Poisson bracket. Quantization is then the procedure of embedding this Lie-algebra into an associative algebra, such that the Poisson-bracket is mapped to $i\hbar$ times the commutator and finding an involution such that the elementary variables are self-adjoint.

The elementary Poisson variables that underlie the Loop Quantization Programme of diffeomorphism invariant gauge theories are holonomies of the connection and fluxes through the conjugated electric field, which seenm to form a closed Poisson system at first sight. Upon closer inspection however, one discovers that the Poisson bracket of smooth cylindrical functions, which by definition depend only on a finite number of holonomies, with an arbitrary flux can not be expressed as a cyindrical function anymore, because the graph underlying the cylindrical function may intersect the surface an infinite number of times, thus resulting in a function that depends on an infinite number of holonomies, which can in general not be reexpressed as a function of a finite number of holonomies.

A simple resolution for this problem is to restrict oneself to analytic edges and analytic surfaces, because an analytic edge that intersects an analytic surface an infinite number of times has to lay inside the surface and is hence not affected by the Poisson action of the flux-variable on this surface, so cylindrical functions defined as functions of finitely many holonomies along piecewise analytic paths and fluxes through piecewise analytic surfaces define a closed Poisson system of variables that separates the points in the classical phase space.

Dirac quantization of a constrained theory requires a unitary representation of the group of gauge transformations generated by the constraints, which in the case of Loop Quantum Gravity amounts to an implementation of the ordinary gauge transformations generated by the Gauss-constraint, the diffeomorphisms generated by the vector constraint and transformations generated by the scalar constraint. Using the group averaging procedure to solve the diffeomorphism constraint leads to the problem that the category of piecewise analytic paths is not left invariant by general C^2-diffeomorphisms, so we need to restrict the group of classical gauge transformations to the subgroup that contains only piecewise analytically invertible piecewise analytic diffeomorphisms.

Using this group, in fact using any subgroup of the classical diffeomrophisms, and solving the diffeomorphism constraint by group averaging yields a nonseparable Hilbert space, due to the existence of cylindrical functions depending on graphs with arbitrarily valent vertices, so the orbits of diffeomorphisms, which by definition can only act as linear transformations on the tangent space at a point, are labeled by continuous moduli built from $GL(3)$-invariants constructed from the tengent vectors adjacent to a vertex[1].

Desiring a separable diffeomorphism-invariant Hilbert space for Loop Quantum Gravity[2], Fairbairn and Rovelli [26] observed that a seemingly harmless extension of the diffeomorphism group yields a separable diffeomorphism-invariant Hilbert space, while allowing for a well defined version of quantum geometry thereon. It is the purpose of this chapter to provide a framework that gives on the one hand a physical reason for this extension and on the other hand to address the issue of a closed system of elementary Poisson variables.

We first give a "quick fix" for the problem by constructing the desired subgroupoid of the double-groupoid of piecewise analytic graphs whose induced action on cylindrical functions on piecewise analytic graphs yields precisely the

[1]Notice that diffeomorphism covariance forces one to allow for cylindrical functions with vertices of arbitrary valence: Given a cylindrical function C_γ depending on closed loops only (so there are no true vertices in the underlaying graph γ), one can pick an arbitrary point in γ and one will find an infinite number of diffeomorphism ϕ_i that fixes this point, but acts nontrivially on all other points. Then $C_\gamma + \sum_{i=1}^N C_{\phi_i(\gamma)}$ will in general have at least one $2N$-valent vertex.

[2]One point made by many critics of Loop Quantum Gravity has been the fact that Loop Quantum Gravity was defined on a non-separable Hilbert space.

Fairbairn-Rovelli diffeomorphism orbits of the respective cylindrical function on a piecewise analytic graph. We present then a physical reason, why precisely this set of transformations should be implemented in the quantum theory. The underlying idea is that the gauge-invariant Hilbert space, that is constructed as a GNS-representation of a configuration algebra (cylindrical functions) using a faithful gauge-invariant Schrödinger type state should be a completion of the gauge-orbits of the associated classical configuration algebra. The Poisson bracket of classical Ashtekar General Relativity does however not support holonomies of the connection, but only three-dimensionally smeared variables. Holonomies on the other hand arise as distributional extensions of the classical connection variables. We therefore consider one-parameter families of classical connection variables that approximate a holonomy when the parameter goes to zero. Applying the orbit argument to this setting means that the diffeomorphism invariant Hilbert space should consist of a completion of the diffeomorphisms of orbits of the one-parameter families of classical observables that approximate the respective cylindrical function. This provides a notion of completeness of the group of quantum gauge transformations, which we impose on Loop Quantum Gravity. It turns out that the completely diffeomorphism-invariant Hilbert space of Loop Quantum Gravity is precisely the Hilbert space that we constructed using the groupoid approach.

3.1.1 Problem of the Fairbairn-Rovelli Construction

Fairbairn and Rovelli consider an arbitrary smoothness class for the edges of a graph, somewhere between C^1 and C^ω without specifying it, because the smoothness class in their paper is assumed to be the same as the one considered for the field configurations in the classical theory. Moreover, they assume that for each vertex v in a chart there is a coordinate length r_o in this chart such that each edge penetrated the coordinate sphere $S(r,v)$ of radius $r < r_o$ around v exactly once. We will prove that this assumption fails for $C^1, ..., C^\infty$-edges by constructing a C^∞ counterexample in this section.

Consider the three curves e_1, e_2, e_3 intersecting only at the vertex $v = (0,0,0)^T$, which are parameterized by:

$$\begin{aligned} e_1(t) &= \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \\ e_2(t) &= \exp(-t^{-2}) \begin{pmatrix} t \\ \sin(\frac{1}{t}) \\ \cos(\frac{1}{t}) \end{pmatrix}, \\ e_3(t) &= \exp(-t^{-2}) \begin{pmatrix} t + \frac{1}{5t} \\ \sin(\frac{1}{t}) + \frac{\sin(1/5t)}{5} \\ \cos(\frac{1}{t}) + \frac{\cos(1/5t)}{5} \end{pmatrix}. \end{aligned} \tag{3.1}$$

e_1 is a coordinate line, e_2 is a coordinate spiral, that winds finer and finer around e_1, such that it winds infinitely often around e_1 as $t \to 0$ and e_3 is a spiral winding finer and finer around e_2 such that it winds infinitely often around e_2 as $t \to 0$. All three curves are however C^∞ on $t \in [0,1]$, but e_3 has an accumulation point of fold-backs at the vertex v, thus violating the Fairbairn-Rovelli assumption. We conclude:

Lemma 1 *The Fairbairn-Rovelli assumptions are inconsistent.*

3.2 Groupoid Approach to the Diffeomorphism Group

The action of a continuously invertible homeomorphism $\phi : \Sigma \to \Sigma$ on a cylindrical function Cyl_γ depending on a graph γ is $U_\phi^* Cyl_\gamma U_\phi = Cyl_{\phi(\gamma)}$. Using the notion of an ordered graph γ, which is a set of oriented edges with a linear order among them, we can investigate the action of any subgroup of the homeomorphism group by its action on ordered graphs, i.e. we consider $\phi : \gamma \to \phi(\gamma)$.

Let us for this purpose denote the set of all ordered graphs by Γ and the double groupoid $\Gamma \times \Gamma$ by $D(\Gamma)$. The elements (γ, γ') of $D(\Gamma)$ are pairs of ordered graphs, the source map is $s(\gamma, \gamma') = \gamma$ while the range map is $r(\gamma, \gamma') = \gamma'$ and the object inclusion map is $e(\gamma) = (\gamma, \gamma)$ together with the composition law $(\gamma, \gamma_o) \circ (\gamma_o, \gamma') = (\gamma, \gamma')$. This groupoid acts on Γ with the moment map $\mu(\gamma) = \gamma$ and the action $(\gamma, \gamma') \triangleright \gamma' = \gamma$.

Each pair of an ordered graph γ and a homeomorphism ϕ defines an element of $D(\Gamma)$ by pairing $(\gamma, \phi(\gamma))$.

Definition 1 *Given a subgroup G of the homeomorphism group, we denote the smallest subgroupoid of $D(\Gamma)$ that contains $\{(\gamma, \phi(\gamma)) : \gamma \in \Gamma, \phi \in G\}$ by $D_G(\Gamma)$.*

Lemma 2 *The orbits of the groupoid action of $D_G(\Gamma)$ in Γ are isomorphic to orbits of the group action of G in Γ.*

proof: If γ_o is in the G-orbit of γ, then there exists a homeomorphism $\phi \in G$ such that $\gamma_o = \phi(\gamma)$ and hence $(\gamma_o, \gamma) \in D_G(\Gamma)$. Since the inverse ϕ^{-1} of a homeomorphism ϕ induces the existence of $(\phi(\gamma), \phi^{-1}(\phi(\gamma))) = (\phi(\gamma), \gamma) = (\gamma, \phi(\gamma))^{-1}$ and the groupoid composition $(\gamma, \phi_o(\gamma)) \circ (\phi_o(\gamma), \phi(\phi_o(\gamma))) = (\gamma, \phi(\phi_o(\gamma)))$ reduces to the composition law for homeomorphisms, we see that $D_G(\Gamma)$ contains precisely the elements $\{(\gamma, \phi(\gamma)) : \gamma \in \Gamma, \phi \in G\}$. $\square$

Using the induced action of $D_G(\Gamma)$ on cylindrical functions by the moment map $\mu(Cyl_\gamma) = \gamma$ and action $(\gamma_o, \gamma) \triangleright Cyl_\gamma = Cyl_{\gamma_o}$, we see:

Corollary 1 *The G-orbit of a cylindrical function coincides with the $D_G(\Gamma)$-orbit for any subgroup G of the homeomorphism group.*

Given any ordered graph γ and any subgroup G of the homeomorphisms, we use the notion G_γ for the subgroup of G whose action on γ coincides with a permutation of the edges of γ and we denote the subgroup of G that acts trivially on γ by TG_γ. Then the group of graph symmetries of γ is $SG_\gamma := G_\gamma / TG_\gamma$. This allows us to define a projection map P_γ through

$$P_\gamma \, Cyl_\gamma := \frac{1}{|SG_\gamma|} \sum_{\phi \in SG_\gamma} Cyl_{\phi(\gamma)},$$

because the action of SG_γ on Cyl_γ is independent of the representative in G.

The G analogue of the antilinear rigging map η used to construct the diffeomrophism invariant Hilbert space of Loop Quantum Gravity is

$$\eta[Cyl_\gamma] : Cyl'_{\gamma'} \mapsto \sum_{\phi \in G/G_\gamma} \langle \phi \triangleright P_\gamma Cyl_\gamma, Cyl'_{\gamma'} \rangle,$$

which is also well defined, because the action of G/G_γ on $P_\gamma Cyl_\gamma$ is again independent of the representative.

This rigging map can be formulated using the respective groupoids only: Let $\mathcal{S}G_\gamma$ be the groupoid whose object set consists of all ordered graphs that have the same edge set as γ and whose morphisms are the permutations of edges, acting on cylindrical functions with graph γ by permuting the arguments. Then $P_\gamma Cyl_\gamma := \frac{1}{|\mathcal{S}G_\gamma|} \sum_{\phi \in \mathcal{S}G_\gamma} Cyl_{\phi(\gamma)}$. Let $\mathcal{D}G$ denote the smallest subgroupoid of the double groupoid of unordered graphs that contains all $\{(\gamma, \phi(\gamma)) : \phi \in G, \gamma = \text{unordered graph}\}$, then $\eta[Cyl_\gamma] : Cyl'_{\gamma'} \mapsto \sum_{\phi \in \mathcal{D}G} \langle \phi \triangleright P_\gamma Cyl_\gamma, Cyl'_{\gamma'} \rangle$, where the action of $\mathcal{D}G$ on Γ needs the definition of a linear "standard" order for every graph (which exists by the axiom of choice). Given any ordered graph γ one needs to use an element $g \in SG_\gamma$ to order γ to obtain the standard order. The action of the unordered graph groupoid is then defined by mapping a standard ordered graph to a standard ordered graph and subsequently acting on the resulting graph with g^{-1}.

Having the rigging map and thus the construction of the diffeomorphism-invariant Hilbert space cast using groupoids only, we have an immediate solution for the problem of reconciling the use of the piecewise analytic category of edges with the desire for a separable diffeomorphism invariant Hilbert space through defining the optimal diffeomorphism groupoid:

Definition 2 *The* **optimal diffeomorphism groupoid** *consists of the smallest subgroupoid of $D(\Gamma)$ that contains all pairs (γ, γ') of piecewise analytic graphs that can be mapped onto each other by a homeomorphism.*

Using the optimal diffeomorphism groupoid, we see that the η-image of the spin network function SNF_γ on the graph γ depends only on the knot class of γ, implying:

Lemma 3 *The optimal diffeomorphism-invariant Hilbert-space of Loop Quantum Gravity is separable.*

proof: Density of the spin network functions in the kinematic Hilbert space of Loop Quantum Gravity allows us to find a dense set in the optimal diffeomorphism invariant Hilbert space by applying the rigging map η using the optimal difeomorphism groupoid. The η-image of a spin network function depends only on the knot-class of the underlying graph which is a countable set as well as the spin labels, so the η-image of the spin network functions is countable as well. □

Notice that the cylindrical functions on piecewise analytic graphs separate the points in configuration space, and that any two lie in the same classical diffeomorphism orbit if and only if the there exists a C^2-diffeomorphism ϕ and hence a groupoid element $(\gamma, \phi(\gamma))$ that relates the underlying graphs. This allows us to factor all classical C^2-diffeomorphisms out, even when we are working in the piecewise analytic category. Thus using the groupoid method we can extend the classical group of diffeomorphisms without having to restrict it first to the piecewise analytic category.

Let us now prove that the optimal diffeomorphism groupoid and the complete quantum extension of the diffeomorphism group yield identical rigged Hilbert spaces.

3.3 Quantum Completion of a Group of Gauge Transformations

Let us take a step back and consider the quantization of a classical field theory defined as a Hamiltonian system $(\Gamma, \{.,.\}, H, \{\chi_i\}_{i\in\mathcal{I}})$ on a phase space $\Gamma = T^*(\mathcal{C})$, which is the cotangent bundle over a configuration space $\mathcal{C}$, with canonical Poisson bracket $\{.,.\}$, Hamiltonian $H : \Gamma \to \mathbb{R}$ and a set of constraints $\{\chi_i\}_{i\in\mathcal{I}}$. The configuration space is often chosen to be some category of field configurations, where the precise choice is due to mathematical convenience but usually not motivated by physical considerations. The elements of $\mathcal{C}$ turn out to be morphisms from a groupoid to a group under rather general circumstances: Given a scalar field theory one can consider the group of modes (with addition as composition), so each field configuration ϕ gives a morphism $\phi : m \mapsto \int d^3x m(x)\phi(x)$. Let us furthermore assume that the constraints are at most linear in the momenta, so their action closes on the configuration space.

The quantum theory requires a Hilbert space representation of the commutative algebra of configuration variables, which is usually constructed as an algebra of cylindrical functions of groupoid morphisms. The quantum configuration space $\mathbb{X}$ is the spectrum of the C^*-completion of the algebra of cylindrical functions. This completion generally enlarges the configuration space $\mathcal{C}$ to $\mathbb{X}$, because although each element of $\mathcal{C}$ defines an element of the spectrum of the quantum configuration algebra by the evaluation functor, there are elements in $\mathbb{X}$ that can only be written as a distributional field configuration. The topology of $\mathbb{X}$ is given as the Gel'fand topology for the spectrum of the configuration algebra. It is natural to implement the gauge transformations that act as homeomorphisms on $\mathcal{C}$ as quantum gauge transformations that act as homeomorphisms on $\mathbb{X}$.

This quantum enlargement $\mathcal{C} \to \mathbb{X}$ however has an important consequence for the representation of the group $\mathcal{T}$ of gauge transformations: Let $q : \mathcal{C} \to \mathbb{X}$ be the appropriate embedding of the classical configuration space into the quantum configuration space, which is assumed to be dense in the Gel'fand topology, since $C(\mathbb{X})$ is a C^*-completion of the classical configuration algebra $\mathfrak{C}$. Notice that the action of the gauge group is so far only defined on $q(\mathfrak{C}) \subset \mathbb{X}$, so the question we have to consider here is how to extend the action of the gauge transformations to all of $\mathbb{X}$. There is however a deeper problem concerned with the quantum gauge transformations:

Following the Dirac procedure we postponed the construction of gauge-invariant so called Dirac observables to the quantum theory, due to technical obstructions that inhibited us to achieve this implementation at the classical level, which would have left us with the gauge-invariant classical configuration space $\mathcal{C}_{inv}$. The implementation of the gauge group is thus only a step in constructing the quantum completion $\mathbb{X}_{inv}$ of $\mathcal{C}_{inv}$. A gauge-invariant classical observable is a function $f : \mathcal{C} \to \mathbb{R}$, such that

$$f(x) = f(\tau x) \,\forall\, x \in \mathcal{C},\, \tau \in \mathcal{T},$$

so classical gauge-invariant observables $f : \mathcal{C}_{inv} \to \mathbb{R}$ are functions that are constant on the gauge orbits $O_x := \{\tau x : \tau \in \mathcal{T}\}$. This means in turn that we are interested in the quantum completion of the space $\mathcal{C}_{inv}$ of gauge orbits O_x. To achieve this, we have to construct a group of quantum gauge transformations $\mathcal{T}_q$ with action on $\mathbb{X}$, such that $\mathbb{X}/\mathcal{T}_q$ is a cylindrical completion of $\mathcal{C}_{inv}$.

Let us now consider a Schrödinger type state on the quantum configuration algebra $C(\mathbb{X})$ given by $\omega(f) := \int_{\mathbb{X}} d\mu(x) f(x)$, and thus a Hilbert-space representation of $C(\mathbb{X})$ on $\mathcal{H} := L^2(\mathbb{X}, d\mu)$ as multiplication operators. It will be our strategy to solve for the gauge-invariant Hilbert space using the group-averaging procedure. Since the Hilbert-space $\mathcal{H}$ is a completion of $C(\mathbb{X})$, we can state the requirement for $\mathcal{T}_q$ for $\mathcal{H}$ and require that the gauge transformations equivalence vectors of $\mathcal{H}$ that can be obtained as limits of a gauge-equivalent sequences of elements of the classical configuration algebra. Formally we define for a unitary representation U of T_q on $\mathcal{H}$ (which assume to be implemented as the pull-back under an isometry of μ) the notion of an incompleteness:

Definition 3 *An* **incompleteness** *is a pair of elements $h_1, h_2 \in \mathcal{H}$ that are the limits of two sequences, which are element-wise gauge-equivalent (at the classical level), for which there is no element $\tau \in \mathcal{T}$ such that $h_1 = U(\tau) h_2$.*

We then require that the group $\mathcal{T}_q$ of quantum gauge transformations contains enough gauge transformations such that there is no incompleteness left.

To put our requirement into a picture, let us consider any pair v_n^1, v_n^2 of sequences, whose limits (for $n \to \infty$) are the Hilbert space elements v^1 and v^2 respectively and assume that for every $n \in \mathbb{N}$ there exists a classical gauge transformation $\tau \in \mathcal{T}$ (the upper line in the diagram), then the requirement implies the existence of an element τ_q that relates the limits (the lower line in the diagram), meaning that the following diagram commutes:

$$\begin{array}{rccc} \tau_n : & v_n^1 & \to & v_n^2 \\ n \to \infty & \downarrow & & \downarrow \\ \tau_q : & v^1 & \to & v^2. \end{array} \tag{3.2}$$

We call the sequences in 3.2 regularizations of their respective limits.

This picture may be misleadingly interpreted as defining an equivalence relation, but this is not the case, since there may be sequences relating v^1 and v^2 as well as v^2 and v^3, but since these sequences may be different, it is not implied that v^1 and v^3 must be related. To obtain an equivalence relation, we can proceed as follows: Let $\mathcal{B}$ be a Hilbert-basis for $\mathcal{H}$ and consider the smallest subgroupoid of the double groupoid $D(\mathcal{B})$ that contains any pair of elements of $\mathcal{B}$ that is related by a regularization 3.2. In terms of groups, we define:

Definition 4 *A group T_q (containing all τ_q) is* **complete***, whenever there exists a regularization such that 3.2 commutes.*

Since we developed our notion of completeness purely on the physical principle that the Dirac procedure of solving the constraints at the quantum level should commute with quantizing the gauge invariant Dirac observables, we have a found a physical requirement that can be applied to diffeomorphisms in Loop Quantum Gravity, which satisfy the technical assumptions made above.

3.4 Regularized Cylindrical Functions

Let us construct a regularization for cylindrical functions by giving a one-parameter family of classical functions, which for each classical field configurations converge to the respective cylindrical function for $\epsilon \to 0$. The regularization sequences are then obtained by taking e.g. $\epsilon_n = \frac{1}{n}$.

3.4.1 Regularized Holonomies

The cylindrical functions used to construct Loop Quantum Gravity are functions of a finite number of holonomies along piecewise analytic curves, which are distributional functionals of the Ashtekar connection and are as such not supported by the classical Poisson bracket. To find an expression for the classical Poisson bracket involving a holonomy $h(e) = P_t\{\exp(\int_0^1 dt \dot{e}^a(t) A_a^i(e(t))\tau^i)\}$, where P_t denotes path-ordering, e an analytic curve, A the pull-back of the Ashtekar connection to Σ and τ^i $\frac{i}{2}$ times the Pauli-matrices, one has to consider the regularized expression:

$$h_e^\epsilon(A) := P_t \left\{ \exp\left(\int_0^1 dt \int_{reg(e)} \delta_0^\epsilon(\sigma)\delta_t^\epsilon(s)\dot{p}^a(s,\sigma)A_a^i(p(s,\sigma))\tau_i \right) \right\}, \qquad (3.3)$$

where $p(s,\sigma)$ denotes a smooth 2-parameter family of mutually non-intersecting path, that coincide with e for $\sigma = (0,0)$ and the expressions $\delta_.^\epsilon$ denote one- resp. two-dimensional regularizations of the Dirac delta. The regularized region of integration is assumed to be open and centered around e: Let us give a precise definition thereof:

Definition 5 *Let e be an edge, then a continuous one-parameter family of open sets σ_e^ϵ is called a* **regulator** *of e, if:*

1. *∂e is in $\partial\sigma_e^\epsilon$ for all $\epsilon > 0$*

2. *the interior of e is inside σ_e^ϵ for all $\epsilon > 0$*

3. *for all x outside e there exists $t > 0$, such that x is outside σ_e^ϵ for all $\epsilon < t$.*

• *Let σ_e^ϵ be a regulator of e, then a continuous one-parameter family $i_\epsilon(\sigma_e^\epsilon)$ is an* **internal approximation** *of e, if:*

1. *for all $\epsilon > 0$: $\partial i_\epsilon(\sigma) \cup i_\epsilon(\sigma)$ is a subset of σ_e^ϵ*

2. *for all x in the interior of e there exists $t > 0$ such that $x \in i_\epsilon(\sigma)$ for all $\epsilon < t$.*

• *A* **regularization** *of an edge is a pair $(\sigma_e^\epsilon, i_\epsilon(\sigma))$ consisting of a regulator σ_e^ϵ of e and an internal approximation $i(\sigma)$ of σ.*

We will now suppose that the region of integration in equation 3.3 is $reg_\epsilon := i_\epsilon(\sigma_e^\epsilon)$ for some regularization of e and that the occurring regularized delta functions vanish at the boundary of reg_ϵ.

Let us now verify that equation 3.3 really converges to the holonomy for every classical connection A. The pull-back to Σ of a classical connection is a smooth function and hence continuous. Thus, using the standard argument that for any continuous function f that is averaged over an open ball of radius r around x_o converges to $f(x_o)$ gives the approximation property for any classical connection. (See figure 3.1 for an illustration of a regularization.) Notice that our definition of a regularization is background independent.

3.4.2 Regularized Cylindrical Functions

In the previous section we gave a prescription for regulating a holonomy; the fundamental configuration variables of Loop Quantum Gravity are however cylindrical functions depending on the holonomies on a graph, so we need a regularization for cylindrical functions that depend on a minimal graph γ, which consists of a compatible regularization of all the edges in γ. The regularized cylindrical function then depends on the regularized holonomies (equ. 3.3), where the regions reg_e satisfy the compatibility that we construct in this section.

A graph $\gamma = (E, V)$ consisting of a set E of edges and a set V of vertices, which satisfy $\forall e \in E : x \in \partial e \Leftrightarrow x \in V$ and $e_i \cap e_j \subset \partial e_i$. The cylindrical functions depend on the connection through the holonomies, which are independent degrees of freedom, so we want that the regularization of a graph is a regularization of each edge, such that two different edges test the connection in two disjoint regions for every $\epsilon > 0$. To be able to approximate the entire graph however, one needs that the vertices are elements of the boundaries of the regulators σ_e^ϵ of the adjacent edges e. This leads us to the definition for a regularization of a graph:

Definition 6 *A* **regularization of a graph** *(E, V) is a set (R, V), where for each edge $e \in E$ there is a regularization $i(\sigma_e)$, such that $\forall \epsilon > 0$:*

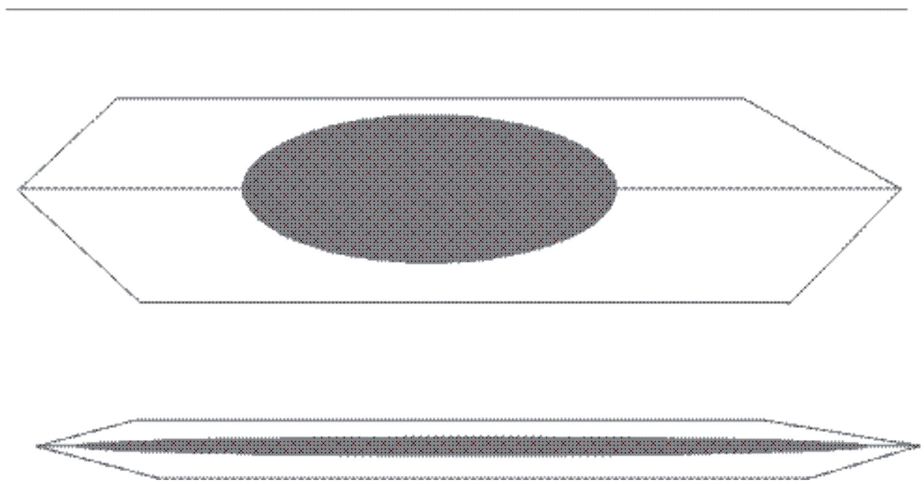

Figure 3.1: Regularization of an edge: the first line depicts an unregularized edge, the second line depicts the regularization of an edge with say $\epsilon = 1$ and the last line depicts the regularization with $\epsilon < 1$. The grey region denotes $i_\epsilon(\sigma)$ and can be seen to spread out over the entire edge towards the boundary points, while at the same time shrinking down to the edge.

1. *for each vertex $v \in V$ and all adjacent edges $e(v)$: $v \in \partial\sigma_e$*
2. *for any two non-parallel edges e_1, e_2, the intersection $(\partial\sigma_{e_1} \cup \sigma_{e_1}) \cap (\partial\sigma_{e_2} \cup \sigma_{e_2})$ contains either one or two common vertices or is empty.*
3. *for each $v \in V$ there is a vertex regularization v^ϵ consisting of an approximation of v by a one-parameter family of open sets*
4. *for a set of edges intersecting parallel at a vertex v there exists a vertex regularization v^ϵ, is such that the σ_e of the parallel edges intersect just inside v^ϵ.*
5. *the $\sigma^\epsilon_e \cap \sigma^\epsilon_{e'} = \partial\sigma^\epsilon_e \cap \partial\sigma^\epsilon_{e'}$ for any two edges e, e'*
6. *for $1 > \epsilon > 0$ blurred graph*[3] *obtained by the union of v^ϵ, σ^ϵ_e has the same iso-knot class as γ*

Lemma 4 *Given a graph $\gamma = (E, V)$ in a manifold, there exists a regularization $\gamma_r = (R, V)$.*

[3]There is an intuitive picture for the construction of the iso-knot class of a blurred graph: (1) For each vertex regularization, fix a point x therein; (2) For each edge regularization span an infinitely thin rubber from the x in the initial vertex regulatization to the x in the final vertex regularization being able to move only within the union of the vertex-regularization and the edge regularization. Letting all rubber bands relax defines a graph whose iso-knot defines the iso-knot class of the blurred graph.

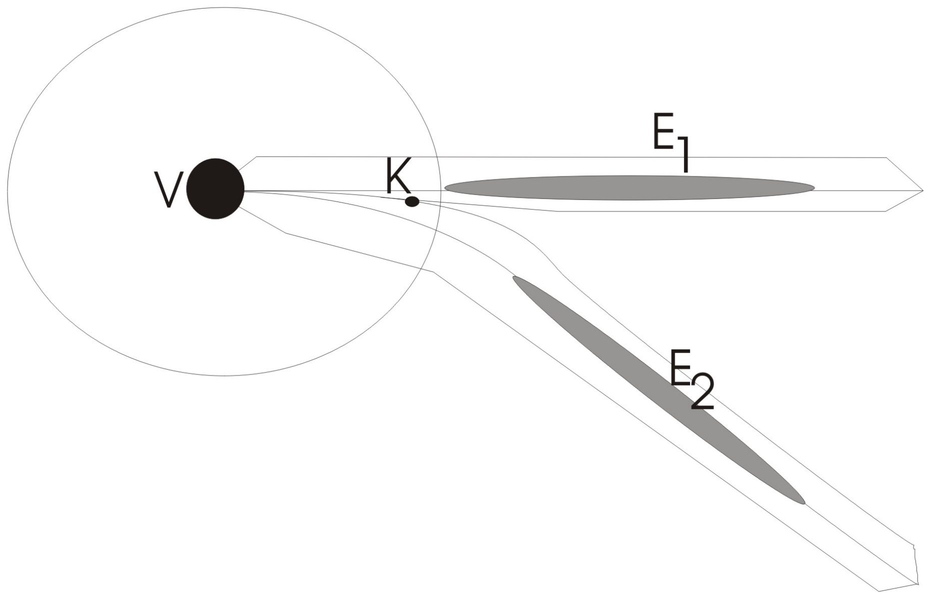

Figure 3.2: Regularization of a graph consisting of two edges: Here we depict the problematic situation, where two edges E_1, E_2 arrive with the same tangent vector at the vertex V. However, due to analyticity, there is no finite region, where E_1 and E_2 coincide. In this case, we allow, that the boundaries of the regions σ_{E_i} partially coincide inside a regularization of the vertex (from K to V), which is indicated by the thin lined circle around the vertex V

proof: The requirements for a regularization can be fulfilled in $\mathbb{R}^3$. Since they are local requirements, one can fulfill them chart by chart and patch them together. $\square$
Now, the obvious definition for a regularized cylindrical function is:

Definition 7 *Given a cylindrical function a representative $f \circ h(\gamma)$, we call $f \circ h_r(\gamma_r)$ a regularization of a cylindrical function, iff h_r is the expression of a regularized holonomy and γ_r is a regularization of γ.*

(See figure)

Notice, that our definition of a regularized cylindrical function does not imply a particular regularization, but leaves way to take any representative of the equivalence class of regularizations, that result in the same cylindrical function in the limit $\epsilon \to 0$.

Notice that this definition of a regularized graph is diffeomorphism covariant, which means that if γ can be mapped onto γ' by a diffeomorphism ϕ, then $\phi(R, V)$ is a regularization of γ'.

3.4.3 Nicely Stratified Diffeomorphisms

Let us consider the regularizations for a given cylindrical function depending on a graph γ and investigate its transformation properties under a special class of γ-nicely stratified diffeomorphisms. These are defined as follows:

Definition 8 *A stratification $\mathcal{M}$ in the d-dimensional manifold in which γ is embedded into is called γ-**nice**, iff*

1. *the interior of each edge is completely contained in a 3-dimensional stratum*
2. *all of γ (including the vertices) lies in the stratification (i.e. the vertices are allowed to lie in less than d-dimensional strata).*

• *A graph γ' is a* **decomposition** *of a graph γ, if the set of points contained in γ and γ' coincide and each edge of γ is equal to the composition of one or more edges of γ'.*

It follows immediately:

Corollary 2 *There exists a minimal subgroupoid of $D(\Gamma)$ that contains all pairs of graphs (γ_1, γ_2) for which there is a pair of decompositions γ_1', γ_2' and a γ_1'-nicely stratified analytic diffeomorphism ϕ mapping γ_1' onto γ_2'.*

Let us now consider the construction of a particular set of γ-nice stratifications that we want to use in this chapter:

Construction 1 *Given an analytic graph γ, let us label its edges $e_1, ..., e_n$. Given a atlas $\mathcal{A}$ of Σ, let us label its charts $C_1, ..., C_k = (U_1, \phi_1), ..., (U_k, \phi_k)$.*

1. *For an edge e_i denote $C_j(e_i) := \{C_n : 0 < n < j \text{ and } e_i \cap C_n \text{nonempty}\}$. Then for each e_i there exists a smallest j_i such that C_{j_i} covers e_i. We denote $A(e_i) := C_{j_i}(e_i)$.*
2. *There exists an analytic coordinate function ϕ_{ij} for each edge e_i and each chart $C_j \in A(e_i)$, such that $\phi_{ij}^{-1}(e_i|_{C_j}) = \{(0,0,t) : 0 < t < 1\}$ due to the analyticity of e_i and $\mathcal{A}$.*
3. *For each vertex v_m there exists take the smallest n, such that $C_n = (U_n, \phi_n) \in \mathcal{A}$ contains a neighborhood of v_m and fix a $\Delta_m > 0$ such that the open coordinate ball (in the chart C_n) $B(v_m, \Delta_m)$ contains only adjacent edges and no other vertices.*
4. *Due to analyticity of γ, there exist $0 < \delta_m < \Delta_m$ for each vertex v_m, such that the coordinate sphere $S(v_m, r)$ is penetrated by each adjacent edge exactly once for all $0 < r < \delta_m$.*
5. *(For illustration only:) For each vertex v_m and each adjacent edge e_i denote the Voronoi-region of points inside $B(v_m, \delta_m) \setminus \{v_m\}$ that are closest of the edge e_i (in the coordinate system given by the chart C_n) by*

V_{mi}. Denote the connected components of the two-dimensional surfaces of points that have equal C_n-coordinate distance to the edges e_i and e_j by $\{S^k_{mij}\}^a_{k=1}$. Denote the one-dimensional curves that have equal C_m coordinate distance to $e_{i_1}, ..., e_{i_n}$ by $L_{mi_1...i_n}$. By choosing a suitable coordinate function, one can achieve that these curves are equidistant to precisely three edges e_i, e_j, e_k, so without loss of generality we can label the equidistant lines by L_{mijk}.

6. *(For illustration only:) Due to the analyticity of the edges adjacent to v_m, one can find $\delta_m > \epsilon_m > 0$ such that the restriction of each S^k_{mij} to $B(v_n, \epsilon) \setminus \{v_m\}$ contains v_m as a boundary point and that the restriction $\tilde{S}^k_{mij}$ of S^k_{mij} to $B(v_n, \epsilon) \setminus \{v_m\}$ is either diffeomorphic to a punctured disk or a triangle. Denote the intersection point of L_{mijk} with the coordinate sphere $S(v_m, \epsilon)$ by M_{mijk}.*

7. *It is the purpose of the $\tilde{S}^k_{mij}$ to separate the edges e_i and e_j in a small neighborhood of v_m. Let us now construct analytic surfaces that achieve the same: For each e_i, e_j adjacent to v_m with $i < j$ we construct a coordinate function ϕ_{mij} such that $e_i(t) = \phi_{mij}(0, 0, t)$ and $e_j(t) = \phi_{mij}(0, c(t), \pm t)$ for some analytic function $c(t)$. One can choose triangular subsets of the surfaces $S^1_{mij} = \{\phi^{-1}_{mij}(a, a, t) : a, t \in \mathbb{R}\}$ and $S^2_{mij} = \{\phi^{-1}_{mij}(a, -a, t) : a, t \in \mathbb{R}\}$ as well as the two surfaces $S^3_{mij} = \{\phi^{-1}_{mij}(a, \pm\frac{1}{2}c(t), t) : a, t \in \mathbb{R}\}$ and $S^4_{mij} = \{\phi^{-1}_{mij}(a, \pm 2c(t), t) : a, t \in \mathbb{R}\}$, so both e_i and e_j are separated by the boundaries of coordinate pyramids. Denote these triangular subsets of the surfaces collectively by S'.*

8. *Let us start with e_i for which the label i is minimal: There is a subset of the set of all surfaces S' such that e_i is separated from all other edges by choosing subsets of the surfaces S'. Denote this set of subsets of the surfaces S' by $\mathcal{S}_i$ and the neighborhood $e_i \cap B(v_m, \epsilon) \setminus \{v_m\}$ that is bounded by $\mathcal{S}_i$ by $\mathcal{N}_i$. Notice that $\mathcal{N}_i$ does not contain the interior of any other edge.*

9. *Proceed with the edge e_j with next higher label j: There is a subset of the set of all surfaces S' such that e_j is separated from all other edges and the region $\mathcal{N}_i$ by choosing subsets of the surfaces S'. Denote this set of subsets of the surfaces S' by $\mathcal{S}_j$ and the neighborhood $e_j \cap B(v_m, \epsilon) \setminus \{v_m\}$ that is bounded by $\mathcal{S}_j$ by $\mathcal{N}_j$. Notice that $\mathcal{N}_i$ does not contain the interior of any other edge or the region $\mathcal{N}_i$. Repeat this step analogously for all $k > j$ until the edge with the highest label is reached. This constructs sets $\mathcal{S}_k$, $\mathcal{N}_k$ for each adjacent edge e_k, which are mutually disjoint.*

10. *For each adjacent edge e_i choose an analytic coordinate function ϕ_{mi}, such that $e_i(t) = (0, 0, t)$. The surface pieces s_{il} in $\mathcal{S}_i$ have the form $s_{il} = \{(a(t, c), b(t, c), t) : (t, c) \in U_{il} \subset \mathbb{R}^2\}$. Consider the surfaces $s'_{il} = \{(\frac{1}{2}a(t, c), \frac{1}{2}b(t, c), t) : (t, c) \in U_{il} \subset \mathbb{R}^2\}$ and collect their set as $\mathcal{S}'_i$. There exists an $\epsilon'_i > 0$ such that the surfaces $\mathcal{S}'_i$ separate the edge e_i form all other adjacent edges inside the ball $B(v_m, \epsilon'_i)$. The minimum of the ϵ'_i is denoted by ϵ'.*

11. *The neighborhoods of* $e_i \cap B(v_m, \epsilon')$ *that are bounded by the* $\mathcal{S}'_i$ *are denoted by* $\mathcal{R}_i$*. The boundary lines of the surfaces in* S'_i *are analytic because they are intersection lines of analytic surfaces and we denote the set containing these by* $\mathcal{L}_i$*. The set of boundary points of the* $\mathcal{L}_i$ *shall be denoted by* $\mathcal{P}_i$*.*

This construction yields an analytic stratification $\mathcal{M}_m$ of the balls $\phi_m^{-1}(B(v_m, \epsilon'))$ around the vertex by taking the $\mathcal{R}_i$, the $\mathcal{S}'_i$, the $\mathcal{L}_i$, the $\mathcal{P}_i$ and the complement of these in $\phi_m^{-1}(B(v_m, \epsilon'))$ and the partition of the sphere $\phi_m^{-1}(S(v_m), \epsilon')$ into analytic surfaces $\mathcal{E}_m$ and analytic lines $\mathcal{J}_m$, that arises by partitioning this sphere, into surfaces that are bounded by elements of the $\mathcal{L}_i$. Let us now construct an adapted stratification of $\Sigma \setminus (\cup_m \phi_m^{-1}(B(v_m, \epsilon')))$ Let us for this purpose consider the restriction γ_o of graph γ to $\Sigma \setminus (\cup_m \phi_m^{-1}(B(v_m, \epsilon')))$, which is a graph that is completely separated:

Construction 2 *For each edge* e_i *in* γ_o *there exists a narrow tubular region* T_i *which are bounded by analytic boundary* B_i *surfaces in* $\Sigma \setminus (\cup_m \phi_m^{-1}(B(v_m, \epsilon')))$ *together with elements of* $\mathcal{L}_i \cap \mathcal{J}_m$ *and elements* $\mathcal{S}'_i \cap \mathcal{E}_m$*. The tubular regions are supposed to be chosen narrow enough, so they do not mutually intersect.*

Definition 9 *The elements of the* $\mathcal{M}_m$ *and the* T_i*,* B_i *together with the complement of all these sets in* Σ *define the adapted stratification* $\mathcal{M}_\gamma$*.*

Since the constructions 1 and 2 are entirely analytic:

Corollary 3 $\mathcal{M}_\gamma$ *is an analytic stratification.*

Construction 3 *For each graph* γ *with graph* γ_o*(as used in construction 2), there is a graph* γ' *containing all the vertices of* γ *and* γ_o *and replacing each edge* $e_i \in \gamma$ *with the three pieces* $e_i^1, e_i^2 \in \gamma_o$ *and* e_i^3*, such that* $e_i = e_i^1 \circ e_i^2 \circ e_i^3$*.*

By construction we have: The stratification $\mathcal{M}_\gamma$ is γ'-nice and γ' is a decomposition of γ.

Corollary 4 *For each graph* γ *there exists a decomposition* γ' *and a* γ'*-nice analytic stratification of* Σ*.*

3.4.4 Adapted Regularization

Given a graph γ, we can reuse the constructions constructions 1 and 2 to furnish a regularization of γ:

Construction 4 1. *Assume that each edge in* γ *is contained in a single chart, if not, then refine* γ *so each edge in the refinement is contained in one chart.*

2. *Define the regulator* σ_e^1 *for each edge* e *as the composition of the two regions* R_i *containing the initial and final part of* e *and the tubular region* T_i *containing the middle part of* e *as well as the boundary surfaces of between the* $\mathcal{R}_i$ *and* $\mathcal{T}_i$*.*

3. *Since e is contained in a single chart, we can choose a coordinate function* ϕ_i*, such that* $e = \{(0,0,t) : 0 < t < 1\}$ *using cylindrical coordinates* (r,θ,t)*. The regulator takes the form* $\sigma_e^1 = \{(a,b,t) : 0 \leq a \leq r(\theta,t), 0 \leq \theta < 2\pi, 0 \leq t \leq 1\}$*.*

4. *Define* $\sigma_e^\epsilon = \{(\epsilon a,b,t) : 0 \leq a \leq r(\theta,t), 0 \leq \theta < 2\pi, 0 \leq t \leq 1\}$ *and* $i'_\epsilon(\sigma_e^\epsilon) := \{(\epsilon a,b,t) : 0 \leq a \leq r(\theta,t)(1-\frac{\epsilon}{3}), 0 \leq \theta < 2\pi, \frac{\epsilon}{3} \leq t \leq 1-\frac{\epsilon}{3}\}$*.*

5. *Using the coordinate balls* $\phi_m^{-1}(B(v_m,\epsilon'_m))$ *of construction 1, we can define vertex regulators* $v^\epsilon := \phi_m^{-1}(B(v_m,\epsilon'_m\epsilon))$*.*

6. *The* σ_e^ϵ*,* $i_\epsilon(\sigma_e^\epsilon) := i'_\epsilon(\sigma_e^\epsilon) \setminus (\cup_m v_m^\epsilon)$ *for each edge in* γ *together with the* v^ϵ *for each vertex in* γ *define the regularization* $R(\gamma)$*.*

Since the σ_e^ϵ are mutually disjoint we have by construction:

Corollary 5 *$R(\gamma)$ is a regularization of γ.*

Combining these constructions and observing that each stratum contains at most the regulator of one edge, we see:

Corollary 6 *For a graph γ there exists a decomposition γ' of γ, a γ'-nice stratification $\mathcal{M}_\gamma$ and a regularization R of γ' such that each stratum contains at most the regulator of one edge.*

3.4.5 Action of Nicely Stratified Analytic Diffeomorphisms

Let us now consider the action of diffeomorphisms on regularized cylindrical functions f_γ, f_δ depending on minimal graphs γ, δ. For f_γ and f_δ and a regularization for γ, δ, we have physical requirements for calling these functions diffeomorphism-equivalent: There exists a diffeomorphism ϕ_ϵ for all $\epsilon > 0$ such that (1) the regularized dependence of f_γ on the connection has to be mapped onto the regularized dependence of f_δ on the connection, so both functions have the same dependence on the connection **and** (2) the regularization of each vertex in γ has to be mapped onto a regularization of the corresponding vertex in δ and the regions σ_e^ϵ outside the vertex regularizations of γ have to be mapped onto the regions σ_e^ϵ outside the vertex regularizations of δ, so the limit $\epsilon \to 0$ can eventually be taken. This leads to the refinement of definition 3:

Definition 10 *We call two graphs γ, δ* **physically diffeomorphic***, if there exists a regularization $R(\gamma)$ of γ and for each $1 > \epsilon_o > 0$ there is a family of diffeomorphisms $\phi_\epsilon : 1 > \epsilon > \epsilon_o$ such that there is a regularization $R(\delta)$ of δ and for all $1 > \epsilon > \epsilon_o$:*

1. $\phi^\epsilon(i_e^\epsilon(\gamma)) = i_e^\epsilon(\delta)$*, for each internal approximation* $i_e^\epsilon(\gamma) \in R(\gamma)$

2. $\phi^\epsilon(\sigma_e^\epsilon \setminus \cup_m(v_m^\epsilon(\gamma))) = \sigma_e^\epsilon \setminus \cup_m(v_m^\epsilon(\delta))$ *for all regulators* $\sigma_e^\epsilon \in R(\gamma)$ *and all vertex regularizations* $v^\epsilon(\gamma) \in R(\gamma)$

3. $\phi^\epsilon(v_m^\epsilon(\gamma)) = v^\epsilon(\delta)$ *for all vertex regularizations* $v^\epsilon(\gamma) \in R(\gamma)$*.*

Given a graph γ, we have the adapted stratification $\mathcal{M}(\gamma)$. Let us denote by $\mathcal{N}(\gamma)$ the stratification that is obtained from $\mathcal{M}(\gamma)$ by combining the stratification of the vertex balls $\phi_m^{-1}(B(v_m, \epsilon'_m))$ into a single 3-dimensional stratum for each vertex. Using the vertex constructions of the previous sections, we are able to show:

Lemma 5 *Given a graph γ and the adapted stratification $\mathcal{N}(\gamma)$, which is γ'-nice, and a let δ be the image of γ under an $\mathcal{N}(\gamma)$-stratified analytic diffeomorphism, then γ and δ are physically diffeomorphic.*

proof: Perform the construction 4 for γ to obtain a regularization $R(\gamma)$, such that each stratum contains at most the regularization of one edge. For any $1 > \epsilon > \epsilon_o > 0$, the regulators σ_e^ϵ are tubular regions containing tubular subregions $i_\epsilon(\sigma_e^\epsilon)$ contained in a single stratum each. The vertex regularizations v^ϵ are balls around the vertices also contained in a single stratum each. Thus, the $\mathcal{M}$ stratified analytic diffeomorphism acts on the regularization $R(\gamma)$ by mapping analytic tubes onto analytic tubes, analytic subtubes onto analytic subtubes and analytic balls onto analytic balls without changing the topological relations amongst these. Since for the regulated graph $R(\gamma)$ with $\epsilon > 0$, there are at most three measurable tangent vectors at each point, because at most an analytic ball touches an analytic tube, one can achieve this action also by a diffeomorphism. $\square$

This lemma does not help us to get rid of the complications associated to the tangent space structure at the vertex, but the observation that the regulated graph has at most three independent tangent vectors at each point hints:

Lemma 6 *If two analytic graphs γ, δ are isomorphic as knots, then they are physically equivalent.*

proof: Because γ and δ are isomorphic as knots, there exists a homeomorphism h that maps γ onto δ. The problematic points are only the vertices; in other words: except for a small region around the vertices there is no obstruction for a diffeomorphism to map an analytic graph onto a graph in the same iso-knot class. The proof rests now on the observation that $R(\gamma)$ is a regularization of every graph γ' and for $\epsilon_n > 0$ as long as γ' lies in $R(\gamma)$, so taking a suitably adapted diffeomorphism ϕ we can achieve $\phi^{-1}(\beta) = \gamma'$ for a suitable γ' that is regularized by $R(\gamma)$ for $\epsilon > \epsilon_o > 0$. $\square$

Let us now focus on the minimal extension of the analytic diffeomorphisms to obtain a physical diffeomorphism group: As a preparation we need:

Lemma 7 *Given a stratification $\mathcal{M}$, a $\mathcal{M}$-stratified analytic diffeomorphism ϕ and an analytic graph γ, there exists a decomposition γ' of γ, a γ'-nice stratification $\mathcal{M}'$ and a $\mathcal{M}'$ stratified analytic diffeomorphism ϕ' such that $\phi'(\gamma') = \phi(\gamma)$.*

proof: Any edge $e \in \gamma$ can be decomposed into a finite number of pieces $\{e_i\}_{i=1}^n$ which are each contained in a single stratum due to analyticity of both e and the strata in $\mathcal{M}$ and the local finiteness of $\mathcal{M}$; moreover one can refine this decomposition, such that each e_i is contained in a single chart. Compactness of the edges yields that this refinement is finite. Then γ' contains the edges $\cup_{e\in\gamma}\{e_i\}_{i=1}^{n_e}$ and vertices $\cup_{e\in\gamma}\{\partial e_i\}_{i=1}^{n_e}$. Since each e_i lies in a single chart we can use an analytic local coordinate function ϕ_i such that $e_i = \phi_i^{-1}(\{(0,0,t) : 0 < t < 1\})$. For every $0 < t < 1$ let $d_i(t)$ denote the coordinate distance in ϕ_i-coordinates of $\phi^{-1}(e_i(t))$ to the closest point in $\gamma \setminus e_i$, which is a piecewise analytic function with $d(t) > 0$ for all $0 < t < 1$ due to analyticity of γ'. Denote the finite number of points t at which $d_i(t)$ is not analytic by $\{t_{ij}\}_{j=1}^{N}$. Construct the region $R_i := \phi_i^{-1}(\{(x,y,t) : x^2 + y^2 < d_i(t)/a_i, 0 < t < 1\})$. ∂R_i can be decomposed into two zero-dimensional strata $Z_{i1} = \phi^{-1}(0,0,0), Z_{i2} = \phi^{-1}(0,0,1)$, N analytic one-dimensional strata O_{ij} (the coordinate circles around $\phi^{-1}(0,0,t_{ij})$ with coordinate radius $d_i/a_i(t_{ij})$ and $N+1$ analytic two-dimensional strata T_{ik} for the rest. Due to analyticity of γ', one can choose the a_i such that these strata do not mutually intersect for different edges in γ'. The stratification $\mathcal{M}'$ contains the Z_{ij}, O_{ij}, T_{ik} and R_i for each edge in γ' as well as the complement of the union of all these sets in Σ. $\mathcal{M}'$ is by construction γ' nice. The action of ϕ on each $e_i \in \gamma'$ is analytic by construction, so for each e_i there exists a coordinate function ψ_i, such that $\phi(e_i) = \psi_i^{-1}(\{(0,0,t) : 0 < t < 1\})$. Since each e_i lies in a 3-dimensional stratum of $\mathcal{M}'$, we can use the restriction of $\psi_i^{-1} \circ \phi_i$ to R_i to patch an $\mathcal{M}'$-stratified analytic diffeomorphism ϕ' together such that $\phi'(\gamma') = \phi(\gamma)$. $\square$

Using this lemma, we can practically drop the attribute nice for stratifications when talking about the action of stratified analytic diffeomorphisms on analytic graphs. The proof of lemma 6 tells us in light of this that the complete diffeomorphism group contains an extension of the stratified analytic diffeomorphisms. Let us give a precise definition of this extension and prove its validity:

Definition 11 *Let $\{p_i\}_{i=1}^{N}$ be a finite set of points and let v_i^ϵ be a vertex-regularization of p_i, such that the v_i^ϵ do not mutually intersect for $\epsilon \leq 1$. Let $\mathcal{M}$ be a stratification of Σ that contains the points p_i as strata. Let $1 > \delta > 0$ and ϕ_ϵ^δ be a family $1 > \epsilon > \delta$ $\mathcal{M}$ stratified analytic diffeomorphism on $\Sigma \setminus (\cup_i v_\epsilon^i)$ satisfying $\phi_\epsilon(x) = \phi_\delta(x)$ for all $1 > \epsilon > \delta > 0$ and all x in $\Sigma \setminus (\cup_i v_\delta^i)$. Then a homeomorphism ψ is called an* **extended stratified analytic diffeomorphism** *if for any $1 > \delta > 0$ there exists a family ϕ_ϵ^δ that coincides with ψ on $\Sigma \setminus (\cup_i v_\epsilon^i)$.*

Lemma 8 *The complete diffeomorphism group for Loop Quantum Gravity contains the extended stratified analytic diffeomorphisms.*

proof: For the set $\{p_i\}_{i=1}^{N}$ and a graph γ there are two possibilities: (1) $p_i \in \gamma$ then we decompose γ to γ' such that p_i is a vertex, (2) otherwise there will be an $\epsilon > 0$ such that $v_i^\epsilon \cap \gamma = \emptyset$, so the action of the extension of ψ does not act on γ. The proof is now given by observing that there is a regularization of γ' such that the v_i^ϵ are vertex regularizations. $\square$

Lemma 9 *If two analytic graphs γ, δ are isomorphic as knots, then there is an extended stratified analytic diffeomorphism ϕ s.t. $\phi(\gamma) = \delta$.*

proof: Construct the adapted stratification $\mathcal{M}(\gamma)$ and regularization $R(\gamma)$ as described in the previous section. Using an adapted coordinate function ϕ_e s.t. each edge $e \in \gamma$ is $e = \phi_e^{-1}\{(0,0,t) : 0 < t < 1\}$ and ϕ_d for each edge $d \in \delta$ analogously. The family ϕ^ϵ is then constructed by patching the $\phi_d^{-1} \circ \phi_e$ together and restricting them to the outside of the vertex regularizations. $\Box$

Corollary 7 *Physical diffeomorphisms can not change the iso-knot class of a graph.*

proof: By definition, one can reconstruct the iso-knot class of a graph γ regularized by $R^\epsilon(\gamma)$ for any $1 > \epsilon > 0$. $\Box$

Putting these lemmata together and observing that an extended stratified analytic diffeomorphism coincides is in particular a homeomorphism, so it can not change the iso-knot class of a graph, we conclude:

Corollary 8 *Two analytic graphs γ, δ are physically equivalent iff there is an extended stratified analytic diffeomorphism ϕ with $\phi(\gamma) = \delta$.*

3.5 Loop Quantum Diffeomorphism Groupoid

Since we are interested in the complete diffeomorphism orbits of analytic graphs, we can use corollary 8 which tells us that we have to only consider extended stratified analytic diffeomorphism for this purpose.

3.5.1 Definition of the System

Using corollary 8 we can define the complete diffeomorphism group for Loop Quantum Gravity as the smallest group generated by the extended stratified analytic diffeomorphisms. At the beginning of this chapter however, we introduced a subgroupoid of the double groupoid $D(\Gamma)$ of the set of all analytic graphs Γ as a replacement for a specific diffeomorphism group. We are thus able to define the complete diffeomorphism groupoid:

Definition 12 *The smallest subgroupoid of $D(\Gamma)$ that contains all pairs $(\gamma, \phi(\gamma))$, where γ is an analytic graph and ϕ is an extended stratified analytic diffeomorphism is the* **complete diffeomorphism groupoid** *of Loop Quantum Gravity.*

Using corollary 8 and the recalling the definition of the optimal diffeomorphism groupoid, we see

Corollary 9 *The complete diffeomorphism groupoid coincides with the optimal diffeomorphism groupoid.*

The spin network functions are a Hilbert-basis for $L^2(\mathbb{X}, d\mu_{AL})$. Using these, we are able to construct a basis for the diffeomorphism invariant Hilbert space. Using corollary 9 and applying lemma 3 yields:

Theorem 1 *The diffeomorphism invariant Hilbert space of Loop Quantum Gravity is separable.*

Chapter 4

Reduction of a Quantum Observable Algebra

In this chapter, we will construct the framework for extracting cosmology from full Loop Quantum Gravity, which we will apply in the next chapter. This chapter is however more general and focuses on the observation that the extraction of a subsystem from a classical system is conveniently described through the pullback under a Poisson embedding of a reduced classical system into the phase space of a full classical system as described in section 2.1.3. A quantum theory on the other hand is a noncommutative analogue of the classical phase space, so the notion of extracting a subsystem translates roughly into the embedding of noncommutative spaces. (Notice that the C-functor is contravariant, so a noncommutative embedding acts as a pullback.)

We construct a general prescription for the quantization of embeddings of integrable classical systems and then focus on transformation group systems, which may not necessarily be Lie-algebroids. The observation behind this is that an integral groupoid of an integrable Lie-algebroid may be constructed through a groupoid induction module. This procedure is analoguous to the embedding of a space through embedding a vector bundle over that space into a larger vector bundle and then recovering the embedded space using the projection in the larger vector bundle. This can be understood as follows: The quantum description of a vector bundle is given by Hilbert-C^*-modules over a C^*-algebra, which is the quantum analogue of a space. We will then provide a construction of a "sub"-module thereof that is compatible with the Hermitian structure and use the Hermitian structure (which is in physicists terms an operator-valued sesquilinear form) to construct a reduced C^*-algebra.

We keep this chapter largely self-contained, further details on representation theory for C^*-algebras, Morita-equivalence and groupoids can be found in appendix A. For more details on the construction presented in this chapter, we refer to [21], which evolved from this work.

4.1 Reduction of Classical Systems

A quantum reduction should admit a classical limit in which it amounts to a the reduction of classical systems. We will therefore review the reduction of classical systems in this section focusing on Poisson systems, whose description is very similar to the canonical description of quantum systems.

4.1.1 Classical Kinematics

Given a smooth manifold Γ used as phase space, one can define a Poisson structure $\{.,.\} : C^\infty(\gamma) \times C^\infty(\Gamma) \to C^\infty(\Gamma)$ thereon as a bilinear, antisymmetric map that satisfies Jacobi identity and defines through $g \mapsto \{f, g\}$ a (Hamiltonian) vector field for each $f \in C^\infty(\Gamma)$. For two Poisson manifolds $(\Gamma_i, \{.,.\}_i)_{i=1,2}$, one calls an immersion $i : \Gamma_1 \to \Gamma_2$ a Poisson map (and Γ_1 a Poisson submanifold) if

$$\{f \circ i, g \circ i\}_2 = \{f, g\} \circ i. \tag{4.1}$$

It follows that the Poisson structure $\{.,.\}_1$ of a Poisson submanifold is determined by the Poisson structure $\{.,.\}_2$ and that the Hamilton vector fields on Γ_1 coincide with the push-forward on Γ_2, implying the physical consequence that the kinematics of the embedded system coincides with the kinematics of the full system, yielding:

Definition 13 *Classical reduction of a full system $(\Gamma_2, \{.,.\}_2)$ to a reduced system $(\Gamma_1, \{.,.\}_1)$ is the pull-back under a a Poisson map.*

There are however many physically uninteresting reductions, which for example are the embedding of a configuration space into the full phase space without embedding the respective momenta. To define the physically interesting Poisson embeddings, it is useful to describe Poisson manifolds through Lie-algebroids.

4.1.2 Lie-algebroids

Lie-algebroids are the generalization of Lie-algebras in very much the same way as groupoids generalize groups. They are dual to Poisson manifolds, so one can describe a Poisson system through a Lie-algebroid:

Definition 14 *A vecor bundle E over a manifold $\mathbb{X}$ together with a bracket $[.,.]_E$ on $\Gamma^\infty(E)$ is called a Lie-algebroid iff there is a vector bundle morphism $\rho : E \to T(\mathbb{X})$ satisfying Leibnitz rule for all $r, s \in \Gamma^\infty(E)$ and $f \in C^\infty(\mathbb{X})$:*

$$[r, fs]_E = f[r, s]_E + (\rho(r)\, f)s,$$

where $\rho(r)$ is the section $\rho \circ r$ in $T(\mathbb{X})$ and ρ is called the anchor map.

The precise formulation of the duality is: Given a Lie-algebroid $(E, [.,.]_E)$ over $\mathbb{X}$ with anchor ρ then E^* is a graded Poisson manifold with $\{Pol_n(E^*), Pol_m(E^*)\} \to$

$Pol_{n+m-1}(E^*)$ with $\{.,.\}$ being determined through linearity, Leibnitz rule and the relations

$$\{\pi^* f, \pi^* g\} = 0,\ \{\pi^* f, I(r)\} = \pi^*(\rho(r)f),\ \{I(r), I(s)\} = I([s,r]_E),$$

where $f, g \in C^\infty(\mathbb{X})$, $r, s \in \Gamma^\infty(E)$ and I denotes the canonical graded morphism between the symmetric tensor product on E and the polynomials of equal degree on E^*.

This duality can be extended to Poisson maps: If ϕ is a Lie-algebroid morphism[1] then ϕ^* is a Poisson-map on E^*. Not all Poisson maps are Lie-algebroid morphisms, but only those that map the base space (or physically speaking the configuration space) into the base space. Moreover, we are able to define the physically interesting embeddings (called full embeddings) in terms of Lie-algebroids:

Definition 15 *A Lie-algebroid morphism $\phi : E_1 \to E_2$ is full, iff every section in E_2, whose anchor is tangential to the image of the base space of E_1, is in the image of E_1.*
A Poisson map i between E_1^* and E_2^* is full, iff it is dual to a full Lie-algebroid morphism.

This notion ensures physically that all momentum variables that are relevant for the embedded system are included tehrein.

4.1.3 Reduction and Poisson Embeddings and Reduced Sensitivity

The isomorphism classes of Lie-algeboids together with full Lie-algebroid morphisms is a partially ordered set, which extends (by duality) to graded Poisson systems. We call the embedded Lie-algebroid "smaller", while the Lie-algebroid that it is embedded into is called "larger". Similarly, we call the to the larger Lie-algebroid dual Poisson system "full system", while we call the to the smaller Lie-algebroid dual Poisson system "reduced system".

Poisson systems are formulated in terms of observable algebras. While there is a freedom of choice at the classical level whether one considers the phase space and Poisson structure thereon or the algebra of smooth functions on the phase space together with the Poisson bracket between them, it is necessary in to consider the observable algebra in quantum theories due to the noncommutativity of the quantum mechanical phase space. Let us now consider the implications of phase space reductions for observable algebras:

Given an observable f of the full system, one can construct an observable of the reduced system (that is embedded into the full system by i) by considering the pull-back of f under i. This can be applied to the entire observable algebra $\mathfrak{A}$

[1] A Lielagebroid morphism is a vector bundle morphism $\phi : E_1 \to E_2$, such that $\rho_2 \circ \phi = \rho_1$ and $\phi([r,s]_{E_1}) = [\phi(r), \phi(s)]_{E_2}$ for all $r, s \in \Gamma^\infty(E_1)$.

and the reduced algebra $\mathfrak{A}_{red}$ can be recovered using the argument from section 2.3 as:

$$\mathfrak{A}_{red} = \{i^* f : f \in \mathfrak{A}\}. \tag{4.2}$$

This is the quotient of the algebra $\mathfrak{A}$ by the ideal of functions that vanish at the embedding, and as such $\mathfrak{A}_{red}$ inherits the algebraic structure from $\mathfrak{A}$.

We argued in section 2.3 that the physical interpretation of an observable is the outcome of a measurement (that corresponds to the observable) on the physical system. The pull-back under a Poisson-embedding of an observable algebra amounts therefore to restricting the sensitivity of our measurements to the measurements that are available in the embedded system. This allows for a simple interpretation of the embedded system as the by the considered set of measurements "accessible" part of the system while the ideal of observables that vanish at the embedding is precisely the by the considered set of measurements "inaccessible" part of the physical system. The active process of performing the pull-back can therefore be interpreted as reducing the sensitivity of the measurements to the respective subsystem.

4.2 Quantization Strategy

Groupoids are a natural link between classical and quantum systems: A classical system, described by an integrable Lie-algebroid, can be integrated to a Lie-groupoid and the appropriately constructed groupoid C^*-algebra serves as the quantum observable algebra, while the groupoid itself encodes the classical integrable system.

4.2.1 Integrable Lie-algebroids

Given a Lie-groupoid $\mathcal{G}$ over $\mathbb{X}$, we define the vector bundle $E = Lie(\mathcal{G})$, whose fibre at x is the tangent space at $e(x)$ of the s-fiber at x. The Lie-bracket of vector fields on $\mathbb{X}$ reduces to a Lie-bracket between the sections in E and the bundle map $\rho : E \to T(\mathbb{X})$ obtained by restricting the canonical map $T(\mathcal{G}) \to T(\mathbb{X})$ to E serves as an anchor map. This "taking the derivative at the source unit" defines the Lie-algebroid associated to the Lie-groupoid $\mathcal{G}$. $\mathcal{G}$ is called s-simply connected, iff the s-fibres of $\mathcal{G}$ are simply connected.

A Lie-algebroid E is called integrable, if it is the Li-algebroid of a Lie-groupoid. Then there exists a unique s-simply-connected Lie-groupoid $\mathcal{G}$ that integrates E. The integrating groupoid of an integrable Lie-algebroid[2] E can be constructed as a certain homotopy quotient of the path groupoid $P(E)$ of E. We will from now on assume that the classical system can be described by an integrable Lie-algebroid, so we have a unique s-simply connected integrating Lie-groupoid at our disposal.

[2]Notice that although this construction can be performed for any Lie-algebroid E, it does not mean that the such obtained groupoid integrates E.

4.2.2 Quantum Algebras

Let $\mathcal{G}$ integrate the classical system described by the Lie-algebroid E. The unit space of $\mathcal{G}$ is the base space of E, which can be physically interpreted as the configuration space $\mathbb{X}$ of the system. The sections in E are momentum vector fields on $\mathbb{X}$, generated by the Poisson action of observables linear in the momenta. The elements of the groupoid are then finite transformations of a point in $\mathbb{X}$ generated by a momentum vector field, which is the generalization of the action of the momentum-Weyl-operators. The quantum algebra is therefore a groupoid C^*-algebra. For finite dimensional Lie-groupoids, one can construct the groupoid C^*-algebra canonically, while one generally needs a Haar system on the groupoid for its definition. Let us focus on the general case:

Let $d\nu$ be a Haar system on a (locally) compact groupoid $\mathcal{G}$ and consider two continuous functions $f_1, f_2 \in C_c(\mathcal{G})$ of compact support. Then

$$f_1 \star f_2 : g \mapsto \int_{g_1 \circ g_2 = g} d\nu f_1(g_1) f_2(g_2) \tag{4.3}$$

defines a noncommutative convolution product. This algebra can be equipped with an involution given by

$$f^* : g \mapsto \overline{f(g^{-1})}. \tag{4.4}$$

One can define a C^*-norm for this $*$-algebra by considering all $*$-representations π on Hilbert spaces $\mathcal{H}_\pi$ of this $*$-algebra and defining:

$$||f|| := \sup_\pi ||\pi(f)||_{\mathcal{H}_\pi}. \tag{4.5}$$

The completion of the above defined $*$-algebra in this norm is a C^*-algebra, wich we define as the groupoid C^*-algebra. As the natural generalization of the Weyl-algebra of ordinary quantum mechanics, we define this C^*-algebra as the quantum observable algebra associated to the classical Lie-algebroid E that was integrated by $\mathcal{G}$. The construction ensures that there exists a classical limit that recovers E.

4.2.3 Requirements for Quantum Embeddings

An embedding i of a locally compact Hausdorff space $\mathbb{X}$ into a locally compact Hausdorff space $\mathbb{Y}$ defines the pull-back $i^* : C(\mathbb{Y}) \to C(\mathbb{X})$, which is the key observation that we use to construct reduced classical systems. Algebraically, one can describe i as an embedding of the spectrum on $C(\mathbb{X})$ into the spectrum of $C(\mathbb{Y})$. Although it is tempting to use this algebraic statement and replace the commutative algebras with noncommutative quantum observable algebras to construct a quantum embedding, such an embedding has certain pathologies that we described in section 2.3, where we considered examples of this type of embedding that produced counterintuitive results.

However, the conditions for a quantum embedding as explained in section 2.3 resolve these pathologies. Quantum systems are not only determined by an observable algebra, but one needs a Hilbert space representation of the observable algebra to do physics. The matching condition for the vacuum expectation values[3] set out in section 2.3 will turn out to resolve the problem of finding an induced representation of the reduced observable algebra.

The classical limit that is needed in section 2.3 will here be obtained as follows:

$$\begin{array}{cccc} & & \mathcal{G}\text{ module }\mathbb{X} & \\ \text{Haar syst.} & C^*(\mathcal{G},\nu) & \rightarrow & \text{representation} \\ \nu & \uparrow & & \hbar \\ & \mathcal{G} & & \downarrow \\ & \downarrow & & 0 \\ & \mathcal{A}(\mathcal{G}) & \rightarrow & \mathcal{P}(\mathcal{G}) \\ & & \text{dualization} & \end{array} \tag{4.6}$$

This diagram starts from a Lie-groupoid $\mathcal{G}$ with a Haar system ν, from which one can immediately construct a C^*-algebra $C^*(\mathcal{G},\nu)$. We assume a faithful representation of this algebra. The classical limit $\hbar \rightarrow 0$ can then be obtained by reversing the top arrow and forgetting about the Haar system ν, while following the downward arrow from the groupoid to arrive at the Lie-algebroid $\mathcal{A}(\mathcal{G})$ and its dual graded Poisson system $\mathcal{P}(\mathcal{G})$. Given a general quantum system $(\mathfrak{A},\pi,\mathcal{H})$ of an observable algebra $\mathfrak{A}$ together with a Hilbert space representation $(\mathcal{H},\pi)$ thereof, we need to assume that $\mathfrak{A}$ is the C^*-algebra of a Lie-groupoid and that $(\mathcal{H},\pi)$ is faithful.

4.3 General Construction

The observation that allows for the general construction outlined in this section is the close link between Morita-equivalence for groupoids and Morita equivalence for C^*-algebras, which was developed in [18]. If two (transitive) locally compact groupoids with Haar systems are Morita equivalent as groupoids, then the respective groupoid C^*-algebras are Morita equivalent as well. The proof in [18] consists of constructing a Morita-equivalence bimodule for the C^*-algebras as a completion of $C_c(\mathbb{X})$, where $\mathbb{X}$ is the equivalence bimodule for the groupoids. This suggests to construct the reduced observable algebra as a reduced groupoid algebra and using the techniques form Morita theory for C^*-algebras to induce a representation. To line out the general construction, we need two additional observations:

First, if $\mathbb{X}$ is a groupoid induction module for a groupoid $\mathcal{G}(\mathbb{X})$ and if i embeds $\mathbb{Y}$ as a subspace of $\mathbb{X}$, then one can induce a subgroupoid $\mathcal{G}(\mathbb{Y})$ of $\mathcal{G}(\mathbb{X})$.

[3]Every representation of a C^*-algebra can be decomposed into cyclic representations. This decomposition is however not unique. Each cyclic representation can be thought of as a GNS-representation, so when discussing representations of C^*-algebras, we do not loose generality if we restrict ourselves to sums of GNS-representations. Since a GNS-representation is (by definition) already completely determined by the vacuum expectation values of the observable algebra, one can ensure that the representation properties are taken into account, when the vacuum expectation values (in each cyclic summand) coincide for a dense set of elements of the quantum observable algebra.

For the cosntruction, let us revisit the induction of $\mathcal{G}(\mathbb{X})$ form $\mathbb{X}$:

Notice that $\mathbb{X}$ is a space with a groupoid $\mathcal{H}$ acting thereon by an action μ with momentum map ρ. $\mathcal{G}(\mathbb{X})$ is then $\mathbb{X} * \mathbb{X}/\mathcal{H}$, where $\mathbb{X} * \mathbb{X} = \{(x_1, x_2) \in \mathbb{X} \times \mathbb{X} : \rho(x_1) = \rho(x_2)\}$ and $\mathcal{H}$ acts thereon by the diagonal action $h \triangleright (x_1, x_2) = (\mu_h(x_1), \mu_h(x_2))$. The unit space is $\mathbb{X}/\mathcal{H}$ and the groupoid composition is $[x_1, x_2]_{\mathcal{H}} \circ [x_2, x_3]_{\mathcal{H}} = [x_1, x_3]_{\mathcal{H}}$. Let us now consider $\mathbb{X}_o = i(\mathbb{Y})$: Denote the largest subgroupoid of $\mathcal{H}$ with $\rho(s(h_o)), \rho(r(h_o)) \in \mathbb{X}_o$ by $\mathcal{H}_o$, which has a well defined action μ_o, ρ_o on X_o defined by restricting μ to $\mathcal{H}_o \times X_o$ and ρ to $\mathcal{H}_o$. We can now construct $\mathcal{G}(\mathbb{Y})$ as $X_o * X_o/\mathcal{H}_o$, where $X_o * X_o = \{(y_1, y_2) \in X_o \times X_o : \rho_o(y_1) = \rho_o(y_2)\}$ and $\mathcal{H}_o$ acts by the diagonal action on $X_o \times X_o$. The unit space is $X_o/\mathcal{H}_o$ and the groupoid composition law $[y_1, y_2]_{\mathcal{H}_o} \circ [y_2, y_3]_{\mathcal{H}_o}$ is well defined, since it is independent of the representatives (y_i, y_j).

Let us now verify that $I : [y_1, y_2]_{\mathcal{H}_o} \to [i(y_1), i(y_2)]_{\mathcal{H}}$ is a well defined embedding of groupoids: *Independence of Representatives:* Assume that $[y_1, y_2]_{\mathcal{H}_o} = [y_3, y_4]_{\mathcal{H}_o}$ then there exists $h_o \in \mathcal{H}_o$ such that $y_1 = h_o \triangleright y_3$ and $y_2 = h_o \triangleright y_4$. Since μ_o and ρ_o are restrictions of μ and ρ and since h_o as an element of a subgroupoid if $\mathcal{H}$ is in particular an element of $\mathcal{H}$, we see that $[i(y_1), i(y_2)]_{\mathcal{H}} = [h_o \triangleright i(y_2), h_o \triangleright i(y_2)]_{\mathcal{H}} = [i(h_o \triangleright y_1), i(h_o \triangleright y_2)]_{\mathcal{H}} = [i(y_3), i(y_4)]_{\mathcal{H}}$. *Matching:* A similar argument can then be used to see that $I(g^o_1 \circ g^o_2) = I(g^o_1) \circ I(g^o_2)$, since $I([y_1, y_2]_{\mathcal{H}_o} \circ [y_2, y_3]_{\mathcal{H}_o}) = I([y_1, y_3]_{\mathcal{H}_o}) = [i(y_1), i(y_3)]_{\mathcal{H}}$ and on the hand $I([y_1, y_2]_{\mathcal{H}_o}) \circ I([y_2, y_3]_{\mathcal{H}_o}) = [i(y_1), i(y_2)]_{\mathcal{H}} \circ [i(y_2), i(y_3)]_{\mathcal{H}} = [i(y_1), i(y_3)]_{\mathcal{H}}$.

Now, since a viable observable algebra can be constructed as an algebra of functions on the groupoid, it is suggestive to construct subalgebras as functions on the subgroupoid.[4]

Second, let $\mathcal{G}$ be a Lie-groupoid integrating a Lie-algebroid E over a base manifold $\mathbb{X}$ and let $\mathbb{Y}$ be a submanifold of $\mathbb{X}$. One can then very often turn $\mathbb{X}$ into an induction module for $\mathcal{G}$. If we now construct the embedded subgroupoid then it will contain all transformations in $\mathcal{G}$ that transform a point of $\mathbb{Y}$ into a point in $\mathbb{Y}$. The associated Lie-algebroid of the embedded groupoid thus contains all vector-fields that close on $\mathbb{Y}$, so the such constructed embedding is full. Meaning that if $\mathbb{X}$ can be turned into an induction-module for $\mathcal{G}$ then the embedding of the classical system obtained by taking the Lie-algebroid that generates the embedded subgroupoid is full, which we considered as the physically interesting kind of embedding.

4.3.1 Reduced Algebra

The groupoid structure served only as link between the classical and quantum structure, which we want to forget about at the end, so the only data for the quantum system that we consider is $(\mathfrak{A}, \pi, \mathcal{K})$, where $\mathfrak{A}$ denotes the quantum observable algebra and $(\pi, \mathcal{K})$ a (faithful) Hilbert-space representation thereof.

[4]In fact for a system of n paricles in one dimension, one obtains a transformation groupoid $\mathcal{G}(\mathbb{R}^n, \mathbb{R}^n)$ of $\mathbb{R}^n$ acting as translations on $\mathbb{R}^n$. This groupoid can be induced using $\mathbb{R}^n$ together with the trivial groupoid as an induction module. Then embedding any $\mathbb{R}^m$ into $\mathbb{R}^n$, one obtains by this procedure a transformation groupoid of m particles, whenever $m < n$. Similar results hold for more complicated classical systems.

The first step into this direction is to follow [18] and turn the groupoid induction module $\mathbb{X}$ into a quantum induction module by completing $C_c(\mathbb{X})$. For its description, let us adopt the notion $\{\nu_x\}_{x\in\mathcal{G}^{(o)}}$ for the Haar system, where ν_x is a translation invariant measure on the s-fibre of x as well as denoting the induction module space by $\mathbb{X}$, the groupoid thereon by $\mathcal{H}$ and the Haar system on $\mathcal{H}$ by $\{\mu_y\}_{y\in\mathcal{H}^{(o)}}$:

The action of $a \in C^*(\mathcal{G})$ on $f \in E := C_c(\mathbb{X})$ is

$$a \triangleright f : x \mapsto \int_{\mathcal{G}} a(g) f(\mu_{g^{-1}} x) d\nu_{\rho(x)}(g).$$

The $C^*(\mathcal{G})$-valued inner product of two elements $f_1, f_2 \in C_c(\mathbb{X})$ turns out to be:

$$\langle f_1, f_2\rangle_A : g \mapsto \int_{\mathcal{H}} \overline{f_1(xh^{-1})} f_2(\mu_g(x)h^{-1}) d\mu_{\rho(y)}(h),$$

where the integration ranges over all composable elements.

The embedding $i : \mathbb{Y} \to \mathbb{X}$ admits a pull-back i^*, which (under the technical assumptions of continuity and properness) extends to a pull-back from $i^* : C_c(\mathbb{X}) \to C_c(\mathbb{Y})$. Since our goal is to forget about the underlying groupoid structure, we will keep the algebraic data $(E, \langle .,. \rangle_A)$, that encodes the quantum algebra through the action of the span of $\langle .,. \rangle_A$ on E. The pull-back i^* is a linear map[5] between the induction module E and a reduced induction module E_o. Since we forgot about the groupoid $\mathcal{H}$, we need to find a structure that allows for the transfer of $\langle .,. \rangle_A$ as a Hermitian inner product on E_o. If $E_o = C_c(\mathbb{Y})$ and $E = C_c(\mathbb{X})$ then one needs a continuous compact extension $P : E_o \to E$ of the functions of compact support on $\mathbb{Y}$, which is a linear map, such that

$$i^* \circ P = id_{E_o} \text{ and } P \circ i^* = id_{img_P(E_o)}, \tag{4.7}$$

where obviously $E_o = img(i^*)$. Using this map P, we can induce a Hermitian operator-valued inner product for $f_1^o, f_2^o, f \in E_o$ through:

$$\langle f_1^o, f_2^o\rangle_{A_o} : f \mapsto i^*(\langle P(f_1^o), P(f_2^o)\rangle_A P(f)). \tag{4.8}$$

We are thus able to recover the reduced algebra through its action on E_o, whereas E_o can ne recovered as the image of i^*. This allows us to define:

Definition 16 *A* **quantum embedding** *of a C^*-algebra $\mathfrak{A}$ obtained through an induction module $(E, \langle .,. \rangle_A)$ is a pair of maps $(i^* : E \to img_{i^*}, P : img_{i^*} \to E)$ satisfying equation 4.7. The embedded pre-C^*-algebra $\mathfrak{A}_o$ is obtained as the span of the operators defined in equation 4.8.*

To complete this subsection, let us establish the operator correspondence $\mathcal{E}$ between the elements of $\mathfrak{A}$ and $\mathfrak{A}_o$ that we postulated in chapter 2.3:

[5] The map i^* is not only linear but in particular a pre-C^*-algebra morphism, which of course ensures the existence of an embedding map i of the spectra in the commutative case.

For $f_1^o, f_2^o \in E_o$, the inner product (equation 4.8) defines an operator $T_{f_1^o, f_2^o} := \langle f_1^o, f_2^o \rangle_{A_o}$ and this set of operators is by construction dense in $\mathfrak{A}_o$. On the other hand:

$$O : (f_1^o, f_2^o) \mapsto O_{f_1^o, f_2^o} := \langle P(f_1^o), P(f_2^o) \rangle_A$$

defines an operator in $\mathfrak{A}$, so we have a natural association:

$$T_{f_1^o, f_2^o} \leftrightarrow O_{f_1^o, f_2^o}. \tag{4.9}$$

While the range of T is dense in $\mathfrak{A}_o$, this is in general not the case for the range of O in $\mathfrak{A}$. However if $f_1, f_2 \in E$, then $U_{f_1, f_2} := \langle f_1, f_2 \rangle_A$ is again by construction dense in $\mathfrak{A}$. On the other hand using i^* one can associate a T with each U by: $U_{f_1, f_2} \leftrightarrow T_{i^* f_1, i^* f_2}$, so the map $\mathcal{E}$ can be defined as the elementary map

$$\mathcal{E} : U_{f_1, f_2} \mapsto T_{i^* f_1, i^* f_2}, \tag{4.10}$$

defining the quantum reduction map.

4.3.2 Induced Representation

The strength of Morita theory is to induce representations; the similarity of the quantum symmetry reduction to Rieffel induction allows for the application of analogous techniques:

Let us consider a (vacuum) state $\omega(a) := \langle \psi, \pi(a)\psi \rangle_{\mathcal{H}}$ for $\mathfrak{A}$, where $(\pi, \mathcal{H})$ is a representation of $\mathfrak{A}$ on $\mathcal{H}$. This defines in particular a representation of the operators $\langle f_1, f_2 \rangle_A$. Hence, one can define the functional for the dense set $T_{f_1^o, f_2^o}$ in $\mathfrak{A}_o$ as:

$$\omega_o(T_{f_1^o, f_2^o}) := \omega(O_{f_1^o, f_2^o}). \tag{4.11}$$

This functional is linear by construction and can be extended to $\mathfrak{A}_o$ by density. To verify positivity, we consider a positive element $a_o^* a_o$, where $a_o \in \mathfrak{A}_o$ as the limit of $a_k = \sum_{n=1}^k \lambda_n T_{f_{n1}^o, f_{n2}^o}$. Let us consider the associated operators $\mathcal{E}(a_k)$, so

$$\omega_o(a_k^* a_k) = \omega(\mathcal{E}(a_k^*)\mathcal{E}(a_k)) \geq 0, \tag{4.12}$$

because $\mathcal{E}(a_k^*) = \sum_{n=1}^k (\lambda_n O_{f_{n1}^o, f_{n2}^o})^* = \sum_{n=1}^k \overline{\lambda_n} O_{f_{n2}^o, f_{n2}^o} = \mathcal{E}(a_k)^*$. Meaning that ω_o extends to a sate on $\mathfrak{A}_o$.

Since any representation of a C^*-algebra can be decomposed into a direct sum of cyclic representations, one can induce a representation of $\mathfrak{A}_o$ from any representation of $\mathfrak{A}$.

Notice that the very definition of the state ω_o ensures that the matching condition (equation 2.15) is satisfied, when the dense set is taken to be the span of $\langle f_1^o, f_2^o \rangle_A$. The verification of the classical limit condition needs the choice of a particular induction module. Will will complete this in section 4.4.3 for the physically important case of transformation group C^*-algebras.

4.4 Construction for Transformation Group Systems

The structure that underlies many physical systems is a group G of momentum transformations (the flow generated by the Poisson action of momenta) that acts freely and properly as translations on a locally compact configuration space $\mathbb{X}$. Let us denote this action by $\triangleright$. The underlying groupoid structure is a transformation groupoid $\mathcal{G}(\mathbb{X}, G)$, whose elements can be denoted as $\mathcal{G} = \{(x, g) : x \in \mathbb{X}, g \in G\}$ (for further details on transformation groupoids and their C^*-algebra see appendix A.3.1).

4.4.1 Reduced Algebra

A very useful induction module for the groupoid C^*-algebra $C^*(\mathbb{X}, G)$ of a transformation groupoid $\mathcal{G}(\mathbb{X}, G)$ is $E = C_c(\mathbb{X})$. The operator-valued inner product for $f_1, f_2 \in C_c(\mathbb{X})$ is

$$\langle f_1, f_2\rangle_A : (x, g) \mapsto \Delta^{-\frac{1}{2}}(g) f_1(x)\overline{f_2(g^{-1} \triangleright x)}, \tag{4.13}$$

where Δ denotes the modular function on G. The action of these operators on $f \in E$ is

$$\langle f_1, f_2\rangle_A f : x \mapsto f_1(x) \int_G d\mu_H(g)\Delta^{-\frac{1}{2}}(g)\overline{f_2(g^{-1} \triangleright x)} f(g^{-1} \triangleright x). \tag{4.14}$$

Let now $i : \mathbb{X}_o \to \mathbb{X}$ be an embedding, and let $i^* : E \to C_c(\mathbb{X}_o)$ and let P satisfy equation 4.7, then the span of the operators defined in equation 4.8 define the reduced pre-C^*-algebra as operators on $E_o = C_c(\mathbb{X}_o)$ for all $f_1^o, f_2^o, f \in E_o$ by:

$$\langle f_1^o, f_2^o\rangle_{A_o} f : x_o \mapsto \langle P(f_1^o), P(f_2^o)\rangle_A P(f)\,(i(x_o)). \tag{4.15}$$

4.4.2 Induced Representation

It is a result of Rieffel induction that any representation of $C^*(\mathbb{X}, G)$ is unitarily equivalent to a direct sum of the fundamental representations of $C^*(\mathbb{X}, G)$ (as convolution operators) on $L^2(\mathbb{X}, d\mu)$, where the measure $d\mu$ is invariant under the action of G. We can therefore provide a very specific discussion of the induced representations by considering these summands, which can be constructed from the vacuum states ω_μ.

It turns out, due to the existence of an approximate identity of the form $id_\epsilon = \sum_i \langle f_i^\epsilon, f_i^\epsilon\rangle_A$ that any positive element $a^* a \in \mathfrak{A}$ can be written as

$$a^* a = \lim_\epsilon \sum_i a^* \langle f_i^\epsilon, f_i^\epsilon\rangle_A a = \lim_\epsilon \langle g_i^\epsilon, g_i^\epsilon\rangle_A,$$

characterizing positive elements of $\mathfrak{A}$ as the closed span of $\langle f, f\rangle_A$, so one can find a representation of the Hilbert-basis for the GNS-representation in terms of a (completion of) E. The analogue argument holds of for $\mathfrak{A}_o$, since positivity is transferred by equation 4.12.

4.4.3 Properties of Quantum Embeddings

The matching condition (equation 2.15) is satisfied for the transformation groupoid case, because it constitutes a special case of groupoid C^*-algebras described in the previous section. The explicit induction module $C_c(\mathbb{X})$ for transformation groupoid C^*-algebras $C^*(\mathbb{X}, G)$ allows us to investigate the quantum embeddings obtained from a classical embedding $i : \mathbb{X}_o \to \mathbb{X}$ very specifically, particularly the classical limit condition (equation 2.14). Since $\mathbb{X}, \mathbb{X}_o$ are assumed locally compact, one can approximate them with a net of increasing compact subsets. The assumption that G acts freely and properly on $\mathbb{X}$ assures that we can define smaller and smaller neighborhoods of the identity in G as those sets of elements of G that transform at least one point in the neighborhood U_x of x into U_x by making these neighborhoods smaller and smaller. This allows for a construction of an approximate identity (first used in [19]) in $C^*(\mathbb{X}, G)$ indexed by a compact set $C \subset \mathbb{X}$, a neighborhood $U \subset G$ of the unit element and $\epsilon > 0$ with the properties:

$$\begin{array}{rcl} id_{C,U,\epsilon}(x,g) & = & 0 \quad \forall g \text{ outside } U \\ |id_{C,U,\epsilon} - 1| & < & \epsilon \quad \forall x \in C \end{array} \tag{4.16}$$

where a triple $(C_1, U_1, \epsilon_1) \geq (C_2, U_2, \epsilon_2)$ if $C_1 \supseteq C_2$ and $U_1 \subseteq U_2$ and $\epsilon_1 \leq \epsilon_2$. It follows that the limit w.r.t. $\geq$ over these triples furnishes an approximate identity in $C^*(\mathbb{X}, G)$. The key is that this approximate identity can be constructed as

$$id_\alpha = \sum_{i=1}^{n_\alpha} \langle f_i^\alpha, f_i^\alpha \rangle_A, \tag{4.17}$$

where $f_i^\alpha \in C_c(\mathbb{X})$. For the construction of the f_i^α it is important to notice that for any C, U there is a covering U_i of C by a finite number of precompact sets, such that g outside U transfroms any point of U_i outside of U_i. Then there exist continuous functions f_i^α with support on U_i, such that $|\sum_i \langle f_i^\alpha, f_i^\alpha \rangle_A - 1| < \epsilon \forall x \in C$.

The transformation groupoid $\mathcal{G}(\mathbb{X}, G)$ can be induced using the groupoid module $\mathbb{X}$ as follows: Denote the trivial groupoid consisting of the G-orbits $[x]_G$ in $\mathbb{X}$ with $s([x]_G) = r([x]_G) = [x]_G$ by $\mathcal{H}$; this groupoid acts trivially on $\mathbb{X}$ using the momentum map $\mu(x) = [x]_G$. Then $\mathbb{X} \star \mathbb{X}/\mathcal{H} = \mathcal{G}$, thus $\mathcal{H}, \mu$ together with the trivial action provides the groupoid induction structure on $\mathbb{X}$. Let us now consider an embedded subspace $i : \mathbb{Y} \to \mathbb{X}$, then $\mathcal{H}$ acts on $i(\mathbb{Y})$ and induces the reduced groupoid over $\mathbb{Y}$ whose arrows are the transformations of G on $i(\mathbb{Y})$ that close on $i(\mathbb{Y})$. Assuming compatibility of the embedding with the groupoid action turns the reduced groupoid into a transformation groupoid $\mathcal{G}(\mathbb{Y}, G_o)$; we will assume this compatibility from now on.

To verify that the desired classical limit is attained, we need to verify that the reduced C^*-algebra is $C^*(\mathbb{Y}, G_o)$. To do this we use the above described approximate identity and apply the quantum reduction $\mathcal{E}$ directly to each summand $\langle f_i^\alpha, f_i^\alpha \rangle_A$. The reduced summands are of the form $\langle f_o, f_o \rangle_{A_o}$ and if P is chosen compatibly, one can use them to construct an approximate identity for $C^*(\mathbb{Y}, G_o)$. The appropriate choice for P is performed in [21], however a complete proof of general existence is still missing.

4.5 Imposing Constraints

From the point of view of a physicist trying to construct a quantum theory, a constrained quantum theory is a replacement of solving a classical constraint system and finding a quantization thereof by finding an annomaly-free quantization of the classical system together with the constraints, as it was first proposed by Dirac. All representations of transformation group systems are unitraily equivalent to direct sums of fundamental representations of $L^2(\mathbb{X}, d\mu)$, which is in turn a completion of the induction module $C_c(\mathbb{X})$.

More specifically, the gauge transformations generated by the constraints are implemented as a group G of unitary transformations on the kinematic Hilbert space $\mathcal{K}$, represented by U. The group-averaging proposal then constructs the inner product by integrating with a translation-invariant measure $d\mu$ over G to obtain a gauge-invariant inner product for $\phi, \psi \in \mathcal{K}$ as:

$$\int_G d\mu(g) \frac{1}{V} \langle \phi, U_g \psi \rangle_{\mathcal{K}} =: \langle \eta(\phi), \eta(\psi) \rangle_{inv.} \tag{4.18}$$

defining the gauge-invariant inner product for the gauge orbits $\eta(\phi), \eta(\psi)$ of ϕ, ψ respectively, where V denotes a normalization constant given by the size of the orbit. However, if $\mathcal{K}$ splits into a direct sum $\oplus_\alpha \mathcal{K}_\alpha$, so $U_g : \mathcal{K}_\alpha \to U_g^\alpha \mathcal{K}_{g(\alpha)}$ and if one fixes precisely one α_o in each G-orbit, and if $R_\alpha(\phi_\alpha)$ is the U_α-average of ϕ_α, then

$$\eta'(\phi_\alpha) := R_{\alpha_o}(\phi_{\alpha_o}) \tag{4.19}$$

allows for a description of the gauge-invariant product in terms of the kinematic inner product:

$$\langle \eta(\phi), \eta(\psi) \rangle_{inv.} = \langle \eta'(\phi), \eta'(\psi) \rangle_{\mathcal{K}}. \tag{4.20}$$

Let us now consider the gauge-invariant matrix elements of $\langle ., . \rangle_A$. The structure of the fundamental representation of $\langle ., . \rangle_A$ is

$$\langle f_1, f_2 \rangle_A \psi : x \mapsto f_1(x) \int_G d\mu_H(g) \overline{f_2(g^{-1} \triangleright x)} \psi(g^{-1} \triangleright x),$$

using equation 4.20 and the fundamental representation on $L^2(\mathbb{X}, d\nu)$ yields:

$$\begin{aligned} &\langle \eta(\phi), \pi_o(\langle f_1, f_2 \rangle_A) \eta(\psi) \rangle_{inv.} \\ &= \textstyle\int_{\mathbb{X}} d\nu(x) \overline{(\eta'(\phi))(x)} f_1(x) \int_G d\mu(g) \overline{f_2(g^{-1} \triangleright x)} (\eta'(\psi))(g^{-1} \triangleright x), \end{aligned} \tag{4.21}$$

which vanishes, whenever $\int_G d\mu(g) \overline{f_2(g^{-1} \triangleright x)} (\eta'(\psi))(g^{-1} \triangleright x) = 0$ for all x. Moreover, if one finds a set $J = \{\psi_i \in C_c(\mathbb{X})\}_{i \in \mathcal{I}}$ dense in $L^2(\mathbb{X}, d\nu)$ and $C_c(\mathbb{X})$ such that $\int_G d\mu(g) \overline{\psi_i(g^{-1} \triangleright x)} (\eta'(\psi_j))(g^{-1} \triangleright x)$ is independent of x, then one obtains that the matrix element vanishes if $\int_{\mathbb{X}} d\nu(x) \overline{\phi(x)} f_1(x)$ vanishes. Having a Hilbert-basis for $L^2(\mathbb{X}, d\mu)$ then allows for the split of the induction module $E = E_1 \oplus E_2$ such that $F_1(f) \in E_1$ is orthogonal to the span of $\eta'(\mathcal{K})$, then one can cnstruct the gauge-invariant observables through the span of

$$\langle F_1(f), \psi_i \rangle_{\mathfrak{A}}, \text{ where } f \in E_1 \text{ and } \psi_i \in J. \tag{4.22}$$

The Gauss- and diffeomorphism- constraint in Loop Quantum Gravity can be treated precisely in this way. The procedure described here can of course be generalized for the case that $\int_G d\mu(g)\overline{\psi_i(g^{-1} \triangleright x)}(\eta'(\psi_j))(g^{-1} \triangleright x)$ is not independent of x, in which case one has a split $E = E_1^i \oplus E_2^i$ depending on the function ψ_i.

Having such a characterization of the gauge-invariant observables, one can construct the gauge invariant reduced algebra by restricting the f_1, f_2 in equation 4.8 to E_1 and J repsectively. The application to the scalar constraint proceeds along the same line, the only difference is that instead of using the group averaging procedure, one inserts the joint kernel projection P for the scalar constraint set explicitly, i.e. one considers $\langle u, P\langle f_1, f_2\rangle_A Pv\rangle_{diff}$. It then turns out that split E into $E_1 \oplus E_2$ resp $J \oplus J_\perp$ is possible.

Chapter 5

Cosmological Reduction of Loop Quantum Gravity

We will now apply the quantum reduction technique developed in the previous chapter to Loop Quantum Gravity and extract a cosmological sector. For this procedure to be applicable to Loop Quantum Gravity, we need to slightly modify Fleischhacks Weyl-algebra. This modification is however only technical, the representations are on the same Hilbert space and any finite number of matrix elements of this algebra coincides with Fleischhacks Weyl-algebra. The resulting theory shares the discrete structure with standard Loop Quantum Cosmology, we are however not able to induce a meaningful dynamics through the procedure described in the previous chapter for the treatment of constraints. We interpret this as a shortcoming of the dynamics of standard Loop Quantum Gravity, which we used for the induction.

5.1 Considerations

Standard Loop Quantum Cosmology can be obtained as follows: One starts with the classical gravitational phase space, imposes Bianchi symmetry on the classical phase space, uses the remaining kinematic gauge symmetries to fix a coordinate system for the reduced phase space and to induce the Poisson structure in terms of these coordinates. The second step is ambiguous: One chooses a set of elementary observables in the full theory, whose classical counterparts separate the points in the reduced phase space and whose Poisson brackets matche the reduced Poisson bracket. Third, one uses the Hilbert-space representation of the full theory and the correspondence of observables between reduced and full theory to induce a Hilbert space representation of the observables of the reduced theory. Finally, one quantizes the scalar constraint for the reduced theory using the same methods as in the quantization of the full theory, which is again an ambiguous procedure.

Goal

The goal of this chapter is to introduce more structure into the construction of the reduced quantum theory and induced Hilbert space representation thereof by applying the construction that we developed in the previous chapter. This construction puts the noncommutative quantum phase (i.e. the observable algebra) into the foreground and therefore separates the process of "quantizing" from the process of "symmetry reducing". We are therefore able to consider the ambiguities in the process of symmetry reduction without having the picture blurred by the effects of quantization (compare subsection 5.4.2). At the end of the construction we want to identify standard Loop Quantum Cosmology with a way of imposing Bianchi symmetry in the quantum theory and thus explicitly see the choices that may be possible.

Strategy

The strategy used in this section differs from the one we presented in [22]: The technical problem that we are faced with is that we need an observable algebra that can be constructed as the induced algebra of an induction module constructed on (a possible enlargement of) an algebra of functions on a quantum configuration space. There is however no known way to construct a viable kinematic observable algebra for Loop Quantum Gravity, that can be induced in this fashion. There is however an algebra of diffeomorphism invariant observables that can be induced in this way. We will therefore first carefully construct this algebra denoted by $\mathfrak{B}_o$ in section 5.3.2. This construction needs a partial gauge fixing of the diffeomorphism symmetry of cylindrical functions, which will allow us to fix them to have graphs that are embedded into a scaffold consisting of a countable set of edges. We are then able to use the span of a subset of the cylindrical functions on the scaffold as an induction module for the diffeomorphism-invariant algebra of Loop Quantum Gravity.

The construction of the quantum embedding is then based on the observation that spin network functions on the scaffold are in particular functions on the classical configuration space. Using the pull-back under the embedding of the reduced configuration space into the full configuration space and partially inverting this linear map lets us thus construct a quantum embedding of the type considered in the previous chapter. We will use this embedding to apply the construction considered in the previous chapter to construct the reduced algebra and an induced representation thereof.

5.2 Adapted Observable Algebra for Loop Quantum Gravity

Fleischhack's Weyl-algebra for Loop Quantum Gravity is generated by elements of the form $f \circ w$, where f is a cylindrical function of the connection and w is a unitary element ($w^* w = 1$) of the group of exponentiated momentum observables. The set $\{f \circ w : f \text{ cylindrical function}, w \text{ momentum Weyl operator}\}$ turns out to be dense in the observable algebra $\mathfrak{A}$, because the momentum Weyl operators act as pull-backs under homeomorphisms in the quantum configuration space $\mathbb{X}$ on cylindrical functions ($w^* f w = \theta^*_w f$). This algebra is represented on $\mathcal{H} = L^2(\mathbb{X}, d\mu_{AL})$, where $d\mu_{AL}$ is the Ashtekar-Lewandowski measure, by representing the cylindrical functions as multiplication operators and the momentum Weyl-operators as the aforementioned pullbacks under homeomorphisms on $\mathbb{X}$. The gauge-variant spin network functions SNF furnish a Hilbert-basis in $\mathcal{H}$. A more detailed description of Fleischhack's algebra can be found in appendix B.3.

This algebra has (heuristically) a resolution of unity in terms of gauge-variant spin network functions given by the observation that for $\psi \in L^2(\mathbb{X}, d\mu_{AL})$ one has $\psi(A) = id\psi(A) = \sum_{T \in SNF} T(A) \int_{\mathbb{X}} d\mu(A) T^*(A)\psi(A) = \sum_{T \in SNF} |T^*\rangle\langle T^*, \psi\rangle$. This resolution of unity however extends over an over-countable sum, making it unusable for our purposes. One can however use this insight to construct an approximate identity using the partially ordered set consisting of the pairs (γ, n), where γ is a graph and n is the maximum spin label and $(\gamma_1, n_1) \geq (\gamma_2, n_2)$ iff $\gamma_1 \geq \gamma_2$ and $n_1 \geq n_2$, so

$$id = \lim_{\leftarrow(\gamma,n)} id_{\gamma,n} = \lim_{\leftarrow(\gamma,n)} \sum_{T \in SNF(\gamma,n)} |T^*\rangle\langle T^*|, \tag{5.1}$$

where $SNF(\gamma, n)$ denotes the spin network functions (including the trivial ones) on γ, with maximal spin label n, which is a finite set for each (γ, n). This approximate identity allows us to write an element $\mathfrak{A}$ as $a = \lim_{\leftarrow(\gamma,n)} a\, id_{(\gamma,n)}$, so:

$$a = \lim_{\leftarrow(\gamma,n)} \sum_{T \in SNF(\gamma,n)} |aT^*\rangle\langle T^*|, \tag{5.2}$$

where we observe that for every element a in the dense subalgebra of $\mathfrak{A}$ consisting of $a = \sum_{i=1}^n f_{\gamma_i} \circ w_i$ every graph γ there exists always a graph $\gamma_a \geq \gamma$, such that $aSNF_{\gamma_a} \subseteq SNF_{\gamma_a}$. Since the action of a on $SNF(\gamma_a)$ reduces to a transformation groupoid action, and due to the convergence of the expansion of cylindrical functions on γ_a in $SNF_{\gamma_a,n}$ for $n \to \infty$, we obtain that we are able to write a as:

$$a = \lim_{\leftarrow\gamma} \sum_{f^1_\gamma, f^2_\gamma \in Cyl(\gamma)} |f^1_\gamma\rangle\langle f^2_\gamma|. \tag{5.3}$$

The algebra $\mathfrak{B}$ of all elements on $\mathcal{H}$ that can be obtained as finite-norm operators through equation 5.3 contains $\mathfrak{A}$, it is however not $\mathfrak{A}$. This is due to the fact that the operators in $\mathfrak{A}$ act cylindrically consistent and the following: The momentum Weyl-group contains Weyl-operators across zero-dimensional quasi-surfaces S_x (at a point x) that can be thought of as elementary. These Weyl-operators act the same on all edges e_x that originate at x and have the same linear structure at x. These two conditions constrain the projective limit $\leftarrow \gamma$. Since $\mathfrak{A}$ is a closed linear subspace of $\mathfrak{B}$, we can formally write $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{C}$ and formally define a linear restriction map $R : \mathfrak{B} \to \mathfrak{A}$, such that $R|_{\mathfrak{A}} = id_{\mathfrak{A}}$ and $R|_{\mathfrak{C}} = 0$[1].

The action of the diffeomorphisms $\mathcal{D}$ on $\mathfrak{A}$ is explained in appendix B.3, and we are able to define an action of $\mathcal{D}$ on $\mathfrak{B}$, whose restriction to $\mathfrak{A}$ coincides with the action on $\mathfrak{A}$, by defining for a diffeomorphism ϕ:

$$\begin{array}{rcl} U_\phi^* b U_\phi & := & \lim_{\leftarrow\gamma} \sum_{f_\gamma^1, f_\gamma^2 \in Cyl(\gamma)} U_\phi^* |f_\gamma^1\rangle\langle f_\gamma^2| U_\phi \\ & = & \lim_{\leftarrow\gamma} \sum_{f_\gamma^1, f_\gamma^2 \in Cyl(\gamma)} |f_{\phi^{-1}(\gamma)}^1\rangle\langle f_{\phi^{-1}(\gamma)}^2|. \end{array} \tag{5.4}$$

This allows us to calculate the matrix-elements of diffeomorphism-invariant observables, which are constructed from elements of $\mathfrak{A}$, but generally lie outside $\mathfrak{A}$, but if bounded within $\mathfrak{B}$. There are many other physically interesting operators that are not in $\mathfrak{A}$, but in $\mathfrak{B}$, e.g. the family of exponentiated volume operators. This is the physical reason, why we will consider $\mathfrak{B}$ in this chapter. In the next section, we will construct diffeomorphism-invariant elements of $\mathfrak{B}$ (i.e. elements in $\mathfrak{B}/\mathcal{D}$) and construct a cosmological quantum embedding for these thereafter.

5.3 Scaffold for Loop Quantum Gravity

We are interested in constructing an embedding for Bianchi I cosmology, so the spatial topology of $\Sigma = \mathbb{R}^3$, and we fix once and for all global homogeneous chart (U, ϕ) in which the generators of the translation invariance are supposed to take the form $\{\partial_i\}_{i=1}^3$ as well as a number $l_o > 0$.

5.3.1 Construction of the Scaffold

The scaffold is a lattice that is large enough, so any knot-class of any graph can be constructed as a combination of edges in the scaffold. Let us use the shorthand

$$((i_1, i_2, i_3), (f_1, f_2, f_3)) := \{\phi(i^a + t(f^a - i^a)) \in \Sigma : 0 \leq t \leq 1\} \tag{5.5}$$

to quickly describe the edges of the scaffold. The first set is

$$\begin{array}{rcl} e_{abc}^1 & := & ((l_o a, l_o b, l_o c), (l_o a + l_o, l_o b, l_o c)) \\ e_{abc}^2 & := & ((l_o a, l_o b, l_o c), (l_o a, l_o b + l_o, l_o c)) \\ e_{abc}^3 & := & ((l_o a, l_o b, l_o c), (l_o a, l_o b, l_o c + l_o)), \end{array} \tag{5.6}$$

where $a, b, c \in \mathbb{Z}$. These edges form a regular cubical lattice and meet at the vertices $v_{abc} = \phi(l_o a, l_o b, l_o c)$. To accommodate for vertices with valence higher than six, we need to introduce "extra bridges" l_{abcn} (where $a, b, c \in \mathbb{Z}$ and $a > n \in \mathbb{N}$), which can be constructed as a rotation of the following concatenation

$$b_{abcn} := e_{abc}^2 \circ e_{a,b+1,c}^1 \circ e_{a+1,b+1,c}^1 \circ ... \circ e_{a+n,b+1,c}^1 \circ e_{a+n,b,c}^2$$

around the axis $x_2 = bl_o, x_3 = cl_o$ with the angle $\alpha_{an} = \frac{\pi}{2}\frac{n}{a^2}$. Clearly, all $\alpha_{an} : 0 < n < a$ are distinct and so none of he bridges l_{abcn} will have interior intersections with one another or with the lattice.

[1]The particular choice of $\mathfrak{C}$ is ambiguous and hence is the choice of R. Knowing a particular map R would reveal considerable insight into the representation theory of $\mathfrak{A}$.

Definition 17 *The* **scaffold** *consists of all vertices* v_{abc} *and of all edges* e^i_{abc}, l_{abcn}*, where* $a, b, c \in \mathbb{Z}$*,* $a > n \in \mathbb{N}$ *and* $i = 1, 2, 3$*.*

We need four important observations about the scaffold: (1) it is not a graph, because it consists of an infinite number of edges and (2) for every $n, m > 0$ there is a set of m vertices in the scaffold with valence greater than m and (3) the scaffold does not contain an accumulation point of edges or vertices and (4) the edges of the path groupoid are oriented, i.e. $((x_i, y_i, z_i), (x_f, y_f, z_f))$ is understood to go from (x_i, y_i, z_i) to (x_f, y_f, z_f).

Let us prove that any knot-class of a graph can be embedded into the scaffold by giving an explicit construction for a graph γ by considering a projection of γ:

1. Choose an explicit enumeration for the vertices and edges of γ, i.e. $V_\gamma = \{v_1, ..., v_n\}$ and $E_\gamma = \{e_1, ..., e_m\}$.
2. Let $k = n + 2m$ and embed the vertices as $i : v_n \mapsto v_{k+n,0,0}$.
3. Split each edge $e_a \in E_\gamma$ into three parts $e_a = e^i_a \circ e^m_a \circ e^f_a$. This splitting can be chosen such that in the considered projection of γ, there are only crossings of sections e^m_a, but no crossings involving e^i_a or e^f_a.
4. Now one can extend i such that the additional vertices v_{ai} and v_{af} (arising as the endpoints of the e^i_a and e^f_a respectively) of additional vertices are embedded into $v_{k+n+1,0,0}, ..., v_{2k,0,0}$. Moreover, one can find unique bridges l_{abcn} to connect $i(v_{ai})$ with $i(v_a)$ and $i(v_{af})$ with $i(v_i)$, which defines the embedding i of the e^i_a and e^f_a.
5. Define a parallel projection $P : (x, y, z) \mapsto (x, y)$ that assigns an overpass of an edge segment that contains (x_1, y_1, z_1) over an edge segment that contains (x_2, y_2, z_2) whenever $x_1 = x_2, y_1 = y_2$ and $z_1 > z_2$. Calculate the projection of the embedding $i(e^i_a), i(e^f_a)$, which generally contains nontrivial overpasses. This defines a braid B_1 of the $i(e^i_a), i(e^f_a)$.
6. Consider the original projection of γ with very small balls around the vertices of γ removed. This defines a braiding B_2 of the e^m_a.
7. Notice that any braid with fixed boundaries can be embedded into a cubical lattice. So attach such an embedding of to the inverse braid of B_1 and then B_2 in the cubical lattice part of the scaffold, thus extending i to an embedding of the pieces e^m_a.

This construction shows that there exists an embedding of the knot-class of any graph into γ. This implies that for any γ there exists a smallest coordinate cube $R(\gamma)$ with center at the coordinate origin that contains a knot-class embedding of γ.

5.3.2 Diffeomorphism-invariant Observable Algebra

Let us now apply the construction described in 4.5 for the diffeomorphism-invariant observables in $\mathfrak{B}$ using the result of chapter 3, i.e. that for any pair (γ_1, γ_2) of graphs there exists an element ϕ of the complete diffeomorphism group mapping $\phi(\gamma_1) = \gamma_2$ if and only if γ_1 and γ_2 are equivalent as knots. Denoting the Hilbert-space completion of the span of nontrivial[2] spin network functions on a graph γ by $\mathcal{K}_\gamma$, one can split the action U of a diffeomorphism ϕ on $\psi_\gamma \in \mathcal{K}_\gamma$ into a graph-changing action $U^c : \mathcal{K}_\gamma \to \mathcal{K}_{\phi(\gamma)}$ and a subsequent graph symmetry $U^s : \mathcal{K}_{\phi(\gamma)} \to \mathcal{K}_{\phi(\gamma)}$, so we can use equation 4.20 to represent the diffeomorphism-invariant matrix element of an element $a \in \mathfrak{B}$ between two diffeomorphism-averaged spin network functions $\eta(T_\gamma)$ and $\eta(T'_{\gamma'})$ by

$$\langle \eta(T_\gamma), a\eta(T'_{\gamma'})\rangle_{inv.} = \langle \eta'(T_\gamma), a\eta'(T'_{\gamma'})\rangle_{\mathcal{K}} \tag{5.7}$$

where we observe that the graph symmetries of γ are a finite group, so η' reduces to averaging over this finite group. Let $\mathcal{H}_\gamma$ denote the η'-image of $\mathcal{K}_\gamma$, then the spin network functions $sSNF_\gamma$, that assign the same spin quantum numbers to each graph-symmetry-related edge (and vertex) in γ, is a Hilbert-basis for $\mathcal{H}_\gamma$. Thus, fixing precisely one γ' in each knot class, one has a construction for the image of η' through:

$$\eta' K_\gamma = \mathcal{H}_\gamma. \tag{5.8}$$

The matrix elements of an element $a = \lim_{\leftarrow\gamma} \sum_{f^1_\gamma, f^2_\gamma \in Cyl(\gamma)} |f^1_\gamma\rangle\langle f^2_\gamma|$ in $\mathfrak{B}$ in the inner product (equation 5.7) coincide with the matrix-elements of $a_\infty = \lim_{n\to\infty} \sum_{f^1, f^2 \in \oplus_{i=1}^n \mathcal{H}_{\gamma'(n)}} |f^1\rangle\langle f^2|$, where we used the observation that the knot classes of graphs are countable, thus allowing for an numberable set of representatives $\gamma'(n)$. Thus, using equation 5.7 for the diffeomorphism-invariant Hilbert space, we can represent the diffeomorphism-invariant elements of $\mathfrak{B}$ as the closure of the finite sums

$$a_n = \sum_{f^1, f^2 \in \oplus_{i=1}^n \mathcal{H}_{\gamma'(n)}} |f^1\rangle\langle f^2| \tag{5.9}$$

in the Hilbert norm defined through the inner product of equation 5.7 as $n \to \infty$. This algebra $\mathfrak{B}_o$ is the diffeomorphism-invariant observable algebra of Loop Quantum Gravity, that we want to consider subsequently. We could have boldly defined the diffeomorphism-invariant algebra $\mathfrak{B}_o$ as a starting point, but then the relation to the standard algebra of Loop Quantum Gravity would not have been clear.

Let us consider an induction module for $\mathfrak{B}_o$: $sSNF_{\gamma'(n)}$ is dense in $\mathcal{H}_{\gamma'(n)}$, so let us define E as the subspace of $\mathcal{K}$ given by the span of the $sSNF_{\gamma'(n)}$ for all $n \in \mathbb{N}$, then E together with the operator-valued inner product:

$$\langle e_1, e_2\rangle_{\mathfrak{B}_o} : e_3 \mapsto e_1\langle e_2, e_3\rangle_{\mathcal{K}} \tag{5.10}$$

[2] I.e. precisely those spin network functions that assign a nontrivial representation to each edge of γ.

is an induction module for $\mathfrak{B}_o$, which follows directly from the construction of $\mathfrak{B}_o$. Notice: The elements of E are in particular functions of the connection.

We will now make use of the scaffold for an explicit construction of η': Let us consider a graph γ, then we saw in the previous section that there exists a smallest coordinate cube $R(\gamma)$ centered around the coordinate origin, such that the knot-class of γ can be embedded into the restriction of the scaffold to $R(\gamma)$. We can therefore fix an embedding i for each knot-class into the scaffold into the vicinity of the coordinate origin. This is practically only feasible for certain small graphs, for large graphs Γ, we will have to assume that we take the average over all possible embeddings $i(\Gamma)_1, ..., i(\Gamma)_{n(\Gamma)}$ into $R(\Gamma)$. The map η' then acts on f_Γ by:

$$\eta'(f_\gamma) := \frac{1}{n(\Gamma)} \sum_{k=1}^{n(\Gamma)} f_{i_k(\Gamma)}, \tag{5.11}$$

which is automatically symmetric under graph symmetries.

5.4 Quantum Embedding for Cosmology

Having an induction module for the diffeomorphism invariant observable algebra of Loop Quantum Gravity, we can apply the construction for quantum embeddings defined in the previous chapter. Moreover having a scaffold depending on straight lines, one can calculate the dependence of the matrix-elements of holonomies along scaffold elements on the isotropic resp. locally rotationally symmetric connection component explicitly. Using (1) the embedding of these symmetric connections into the full configuration space and (2) viewing the elements of the induction module for the diffeomorphism-invariant algebra in the previous section as functions of the connection, we are able to construct a quantum embedding. We will construct the simplest one of these embeddings.

5.4.1 Embedding Maps

We continue to use the global homogeneous chart. In this chart, a symmetric connection takes the form:

$$A_{hom} = \Lambda^I_a dx^a \tau_I. \tag{5.12}$$

Given a straight line segment $e = ((i_1, i_2, i_3), (f_1, f_2, f_3))$, we calculate the holonomy along this line as

$$h_e(A_{hom}) = \mathbb{I}\cos(\frac{L}{2}) + 2\hat{n}^I \tau_I \sin(\frac{L}{2}), \tag{5.13}$$

where

$$\begin{array}{rcl} L &=& \sqrt{\sum_{I=1}^3 \left((f^a - i^a)\Lambda^I_a\right)^2} = ||e^a \Lambda_a|| \\ \hat{n}^I &=& \frac{(f^a - i^a)\Lambda^I_a}{L} = \hat{e}^a \Lambda^I_a. \end{array}$$

This expression can be used to evaluate the dependence of all holonomies along elements of the scaffold. Given a general homogeneous connection, we can always rotate the chart and apply a gauge transformation, such that it takes diagonal form:

$$A^{diag}_{hom} = a\,dx^1\tau_1 + b\,dx^2\tau_2 + c\,dx^3\tau_3.$$

Assuming local rotational symmetry around the x_1-axis resp. isotropy, one can further simplify the connection:

$$\begin{array}{rcl} A_{LRS} & = & a\,dx^1\tau_1 + c\,dx^2\tau_2 + c\,dx^3\tau_3, \\ A_{iso} & = & c\,dx^1\tau_1 + c\,dx^2\tau_2 + c\,dx^3\tau_3. \end{array}$$

Notice that the graphs in the scaffold can be decomposed into a finite set of straight elementary pieces of coordinate length l_o that are either parallel to the x_1-axis or inside a $x_1 = const.$ plane. The matrix-elements calculated in equation 5.13 for these elementary pieces are then linear combinations of products of $e^{\pm\frac{i}{2}l_o c}$ and $e^{\pm\frac{i}{2}l_o a}$. Since the holonomy along the concatenation of elementary paths is matrix product of the holonomies along the elementary paths and the holonomy along the inverse path is the inverse matrix of the holonomy, we obtain that the dependence of a holonomy along any finite path e in the scaffold depends on the symmetric connections as a finite sum of the form

$$\begin{array}{rcl} h_e(A_{LRS}) & = & \sum_{n,m=1}^{N,M} \xi_{nm} e^{\frac{i}{2}l_o(ma+nc)} \\ h_e(A_{iso}) & = & \sum_{n=1}^{N} \xi_n e^{\frac{i}{2}l_o\,nc}, \end{array} \tag{5.14}$$

with normalization constants ξ and $n, m \in \mathbb{Z}$.

These preparations allow us to construct a map p, for the cylindrical functions on the scaffold. The quantum embedding map p will then be the restriction of this map to the induction module E for the algebra $\mathfrak{B}_o$ of diffeomorphism invariant observables of Loop Quantum Gravity, using that E is a subset of the cylindrical functions on the scaffold. We will work with the dense set of gauge-variant spin network functions on the scaffold. We defined a spin network function to be a function on the connection of the form $T_\gamma = \prod_{e\in\gamma}\rho^{j_e}_{m_e n_e}(h_e(A))$. Since any matrix element of representation ρ^j_{mn} is a polynomial of the matrix elements of the fundamental representation, we can use equation 5.14 to determine the dependence of T_γ on the symmetric connection, whenever γ is in the scaffold as finite sums:

$$\begin{array}{rcl} T_\gamma(A_{LRS}) & = & \sum_{n,m=1}^{N,M} \xi_{nm} e^{\frac{i}{2}l_o(ma+nc)} \\ T_\gamma(A_{iso}) & = & \sum_{n=1}^{N} \xi_n e^{\frac{i}{2}l_o nc}, \end{array} \tag{5.15}$$

which is of the same form as equation 5.14. We may however have restrictions in the normalization constants ξ. It will turn out to be convenient to use an embedding of the knot classes of graphs into the scaffold, where the ξs in equation 5.15 are unity. This is easily achieved for the isotropic connection by using any embedding of the simplest nontrivial graph that consists of one edge only into $e_o = ((0,0,0),(0,0,1))$, because $\rho^n_{\pm n,\pm n}(h_{e_o}(A)) = e^{\pm\frac{i}{2}l_o nc}$. Here, we used the indexing of the matrix elements of the $SU(2)$-representations ranging over

$n, m = -j, -j+1, ..., +j$. The situation for the locally rotationally symmetric connection is a little more complicated. Here we need to use a totally disconnected graph of two edges, which we embed as $e_o = ((0,0,0),(0,0,1))$ and $e_1 = ((0,0,2),(1,0,2))$ respectively. Whereas $e^{\pm \frac{i}{2} n l_o c}$ is then represented through the matrix-elements of the n-th representation of h_{e_o} as before, and $e^{\pm \frac{i}{2} m l_o a}$ is presented as a more complicated linear combination $\sum_{kl} a_{kl} \rho^m_{kl}(h_{e_1})$ of the matrix elements of $\rho^m(h_{e_1})$, which can be obtained as $\rho^m_{\pm m, \pm m}(U^\dagger h_{e_1}(A) U)$, where U is the internal $SU(2)$-rotation of the 1-direction into the 3-direction. We want to remark that these spin network functions are not yet elements of E, because we still have to average over the graph symmetries, which we will do in section 5.4.3, where we consider gauge- and diffeomorphism- invariance. For a cylindrical function T_γ with γ in the scaffold, we define p as the pull-back under the embedding of the symmetric connection into the phase space, wich we define explicitly as the extension by density of:

$$\begin{array}{rcl} p_{LRS} T_\gamma : (a, c) & \mapsto & T_\gamma(A_{LRS}(a,c)) \\ p_{iso} T_\gamma : c & \mapsto & T_\gamma(A_{iso}(c)), \end{array} \tag{5.16}$$

where we used the explicit expressions for A_{LRS} and A_{iso} in the symmetric chart. Equation 5.15 then yields the image of p, so we can construct the quantum embedding by constructing an inverse q, such that $p \circ q = id$ and $q \circ p = id_{img(q)}$. We construct this map first for the exponential functions and then extend it by linearity, for the isotropic case we may choose:

$$q_{iso} : e^{\frac{i}{2} l_o n c} \mapsto \rho^n_{\pm n, \pm n}(h_{e_o}), \tag{5.17}$$

using the above observations. Using $\sum_{kl} a_{kl} \rho^m_{kl}(h_{e_1})$ as above, we can define the locally rotationally symmetric embedding as the extension by linearity of:

$$q_{LRS} : e^{\frac{i}{2} l_o (ma + nc)} \mapsto \rho^n_{\pm n, \pm n}(h_{e_o}) (\sum_{kl} a_{kl} \rho^m_{kl}(h_{e_1})), \tag{5.18}$$

which satisfies the consistency conditions for quantum embeddings.

5.4.2 Ambiguities

The construction of the scaffold is ambiguous and with this the construction of the algebra $\mathfrak{B}_o$, because the only demand for the the scaffold is that a graph of any knot class can be embedded into it. The construction presented here was guided by computability and simplicity. The holonomy differential equation

$$\dot{h}(t) = (e^* A)(t) h(t) \tag{5.19}$$

for $SU(2)$ reduces to a set of two second order differential equations for the two independent matrix elements of a special unitary 2×2-matrix. Apart from a few special cases, one can not find exact solutions to these. The reason, why we restricted ourselves to piecewise linear edges is that equation 5.19 can be exactly solved in this case. If we had used a different scaffold, then we would most likely not have had a chance to solve equation 5.19, meaning that we would have obtained a different dependence of the spin network functions on the scaffold on the connection degrees of freedom[3].

This raises the question: What is the physical significance of this ambiguity?

Let us argue that the ambiguity is the choice of gauge for the diffeomorphisms: We have chosen a global chart (U, ϕ) in which the homogeneous connection takes the form of equation 5.12. If we reexpress this the homogeneous connection in a chart (U, ψ), then we have to pull-back the expression for the connection under the diffeomorphism $\rho = \psi \circ \phi^{-1} : U \to U$. Since $h_e(\rho^* A) = h_{\rho(e)}(A)$, we see that this new choice of chart has the same effect as mapping the scaffold under the diffeomorphism ρ.

The procedure presented here can also be understood as a method to impose Bianchi symmetry in a quantum theory. Since there are many families $\{X_i\}_{i=1,2,3}$ of globally commuting vector fields, each related as the push-forward under a diffeomorphism ρ, we can state the observation that different ρ lead to different quantum theories as: Different choices for $\{X_i\}_{i=1,2,3}$ may lead to different quantum theories. As of now, we do not see fundamental reasons (other than computability) that imply a "correct choice" or at least point out some "wrong choices".

One fundamental reason could turn out to be the topology of the reduced configuration space. We simply assumed here that it was irrelevant, but a preferred topology would put very tight restrictions on the gauge fixing of the diffeomorphisms, because the topology should arise as the Gel'fand topology of the spectrum of the reduced configuration algebra. Finding a preferred topology is however not simple: The classical theory assumes the topology of $\mathbb{R}$, which is however noncompact, so it can not be continuously embedded into the compact quantum configuration space. The universal Stone-Čech topology $\beta\mathbb{R}$, also seems to be ruled out, because the Liouville-Green-expansion[24] of the holonomy differential equation has only solutions that become almost periodic in c as $c \to \infty$, which does not allow for the constructions of functions like $c \mapsto \sin(c^2)$ as a uniform limit of linear combinations thereof.

5.4.3 Gauge- and Diffeomorphism-Invariance

We will implement the gauge- and diffeomrophism invariant observable algebra using equation 4.22. Its implementation is hugely simplified due to the orthogonality and density of both the gauge-variant and gauge-invariant spin network functions in the kinematic Hilbert space and the gauge-invariant Hilbert space respectively. The image of a cylindrical function under the map η' for SU(2)-gauge- transformations and diffeomorphisms[4], can be expanded in gauge-invariant spin network functions on the scaffold, which are (1) on the graphs in

[3] If we had used a scaffold with two incommensurable length l_o, l_1 for each direction, then, using a result of Velinho[27], we would have obtained that the spin network functions would be almost periodic functions of the symmetric connection degrees of freedom.

[4] We defined the map η' in equation 4.19 and endowed with a Hermitian structure in equation 4.20.

the image of the embedding of knot-classes and (2) the spin labels are symmetric under graph symmetries. We denote the set of spin network functions that satisfies these conditions by $\mathfrak{S}$. Hence it is sufficient for the implementation of equation 4.22 to investigate the matrix elements between gauge-invariant spin network functions $S_1, S_2, T_1, T_2 \in \mathfrak{S}$ that satisfy these two conditions:

$$\langle S_1, \langle T_1, T_2\rangle_{\mathfrak{B}_o} S_2\rangle = \langle S_1, T_1\rangle\langle T_2, S_2\rangle, \tag{5.20}$$

which shows that the spaces E_1 and J in equation 4.22 are one-dimensional complex linear spaces. The implementation of the kinematic constraints can therefore be applied to the observable algebra and we obtain that the gauge-invariant elements of $\mathfrak{B}_o$ are obtained as (limits of sequences of) the sums:

$$a = \sum_{n=1}^{N} \langle T_1^n, T_2^n\rangle_A \Big|_{T_1^i, T_2^j \in \mathfrak{S}}. \tag{5.21}$$

Let us now investigate the structure of the elements of $\mathfrak{S}$, so we are able to explicitly calculate their dependence on the symmetric connection, to be able to construct the gauge- and difeomorphism invariant quantum embedding.

The gauge-invariant spin network functions are linear combinations of products of traces of holonomies of closed loops. Thus, we calculate the holonomies around closed loops in the scaffold. These can all be generated by the three elementary loops $((0,0,0),(1,0,0),(1,1,0),(0,1,0),(0,0,0))$, $((0,0,0),(0,1,0),(0,1,1),(0,0,1),(0,0,0))$ and $((0,0,0),(0,0,1),(1,0,1),(1,0,0),(0,0,0))$; the other elementary loops coincide due to homogeneity. The traces of these holonomies are all even periodic functions in a, c with periodicity l_o:

$$\begin{aligned} Tr(h_{((000)(100)(110)(010)(000))}) &= 2\cos(al_o) + \sin^2(al_o) \\ Tr(h_{((000)(010)(011)(001)(000))}) &= 2\cos^2(al_o/2) + 2\cos(cl_o)\sin^2(al_o/2) \\ Tr(h_{((000)(001)(101)(100)(000))}) &= 2\cos^2(al_o/2) + 2\cos(cl_o)\sin^2(al_o/2). \end{aligned} \tag{5.22}$$

But the "Wilson loops around these elementary plaquettes" contain all the gauge invariant information of the homogeneous connection, because there are no "smaller plaquettes" in our scaffold. This means that all solutions to the Gauss constraint are even functions of periodicity l_o. This suggests to fix the embedding of "disjoint loop graphs" l_k, which are graphs that consists of k disjoint unknotted loops α_k^n, into the scaffold as elementary loops as k disjoint elementary loops around $((00n)(10n)(11n)(01n)(00n))$. Since we restrict the domain of p_{iso} to $\mathfrak{S}$ to obtain the gauge-invariant p_{iso}^{inv}, it suffices to define q_{iso}^{inv} as the extension by linearity of:

$$q_{iso}^{inv} : (2\cos(cl_o) + \sin^2(cl_o))^k \mapsto N \prod_{n=1}^{k} Tr(\rho^{\frac{1}{2}}(h_{\alpha_k^n})), \tag{5.23}$$

where N is a normalization constant. This choice is already invariant under graph symmetries, since all loops have the same spin quantum number assigned[5].

The analogue construction for LRS-symmetry is a little more involved and requires to fix an embedding of disconnected one- and two-loop graphs: Notice that two loop graphs, apart from a possible knotting, come in two different diffeomorphism classes: the 8-configuration joins the two individual loops at a vertex and the B-configuration joins the two loops at an edge. We want to implement $\left(2\cos(al_o)+\sin^2(al_o)\right)^n\left(2\cos^2(al_o/2)+2\cos(cl_o)\sin^2(al_o/2)\right)^m$, so we use for $m \geq n$ a totally disjoint graph that consists of n B-type graphs and $m-n$ single loops, whereas for $n > m$ we use a totally disjoint graph that contains m 8-type loops and $n-m$ single loops, so the graph can in both cases be decomposed into the single loops $(\alpha^i_{nm})_{i=1}^n(\beta^j_{nm})_{i=1}^m$, which are lattice translations of ((000)(100)(110)(010)(000)) and ((000)(010)(011)(001)(000)) respectively. We can therefore construct q^{inv}_{LRS} as the extension by linearity of:

$$\begin{array}{rcl} q^{inv}_{LRS} & : & \left(2\cos(al_o)+\sin^2(al_o)\right)^n\left(2\cos^2(al_o/2)+2\cos(cl_o)\sin^2(al_o/2)\right)^m \\ & \mapsto & N\prod_{r=1}^n Tr(\rho^{\frac{1}{2}}(h_{\alpha^r_{nm}}))\prod_{s=1}^m Tr(\rho^{\frac{1}{2}}(h_{\beta^s_{nm}})), \end{array} \tag{5.24}$$

where N is a normalization constant. To obtain a readable computation of the reduced algebra and its induced representation in the next section, we will introduce the short-hands (k) for $(2\cos(cl_o)+\sin^2(cl_o))^k$ and (n,m) for $\left(2\cos(al_o)+\sin^2(al_o)\right)^n\left(2\cos^2(al_o/2)+2\cos(cl_o)\sin^2(al_o/2)\right)^m$ respectively.

5.4.4 Embeddable Loop Quantum Cosmology

In the previous section, we obtained a quantum embedding of the LRS- resp. isotropic Bianchi I cosmology using induction modules given by the span of $\{(n,m) : n,m \in \mathbb{N}\}$ and $\{(n) : n \in \mathbb{N}\}$ respectively. Let us now calculate the reduced algebra using equation 4.8. Using the orthogonality of cylindrical functions on a different graph w.r.t. integration over the Ashtekar-Lewandowski measure, we obtain

$$\begin{array}{rcrcl} \langle (n_1,m_1),(n_2,m_2)\rangle_{LRS}: & (n_3,m_3) & \mapsto & \delta_{n_2,n_3}\delta_{m_2,m_3}(n_1,m_1) \\ \langle (k_1),(k_2)\rangle_{iso}: & (k_3) & \mapsto & \delta_{k_2,k_3}(k_1), \end{array} \tag{5.25}$$

which are precisely the Fourier-decomposition induction modules of quantum mechanics on a 2-torus and on a circle respectively.

Using equation 4.12, we can use these operators to induce a vacuum state through the vacuum state on full Loop Quantum Gravity. Using the above

[5] There is a subtle point about the graph symmetries that we did not address yet, because two loops are also diffeomorphic, if their orientation is opposite, but:

Lemma 10 *$Tr(h_e) = Tr(h_{e^{-1}})$ for all edges e.*

proof: an element of $SU(2)$ can be written as $h_e = \begin{pmatrix} a_e & b_e \\ -\bar{b}_e & \bar{a}_e \end{pmatrix}$, so $Tr(h_{e^{-1}}) = Tr((h_e)^{-1}) = Tr((h_e)^\dagger) = a_e + \bar{a}_e = Tr(h_e)$. □

orthogonality again, we obtain:

$$\begin{array}{rcl} \omega_{LRS}(\langle(n_1,m_1),(n_2,m_2)\rangle_{LRS}) & = & \delta_{n_1,0}\delta_{n_2,0}\delta_{m_1,0}\delta_{m_2,0} \\ \omega(\langle(k_1),(k_2)\rangle_{iso}) & = & \delta_{k_1,0}\delta_{k_2,0}, \end{array} \tag{5.26}$$

which coincides with the canonical vacuum state of quantum mechanics on the 2-torus resp. the circle. Since the Hilbert-space representation of Loop Quantum Gravity is cyclic, we conclude that the induced LRS-symmetric sector is equivalent with quantum mechanics on the 2-torus and the induced isotropic sector is equivalent with quantum mechanics on the circle.

5.5 Tentative Dynamics

Let us now use the set of scalar constraints to induce a dynamics for the reduced system constructed in the previous subsection. We thus search for those elements $a_{LRS} = \sum_i a_i \langle(n_1^i, m_1^i),(n_2^i, m_2^i)\rangle_{LRS}$ of the induced algebra, for which the corresponding operator $a = \sum_i a_i \langle q(n_1^i, m_1^i), q(n_2^i, m_2^i)\rangle_{\mathfrak{B}_o}$ commutes with the set of scalar constraints (we denote the elements of this set by H here). We therefore calculate matrix elements between any two cylindrical functions ψ, ϕ on the scaffold:

$$\begin{array}{rcl} \langle\psi,[H,a]\phi\rangle & = & \sum_i a_i \langle \psi, (H\langle q(n_1^i,m_1^i), q(n_2^i,m_2^i)\rangle_{\mathfrak{B}_o} - \langle q(n_1^i,m_1^i), q(n_2^i,m_2^i)\rangle_{\mathfrak{B}_o} H)\phi\rangle \\ & = & \sum_i a_i \langle\psi, Hq(n_1^i,m_1^i)\rangle\langle Hq(n_2^i,m_2^1),\phi\rangle, \end{array}$$

where we used that the set of scalar constraints is Hermitian. However, any matrix-element of any scalar constraint H with graph that has at most trivalent vertices vanishes. Since the $q(n,m)$ contain only such graphs, we conclude

$$\langle\psi,[H,a]\phi\rangle = 0 \, \forall \psi, \phi, \tag{5.27}$$

which implies that any a in the reduced algebra commutes with any scalar constraint H. This means that the scalar constraint is empty for the reduced algebra, which is the reason, why we titled this section "tentative" dynamics. This pathology is imposed to the vanishing of the constraint operator in full Loop Quantum Gravity on graphs with vertices with valence of at most three.

5.6 Meaning for Standard Loop Quantum Cosmology

The differences between the cosmology constructed in this chapter and standard Loop Quantum Cosmology is mainly due to the different treatment of the diffeomorphism constraint: Standard Loop Quantum Cosmology is constructed as a quantization of a classically reduced model, in which the diffeomorphism constraint is empty. The construction here uses the full quantum theory and imposes the full diffeomorphism constraint. The use of the full quantum completion of the group of diffeomorphisms gave us the freedom to choose a representative graph for every knot class of graphs, which is the technical reason for any difference with standard Loop Quantum Cosmology, which are obvious since we obtain an isotropic sector that is equivalent to quantum mechanics on a circle, while standard Loop Quantum Cosmology is equivalent to quantum mechanics on the Bohr-compactification of the real line.

We see this work however as a strengthening of the results of standard Loop Quantum Cosmology, because the super-selection sectors of standard Loop Quantum Cosmology, i.e. those spaces that are left invariant by its Hamilton constraint and the fundamental momentum operator, are equivalent to quantum mechanics on a circle. The argument is therefore as follows: The construction presented here shows how one of these sectors can be induced from the full quantum theory without going the intermediate step of classical symmetry reduction. We thus propose to view standard Loop Quantum Cosmology, which is a direct sum of these super-selection sectors, as a direct sum of quantum reductions of full Loop Quantum Gravity. One of the main technical results of standard Loop Quantum Cosmology is its kinematic discreteness. This discreteness allows for the decomposition of standard Loop Quantum Cosmology into its super-selection sectors. Many physical results about the evolution through the big bang rely on this discreteness and the possibility to perform this decomposition. We thus understand our construction as a strengthening of these particular results.

Chapter 6

Smooth Geometries for Loop Quantum Gravity

This chapter and the following take a different view of constructing a relation between Loop Quantum Gravity and cosmological sectors thereof. Previously, we focused on constructing a reduced quantum observable algebra from the full observable algebra of Loop Quantum Gravity and we induced a Hilbert space representation thereof from full Loop Quantum Gravity. Now we will consider an adjustment of the observable algebra of Loop Quantum Gravity and construct states thereon that have the smooth geometry of classical spaces. Using the GNS construction, we construct a Hilbert space representation of the observable algebra underlying Loop Quantum Gravity and impose the Gauss- and diffeomorphism constraint on states in this representation.

6.1 Mathematical Setup and Ideas

This section contains the lemmata and proofs that we refer to in the subsequent sections.

6.1.1 Definition of the C^*-algebra

Let us slightly generalize the notation of [23] and let us consider a compact Hausdorff space $\mathbb{X}$ and a regular Borel probability measure μ thereon. The (possibly distributional) integral kernels K on $\mathbb{X}$, whose action on $C(\mathbb{X})$ defined through $(Kf)(x) = \int d\mu(x')K(x,x')f(x')$ leaves $C(\mathbb{X})$ invariant and is invertible (on a common dense domain) in $C(\mathbb{X})$ form a group $\mathcal{G}$. We denote the subgroup of elements of $\mathcal{G}$, that leave μ invariant, i.e. $\int d\mu(x)(Kf)(x) = \int d\mu(x)f(x)$ for all $f \in C(\mathbb{X})$, by $\mathcal{G}(\mu)$. Moreover, we denote the "unitary" subgroup of $\mathcal{G}(\mu)$ (called unitary due to their unitary action on $L^2(\mathbb{X}, d\mu)$), i.e. elements of $\mathcal{G}$ with $\int d\mu(x)\overline{(Kf)(x)}(Kg)(x) = \int d\mu(x)\overline{f(x)}g(x)$ for all pairs $f, g \in C(\mathbb{X})$, by $\mathcal{U}(\mu)$. For $\psi \in L^2(\mathbb{X}, d\mu)$ and $f \in C(\mathbb{X}$ acting as multiplication operators in

$\mathcal{B}(L^2(\mathbb{X}, d\mu))$, we denote $w(f)w(\psi)$ by $w(f\psi)$ for all $w \in \mathcal{U}(\mu)$. As operators in $\mathcal{B}(L^2(\mathbb{X}, d\mu))$, we have for $w \in \mathcal{U}(\mu)$ and $f \in C(\mathbb{X})$: $wfw^* = w(f)$ and similarly for $w_1, w_2 \in \mathcal{U}(\mu)$: $w_1 w_2 w_1^* = w_1(w_2)$.

Canonical $*$-algebra

Given a compact Hausdorff space $\mathbb{X}$ and a regular Borel probability measure μ thereon and a subgroup $\mathcal{W}$ of $\mathcal{U}(\mu)$, we are able to define a $*$-algebra:

Definition 18 *Given $\mathbb{X}, \mu, \mathcal{W}$, we denote the finite sums of ordered elements $f \circ w$, where $f \in C(\mathbb{X})$ and $w \in \mathcal{W}$, by $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$.*

Generalizing lemma 2.1 of [23] to this case yields:

Lemma 11 *Given $\mathbb{X}, \mu, \mathcal{W}$, then $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ is an algebra generated by elements of the form $f \circ w$, where $f \in C(\mathbb{X})$ and $w \in \mathcal{W}$.*

proof: Notice that $w \circ f = w(f)w$ for all $f \in C(\mathbb{X})$ and $w \in \mathcal{W}$. Moreover, $(f_1 \circ w_1)(f_2 \circ w_2) = f_1 w_1(f_2)(w_1 w_2)$ implies that for any $a_1, a_2 \in \mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ $a_1 a_2$ is again in $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$, because all $a \in \mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ are of the form

$$a_i = \sum_{j=1}^{n} f_{ij} w_{ij}$$

which is preserved, since $w_{ij}(f_{kl}) \in C(\mathbb{X})$ and $(w_{ij} w_{kl}) \in \mathcal{W}$. $\square$

Using the relations that are induced for the elements of $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$, we equip $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ with an involution:

Lemma 12 *For $a = \sum_{i=1}^n f_i w_i \in \mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ we have an involution given by $a^* = \sum_{i=1}^n w_i^{-1}(\overline{f_i}) w_i^*$.*

proof: Using the adjoint in $\mathcal{B}(L^2(\mathbb{X}, d\mu))$ gives

$$\begin{array}{rclcl} (\sum_{i=1}^n f_i w_i)^* &=& \sum_{i=1}^n (f_i w_i)^* &=& \sum_{i=1}^n w_i^* f_i^* \\ &=& \sum_{i=1}^n w_i^* \overline{f_i} w_i w_i^* &=& \sum_{i=1}^n w_i^{-1}(\overline{f_i}) w_i^*. \end{array}$$

Since the adjoint is an involution that closes on $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$, it defines an involution on $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$. $\square$

Corollary 10 *$\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ is a $*$-subalgebra of $\mathcal{B}(L^2(\mathbb{X}, d\mu))$.*

Given $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ there is a natural C^*-algebra $\mathfrak{A}(\mathbb{X}, \mu, \mathcal{W})$, which is the C^*-closure of $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ in $\mathcal{B}(L^2(\mathbb{X}, d\mu))$.

Definition 19 *Given $\mathbb{X}, \mu, \mathcal{W}$, we call $\mathfrak{A}(\mathbb{X}, \mu, \mathcal{W})$ the canonical C^*-algebra associated to $(\mathbb{X}, \mu, \mathcal{W})$ and the fundamental representation on $L^2(\mathbb{X}, d\mu)$ the canonical representation of $\mathfrak{A}(\mathbb{X}, \mu, \mathcal{W})$, which we denote by π_o.*

Lemma 13 *If the canonical representation π_o of $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ is non-degenerate, then $||\pi_o(f \circ id)||_{\mathcal{H}} = ||f||_\infty$ and $||\pi_o(id \circ w)||_{\mathcal{H}} = 1$.*

proof: $||f||_{\mathcal{H}} = ||f||_\infty$ for a non-degenerate representation of $C(\mathbb{X})$. The unitarity of the representation of $\mathcal{W}$ gives the second assertion. $\square$
We will later require this equality to hold for non-canonical representations.

6.1.2 Unitaries

Let us now consider the definition of unitaries. Recall the definition of a continuous measure generating system from [23]: A subset E of $C(\mathbb{X})$ is called a continuous μ generating system, if the span of E is dense in $C(\mathbb{X})$ and $L^2(\mathbb{X}, d\mu)$ and if $id \in E$ and all other elements of e are orthogonal to id in $L^2(\mathbb{X}, d\mu)$. It follows that if $\langle e, \psi \rangle_{L^2(\mathbb{X}, d\mu)} = 0$ for all $e \in E \setminus \{id\}$, then $\psi = e^{i\phi} ||\psi|| id$. Let us refine this definition:

Definition 20 *E is called a Hermitian measure generating system, iff E is a continuous measure generating system and if there is a set H of Hermitian operators on $\mathcal{B}(L^2(\mathbb{X}, d\mu))$ such that all the elements of E are mutual eigenvectors of the elements of H and any elements of E can be distinguished by the eigenvalues w.r.t. elements of H. We call H the labeling set.*

Notice, that without loss of generality we are able to assume that $h\, id = 0$ for all $h \in H$, since we would simply redefine $h' := h - h_{id} 1_{\mathcal{B}(L^2(\mathbb{X}, d\mu))}$, where h_{id} denotes the eigenvalue of id. We assume from now on that this re-normalization is carried out. We call the collection of all eigenvalues of an element $e \in E$ w.r.t the labeling set H the label of e and denote it by $H(e)$.

Definition 21 *Given a function f from the set of all labels to $\mathbb{R}$. Define the transformation $\tau_f : L^2(\mathbb{X}, d\mu) \to L^2(\mathbb{X}, d\mu)$ as the linear extension from E to $L^2(\mathbb{X}, d\mu)$ of:*

$$\tau_f e := \exp(i\, f(H(e)))e.$$

Lemma 14 *τ_f leaves μ invariant, if $f(H(id)) = 0$.*

proof: E is dense in $C(\mathbb{X})$, hence for all $g \in C(\mathbb{X})$ there is a uniformly convergent series g_i such that $g(x) = \sum_i g_i e_i(x)$.

$$\int d\mu(x) g(x) = \sum_i g_i \int d\mu(x) e_i(x) = g_{H(id)},$$

where we used the uniform convergence. On the other hand:

$$\int d\mu(x)(\tau_f g)(x) = \sum_i g_i \int d\mu(x) \exp(i\, f(H(e_i)))e_i = g_{H(id)} \exp(i\, f(H(id))),$$

which gives the assertion by comparing the right hand sides. □

Lemma 15 *If $f(H(id)) = 0$, then τ_f extends to a unitary operator on $L^2(\mathbb{X}, d\mu)$.*

proof: Notice that $f(\hat{H})$ is an operator with real eigenvalues on the domain E. Since E is dense in $L^2(\mathbb{X}, d\mu)$, we consider the Hermitian extension of f to $L^2(\mathbb{X}, d\mu)$. Then $f(\hat{H})$ is the generator of a unitary one-parameter group on $L^2(\mathbb{H}, d\mu)$, which coincides on E with $\tau_{\lambda f}$ for $\lambda = 1$. This defines the desired unitary extension. □

Corollary 11 *If $f(H(id)) = 0$ then τ_f is in $\mathcal{U}(\mu)$.*

Let us consider the integral kernel for the τ_f.

Lemma 16 *The integral kernel $K_{\tau_f}(x,x')$ of τ_f has the form $K_{\tau_f}(x,x') = \sum_{i\in\mathcal{I}} g_i(x)f_i(x')$.*

proof: Since E is dense in $L^2(\mathbb{X}, d\mu)$, we can find a dense subset E_o of E that is linearly independent; this generally reduces the label set to a subset, that we denote by $\mathcal{I}$. E_o can then be orthonormalized (using Grahm-Schmidt). We denote the elements of E_o by e^o. Then $(\tau_f e)(x) = \sum_{i\in\mathcal{I}}(\tau_f e_i^o)(x)\langle e_i^o, e\rangle_{L^2(\mathbb{X},d\mu)}$; the extension by density gives the integral kernel. □

6.1.3 States and C^*-algebra

For the practical definition of a C^*-closure of a $*$-algebra $\mathfrak{A}_o(\mathbb{X},\mu,\mathcal{W})$, it is useful define a state on $\mathfrak{A}_o(\mathbb{X},\mu,\mathcal{W})$, to calculate the GNS representation and then to calculate the C^*-closure in this representation. We will also require the weakening of the non-degeneracy condition $||\pi_{GNS}(f\circ id)||_{\mathcal{H}} = ||f||_\infty$, $||id\circ w||_{\mathcal{H}} = 1$.

Definition 22 *For a map $F:\mathcal{W}\to U(1)$ and regular Borel probability measure ν on $\mathbb{X}$ define the functional $\omega_F:\mathfrak{A}_o(\mathbb{X},\mu,\mathcal{W})\to\mathbb{C}$ by*

$$\omega_{F,\nu}(\sum_{i=1}^n f_i w_i) := \sum_{i=1}^n F(w_i)\int d\nu(x) f_i(x).$$

Lemma 17 *If F is a group morphism, then $\omega_{F,\mu}$ is a state on $\mathfrak{A}_o$.*

proof: We need to show, that ω is a positive linear functional.
bound: For all $f\in\mathfrak{C}$, $||f||_\infty$ is finite by construction, thus for all $\{f_i\}_{i=1}^n$ there exists a finite M s.t. $\sum_{i=1}^n ||f_i||_\infty \le M$. Then

$$\begin{array}{rcl} |\omega_{E_o}(b)| & = & |\sum_{i=1}^n F(w_i)\int d\mu(A) f_i(A)| \\ & \le & |\sum_{i=1}^n F(w_i)\int d\mu(A) ||f_i||_\infty| \\ & \le & |\sum_{i=1}^n F(w_i)||f_i||_\infty| \\ & \le & \sum_{i=1}^n |F(w_i)||f_i||_\infty| = \sum_{i=1}^n ||f_i||_\infty \le M. \end{array}$$

We used that $d\mu$ is a probability measure on $\bar{\mathcal{A}}$ for the second inequality. **linearity:**

$$\begin{array}{rcl} \omega_{F,\mu}(b_1+\alpha b_2) & = & \sum_{i=1}^{n_1} F(w_{1,i})\int d\mu_{AL}(A) f_{1,i}(A) \\ & & +\alpha\sum_{i=1}^{n_2} F(w_{2,i})\int d\mu_{AL}(A) f_{2,i}(A) \\ & = & \omega_{F,\mu}(b_1) + \alpha\omega_{F,\mu}(b_2). \end{array}$$

positivity:

$$\begin{aligned}
\omega_{F,\mu}(b^*b) &= \omega_{F,\mu}\left((\textstyle\sum_{i=1}^n f_i w_i)^*(\sum_{i=1}^n f_i w_i)\right)\\
&= \textstyle\sum_{i,j=1}^n \omega_{F,\mu}\left((w_j^*(f_j)^* f_i w_i\right)\\
&= \textstyle\sum_{i,j=1}^n \omega_{F,\mu}\left((w_j)^*(f_j)^* f_i w_i\right)\\
&= \textstyle\sum_{i,j=1}^n \omega_{F,\mu}\left((w_j)^*\overline{f_j} w_j (w_j)^* f_i w_j (w_j)^* w_i\right)\\
&= \textstyle\sum_{i,j=1}^n \omega_{F,\mu}\left((K_j^{-1}\overline{f_j})(K_j^{-1} f_i)(w_j)^* w_i\right)\\
&= \textstyle\sum_{i,j=1}^n F_j^* F_i \int d\mu(x)(K^{-1}(\overline{f_j} f_i))(x)\\
&= \textstyle\sum_{i,j=1}^n F_j^* F_i \int d\mu(x)\overline{f_j}(x) f_i(x)\\
&= \textstyle\int d\mu(x) \sum_{i,j=1}^n \overline{(F_j f_j(x))}(F_i f_i(x))\\
&= \textstyle\int d\mu(x) |\sum_{i=1}^n F_i f_i(x)|^2 \geq 0,
\end{aligned}$$

where we used that F is a group morphism of $\mathcal{W}$ into $U(1)$ and that for all $F \in U(1) : F^* = F^{-1}$ as well as that $w \in \mathcal{W}$ leaves μ invariant. □

Definition 23 *The GNS-representation of* $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ *is denoted by* $(\mathcal{H}_F, \pi_F)$.

It is in general rather difficult to find the Gel'fand ideal for a general state ω, it is in general much simpler to construct the Gel'fand ideal for a Schrödinger state ω_ν on $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ defined using the measure ν:

$$\omega_\nu(\sum_{i=1}^n f_i w_i) := \sum_{i=1}^n \int d\nu(x) f_i(x). \tag{6.1}$$

Definition 24 *Given a morphism* F *from* $\mathcal{W}$ *to* $U(1)$, *we define the map* κ_F : $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W}) \to \mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ *by:*

$$\kappa_F : \sum_{i=1}^n f_i w_i \mapsto \sum_{i=1}^n F_i^* f_i w_i.$$

Lemma 18 κ_F *is an automorphism of* $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$. *The inverse is given by*

$$\kappa_F^{-1} : \sum_{i=1}^n f_i w_i \mapsto \sum_{i=1}^n F_i f_i w_i.$$

proof: $\mathbb{C}$-linearity of κ_F as well as $\kappa_F^{-1}\kappa_F b = \kappa_F \kappa_F^{-1} b = b$ for all $b \in \mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ follows immediately. Moreover simple insertion reveals $\kappa_F^{-1}(\kappa_F(b_1)\kappa_F(b_2)) = b_1 b_2$. □

Lemma 19 *The algebra element* $\kappa_F(b)$ *is in the Gel'fand ideal of* $\omega_{F,\mu}$ *if and only if* b *is in the Gel'fand ideal of* ω_ν.

proof: $\omega_\nu(\sum_{i=1}^n f_i w_i) = \sum_{i=1}^n \int d\mu(x) f_i(x) = \sum_{i=1}^n F_i \int d\mu(x) F_i^* f_i(x) = \omega_{F,\mu}(\kappa_F \sum_{i=1}^n f_i w_i)$.
□

Using this simple relation between the Gel'fand ideals, we find the relation between the GNS representations. Recall that the GNS-Hilbert space $\mathcal{H}_\omega$ consists of the equivalence classes of limits of those sequences $(a_n)_{n=1}^\infty$ of elements of $\mathfrak{A}_o$ for which $\lim_{n\to\infty} \omega(a_n^* a_n)$ converges, where equivalence is taken by factoring the Gel'fand ideal.

Lemma 20 *If $\mathcal{N}$ is a of representatives of $\mathcal{H}_{\omega_\mu}$ in $\mathfrak{A}_o$ (corresponding to the Schrödinger functional ω_μ), then $\kappa_F^{-1}(\mathcal{N})$ is a set of representatives of $\mathcal{H}_{\omega_{F,\mu}}$.*

proof: We have for $a, a' \in \mathfrak{A}_o$ and $i_F \in \mathcal{I}_{\omega_{F,\mu}}, i \in \mathcal{I}_{\omega_\mu}$: $a \sim a' + i_F = a' + \kappa_F(i)$, hence $\kappa_f^{-1}(a) \sim \kappa_F^{-1}(a') + i$. □

Using the observation that $\mathcal{H}_\omega$ can be represented as a completion of $C(\mathbb{X})$ for any Schrödinger functional ω_μ, we conclude:

Corollary 12 *There is a dense set of representatives of $\mathcal{H}_{\omega_{F,\mu}}$, given by elements of $C(\mathbb{X})$.*

Using the density of a Hermitian measure generating system E in $L^2(\mathbb{X}, \mu)$ and the linearity of κ_F, we conclude:

Corollary 13 *There is a dense set of representatives of $\mathcal{H}_{\omega_{F,\mu}}$, given by elements of E.*

Definition 25 *The completion of the $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ in the GNS-representation w.r.t. the state $\omega_{F,\mu}$ is the C^*-algebra $\mathfrak{A}_F(\mathbb{X}, \mu, \mathcal{W})$.*

Lemma 21 *If the canonical representation π_o of $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ is non-degenerate, then $||\pi_F(f \circ id)||_{\mathcal{H}_F} = ||f||_\infty$ and $||\pi_F(id \circ w)||_{\mathcal{H}_F} = 1$.*

proof: Using the representation on the Hermitian generating system E gives the equality. □

Let us collect the results of this subsection:

Theorem 2 *Given an algebra $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$, a Hermitian generating system E for μ and a $F : H(E) \to \mathbb{R}$ with $F(H(id)) = 0$, then $\omega_{F,\mu}$ is a state on $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$, there is a C^*-completion $\mathfrak{A}_F(\mathbb{X}, \mu, \mathcal{W})$ and a GNS-representation $(\mathcal{H}_F, \pi_F)$, that satisfies $||\pi_F(f \circ id)||_{\mathcal{H}_F} = ||f||_\infty$, $||\pi_F(id \circ w)||_{\mathcal{H}_F} = 1$, and E is dense in $\mathcal{H}_F$.*

6.1.4 Regularity

We recall the regularity definition (definitions 2.7-2.9 in [23]): The homeomorphisms $\mathbb{R} \to \mathcal{W}$, that lie in a subset $\mathcal{R}$ are one parameter $\mathcal{R}$ subgroups of $\mathcal{W}$. We call a one parameter subgroup of $\mathcal{W}$ regular iff it is weakly continuous. Let π be representation of $\mathfrak{A}_o(\mathbb{X}, \mu, \mathcal{W})$ on a Hilbert space $\mathcal{H}$. We call π regular w.r.t. $\mathcal{R}$, iff π maps regular one parameter subgroups of $\mathcal{R}$ to weakly continuous one parameter subgroups $\pi(\mathcal{R}) \subset \pi(\mathcal{W}) \subset \mathcal{B}(\mathcal{H})$.

It is practically easier to check regularity in a Schrödinger representation than in a more general GNS representation.

Lemma 22 *If $F|_\mathcal{R}$ is continuous and if the GNS representation through ω_μ is regular w.r.t. $\mathcal{R}$, then $\omega_{F,\nu}$ is regular w.r.t. $\mathcal{R}$.*

proof: use a continuous measure generating system E, such that for all $e_1, e_2 \in E$ and all one parameter subgroups $(t \mapsto w_t) \in \mathcal{R}$:

$$\begin{aligned}
& \lim_{t\to t_o} |\langle \pi(e_1)\Omega_{F,\mu}, (w_t - w_{t_o})\pi(e_2)\Omega_{F,\mu}\rangle_{F,\mu}| \\
= & \lim_{t\to t_o} |\omega_{F,\mu}(e_1^*(w_t - w_{t_o})e_2)| \\
= & \lim_{t\to t_o} |\omega_{F,\mu}(e_1^* w_t e_2) - \omega_{F,\mu}(e_1^* w_{t_o} e_2)| \\
= & \lim_{t\to t_o} |\omega_{F,\mu}(e_1^* K_t(e_2) w_t) - \omega_{F,\mu}(e_1^* K_{t_o}(e_2) w_{t_o})| \\
= & \lim_{t\to t_o} |F(w_t) \int d\mu(x)\overline{e_1(x)}(K_t e_2)(x) - F(w_{t_o}) \int d\mu(x)\overline{e_1(x)}(K_{t_o} e_2)(x)| \\
= & \lim_{t\to t_o} |F(w_t)\omega_\mu(e_1^* w_t e_2) - F(w_{t_o})\omega_\mu(e_1^* w_{t_o} e_2)| \\
= & \ 0,
\end{aligned}$$

where we used the continuity of F and the regularity of ω_μ w.r.t. $\mathcal{R}$, which implies $\lim_{t\to t_o} |\omega_\mu(e_1^* w_t e_2) - \omega_\mu(e_1^* w_{t_o} e_2)| = 0$, as well as the continuity of the product of two continuous functions on $\mathbb{R}$. □

6.2 Adjusted Observable algebra of Loop Quantum Gravity

Although Fleischhacks formulation of the Weyl algebra for Quantum Geometry[23] arises naturally from Loop Quantum Gravity, it turns out to be more convenient to work with a slightly modified version of the Weyl group $\mathcal{W}$. The usual definition of the Schrödinger state ω_o of Loop Quantum Gravity is only well defined, because it does not depend on the ordering of the Weyl operators. As soon as one wants to construct a a state on the Weyl algebra that gives nontrivial vacuum expectation values to the electric field, one needs to fix this operator ordering ambiguity if one wants to be able to define the state in the same way as introduced in section 6.1. This is a nontrivial task, and we circumvent it by using a reformulation of the Weyl algebra that is such that the commutators of the new Weyl operators obtain vanishing vacuum expectation values.[1]

6.2.1 Quasi-Surfaces

Although the exponential action of fluxes through two-dimensional surfaces generates the Weyl group of Loop Quantum Geometry, it turns out that the Weyl group contains more general objects, which correspond to exponential actions through quasi-surfaces. Let us recall the definition of a quasi-surface form [23]:

A decomposition of an edge e is a finite collection of edges $(e_1, ..., e_n)$, such that $e_1 \circ ... \circ e_n = e$. Given a subset S of Σ, we call an edge e S-admissible, iff $e \cap S = \emptyset$ or $e \cap S = e$ or $e \cap S = i(e)$ or $e \cap S = f(e)$. An element of this decomposition is called S-simple. A quasi-surface is a subset S of Σ, such that every edge $e \in \mathcal{P}(\Sigma)$ has a minimal decomposition into S-admissible pieces e_i. The particular category of 2-dimensional surfaces, that we have in mind

[1] Because the configuration algebra and the action of the diffeomorphisms thereon is very similar to the observable algebra of Loop Quantum Gravity, we omit those elements here and refer to chapter 8.

consists of stratified analytic surfaces. The orientation of a surface is captured by its (incoming) intersection function with paths: An intersection function σ_S for a surface S is a function $\sigma_S : \mathcal{P}(\Sigma) \to \{-1, 0, +1\}$ that vanishes, whenever $\{i(e), f(e)\} \cap S = \emptyset$, satisfying $\sigma_S(e_1) = \sigma_S(e_2)$, whenever either $i(e_1) = i(e_2)$ and $e_1 \uparrow\uparrow e_2$ or $f(e_1) = f(e_2)$ and $e_1 \downarrow\downarrow e_2$. The orientation of an oriented surface S in Σ defines a natural intersection function through: $\sigma_S(e) = 0$, whenever $\{i(e), f(e)\} \cap S = \emptyset$ or $\dot{e}$ is tangent to S at the respective boundary point and $\sigma_S(e) = +1$ if $i(e) \in S$ and $\dot{e}$ above S or $b(e) \in S$ and $\dot{e}$ beneath S. Furthermore, $\sigma_S(e) = -1$ whenever $\sigma_S(e) = +1$ if $i(e) \in S$ and $\dot{e}$ beneath S or $b(e) \in S$ and $\dot{e}$ above S. This is precisely the way in which the orientation of S is encoded in the intersection function σ_S. To generalize the concept of an oriented surface, one defines: An oriented quasi-surface is a quasi-surface together with an intersection function.

Given a subset of a quasi-surface that is itself a quasi-surface, we call it a quasi-subsurface and if the intersection function of an oriented quasi-surface coincides with the restriction of the intersection function of the quasi-surface then we call the orientation induced. An important lemma form [23] is that given two quasi surfaces S_1 and S_2, then there are is an induced intersection function $\sigma_{S_1 \cap S_2}$ for $S_1 \cap S_2$ and $\sigma_{S_1 \cup S_2}$ for $S_1 \cup S_2$.

6.2.2 Weyl-operators

Having quasi-surfaces at our disposal, we are in the position to recall the definition of quasi-fluxes and Weyl operators [23]. Given an oriented quasi-surface (S, σ_S), we define for each S-simple $e \in \mathcal{P}(\Sigma)$ and each map $\mu : \Sigma \to G$ the map $\kappa_{S,\sigma,\mu}$:

$$(\kappa_{S,\sigma,\mu} A)(e) := \begin{cases} (\mu(i(e)))^{\sigma_S(e)} A(e) (\mu(f(e)))^{\sigma_S(e)} & \text{if} \quad \{i(e), f(e)\} \subset S \cap e \\ A(e) & \text{if} \quad e \subset S \vee e \cap S = \emptyset \end{cases}$$

Since all $e \in \mathcal{P}$ can have a minimal decomposition $(e_1, ..., e_n)$ into S-simple pieces, we extend the map $\kappa_{S,\sigma,\mu}$ to all elements of $\mathcal{A}$ by defining the map $\Theta : \mathcal{A} \to \mathcal{A}$:

$$\Theta_{S,\sigma,\mu}(A) : e \mapsto (\kappa_{S,\sigma,\mu} A)(e_1) ... (\kappa_{S,\sigma,\mu} A)(e_1).$$

This defines the action of the Quasi-flux Θ on $\mathcal{A}$. This action is a homeomorphism on $\mathcal{A}$, that leaves the canonical measure μ_o invariant. It is important to notice, that given two quasi-surfaces S_1 and S_2 and two functions μ_1, μ_2, that commute on $S_1 \cap S_2$, then using

$$\mu := \begin{cases} \mu_1 & \text{for} \quad S_1 \setminus S_2 \\ \mu_1 \mu_2 & \text{for} \quad S_1 \cap S_2 \\ \mu_2 & \text{for} \quad S_2 \setminus S_1 \end{cases}$$

we obtain that $\Theta_{S_1,\sigma_1,\mu_1} \Theta_{S_2,\sigma_2,\mu_2} = \Theta_{S_2,\sigma_2,\mu_2} \Theta_{S_1,\sigma_1,\mu_1} = \Theta_{S_1 \cup S_2, \sigma_{S_1 \cup S_2}, \mu}$.

Since each Θ is a continuous map $\mathcal{A} \to \mathcal{A}$, one obtains a pull-back under this homeomorphism $\Theta^* : C(\mathcal{A}) \to C(\mathcal{A})$. Θ^* is surjective, since $\mathcal{A}$ is compact, and extends to a unitary operator on $L^2(\mathcal{A}, d\mu_o)$, since it leaves the canonical measure μ_o invariant. This is the definition of the usual Weyl-operators $W_{S,\sigma,\mu}$:

$$W_{S,\sigma,\mu} := (\Theta_{S,\sigma,\mu})^*.$$

Given a constant function μ, it turns out, that this representation of the Weyl-operators is regular w.r.t. the topology induced by the one on G. The action of a diffeomorphism ϕ on a Weyl-operator $W_{S_o,\mu}$, where S_o denotes an oriented surface, is $\alpha_\phi(W_{S_o,\mu}) := W_{\phi(S_o),\mu}$, such that the unitary action of the diffeomorphism becomes:

$$U^*_\phi W_{S_o,\mu} U_\phi := \alpha_\phi(W_{S_o,\mu}) = W_{\phi(S_o),\mu}.$$

Although the fluxes generate the Weyl group of Loop Quantum geometry, it is difficult to describe the Weyl group, due to the non-commutativity of the Weyl operators. To be able to construct a modified (almost commutative) Weyl group, we need to introduce the area operator of an oriented quasi-surface S, σ_S[2]: The area operator is Hermitian and it is our strategy to define an operator first on a dense domain given by the spin network functions and to define the Hermitian completion of this operator as the area Weyl-operator.

For the definition of the area operator see appendix C. This area operator A_S is self-adjoint on the gauge-variant spin network functions, so we can define its unique Hermitian extension by density in $L^2(\mathcal{A}, d\mu_o)$. It is important to notice, that the gauge-variant spin network functions based on admissible graphs[3] are eigenfunctions of A_S! Let us now consider the exponentiated action W^A_S of this operator on gauge-variant spin network functions T_γ (viewed as an element of $L^2(\mathcal{A}, d\mu_o)$):

$$W^A_S(\lambda) T_\gamma := e^{i\lambda A_S} T_\gamma = \sum_i a_i T_{\gamma_S,i} e^{i\lambda A_S(T_{\gamma_S,i})} =: \alpha^A_{S,\lambda}(T_\gamma), \tag{6.2}$$

where $A_S(T_{\gamma_S,i})$ denotes the S-area eigenvalue of $T_{\gamma_S,i}$. Notice, that the second equality ensures that the first one is well defined. Notice moreover, that the action of the operator $W^A_S(\lambda)$ is unitary on gauge-variant spin network functions and weakly continuous in λ, since W^A_S furnishes a unitary representation of $\mathbb{R}$ generated by a Hermitian operator. Thus, we can consider the unique unitary extension to all of $L^2(\mathcal{A}, d\mu_o)$. Using the unitarity of the representation of the diffeomorphism group on $L^2(\mathcal{A}, d\mu_o)$, we find the expected diffeomorphism transformation properties for the area operators:

$$U^*_\phi W^A_S U_\phi = W^A_{\phi(S)}.$$

Definition 26 *The abstract operators $W^A_S(\lambda)$, that are defined by $(W^A_S(\lambda))^* = W^A_S(-\lambda)$ together with the action on elements of $C(\mathcal{A})$, given by $(W^A_S)^* F W^A_S := \alpha^A_S(T)$ and vanishing commutation relations with the (ordinary) Weyl-operators are the* **area Weyl Operators***, which generate the area Weyl group.*

[2] Although the area of a surface S is independent of its orientation, it is useful to have the intersection function σ_S available.

[3] A graph is admissible to a surface, if the only transversal intersections of the surface with the graph are bi-valent vertices with gauge-invariant intertwiner.

Let us now include the operators $W^A_S(\lambda)$ as additional elements of the C^*-algebra of Loop Quantum Geometry.

Lemma 23 *The unitary extension of the action defined in equation 6.2 of the area Weyl operators is regular.*

proof: We already established weak continuity in the parameter λ, so using the topology induced by $\mathbb{R}$ gives regularity. $\Box$

It may be useful to have an explicit expression for the action of W^A_S on $L^2(\mathcal{A}, d\mu_o)$:

$$W^A_S : \Psi \mapsto \sum_{T \in SNF} \alpha^A_S(T) \int_{\mathcal{A}} d\mu(A)\overline{T(A)}\Psi(A),$$

where the sum extends over all gauge variant spin network functions.

6.2.3 Definition of the adjusted Algebra

Let us collect some data, which we need to give a precise definition of Loop Quantum Geometry: First, we assume an analytic three-dimensional spin manifold Σ, which provides the classical topology. The gauge group for Loop Quantum Geometry is $SU(2)$, which acts naturally on the spin bundles over Σ. Given Σ, we consider only piecewise analytical path, hence the underlying path groupoid $\mathcal{P}(\Sigma)$ consists is generated by analytical path in Σ and consists of all finite piecewise analytical paths in Σ. We assume a unitary representation of the extended stratified analytical diffeomorphisms, which we denote by $\mathcal{D}$[4], which leaves the groupoid $\mathcal{P}(\Sigma)$ invariant. Constructing $\mathcal{A}$ as well as the canonical representation of $C(\mathcal{A})$ on $L^2(\mathcal{A}, d\mu_o)$ and the canonical action of the diffeomorphisms on $L^2(\mathcal{A}, d\mu_o)$ from the data $(\Sigma, \mathcal{P}(\Sigma), \mathcal{D}')$, we obtain the canonical representation of the configuration algebra of Loop Quantum Gravity. Form now on, we will depart from this canonical representation and assume structural data that differs from [23]:

Let e_o be a triad field on Σ, stemming from a (classical) Riemannian metric q. Denote the the densitized inverse of the triad e by E_o. Moreover, let τ be a (not necessarily continuous) global section in the trivial Lie-algebra vector bundle of $SU(2)$ over Σ, that is normalized at each point: $k^{ij}\tau_i\tau_j = 1$. These two structures provide the classical background, that we will incorporate in the next section. Denote the set of stratified analytic two-surfaces in Σ by $\mathcal{S}(\Sigma)$ and associate to each oriented $S \in \mathcal{S}(\Sigma)$ its natural intersection function σ_S. Denote the set of piecewise constant real functions on $f : \Sigma \to \mathbb{R}$ by K[5]. Let us now use the data $(E_o, \tau, \mathcal{S}, K)$ to define the Weyl group, that we want to consider:

[4] If necessary, we complete $\mathcal{D}$ to a groupoid $\mathcal{D}'$ by considering the action on $\mathcal{P}(\Sigma)$ and taking the smallest subgroupoid of the double groupoid of the set $\mathcal{P}(\Sigma)$, that contains all elements of $\mathcal{D}$. This definition ensures, that $\mathcal{D}'$ still preserves $\mathcal{P}(\Sigma)$, but providing the inverses whenever necessary. The precise construction is explained in chapter 3.

[5] With piecewise constant we mean that when restricted to any quasi-surface, we demand that there is a locally finite decomposition of the quasi-surface into quasi-subsurfaces, such that f is constant on the quasi-subsurfaces in this decomposition.

For each $S \in \mathcal{S}$ and each constant $f \in K$[6] consider the three unitary representations of $\mathbb{R}$ on $L^2(\mathcal{A}, d\mu_o)$, denoted by their action on gauge variant spin network functions considered as a dense set in $L^2(\mathcal{A}, d\mu_o)$:

$$\begin{array}{lcccl} W_S^A(f) & : & T & \mapsto & \alpha^A_{S,\lambda(f)}(T) \\ W_S^+(f) & : & T & \mapsto & (\Theta_{S,\sigma,\exp(\lambda(f)\tau)})^* T \\ W_S^-(f) & : & T & \mapsto & (\Theta_{S,\overline{\sigma},\exp(\lambda(f)\tau)})^* T, \end{array} \tag{6.3}$$

where $\lambda(f)$ denotes the constant value of f on S and $\overline{\sigma}$ denotes the intersection function corresponding to the opposite orientation of S [7]. We notice, that these almost commute: First, any two the operators on any two disjoint surfaces commute as well as the operators on coinciding surfaces, because the area operator commutes with with the flux operators on the same surface and there is only one flux operator per surface. Given any two surfaces two-dimensional S_1 and S_2, then $[W_{S_1}, W_{S_2}]$ is an operator that acts nontrivially only on a quasi-surface that is contained in a one-dimensional quasi-surface. Thus measuring the area of the base space support of the commutator with and classical E_o will give a vanishing result, hence commutativity.

Definition 27 *The group generated by the elements $W_S^A(f), W_S^+(f), W_S^-(f)$, where f varies over K, S varies over $\mathcal{S}(\Sigma)$ is called the Weyl group of Loop Quantum Geometry, denoted by $\mathcal{W}$.)*

We define the involution for $W(\lambda) \in \mathcal{W}$ by $W^*(\lambda) = W(-\lambda)$. A simple application of the definition of the three families of Weyl operators reveals $W(-\lambda)W(\lambda)T = T$ for all gauge variant spin network functions and hence:

Corollary 14 $W^*(\lambda) = (W(\lambda))^{-1}$.

Moreover:

Corollary 15 *$\mathcal{W}$ is commutative.*

This definition, which depends on τ differs from the definition of Loop Quantum Geometry in [23]. Although we are not able to recover all Weyl-operators in Fleischhacks definition, we claim that this definition contains "enough" all gauge-invariant quantum geometry:

Lemma 24 *The generators of the one parameter subgroups of the Weyl group furnish a Hermitian labeling set.*

proof: Recall from [23], that the gauge variant spin network functions are a generating set for μ_o. Given a spin network function T_γ based on a graph γ, notice that the geometry of each edge can be determined through the area operators: For any edge $e \in \gamma$, choose an affine parametrization $e(t)$, consider

[6] The operators corresponding to nonconstant $f \in K$ arise as products of operators defined on quasi-subsurfaces in the decomposition of S in which f is constant in each quasi-subsurface.

[7] Of course, W_S^+ is the same operator as $W_{\overline{S}}^-$ and $W_S^A = W_{\overline{S}}^A$ and we will identify them with each other.

two one-parameter families of surfaces $S_e(t)$ and $S_e^o(t)$, where $S_e(t)$ intersects e only at $e(t)$, and does not intersect any other edge in γ; $S_e^o(t) := S_e(t) \setminus \{e(t)\}$. Since e carries a nontrivial representation, $A_{S_e(t)}T \neq 0$, but $A_{S_e^o}T = 0$. Moreover the representation on e is unambiguously determined by $A_{S_e(t)}$, for any t.

For each analytic edge $e \in \gamma$ consider the two stratified analytic "umbrella-shaped" surfaces S_e^i and S_e^f at $i(e)$ and $f(e)$ respectively, which are constructed such that only e is "above" them and all other other edges $e \in \gamma$ are either completely outside or tangential to $S_e^{i,f}$. These act as left- and right-invariant vector fields on the matrix element corresponding to e. Thus, all quantum numbers carried on T can be determined. □

Definition 28 *Given the data $(\Sigma, \mathcal{P}(\Sigma), \mathcal{D}', E_o, \tau, \mathcal{S}, K)$, and after constructing the canonical representation of $C(\mathcal{A})$, $\mathcal{W}$, $\mathcal{D}'$ on $L^2(\mathcal{A}, d\mu_o)$, we define the C^*-algebra of Loop Quantum Geometry as the closure of the span of the representation $f \circ w \circ \phi$ in the canonical representation on $L^2(\mathcal{A}, d\mu_o)$, where $f \in C(\mathcal{A})$, $w \in \mathcal{W}$ and $\phi \in \mathcal{D}'$. We denote this algebra by $\mathfrak{A}$ or $\mathfrak{A}(E_o, \tau)$, whenever we want to point to the explicit dependence.*

Although this algebra is smaller than the standard definition, we saw by lemma 24, that it is large enough to encode quantum geometry. It will however turn out that it is very useful to have all flux operators available; let us therefore construct them:

The construction will not be through homeomorphisms on $\mathcal{A}$, but to define the fluxes that generate them through the commutators with gauge variant spin network functions. It will turn out that these form a dense set in each summand of the Hilbert space that we will construct and we are thus later able to to extend their action by density. The important initial observation is that the generators of the $W_S^+(\lambda), W_S^-(\lambda), W_S^A(\lambda)$, i.e. the flux-operators parallel and antiparallel to τ and the area operators, act on a gauge variant spin network function T_γ by splitting the edges $e \in \gamma$ at the intersections $e \cap S$ and inserting the respective vertex operators $J_{v,e}^{(u)}, J_{v,e}^{(d)}, \sqrt{J_{v,e}^2}$ at all intersection vertices. Thus the description of their action involves two steps: first the splitting and second the action as the respective $SU(2)$-operator on the pieces of the splitting.

Let us first consider the splitting of an edge $e \in \gamma$: The splitting occurs either at a boundary point or in the interior of the edge. If $S \cap interior(e) = \emptyset$, then the splitting is trivial, i.e. the edge e is not changed. If the splitting occurs in the interior of e, then the edge e is replaced with $e \mapsto (e_1, e_2)$, where $e_1 \circ e_2 = e$ and $S \cap e = f(e_1) = i(e_2)$, and $u^j_{nm}(h_e(A))$, where u^j_{nm} is the nm-matrix element in the j representation of $SU(2)$, is replaced with

$$u^j_{nm}(h_e(A)) \mapsto \sum_{k=-j}^{j} u^j_{nk}(h_{e_1}(A)) u^j_{km}(h_e(A)).$$

This splitting procedure has to be performed for all edges.

Let us now consider the insertion of the operators $J^{(u)}_{v,e}, J^{(d)}_{v,e}, \sqrt{J^2_{v,e}}$: Consider the insertion into the splitting of $u^j_{nm}(h_e(A))$. This amounts to inserting the respective matrix:

$$\sum_{k,l=-j}^{j} u^j_{nk}(h_{e_1}(A)) M^j_{kl} u^j_{lm}(h_e(A)).$$

Since the necessary quantum number j, that we need to determine the representation matrix M^j of $J^{(u)}_{v,e}, J^{(d)}_{v,e}, \sqrt{J^2_{v,e}}$ is determined by $\sqrt{J^2_{v,e}}$ and the necessary quantum numbers, that we need to determine k, l in the splitting of $u^j_{nm}(h_e(A))$ are determined by $J^{(u)}_{v,e}, J^{(d)}_{v,e}$, which are available as fundamental operators in our theory, we can construct an arbitrary flux operator as a self-adjoint operator whose domain is given by the gauge-variant spin network functions. The Hermitian extension defines the respective flux operator, which is constructable due to the density of the gauge-variant spin network functions in each summand of the representation, that we will construct. Exponentiating these operators then gives the desired Weyl-operators, which act precisely as Fleischhack's homeomorphisms on the gauge-variant spin network functions. So, all Weyl-operators used by Fleischhack are present in our algebra. But when we want to represent them, then we have to first reexpress them in terms of the fundamental Weyl operators.

6.3 Definition of DQG States, GNS-Representation and kinematic constraints

Using the mathematical preparations of section 6.1 we define a state on the modified algebra of Loop Quantum Geometry that we defined in section **??**. These sates are labeled by 3-geometries, which we encoded in a densitized inverse triad E_o in the structure data quoted in the previous section, particularly the map τ. This amounts to an application of the results of section 6.2.3, so we are only left with the performing the work programme for the algebra described in definition 28.

6.3.1 Definition of DQG states

The finite sums of $a = \sum_{i=1}^n f_i \circ w_i \circ \phi_i$, where $f_i \in C(\mathcal{A})$, $w_i \in \mathcal{W}$ and $\phi_i \in \mathcal{D}'$ are dense in $\mathfrak{A}$ by definition 28. Let us define the state ω_{E_o} on these finite sums as:

$$\omega_{E_o}(a) := \sum_{i=1}^{n} F_{E_o}(w_i) \int_{\mathcal{A}} d\mu_o(A) f(A), \tag{6.4}$$

where $F_{E_o}(W_S) = 1$ whenever S is contained in a less than two-dimensional subset of Σ and otherwise

$$F_{E_o} : \begin{cases} W_S^A(\lambda) & \mapsto & \exp(i\lambda \int_S |E_o|) \\ W_S^+(\lambda) & \mapsto & \exp(i\lambda \int_S E_o) \\ W_S^-(\lambda) & \mapsto & \exp(i\lambda \int_{\overline{S}} E_o). \end{cases} \tag{6.5}$$

Lemma 25 *$\mathcal{W}$ leaves $d\mu_o$ invariant.*

proof: The gauge-variant spin network functions furnish a Hermitian measure generating system with labeling set given by the area- and "allowed flux"-operators[8] by lemma 24. Moreover, for all elements H in the labeling set, we have $H\, id = 0$. Noticing that all $W \in \mathcal{W}$ act on the gauge variant spin network functions as $\exp(i\lambda H)$ lets us apply lemma 14. $\Box$

Lemma 26 *ω_{E_o} is a state on $\mathfrak{A}$ (for now defined without including diffeomorphisms).*

proof: To apply lemma 17, we need to prove that F_{E_o} is a group morphism: From lemma 15 we see that all elements of $\mathcal{W}$, that arise as commutators are mapped to 1. Moreover, $W(\lambda)^{-1} = W(-\lambda)$ and equation 6.5 reveals that $F(W(\lambda))^{-1} = F(W(-\lambda))$. The rest of the group morphism follows from a direct check of $F(W_1)F(W_2) = F(W_1W_2)$ by inserting the nine qualitatively different possibilities into equation 6.5. $\Box$

It is quite simple to verify that the W extend to unitary operators in the GNS representation constructed from ω_{E_o} using $W^*(\lambda) = W(-\lambda)$, because for any $a, b \in \mathfrak{A}$ one has:

$$\omega_{E_o}((W(\lambda)a)^*(W(\lambda)b)) = \omega_{E_o}(a^*W(-\lambda)W(\lambda)b) = \omega_{E_o}(a^*b).$$

Having a state on $\mathfrak{A}$ lets us construct the corresponding GNS representation:

6.3.2 GNS-representation

Using the Schrödinger state $\omega_o(\sum_i f_i \circ w_i) = \sum_i \int_{\mathcal{A}} d\mu_o(A) f_i(A)$ for $f_i \in C(\mathcal{A})$, $w_i \in \mathcal{W}$ and knowing that the gauge-variant spin network functions are a Hermitian generating set for $d\mu_o$, we explicitly construct the GNS-representation for ω_{E_o}. Let us now apply definition 24 to F given by equation 6.5, then lemma 18 tells us κ_F is an automorphism of $\mathfrak{A}$.

Our strategy is to use this automorphism to construct the GNS-representation corresponding to ω_{E_o} directly from the GNS representation corresponding to ω_o, along the lines outlined in lemma 20 and the subsequent construction. Let us therefore describe the canonical GNS-representation. Let $a_j = \sum_{i=1}^N f_{ij} w_{ij}$ be an element of $\mathfrak{A}$, then $\eta_o : \mathfrak{A} \to (\mathfrak{A})/\mathcal{N}_o \subset \mathcal{H}_o$, where $\mathcal{N}_o$ denotes the Gel'fand ideal corresponding to ω_o, $\pi_o : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and $\langle ., . \rangle_o$ are determined by requiring

$$\omega_o(a_1^* a_2 a_3) = \langle \eta_o(a_1), \pi_o(a_2)\eta_o(a_3) \rangle_o$$

[8] We call a flux operator "allowed" if it is parallel to the restriction of τ to the respective quasi-surface.

to hold for all a_i. The canonical Schrödinger representation satisfies this condition and can be denoted as the dense extension of:

$$\begin{array}{rcl} \eta_o(a): & A\mapsto & \sum_i f_i \\ \pi_o(a)\phi: & A\mapsto & \sum_i f_i(A)(\alpha_{w_i}(\phi))(A) \\ \langle\phi,\phi'\rangle_o & := & \int_{\mathcal{A}}\overline{\phi}(A)\phi'(A), \end{array}$$

where $a=\sum_{i=1}^N f_i\circ w_i$ with $f_i\in C(\mathcal{A})$, $w_i\in\mathcal{W}$ and $\phi,\phi'\in C(\mathcal{A})\sim(\mathfrak{A})/\mathcal{N}_o$. Let us consider $\omega_{E_o}(\kappa_F(a))$:

$$\begin{array}{rcl} \omega_{E_o}(\kappa_F(a)) & = & \omega_{E_o}(\sum_i f_iF^{-1}(w_i)w_i) \\ & = & \sum_i F(w_i)\int d\mu_o(A)f_i(A)F^{-1}(w_i) \\ & = & \sum_i\int d\mu_o(A)f_i(A)=\omega_o(a), \end{array}$$

from which we deduce that we have to only insert κ_F into the definitions of η_o and π_o to obtain η_{E_o} and π_{E_o}, because we have

$$\omega_{E_o}(\kappa_F(a_1^*)\kappa_F(a_2)\kappa_F(a_3))=\omega_{E_o}(\kappa_F(a_1^*a_2a_3))=\omega_o(a_1^*a_2a_3).$$

So we can construct the the GNS-representation by using this identity and $\kappa_F^{-1}(\kappa_F(a))=a$ for all $a\in\mathfrak{A}$ and obtain:

$$\begin{array}{rlcl} \eta_{E_o}(a) & =\eta_o(\kappa_F(a)): & A\mapsto & \sum_i F(w_i)f_i \\ \pi_{E_o}(a)\phi & =\pi_o(\kappa_F(a)): & A\mapsto & \sum_i F(w_i)f_i(A)(\alpha_{w_i}(\phi))(A) \\ \langle\phi,\phi'\rangle_{E_o} & =\langle\phi,\phi'\rangle_o & := & \int_{\mathcal{A}}\overline{\phi}(A)\phi'(A). \end{array}$$

We denote the E_o-GNS-vacuum state corresponding to the cyclic vector $\eta(id)$ by $\eta_{E_o}(id)=:\Omega_{E_o}$. We will denote the GNS-representation constructed from ω_{E_o} by $(\mathcal{H}_{E_o},\pi_{E_o})$.

6.3.3 Implementation of the Diffeomorphisms

We know by lemma 22 that this representation is regular, because F is continuous and the canonical representation of the algebra $\mathfrak{A}$ of Loop Quantum Geometry is regular. Let us now introduce a unitary representation of $\mathcal{D}$. This means, that we seek a U_ϕ for all $\phi\in\mathcal{D}$, such that:

$$\begin{array}{rcl} \omega_{E_o}(a) & = & \langle\Omega_{E_o},\pi_{E_o}(a)\Omega_{E_o}\rangle_{E_o} \\ & = & \langle U_\phi\Omega_{E_o},U_\phi\pi_o(a)\Omega_{E_o}\rangle_{E_o} \\ & = & \langle U_\phi\Omega_{E_o},U_\phi\pi_o(a)U_\phi^*U_\phi\Omega_{E_o}\rangle_{E_o} \\ & = & \langle U_\phi\Omega_{E_o},\pi_o(\alpha_\phi(a))U_\phi\Omega_{E_o}\rangle_{E_o} \\ & = & \omega_{E_o}^\phi(\alpha_\phi(a)), \end{array}$$

where we assumed a covariant action α_ϕ of the diffeomorphisms $\phi\in\mathcal{D}$ on the elements of $\mathfrak{A}$, defined through the extension by density of:

$$\alpha_\phi(a)=\alpha_\phi(\sum_i f_{i,\gamma_i}w_{i,S_i})=\sum_i f_{i\phi(\gamma_i)}w_{i,\phi(S_i)}.$$

Let us now determine the state $\omega^{\phi}_{E_o}$ by considering:

$$\begin{aligned}
\omega_{\phi^{-1}(E_o)}(\alpha_{\phi^{-1}}(a)) &= \omega_{\phi(E_o)}(\textstyle\sum_i f_{i,\gamma_i} w_{i,S_i}) \\
&= \textstyle\sum_i F_{\phi^{-1}(E_o)}(w_{i,S_i}) \int_{\mathcal{A}} d\mu_o(A) f_{i,\gamma_i}(A) \\
&= \textstyle\sum_i F_{(E_o)}(w_{i,\phi(S_i)}) \int_{\mathcal{A}} d\mu_o(A) f_{i,\phi(\gamma_i)}(A) \\
&= \omega_{E_o}(\alpha_\phi(a)) = \langle \Omega_{E_o}, \pi_{E_o}(\alpha_\phi(a)) \Omega_{E_o} \rangle_{E_o} \\
&= \omega_{E_o}(\alpha_\phi(a)) = \langle \Omega_{E_o}, U^*_\phi \pi_{E_o}(a) U_\phi \Omega_{E_o} \rangle_{E_o} \\
&= \omega_{E_o}(\alpha_\phi(a)) = \langle U_\phi \Omega_{E_o}, \pi_{E_o}(\alpha_\phi(a)) U_\phi \Omega_{E_o} \rangle_{E_o},
\end{aligned}$$

where we used the diffeomorphism invariance of $d\mu_o$ in the third line. Comparing these two calculations lets us conclude that

$$U_\phi \Omega_{E_o} = \Omega_{\phi(E_o)},$$

so we have to consider a direct sum of GNS-representations, one for each distinct inverse triad $\phi(E_o)$ in the diffeomorphism orbit of E_o.

6.3.4 Implementation of $SU(2)$-gauge Transformations

Let us now consider $SU(2)$-gauge transformations: Let there be a map $\Lambda : \Sigma \to SU(2)$, then for each $A \in \mathcal{A}$ we have an action of the gauge transformation α_Λ acting as:

$$\alpha_\Lambda A : e \mapsto \Lambda^{-1}(i(e)) h_e(A) \Lambda(f(e)).$$

The action on the triads E is similarly:

$$\alpha_\Lambda E = \Lambda^{-1} E \Lambda,$$

where the action is taken pointwise in Σ. Using the linearity of the of the integrals we obtain the gauge action on the area- and flux- Weyl operators. Moreover, using the precise analogue of the two previous calculations we obtain that requiring a unitary covariant representation of the gauge transformations, i.e. for all Λ there is a $U_\Lambda = (U^*_\Lambda)^{-1}$ with $U^*_\Lambda \pi(a) U_\Lambda := \alpha_\Lambda(a)$, means that we have to extend the representation constructed from E_o to a direct sum of GNS-representations for every distinct $\alpha_\Lambda(\phi(E_o))$ in the gauge$\times$diffeomorphism orbit of E_o. Let U_Λ denote the unitary representation of the $SU(2)$-gauge transformation Λ, then the unitary action on the vacuum vector Ω_{E_o} is

$$U_\Lambda \Omega_{E_o} = \Omega_{\alpha_\Lambda(E_o)}.$$

There is one caveat stemming from the $SU(2)$-gauge transformation properties of the electric fields: We constructed the quantum algebra using only those flux-Weyl-operators, that correspond to exponentiated fluxes parallel to the structure τ. This structure τ then transforms under the $SU(2)$-gauge transformations as

$$\alpha_\Lambda(\tau) : x \in \Sigma \mapsto \Lambda(x)\tau(x),$$

so τ varies among the summands. Since the definition of the algebra $\mathcal{A}$ depends on τ, one may worry whether the representation is then well defined at all. But as we saw at the end of subsection 6.2.3, there are Flux-Weyl-operators available for any internal direction in the gauge group not just those parallel to τ. Thus, there is no problem with the definition of the direct sum of representations[9].

Taking the direct sum of GNS-representations is nothing else than to say that the gauge- and diffeomorphism variant representations are labeled by classical Riemannian geometries $\mathcal{G}$ and we construct them as the direct sum of GNS-representations of ω_{E_o} states, where we take the sum over all E_o that describe $\mathcal{G}_o$, i.e. $\mathcal{G}(E_o) = \mathcal{G}_o$. Since the representation depends on the structure τ, which needs to be provided for precisely one E_o, we will spell this out explicitly by writing $\pi_{\mathcal{G}_o,\tau}$. This is to say that we use the following Hilbert space representation:

$$(\mathcal{H}_{\mathcal{G}_o}, \pi_{\mathcal{G}_o,\tau}) := \oplus_{\{E_o:\mathcal{G}(E_o)=\mathcal{G}_o\}}(\mathcal{H}_{E_o}, \pi_{E_o,\tau^G}),$$

where τ^G denotes the gauge transformed structure τ.

Since the gauge-variant spin network functions are a Hermitian generating set for the measure μ_o, we can apply corollary 12 and obtain that the gauge-variant spin network functions are dense in $\mathfrak{A}$. But the gauge-variant spin network functions are also orthogonal in the E_o-representation, because for any two gauge-variant spin network functions T_1, T_2 we have:

$$\begin{array}{rcl} \langle T_1, T_2\rangle_{E_o} & = & \langle \pi(T_1)\Omega_{E_o}, \pi(T_2)\Omega_{E_o}\rangle_{E_o} \\ & = & \omega_{E_o}(T_1^* T_2) = \int_{\mathcal{A}} d\mu_o(A)\overline{T_1(A)}T_2(A) \\ & = & \omega_o(T_1^* T_2) = \langle T_1, T_2\rangle_o \end{array}$$

together with the orthogonality of the gauge-variant spin network functions in the LOST/F-representation. Moreover, we saw that one can include the diffeomorphisms and $SU(2)$-gauge transformations as unitary transformations, if we take the direct sum of ω_{E_o} -GNS-representations, with $\mathcal{G}(E_o) = E_o$. Since the vectors in different summands are orthogonal to each other, we find that a dense orthogonal set of $\mathcal{H}_{\mathcal{G}_o}$ is given by the gauge variant spin network functions $T \in SNF$ times $E_o \in \{E_o : \mathcal{G}(E_o) = \mathcal{G}_o\}$, which we denote by $T \circ E_o$. The $\mathcal{H}_{\mathcal{G}_o}$ inner product for two elements $T_1 \circ E_o^1, T_2 \circ E_o^2$, where T_i is a normalized gauge-variant spin network function, is:

$$\langle T_1 \circ E_o^1, T_2 \circ E_o^2\rangle_{\mathcal{G}_o} = \left\{ \begin{array}{ll} 1 & \text{for: } T_1 = T_2 \wedge E_o^1 = E_o^2 \\ 0 & \text{otherwise.} \end{array} \right.$$

Let us summarize these observations:

Lemma 27 *Let $nSNF$ denote the set of normalized gauge variant spin network functions, then $nSNF \times \{E_o : \mathcal{G}(E_o) = \mathcal{G}_o\}$ is a dense orthonormal set in $\mathcal{H}_{\mathcal{G}_o}$.*

6.3.5 Solution of the kinematic Constraints

Having a unitary representation of of the $SU(2)$-gauge transformations and diffeomorphisms on $\mathcal{H}_{\mathcal{G}_o}$ at our disposal we can construct the gauge- and diffeomorphism invariant states. The knowledge of a dense orthonormal set saves

[9] These operators are however in general not generated by Hermitian operators.

a considerable amount of work, because we have to only construct the group average of these vectors and extend the construction by density. It will turn out that the coupling between the spin network function and E_o leads to nontrivial modifications compared to the LOST/F-representation.

The construction of the solutions to the Gauss constraint uses the observation that $SNF \times \{E_o : \mathcal{G}(E_o) = \mathcal{G}_o\}$ is dense in $\mathcal{H}_{\mathcal{G}_o}$ by splitting the construction of the solutions to the Gauss constraint into three steps:

1. We classify all gauge invariant couplings of $T_\gamma \in SNF$ to the background E_o.

2. For each possible gauge invariant coupling between the T_γs and E_o, we write down the gauge invariant linear combinations of gauge variant spin network functions. This amounts to "closing the color lines in γ" that do not end at a gauge invariant coupling. This yields "partial solutions" to the Gauss constraint.

3. We restrict ourselves to the partial solutions and solve the Gauss constraint using the group averaging procedure.

Given a gauge variant spin network function T_γ depending on a graph γ, in particular:

$$T_\gamma : A \mapsto \prod_{e \in \gamma} u^{j_e}_{m_e,n_e}(h_e(A)).$$

The basic observation that we need is that for edges $(y,x),(x,z)$ with $f(y,x) = x = i(x,z)$ objects of the form:

$$O^a := Tr\left(((someth.)(A,E_o))(y,z)(h_{(y,x)}(A)){E_o}^a_i(x)\tau^i(h_{x,z}(A))\right)$$

are gauge invariant, because $(h_e(A))_{n,m} \mapsto \left(\Lambda^{-1}(i(e))h_e(A)\Lambda(f(e))\right)_{n,m}$ and $E_o(x) \mapsto \Lambda^{-1}(x)E_o(x)\Lambda(x)$, but obviously not diffeomorphism invariant. Moreover, all nontrivial representations of $SU(2)$ arise as (symmetrized) tensor products of the spin-1/2 representation, which can be coupled to the appropriate tensor product of $E_o(x)$. This is however only a special case of the general picture: Any function $F_x(E_o)$ built from $E_o(x)$ that transforms under some representation of $SU(2)$ can be gauge invariantly coupled to a spin network function T with vertex x by constructing a gauge-invariant intertwiner between the representation of $F_x(E_o)$ and representations adjacent to x in T. Thus, given a gauge-variant spin network T_γ on a graph γ, then we can couple it gauge invariantly to E_o by assigning a function $F_v(E_o)$ and an gauge-invariant intertwiner M_v between the representation of $F_v(E_o)$ and adjacent spins to each vertex v of γ.

Having a gauge-invariant coupling $\{(F_v(E_o), M_v)\}_{v \in V(\gamma)}$ for spin networks to E_o, we can easily classify the gauge-invariant spin networks on γ that couple gauge invariantly to E through $\{(F_v(E_o)\}_{v \in V(\gamma)}$: These are labeled by an irreducible representation of $SU(2)$ for each edge $e \in \gamma$ and a gauge-invariant intertwiner between all representations at the vertices for all vertices $v \in V(\gamma)$. We call the set of these functions coupled gauge invariant spin network functions.

Notice that the inner product of two coupled gauge invariant spin network functions $T_1(A, E_o), T_2(A, E_o)$ depends only on the representation space elements evaluated at E_o of representations under which the $F_v(E_o)$ and the value $\Pi_{v \in V(\gamma)} F_v(E_o)$ transform and not on the precise function, because:

$$\langle T_1(A, E_o), T_2(A, E_o)\rangle_{E_o} = \int_{\mathcal{A}} d\mu_o(A) \overline{\left(\prod_{v \in V(\gamma_1)} F_v(E_o) \right)_{\vec{m},\vec{n}} T_1(A)_{\vec{m},\vec{n}}} \left(\prod_{v \in V(\gamma_2)} F_v(E_o) \right)_{\vec{k},\vec{l}} T_2(A)_{\vec{k},\vec{l}},$$

which leads to new normalizations of the coupled gauge invariant spin network functions.

Let us now solve the remaining gauge-variance of the states $\pi(T(E_o))\Omega_{E_o}$, where T_{E_o} is a coupled gauge invariant spin network function[10], using the group averaging procedure: The group of all gauge transformations $\mathcal{G}$ splits into a subgroup $t\mathcal{G}_{T,E_o}$, consisting of the gauge transformations that act trivially on $\pi(T(E_o))\Omega_{E_o}$ and the quotient group $n\mathcal{G}_{T,E_o} = \mathcal{G}/t\mathcal{G}_{T,E_o}$, i.e. the group of gauge transformations that act nontrivially on $\pi(T(E_o))\Omega_{E_o}$. This allows us to define the anti-linear rigging map η:

$$\eta(\pi(T(E_o))\Omega_{E_o}) : \pi(T'(E'_o))\Omega_{E'_o} \mapsto \sum_{\Lambda \in n\mathcal{G}_{T,E_o}} \langle U_\Lambda \pi(T(E_o))\Omega_{E_o}, \pi(T'(E'_o))\Omega_{E'_o}\rangle_{\mathcal{G}_o}.$$

Notice that the inner product in $\mathcal{H}_{\mathcal{G}_o}$ vanishes, whenever $E_o \neq E'_o$ due to the construction of $\mathcal{H}_{\mathcal{G}_o}$ as a direct sum of GNS-Hilbert spaces $\mathcal{H}_{E_o}$. Since the nontrivial action of $n\mathcal{G}_{T,E_o}$ transforms the background only, there is only one contributing summand in the rigging map. Hence, the effect of the group averaging is to map the label E_o of the state $\pi(T(E_o))\Omega_{E_o}$ into its gauge orbit $\mathcal{O}(E_o)$. The dependence of any of the functions $F_v(E)$ on the gauge orbit is the same as the dependence of F_v on the group average of E. Thus using the group average $G(F(E)) = \int_{SU(2)} d\mu_H(g) F_v(g^{-1}Eg)$ of the vertex functions and defining the action on a coupled spin network function as the direct tensor product of the action on all occurring vertex functions, the gauge-invariant inner product becomes:

$$\begin{aligned} &\langle \eta(\pi(T(E_o))\Omega_{E_o}), \eta(\pi(T'(E'_o))\Omega_{E'_o})\rangle_{\mathcal{O}(E_o)} \\ &:= \eta(\pi(T(E_o))\Omega_{E_o})[\pi(T'(E'_o))\Omega_{E'_o}] \\ &= \begin{cases} \int_{\mathcal{A}} d\mu_o(A) \overline{G(T(E_o, A))} G(T'(E_o, A)) & \text{for} E'_o \in \mathcal{O}(E_o) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence this Hilbert space is a direct sum of Hilbert spaces indexed by gauge orbits $\mathcal{O}(E_o)$, so we denote a dense set by the elements $\pi(T(\mathcal{R}(E_o)))\Omega_{\mathcal{O}(E_o)}$, where $\Omega_{\mathcal{O}(E_o)}$ denotes the $SU(2)$-gauge group average of Ω_{E_o}.

[10] This of course implies that the gauge transformations act nontrivially only on the background E_o.

Let us conclude the discussion of the $SU(2)$-gauge invariant Hilbert space with noticing that the area-Weyl operator is gauge invariant and is hence "unchanged by the group averaging".

Let us now construct the solutions to the diffeomorphism constraint: This construction will be split into two steps, first we construct diffeomorphism-scalar couplings to the background and then we solve the diffeomorphism constraint by group-averaging the diffeomorphism-scalar coupled gauge-invariant spin network functions over the diffeomorphism group:

Noticing that the E is a vector valued density of weight one, or a dual two form, we know the transformation properties of any $F_v(E)$. We call those $F_v(E)$, that transform as scalars, scalar couplings. We will from now on restrict our attention to those coupled gauge invariant spin network function for which all couplings to the background are scalar couplings, even if it is not explicitly stated.

The action of a diffeomorphism $\phi \in \mathcal{D}$ on a Hilbert space element $\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)}$, where T_γ is a coupled gauge invariant spin network function based on the graph γ, is $U_\phi \pi(T_\gamma)\Omega_{\mathcal{O}(E_o)} = \pi(T_{\phi(\gamma)})\Omega_{\phi(\mathcal{O}(E_o))}$. Denote by $t\mathcal{D}_{\gamma,O(E_o)}$ the subgroup of diffeomorphisms that act trivially on $\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)}$, and denote by $\mathcal{D}_{\gamma,\mathcal{O}(E_o)}$ the group of diffeomrphisms that maps γ into itself and at the same time $\mathcal{O}(E_o)$ into itself, i.e. those invariances of $\mathcal{O}(E_o)$, that map γ onto itself[11]. Then

$$Sym_{\gamma,\mathcal{O}(E_o)} := \mathcal{D}_{\gamma,\mathcal{O}(E_o)}/t\mathcal{D}_{\gamma,\mathcal{O}(E_o)}$$

is a finite group consisting of the graph symmetries of γ that leave $\mathcal{O}(E_o)$ invariant. Then define $P_{\gamma,\mathcal{O}(E_o)}$ through its action:

$$P_{\gamma,\mathcal{O}(E_o)} : \pi(T_\gamma)\Omega_{\mathcal{O}(E_o)} \mapsto \frac{1}{|Sym_{\gamma,\mathcal{O}(E_o)}|} \sum_{\phi \in Sym_{\gamma,\mathcal{O}(E_o)}} T_{\phi(\gamma)}\Omega_{\mathcal{O}(E_o)}.$$

With these preparations, we are able to define the antilinear rigging map $\eta_{diff}(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)})$ through its action on any coupled gauge invariant spin network function $\pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)}$:

$$\eta_{diff}(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)}) : \pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)} \mapsto \sum_{\phi \in \mathcal{D}/\mathcal{D}_{\gamma,\mathcal{O}(E_o)}} \langle \pi(T_{\phi(\gamma)})\Omega_{\phi(\mathcal{O}(E_o))}, \pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)}\rangle_{\mathcal{G}_o}.$$

Notice that, there is only one summand that can contribute, i.e. when $\phi(\gamma) = \gamma' \wedge \phi(\mathcal{O}(E_o)) = \mathcal{O}(E'_o)$, due to orthogonality in $\mathcal{H}_{\mathcal{G}_o}$ and the fact that we factored the subgroup of diffeomorphisms that preserve $\mathcal{O}(E_o), \gamma$ out in our averaging sum. The diffeomorphism invariant inner product between $\eta(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)})$ and $\eta(\pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)})$ is

$$\begin{array}{l} \langle \eta_{diff}(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)}), \eta_{diff}(\pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)})\rangle_{\mathcal{G}_o} \\ := \eta_{diff}(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)})[\eta_{diff}(\pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)})]. \end{array} \tag{6.6}$$

[11]Notice that there may be no diffeomorphisms that act trivially on a given state, e.g. for nontrivial $\mathcal{O}(E_o)$.

The justification for indexing this inner product by three geometries $\mathcal{G}_o$ rests on calculating the diffeomorphism group averaged inner product explicitly:

$$\langle \eta_{diff}(\pi(T_\gamma)\Omega_{\mathcal{O}(E_o)}), \eta_{diff}(\pi(T'_{\gamma'})\Omega_{\mathcal{O}(E'_o)})\rangle_{\mathcal{G}_o}$$
$$= \left\{ \begin{array}{ll} \sum_{\phi \in Sym(\mathcal{O}(E_o))} \int_{\mathcal{A}} d\mu_o(A) \overline{T_{\phi(\gamma)}(A)} T'_{\gamma'}(A) & \text{for: } \mathcal{O}(E'_o) \in \mathcal{G}(\mathcal{O}(E_o)) \\ 0 & \text{otherwise,} \end{array} \right.$$

where $Sym(\mathcal{O}(E_o))$ denotes the subgroup of $\mathcal{D}$ that contains the symmetries of $\mathcal{O}(E_o)$. These results allow us to characterize the elements of a dense set in the gauge- and diffeomorphism invariant Hilbert space:

Lemma 28 *Given a three-geometry $\mathcal{G}_o$, there is a dense set in the gauge- and diffeomorphism invariant Hilbert space $\mathcal{H}_{\mathcal{G}_o}$ given by the gauge invariantly scalar coupled gauge invariant spin network functions and the inner product of $\mathcal{H}_{\mathcal{G}_o}$ coincides with equation 6.6 on this set.*

Let us close by commenting on spaces with symmetry, e.g. spaces with constant curvature: The diffeomorphism average procedure yields embedded spin networks. These are however "smeared" by the symmetry group of the geometry. It may thus be practically useful to divide the diffeomorphisms by this symmetry group, so one can work with embedded spin network functions and to keep in mind that the solutions to the diffeomorphism constraint are constructed by averaging over $Sym(\mathcal{O}(E_o))$.

6.4 Essential Geometry

We define the essential geometry of a representation of the algebra of Loop Quantum Geometry, which is similar to a "static semiclassical background geometry". Our definition does not need the notion of a vacuum vector, i.e. it can be recovered from the representation itself. We go on to show that the essential geometry of the GNS-representation of ω_{E_o} is precisely given by the Riemannian structure corresponding to E_o. Moreover, the essential geometry of the analogous state ω_o to the LOST/F-vacuum state has the degenerate essential geometry $g = 0$, and can thus be regarded as a degenerate extension of the family of ω_{E_o} states.

6.4.1 Definition of the Essential Geometry

Classical areas and volumes on a three-dimensional manifold Σ are integrals over two- and three-forms (e.g. the induced area- and volume-form on a Riemannian manifold) respectively and their value does not change when a less than two-dimensional submanifold is removed from Σ, due to the continuity of the forms. The Riemannian structure can on the other hand be recovered form the knowledge of the areas and volumes of each surface and region respectively. The situation in the LOST/F-representation of Loop Quantum Geometry is quite different: The quanta of area are distributions concentrated on the edges of a graph of a spin network state and the quanta of volume are distributions concentrated on the respective vertices. Both sets are of Riemannian measure zero for any Riemannian structure on Σ and there seems to be an "excess of geometry" concentrated on the edges and vertices.

Definition 29 *Given a state ω on the algebra of Loop Quantum Geometry, the* **excess** $\mathfrak{E}(\omega)$ *of ω is the smallest set of zero- and one-dimensional embedded submanifolds of Σ, such that the expectation values of the area and volume operators measuring surfaces and regions in $\Sigma \setminus \mathfrak{E}(\omega)$ do not change whenever countable number of zero- and one-dimensional subsets is removed from them.*

The stability of classical areas and volumes under the removal of zero- and one-dimensional subsets motivates:

Definition 30 *The* **stable geometry** *of a state ω is given by the geometry reconstructed from the expectation values $\omega(A(S \setminus \mathfrak{E}(\omega)))$ and $\omega(V(R \setminus \mathfrak{E}(\omega)))$ of all surfaces S resp. regions R.*

This motivates the definition of something similar to the "static classical background" for any representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$:

Definition 31 *For any representation $(\mathcal{H}, \pi)$ we define the* **essential geometry** *as the geometry reconstructed from the areas and volumes*

$$\begin{aligned} A(S) &= \sup_{v \in \mathcal{H}} \langle v, \pi(A(S \setminus \mathfrak{E}(\omega)))v \rangle \\ V(R) &= \sup_{v \in \mathcal{H}} \langle v, \pi(V(R \setminus \mathfrak{E}(\omega)))v \rangle, \end{aligned}$$

where S varies over all surfaces and R over all regions in Σ.

6.4.2 Essential Vacuum Expectation Values

Let us consider the essential expectation values of $\pi(T_1)\Omega_{E_o}$ and $\pi(T_2)\Omega_{E_o}$ for two distinct spin network functions T_1, T_2. Let us now prove that the essential geometry coincides for these two states:

The excess of these two states is precisely given by the minimal graphs γ_1, γ_2 upon which the two spin-network functions depend. So we can calculate the essential area expectation values of a surface S' in Σ as the area expectation value of $S \setminus \gamma_i$ of:

$$\langle \pi(T_i)\Omega_{E_o}, A(S)\pi(T_i)\Omega_{E_o} \rangle = \langle \pi(T_i)\Omega_{E_o}, A(S \setminus \gamma_i)\pi(T_i)\Omega_{E_o} \rangle = \langle \Omega_{E_o}, A(S \setminus \gamma_i)\Omega_{E_o} \rangle, \tag{6.7}$$

where we used the regularity, i.e. non-distributionality, of E_o. Let us calculate these vacuum expectation values:

$$\begin{aligned} \langle A(S) \rangle_{E_o} &= \lim_{t \to 0} \tfrac{1}{2it} \left(\omega_{E_o}(W_A^S(t)) - \omega_{E_o}(W_A^S(-t)) \right) \\ &= \tfrac{\partial}{\partial t} \exp(t A_{E_o(S)})|_{t=0} = A_{E_o}(S), \end{aligned}$$

so the essential expectation values of the area operators coincide with the classical areas of S calculated in the geometry described through E_o. Calculating higher derivatives reveals that there are no fluctuations in the essential expectation values for the area operators. Moreover it turns out that the action on the ground state Ω_{E_o} of any two area operators commutes. This allows us to calculate the essential expectation values of the volume operator without further effort: The expectation values for the volume operator of a region is

$$\begin{array}{rcl} \langle V(R)\rangle_{E_o} & = & V(R) = \langle \Omega_{E_o}, \lim_{\epsilon\to 0}\sum_{C\in L_{U,\epsilon}} V(A_a(C),...,B_c(C))\Omega_{E_o}\rangle \\ & = & V_{E_o}(R), \end{array}$$

which is independent of the choice of chart (U,ϕ). Thus, the essential geometry turns out to be precisely the geometry that is described through the classical densitized inverse triad E_o.

Since the essential geometry can be recovered from any state in the GNS-representation and is fixed by the E_o-geometry, we have a geometric background in the E_o-geometry that can be determined operationally, since the effect of a state can be determined operationally and coincides with the geometry reconstructed from the area- and volume expectation values of the vacuum vector, which is precisely the geometry encoded in E_o.

6.4.3 Relation to the LOST/F-Representation

Let us compare this situation with the GNS-representations constructed from grand canonical equilibrium states on the observable algebra of a free Klein-Gordon quantum field theory: These states are distinguished by the inverse temperature β and chemical potential μ. If the values of β,μ are sufficiently small (compared to the energy gap), then the state $\omega_{\beta,\mu}$ describes a Bose-Einstein condensate in which the ground state is macroscopically occupied, such that in the thermodynamical limit there is no possibility that fluctuations can change the overall occupation density ρ_o of the ground state. This in turn means that a similar definition of essential-ρ_o distinguishes these thermodynamical states and representations of the CCR constructed from them. Thus, one can compare the nontrivial essential geometry of a state to a "condensate of Loop Quantum Geometry".

This insight together with the result of subsection 6.4.2, i.e. that the vacuum expectation values of the geometric operators coincide with the classical values in the geometry of E_o, suggests to consider the $\mathcal{G}(E_o)$ states as a particular (non-normalizable) limit of states in the LOST/F representation: Let $\mathcal{T}(M)$ be a one parameter family of scale M-dependent simplicial decompositions of Σ, such that the mean length of each edge $\langle L(M)\rangle = M^{-1}$ and the standard deviation of the length is minimized. Calculate the classical area $A(f_i)$ of all faces f_i and the classical volumes $V(c_j)$ for each cell c_j for a value of M given the geometry $\mathcal{G}(E_o)$. Then consider the state:

$$\omega_{\mathcal{O}(E_o),M}(a) := \lim_{\leftarrow\gamma}\lim_{i\to\infty}\frac{1}{|GeomSNF^i_\gamma|}\sum_{T_\gamma\in GeomSNF^i_\gamma} C_{T_\gamma}\langle T_\gamma,\pi_o(a)T_\gamma\rangle_o, \quad (6.8)$$

where the geometric constraint is that the expectation value of the areas of all faces f_i match $A(f_i)$ and the expectation value of the volumes of each cell is $V(c_j)$ in a compact region of Σ. The finite set $GeomSNF^i_\gamma$ is a set that contains "enough" cylindrical functions T_γ with base graph γ that carry at most representation i and such that the vacuum expectation values satisfy the geometrical constraint. These states lead outside the LOST/F-representation, but approximate ω_{E_o} at least on the lattice $\mathcal{T}(M)$. One can of course put this approximation statement on its head and assert: The $\mathcal{G}(E_o)$-states approximate a one parameter family of LOST/F states, which arise, when both limits in equation 6.8 are truncated.

Chapter 7

Smooth Loop Quantum Cosmology

Having the states of chapter 6 which describe a smooth geometry at our disposal, we can naturally view them as the states describing an extended geometry, e.g. the universe. We will construct a correspondence between standard Loop Quantum Cosmology and a mini-superspace constructed from Ω_{E_o}-states. We implement Bojowalds dynamics for Loop Quantum Cosmology thereon and explain a road map towards a fundamental dynamics for Ω_{E_o} states.

7.1 General Idea

Bojowald initially constructed symmetric spin network functions [12] as spin network functions that depend on a symmetric connection. The interpretation of the states constructed this way was explained in [45] as equivalence classes of spin network functions on a piecewise straight graph that depend, where the equivalence was taken with respect to the dependence on symmetric connections. The general idea that we propose in this short chapter is to depart from considering explicit functionals of the symmetric connection resp. equivalence classes of functionals of the complete connection and to consider the Ω_{E_o}-states as the states that correspond to a symmetry reduction:

We saw that the essential geometry of a vacuum vector Ω_{E_o} is precisely the geometry described by the classical densitized inverse dreibein E_o. This suggests the use of these states with symmetric E_o as states describing spatially symmetric geometries as they are usually used in a minisuperspace construction. Establishing a correspondence between the symmetric E_o and Bojowalds states gives these an interpretation as states on full Loop Quantum Geometry. In turn, one can use this correspondence to use the well understood dynamics of Loop Quantum Cosmology as a "cosmologically induced" dynamics for the E_o-states. The construction of such a correspondence does however require the introduction of degrees of freedom that correspond to Bojowalds inflaton field in the framework introduced in the previous chapter, which we will construct at the end of the next section.

7.2 Construction of a Cosmological Quantum Minisuperspace

Neglecting the discussion of a differentiability class, the construction of DeWitt's superspace starts with fixing a topology for the Cauchy surface Σ and considering the set $\mathcal{Q}$ of all Euclidean 3-metrics q on Σ. The superspace is then the quotient of $\mathcal{Q}$ by the group $\mathcal{D}$ of diffeomorphisms that are generated by the diffeomorphism constraints. One then expects that a quantum theory can be constructed using wave functions on the superspace; a (geometrically) semiclassical wave package is then one that is strongly peaked around a particular classical geometry, so neglecting fluctuations it is very close to an eigenstate of geometry. Since the Ω_{E_o}-states are eigenstates of the geometry, we can naturally replace the semiclassical states with Ω_{E_o}-states. Let us now revisit the construction of the minisuperspace of standard Loop Quantum Cosmology and then apply this replacement:

For the construction of a minisuperspace in connection variables (compare appendix B.4), we assumed a symmetry group and worked the symmetric phase space out, which is constructed as follows: Given a symmetry group $\mathcal{S}$, we call a phase space point (A, E) symmetric, if for all $s \in \mathcal{S}$ there exists a local gauge transformation $g_s : \Sigma \to SU(2)$, such that

$$s^*(A, E) = (g_s^{-1} A g_s + g_s^{-1} d g_s, g_s^{-1} E g_s).$$

For the purpose of isotropic cosmology we fix the gauge- and diffeomorphism invariance by introducing a triad e_i^a and a co-triad ω_a^i which are invariant under $\mathcal{S}$. Then every symmetric point (A_o, E_o) that satisfied the Gauss- and diffeomorphism constraint can be mapped into an equivalent pair of the form:

$$(A_s, E_s) = (c\omega^i \tau_i, p\sqrt{|q(e)|} e_i \tau^i).$$

In standard Loop Quantum Cosmology, one builds wave functions of the symmetric connection and constructs an inner product for these from the inner product of Loop Quantum Gravity. Bojowalds procedure yields a Hilbert space that is the span of states $\psi_\mu : \mu \in \mathbb{R}$, which turn out to be eigenstates of the symmetric triad, and he induces an inner product $\langle \psi_\mu, \psi_{\mu'} \rangle = \delta_{\mu,\mu'}^{Kron.}$. The Ω_{E_o}-states with symmetric $E_o(p) = p\sqrt{|q(e)|} e_i \tau^i$ are eigenstates of the geometric observables that satisfy these symmetry relations. The inner product is, according to the construction in the previous chapter, induced as:

$$\langle \Omega_{E_o(\mu)}, \Omega_{E_o(\mu')} \rangle = \delta_{\mu,\mu'}^{Kron.},$$

which matches the inner product of standard Loop Quantum Cosmology if we use the replacement:

$$\psi_\mu \to \Omega_{E_o(\mu)}. \tag{7.1}$$

We thus construct the geometric sector of the cosmological quantum Hilbert space $\mathcal{H}^{cosm.}_{grav}$ as the span of the states $\Omega_{E_o(\mu)} : \mu \in \mathbb{R}$ and use their inner product as constructed in the previous chapter.

To construct a nontrivial cosmological model, one needs to introduce a symmetric matter degree of freedom and construct a cosmological Hilbert space $\mathcal{H}^{cosm.} = \mathcal{H}^{cosm.}_{grav.} \otimes \mathcal{H}^{cosm.}_{matter}$. Since isotropic matter distributions are not included in the matter states developed for standard Loop Quantum Gravity, we have to apply the construction of the previous chapter to the matter degrees of freedom:

7.2.1 Smooth Matter for Loop Quantum Gravity

Loop Quantum Gravity naturally incorporates Higgs fields with compact gauge group. A single real scalar field ϕ can be modeled as the gauge-variant sector of a Higgs field theory with gauge group $\bar{\mathbb{R}}_{Bohr}$. The fundamental observables are the point holonomies $U_x(\lambda) := \exp(i\lambda\phi(x))$ and the conjugated momenta, which are densities of weight one, can be integrated over regions R to give the fundamental momentum variables $\pi_R := \int_R \pi$, where π denotes the field momentum. The fundamental Weyl-operators $V(R)$ are constructed from the exponentiated Poisson action of the π_R on the $U_x(\lambda)$[1] corresponding to $V_R(\mu) := \exp(i\mu\pi_R)$. This yields the Weyl commutation relations $V^*_R(\mu)U_x(\lambda)V_R(\mu) = e^{i\lambda\mu\delta_{x\in R}}U_x(\lambda)$. Thus, the fundamental observable algebra consists of the finite sums $a := \sum_{i=1}^N a_i U_{x_i}(\lambda_i)V_{R_i}(\mu_i)$. Since the classical ϕ, π Poisson-commute with the gravitational degrees of freedom, we are lead to implement the sums a as operators that commute with all gravitational observables. The fundamental observable algebra for the gravity+scalar system is thus generated by the finite sums $b := \sum_{i=1}^N Cyl_i U_{x_i^1}(\lambda_i^1)...U_{x_i^{n_i}}(\lambda_i^{n_i})W_i V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})$, where Cyl_i denotes cylindrical functions of the Ashtekar connection and W_i the adjusted Weyl-operators. Using matter-cylindrical functions $Cyl_{matt.}$ as functions of the scalar field that can be written as $Cyl_{matt.} = F(U_{x_1}(\lambda_1), ..., U_{x_n}(\lambda_n))$, where $F : \bar{\mathbb{R}}^n_{Bohr} \to \mathbb{C}$ is continuous and almost periodic, we can extend the fundamental observable algebra to finite sums $b := \sum_{i=1}^N Cyl_i Cyl_{matt.i}(\lambda_i)W_i V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})$.

The canonical Ashtekar-Lewandowski state ω_o on these sums b is defined (analogous to a lattice gauge theory with scalars) as

$$\begin{array}{rcl} \omega_o(b) & := & \sum_{i=1}^N \int d\mu_{AL}(A)d\mu_{Bohr}(y_{1_i})...d\mu_{Bohr}(y_{n_i})Cyl_i(A)F_i(y_{1_i}, ..., y_{n_i}) \\ & =: & \sum_{i=1}^N \int d\mu_{AL}(A,\phi)Cyl_i(A)Cyl_{matt.i}(\phi), \end{array} \tag{7.2}$$

defining the extended Ashtekar-Lewandowski measure. Using this measure we are in the position to apply lemma 17 after noticing that the operators W_i and $V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})$ commute with each other, so the combined momentum Weyl-group is the direct product of the gravitational momentum Weyl-group and the matter momentum Weyl-group. This means that we can define a group morphism $F : \mathcal{W} \to U(1)$ as the product of $F_{grav.}$ acting on the gravitational

[1] The considered Poisson bracket is $\{\pi_R, U_x(\lambda)\} = i\lambda U_x(\lambda)$ if $x \in R$ and vanishes otherwise.

part only and $F_{matt.}$ acting on the matter part only. We have specified $F_{grav.}$ already in the construction of the Ω_{E_o}-states, so we only have to construct $F_{matt.}$ here.

The nontrivial group relations are: $V_R(\mu_1)V_R(\mu_2) = V_R(\mu_1+\mu_2)$, $(V_R(\mu))^{-1} = (V_R(\mu))^* = V_R(-\mu) \Rightarrow V_R(0) = id$ and $V_{R_1}(\mu)V_{R_2}(\mu) = V_{R_1 \cup R_2}(\mu)$ for all disjoint regions R_1, R_2. Thus, defining $F_{matt.}$ as

$$F_{matt.}(V_{R^1}(\mu^1)...V_{R^m}(\mu^m)) := \exp\left(i\left(\mu^1\int_{R^1}\pi_o + ... + \mu^m\int_{R^m}\pi_o\right)\right) \quad (7.3)$$

for some classical density π_o on Σ satisfies the group relations and thus furnishes a group morphism into $U(1)$.

To define a state using this relation, we need to verify that the extended Ashtekar-Lewandowski measure is left invariant under the extended momentum Weyl-group. This is most easily verified using the observation that the functions of the form $U_{x_1}(\lambda_1)...U_{x_n}(\lambda_n)$ are an orthonormal set in the extended Hilbert space and dense in the matter configuration variables and thus in the tensor factor $\mathcal{M}_{matt.}$ of the extended Hilbert space. Using these "scalar network functions" and that the gravitational operators commute with the matter operators, we calculate

$$\int d\mu_o(A,\phi)U_{x_1}(\lambda_1)...U_{x_n}(\lambda_n) = \begin{cases} 1 & \text{if all } \lambda_i \text{ vanish} \\ 0 & \text{otherwise.} \end{cases}$$

Using the density of the scalar network functions we can expand any configuration observable as $f(\phi) = \sum_{i=1}^N f_i U_{x_i^1}(\lambda_i^1)...U_{x_i^{n_i}}(\lambda_i^{n_i})$ and calculate the action of any momentum-Weyl-operator

$$\begin{array}{rcl} \int d\mu_o(A,\phi)V_R(\mu)^* f(\phi)V_R(\mu) & = & \int d\mu_o(A,\phi)V_R(\mu)^* f_1 V_R(\mu) \\ & = & f_1 \\ & = & \int d\mu_o(A,\phi)f(\phi), \end{array} \quad (7.4)$$

where we have assumed without loss of generality that there is precisely one constant term, which is the first summand f_1 in the expansion of f; yielding the invariance of the extended measure under the extended momentum-Weyl-group.

Using lemma 17 for the matter part, lemma 26 for the gravitational part we can use equation 7.3 to define a state ω_{E_o,π_o} for any classical densitized inverse triad E_o and any density π_o through:

$$\omega_{E_o,\pi_o}(b) := \sum_{i=1}^N F(W_i)F_{matt.}(V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i}))\int d\mu_o(A,\phi)Cyl_i(A)Cyl_{matt.i}(\phi) \quad (7.5)$$

Inserting $\pi(R) := -i\frac{\partial}{\partial t}V_R(\mu)|_{t=0}$ and using linearity of the state ω_{E_o,π_o}, we obtain the vacuum expectation values for $\pi(R)$:

$$\omega_{E_o,\pi_o}(\pi(R)) = \int_R \pi_o, \quad (7.6)$$

which coincides with the classical values $\pi_o(R)$, which is precisely the desired behavior as a quasilocal eigenstate of the field momentum corresponding to a classical field momentum π_o.

Let us construct the matter part of the GNS-representation out of this state using the κ_F-map for the matter extended F to relate this representation to the canonical Schrödinger representation (constructed as the GNS-representation of ω_o) for the matter part. The Gel'fand ideal reduces for pure matter observables $a = \sum_{i=1}^N Cyl_{matt.i} V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})$ to the set $\{a : \sum_{i=1}^N Cyl_{matt.i} = 0\}$, so we have the canonical GNS-representation for the matter part as the extension of the representation π_o:

$$
\begin{array}{rcl}
\eta_o(a): & A \mapsto & \sum_{i=1}^N Cyl_{matt.i} \\
\pi_o(a)\chi: & A \mapsto & \sum_{i=1}^N Cyl_{matt.i} \alpha_{V_{R_i^1}(\mu_i^1)}(...\alpha_{V_{R_i^N}(\mu_i^N)}(\chi)...)(A) \\
\langle \chi_1, \chi_2 \rangle_o & = & \int d\mu_o(A,\phi) \overline{\chi_1(\phi)} \chi_2(\phi).
\end{array}
$$

Using lemma 20, we can immediately construct the GNS-representation π_{π_o} for the matter part of ω_{E_o,π_o} and obtain a kinematic matter Hilbert space $\mathcal{H}^{kin.}_{matt.}(\pi_o)$ with the following representation of the matter observables:

$$
\begin{array}{rclcl}
\eta_{\pi_o}(a) & = & \eta_o(\kappa_F(a)) & A \mapsto & \sum_{i=1}^N Cyl_{matt.i} F(V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})) \\
\pi_{\pi_o}(a)\chi & = & \pi_o(\kappa_F(a))\chi: & A \mapsto & \sum_{i=1}^N Cyl_{matt.i} F(V_{R_i^1}(\mu_i^1)...V_{R_i^{m_i}}(\mu_i^{m_i})) \\
 & & & & \alpha_{V_{R_i^1}(\mu_i^1)}(...\alpha_{V_{R_i^N}(\mu_i^N)}(\chi)...)(A) \\
\langle \chi_1, \chi_2 \rangle_{\pi_o} & = \langle \chi_1, \chi_2 \rangle_o & = & & \int d\mu_o(A,\phi) \overline{\chi_1(\phi)} \chi_2(\phi).
\end{array}
\tag{7.7}
$$

Using the action of a diffeomorphism φ on π_R induces the action on $V_R(\mu)$. The same reasoning as in the previous chapter then yields that one has to sum over all GNS-representations constructed form all distinct smooth densities $\rho = \varphi^* \pi_o$ in the diffeomorphism class of π_o to obtain a unitary action of the diffeomorphisms. The action of the Gauss constraint on the matter part is trivial, since ϕ is a scalar under the $SU(2)$-gauge transformations.

To put it in a nutshell, we obtain a matter Hilbert space $\mathcal{H}_{matt.}$ that is the direct sum of the GNS-Hilbert spaces $\mathcal{H}_{matt.} = \oplus_{\rho \in \mathcal{D}(\pi_o)} \mathcal{H}^{kin.}_{matt.}(\rho)$ for all distinct ρ in the diffeomorphism class of π_o. The Hilbert space for the matter-gravity system is then the tensor product of the gravitational and the matter Hilbert space and the representation of the gravity and matter observables act on the respective tensor factor as constructed above, while leaving the other factor invariant.

7.2.2 Matter-Gravity Minisuberspace

Following the strategy outlined at the beginning of this section, we construct a minisuperspace consisting of precisely those GNS-vacuum vectors that are symmetric eigenstates of the geometric operators and the matter-momenta. Thus, taking π_o to be homogeneous, i.e. $\pi_o(x) = \pi_o(x_o) = \nu$ in the homogeneous chart, we build the matter part of the cosmological minisuperspace by taking the direct sum of the matter GNS-vacua Ω_{π_o} for all homogeneous π_o. Using the construction in the previous section, this induces the inner product for the matter part:

$$\langle \Omega_{\pi_o(\nu_1)}, \Omega_{\pi_o(\nu_2)} \rangle = \delta^{Kron.}_{\nu_1,\nu_2}. \tag{7.8}$$

This induces the representation of the homogeneous matter observables on $\mathcal{H}^{cosm.}_{matter} = L^2(\bar{\mathbb{R}}_{Bohr}, d\mu_{Bohr}) \sim \oplus_{\nu \in \mathbb{R}} \Omega_{\pi_o(\nu)}$. The combined gravity-matter system is then represented on the cosmological Hilbert space $\mathcal{H}^{cosm.} = \mathcal{H}^{cosm.}_{grav.} \otimes \mathcal{H}^{cosm.}_{matter}$.

Choosing the isotropic cosmological chart, we can take any geometric observable that coincides classically with the volume density as a representative operator for the volume density. Since all classical representatives yield the same volume density (as we see by applying the results of section 6.4.2), we are able to introduce an unambiguous volume density operator $\hat{\rho}$. Evaluating this using any representative yields:

$$\langle \mu \otimes \nu, \hat{\rho} \mu \otimes \nu \rangle = \rho_o \mu. \tag{7.9}$$

The same argument of course applies for the field momentum density, so we are able to introduce an unambiguous operator $\hat{\pi}$ with expectation values

$$\langle \mu \otimes \nu, \hat{\pi} \mu \otimes \nu \rangle = c_o \nu. \tag{7.10}$$

This argument can of course not be applied for the construction of connection respectively field strength operators.

Using these expectation values and their spatial symmetry, we have a precise correspondence between the states constructed here and the standard Loop Quantum Cosmology states:

$$\psi_{\mu,\nu} \leftrightarrow \Omega_{E_o(\mu),\pi_o(\nu)}. \tag{7.11}$$

7.3 Implementation of the Loop Quantum Cosmology Dynamics

Using the correspondence (equation 7.11), we can relate Bojowalds states to states on the full observable algebra of Loop Quantum Gravity. Using this relation from left to right allows us to investigate generic perturbations around these states without having to model the perturbation before, because the states on the right are states on the full observable algebra of Loop Quantum Gravity. Using the relation from the right to the left allows us on the other hand to define a dynamics for the cosmologically interesting states by using the established Loop Quantum Cosmology dynamics. Using this cosmological dynamics is an approximation only, but may lead to new insight since the dynamics of full Loop Quantum Gravity is an unresolved issue.

Let us put the induction of a dynamics for the Ω_{E_o,π_o}-states on a firmer basis by comparing them with coherent states: Let q_i, p_i be canonically conjugate phase space coordinates and let ω_o be a Gaussian ground state with vanishing vacuum expectation values for all q_i and p_i. Then one can define a coherent state $\omega_{\vec{x}_o,\vec{p}_o}$ by setting for the elementary Weyl operators $U_i(\lambda) = e^{i\lambda x_i}, V_i(\mu) = e^{i\mu p_i}$:

$$\omega_{\vec{x}_o,\vec{p}_o}(U_i(\lambda_i)V_i(\mu_i)) := e^{i\mu_i x^i_o} e^{i\lambda_i p^i_o} \omega_o(U_i(\lambda_i)V_i(\mu_i))e^{i\phi}, \tag{7.12}$$

where ϕ depends on the chosen operator ordering. The definition of this state as the linear extension of equation 7.12 is completely analogous to the definition of the E_o, π_o-states, which we can thus view as kinematic coherent states for Loop Quantum Gravity. In may systems, one has a time evolution of coherent states that is very close to a classical evolution and some decoherence. Assuming such an action of the constraint operator(s) on the E_o, π_o-states justifies the use of the right-to-left direction of the correspondence 7.11.

Standard (flat space) Loop Quantum Cosmology uses a single scalar constraint acting on the basis vectors $\psi_{\mu,\nu}$ as:

$$\begin{array}{rcl} \hat{C}_{grav}\psi_{\mu,\nu} & = & (V_{\mu+5\mu_o} - V_{\mu+3\mu_o})\psi_{\mu+4\mu_o,\nu} - 2(V_{\mu+\mu_o} - V_{\mu-\mu_o})\psi_{\mu,\nu} \\ & & +(V_{\mu-3\mu_o} - V_{\mu-5\mu_o})\psi_{\mu-4\mu_o,\nu} \\ & = & -\frac{1}{3} 8\pi G \iota^3 \mu_o^3 l_{Pl}^2 \hat{C}_{matt}(\mu)\psi_{\mu,\nu}, \end{array} \tag{7.13}$$

where $V_\mu = \left(|\mu|\frac{\iota}{6}\right)^{\frac{3}{2}}$ is the volume eigenvalue for ψ_μ and $\hat{C}_{matt}$ is the matter part of the scalar constraint, which depends on the matter and potential considered in the specific model. Using the correspondence, we find the from standard Loop Quantum Cosmology induced dynamics for the cosmologically interesting Ω_{E_o,π_o} (by simple substitution):

$$\begin{array}{rcl} \hat{C}_{grav}\Omega_{E_o(\mu),\pi_o(\nu)} & = & (V_{\mu+5\mu_o} - V_{\mu+3\mu_o})\Omega_{E_o(\mu+4\mu_o),\pi_o(\nu)} \\ & & -2(V_{\mu+\mu_o} - V_{\mu-\mu_o})\Omega_{E_o(\mu),\pi_o(\nu)} \\ & & +(V_{\mu-3\mu_o} - V_{\mu-5\mu_o})\Omega_{E_o(\mu-4\mu_o),\pi_o(\nu)} \\ & = & -\frac{1}{3} 8\pi G \iota^3 \mu_o^3 l_{Pl}^2 \hat{C}_{matt}(\mu)\Omega_{E_o(\mu),\pi_o(\nu)}, \end{array} \tag{7.14}$$

The phenomenology of this dynamics by itself is of course the same as the phenomenology of standard Loop Quantum Cosmology. It is however applied to a subspace of a full theory of quantum gravity and may as such be used to shed light on the dynamics thereof.

7.4 Towards a Fundamental Dynamics for DQG

The main problem of constructing a complete anomaly-free implementation of the scalar constraint (resp. the master constraint) for full Loop Quantum Gravity is the difficulty in finding phenomenologically acceptable diffeomorphism covariant regulators for the classical constraints in terms of loop variables. These regulators are necessary due to the singular smearing of the loop variables and anomaly-freeness rules out most regulators. In ordinary quantum field theory on a background on the other hand one constructs various regulators through point-splitting the operator products that depend in the same points. In standard Loop Quantum Gravity there is however no intrinsic notion of distance outside the graph, so one can not split points, let them approach each other again and then renormalize the observables that emerge in the limit of coinciding points. Using an E_o-representation however, one has an intrinsic notion of distance (w.r.t. the essential geometry), suggesting that one can use very similar techniques as in a background dependent field theory. In short: The E_o-geometry opens the door for the application of a wide range of (common) regularization techniques.

Other than background dependent quantum field theories however, one does not have to implement the entire set of Hamiltonian constraints, but it is sufficient for the construction of a physical theory to construct the kernel. Since we are using the idea of Dirac to postpone the imposition of the constraints until after quantization, we have to ensure that the so constructed kernel-Hilbert space is a quantization of the classical constraint surface. This means that for each point (A, E) on the classical constraint surface, there is a coherent state peaked around this point. We saw in the previous section that the Ω_{E_o} states are coherent states, which are due to diffeomorphism covariance sharp in the densitized inverse triad E and infinitely wide in the connection A. Moreover, all spin network excitations around Ω_{E_o} are still sharp in E, but with distributional spatial eigen-geometries[2]. This suggests the construction of the constraint kernel through regularizing the distributional spatial eigen-geometries stemming from the spin-network excitations by point splitting techniques in the essential E_o-geometry.

Viewing the Ω_{E_o} states not only as kinematic coherent states, but as dynamic coherent states[3], one can describe a programme to construct the kernel

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[2]Using the volume operator defined in the next chapter, we have a set of geometric observables that is (1) large enough to reconstruct the geometry and (2) when using an external regularization has the spin network functions as eigenfunctions.

[3]There is a nice motivation for considering the E_o-states as dynamic coherent states: harmonic oscillator coherent states $|(x_o, p_o)_w\rangle$ with the matching width parameter w transform classically under time evolution, i.e. $U_t|(x_o, p_o)_w\rangle = e^{i\phi(t)}|(x(x_o, p_o, t), p(x_o, p_o, t))_w\rangle$, where $x(x_o, p_o, t), p(x_o, p_o, t)$ are the classical trajectories, otherwise they obtain a time-dependent width $w(t)$. The E_o-states transform classically under the spatial diffeomorphisms, i.e. $U_\phi|\psi_\gamma\rangle_{E_o} = |\psi_{\phi(\gamma)}\rangle_{E_o}$, which is a subset of the four-diffeomorphisms chosen by the foliation of space-time. To put all four-diffeomorphisms on an equal footing implies to consider the E_o-states as dynamical coherent states as well as kinematic coherent states.
To show that this argument does not only work for a harmonic oscillator, let us assume that $|\alpha\rangle$ are coherent states, i.e. $a|\alpha\rangle = \alpha|\alpha\rangle$ with the phase space peakedness property, i.e. $\alpha \mapsto |\langle\beta, \alpha\rangle|^2$ falls off rapidly as β departs from α. These states are assumed to evolve classically under the phase space flow σ_t, so $|\alpha(t)\rangle = |\sigma_t(\alpha)\rangle$. The condition for the scalar product so this action is unitary is: $\langle\sigma_{-t}(\alpha), \beta\rangle = \langle\alpha, \sigma_t(\beta)\rangle$. A scalar product that has the phase space peakedness property and supports this flow unitarily can be defined as follows: Denote the σ-trajectory that contains α by $T(\alpha)$ and choose a hypersurface that intersects each trajectory precisely once at a base point α_o. Then each point α can be given coordinates $(\alpha_o(\alpha), t(\alpha))$, where $t(\alpha)$ is defined through $\sigma_t(\alpha_o(\alpha)) = \alpha$ and $\alpha_o(\alpha)$ is the base point of the trajectory that contains α. Choose a metric d on the hypersurface, which extends to a metric on the trajectories. Then for suitable flows,

$$\langle\alpha, \beta\rangle := \exp\left(-d^2(\alpha_o, \beta_o) + i\phi(\alpha_o, \beta_o, t(\alpha), t(\beta)) - c^2(t(\alpha) - t(\beta))\right)$$

projection operator as follows:

For this purpose, we first define a notion of closeness for regions R, R' using the essential geometry E_o of the vacuum state Ω_{E_o} by defining a distance between them through the difference of essential areas:

$$d_{E_o}(R, R') := (V_{E_o}(R) + V_{E_o}(R')) - 2V_{E_o}(R \cap R'). \tag{7.15}$$

For every gauge-invariant embedded spin network function $\psi = \pi_{E_o}(SNF)\Omega_{E_o}$, we define a sequence of smooth densitized inverse triads $\{E_n\}_{n=1}^{\infty}$ to be a regulator of ψ, if for every region R in Σ and every $\epsilon > 0$ there exists an $n_o > 1$ and a region R' with $d_{E_o}(R, R') \leq \epsilon$, such that

$$\langle \psi, V(R')\psi \rangle - V_{E_n}(R) \leq \epsilon l_{Pl}^3 \, \forall \, n > n_o. \tag{7.16}$$

The regulator sequences $\{E_n\}_{n=1}^{\infty}$ tend towards the distributional quantum geometry of ψ in the limit $n \to \infty$, but each element is a smooth densitized inverse triad. This allows us to evaluate functions of each element of the sequence classically in a way very analogous to point-splitting techniques.

Let us now consider sequences of phase space points $\{(A_n, E_n)\}_{n=1}^{\infty}$ consisting of pairs of smooth connections and smooth densitized inverse triads. We define this sequence to be a constraint solution for ψ, if all elements (A_n, E_n) are solutions to the (diffeomorphism-extended) master constraint

$$M = \int_\Sigma \frac{d^3\sigma}{\sqrt{|q|}} \left(C^2(\sigma) + q_{ab} D^a(\sigma) D^b(\sigma) \right)$$

and $\{E_n\}_{n=1}^{\infty}$ by itself is a regulator of ψ.

Using the density of the gauge-invariant embedded spin network functions in the gauge-invariant Hilbert space, we are able to define the kernel of the master constraint operator through the extension by density of the action on gauge-invariant embedded spin network functions ψ:

$$\hat{P}\psi := \psi \begin{cases} 1 & \text{if } \exists \text{ a constraint solution } \{(A_n, E_n)\}_{n=1}^{\infty} \text{ for } \psi \\ 0 & \text{otherwise.} \end{cases} \tag{7.17}$$

It follows immediately from the construction that there is a state Ω_{E_o} for each classical solution to the master constraint (A_o, E_o), that is peaked around the classical 3-geometry of the solution.

---

defines a scalar product on the span of the states $|\alpha\rangle$ whenever ϕ is antisymmetric and the inner product satisfies positivity, which supports $U_t|\alpha\rangle := |\sigma_t(\alpha)\rangle$ as a unitary representation of the flow σ_t. The commutation relation of a, a^* will however in general fail to be canonical.

Chapter 8

Loop Quantum Geometry based on a fundamental Area Operator

This chapter is devoted to a version of quantum geometry that is based entirely on the existence of a fundamental area operator. Since the area operator is viewed as fundamental, it remains to construct a volume and a length operator; we start with the construction of the volume operator. This work was triggered by the discovery of the smooth geometry sates (chapter 6) and it is due to the existence of this version of quantum geometry that the algebra used in chapter 6 is physically complete.

8.1 Volume Operator with a Fundamental Area Operator

The quantization strategy is analogous to the one applied in section C.3: We use the classical expression for the volume functional of a region, reexpress it as a limit of cell volumes of a family of partitions of the region, which we then reexpress using classical areas. Then we replace the classical area variables with the corresponding area operators and take the limit on spin network functions. We then use a similar averaging procedure to obtain a background independent volume operator for spin network functions. The final operator is then the Hermitian extension of the essentially self-adjoint operator defined on spin network functions.

8.1.1 Classical Volume Functional

Given a three-dimensional Riemannian manifold (Σ, q) with metric q, we derive a functional for the volume of a three-dimensional region $R \subset \Sigma$ as the limit of Riemann sums over volumes of cells in a homogeneous metric. Each cell is a parallelepiped and its volume is expressed through the six independent area measurements in the parallelepiped. This is a three-dimensional generalization of Heron's formula, however the derivation is much more involved.

Volume of a Parallelepiped in a homogeneous Metric

Consider a parallelepiped P_o in a homogeneous metric background on $\mathbb{R}^3$: More concretely, consider a homogeneous metric q^o_{ab}. Then due to homogeneity we can assume without loss of generality that the parallelepiped $P(\vec{a}_o, \vec{b}_o, \vec{c}_o)$ is spanned by three (linearly independent) vectors at the origin. Let us change into the Riemannian normal coordinate system at the origin, so the transformed metric is Euclidean $q_{ab} = \delta_{ab}$ due to homogeneity. The coordinate transform amounts to the multiplication by a constant invertible matrix. The transformed parallelepiped $P(\vec{a}, \vec{b}, \vec{c})$ is then spanned by the three (linearly independent) transformed vectors $\vec{a}, \vec{b}, \vec{c}$.

Using the rotational freedom of Euclidean $\mathbb{R}^3$, we can assume without loss of generality that the three spanning vectors take the form:

$$\vec{a} = (a_1, 0, 0), \qquad \vec{b} = (b_1, b_2, 0), \qquad \vec{c} = (c_1, c_2, c_3). \tag{8.1}$$

The volume of $P(\vec{a}, \vec{b}, \vec{c})$ is

$$V(P) = det(\vec{a}, \vec{b}, \vec{c}) = a_1 b_2 c_3. \tag{8.2}$$

There are six independent area measurements at this parallelepiped, the three independent face areas (denoted by $\sqrt{A_a}$)[1] and three independent diagonal cross cuts (denoted by $\sqrt{B_a}$). There is an ambiguity about which three of the six possible diagonal cross cuts one should choose, which we fix by choosing the three that contain the origin. Having fixed the three diagonals, we may move the surface areas to the center, which is an isometry due to homogeneity, so all six areas intersect at the center of P. The resulting surfaces are depicted in figure 8.1. The squares of the areas have the following expression on terms of the components of $\vec{a}, \vec{b}, \vec{c}$:

$$\begin{array}{rclcl}
A_a &=& |\vec{b} \times \vec{c}|^2 &=& (b_2 c_1 - b_1 c_2)^2 + \left(b_1^2 + b_2^2\right) c_3^2 \\
A_b &=& |\vec{a} \times \vec{c}|^2 &=& a_1^2 \left(c_2^2 + c_3^2\right) \\
A_c &=& |\vec{a} \times \vec{b}|^2 &=& a_1^2 b_2^2 \\
B_a &=& |(\vec{b} + \vec{c}) \times \vec{a}|^2 &=& a_1^2 \left((b_2 + c_2)^2 + c_3^2\right) \\
B_b &=& |(\vec{a} + \vec{c}) \times \vec{b}|^2 &=& (b_2 (a_1 + c_1) - b_1 c_2)^2 + \left(b_1^2 + b_2^2\right) c_3^2 \\
B_c &=& |(\vec{a} + \vec{b}) \times \vec{c}|^2 &=& (b_2 c_1 - (a_1 + b_1) c_2)^2 + \left((a_1 + b_1)^2 + b_2^2\right) c_3^2.
\end{array} \tag{8.3}$$

[1]It turns out to much more convenient to express the square of the area rather than the area itself, which is the reason for the notation $\sqrt{A_a}$ and $\sqrt{B_a}$ for the respective areas.

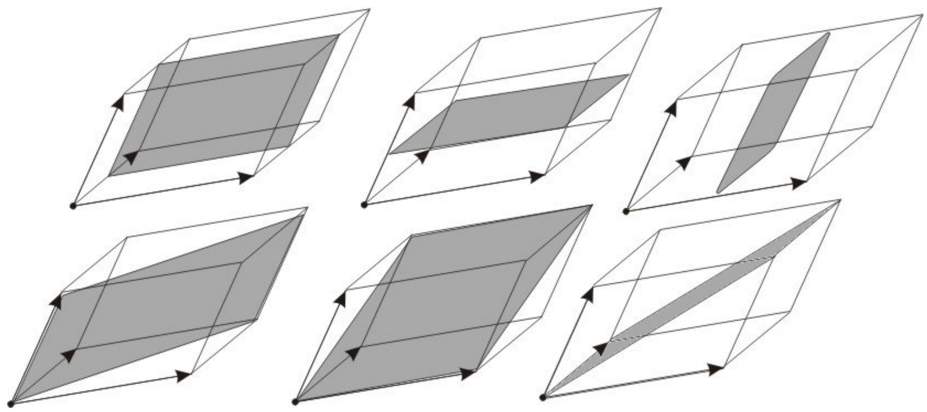

Figure 8.1: The six surfaces of the parallelepiped spanned by the vectors (arrows) containing the origin (solid dot) with independent areas. The top line contains the translated faces, the bottom line the diagonal cross sections.

Inverting equation 8.2 for $a_1 = \frac{V}{b_2 c_3}$ and inserting into equation 8.3 yields the system of six independent equations, which we want to solve for V while eliminating the remaining vector components b_1, b_2, c_1, c_2, c_3:

$$\begin{aligned}
A_a &= (b_2 c_1 - b_1 c_2)^2 + \left(b_1^2 + b_2^2\right) c_3^2 \\
A_b &= \frac{\left(c_2^2 + c_3^2\right) V^2}{b_2^2 c_3^2} \\
A_c &= \frac{V^2}{c_3^2} \\
B_a &= \frac{\left((b_2 + c_2)^2 + c_3^2\right) V^2}{b_2^2 c_3^2} \\
B_b &= \left(b_1^2 + b_2^2\right) c_3^2 + \frac{(b_2 c_1 c_3 - b_1 c_2 c_3 + V)^2}{c_3^2} \\
B_c &= \left(b_2 c_1 - c_2 \left(b_1 + \frac{V}{b_2 c_3}\right)\right)^2 + {c_3}^2 \left(b_2^2 + \left(b_1 + \frac{V}{b_2 c_3}\right)^2\right)
\end{aligned} \tag{8.4}$$

Solving this system of equations is possible and was carried out, however it amounts to very messy algebraic manipulations. (It turned out to be necessary to map out the solution strategy and to implement it in computer algebra, due to the size of the appearing terms, which seemed unfeasible on paper.) The final result is:

$$\begin{aligned}
V = \Bigg| & \frac{a\, b^2\, c^2\, d\, A_b}{4\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} + \frac{b^4\, d\, e^2\, A_b}{16\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} \\
& - \frac{b^4\, d^2\, f\, A_b}{16\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} - \frac{a\, c^4\, A_c}{b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2} \\
& + \frac{b^2\, c^2\, e^2\, A_c}{4\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} + \frac{b^2\, c^2\, d\, f\, A_c}{4\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} \\
& + \frac{\sqrt{a\, b^6\, c^2\, d^2\, e^2\, {A_b}^2 - 4\, a\, b^4\, c^4\, d\, e^2\, A_b\, A_c + b^6\, c^2\, d\, e^4\, A_b\, A_c - b^6\, c^2\, d^2\, e^2\, f\, A_b\, A_c + 4\, b^4\, c^4\, d\, e^2\, f\, {A_c}^2}}{4\,(b^4\, d^2\, {A_b}^2 - 8\, b^2\, c^2\, d\, A_b\, A_c + 16\, c^4\, {A_c}^2)} \Bigg|^{\frac{1}{4}}
\end{aligned} \tag{8.5}$$

where:

$$
\begin{aligned}
a &= \left(-{A_b}^2 - (A_c - B_a)^2 + 2\,A_b\,(A_c + B_a)\right)\left({A_a}^2 + (A_c - B_b)^2 - 2\,A_a\,(A_c + B_b)\right)\\
b &= \sqrt{{A_b}^2 + (A_c - B_a)^2 - 2\,A_b\,(A_c + B_a)}\\
c &= A_b\left({A_b}^2 + (A_c - B_a)^2 - 2\,A_b\,(A_c + B_a)\right)\\
d &= (A_b + A_c - B_a)^2\left({A_b}^2 + (A_c - B_a)^2 - 2\,A_b\,(A_c + B_a)\right)\\
e &= \left(-{A_b}^2 - (A_c - B_a)^2 + 2\,A_b\,(A_c + B_a)\right)^{\frac{3}{2}}(A_a + A_b - B_c)\\
f &= \left(-{A_b}^2 - (A_c - B_a)^2 + 2\,A_b\,(A_c + B_a)\right)\left({A_a}^2 + (A_b - B_c)^2 - 2\,A_a\,(A_b + B_c)\right)
\end{aligned}
$$

Classical Volume of a Region

Let (Σ, q) be a three-dimensional Riemannian manifold with analytic atlas $\{(U_i, \phi_i)\}_{i=1}^N$ for Σ and regular metric q thereon. With regular, we mean that given any analytic chart (U, ϕ) and any coordinate cube $C(\epsilon)$ of coordinate size ϵ in this chart that the expression $\epsilon^{-3}V(C(\epsilon))$, where V denotes the volume, converges with a positive power of ϵ as $\epsilon \to 0$. This condition is obviously satisfied by smooth classical metrics. Our task is to calculate the volume of a given pre-compact region $R \subset \Sigma$,i.e. an open region that is contained in a compact set, which we assume to be an open sub-manifold, so there is a neighborhood of each point $x \in R$, that is isomorphic to a simply connected open subset of $\mathbb{R}^3$. So, we are classically led to consider:

$$V(R) := \int_R \sqrt{|q|}d^3x. \tag{8.6}$$

Since the density $\sqrt{|q|}d^3x$ is bounded, i.e. for any compact set C the functional $V(C) < \infty$, we can insert a partition of unity $\mathcal{R} = \{R_i, \rho_i\}_{i=1}^k$ for the region R to obtain:

$$V(R) = \sum_{i=1}^k V(R_i). \tag{8.7}$$

Without loss of generality, the resolution of unity $\mathcal{R}$ can be chosen to consist of sets that are topologically equivalent to open balls in $\mathbb{R}^3$, which additionally can be assumed to be completely contained in one chart of U_i of the given atlas of Σ. So, using equation 8.7 we have to only consider equation 8.6 for regions E that are topologically equivalent an open ball and which are coordinized using a single chart.

Our strategy to construct an equivalent expression to equation 8.6 using a limit of Riemann sums over a cell decomposition of E, where each cell is assumed to be a parallelepiped in the chart containing E. The summands in the Riemann sums are then expressed, using the result of the previous section, in terms of the six independent areas of each cell. To do this, we need to calculate the q-volume of a parallelepiped $P(x_o, \vec{a}, \vec{b}, \vec{c})$ at base-point x_o, spanned by $\vec{a}, \vec{b}, \vec{c}$:

$$\begin{aligned} &\epsilon^{-3} V(P(x_o, \epsilon\vec{a}, \epsilon\vec{b}, \epsilon\vec{c})) = \epsilon^{-3} \int_{P(\epsilon)} d^3u \sqrt{|\phi^* q|(u)} \\ &\longrightarrow \epsilon^{-3} \int_{P(\epsilon)} d^3u \sqrt{|\phi^* q|(x_o)} \text{ as: } \epsilon \to 0 \end{aligned} \tag{8.8}$$

where the constant approximation for $\epsilon \to 0$ is due to the regularity assumption on our metric. The number of cubical cells in the decomposition of E on the other hand grows as $C(E)\epsilon^{-3}$ as $\epsilon \to 0$, where $C_U(E)$ denotes the coordinate volume of E in the chart (U, ϕ).

Let us be more specific about the cubical decomposition of Σ: Given a region E that is topologically equivalent to an open ball, $\epsilon > 0$ and a chart (U, ϕ) containing E, we can apply a translation in U, moving the coordinate center of mass of E to the origin of U. We then use a piecewise stratified diffeomorphism that makes E to a coordinate cube of side length $2n\epsilon$ with $n \in \mathbb{N}$ large enough such that the coordinate center of mass is again at the origin of the transformed chart. We denote the transformed chart again by (U, ϕ). Then we take the cubical lattice fixed at the origin of U, such that for all $(n_1, n_2, n_3) \in \mathbb{Z}$ we have a coordinate cell:

$$C^{\epsilon}_{n_1,n_2,n_3} = \{(u_1, u_2, u_3) : n_1 < \epsilon^{-1}u_1 < n_1{+}1, n_2 < \epsilon^{-1}u_2 < n_2{+}1, n_3 < \epsilon^{-1}u_3 < n_3{+}1\}.$$

Denote the set of all cells C for which $C \cap \phi(E) = C$ by D^{ϵ}_U. Clearly, $L^{\epsilon}_U := \phi^{-1}(D^{\epsilon}_U)$ defines a cubical decomposition of E, and the number of elements of grows as

$$|L^{\epsilon}_U| \to C_U(E)\epsilon^{-3} \text{ as: } \epsilon \to 0.$$

So, we are able to approximate the volume as a sum over the volumes of $C \in L^{\epsilon}_U$ with homogeneous metric inside C, such that this approximation becomes exact in the limit $\epsilon \to 0$.

Internal versus External Regularization

Given a cellular decomposition L^{ϵ}_U of E, we need to specify which surfaces we want to use in equation 8.5, to calculate the volume of a cell. There are two very distinct ways to proceed and a few "middle-roads":

The first, which is closest to [40] can be called internal regularization, because one chooses only surfaces "inside" the cell. This can be achieved by moving the three faces, that one uses to calculate the volume, to the "coordinate center of mass" (see figure 8.2, left). Since all areas are measured using open sets, there is no measurement on the boundary of the cell and hence the approximation is "internal". This divides the cell into 24 distinct regions (see figure 8.2, right). Gauge invariance seems to have been the reason for using an internal regularization in [40]. The area measurements that we use here are however already gauge invariant and we are thus free to use a more general approach.

The second, which is closest to [41] can be called external regularization, because there is an open region around the "coordinate center of mass" of the cell that contains no surface. The construction can be described as follows: cut

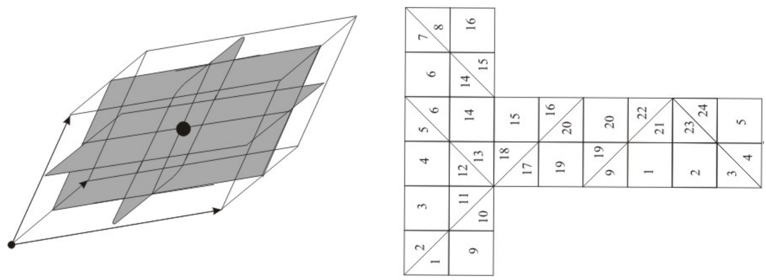

Figure 8.2: The left side shows areas, which are isometric to the faces in a homogeneous metric, that are used in the internal regularization. All six surfaces in the internal regularization meet at the "coordinate center of mass" of C (solid dot). The right side shows the net of the cell with the 24 distinct regions in the internal regularization.

the cubical cell into the six square pyramids that meet at the "coordinate center of mass" of the cell. Each square pyramid has one face as boundary. Move the tip of the square pyramids to the opposite face, so the center of the bottom square of the square pyramid is at the "coordinate center of mass" of the cell. The set of the six times four triangular faces of the six moved square pyramids will be referred to as "moved diagonal cross sections". Figure 8.3 shows the six times four resulting surfaces on the left of each half of the figure. The relation to the diagonal cross cuts is indicated on the right hand side of figure 8.3. As such figure 8.3 provides the graphical proof of the equality between the area of the shaded regions of the moved diagonals and the diagonal cross section, which was already evident from our construction.

It turns out that the treatment of the diffeomorphisms is less ambiguous in the external regularization, because in the quantum theory we want to capture vertices of a spin network at the coordinate center of mass to ensure convergence of the limit of Riemann sums. In the internal regularization one uses surfaces which intersect precisely at the "coordinate center of mass" and we have to carefully work the action of the diffeomorphisms on the vertex out, which on the other hand depends heavily on the class of diffeomorphisms that one chooses. Hence, let us describe the external regularization in a little more detail:

Given a cell $\phi(C^{\epsilon}_{k,l,m})$ in a chart (U, ϕ), then additional faces of the external square pyramids that are added are denoted by $\phi(F^{\epsilon}_{k,l,m,i,j})$, where the indices k, l, m denote the cell that they are attached to, $i = 1, ..., 6$ is a label that denotes the face that the pyramid originated from and $j = 1, ..., 4$ is a label that labels the triangular face of the the moved pyramid. A particular labeling (i, j) is provided in figure 8.4. Since one only needs three out of the six diagonal cross sections to calculate the volume in a homogeneous metric, one will also only need twelve out of the 24 faces to calculate the volume of the respective cell in a homogeneous metric. The correspondence between the faces and the respective diagonals is illustrated in figure 8.4.

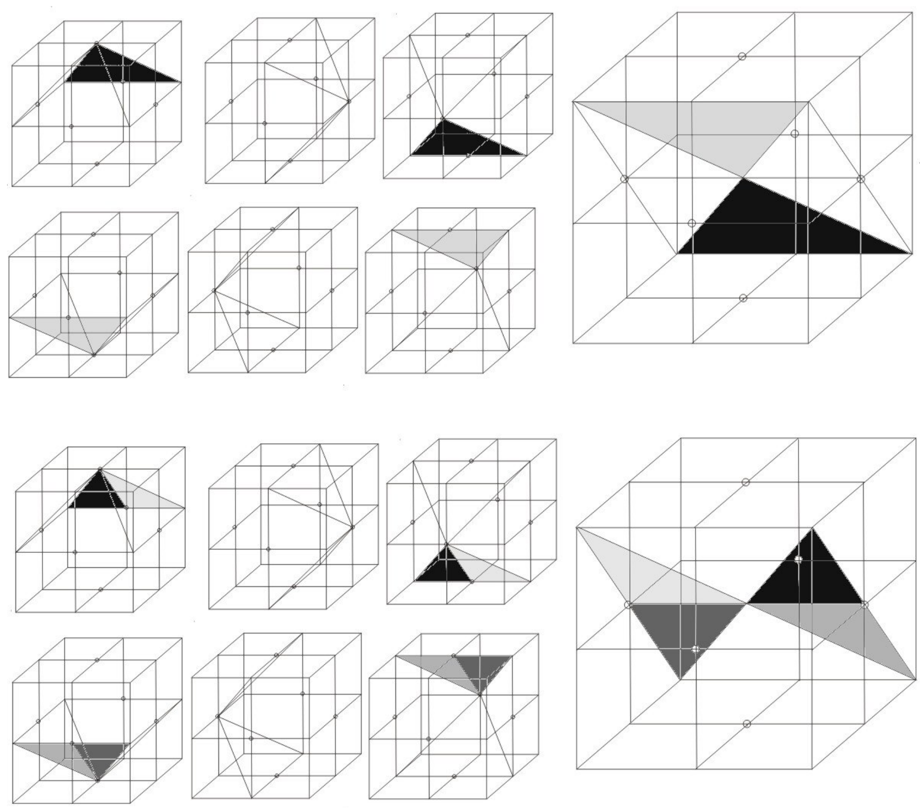

Figure 8.3: This figure shows the six times four pieces of the "moved" diagonal cross sections on the left and provides a graphical proof of the equality of the area of a diagonal cross section and the sum of the shaded areas of the "moved diagonals": Both lines illustrate the equality between twice the respective half of the diagonal cross section area and the shaded surfaces of the "moved" diagonals on the left.

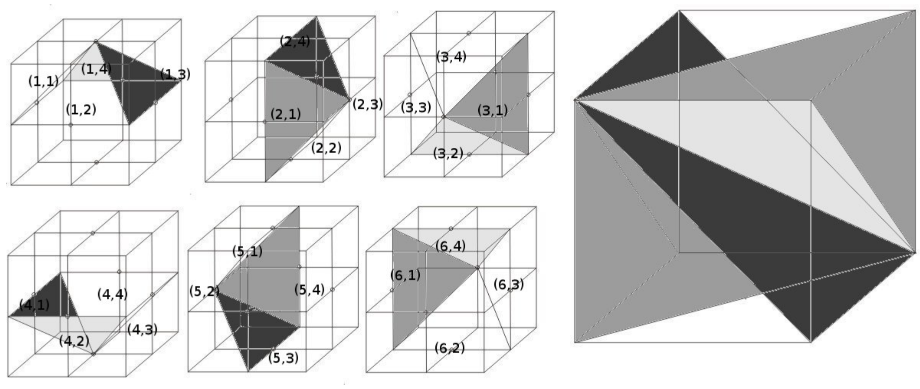

Figure 8.4: This figure illustrates the relation between the "used diagonal cross sections" and the "used faces" in the moved diagonals. The numbers indicate the pair (i, j) in the labeling of the face $F_{k,l,m,i,j}$. The coordinate chart U is assumed to be right-handed cartesian and the 1-direction is assumed to be going from left to right.

There are six distinct diagonal cross sections, however one only needs three to calculate the volume. Thus there are $\left(\begin{array}{c} 6 \\ 3 \end{array} \right) = 20$ choices, but only those choices that provide a cross section for each face of the cubical cell are permissible, thus there are only $2^3 = 8$ permissible choices, because after fixing one vertex of the cell there are two possibilities for the cross sections through the three faces adjacent to the vertex. These possibilities are illustrated in figure 8.5. The choice for the diagonal cross section that contains the chosen vertex is labeled with 0, the one that avoids the respective vertex is labeled by 1. After choosing an ordering for the adjacent faces, one obtains a binary number that encodes the permissible choice of diagonal cross sections.

The particular choice of faces is irrelevant for the classical volume functional. For the quantum theory it will however turn out to be important, so let us fix the choice of faces to be 000 form now on.[2] We will use the arithmetic mean for the surface areas $A_a = \frac{1}{2}(A^1_a(C) + A^2_a(C))$, where $A^1_a(C)$ denotes the C-face with the lower coordinate value for the a-component and A^2_a the component with the higher coordinate value, and the moved C-diagonals $B_{a,000}(C)$, which we add to the respective area of the diagonal cross section as indicated in figure 8.5. Inserting this into equation 8.5 yields the regulated expression:

$$V(C) := V(\frac{1}{2}(A^1_1(C) + A^2_1(C)), ..., B_{3,000}(C)). \tag{8.9}$$

It will turn out that diffeomorphism covariance of the quantum operator requires a kind of averaging that removes the particular choice of faces.

[2]We have provided a volume functional only for choices that are labeled by a "Y", but it is not difficult to reexpress the volume functional for a "Δ" choice.

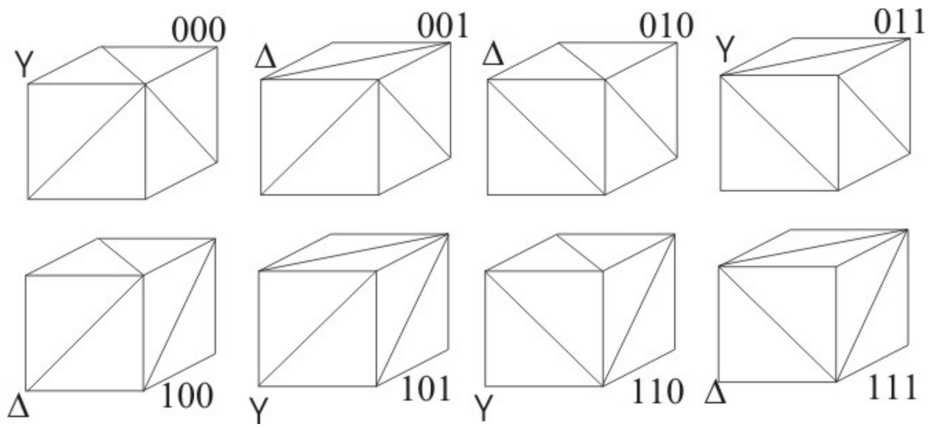

Figure 8.5: This figure illustrates the relation between the "used diagonal cross sections" and the "used faces" in the moved diagonals. The coordinate chart U is assumed to be right-handed cartesian and the 1-direction is assumed to be going from left to right.

Approximation of the Classical Volume

Classical densities are smooth and so their restriction to compact sets are in particular continuous and bounded and is thus Riemann integrable. So, if we have a chart (U, ϕ) and a cubical partition L^ϵ_U, which has the property that the maximal coordinate volume of a cell C in L^ϵ_U goes to zero as $\epsilon \to 0$, we are able to define the Riemann integral as the limit of the Riemann sums, labeled by the partition L^ϵ_U:

$$V(L^\epsilon_U) := \sum_{C \in L^\epsilon_U} V(C) = \sum_{C \in L^\epsilon_U} V(\frac{1}{2}(A^1_1(C) + A^2_1(C)), ..., B_{3,000}(C)), \quad (8.10)$$

where the volume of a cell is still given by equation 8.9. Continuity and compact support of the integral ensure the convergence for all classical metrics, we are thus able to identify the classical relation between the volume of a region R, which is still assumed to be contained in a single chart:

$$V(R) = \lim_{\epsilon \to 0} V(L^\epsilon_U) = \lim_{\epsilon \to 0} \sum_{C \in L^\epsilon_U} V(C). \quad (8.11)$$

For regions that are not contained in a single chart, we will use a partition of unity $\mathcal{R}$ that consists of regions that are contained in a single chart and write the volume as a sum over the volumes of the elements of $\mathcal{R}$.

Notice that this classical definition of a volume depends on a particular atlas $\{(U_i \phi_i)\}^n_{i=1}$ for Σ. While differential geometry and convergence of the Riemann integral imply that volume functional (equation 8.11) is independent of the atlas for any classical Riemann metric, we will have to pay special attention to this issue when quantizing equation 8.11, due to the singular nature of quantum geometries.

Averaging

Although equation 8.11 converges to the volume of R independent of the chart, we have to consider how to get rid of the chart dependence. The approach taken in [40] is to take a given chart and average over a sufficient set of transformations of this chart so the average becomes independent of the particular choice made for the initial chart. Due to the tangent space sensitivity of the Ashtekar-Lewandowski volume operator they where forced to average over the action of diffeomorphisms on the tangent space of the intersection point of the internal regularization. Our regularization procedure however uses an external regularization, which is not sensitive to the tangent space structure.

So let us consider a possible averaging procedure, although the averaging procedure is trivial at the classical level: Let (U, ϕ) be a chart containing R and let L_U^ϵ be the cubical partitioning of R induced by the chart. Given a finite set of piecewise analytic diffeomorphisms $\phi_i : \mathbb{R}^3 \to \mathbb{R}^3$, then the volume defined in equation 8.11 can be modified to

$$V(R) = \lim_{\epsilon \to 0} \frac{1}{k} \sum_{i=1}^{k} V(L^\epsilon_{\phi_i(U)}). \tag{8.12}$$

Equation 8.12 will still converge to the classical volume of R.

Let us consider a particular set of diffeomorphisms: Let v_j be a finite set of points in R and let c_j be a sphere of coordinate radius $\frac{\epsilon}{3}$ around v_j and let p_{ij} be a set of points on the coordinate sphere c_j. For each pair of sets $\{v_j, \{p_{ij}\}_{i=1}^{m_j}\}_{j=1}^n$ and $\{v_j, \{p_{ij}^1\}_{i=1}^{m_j}\}_{j=1}^n$ there is a piecewise analytic diffeomorphism ϕ leaving the v_j invariant and mapping $\phi : p_{ij} \mapsto p_{ij}^1$, and given k sets $\{v_j, \{p_{ij}^r\}_{i=1}^{m_j}\}_{j=1}^n$ there are k piecewise analytic diffeomorphisms leaving the v_j invariant while mapping $\phi^r : p_{ij} \mapsto p_{ij}^r$. We will use a similar set of k such diffeomorphisms in the operator version of equation 8.12 to perform the averaging in the quantum theory that results in removing the dependence of the operator version of equation 8.12.

8.1.2 Volume Operator

We will now give a definition of a volume operator based on equation 8.12. We will give the definition of a self-adjoint operator on the domain of spin network functions and consider its Hermitian completion from this dense domain.

Adaption to the Graph

One has the freedom to adapt the precise action of the volume operator to the graph of the spin network, but such an adaption must be diffeomorphism covariant, so we obtain a diffeomorphism covariant operator in the limit $\epsilon \to$

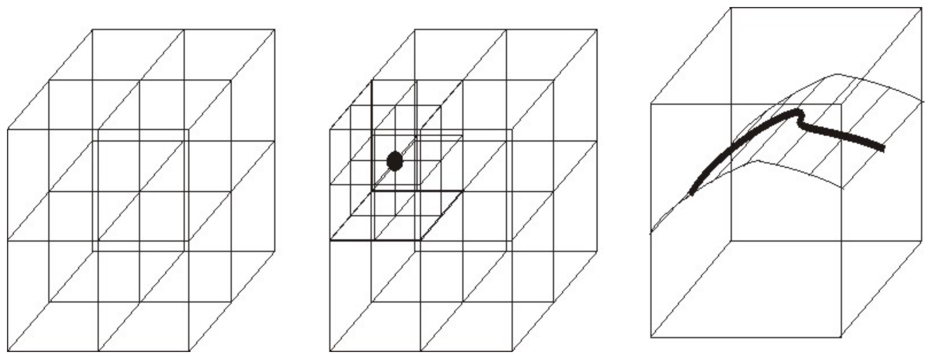

Figure 8.6: *Adaption to a graph:* left: a sample cubical decomposition, center: a sample refinement of a cell, right: a sample split of a cubical cell into two cells that are diffeomorphic to a cube.

0. The definition of diffeomorphism covariance, that we use here is due to J. Lewandowski and first appeared in [44]: If there is a diffeomorphism relating two cylindrical functions Cyl_γ, $Cyl_{\gamma'}$, then there must be a diffeomorphism that maps the adaption to Cyl_γ to the adaption of $Cyl_{\gamma'}$. We can construct such an adaption as follows:

We assume a chart (U, ϕ) that contains R and a suitably small $\epsilon > 0$. For each vertex $v \in V(\gamma)$ remove a coordinate cube c_v of size ϵ^3 with coordinate center of mass v from R. ϵ is assumed to be small enough, such that all c_v are disjoint. Then decompose $R \setminus (\cup_v c_v)$ into coordinate cubes L^ϵ_U as described in section 8.1.1[3]. The left picture in figure 8.6 depicts an example cubical decomposition. The cells in L^ϵ_u may still contain more than one edge. Since the neighborhoods c_v of the vertices v of γ are removed from R, we can always find a refinement, decomposing a cubical cell of L^ϵ_u into several cubical cells, such that each cell contains only one edge. A sample refinement is depicted in central picture in figure 8.6. These refined cells contain at most one edge. Each cell that contains an edge can be split into two cells, that are each diffeomorphic to a cube and which contain the edge in their mutual boundary. A sample decomposition is depicted in the right picture of figure 8.6. This refined decomposition of $R \setminus (\cup_v c_v)$ together with the set $\{c_v\}_{v \in V(\gamma)}$ defines the partition $P^\gamma_o(R)$.

We then define the refinement process of the partition $P^\gamma_n \to P^\gamma_{n+1}$ as follows: For each cell that does not contain a vertex is split into $3 \times 3 \times 3$ cells by dividing it at the coordinate thirds in each direction in the chart used to define the parent cell. Each cell that contains a vertex is refined in the same way, such that the vertex lies in the central cell. However, the adjacent cells may

[3] In section 8.1.1, we assumed that the considered region is topologically equivalent to an open ball. This is in general not the case for $R \setminus (\cup_v c_v)$. We will therefore have to decompose $R \setminus (\cup_v c_v)$ into regions that are topologically equivalent to an open ball to apply this construction.

contain edges going through their interior, so we subsequently apply the step previously described for L_U^ϵ, such that cells that do not contain a vertex have edges only running through their boundary. Packaging the partitions $P_n^\gamma(R)$ together defines the family $\mathcal{F}_\gamma(R)$. For each family $\mathcal{F}$ we have the classical identity:

$$V_{\mathcal{F}}(R) = \lim_{n\to\infty} \sum_{c\in P_n} V(c). \tag{8.13}$$

Due to the sensitivity of the quantum volume cell $\hat{V}(c_v)$ to the topological relations between the edges adjacent to the vertex v and the surfaces used to measure the cell volume, we have to constrain the partition process to a process that preserves these topological relations for all $n \geq n_o$ to ensure convergence. This is possible due to piecewise analyticity of γ, which ensures that all edges are outgoing form v for small enough $\epsilon > 0$.

Averaging

Using the quantum version of equation 8.13 as a basis for our quantization, we see that we have to only consider the volume of an individual cell. Due to the sensitivity of the quantum volume cell $\hat{V}(c_v)$ to the topological relations between the edges adjacent to the vertex v and the surfaces used to measure the cell volume, we have a sensitivity to the chart used to define the cell, which introduces a background dependence.

Using the piecewise analyticity of γ we know that all edges adjacent to v are outgoing from v for $\epsilon > 0$ small enough. Thus, we can get rid of this background dependence by averaging over all possible topological relations between purely outgoing edges and the surfaces used to measure the cell volume. To be more specific: Let $\mathcal{T}(v)$ denote the set of topological relations that exist for any $\epsilon > 0$. Since the edges are all outgoing for small enough ϵ, there are only three possible relations between a measuring surface and an edge: (1) the edge avoids the surface, (2) the edge penetrates the surface transversally or (3) the edge is tangent to the surface. Since there is only a finite number of edges, only six surfaces and a finite number of relations between each edge and each surface, we conclude that $\mathcal{T}(v)$ is a finite set, moreover the number of topological relations is bounded by $3^{6\times \text{valence of } v}$. However, not all topological relations can be achieved by a diffeomorphism[4], leaving us with a nontrivial combinatorial problem. Let $\mathcal{A}(v) \subset \mathcal{T}(v)$ denote the set of possible topological relations that can be achieved between $\phi(\gamma)$ and the set of measuring surfaces by applying an appropriate diffeomorphism ϕ. We can therefore equivalently describe $\mathcal{A}(v)$ by a set of representative diffeomorphisms that achieve these topological relations.

[4]The sensitivity to the tangent space structure in the internal regularization is the source for additional complications because a diffeomorphism acts only as a linear transformation on the tangent space yielding additional constraints for the topological relations that can be achieved with a diffeomorphism. Using extended diffeomorphisms, that do not act as linear transformations on the tangent space, we can remove some of these constraints.

We are now able to refine the definition of the cell volume $V_\gamma(c_v)$ by using the adaption to the graph and the averaging over the possible topological relations:

$$V_\gamma(c_v) := \frac{1}{|\mathcal{A}(v)|} \sum_{\phi \in \mathcal{A}(v)} V(c_v) \circ \phi. \tag{8.14}$$

Definition of the Volume Operator

We have now collected all the ingredients needed for the definition of a volume operator, so we can finally insert equation 8.14 into equation 8.13 and insert equation 8.5, where we replace the area functionals with the respective area operator. To be able to use equation 8.5 however, we have to ensure that there exists a parallelepiped with the squared surface areas $A_a, ..., B_c$. To truncate the data $A_a, ..., B_c$ correctly, we define the function

$$\theta(A_a, ..., B_c) := \begin{cases} 1 & if \quad \exists \text{ paralleepiped with suqred areas} A_a, ..., B_c \\ 0 & \quad\;\; \text{otherwise} \end{cases} \tag{8.15}$$

This makes $V(A_a, ..., B_c)\theta(A_a, ..., B_c)$ a positive semi-definite function for all 6-tuples $(A_a, ..., B_c)$. Using the observation that the gauge-variant spin network functions are eigenfunctions of the area operator, we are able to define for any gauge-variant spin network function SNF_γ on graph γ:

$$\hat{V}(R)SNF_\gamma := \hat{V}_{\mathcal{F}_\gamma}(R)SNF_\gamma = \lim_{n\to\infty} \sum_{c \in P_n} \hat{V}(c). \tag{8.16}$$

Using the observation that all area operators vanish when there is no edge in the cell, we observe that the sum has to be taken only over the calls containing vertices, so:

$$\hat{V}(R)SNF_\gamma := \hat{V}_{\mathcal{F}_\gamma}(R)SNF_\gamma = \lim_{n\to\infty} \sum_{v \in V(\gamma \cap R)} \hat{V}(c_v), \tag{8.17}$$

where:

$$\begin{aligned} \hat{V}(c_v)SNF_\gamma \quad := \quad & \textstyle\sum_{\phi\in\mathcal{A}(v)} \frac{1}{|\mathcal{A}(v)|} \theta((\hat{A}(A_a(c_v)))^2, ..., (\hat{A}(B_c(c_v)))^2) \\ & V((\hat{A}(A_a(c_v)))^2, ..., (\hat{A}(B_c(c_v)))^2) SNF_{\phi(\gamma)}, \end{aligned} \tag{8.18}$$

and where $V(...)$ is given by equation 8.5. The gauge-variant spin network functions are eigenfunctions of the volume operator, since the gauge-variant spin network functions are eigenfunctions of the mutually commuting area operators, so we can make sense of equation 8.5 through a spectral definition. We also see that this operator is positive semi-definite for all gauge-variant spin network functions, because it is the sum over averages of positive semi-definite values.

8.1.3 Properties of the Volume Operator

Let us now collect some important properties that we need to prove that equation 8.17 defines a positive semi-definite Hermitian operator, that is gauge invariant and transforms covariantly under diffeomorphisms.

Cylindrical Consistency

Any cylindrical function can be expanded in gauge-variant spin network functions, so using linearity of the volume operator, we can prove that $\hat{V}(R)$ is non-graph-changing and cylindrically consistent by showing this for gauge-variant cylindrical functions:

non graph-changing: The gauge-variant spin network functions are eigenfunctions of the volume operator, thus the action is non-graph changing.

cylindrical consistency: A gauge-variant cylindrical function $SNF_\gamma \propto \prod_{e\in\gamma} \rho^{j_e}_{m_e n_e}(h_e(A))$ has a minimal graph γ_o, which consists precisely of the edges e for which $j_e \neq 0$. So enlarging the graph γ_o adds only edges with representation 0, which do not contribute to an area measurement. Hence $V(R)SNF_\gamma = V(R)SNF_{\gamma_o}$, since we expressed the volume operator only through area operators.

Positivity, Symmetry and Hermitian Extension

We already saw that the gauge-variant spin network functions are eigenfunctions of our volume operators $\hat{V}(R)$ with positive semi-definite eigenvalues. Moreover, since the gauge-variant spin network functions form a complete orthogonal set in $L^2(\mathbb{X}, d\mu_{AL})$, we see that all operators $\hat{V}(R)$ are real-symmetric and positive semi-definite in this dense domain. Hence there is a unique unbounded Hermitian operator on $L^2(\mathbb{X}, d\mu_{AL})$, that coincides with our definition of $V(R)$ for any gauge-variant spin network function.

This Hermitian extension is the volume operator $\hat{V}(R)$ for any region R.

Covariance with respect to the Kinematical Constraints

Gauge-invariance: The volume operator is a spectral function of area operators, which are gauge-invariant, implying its gauge invariance.

Covariance under Diffeomorphisms: We have to show that $U^*_\phi \hat{V}(R) U_\phi = V(\phi^{-1}(R))$ for any diffeomorphism ϕ. Consider therefore two spin network functions f_γ and g_δ depending on the minimal graphs γ and δ respectively:

$$\begin{aligned} \langle f_\gamma, U^*_\phi \hat{V}(R) U_\phi g_\delta \rangle &= \langle U_\phi f_\gamma, \hat{V}(R) U_\phi g_\delta \rangle \\ &= \langle f_{\phi(\gamma)}, \hat{V}(R) g_{\phi(\delta)} \rangle \\ &= V_R^{g_{\phi(\delta)}} \langle f_{\phi(\gamma)}, g_{\phi(\delta)} \rangle, \end{aligned}$$

since the gauge-variant spin network functions are eigenfunctions of our volume operator. The eigenvalue however is a sum over eigenvalues associated with the vertices of δ, so we have to only consider the eigenvalues $\hat{V}(c_{v_\phi}) g_{\phi(\delta)}$ associated to the vertices $v_\phi \in V(\phi(\delta))$. Notice that $\phi^{-1} v_\phi = v \in V(\gamma)$. There are now two cases (1) $v_\phi \cap R = \emptyset$ then the eigenvalue vanishes, which is also implied by $v \cap \phi^{-1}(R) = \emptyset$; (2) $v_\phi \cap R = v_\phi$, which is equivalent to $v \cap \phi^{-1}(R) = v$, in which case the result is non vanishing. The volume eigenvalue of the unaveraged cell volume depends on the germ of edges at the vertex, which is altered by the diffeomorphism. However, since we averaged over all possible topological relations that can be achieved using a diffeomorphism, we see that the eigenvalue is unchanged. This implies the covariance of the volume operator under diffeomorphisms.

8.2 Length Operator

Having a volume operator based on area measurements at our disposal, we apply our presentation of Thiemanns length operator (section C.4) with one small modification, which is due to the unknown behavior of our volume operator on tri-valent vertices. Since Thiemanns argument (footnote 3) can be applied to the trivalent as a special case of case 5, we can immediately use the gauge invariance of our volume operator and insert this volume operator into the formula equation C.8 for Thiemanns length operator to obtain a length operator in terms of area operators. Since our volume operator acts nontrivial at vertices only, we see that the simplification (equation C.9) holds as well.

Chapter 9

Conclusion

Standard Loop Quantum Cosmology has turned out to be a valuable tool to investigate Quantum Gravity effects. It is a quantum cosmological model that has many features induced from Loop Quantum Gravity and shares in particular its discreteness. The results of this thesis, which is concerned with broadening the understanding of the relation between macroscopic models and Loop Quantum Gravity itself, can therefore be used in particular to "sharpen" this tool. The technical problem attacked in this thesis is twofold:

(1) Loop Quantum Gravity is a continuum theory arising from a lattice gauge theory on a floating or changing lattice therefore inherits lattice discreteness for the geometric observables. This discreteness obstructs the construction of a smooth classical geometry, while it is widely assumed to provide mechanisms for singularity avoidance.

(2) The states of Loop Quantum Gravity depend on particular classes of lattices, which generically resolve only a compact spatial topology and make the predictions vulnerable to lattice effects.

To overcome these technical problems, we looked at the relation of classical cosmology and General Relativity and tried to read off mechanisms that could be applied to the quantum theory, very often using hints from noncommutative geometry. This lead to the following main results:

1. *Construction of the quantum analogue of a phase space embedding*: In analogy to using the pull-back under a Poisson-embedding of a reduced system into a full classical system, we demanded that a quantum embedding provides (1) reduces to the pull-back under such an embedding in a suitable classical limit and that (2) the expectation values of the reduced system are matched by expectation values in the full system. Using the noncommutative analogue of embedding a vector bundle over the reduced phase space into a vector bundle over the full phase space and recovering the embedding using the bundle projection, we construct a quantum embedding as follows: The noncommutative analogue of a vector bundle is a Hilbert-C^*-module (induction module) and for transformation group systems there exists an induction module given by func-

tions on the configuration space. Having a pair of linear maps p, q[1] from the reduced configuration observables into the full configuration observables, that satisfy certain consistency conditions, we are able to use methods similar to Rieffel induction to construct a quantum embedding prescription that satisfies our demands.

2. *The extended Diffeomorphism-group is physically relevant*: It was conjectured by Fairbairn and Rovelli [26] that the gauge-group of Loop Quantum Gravity contains certain extended diffeomorphisms. The authors did however not consider problems arising form piecewise analyticity of the path groupoid used to construct Loop Quantum Gravity, which is vital to obtain a closed observable algebra. Since the existence of this extension was important for the application of quantum embeddings to Loop Quantum Gravity, we formulated a physical completeness argument for the gauge group of a quantum theory. Using this completeness argument, we showed that the diffeomorphism-orbits of spin network functions are labeled by the knot-class of the underlying graph.

3. *There is an embeddable cosmological sector in Loop Quantum Gravity describing discrete cosmology*: Using the previous two results, we constructed an algebra for diffeomorphism invariant observables in Loop Quantum Gravity. We carefully constructed an observable algebra for diffeomorphism-invariant observables in Loop Quantum Gravity and constructed an induction module therefore using the span of a subset of spin network functions. Since the spin network functions are particular functions on the configuration space, that underlies Loop Quantum Gravity, we used an explicit expression for homogeneous connections to construct a quantum embedding from this induction module. The reduced observable algebra and its induced representation turned out to be equivalent to a super selection sector of standard Loop Quantum Cosmology. The construction presented in this thesis is however not unique and we traced the ambiguities back to finding a gauge for the diffeomorphisms.

4. *There are states on the algebra of Loop Quantum Gravity describing smooth spatial geometries*: For each classical geometry, we constructed positive linear functionals on a tailored version of the Weyl-algebra of Loop Quantum Gravity, which turn out to be eigenstates of the geometric observables with eigenvalues that match the classical values of the corresponding geometric observables in this given geometry. Using these states to perform a GNS-construction we find new families of representations of the algebra of Loop Quantum Gravity with diffeomorphism-variant vacua, which is an instance in which the celebrated LOST/F uniqueness result does not hold. These representations turn out to be labeled by the classical geometry they where constructed from. Summing over orbits of these GNS-representations, we construct a Hilbert space that carries a unitary covariant representation of the $SU(2)$-gauge transformations and diffeomorphisms. Using the group-averaging procedure, we constructed the gauge-

[1] The map p can be constructed as the pull-back under an embedding of the reduced configuration space, while q is a partial inverse of p.

and diffeomorphism-invariant Hilbert space, which turns out to contain a basis given by gauge-invariantly coupled spin-networks that are embedded modulus isometries.

5. *One can interpret standard Loop Quantum Cosmology in terms of the aforementioned states*: Using the aforementioned states on Loop Quantum Gravity, one can build mini-Hilbert spaces to model symmetric geometries by constructing the Hilbert-completion of the span of the symmetric vacuum states. Since the vacuum states transform coherently under spatial diffeomorphisms, we conjecture coherence under all diffeomorphisms, allowing us to conjecture a dynamics that has a viable classical limit.
   Constructing the mini-Hilbert space for standard cosmology, we obtain a kinematic equivalence with standard Loop Quantum Cosmology. This was used to transfer Bojowald's dynamics to our model. This gives an interpretation to standard Loop Quantum Cosmology in terms of states on the full theory, opening the door for the study of fluctuations.

6. *We constructed a volume operator based on fundamental area operators:* This volume operator is a quantization of the classical volume functional when this is reexpressed as the limit of Riemann sums which extend over cell volumes expressed as functions of areas in cells and thus physically justifies the adjustment of the observable algebra of Loop Quantum Gravity.

This work is obviously only a first step towards the understanding of the relation between Loop Quantum Gravity and cosmology, leading to further questions:

1. *Can one invert the quantum symmetry reduction to learn lessons for the full theory?* The suggestion that the lessons learned from Loop Quantum Cosmology could be applied to complete the full theory is a widely used motivation for the study of standard Loop Quantum Cosmology. Since our construction presented in this thesis establishes a concrete link between the full theory and the reduced model, one has reasonable hope to "invert" this link so one can apply results from cosmology (or other symmetric models) to the full theory.

2. *Is there a criterion for the selection for "the correct way" to impose Bianchi symmetry in the quantum theory?* Particularly: (1) Is there a mathematical reason for ruling some choices out and (2) is there a physical reason ruling other choices out?

3. *Are there even more states on the observable algebra of Loop Quantum Gravity?* The states constructed in chapter 6 are eigenstates of the momentum operators of Loop Quantum Gravity. They are semiclassical in the sense that they describe classical spatial geometries as opposed to the "bumpy" weave geometries of spin network states. There are ansätze for the construction of further states that are also labeled by classical fields, which do not describe spatial geometries. It seems that if these conjectured states exists then they are

due to lattice effects of the floating lattice of Loop Quantum Gravity, which could give new ansätze for the application of Smolin et al.'s programm of constructing a standard model as lattice effects in Loop Quantum Gravity.

4. *Can the dynamics conjectured in chapter 7 be implemented successfully?*

5. *Can one find a way to induce a nontrivial dynamics for the reduced model?*

There are many more questions that we omit for the sake of brevity.

Appendix A

C^*-algebras and strong Morita Equivalence

This is appendix is intended to serve as a lexicon that fixes the notation that we use in the treatment of C^*-algebras. In section A.1 we included only definitions and theorems, because the field of C^*-algebras is readily available in textbooks [29, 30, 31] and [1]. The following section, which is concerned with Morita equivalence and Rieffel induction for C^*-algebras, includes instructive proofs that are meant to illustrate methods that are used in this field. This is necessary, beacuse this field is not very much known among physicists and most of the background has to be extracted from original works in this field of mathematics, e.g. [35, 36, 19, 37, 38]. The final section contains two applications of the Morita theory for C^*-algebras, which have physical significance. These are directly taken from original literature [19, 18, 32] and include the important steps of the proofs, that play a role in the construction of quantum embeddings and in proving their properties.

A.1 Preparations

This section quotes basic definitions and central theorems about the theory of C^*-algebras without proving these. It is intended to fix notation and to quote theorems for so we can refer to them elsewhere.

A.1.1 Foundations

We introduce the abstract concept of a C^*-algebra without referring to a particular representation as bounded operators on a Hilbert space. This allows us to abstractly discuss representations of C^*-algebras.

Definition 32 *1. An operator on a Banach space $\mathcal{B}$ is* **bounded***, if it is a* *linear map $O : \mathcal{B} \to \mathcal{B}$ with finite operator norm:*

$$||O|| := \sup\{||Ob|| : b \in \mathcal{B}; |b| \leq 1\}.$$

2. *A Banach space $\mathcal{B}$ is a* **Banach algebra** *if $\mathcal{B}$ is an algebra and if for all $b_1, b_2 \in \mathcal{B}$:*

$$||b_1 b_2|| \leq ||b_1|| \, ||b_2||.$$

3. *An involution on an algebra $\mathfrak{A}$ is a antilinear map $* : \mathfrak{A} \to \mathfrak{A}$, such that for all $a, a_1, a_2 \in \mathfrak{A}$:*

$$(a_1 a_2)^* = a_2^* a_1^* \text{ and } a^{**} = a.$$

4. *A* **C^*- algebra** *is a Banach algebra with involution satisfying*

$$||aa^*|| = ||a||^2$$

for all $a \in \mathfrak{A}$.

5. *A linear map $l : \mathcal{B} \to \mathbb{C}$ on a Banach algebra $\mathfrak{B}$ is called a* **functional***, iff it has finite norm:*

$$||l|| := \sup\{|l(b)| : b \in \mathcal{B}; ||b|| = 1\}.$$

The **dual** *$\mathcal{B}^*$ of a Banach space $\mathcal{B}$ is the space of all functionals.*

6. *A linear map $m : \mathfrak{A} \to \mathfrak{B}$ between two C^*-albegras $\mathfrak{A}$ and $\mathfrak{B}$ is called a* **morphism** *if for all $a, a_1, a_2 \in \mathfrak{A}$*

$$m(a_1 a_2) = m(a_1) m(a_2) \text{ and } m(a^*) = m(a)^*$$

7. *Given a $b \in \mathcal{B}$, the set $\sigma(b) := \{z : b - z\mathbb{I}$ is not invertible in $\mathcal{B}\}$ is called the* **spectrum** *of the element b of the unitial Banach algebra $\mathcal{B}$.*

8. *A closed linear subspace $\mathcal{I}$ of a Banach algebra $\mathcal{B}$ is an* **ideal** *if for any $b \in \mathcal{B}$ and all $i \in \mathcal{I}$: $bi \in \mathcal{I}$ and $ib \in \mathcal{I}$. An ideal $\mathcal{I}$ of $\mathcal{B}$ is* **maximal***, if there is no proper ideal of $\mathcal{B}$ that contains $\mathcal{I}$ as a proper subspace.*

Let us quote some important theorems about Banach spaces and C^*-algebras whose proofs can be found in various textbooks:

Theorem 3 1. **Hahn-Banach:** *Each functional on a linear subspace $\mathcal{B}_o$ of a Banach space $\mathcal{B}$ has a norm-equivalent extension to $\mathcal{B}$.*

2. *Every morphism m of C^*-algebras $\mathfrak{A}, \mathfrak{B}$ satisfies $||m(a)|| \leq ||a||$ for all $a \in \mathfrak{A}$.*

3. *For each Banach algebra $\mathcal{B}$ there exists a morphism into a unitial Banach algebra $\mathcal{B}_u$ such that $\mathcal{B}_u / \mathcal{B} = \mathbb{C}$.*

4. *The spectrum $\sigma(b)$ is a nonempty, compact subset of $\{z \in \mathbb{C} : |z| \leq ||b||\}$ for each element b of a Banach algebra.*

5. **Gel'fand-Mazur:** *If every nontrivial element of a unitial Banach algebra $\mathcal{B}$ is invertible, then $\mathcal{B}$ is isomorphic to $\mathbb{C}$.*

6. *The spectral radius $r(b) := \sup\{|z| : z \in \sigma(a)\}$ of an element b of a unitial Banach algebra is:*
$$r(b) = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}}.$$

7. *Given an ideal $\mathcal{I}$ of a Banach algebra $\mathcal{B}$, then $\mathcal{B}/\mathcal{I}$ is a Banach algebra with norm and multiplication:*
$$||[b]|| := \inf\{||b+i|| : i \in \mathcal{I}\} \text{ and } [b_1][b_2] = [b_1 b_2],$$
where $[b]$ denotes the equivalence class of elements of $\mathcal{B}$ under $b \sim b + i$if $i \in \mathcal{I}$.

One can unitialize C^*-algebras $\mathfrak{A}$ in the following way: Consider each element $a \in \mathfrak{A}$ as an operator $O(a) : \mathfrak{A} \to \mathfrak{A}$ by $b \mapsto ab$. Then the algebra $\mathfrak{A}_1 = \{a + z\mathbb{I} : a \in \mathfrak{A}, z \in \mathbb{C}\}$ with multiplication $(a_1 + z_1\mathbb{I})(a_1 + z_2\mathbb{I}) = (a_1a_2 + z_1a_2 + z_2a_1) + (z_1z_2)\mathbb{I}$ and natural involution and operator norm is a unitial C^*-algebra, whose spectrum $\Delta(\mathfrak{A}_1)$ is the one point compactification of $\Delta(\mathfrak{A})$.

Lemma 29 *To each C^*-algebra $\mathfrak{A}$ there exists a C^*-algebra $\mathfrak{A}_1$ with $\mathfrak{A}_1/\mathfrak{A} = \mathbb{C}$.*

To each element a of a C^*-algebra $\mathfrak{A}$, there exists a smallest C^*-algebra $C(a, \mathbb{I})$, which is generated by a and the unit element $\mathbb{I}$, which turns out to be the norm closure of the polynomials in a. $C(a, \mathbb{I})$ is commutative for all normal elements a.

Definition 33 1. *An element a of a C^*-algebra is called* **normal** *if it commutes with a^*. It is* **self-adjoint** *if $a = a^*$.*

2. *An element a of a C^*-algebra is called* **positive***, if it is self-adjoint and if its spectrum $\sigma(a)$ is positive. Positive elements are denoted by $a \geq 0$ and $\mathfrak{A}^+ := \{a \in \mathfrak{A} : a \geq 0\}$.*

Theorem 4 1. *Given a self-adjoint element a in a C^*-algebra $\mathfrak{A}$, then its spectrum is real and coincides with the spectrum of a in $C(a, \mathbb{I})$. Moreover, $\Delta(C(a, \mathbb{I}))$ is homeomorphic to $\sigma(a)$, such that the Gel'fand transform becomes the identity.*

2. *For each self-adjoint element a of a C^*-algebra $\mathfrak{A}$ and for each $f \in C(\sigma(a))$, there exists an operator $F(a) \in \mathfrak{A}$, such that $\sigma(F(a)) = f(\sigma(a))$ and $||F(a)|| = ||f||_{sup}$.*

3. *Given a C^*-algebra $\mathfrak{A}$ then each norm $||.||_s$ on $\mathfrak{A}$ for which $||a^*a|| = ||a||^2$ coincides with the C^*-norm on $\mathfrak{A}$.*

4. *The set of positive elements $\mathfrak{A}^+$ is a convex cone for each C^*-algebra $\mathfrak{A}$.*

5. *$\mathfrak{A}^+ = \{a^*a : a \in \mathfrak{A}\}$ for each C^*-algebra $\mathfrak{A}$.*

A.1.2 Commutative C^*-algebras

Let us now consider commutative algebras, i.e. $ab = ba$ for all elements of these algebras. We start with some standard definitions:

Definition 34 *1. Given a Banach algebra $\mathcal{B}$, the set $\Delta(\mathcal{B})$ consisting of the nontrivial functionals ω which satisfy for all $b_1, b_2 \in \mathcal{B}$ is called its spectrum:*

$$\omega(b_1 b_2) = \omega(b_1)\omega(b_2).$$

2. *The* **Gel'fand topology** *of $\Delta(\mathcal{B})$ is the restriction of weak $*$-topology to the spectrum $\Delta(\mathcal{B})$ of the Banach algebra $\mathcal{B}$. I.e. $\omega_n \to \omega$ in the Gel'fand topology iff $\omega_n(b) \to \omega(b)$ for all $b \in \mathcal{B}$.*

3. *Given a locally compact Hausdorff space $\mathbb{X}$, the $*$-algebra of all continuous complex valued functions on $\mathbb{X}$, which vanish at infinity, with pointwise multiplication, pointwise addition and pointwise involution is denoted by $C_o(\mathbb{X})$. If $\mathbb{X}$ is compact then vanishing at infinity is waived and the algebra is denoted by $C(\mathbb{X})$.*

4. *The* **sup-norm** *on $C_o(\mathbb{X})$ is given by*

$$||f|| := \sup\{|f(x)| : x \in \mathbb{X}\}.$$

Given a Banach algebra $\mathcal{B}$, we can embed $\mathcal{B}$ into $\mathcal{B}^{**}$ by the following construction:

$$\hat{.} : \mathcal{B} \to \mathcal{B}^{**} \text{ by } \hat{b} : \omega \mapsto \omega(b). \tag{A.1}$$

Taking $\omega \in \Delta(\mathcal{B})$, then $\hat{b}$ defines a function on $\Delta(\mathcal{B})$, which is continuous in the Gel'fand topology on $\Delta(\mathcal{B})$. This lets us define the **Gel'fand transform** as a map from $\mathcal{B}$ to $C(\Delta(\mathcal{B}))$ by:

$$\hat{.} : \mathcal{B} \to C(\Delta(\mathcal{B})) : b \mapsto \hat{b}.$$

There are various consequences of this construction, we quote some here and refer to standard literature for the proofs:

Theorem 5 *1. Each $\omega \in \Delta(\mathcal{B})$ is continuous and of unit norm and satisfies for unitial $\mathcal{B}$: $\omega(\mathbb{I}) = 1$, $||\omega|| = 1 \Rightarrow |\omega(b)| \leq ||b|| \forall b \in \mathcal{B}$.*

2. *Given a commutative unitial Banach algebra $\mathcal{B}$: For each $\omega \in \Delta(\mathcal{B})$ there is a maximal ideal $\mathcal{I}_\omega := ker(\omega)$ and for each maximal ideal $\mathcal{I}_m$ there is an $\omega_{\mathcal{I}_m} \in \Delta(\mathcal{B})$ such that $\mathcal{I}_m = ker(\omega_{\mathcal{I}_m})$. Moreover, $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $\omega_1 = \omega_2$.*

3. *For any commutative Banach algebra $\mathcal{B}$: $\Delta(\mathcal{B})$ is a compact Hausdorff space in the Gel'fand topology.*

4. *Given a locally compact Hausdorff space $\mathbb{X}$, then $C_o(\mathbb{X})$ is a C^*-algebra whose norm is the sup-norm.*

5. *Given a commutative unitial Banach algebra $\mathcal{B}$, the Gel'fand transform is a homomorphism from $\mathcal{B}$ to $C(\Delta(\mathcal{B}))$, whose image separates points in $\Delta(\mathcal{B})$. Moreover, $||\hat{b}||_{sup} \leq ||b||$ for all $b \in \mathcal{B}$.*
   For a nonunitial commutative Banach algebra $\mathcal{B}$ $\Delta(\mathcal{B})$ is locally compact and Hausdorff and the one point compactification of $\Delta(\mathcal{B})$ for $\mathcal{B}_u$. The Gel'fand transform is a homomorphism from $\mathcal{B}$ to $C_o(\Delta(\mathcal{B}))$ whose image separates points in $\Delta(\mathcal{B})$ and $||\hat{b}||_{sup} \leq ||b|| \forall b \in \mathcal{B}$.

6. *Given a commutative C^*-algebra $\mathfrak{A}$ then if $\mathfrak{A}$ is unitial there is a compact Hausdorff space $\mathbb{X}$ such that $\mathfrak{A} = C(\mathbb{X})$ and if $\mathfrak{A}$ is nonunitial, then there is a locally compact Hausdorff space $\mathbb{X}$ such that $\mathfrak{A} = C_o(\mathbb{X})$.*

7. **Stone-Weierstrass:** *Given a compact space $\mathbb{X}$ and a unitial C^*-algebra $\mathfrak{A}$ of functions on $\mathbb{X}$, which separates points in $\mathbb{X}$, then $\mathfrak{A} = C(\mathbb{X})$.*

8. *Any locally compact Hausdorff space $\mathbb{X}$ is homeomorphic to $\Delta(C(\mathbb{X}))$ with its Gel'fand topology.*

We refer to the one-one correspondence between commutative C^*-algebras and locally compact Hausdorff spaces as the Gel'fand theory for C^*-algebras.

A.1.3 Approximate Units and Ideals

C^*-algebras do not generally have units, neither do proper ideals of a C^*-algebra contain unit elements. The study of ideals of commutative C^*-algebras is equivalent to the study of functions vanishing on open sets. A useful concept for this study is given by approximate identities, which exist also for nonunitial C^*-algebras and proper ideals.

Definition 35 1. *Given a nonunitial C^*-algebra $\mathfrak{A}$, a directed set $(N, \leq)$ and a family $\{\mathcal{I}_n\}_{n\in N}$ of elements of $\mathfrak{A}$ indexed by elements of the directed set, then we call $\{\mathbb{I}_n\}_{n\in N}$ an* **approximate identity** *if each $\mathbb{I}_n$ is self-adjoint with*

$$\sigma(\mathbb{I}_n) \subset [0, 1]$$

and for each $a \in \mathfrak{A}$:

$$\lim_{\leftarrow n} ||a - \mathbb{I}_n a|| = 0 = \lim_{\leftarrow n} ||a - a\mathbb{I}_n||.$$

2. *A C^*-algebra is called separable, if it contains a countable dense set.*

Approximate units always exist:

Lemma 30 1. *A nonunitial C^*-algebra has an approximate identity.*

2. *A separable C^*-algebra has an approximate identity with countable directed set $(N, \leq)$.*

Using approximate identities one can prove important results for ideals of C^*-algebras, which we quote without proof:

Theorem 6 *1. Every ideal of a C^*-algebra is self-adjoint, i.e. it contains the adjoints of all its elements.*

2. *The quotient of a C^*-algebra by an ideal $\mathcal{I}$ is a C^*-algebra with the induced algebraic operations $[a_1][a_2] := [a_1 a_2]$, induced norm $||[a]|| := \inf\{||a+i|| : i \in \mathcal{I}\}$ and induced involution $[a]^* := [a^*]$.*

3. *For any ideal $\mathcal{I}$ of a C^*-algebra $\mathfrak{A}$ and $a \in \mathfrak{A}$ and any approximate identity $\{\mathbb{I}_n\}_{n \in N}$ for $\mathcal{I}$, one has*
$$||[a]|| = \lim_{\leftarrow n} ||a - a\mathbb{I}_n||.$$

4. *Every ideal in a C^*-algebra $\mathfrak{A}$ is the kernel of some C^*-algebra morphism $m : \mathfrak{B} \to \mathfrak{A}$.*

5. *An injective morphism of C^*-algebras is isometric.*

6. *Let $m : \mathfrak{A} \to \mathfrak{B}$ be a morphism of C^*-algebras, then $m(A)$ is a C^*-subalgebra of $\mathfrak{B}$.*

A.1.4 Representations and GNS-construction

For the purpose of doing quantum mechanics, one needs to construct particular Hilbert space representation for a given C^*-algebra of quantum observables.

Definition 36 *1. Given a C^*-algebra and a Hilbert space $\mathcal{H}$, we call a linear map $\pi : \mathfrak{A} \to B(\mathcal{H})$ a* **representation** *of $\mathfrak{A}$ on $\mathcal{H}$ if for all $a, a_1, a_2 \in \mathfrak{A}$:*
$$\pi(a_1 a_2) = \pi(a_1)\pi(a_2) \text{ and } \pi(a^*) = \pi(a)^*.$$

2. *Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of a C^*-algebra $\mathfrak{A}$ are called* **equivalent**, *iff there is a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $a \in \mathfrak{A}$:*
$$U\pi_1(a)U^* = \pi_2(a).$$

3. *A representation $(\mathcal{H}, \pi\mathfrak{A})$ is called* **cyclic**, *if there exists an element $\Omega \in \mathcal{H}$ such that $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. The vector Ω is then called cyclic vector.*

4. *A functional ω on a C^*-algebra $\mathfrak{A}$ is called a* **state**, *iff it is (1) positive, i.e. $\omega(a^+) \geq 0 \forall a^+ \in \mathfrak{A}^+$, and (2) normalized, i.e. $||\omega|| = 1$. The set of all states is denoted by $\mathcal{S}(\mathfrak{A})$.*

5. *A linear map $m : \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras is* **positive**, *iff for all $a^+ \in \mathfrak{A}^+ : m(a^+) \in \mathfrak{B}^+$.*

Let us recall some results about representations and states:

Theorem 7 *1.* **Riesz:** *Every state on $C(\mathbb{X})$ is a probability measure on $\mathbb{X}$.*

2. *A bounded functional ω on a unitial C^*-algebra is positive, iff $||\omega|| = \omega(\mathbb{I})$.*

3. *Every positive map between C^*-algebras is continuous.*

4. **Cauchy-Schwartz:** *A positive linear functional on a C^*-algebra $\mathfrak{A}$ satisfies: $|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$ for all $a, b \in \mathfrak{A}$.*

5. *Given a nonunitial C^*-algebra $\mathfrak{A}$ and a state ω thereon, then $\omega_1 : a + z\mathbb{I} \mapsto \omega(a) + z$ is the unique extension of ω to $\mathfrak{A}_1$.*

6. *Given an element a of a C^*-algebra $\mathfrak{A}$ and $z \in \sigma(a)$, then there exists a state ω_a on $\mathfrak{A}$ with $\omega_a(a) = z$.*

7. *The set of states on a C^*-algebra $\mathfrak{A}$ is a convex space and is compact, if $\mathfrak{A}$ is unitial.*

8. *Every nondegenerate representation of a C^*-algebra is a direct sum of cyclic representations.*

A very important tool in the representation theory of C^*-algebras is the GNS construction which works as follows:
Given a state ω on a C^*-algebra $\mathfrak{A}$, one can define a sesquilinear form on $\mathfrak{A}$ by setting for $a_1, a_2 \in \mathfrak{A}$:

$$(a_1, a_2) := \omega(a_1^* a_2). \tag{A.2}$$

For all $a \in \mathfrak{A}$: $(a, a) = \omega(a^*a) \geq 0$, i.e. $(.,.)$ is positive semi-definite. Consider the null space:

$$\mathfrak{I}_\omega = \{a \in \mathfrak{A} : \omega(a^*a) = 0\}, \tag{A.3}$$

which turns out to be a closed left ideal of $\mathfrak{A}$, which we call the Gel'fand ideal of ω. The space $\mathcal{H}_o := \mathfrak{A}/\mathcal{I}_\omega$ consists of the equivalence classes $[a]$ under the equivalence relation $a \sim b$ iff $\exists i \in \mathcal{I}_\omega : a = b + i$. The sesquilinear structure $(.,.)$ induces an inner product $(.,.)_\omega$ on $\mathcal{H}_o$ defined for any $a_1, a_2 \in \mathfrak{A}$ by:

$$([a_1], [a_2])_\omega := (a_1, a_2)_o. \tag{A.4}$$

The completion of $\mathcal{H}_o$ in the inner product $(.,.)_\omega$ is a Hilbert space denoted by $\mathcal{H}_\omega$. This Hilbert space carries a natural representation π_ω of $\mathfrak{A}$ defined for $a \in \mathfrak{A}$ and $[b] \in \mathcal{H}_o$ by:

$$\pi_\omega(a)[b] := [ab], \tag{A.5}$$

which is continuous and can thus be extended by density to all of $\mathcal{H}_\omega$. Finally, $\Omega_\omega := [\mathbb{I}]$ obviously defines a cyclic vector and hence for all $a \in \mathfrak{A}$:

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega)_\omega. \tag{A.6}$$

Thus, given a state ω on a C^*-algebra $\mathfrak{A}$, we have a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{A}$ on $\mathcal{H}_\omega$.
This construction can be used to prove very important properties of representations of C^*-algebras:

Theorem 8 *1. Given a C^*-algebra and two cyclic representations $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ which satisfy $\omega_1(a) := (\Omega_1, \pi_1(a)\Omega_2) = \omega_2(a)$, then the map $U : \pi_1(a)\Omega_1 \mapsto \pi_2(a)\Omega_2$ extends to a unitary equivalence between the two representations.*

2. Every C^*-algebra is isomorphic to a norm closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Particulary, there exists a faithful representation for each C^*-algebra.

3. An element a of a C^*-algebra is positive iff $\pi_\omega(a) \geq 0$ for all cyclic representations.

There is an important generalization to the GNS construction, which needs the following notation:

Definition 37 *1. Given a C^*-algebra $\mathfrak{A}$, for each $n \in \mathbb{N}$, the C^*-algebra $M_n(\mathfrak{A})$ denotes the matrix algebra of $n \times n$ -matrices with entries in $\mathfrak{A}$.*

2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $m : \mathfrak{A} \to \mathfrak{B}$ be linear. m is called **completely positive***, iff for all $n \in \mathbb{N}$ the induced map $m_n(M)_{ij} = m(M_{ij})$ between the matrix algebras $M_n(\mathfrak{A})$ and $M_n(\mathfrak{B})$ is positive.*

3. A linear map W between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is called a **partial isometry***, iff the W pre-image of $\mathcal{H}_2$ is closed in $\mathcal{H}_1$ and for all elements v_1, w_1 in the W-pre-image: $(v_1, w_1)_1 = (W(v_1), W(w_1))_2$ and $W = 0$ outside the W-pre-image.*

Let us now assume a completely positive map $m : \mathfrak{A} \to \mathfrak{B}$ between two C^*-algberas and given a faithful representation $(\mathcal{H}, \rho)$ of $\mathfrak{B}$, we can modify the GNS-construction to the Stinespring construction to construct a partially isometric representation of $\mathfrak{A}$ as follows:
We start with defining a sesquilinear form $(.,.)_o$ on $\mathfrak{A} \otimes \mathcal{H}$ by setting for $a_1, a_2 \in \mathfrak{A}$ and $h_1, h_2 \in \mathcal{H}$:

$$(a_1 \otimes h_1, a_2 \otimes h_2)_o := (h_1, \pi(m(a_1^* a_2))h_2), \tag{A.7}$$

which is positive semidefinite since m is completely positive. The null space

$$I := \{v \in \mathfrak{A} \otimes \mathcal{H} : (v, v)_o = 0\} \tag{A.8}$$

can be factored out of $\mathfrak{A} \otimes \mathcal{H}$, i.e. we define the space $\mathcal{H}_o := (\mathfrak{A} \otimes \mathcal{H})/\mathcal{I}$, on which the sesquilinear structure $(.,.)_o$ is nondegenerate through the equivalence classes $[v]$ of the equivalence relation $v_1 \sim v_2$ iff $\exists i \in \mathcal{I}$ such that $v_1 = v_2 + i$. This is used to define the inner product $\langle .,. \rangle$ for all $a_1, a_2 \in \mathfrak{A}$ and all $h_1, h_2 \in \mathcal{H}$ by:

$$\langle [a_1 \otimes h_1], [a_2 \otimes h_2] \rangle := (a_1 \otimes h_1, a_2 \otimes h_2). \tag{A.9}$$

The Hilbert space $\mathfrak{H}$ is the completion of $\mathcal{H}_o$ in the sesquilinear form $\langle .,. \rangle$. The representation π of $\mathfrak{A}$ on $\mathfrak{H}$ is defined on $\mathcal{H}_o$ for all $a, b \in \mathfrak{A}$ and all $h \in \mathcal{H}$ through:

$$\pi(a)[b \otimes h] := [ab \otimes h], \tag{A.10}$$

which is well defined because $\pi(a)\mathcal{I}$ is contained in $\mathcal{I}$. Moreover π is continuous, since $||\pi(a)|| \leq ||a||$ for all $a \in \mathfrak{A}$, such that we can extend π by density to all of $\mathfrak{H}$. We can build a partial isometry W between $\mathcal{H}$ and $\mathfrak{H}$, which is defined for all $h \in \mathcal{H}$ by:

$$Wh := [\mathbb{I} \otimes h]. \tag{A.11}$$

The adjoint W^* of W is given by the extension of the structure that acts for each $a \in \mathfrak{A}$ and each $h \in \mathcal{H}$ as:

$$W^*[a \otimes h] := m(\rho(a))h. \tag{A.12}$$

The extension by density clearly satisfies: $W^*W = \mathbb{I}$. This can be summarized as:

Theorem 9 *Given a completely positive map $m : \mathfrak{A} \to \mathfrak{B}$ between two unitial C^*-algebras with $m(\mathbb{I}) = \mathbb{I}$ and given a representation $(\mathcal{H}, \rho)$ of $\mathfrak{B}$, Then there exists a Hilbert space representation $(\mathfrak{H}, \pi)$ of $\mathfrak{A}$ and a partial isometry W such that $\rho(m(a)) = W^*\pi(a)W$.*

A.1.5 C^*-algebra of compact operators on a Hilbert space

An important C^*-algebra is the algebra of compact operators on a Hilbert space. This very special C^*-algebra is explained in this section, because Morita theory for C^*-algebras views general C^*-algebras in a similar way.

Definition 38
1. *The* **finite rank** *operators on a Hilbert space $\mathcal{H}$ is the finite span of the rank one projections in $\mathcal{H}$, i.e. those projections on $\mathcal{H}$ that do not have a proper subprojection. The algebra of finite rank operators is denoted by $B_f(\mathcal{H})$.*
2. *The norm closure of $B_f(\mathcal{H})$ is the C^*-algebra of* **compact operators** *on $\mathcal{H}$, i.e. the smallest C^*-algebra that contains $B_f(\mathcal{H})$. The algebra of compact operators is denoted by $B_o(\mathcal{H})$.*
3. *The algebra of* **Hilbert-Schmidt operators** *consists of the operators with finite Hilbert-Schmidt norm: $||a||_2 := \sqrt{\sum_i ||ae_i||^2}$, where e_i is an arbitrary orthonormal basis of $\mathcal{H}$. The algebra of Hilbert-Schmidt operators is denoted by $B_2(\mathcal{H})$.*
4. *The algebra of* **trace class operators** *consists of the operators for which the trace norm $||a||_1 := \sum_i ((a^*a)^{\frac{1}{2}} e_i, e_i)$, where e_i is an arbitrary orthonormal basis of $\mathcal{H}$, is finite. The algebra of trace class operators is denoted by $B_1(\mathcal{H})$.*

Let us now review some properties of the algebra of compact operators and its subalgebras B_2, B_1, B_f:

Theorem 10
1. *An operator lies in $B_o(\mathcal{H})$, iff it can be norm-approximated by finite rank operators.*

2. *The unit operator lies in $B_o(\mathcal{H})$, iff $\mathcal{H}$ is finite dimensional.*

3. *The $B_o(\mathcal{H})$ is an ideal of $B(\mathcal{H})$.*

4. *Let $\mathcal{B}$ be the unit ball in $\mathcal{H}$ and let $a \in B_o(\mathcal{H})$, then $a\mathcal{B}$ is compact.*

5. *Denote the rank one projection with image v by P_v. An self-adjoint operator a is compact, iff there is a sequence $\lambda_n \in \mathbb{R}$ whose only accumulation point is zero and $a = \sum_n \lambda_n P_{v_n}$, where v_n is an orthonormal set of elements of $\mathcal{H}$.*

6. *Every bounded operator a on $\mathcal{H}$ has a polar decomposition*
$$a := U|a| = U\sqrt{a^*a},$$
where U is a partial isometry, whose kernel coincides with the kernel of a.

7. *For $\mathcal{H}$ infinite dimensional one has*
$$B_f(\mathcal{H}) \subset B_1(\mathcal{H}) \subset B_2(\mathcal{H}) \subset B_o(\mathcal{H}) \subset B(\mathcal{H}).$$

8. *The state space of $B_o(\mathcal{H})$ consists of all positive elements of the algebra of trace class operators $\rho \in B_1(\mathcal{H})$ with unit trace. (density matrices)*

9. *The pure states of the algebra of compact operators consists of all rank one projections.*

10. *There is exactly one unitary equivalence class of irreducible representations of the algebra of compact operators. I.e. every representation of the algebra of compact operators is unitarily equivalent to the fundamental representation on $\mathcal{H}$.*

A.2 Morita Equivalence of C^*-algebras and Rieffel Induction

This section serves as a briefing on Hilbert C^*-modules and related structures. This subject is not widely known amongst physicists, thus although the proofs are all available in the literature, we include the proofs of certain fundamental theorems, mainly to explain the techniques used in this field of mathematics. If it becomes too technical, we rather sketch the proof and focus on the underlying construction.

A.2.1 Preparations

Gel'fand theory for commutative C^*-algebras tells us that commutative C^*-algebras are equivalent to locally compact Hausdorff spaces, since $\Delta(C(\mathbb{X})) = \mathbb{X}$ as a topological space and $C(\Delta(\mathfrak{A})) = \mathfrak{A}$ as a C^*-algebra for any locally compact Hausdorff space $\mathbb{X}$ and any commutative C^*-algebra $\mathfrak{A}$. There is an analogous correspondence for vector bundles:

Definition 39 1. *A* **vector bundle** *is a bundle $E(\pi, \mathbb{X}, F)$ in which each fibre is a finite dimensional vector space, such that the subspace topology of each fibre is the topology of this linear space, particularly that each trivialization $T : \pi^{-1}(x) \to F$ is a linear map.*

2. *A vector bundle is complex, if the typical fibre $F = \mathbb{C}^m$ for some finite number m.*

The space of sections $\Gamma(E)$ of a a complex vector bundle E over the base space $\mathbb{X}$ has the remarkable property that it can be written as a finitely generated projective module:

$$\Gamma(E) = p(\oplus^m C(\mathbb{X})) \tag{A.13}$$

for some idempotent matrix $p^2 = p$ in $M_{mm}(C(\mathbb{X}))$. On the one hand each finitely generated projective module $p(\oplus^m C(\mathbb{X}))$ over $C(\mathbb{X})$ corresponds to a space of sections in a complex vector bundle Γ in a canonical way. Each complex vector bundle on the other hand defines a finitely generated projective module. Thus:

Theorem 11 Serre-Swan: *There is the analogue correspondence between the finitely generated projective modules $\mathcal{M} = p(\oplus^m \mathfrak{A})$ over a commutative C^*-algebra $\mathfrak{A}$ and the space of sections $\Gamma(E)$ of fibre bundles E over $\Delta(\mathfrak{A})$, which means that every finitely generated projective module over a commutative C^*-algebra is a space of sections in a complex vector bundle over its spectrum.*

A **Hermitian structure** over a complex vector bundle E over $\mathbb{X}$ is a map $(.,.)_x$ that defines an inner product on each fibre $\pi^{-1}(x)$, such that $x \mapsto (f_x, g_x)_x \in C(\mathbb{X})$.

Definition 40 *A* **Hilbert bundle** *H is a projective module over a commutative C^*-algebra $C(\mathbb{X})$ such that the typical fibre F is a Hilbert space with Hermitian structure $x \mapsto \langle .,. \rangle_x$, with values in $C(\mathbb{X})$.*

Finitely generated projective modules with Hermitian structure are clearly Hilbert bundles with a canonical Hermitian structure; general Hilbert bundles do however not necessarily possess a finite dimensional fibre, but their fibres are allowed to be arbitrary Hilbert spaces. An important step in the understanding of noncommutative geometry is done by generalizing Hilbert-bundles to modules over not necessarily commutative C^*-algebras:

A.2.2 Hilbert bundles and Hilbert C^*-modules

A Hilbert-bundle is a generalization of a complex vector bundle, in the sense that we take an arbitrary Hilbert space as fibre.
In a similar way, one can view a Hilbert C^*-module as a generalization of a Hilbert bundle, just that its base algebra is not commutative $C(\mathbb{X})$ anymore, but a generally noncommutative C^*-algebra. Let us prepare the definition of a Hilbert C^*-module:

Definition 41 *We call a triple* $(E, \langle .,. \rangle_{\mathfrak{A}}, \pi)$ *consisting of a complex linear space* E *together with a right action* $(\pi, \mathfrak{A})$ *of a* C^**-algebra* $\mathfrak{A}$ *on* E *(i.e.* $\pi(ab)e = eab = \pi(b)\pi(a)e$*) and a bilinear structure* $\langle .,. \rangle_{\mathfrak{A}} : E \times E \to \mathfrak{A}$ *a* **semi Hilbert** C^***-module***, iff for all* $e, e_1, e_2 \in E$*,* $a \in \mathfrak{A}$*:*

1. $\langle .,. \rangle_{\mathfrak{A}}$ *is linear in the second slot*
2. $\langle e_1, e_2 \rangle^*_{\mathfrak{A}} = \langle e_2, e_1 \rangle_{\mathfrak{A}}$
3. $\langle e_1, e_2 a \rangle_{\mathfrak{A}} = \langle e_1, e_2 \rangle_{\mathfrak{A}} a$
4. $\langle e, e \rangle_{\mathfrak{A}} \geq 0$,

where we used the notation $\pi(a)e = ea$.

For a semi Hilbert C^*-module, one can introduce a semi-norm, which is defined for $e \in E$:

$$||e|| := ||\langle e, e \rangle_{\mathfrak{A}}||^{\frac{1}{2}} = \sqrt{\sup\{\omega(\langle e, e \rangle_{\mathfrak{A}}) : \omega \in \mathcal{E}(\mathfrak{A})\}}, \tag{A.14}$$

which is obviously a semi-norm, since $\langle e, e \rangle_{\mathfrak{A}} \geq 0$. By factoring out the zero-space of this norm, we obtain a pre-Hilbert C^*-module:

Definition 42 *A semi Hilbert* C^**-module* $(E, \langle .,. \rangle_{\mathfrak{A}}, \pi)$ *is called a* **pre-Hilbert** C^***-module***, if for all* $e \in E$*:* $\langle e, e \rangle_{\mathfrak{A}} = 0 \Leftrightarrow e = 0$.

In the definition of a semi Hilbert C^*-module, we can actually allow any $\mathfrak{A}_o$ to be a pre-C^*-algebra and use the C^*-norm to complete this algebra to $\mathfrak{A}$.

Lemma 31 *If* E *is a pre-Hilbert* C^**-module, then* $||.||$ *is a norm on* E.

proof: $||e||$ is clearly a semi-norm, since $\langle e, e \rangle_{\mathfrak{A}} \geq 0$. Since $\langle e, e \rangle_{\mathfrak{A}} = 0 \Leftrightarrow e = 0$ and since for any $a > 0$ there exists a state s.t. $\omega(a) > 0$, we see that $||e|| = 0$ implies $e = 0$. □

Definition 43 *A pre-Hilbert* C^**-module is a Hilbert-*C^**-module* $(E, \langle .,. \rangle_{\mathfrak{A}}, \pi)$ *over a* C^**-algebra* $\mathfrak{A}$*, such that* E *is complete in* $||.||$.

Lemma 32 *Any pre-Hilbert* C^**-module over a pre-*C^**-algebra* $\mathfrak{A}_o$ *can be completed to a Hilbert* C^**-module over the* C^**-completion* $\mathfrak{A}$.

proof:

1. E can be completed in $||.||$.
2. Since $||ea|| = \sqrt{||\langle ea, ea \rangle_{\mathfrak{A}}||} = \sqrt{||a^* \langle e, e \rangle_{\mathfrak{A}} a||}$, and since $\langle e, e \rangle \geq 0$, we can use $a^* b^* b a \leq ||b||^2 a^* a$, which implies:

$$||ea|| \leq \sqrt{||||\langle e, e \rangle_{\mathfrak{A}}||^2 a^* a||} = \sqrt{||\langle e, e \rangle_{\mathfrak{A}}||||a^* a||} = ||e||||a||$$

. Now suppose any pair of sequences $\mathfrak{A}_o \ni a_n \to a \in \mathfrak{A}$ and $E_o \ni e_n \to e \in E$, we can extend $\pi(a_n)e_n = e_n a_n$ from $\mathfrak{A}_o$ and E_o to $\mathfrak{A}$ and E by continuity, since the inequality implies convergence.

3. Consider $e, f \in E_o, a \in \mathfrak{A}_o$: $0 \leq \langle e - fa, e - fa\rangle = \langle e, e\rangle_{\mathfrak{A}} - a^*\langle f, e\rangle_{\mathfrak{A}} - \langle e, f\rangle_{\mathfrak{A}} a + a^*\langle f, f\rangle_{\mathfrak{A}} a$, since setting $a = \langle f, e\rangle_{\mathfrak{A}} ||f||^{-2}$ is in $\mathfrak{A}_o$ for $f \neq 0$, we have

   $$\langle e, f\rangle_{\mathfrak{A}} ||f||^{-2}\langle f, e\rangle_{\mathfrak{A}} + \langle e, f\rangle_{\mathfrak{A}}\langle f, e\rangle_{\mathfrak{A}} ||f||^{-2} - \langle e, e\rangle_{\mathfrak{A}} \geq ||f||^{-4}\langle e, f\rangle_{\mathfrak{A}}\langle f, f\rangle_{\mathfrak{A}}\langle f, e\rangle_{\mathfrak{A}}$$

   Since $\langle f, f\rangle \geq 0$: $c^*\langle f, f\rangle_{\mathfrak{A}} c \geq c^* ||\langle f, f\rangle_{\mathfrak{A}}||$, which implies:

   $$\langle e, f\rangle_{\mathfrak{A}}\langle f, e\rangle_{\mathfrak{A}} \leq \langle e, e\rangle_{\mathfrak{A}} ||f||^2,$$

   which implies $\langle e, f\rangle_{\mathfrak{A}} \leq ||e||||f||$.
   Using this analogue to the Chauchy-Schwarty inequality, we see that for any two $E_o \ni e_n, f_n \to e, f \in E$ the $\langle e_n, f_n\rangle_{\mathfrak{A}} \to \langle e, f\rangle_{\mathfrak{A}}$ lies in the norm-completion $\mathfrak{A}$ of $\mathfrak{A}_o$.

4. Obviously for any two sequences $E_o \ni e_n, f_n \to$, $\langle e, f\rangle^*_{\mathfrak{A}} = \langle f, e\rangle_{\mathfrak{A}}$ holds by continuity as well as for $a_o \ni a_n \to a \in \mathfrak{A}$ $\langle e, fa\rangle_{\mathfrak{A}} = \langle e, f\rangle_{\mathfrak{A}} a$ holds by continuity. Moreover, the positivity and nondegeneracy of $\langle e, e\rangle_{\mathfrak{A}}$ also obviously hold.

□
If we allow the base algebra of a semi Hilbert C^*-module to be a pre-C^*-algebra $\mathfrak{A}_o$, then we can again complete it in the norm to a C^*-algebra and conclude:

Corollary 16 *Any semi Hilbert C^*-module E_o over a pre C^*-algebra $\mathfrak{A}_o$ can be turned into a Hilbert C^*-module E over the C^*-completion $\mathfrak{A}$.*

proof: First factor the zero-space of the E_o-norm out to obtain a pre C^*-module E and use then the previous lemma. □
The practical value of this is, that one can construct Hilbert C^*-modules from dense subalgebras and that one does not have to worry about nondegeneracy and completeness, which can be achieved after a module with the desired properties is constructed.

A.2.3 Adjoinable Maps

One important property of a Hilbert C^*-module E is, that it defines a certain C^*-algebra, which we want to describe in the following: Let us consider the linear maps $A : E \to E$, then we can look for the subset of these maps, that is adjoinable in $\langle ., .\rangle_{\mathfrak{A}}$:

Definition 44 *A linear map $A : E \to E$ is called* **adjoinable**, *iff there exists a linear map $A^* : E \to E$, such that for all $e, f \in E$:*

$$\langle e, Af\rangle_{\mathfrak{A}} = \langle A^* e, f\rangle_{\mathfrak{A}}.$$

Since E is a Banach space, we can define a norm on for the adjoinable maps by

$$||A|| := \sup\{||Ae|| : e \in E, ||e|| \leq 1\}. \tag{A.15}$$

This algebra is actually a C^*-algebra, whose involution is given by the adjoining. We call this algebra $C^*(E, \mathfrak{A})$ and note the following properties:

Lemma 33 *Let $C^*(E,\mathfrak{A})$ be the algebra of adjoinable maps on a Hilbert C^*-module E over $\mathfrak{A}$, together with the above defined norm and the adjoining as involution, then*

1. *The $C^*(E,\mathfrak{A})$-action on E is compatible with the action of $\mathfrak{A}$, i.e. $(Ae)a = A(ea) \forall a \in \mathfrak{A}, e \in E$.*
2. *The action of $C^*(E,\mathfrak{A})$ is bounded.*
3. *The adjoint of an element A is unique and defines an involution.*
4. *$C^*(E,\mathfrak{A})$ is a C^*-algebra.*
5. *All $A \in C^*(E,\mathfrak{A})$ satisfy $\langle Ae, Ae\rangle_{\mathfrak{A}} \leq ||A||^2 \langle e, e\rangle_{\mathfrak{A}}$ for all $e \in E$.*
6. *The action of $C^*(E,\mathfrak{A})$ on E is nondegenerate.*

proof:

1. To show that $A(ea) = (Ae)a$ for all $e \in E, a \in \mathfrak{A}$ and $A \in C^*(E,\mathfrak{A})$ consider:

   $$\langle f, A(ea)\rangle_A = \langle A^* f, ea\rangle_A = \langle A^* f, e\rangle_A a = \langle f, (Ae)\rangle_A = \langle f, (Ae)a\rangle_A,$$

   which holds for all $e, f \in E$ and thus implies $A(ea) = (Ae)a$ due to $||\langle f, e\rangle_A|| = 0 \forall f in E \Rightarrow e = 0$.

2. To show that A is bounded we define the form $T_f : E \to A : e \mapsto \langle A^* A f, e\rangle_A$. Then:

   $$||T_f|| = sup\{||T_f e|| : ||e|| \leq 1\} = \sup\{||\langle A^* A f, e\rangle_A|| : ||e|| \leq 1\}$$

   $$\leq \sup\{||A^* A f||\, ||e|| : ||e|| \leq 1\} = ||A^* A f|| < \infty,$$

   which shows the boundedness of T_f from the fact that $A^* A f \in E$.
   On the other hand:

   $$||T_f|| = \sup\{||\langle f, A^* A e\rangle_A|| : ||e|| \leq 1\} \leq ||f|| \sup\{||A^* A e|| : ||e|| \leq 1\}$$

   Using $||T_f|| < \infty$ implies $\sup\{||A^* A e|| : ||e|| \leq 1\} \leq 1$ and the Banach Steinhaus theorem[1], we see that $\sup\{||T_f|| : ||f|| \leq c\} < \infty$ for all finite values of $c \geq 0$. Thus,

   $$||A|| = \sup\{||Ae|| : ||e|| \leq 1\} = \sup\{||\langle Ae, Ae\rangle_A|| : ||e|| \leq 1\}$$

   $$\leq \sup\{||\langle Af, Ae\rangle_A|| : ||e|| \leq 1, ||f|| \leq 1\} = \sup\{||T_f|| : ||f|| \leq 1\} < \infty.$$

[1] The Banach Steinhaus Theorem states that if $\mathbb{X}$ is a Banach space, $\mathbb{Y}$ is a normed linear space and if $\{T_i : \mathbb{X} \to \mathbb{Y}\}_{i \in \mathcal{I}}$ is a family of bounded linear maps such that for each $i \in \mathcal{I}$: $\sup\{||T_i(x)|| : x \in \mathbb{X}, ||x|| \leq 1\}$ is bounded, then $\sup\{||T_i|| : i \in \mathcal{I}\}$ is bounded.

3. • Assume $\exists B \neq A$ s.t. $\langle Be, f\rangle_A = \langle e, Af\rangle_A \forall e, f \in E$. Then $\langle (B - A^*)e, f\rangle_A \neq 0$ for some $e, f \in E$

$$\Rightarrow \langle Be, f\rangle_A - \langle A^* e, f\rangle_A = \langle e, Af\rangle_A - \langle e, Af\rangle_A = 0$$

$\Rightarrow$ contradiction! $\rightarrow A^*$ is unique for each A.

•

$$\langle Ae, f\rangle_A = (\langle f, Ae\rangle_A)^* = \langle e, A^* f\rangle_A = \langle A^{**} e, f\rangle_A,$$

which is true for all $e, f \in E$ and hence $A^{**} = A$.

•

$$\langle e, A(Bf)\rangle_A = \langle A^* e, Bf\rangle_A = \langle B^* A^* e, f\rangle_A,$$

which is true for all $e, f \in E$ and hence $(AB)^* = B^* A^*$.
$\Rightarrow$ the adjoint map defines an involution.

4. We first show that $C^*(E, \mathfrak{A})$ is a Banach algebra that is closed under the involution:
For a Cauchy sequence $\{A_n\}$ there and each $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$\sup\{||A_n e - A_m e|| : ||e|| \leq 1\} < \epsilon,$$

which implies that $e_m := A_m e$ converges in E to Ae due to completeness of E. Hence there is an map $A : E \rightarrow E$ defined by the linear extension of Ae for all e in the unit ball $\Rightarrow$

$$\sup\{||A_n e - Ae|| : ||e|| \leq 1\} < \epsilon \forall n > N(\epsilon).$$

Thus, $A_n \rightarrow A$. Let us consider

$$||\langle e, A_n f\rangle_A - \langle e, Af\rangle_A|| < \epsilon \ \forall n > N(\epsilon)$$

$$\Rightarrow ||\langle e, (A_n - A)f\rangle_A|| < \epsilon$$

$$\Rightarrow ||\langle (A_n^* - A^*)e, f\rangle_A|| < \epsilon,$$

thus A_n^* converges to A^*. To prove $||A||^2 \leq ||A^* A||$, consider

$$\begin{aligned} ||A||^2 = & \ \sup\{||\langle Ae, A_e\rangle_A|| : ||e|| \leq 1\} \\ = & \ \sup\{\langle e, A^* Ae\rangle_A : ||e|| \leq 1\} \\ \leq & \ \sup\{||A^* Ae|| \, ||e|| : ||e|| \leq 1\} = ||A^* A||, \end{aligned}$$

which makes $C^*(E, \mathfrak{A})$ a C^*-algebra.

5. To show the bound $\langle Ae, Ae\rangle_A \leq ||A||^2 \langle e, e\rangle_A$, we consider $A \geq 0 \Rightarrow \exists B$ s.t. $A = B^* B$:

$$\Rightarrow \forall A \geq 0 : \langle e, Ae\rangle_A = \langle Be, Be\rangle_A \geq 0,$$

which considering that $||A^* A||\mathbb{I} - A^* A \geq 0$ implies:

$$\langle e, (||A^* A||\mathbb{I} - A^* A)e\rangle_A = ||A^* A||\langle e, e\rangle_A - \langle Ae, Ae\rangle_A \geq 0.$$

6. To prove nodegeneracy, assume there is $e \neq 0 \in E$ such that e is annihilated by all $A \in C^*(E, \mathfrak{A})$ and consider $A : e \mapsto e_1\langle e_2, e\rangle_A$:

$$\begin{aligned} &\Rightarrow \quad e_1\langle e_2, e\rangle_A = 0 \, \forall e_1, e_2 \in E \\ &\Rightarrow \quad \langle e_2, e\rangle_A = 0 \, \forall e_2 \in E \\ &\Rightarrow \quad \langle e, e\rangle_A = 0 \\ &\Rightarrow \quad e = 0. \end{aligned}$$

□

Given a Hilbert C^*-module E over $\mathfrak{A}$, there is an obvious subalgebra of $C^*(E, \mathfrak{A})$, that is particularly easy to construct. Let us consider the linear maps $t : E \to E$, which are labeled by two elements $e, f \in E$ and act on $g \in E$ as:

$$t_{e,f} : g \mapsto e\langle f, g\rangle_{\mathfrak{A}}, \tag{A.16}$$

which is obviously an adjoinable map. The similarity to the rank one-operators on a Hilbert-space suggests:

Definition 45 *The C^*-subalgebra $C_o(E, \mathfrak{A})$ of $C^*(E, \mathfrak{A})$, which is the subalgebra generated by the operators t_{ef}, is called the algebra of* **"compact operators"***.*

From the properties of Hilbert-C^*-modules, we obtain the following properties for the operators t, by simply inserting the definition of t_{ef} and using the properties quoted above:

$$\begin{array}{rcl} t^*_{ef} & = & t_{fe} \\ a t_{ef} & = & t_{ae,f} \\ t_{ef} a & = & t_{e,a^* f} \\ ||t_{ef}|| & \leq & ||e|| ||f||. \end{array} \tag{A.17}$$

With these two constructions we are able to associate two important C^*-algebras to each Hilbert C^*-module E over $\mathfrak{A}$, namely $C^*(E, \mathfrak{A})$ and $C_o(E, \mathfrak{A})$, where it is obvious, that the later is a two-sided ideal of the first.

A.2.4 Full Hilbert C^*-modules

In order to become able to start viewing Hilbert C^*-modules E as structures, that mediate between the C^*-algebras $\mathfrak{A}$ and $C^*(E, \mathfrak{A})$ resp. $C_o(E, \mathfrak{A})$, let us apply the concept of denseness to Hilbert C^*-modules:

Definition 46 *We call a pre-Hilbert C^*-module E over $\mathfrak{A}$* **full***, iff span$\{\langle e, f\rangle_{\mathfrak{A}} : e, f \in E\}$ is dense in $\mathfrak{A}$.*

To a given Hilbert C^*-module, we can associate its conjugate $\bar{E}$:

Definition 47 *To a given Hilbert C^*-module $(E, \langle ., .\rangle_{\mathfrak{A}}, \pi)$ we associate the* **conjugate** *Hilbert C^*-module $\bar{E}$ consisting of the complex conjugate bundle together with the right action $\bar{\pi}(a)e := a^* e$and the $C_o(E, \mathfrak{A})$-valued bilinear form $\langle e, f\rangle_{C_o(E,\mathfrak{A})} := t_{e,f}$.*

Lemma 34 *For a given full Hilbert C^*-module E over $\mathfrak{A}$, $\bar{E}$ is a full Hilbert C^*-module over $C_o(E,\mathfrak{A})$ with compatible left $\mathfrak{A}$ action $\pi_l(a)e := ea^*$.*

The proof consists of checking the various properties of full Hilbert-C^*-modules through the already established properties of the operators T_{ef}, e.g.:

$$\begin{array}{rcrl} T^*_{ef} = T_{fe} & \Rightarrow & \langle e,f\rangle^*_{C_o} = & \langle f,e\rangle_{C_o} \\ T^*_{e,fa} = T_{e,f}a & \Rightarrow & \langle e,fa\rangle_{C_o} = & \langle e,f\rangle_{C_o}a \\ ||T^*_{ee}|| = 0 \leftrightarrow e = o & \Rightarrow & ||\langle e,e\rangle_{C_o}|| = 0 & \leftrightarrow e = 0 \end{array} \tag{A.18}$$

We will not quote the entire proof here, since another more adapted technique is more useful in the cases that we consider later.
The proof of the next corollary is rather technical such that we omit it here.

Corollary 17 *For a given full Hilbert C^*-module E over $\mathfrak{A}$, $C_o(\bar{E},C_o(E,\mathfrak{A}))$ is isomorphic to $\mathfrak{A}$.*

This establishes full Hilbert C^*-modules over a C^*-algebra $\mathfrak{A}$ as structures linking this algebra with $C_o(E,\mathfrak{A})$ in the sense, that both algebras can be calculated from this module and its conjugate respectively.
We are now in the position to define Morita equivalence for C^*-algebras:

Definition 48 *Let E be a full Hilbert C^*-module over $\mathfrak{A}$. Then E is called a* **Morita equivalence bimodule** *between $\mathfrak{A}$ and $C_o(\mathfrak{A},E)$. Moreover two C^*-algebras $\mathfrak{A},\mathfrak{B}$ are called* **Morita equivalent** *if there exists a Morita equivalence bimodule E linking $\mathfrak{A}$ with $\mathfrak{B} = C_o(\mathfrak{A},E)$.*

The following lemma lets us construct an equivalence relation given by the existence of a full Hilbert C^*-module linking two algebras:

Lemma 35 *Given two full Hilbert C^*-modules E_1, E_2 linking the C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{B}$ and $\mathfrak{C}$, one can construct a full Hilbert C^*-module E linking $\mathfrak{A}$ and $\mathfrak{C}$.*

Rather working through the proof of this lemma, let us consider the main idea: Given a Morita equivalence bimodule E_1 linking $\mathfrak{A}$ to $\mathfrak{B}$ and given a Morita equivalence bimodule E linking $\mathfrak{B}$ with $\mathfrak{C}$, we can build the tensor product $E_1 \otimes_B E_2$ as the quotient of $E = E_1 \otimes E_2$ by the ideal $\mathcal{I} = \{e_1 b \otimes e_2 - e_1 \otimes b e_2 : b \in \mathfrak{B}\}$. The space E carries a left action of $\mathfrak{A}$ which is given by:

$$\pi_l(a) e_1 \otimes_B e_2 := a e_1 \otimes_B e_2. \tag{A.19}$$

Let us now define the sesquilinear map with values in $\mathfrak{A}$ by

$$\langle e_1 \otimes e_2, f_1 \otimes f_2\rangle_o := \langle e_1 \langle f_2, e_2\rangle_B, f_1\rangle_A. \tag{A.20}$$

It is easily checked that E together with the structure $\langle .,.\rangle_o$ is a full Hilbert C^*-module over $\mathfrak{A}$. Using the analogue of the rank one operators $T_{f,g}$ on E:

$$T_{e,f} h := e\langle f,h\rangle_o \tag{A.21}$$

it turns out that the linked C^*-algebra $C_o(\mathfrak{A},E)$ is indeed $\mathfrak{C}$. This tensor product construction is essential for the proof of:

Theorem 12 *Morita equivalence is an equivalence relation for C^*-algebras.*

proof: Reflexivity: Given a C^*-algebra $\mathfrak{A}$, it is easy to verify that $E = \mathfrak{A}$ together with $\langle a_1, a_2 \rangle_A := a_1^* a_2$ satisfies the axioms for a Hilbert C^*-module with values in $\mathfrak{A}$ and that the rank one operators $T_{a_1,a_2} : b \mapsto a_1 a_2^* b$ span exactly $\mathfrak{A}$.
Symmetry is clear from the construction of $\bar{E}$, which links $C_o(\mathfrak{A}, E)$ with $\mathfrak{A}$ exactly inverse to E which links $\mathfrak{A}$ with $C_o(\mathfrak{A}, E)$.
Transitivity is due to the previous lemma and is achieved through the tensor product construction. □
As we have seen earlier, we can construct Hilbert C^*-modules over a C^*-algebra from pre-Hilbert-C^*-modules over a dense pre-C^*-algebra. Let us now collect the analogous properties, that are necessary to link two C^*-algebras with a full pre-Hilbert-C^*-module. This is a very practical way to construct Morita equivalence bimodules.

Lemma 36 *Given*

1. *a pre-Hilbert C^*-module E over a C^*-algebra $\mathfrak{A}$,*

2. *a left action of a C^*-algebra $\mathfrak{B}$ on $\bar{E}$, such that E is a full Hilbert-pre C^*-module over $\mathfrak{B}$, such that for all $e_1, e_2, e_3 \in E$:*

$$\langle e_1, e_2 \rangle_B e_3 = e_1 \langle e_2, e_3 \rangle_A$$

3. *which satisfy for all $e \in E$, $e \in \mathfrak{A}$ and $b \in \mathfrak{B}$:*

$$\begin{array}{rcl} \langle ea, ea \rangle_B & \leq & ||a||^2 \langle e, e \rangle_B \\ \langle be, be \rangle_A & \leq & ||b||^2 \langle e, e \rangle_A, \end{array}$$

then E can be completed to a Morita equivalence bimodule linking the C^*-algebras $\mathfrak{A}, \mathfrak{B}$.

The proof consists of checking the properties of a Morita equivalence bimodule for the completion of E and its conjugate $\bar{E}$ directly and is not instructive.

A.2.5 Induced Representations for C^*-algebras (Rieffel Induction)

Given a Hilbert space representation $(\mathcal{H}, \pi)$ of C^*-algebra $\mathfrak{A}$, it is the purpose of this subsection is to construct induced representations for a Morita equivalent C^*-algebra $\mathfrak{B}$ using the Morita equivalence bimodule E that links the two C^*-algebras. This is done by a generalization of the GNS construction known as Rieffel induction:

Given a full Hilbert C^*-module E over $\mathfrak{A}$, let us first describe the induction out of a state ω on $\mathfrak{A}$ to a representation of $C_o(E, \mathfrak{A})$:

For a given state ω on $\mathfrak{A}$, we define the sesquilinear form $(.,.)_o$ on E by defining for $e_1, e_2 \in E$:

$$(e_1, e_2)_o := \omega(\langle e_1, e_2 \rangle_A). \tag{A.22}$$

This sesquilinear form is positive semidefinite, since $\langle e, e\rangle_A \geq 0$ for all $e \in E$ and since $\omega : \mathfrak{A}^+ \to \mathbb{R}_o^+$. For each element E in its null space

$$N_\omega = \{e \in E : (e, e)_o = 0\} \tag{A.23}$$

it is implied that $(e, f)_o = 0$, since

$$0 = \omega(\langle e, e\rangle_A)\omega(\langle f, f\rangle_A)_o \geq |(e, f)_o|^2 \tag{A.24}$$

by the Cauchy Schwartz inequality for states.

The form $(.,.)_o$ can be turned into an inner product $(.,)$ on E/N_ω, which using the equivalence classes $[e] \in E/N_\omega$ is defined by:

$$([e], [f]) := (e, f)_o. \tag{A.25}$$

This structure is actually well defined since for all $e_1, e_2 \in N_\omega$ and all $f_1, f_2 \in E$:

$$(e_1 + f_1, e_2 + f_2)_o = \omega(\langle e_1 + f_1, e_2 + f_2\rangle_A) = \omega(\langle f_1, f_2\rangle_A) + 0 + 0 + 0 = (f_1, f_2)_o. \tag{A.26}$$

The closure of E/N_ω defines the Hilbert space $\mathcal{H}$.

The induced representation ρ of $C_o(E, \mathfrak{A})$ on $\mathcal{H}$ is first defined on the dense subspace E/N_ω through:

$$\rho(A)[e] := [Ae], \tag{A.27}$$

which turns out to be continuous because $||\rho(A)|| = sup\{||\rho(A)[e]|| : ||e|| \leq 1\}$, which can be estimated to be $||\rho(A)|| \leq ||A||$, since

$$||\rho(A)[e]||^2 \leq \sup_{e\in[e]} \omega(\langle\rho(A)e, \rho(A)e\rangle_A) \leq \sup_{e\in[e]} \omega(||A||^2\langle e, e\rangle_A) \leq ||A|| \sup_{e\in[e]} ||e||^2. \tag{A.28}$$

The continuous extension from the dense set E/N_ω to $\mathcal{H}$ defines the representation ρ, which clearly satisfies $\rho(a)\rho(b) = \rho(ab)$ and $\rho(a^*) = \rho(a)^*$ as can be checked by direct computation. This completes the induction of a representation $(\mathcal{H}, \rho)$ of $C_o(E, \mathfrak{A})$ from a state ω on $\mathfrak{A}$.

This construction can be extended to general representations $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, which are necessarily direct sums of cyclic representations:

Given a representation $(\mathcal{K}, \pi)$ of $\mathfrak{A}$ and given a full Hilbert C^*-module E over $\mathfrak{A}$, let us define the sesquilinear form $(.,.)_o$ on $E \otimes \mathcal{K}$ for $e_1, e_2 \in E$ and $v_1, v_2 \in \mathcal{K}$ by the linear extension of:

$$(e_1 \otimes v_1, e_2 \otimes v_2)_o := \langle v_1, \pi(\langle e_1, e_2\rangle_a)v_2\rangle. \tag{A.29}$$

This form is again positive semi-definite since the inner product of $\mathcal{K}$ and $\langle.,.\rangle_A$ are positive semi-definite. Its null space

$$N_\mathcal{K} := \{t \in E \otimes \mathcal{K} : (t, t)_o = 0\} = \{t \in E \otimes \mathcal{K} : (t, s) = 0 \forall s \in E \otimes \mathcal{K}\}, \tag{A.30}$$

where the second equation follows form the Cauchy Schwartz inequality.

Using the canonical equivalence classes $[e]$ in $E \otimes \mathcal{K}/N_{\mathcal{K}}$, one can define an inner product $(.,.)$ by setting for $t, s \in E \otimes \mathcal{K}$:

$$([t],[s]) := (t,s)_o, \tag{A.31}$$

which turns out to be well defined using a similar argument as in the GNS construction. Thus, the completion of $E \otimes \mathcal{K}/N_{\mathcal{K}}$ in $(.,.)$ defines a Hilbert space, say $\mathcal{H}$.

This Hilbert space $\mathcal{H}$ carries a natural representation ρ of $C^*(E, \mathfrak{A})$, which we define for $t \in E \otimes \mathcal{K}$ as:

$$\rho(A)[t] := [(A \otimes \mathbb{I})t]. \tag{A.32}$$

The extension to $\mathcal{H}$ by density is well defined and possible, since $||\rho(A)|| \leq ||A||$ using a similar argument as before.

Rieffel induction has an important consequence:

Theorem 13 *Given two Morita equivalent C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, then:*

1. *Each representation of $\mathfrak{A}$ induces a representation of $\mathfrak{B}$ by Rieffel induction and vice versa.*

2. *The from $(\mathcal{H}, \pi, \mathfrak{A})$ induced representation of $\mathfrak{B}$ using the equivalence module E induces a representation of $\mathfrak{A}$ which is unitarily equivalent to $(\mathcal{H}, \pi)$ using Rieffel induction by the conjugate equivalence module $\bar{E}$.*

3. *The induced representation of an irreducible representation is irreducible.*

4. *The induced representation of a direct sum of representations is the direct sum of the induced representations.*

The proof consists of direct construction of the unitary intertwiners.

The irreducible representations of a commutative algebra $\mathfrak{A}$ are given by the GNS constructions through the states that are given by the evaluation at points in the spectrum $\mathbb{X} = \Delta(\mathfrak{A})$. This lets us consider Rieffel induction as maps between the spectra, which also applies to noncommutative C^*-algebras. This is the idea behind the construction of noncommutative embeddings using a construction similar to Rieffel induction.

A.2.6 Linking Algebra

Given two Morita equivalent C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, there is always a C^*-algebra $\mathfrak{C}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are complementary full corners of $\mathfrak{C}$. Let us introduce some notation:

Definition 49 1. *A subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is* **hereditary** *if for all $b_1, b_2 \in \mathfrak{B}$ and all $a \in \mathfrak{A}$: $b_1 a b_2 \in \mathfrak{B}$.*

2. *A subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is called* **full***, if there is no proper two-sided ideal in $\mathfrak{A}$ that contains $\mathfrak{B}$.*

3. *A subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is a* **corner***, if there is a projection p in the multiplier algebra $M(\mathfrak{A})$, such that $\mathfrak{B} = \{pap : a \in \mathfrak{A}\}$. $p\mathfrak{A}p$ is dense in $\mathfrak{A}$ if the corner is full.*

4. *Two corners $p\mathfrak{A}p$ and $p'\mathfrak{A}p'$ are* **complementary***, if $p + p' = \mathbb{I}$.*

5. *Two C^*-algebras $\mathfrak{A}, \mathfrak{B}$ are* **stably** *isomorphic if $\mathfrak{A} \otimes \mathcal{K}$ is isomorphic to $\mathfrak{B} \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space.*

Given a full hereditary subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$, one can construct

$$\mathfrak{B}\mathfrak{A} = \{ba : b \in \mathfrak{B}, a \in \mathfrak{A}\},$$

which is a left-$\mathfrak{B}$-right-$\mathfrak{A}$-module. It is easily seen that the closed span $\overline{\mathfrak{A}\mathfrak{B}}$ extends to a Morita-equivalence bimodule between $\mathfrak{B}$ and $\mathfrak{A}$, since

$$\begin{aligned} \langle b_1 a_1, b_2 a_2 \rangle_A &:= a_1^* b_1^* b_2 a_2, \\ \langle b_1 a_1, b_2 a_2 \rangle_B &:= b_1 a_1 a_2^* b_2^* \end{aligned} \tag{A.33}$$

are dense in $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

Theorem 14 1. *Two C^*-algebras are Morita equivalent if and only if there exists a C^*-algebra such that the two Morita equivalent algebras are complementary full corners.*

2. *If two C^*-algebras are stably isomorphic, then they are Morita equivalent.*

proof:

Definition 50 *The C^*-algebra that contains two Morita equivalent C^*-algebras is called their* **linking algebra***.*

If two C^*-algebras are complementary full corners, then there exist a projections p, q in the multiplier algebra $M(\mathfrak{A})$ of the linking algebra such that the first $\mathfrak{B} = p\mathfrak{A}p$ and the second is $\mathfrak{C} = q\mathfrak{A}q$. The module

$$E = q\mathfrak{A}p := \{qap : a \in \mathfrak{A}\}$$

is easily verified to be a left-$\mathfrak{C}$-right-$\mathfrak{B}$-module, with the inner products:

$$\begin{aligned} \langle qa_1p, qa_2p \rangle_B &:= pa_1^* q q a_2 p, \\ \langle qa_1p, qa_2p \rangle_A &:= qa_1 p p a_2^* q, \end{aligned} \tag{A.34}$$

which are easily verified to satisfy the conditions for a Morita equivalence bimodule.

On the other hand, if there is a Morita equivalence bimodule E between $\mathfrak{C}$ and $\mathfrak{B}$, then one can construct the linking algebra $\mathfrak{A}$ directly as follows:

Let $\bar{E}$ be the adjoint equivalence bimodule, and consider the matrix algebra $\mathfrak{A}_o$ of the following form:

$$\mathfrak{A}_o := \left\{ \begin{pmatrix} c & e \\ \bar{f} & b \end{pmatrix} : c \in \mathfrak{C}, b \in \mathfrak{B}, e \in E, \bar{f} \in \bar{E} \right\}, \tag{A.35}$$

together with the matrix product:

$$\begin{pmatrix} c & e \\ \bar{f} & b \end{pmatrix} \begin{pmatrix} c' & e' \\ \bar{f}' & b' \end{pmatrix} := \begin{pmatrix} cc' + \langle e, f' \rangle_C & ce' + eb' \\ \bar{f}e' + b\bar{f}' & bb' + \langle f, e' \rangle_B \end{pmatrix} \tag{A.36}$$

and the involution:

$$\begin{pmatrix} c & e \\ \bar{f} & b \end{pmatrix}^* := \begin{pmatrix} c^* & f \\ \bar{e} & b^* \end{pmatrix}. \tag{A.37}$$

The Hilbert-C^*-module $E \oplus \mathfrak{B}$ carries a natural $\mathfrak{B}$-valued inner product

$$\langle (e_1, b_1), (e_2, b_2) \rangle_B := \langle e_1, e_2 \rangle_B + b_1^* b_2,$$

upon which $\mathfrak{A}$ acts faithfully by

$$\begin{pmatrix} c & e \\ \bar{f} & b \end{pmatrix} (e', b')^T := (ce' + eb', bb' + \langle f, e' \rangle_B).$$

It is easily seen that this action is bounded by verifying this for the for matrices with one nonvanishing entry separately. Using the operator norm for this representation lets us complete $\mathfrak{A}_o$ to the linking algebra $\mathfrak{A}$.

One can then verify that the two operators

$$p := \begin{pmatrix} \mathbb{I}_E & 0 \\ 0 & 0 \end{pmatrix}, \; q := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_B \end{pmatrix}$$

are projections in the multiplier algebra of $\mathfrak{A}$, such that $p\mathfrak{A}p$ and $q\mathfrak{A}q$ are corners of $\mathfrak{A}$, which turn out to be full.

The second statement follows from considering the linking algebra $\mathfrak{B} \otimes \mathcal{K}$, such that $\mathfrak{B}$ is isomorphic to $\mathfrak{B} \otimes p$ for any rank-one projection p in $\mathcal{K}$, which makes it into a full corner of the linking algebra. This makes $\mathfrak{B} \otimes \mathcal{K}$ Morita equivalent to $\mathfrak{B}$. The same argument applies to $\mathfrak{C}$, such that the stable isomrphism implies that $\mathfrak{B}$ is Morita equivalent to $\mathfrak{C}$. □

If we consider a locally compact space $\mathbb{X}$ and hence a commutative C^*-algebra $C(\mathbb{X})$ and we want to factor the free and proper action of a transformation group $\mathbb{G}$ out, then we can consider the transformation group algebra $C^*(\mathbb{X}, \mathbb{G})$ and try to find the unique commutative C^*-algebra that is Morita equivalent to $C^*(\mathbb{X}, \mathbb{G})$, which is precisely $C(\mathbb{X}/\mathbb{G})$.

A.3 Two Important Examples of Morita Equivalence of C^*-algebras

This section contains two examples of explicit Morita equivalence bimodules, that have significance for physical systems.

The first are transformation group systems. A transformation group C^*-algebra $C^*(\mathbb{X}, \mathbb{G})$ of a locally compact space $\mathbb{X}$ and a locally compact group $\mathbb{G}$ is particularly interesting, since $C(\mathbb{X})$ can serve as the algebra of configuration variables for a physical system and the contained unitary action of $\mathbb{G}$ can be interpreted as the Weyl-group consisting of an extension of the exponentiated Poisson actions of momentum vector fields.

Groupoid C^*-algebras on the other hand arise naturally as quantum algebras in the quantization of integrable Poisson systems.

A.3.1 Transformation Group algebras

Let us consider two locally compact groups $\mathbb{G}, \mathbb{H}$, which are both supposed to act freely on a locally compact Hausdorff space $\mathbb{X}$, such that the actions commute $(gx)h = g(xh)$ for all $g \in \mathbb{G}, x \in \mathbb{X}$ and $h \in \mathbb{H}$ and wandering, which means that the set $W_C := \{g \in \mathbb{G} : (x\mathbb{G}) \cap \mathbb{G} \neq \emptyset\}$ for any compact set $C \subset \mathbb{X}$ is precompact and similarly if $\mathbb{G}$ is replaced by $\mathbb{H}$. The wandering condition has the consequence that $\mathbb{X}/\mathbb{G}$ and $\mathbb{X}/\mathbb{H}$ are both locally compact and Hausdorff. The transformation group C^*-algebras based on the occurring spaces have physical interpretations: let $\mathbb{G}$ be the group of gauge transformations and $\mathbb{H}$ be a group of gauge invariant Weyl operators, then $C^*(\mathbb{X}/\mathbb{G}, \mathbb{H})$ is a gauge invariant quantum algebra. For these algebras there is a theorem by Green:

Theorem 15 *Give a locally compact Hausdorff space* $\mathbb{X}$ *and two groups* $\mathbb{G}, \mathbb{H}$ *whose actions on* $\mathbb{X}$ *are free and wandering and commute with each other, then* $C^*(\mathbb{X}/\mathbb{G}, \mathbb{H})$ *is Morita equivalent to* $C^*(\mathbb{X}/\mathbb{H}, \mathbb{G})$.

proof: We construct an explicit pre-Hilbert-C^*-bimodule between $\mathfrak{A} := C_c(\mathbb{X}/\mathbb{G}, \mathbb{H})$ and $\mathfrak{B} := C_c(\mathbb{X}/\mathbb{H}, \mathbb{G})$ and prove the conditions in lemma 36.

Consider the function space $E = C_c(\mathbb{X})$, upon which both $\mathbb{G}$ and $\mathbb{H}$ act "unitarily" by:

$$\begin{array}{ll} U_g e: & x \mapsto \Delta_G(g)^{\frac{1}{2}} e(g^{-1}x) \\ V_h e: & x \mapsto \Delta_H(h)^{-\frac{1}{2}} e(hx), \end{array} \tag{A.38}$$

where $g \in \mathbb{G}$, $h \in \mathbb{H}$, $e \in E$ and $x \in \mathbb{X}$ and Δ_G and Δ_H denote the modular function in $\mathbb{G}$ and $\mathbb{H}$ respectively. A function $f \in C(\mathbb{X}/\mathbb{G})$ defines a function on $F \in C(\mathbb{X})$ as the constant extension of f along the orbits of $\mathbb{G}$. The same applies to $f \in C(\mathbb{X}/\mathbb{H})$. Thus, there is a "covariant" representation of $\mathfrak{A}$ and of $\mathfrak{B}$ on E given by:

$$\begin{array}{ll} ae: & x \mapsto \int_H d\mu_H(h) e(hx) a(x,h) \Delta_H(h)^{-\frac{1}{2}} \\ be: & x \mapsto \int_G d\mu_G(g) b(x,g) \Delta_G(g)^{\frac{1}{2}} e(g^{-1}x), \end{array} \tag{A.39}$$

where $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $g \in \mathbb{G}$, $h \in \mathbb{H}$, $e \in E$ and $d\mu_H, d\mu_G$ denote the Haar measure in $\mathbb{H}$ and $\mathbb{G}$ respectively. The commutativity of the actions of $\mathbb{G}$ and $\mathbb{H}$ then implies that the actions of $\mathfrak{A}$ and $\mathfrak{B}$ commute, which makes E into an $\mathfrak{A}$-$\mathfrak{B}$-bimodule. The algebra-valued inner products are defined by:

$$\begin{array}{ll} \langle e_1, e_2 \rangle_A(h, [x]_G) := & \Delta_H(h)^{-\frac{1}{2}} \int_G d\mu_G(g) e_1(g^{-1}x) \overline{e_2(h^{-1}g^{-1}x)} \\ \langle e_1, e_2 \rangle_B(g, [x]_H) := & \Delta_G(g)^{-\frac{1}{2}} \int_H d\mu_H(h) \overline{e_1(h^{-1}x)} e_2(h^{-1}gx), \end{array} \tag{A.40}$$

where $g \in \mathbb{G}$, $h \in \mathbb{H}$, $x \in \mathbb{X}$, $e_1, e_2 \in E$ and $[x]_G$ and $[x]_H$ denote the $\mathbb{G}-$ resp. $\mathbb{H}-$orbit that contains x. It is clear that $\langle e_1, e_2 \rangle_A e_3 = e_1 \langle e_2, e_3 \rangle_B$. Thus one needs to check that the inner products are positive and dense in $\mathfrak{A}$ and $\mathfrak{B}$ respectively and the continuity, i.e. $\langle ae, ae\rangle_B \leq ||a||^2\langle e, e\rangle_B$ for $a \in \mathfrak{A}$, $e \in E$ and the converse for $\langle ., .\rangle_A$. Since the roles of $\mathfrak{A}$ and $\mathfrak{B}$ can be interchanged by switching the groups $\mathbb{G}$ and $\mathbb{H}$, we have to only prove the situation for $\mathfrak{A}$:

The tool for proving these properties is an approximate identity of the form $id_i = \sum_j \langle e^i_j, e^i_j\rangle_A$. This approximate identity shall be of the form $id_{C,U(e),\epsilon}$, where $U(e)$ are decreasing neighborhoods of the identity in $\mathbb{H}$, C are increasing compact subsets of $\mathbb{X}/\mathbb{G}$ and decreasing $\epsilon > 0$, such that:

$$\begin{array}{lll} id_{C,U(e),\epsilon} & = 0 & \forall h \in \mathbb{G} \setminus U(e) \\ |\int_H d\mu_H(h)\Delta(h)\imath 2 id_{C,U(e),\epsilon}(h, [x]_G) - 1| & \leq \epsilon & \forall [x]_G \in C \end{array} \tag{A.41}$$

Such an net will converge in the inductive limit topology on $\mathfrak{A}$.

A key ingredient in the construction of this approximate identity is the observation that the freeness and wandering property of the action of $\mathbb{H}$ implies that for each $x \in \mathbb{X}$ and neighborhood $U(e)$ of the identity of $\mathbb{H}$, there is a neighborhood $N_U(p)$ such that the wandering set $W_N := \{h \in \mathbb{H} : hN_U(p) \cap N_U(p) \neq \emptyset\}$ is a subset of $N_U(p)$. In other words if we shrink the neighborhood of a point x closer and closer to x itself then the set elements of $\mathbb{H}$ that transform at least one element of this neighborhood inside the neighborhood has to shrink closer and closer to the identity element of $\mathbb{H}$.

Thus, for each C and $U(e)$, we can find a finite covering of C by open sets $N_i := N_U(p_i)$. Moreover we can choose $e_i \in C^+_c(N_U(p_i))$, such that $\sum_i e_i \in C_c(C)$ is positive definite on C. By group-averaging with respect to $\mathbb{G}$, we can turn these functions into functions on $\mathbb{X}/\mathbb{G}$. Thus, we have a net that satisfies $id_{C,U(e),\epsilon} = 0 \forall h \in \mathbb{G} \setminus U(e)$.

Using the observation that of the form $x \mapsto f(x) \int_G d\mu_H(g) f(g^{-1}x)$ with $g \in C^+_c(\mathbb{X})$ are dense in $C^+_c(\mathbb{X})$, we can for every $\epsilon > 0$ approximate e_i by functions of this form. We can particularly find continuous regularizations e_i of the characteristic functions of $N_U(p_i)$ such that $\int_G d\mu_h(g) \sum_i e_i(g^{-1}x) = 1$ inside C and approximate them by functions of this form. Thus using these $f_i(x)$, we have an approximate identity $\sum_i \langle f_i, f_i\rangle_A$ that satisfies all three conditions.

The convergence of id in the inductive limit topology implies that $id\, e$ converges to e for all $e \in E$, such that

$$\begin{array}{lll} \langle ide, e\rangle_B = & \sum_i \langle\langle f_i, f_i\rangle_A e, e\rangle_B = & \sum_i \langle f_i \langle f_i, e\rangle_B, e\rangle_B \\ & = \sum_i \langle f_i, e\rangle^*_B \langle f_i, e\rangle_B & \geq 0, \end{array} \tag{A.42}$$

which implies the positivity of the inner product.
Moreover, since for all $a \in \mathfrak{A}$: $a\, id \to a$, we have

$$a\, id = \sum_i \langle af_i, f_i\rangle_A \to a,$$

which implies the density of the inner product in $\mathfrak{A}$.
The operators V_h are "unitary" for $\langle ., .\rangle_A$ as seen by direct calculation.
Moreover using the positivity of $\langle e, e\rangle_A$, we can reuse the proof of the continuity

of the representation of $C(\mathbb{X})$ on $L^2(\mathbb{X})$, since the operator $||f||^2\mathbb{I} - f^*f$ has a positive square root as a multiplication operator on $C(\mathbb{X})$, which implies that $\langle fe, fe\rangle_A \leq ||f||^2\langle e,e\rangle_A$, when $\mathbb{X}$ is replaced by $\mathbb{X}/\mathbb{G}$. Combining these two to an integrated $a = \int_H d\mu(h)a(x,h)V_h$, we obtain that for all $a \in \mathfrak{A}$ and $e \in E$: $\langle ae, ae\rangle_A \leq ||a||^2\langle e,e\rangle_A$. □

We saw the important role of the approximate identity in the above proof. Reusing exactly the same reasoning as in the proof above, we obtain the corollary:

Corollary 18 *Let E be a pre-Hilbert-C^*-bimodule between two pre-C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with two sesquilinear forms: $\langle .,.\rangle_A : E \times E \to \mathfrak{A}$ and $\langle .,.\rangle_B : E \times E \to \mathfrak{B}$, which satisfy:*

$$\langle e_1, e_2\rangle_A e_3 = e_1\langle e_2, e_3\rangle_B$$

for all $e_1, e_2, e_3 \in E$, and let there be approximate identities $id_A = \sum_i \langle e_i, e_i\rangle_A$ and $id_B = \sum_j \langle f_j, f_j\rangle_B$, where $e_i, f_j \in E$ which are both approximate identities for the C^*completions and the action of the C^*-completions on E, then:
E extends to an Morita-equivalence bimodule between the C^*-completions of $\mathfrak{A}$ and $\mathfrak{B}$.

This way of constructing an approximate identity makes this example very useful.

A.3.2 Groupoid C^*-algebras

It was proven by Muhly, Renault and Williams [18]that a C^*-algebra of two Morita equivalent groupoids with Haar system are themselves Morita equivalent. This example is actually a generalization of the previous example, which can be viewed as the special case of a transformation groupoid $\mathcal{G}(\mathbb{X}, \mathbb{G})$.

We will focus on the Muhly-Renault-Williams theorem in the context of finite dimensional Lie groupoids, which always posses Haar systems and follow[32].

Definition 51
1. *A* **Lie groupoid** *is a topological groupoid $\mathcal{G}$ such that $\mathcal{G}$ and the unit space $\mathcal{G}^{(o)}$ are manifolds and such that the range and source map are surjective submersions.*

2. *A* **left action** *of a Lie groupoid $\mathcal{G}$ on a manifold $\mathbb{X}$ is given by a smooth momentum map $\mu : \mathbb{X} \to \mathbb{G}^{(o)}$ and a smooth map $\alpha : \mathcal{G} \star \mathbb{X} := \{(g,x) \in \mathcal{G} \times \mathbb{X} : s(g) = \mu(x)\} \to \mathbb{X}$, such that $\alpha_{g_1g_2}x = \alpha_{g_1}(\alpha_{g_2}x)$, whenever $(g_1, \alpha_{g_2}x) \in \mathcal{G} \star \mathbb{X}$ and $\mu(\alpha_g x) = t(g) forall g \in \mathcal{G} \star \mathbb{X}$. A* **right action** *is defined similarly with reversed order of multiplication and toggled roles of s and t. Left and right actions are shorthanded by $\triangleright, \triangleleft$ respectively.*

3. *A manifold $\mathbb{X}$ is a $\mathcal{G} - \mathcal{H}$-***bimodule***, if it carries a left $\mathcal{G}$-action and a right $\mathcal{H}$-action, such that $(g \triangleright x) \triangleleft h = g \triangleright (x \triangleleft h)$, whenever $(g,x) \in \mathcal{G} \star \mathbb{X}$ and $(x,h) \in \mathbb{X} \star \mathcal{G}$.*

4. *A left action (μ, α) of $\mathcal{G}$ on $\mathbb{X}$ is* **principal***, if the momentum map μ is a surjective submersion and the action α is free, i.e. $g \triangleright x = x$ implies that*

$g \in \mathcal{G}^{(o)}$, and proper as a map form $\mathcal{G} \star \mathbb{X} \to \mathbb{X} \times \mathbb{X}$. Reversing the order of the components gives the definition for a principal right action.

5. *A left- and right principal $\mathcal{G}-\mathcal{H}$-bimodule $\mathbb{X}$ is a* **equivalence bimodule***, if the left momentum map reduces to an isomorphism of $\mathbb{X}/\mathcal{H} \to \mathcal{G}^{(o)}$ and if the right momentum map reduces o an isomorphism of $\mathcal{G} \setminus \mathbb{X} \to \mathcal{H}^{(o)}$.*

6. *Two Lie-groupoids are called* **Morita equivalent** *as Lie groupoids, if there is an equivalence bimodule between them.*

Let us review some facts about Lie groupoids before we state the main theorem of this section:

Theorem 16 *Let $\mathcal{G}$ be a Lie groupoid, then:*

1. *The object inclusion map e is an immersion of $\mathcal{G}^{(o)} \to \mathcal{G}$ and the inversion $g \mapsto g^{-1}$ is a diffeomorphism of $\mathcal{G}$.*

2. *$\mathcal{G} \star \mathcal{G}$ is a closed submanifold of $\mathcal{G} \times \mathcal{G}$ and the fibres $s^{-1}(u)$ and $r^{-1}(u)$ are submanifolds of $\mathcal{G}$ for each $u \in \mathcal{G}^{(o)}$.*

The Muhly-Renault-Williams theorem states that if two Morita equivalent groupoids, mediated by $\mathbb{X}$, have Haar systems, i.e. a system of measures μ^u on each fibre $s^{-1}(u) : u \in \mathcal{G}^{(o)}$ which is invariant under the groupoid operations, then the convolution C^*-algebras, i.e. the algebra functions on the groupoids with product $f_1 * f_2(g) := \int \mu(g') f_1(g') f_2((g')^{-1} \circ g)$ and involution $f^*(g) := \overline{f(g^{-1})}$, of the two groupoids are Morita equivalent as C^*-algebras. The pre-equivalence bimodule for this case is $C_c(\mathbb{X})$. Lie groupoids allow for the definition of half densities [1] and thus possess a canonical convolution algebra. This allows the restatement of the theorem as[32]:

Theorem 17 *Given two Morita equivalent Lie groupoids, then their convolution C^*-algebras are Morita equivalent.*

In order to prove this theorem, let us review the theory of half-densities on a Lie groupoid:

Definition 52

1. *Given a finite dimensional (n dimensional) vector bundle E over a manifold $\mathbb{X}$, we denote the n-fold antisymmetric tensor product without the zero section of E by $\mathcal{A}(E)$, on which $\mathbb{C} \setminus \{0\}$ acts by pointwise multiplication.*

2. *An* α**-density** *is a section in the line bundle $\wedge^\alpha(E)$ associated to $\mathcal{A}(E)$ through the representation of $\mathbb{C} \setminus \{0\}$ by: $c \mapsto |z|^\alpha$. These sections define maps $s : \mathcal{A}(E) \to \mathbb{C}$, which satisfy $s(ca) = |c|^\alpha s(a)$, for any $c \in \mathbb{C} \setminus \{0\}, a \in \mathcal{A}(E)$.*

Notice the isomorphisms $\wedge^\alpha(E) \otimes \wedge^\beta(E) = \wedge^{\alpha+\beta}(E)$ and $\wedge^\alpha(E_1 \oplus E_2) = \wedge^\alpha(E_1) \otimes \wedge^\alpha(E_2)$.

Theorem 18 *1. Special case: using $\wedge^{\frac{1}{2}}(T(\mathbb{X}))$ and consider two smooth sections s_1, s_2 of compact support therein. Then $\int_{\mathbb{X}} s_1 s_2$ is well defined and independent of a measure on $\mathbb{X}$.*

2. Let $\pi^{-1} : \mathbb{X} \to E$ be a surjectively submersing fibration and $T_\pi(\mathbb{X})$ be the subbundle of $T(\mathbb{X})$ that is tangent to the fibres π^{-1}, then for each smooth section s in $\wedge^1 T_\pi(\mathbb{X})$ of compact support in $\mathbb{X}$, $\int_{\mathbb{X}} s$ is well defined and for each pair of smooth sections sections s_1, s_2 in $\wedge^{\frac{1}{2}} T_\pi(\mathbb{X})$ with compact support in $\mathbb{X}$, $\int_{\mathbb{X}} s_1 s_2$ is well defined.

Let us denote the smooth sections of $\wedge^\alpha(T_\pi(\mathbb{X}))$ with compact support in $\mathbb{X}$ by $C_c^\infty(\mathbb{X}, \wedge^\alpha(T_\pi(\mathbb{X})))$ and for short by $\Gamma_\pi^\alpha(\mathbb{X})$.

Let us define the category of principal $\mathcal{G}$-bundles. Given a principal left $\mathcal{G}$-bundle $\mathbb{X}$ with momentum map μ and action α, we consider the pull-back under the action of $\mathcal{G}$ on sections in $\wedge^{\frac{1}{2}} T(\mathbb{X})$, which turns $\Gamma^\alpha(\mathbb{X})$ into a principal left-$\mathcal{G}$-space itself. Let us consider $C_{cG}^\infty(\mathbb{X}, \wedge^{\frac{1}{2}} T^\mu(\mathbb{X}))$ the $\mathcal{G}$-equivariant sections of compact support on the space of $\mathcal{G}$-orbits in $\mathbb{X}$ in the part of $T(\mathbb{X})$ that is tangent to the momentum map μ. It is canonically isomorphic to $C_c^\infty(\mathcal{G} \setminus \mathbb{X}, \mathcal{G} \setminus (\wedge^{\frac{1}{2}} T^\mu(\mathbb{X})))$ by identifying a section constant along a $\mathcal{G}$-orbit with the section in the bundle over the orbit space.

Given a second principal left-$\mathcal{G}$-bundle $\mathbb{Y}$ with momentum map ν, then we can construct $\mathbb{X} \star \mathbb{Y} := \{(x, y) \in \mathbb{X} \times \mathbb{Y} : \mu(x) = \nu(y)\}$, which carries a diagonal action of $\mathcal{G}$ given by $g \triangleright (x, y) := (g \triangleright x, g \triangleright y)$. Let us now define $(\mathbb{X}, \mathbb{Y})_G$ by

$$(\mathbb{X}, \mathbb{Y})_G := C_c^\infty(\mathbb{X} \star \mathbb{Y}, \wedge^{\frac{1}{2}} T^\mu(\mathbb{X}) \otimes T^\nu(\mathbb{Y})). \tag{A.43}$$

Using the isomorphism $T_{(x,y)}^{\mu(x)=\nu(y)}(\mathbb{X} \star \mathbb{Y}) = T_x^\mu(\mathbb{X}) \oplus T_y^\nu(\mathbb{Y})$, we obtain the induced isomorphism for the space of sections:

$$(\mathbb{X}, \mathbb{Y})_G = C_{cG}^\infty(\mathbb{X} \star \mathbb{Y}, \wedge^{\frac{1}{2}} T^{\mu=\nu}(\mathbb{X} \star \mathbb{Y})). \tag{A.44}$$

The spaces $(.,.)_G$ are the morphisms in the category of principal $\mathcal{G}$ bundles, and their composition is defined as follows:

Having a principal left-$\mathcal{G}$-space $\mathbb{Z}$, we can compose $(\mathbb{X}, \mathbb{Y})_G$ with $(\mathbb{Y}, \mathbb{Z})_G$ by composing the sections $s_1 \in (\mathbb{X}, \mathbb{Y})_G$ and $s_2 \in (\mathbb{Y}, \mathbb{Z})_G$ to obtain $s_1 * s_2 \in (\mathbb{X}, \mathbb{Z})_G$ as:

$$s_1 * s_2(x, z) := \int_{\nu^{-1}(\mu(x))} s_1(x, .) \otimes s_2(., z). \tag{A.45}$$

This category is actually involutive; using the toggle map $\tau : E_1 \otimes E_2 \to E_2 \otimes E_1$ by $e_1 \otimes e_2 \mapsto e_2 \otimes e_1$ we define the involution $.^*$ mapping $(\mathbb{X}, \mathbb{Y})_G$ into $(\mathbb{Y}, \mathbb{X})_G$ by:

$$s^*(x, y) := \tau(\overline{s(x, y)}). \tag{A.46}$$

The pre-C^*-algebra $C_c^\infty(\mathcal{G})$ arises as a special element of this category. First we consider $\mathcal{G}$ as a left $\mathcal{G}$-bundle with the range map providing a momentum map and the $\mathcal{G}$-action given by left translation in the groupoid. Then, using again $T_{(x,y)}^{\mu(x)=\nu(y)}(\mathbb{X} \star \mathbb{Y}) = T_x^\mu(\mathbb{X}) \oplus T_y^\nu(\mathbb{Y})$, we apply the observation $\mathcal{G} \setminus (\mathcal{G} \star \mathbb{X}) = \mathbb{X}$ by the map $[g,x]_G \mapsto g^{-1} \triangleright x$ for any principal $\mathcal{G}$-bundle $\mathbb{X}$. Taking the derivative maps this yields

$$(\mathcal{G},\mathbb{X})_G = C^\infty_{cG}(\mathcal{G}\star X, \wedge^{\frac{1}{2}}\mathcal{G}\star\mathbb{X}) = C^\infty_c(\mathbb{X}, \wedge^{\frac{1}{2}}T^{\mathcal{G}}(\mathbb{X})\otimes\wedge^{\frac{1}{2}}T^{\mu}(\mathbb{X})), \qquad \text{(A.47)}$$

where $T^{\mathcal{G}}(\mathbb{X})$ denotes the fibering induced by the derivative of the action of $\mathcal{G}$ on $\mathbb{X}$. The special case of this isomorphism is given by considering:

$$\begin{aligned}(\mathcal{G},\mathcal{G})_G = &\quad C^\infty_{cG}(\mathcal{G}\star\mathcal{G}, \wedge^{\frac{1}{2}}T^{r=r}(\mathcal{G}\star\mathcal{G}))\\ = &\quad C^\infty_c(\mathcal{G}\setminus(\mathcal{G}\star\mathcal{G}), \mathcal{G}\setminus\wedge^{\frac{1}{2}}T^{r=r}(\mathcal{G}\star\mathcal{G}))\\ = &\quad C^\infty_c(\mathcal{G}, \wedge^{\frac{1}{2}}T^{r}(\mathcal{G})\otimes\wedge^{\frac{1}{2}}T^{r}(\mathcal{G}))\\ = &\quad C^\infty_c(\mathcal{G}),\end{aligned} \qquad \text{(A.48)}$$

which turns out to be the convolution algebra of smooth half-densities with compact support on $\mathcal{G}$, which itself is dense in the C^*-algebra $C(\mathcal{G})$ if the Lie-groupoid is locally compact.

A principal $\mathcal{G}$-module $\mathbb{X}$ with surjective momentum map induces a Morita equivalent groupoid $\mathcal{H}$ in a canonical way, i.e. one can construct a groupoid $\mathcal{H}$ such that $\mathbb{X}$ is a left-$\mathcal{G}$-right-$\mathcal{H}$-module, which is principal for both actions with surjective momentum maps. This groupoid is constructed from the double space $\mathbb{X}\star\mathbb{X} := \{(x,y)\in\mathbb{X}\times\mathbb{X} : \mu(x)=\mu(y)\}$, where the left $\mathcal{G}$-orbits are factored out by the diagonal action of $\mathcal{G}$ on $\mathbb{X}\star\mathbb{X}$. The orbits $[x,y]_G$ carry a natural composition law $[x,y]_G\circ[y,z]_G := [x,z]_G$, which turns $\mathcal{H} := \mathbb{X}\star\mathbb{X}/\mathcal{G}$ into a groupoid over $\mathcal{H}^{(o)} = \mathbb{X}/\mathcal{G}$. The right action of $\mathcal{H}$ on $\mathbb{X}$ is $x\triangleright[x,y]_G := y$. In the case of Lie groupoids it turns out that $\mathcal{H}$ is again a Lie groupoid.

We can now rerun the steps that we have constructed for left-$\mathcal{G}$-bundles for right-$\mathcal{H}$-bundles, which done by only changing names and reversing the order of the action of the groupoid. Thus, we can construct:

$$\begin{aligned}(\mathbb{X},\mathcal{H})_H = &\quad C^\infty_{cH}(\mathbb{X}\star\mathbb{H}, \wedge^{\frac{1}{2}}T^{\nu=s}(\mathbb{X}\star\mathcal{H}))\\ = &\quad C^\infty_c(\mathbb{X}\wedge^{\frac{1}{2}}T^{\nu}(\mathbb{X})\otimes\wedge^{\frac{1}{2}}T^{\mathcal{H}}(\mathbb{X})),\end{aligned} \qquad \text{(A.49)}$$

for which the definition of a $\mathcal{G}$-$\mathcal{H}$-bimodule $\mathbb{X}$ imply that $T^{\mu}(\mathbb{X}) = T^{\mathcal{H}}(\mathbb{X})$ and $T^{\nu}(\mathbb{X}) = T^{\mathcal{G}}(\mathbb{X})$. The pull-back under this equivalence lets us identify:

$$(\mathcal{G},\mathbb{X})_G = (\mathbb{X},\mathcal{H})_H, \qquad \text{(A.50)}$$

which is indeed a bimodule connecting the pre-C^*-algebras $C^\infty_C(\mathcal{G})$ and $C^\infty_C(\mathcal{H})$ by defining the operator-valued inner product $\langle.,.\rangle_G$ and $\langle.,.\rangle_H$ by:

$$\begin{aligned}\langle s_1,s_2\rangle_G := &\quad s_1 * s_2^*\\ \langle s_1,s_2\rangle_H := &\quad s_1^* * s_2,\end{aligned} \qquad \text{(A.51)}$$

which are valued in $(\mathcal{G},\mathcal{G})_G$ and $(\mathcal{H},\mathcal{H})_H$ respectively and thus have a canonical interpretation as elements of $C(\mathcal{G})$, $C(\mathcal{H})$ respectively. The compatibility of elements of the two inner products and the compatibility with the action of the two C^*-algebras are easily calculated using the associativity of the convolution product. Positivity and density of the inner product are proven in a similar way as in the case of transformation group C^*-algebras. Continuity of the inner product requires a further technical step that we omit here.

Thus, we have established that $(\mathcal{G}, \mathbb{X})_G$ is indeed a pre-Morita equivalence bimodule between $C(\mathcal{G})$ amd $C(\mathcal{H})$.

Finally, we want to remark that the result in the previous subsection are obtained whenever $\mathbb{X}$ is a manifold and $\mathbb{G}$ is a Lie group acting freely and properly by diffeomorphisms on $\mathbb{X}$, so we have the associated transformation groupoid $\mathcal{G}(\mathbb{X}, \mathbb{G})$. Given $\mathbb{X} = \mathbb{Y}/\mathbb{H}$ for some free and proper action of a Lie group $\mathbb{H}$ that commutes with the action of $\mathbb{G}$, we turn $\mathbb{Y}$ into a Lie groupoid bimodule connecting $\mathcal{G}(\mathbb{Y}/\mathbb{H}, \mathbb{G})$ and $\mathcal{G}(\mathbb{Y}/\mathbb{G}, \mathbb{H})$, which using the result of this subsection implies the Morita equivalence in the previous subsection.

Appendix B

Background on Ashtekar Variables, Quantum Field Theory and Loop Quantum Gravity and Cosmology

This appendix shall serve as a gentle briefing on the ideas that underlie standard Loop Quantum Gravity and standard Loop Quantum Cosmology. Loop Quantum Gravity is formulated in terms of the Ashtekar variables for General Relativity, which are introduced in section B.1. A fruitful approach to constructing quantum field theories is to construct theories of groupoid morphisms explained in section B.2, which provides the construction principle used in this thesis. Loop Quantum Gravity viewed as a theory of morphisms form the path groupoid into the Ashtekar-Barbero gauge group is very analogous (section B.3, which however present using the standard approach used in most of the literature). We then present standard Loop Quantum Cosmology in section B.4.

B.1 General Relativity and Connection Dynamics

We gave the Hamiltonian formulation of GR in chapter 2, but the formulation of LQG rests on Ashtekar's [2] connection formulation of GR, which we review in this appendix. The derivation of the connection dynamics from the usual metric dynamics is not important for this thesis, so we present only the final result and explain its relation to the metric formalism.

We used the foliation of $\mathbb{X}^4 = \mathbb{R} \times \Sigma$ in section 2.1.1 to express the metric g on $\mathbb{X}^4$ in terms of a lapse function N, a shift vector field N^a and a spatial metric q on the Cauchy surface Σ (see equation 2.1). Let us consider a dreibein

e on Σ, such that $q_{ab} = \delta_{ij} e^i_a e^j_b$, so we can define the densitized inverse dreibein:

$$E^a_j := \frac{1}{2\iota} \epsilon^{abc} \epsilon_{jkl} e^k_b e^l_c, \tag{B.1}$$

where ι denotes the Immirzi parameter. Using the spin connection

$$\Gamma^i_a = \frac{1}{2} \epsilon^{ijk} e^b_k \left(\partial_{[b} e^j_{a]} - e^c_j e^l_a \delta_{lm} \partial_b e^m_c \right), \tag{B.2}$$

we find that the Ashtekar-gauge field which is a linear combination of the spin connection and the extrinsic curvature K

$$A^i_a = \Gamma^i_a + \iota K^i_a \tag{B.3}$$

is canonically conjugate to the desnitized inverse dreibein. The ADM action (equation 2.2) reads in these variables:

$$S = \frac{1}{\kappa} \int dt d^3\sigma \left(\dot{A}^i_a E^a_i - \left(\Lambda^i G_i + N^a V_a + NC \right) \right), \tag{B.4}$$

where Λ is a Lagrange multiplier for the $SU(2)$-gauge transformations generated by G_i. The first term is the symplectic potential implying the only non vanishing canonical Poisson bracket among the Ashtekar variables (A, E):

$$\{E^a_j(x), A^k_b(y)\} = \kappa \delta^a_b \delta^k_j \delta(x, y), \tag{B.5}$$

where κ is the coupling constant of GR. The second summand is the total Hamiltonian $H = \int d^3\sigma (\Lambda^i G_i + N^a V_a + NC)$, which is a linear combination of the three sets of constraints:

$$\begin{array}{rcl} G_j & = & D_a E^a_j \\ V_a & = & F^i_{ab} E^b_i \\ C & = & \left(F^j_{ab} + (\iota^2 + \frac{1}{4}) \epsilon_{jmn} K^m_a K^n_b \right) \frac{\epsilon_{jkl} E^a_k E^b_l}{\sqrt{|q|}}, \end{array} \tag{B.6}$$

where D denotes the covariant derivative w.r.t. the connection A, $F = dA + A \wedge A$ denotes the curvature two-form of A. The constraint G is the Gauss constraint generating ordinary $SU(2)$-gauge transformations, the constraint V is the diffeomorphism constraint generating the spatial diffeomorphisms and the constraint C is the scalar (or Hamiltonian) constraint constraining the dynamics. Notice that the extrinsic curvature K is expressed in terms of E.

The classical smearing is achieved using a densitized $su(2)$-valued vector field F^a_i and an $su(2)$-valued co-vector field f^j_b, such that the smearing

$$F(A) := \int d^3\sigma A^i_a(\sigma) F^a_i(\sigma)\, ,\; E(f) := \int d^3\sigma E^b_j(\sigma) f^j_b(\sigma) \tag{B.7}$$

can be used to give a precise definition of the Poisson bracket:

$$\{E(f), F(A)\} = F(f). \tag{B.8}$$

However, since A transforms as a connection, we see that this smearing is not gauge covariant, although it is diffeomorphism covariant. This is the reason, why quantization is based on the holonomy variables, which are one-dimensionally smeared.

B.2 Field Theories of Groupoid Morphisms

Let us recall the definition of a groupoid as a small category in which each morphism is invertible. This category theoretic definition can be made more explicit:

Definition 53 *A pair of two collections G and U respectively, together with two maps $r, s : G \to U$, a map $e : U \to G$ and a composition map $\circ : G^{(2)} := \{(g_1, g_2) \in G^2 : r(g_1) = s(g_2)\} \to G$ is called a* **groupoid**, *iff*

$$\begin{array}{rclrcl} r(g_1 \circ g_2) & = & r(g_2) & s(g_1 \circ g_2) & = & s(g_1) \\ r(e(u)) & = & u & s(e(u)) & = & u \\ g \circ e(r(g)) & = & g & e(s(g)) \circ g & = & g \end{array}$$

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$$
$$\forall g \in G : \exists\, g^{-1} s.t.\, g^{-1} \circ g = e(r(g))\,,\; g \circ g^{-1} = e(s(g))$$

Each group H is a groupoid with $G = H$ $U = \{e\}$, $r(g) = e = s(g)$, $g_1 \circ g_2 = g_1 g_2$ and $e(e) = e$. Given a groupoid $\mathcal{G}$ and a group H, one has thus a natural notion of groupoid morphisms, i.e. maps such that $A : \mathcal{G} \to H$ satisfy $A(e) = e$, $A(g_1 \circ g_2) = A(g_1)A(g_2)$ and $A(g^{-1}) = A(g)^{-1}$. We will now associate a quantum field theory for each pair $\mathcal{G}, H$.

B.2.1 Decompositions

Elements of groupoids are naturally decomposed into into sequences of groupoid elements: A **decomposition** of a groupoid element g is a finite ordered set $(g_1, ..., g_n)$ such that $g = g_1 \circ ... \circ g_n$. The set of all finite decompositions of elements of a groupoid $\mathcal{G}$ is denoted by $Dec(\mathcal{G})$. A decomposition d_1 of g is finer than a decomposition d_2 of g if for each element of $d_2 g$ there is a subset of elements in $d_1 g$ that furnish a decomposition of this element.

Definition 54 *Given a groupoid $\mathcal{G}$, we call a function $d : \mathcal{G} \to Dec(G)$ a* **decomposition function** *if $\circ dg = g$ for all $g \in \mathcal{G}$ and $dg_1 \cup dg_2$ contains all elements of $d(g_1 g_2)$.*

Given two decomposition functions d_1, d_2, we call d_1 finer than d_2 (denoted by $d_1 \geq d_2$) if $d_1 g$ is finer than $d_2 g$ for all $g \in \mathcal{G}$. Notice that $\geq$ defines a partial order on the set of decomposition functions. Moreover, the set of decomposition functions defines a semigroup through their action on $\mathcal{G}$. Given a semigroup $\mathcal{D}$ of decomposition functions, it turns out handy to consider $\mathcal{D}$-hereditary subsets of $\mathcal{G}$, which are subsets $\mathcal{H}$ of $\mathcal{G}$, such that each $\mathcal{D}$-decomposition of an $\mathcal{H}$ element consists of $\mathcal{H}$-elements only. A $\mathcal{D}$ hereditary set $\mathcal{H}$ is called $\mathcal{G}$-complete, if every element of $\mathcal{G}$ can be decomposed into elements of $\mathcal{H}$.

Let us now turn $Dec(\mathcal{G})$ into a group by considering the enlarged groupoid associated to $\mathcal{G}$, that is constructed as follows: To each element $u \in U$, we associate the set of elements $\{(t, u, t) : t \in \mathbb{R}\}$ and the enlarged unit set consists of the union of all $\{(t, u, t) : t \in \mathbb{R}\}$[1]. To each groupoid element g, we associate the set $\{(t_1, g, t_2) : t_1, t_2 \in \mathbb{R}\}$ and the enlarged groupoid set consists of the union of all these sets. The source and range maps of the enlarged groupoid are defined by $s(t_1, g, t_2) := (t_1, s(g), t_1)$, $r(t_1, g, t_2) := (t_2, r(g), t_2)$ and the object inclusion map is $e(t, u, t) := (t, e(u), t)$.This is compatible with the composition:

$$(t_1, g_1, t_2) \circ (t_2, g_2, t_3) := (t_1, g_1 \circ g_2, t_3),$$

whenever g_1, g_2 are composable. This enlarged groupoid is called the ($\mathbb{R}$-) **weighted groupoid**. Let us now define weighted decomposition functions, whose action on the weighted groupoid naturally form groups:

Given a decomposition d of a groupoid element g and a bounded function $f : U \to \mathbb{R}$, we have an associated weighted decomposition function df defined through:

$$df(t_1, g, t_2) := ((t_1 + f(s(dg_1)), dg_1, -f(r(dg_1))), ..., (f(dg_n), dg_n, t_2 - f(r(dg_n)))).$$

This definition extends to decomposition functions D in a natural way, since Dg is a decomposition of g to which we can apply the definition. This defines a **weighted decomposition function** Df associated to each pair of a decomposition function D and bounded function $f : U \to \mathbb{R}$. Let us now define an equivalence relation between weighted decompositions through:

$$\begin{aligned} & ((t_1, g_1, s_1), ..., (t_k, g_k, 0), (0, g_{k+1}, s_{k+1}), ..., (t_n, g_n, s_n)) \\ \sim \quad & ((t_1, g_1, s_1), ..., (t_k, g_k \circ g_{k+1}, s_{k+1}), ..., (t_n, g_n, s_n)), \end{aligned}$$

whenever g_k and g_{k+1} are composable. This relation implies further equivalences and moreover it implies that there is a shortest representative for each weighted decomposition. The action of weighted decomposition functions on these equivalence classes of weighted decompositions extends to a group action. We will use this group as the momentum group.

B.2.2 Configuration Space

The quantum configuration space $\mathbb{X}$ is defined as the space of all groupoid morphisms $Hom(\mathcal{G}, H)$. To define a suitable topology we need the notion of graphs and to define an action of the group of weighted decomposition functions, we need the definition of a weighted graph.

A finite set γ of groupoid elements (which are not necessarily composable) is a **graph**. An element of γ may be called an edge of γ. The set $V = \{r(g) : g \in \gamma\} \cup \{s(g) : g \in \gamma\}$ is called the vertex set of γ. This allows for the definition of weighted graphs: A weighted graph is a triple consisting of a graph γ and two functions $f_1, f_2 : V \to \mathbb{R}$, so to each edge g in γ there is a triple $(f_1(s(g)), g, f_2(r(g)))$. A vertex is called bi-valent, if there are precisely two adjacent edges. A weighted graph is called reducible, if it contains a bi-valent

[1] Instead of $\mathbb{R}$, one could take a different group, but $\mathbb{R}$ is the only case necessary to be considered here.

vertex v with adjacent weights $f_1(v) = 0 = f_2(v)$. A reducible graph is defined to be equivalent to a graph that is obtained by composing the adjacent edges and removing the vertex (this may require the inversion of one edge!). The action of weighted decomposition functions on equivalence classes of weighted graphs is obtained by applying the decomposition function to each weighted edge $(f_1(s(g)), g, f_2(r(g)))$ separately modulus graph equivalences.

Let now H be a Lie group, such that a map $\lambda : U \to \mathcal{L}(G)$ together with a weighted decomposition function D defines a transformation of groupoid morphisms through:

$$\lambda D\,A : g \mapsto \exp(f\lambda(s(g_1)))A(g_1)\exp(-f\lambda(r(g_1)))...\exp(f\lambda(s(g_n)))A(g_n)\exp(-f\lambda(r(g_n))), \tag{B.9}$$

which will be the action of a momentum variable on a groupoid morphism $A \in Hom(\mathcal{G}, H)$.

Assuming that H is compact and Hausdorff, we can apply Tychonov theory to equip $\mathbb{X}$ with a compact Hausdorff topology, even when $\mathcal{G}$ is not countable: A graph γ is called larger than γ' if each edge of γ' is contained in γ and we denote $\gamma \geq \gamma'$. The set of all graphs on a groupoid is clearly a directed set w.r.t. $\geq$.
Given a directed set S, a projective family (X_s, p_{rs}) consists of a collection of sets X_s (one for each $s \in S$) together with a collection of surjective functions $p_{rs} : X_r \to X_s$, whenever $r \geq s$, which satisfy the compatibility condition $p_{rs} \circ p_{tr} = p_{ts}$. Having a projective family, one can define the projective limit $\bar{X}$ of (X_s, p_{rs}) as the space that contains all equivalence classes that are defined through $x_r \sim x_s$ whenever $\exists x_t$ with $x_r = p_{rt}x_t$ and $x_s = p_{st}x_t$.

Since the set of graphs is a directed set, one can follow the following strategy to equip $\mathbb{X}$ with a Tychonov topology by using the graphs as a directed set. The spaces X_γ are then the morphisms form the subgroupoid of $\mathcal{G}$ generated by the elements of the graph γ to the gauge group H. The compactness and Hausdorffness of the gauge group then implies that the direct product of spaces $X_\infty = \times_{g \in \mathcal{G}} H$ is compact and Hausdorff in the Tychonov topology. This topology is characterized as the weakest topology for X_∞ for which all p_γ are continuous, where $p_\gamma A : g \in \gamma \mapsto A(g)$ with $A \in X_\infty$ and $g \in \gamma$ is the restriction of a map A to a domain given by the groupoid generated by γ. This topologizes X_∞ as a compact Hausdorff space, but X_∞ is generally much larger than $\mathbb{X} = Hom(\mathcal{G}, H)$, because not all elements of X_∞ have the groupoid morphism structure.

Let us consider $\bar{X}$, the projective limit of the X_s, where s runs over all graphs. Then the map $M : \mathbb{X} = Hom(\mathcal{G}, H) \to \bar{X}$ given by $M : A \mapsto (\gamma \mapsto A(\gamma))$ is a bijection by construction, so $\bar{X} = \mathbb{X}$. By considering a net A_n of elements of $\bar{X}$ that converges to a limit element $A \in \bar{X}$ and using the continuity of the p_{rs} in the Tychonov topology reveals that $\bar{X}$ is a closed subspace of X_∞ and hence $\mathbb{X}$ is a closed subspace of X_∞. This means that the quantum configuration space $\mathbb{X}$, given by the groupoid morphisms form $\mathcal{G}$ to the gauge group, is a compact Hausdorff space in the Tychonov topology. This compactness and Hausdorffness is necessary for the construction of a C^*-algebra of configuration variables.

Using the Gel'fand Naimark correspondence between commutative C^*-algebras and the algebra of continuous functions on a locally compact Hausdorff space or in turn the spectrum of a commutative C^*-algebra in the Gel'fand topology and a locally compact Hausdorff space, we could have defined the C^*-algebra, which will serve as the configuration algebra of the quantum theory, and then calculated its spectrum as the quantum configuration space. We will here only show how this C^*-algebra can be constructed but not prove the isomorphism of its spectrum and $\mathbb{X}$, which is an adaption of the proofs found in standard literature (e.g. [3]) to this situation.

We achieve this by constructing cylindrical functions and completing their algebra in a suitable C^*-norm. Let Γ denote the set of all graphs, then we define the space of superficial functions as

$$Supf := \cup_{\gamma\in\Gamma} C(X_\gamma).$$

This means a superficial function can be viewed as a function on $\bar{X}$ of the form $f_\gamma = f \circ p_\gamma$, where $f \in C(G^{|\gamma|})$. To factor the redundancies of these functions that have equal dependence on the elements of $\bar{X}$, we define the equivalence relation $f^1_\gamma \sim f^2_{\gamma'}$ iff $\forall \delta \geq \gamma, \gamma'$: $p^*_{\gamma\delta} f^1_\gamma = p^*_{\gamma'\delta} f^2_\delta$.

Definition 55 *A* **cylindrical function** *is a $\sim$ equivalence class of superficial functions.*

In other words: The space of cylindrical functions is $Cyl(\bar{X}) = Supf/\sim$. The algebraic operations for cylindrical functions are given through the operations of superficial functions, particularly pointwise addition, pointwise multiplication, pointwise scalar multiplication and pointwise complex conjugation. These operations are cylindrically consistent, i.e. they respect the $\sim$-equivalence classes. the unit element of this algebra is the equivalence class of functions $id : A_\gamma \mapsto 1$. A C^*-norm is given by the sup-norm:

$$||f|| := \sup_{\gamma\in\Gamma}\{||f_\gamma||_\infty\}. \tag{B.10}$$

The completion of this algebra in this norm is a commutative C^*-algebra and it turns out that its spectrum is isomorphic to $\mathbb{X}$. The cylindrical functions are by construction a dense set in this C^*-algebra, which serves as the quantum configuration algebra.

B.2.3 Quantum Observable Algebra

The algebra of configuration variables is $C(\mathbb{X})$, but we have not yet specified the action of the momentum Weyl-algebra. As in ordinary quantum mechanics, we will consider a subset of the homeomorphisms h on the quantum configuration space and define the unitary momentum Weyl-operators w_h to act adjointly as pull-backs under this homeomorphism, i.e. $w^*_h w_h = 1 = w_h w^*_h$ and $w^*_h f w_h = h^* f$ for $f \in C(\mathbb{X})$. We saw how the finite weighted decomposition functions D

where turned into a transformation of elements of $\mathbb{X}$ using a map $\lambda : U \to \mathcal{L}(H)$ (equation B.9). It turns out that the transformations λD are homeomorphisms of $\mathbb{X}$ for any finite weighted decomposition function D and any bounded λ, as is easily verified using the continuity of the group product and considering the transformation of a net A_n converging to A. We are thus able to define (a subgroup of) the finite weighted decomposition functions λD as the momentum Weyl-group $\mathcal{W}$. For any $\xi \in \mathcal{W}$ we have the canonical action on $f \in C(\mathbb{X})$ defined through:

$$w_\xi^{-1} f w_\xi = w_\xi(f) := \xi^* f. \tag{B.11}$$

This allows for the definition of an action of elements of the form $f \circ w$ on elements $g \in C(\mathbb{X})$:

$$f \circ w \triangleright g : A \mapsto f(A)(\xi^* g)(A).$$

The involution for these elements is dictated by the involution in $C(\mathbb{X})$ and the "unitarity" of the action of $\mathcal{W}$:

$$f^* \quad := \quad \overline{f} \qquad w^* \quad = \quad w^{-1}.$$

Notice that the algebra generated by the finite sums $a = \sum_{i=1}^n f_n \circ w_n$ does again contain only finite sums of this form, because $(f_1 \circ w_1)(f_2 \circ w_2) = f_1 w_1^{-1}(f_2) \circ w_1 w_2$. We thus have a closed noncommutative *-algebra $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$ of finite sums a with a norm that is constrained by

$$||f \circ 1|| \quad = \quad ||f|| \qquad ||1 \circ w|| \quad = \quad 1.$$

Let us now define the canonical representation of this algebra to explicitly complete $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$ to a C^*-algebra.

B.2.4 Canonical Representation

Given a compact group H, there is a canonical representation of $C(H^n)$ on $L^2(G^n, \otimes^n d\mu_H)$, where $d\mu_H$ is the unique normalized Haar measure. This allows for the definition of a state ω_o on $C(\mathbb{X})$ through

$$\begin{array}{rcl} f & = & f_\gamma p_\gamma \\ \omega_o(f) & := & \int \otimes^{|\gamma|} d\mu_H(g_1, ..., g_{|\gamma|}) f_\gamma(g_1, ..., g_{|\gamma|}). \end{array} \tag{B.12}$$

This definition is independent of the cylindrical representative $f_\gamma p_\gamma$ of f due to the normalization of the Haar measure. Performing the GNS-construction from this Schrödinger-type state yields a Hilbert space $\mathcal{H}$, which turns out to be $L^2(\mathbb{X}, d\mu_o)$, where $d\mu_o$ is the canonical Ashtekar-Lewandowski measure on $\mathbb{X}$ (compare e.g. [3]). This measure is characterized as the unique measure in $\mathbb{X}$, whose push-forward under any p_γ coincides with $\otimes^{|\gamma|} d\mu_H$. We thus have a C^*-representation of $C(\mathbb{X})$ on $L^2(\mathbb{X}, d\mu_o)$ as multiplication operators, which turns out to be faithful.

The state ω_o can be extended to a state ω on $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$:

$$\begin{array}{rcl} a & = & \sum_{i=1}^n f_i \circ w_i \\ \omega(a) & = & \sum_{i=1}^n \int d\mu_o(A) f_i(A). \end{array} \tag{B.13}$$

Since the Haar measure is invariant under group translations, we obtain that the Ashtekar-Lewandowski measure is invariant under the transformations λD, which is necessary for the positivity proof of ω. Performing the GNS-construction from this state yields the **canonical representation** π of $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$ on $L^2(\mathbb{X}, d\mu_o)$, which can be characterized through:

$$\begin{array}{rcl} \pi(f)\psi & = & (A \mapsto f(A)\psi(A)) \\ \pi(w)\psi & = & \xi^*\psi. \end{array} \tag{B.14}$$

The norm of $f \circ 1$ as well as $1 \circ w$ is clearly satisfied as well as the involution by this representation and we thus have a $*$-representation of $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$ on $L^2(\mathbb{X}, d\mu_o)$. The closure of this algebra in the Hilbert-space norm then defines the C^*-completion $\mathfrak{A}(\mathbb{X}, \mathcal{W})$ of quantum observables.

Let us summarize: For any groupoid $\mathcal{G}$ and compact Lie group H, we have a canonical compact Hausdorff quantum configuration space $\mathbb{X}$. For any subset S of the finite weighted decompositions, we have a momentum Weyl group $\mathcal{W}$ generated by the transformations λD, which act as homeomorphisms on $\mathbb{X}$. Thus, we have a canonical pre-C^*-algebra $\mathfrak{A}_o(\mathbb{X}, \mathcal{W})$ which is faithfully represented on $L^2(\mathbb{X}, d\mu_o)$, where $d\mu_o$ is the canonical measure on $\mathbb{X}$ induced from the Haar measure on H. Thus, the structural data for this quantum field theory consists of the triple $(\mathcal{G}, H, S)$.

B.2.5 Unitary Transformations

The momentum Weyl-transformations λD where implemented as unitary operators on $L^2(\mathbb{X}, d\mu_o)$. The deeper reason for their unitarity is twofold: (1) these transformations are homeomorphisms of $\mathbb{X}$ and (2) these transformations leave the canonical Ashtekar-Lewandowski measure invariant:

$$\begin{array}{rcl} \langle w\phi, w\psi\rangle & = & \int d\mu_o(A)\overline{\phi(\xi^{-1}(A))}\psi(\xi^{-1}(A)) \\ & = & \int d\mu_o(\xi(A))\frac{d\mu_o(A)}{d\mu_o(\xi(A))}\overline{\phi(A)}\psi(A) \\ & = & \int d\mu_o(A)\overline{\phi(A)}\psi(A) \qquad = \quad \langle \phi, \psi\rangle, \end{array} \tag{B.15}$$

where $\frac{d\mu_o(A)}{d\mu_o(\xi(A))}$ denotes the Radon derivative of $d\mu_o$ and we used the invariance of $d\mu_o$ under the momentum Weyl-transformations.

Given the structure $\mathbb{X} = Hom(\mathcal{G}, H)$, we can find two other sets of transformations that act as homeomorphisms on $\mathbb{X}$, that leave $d\mu_o$ invariant, so we can use the calculation equation B.15 to show that their pull-backs act as unitary operators on $L^2(\mathbb{X}, d\mu_o)$:

For the first set of transformations consider an automorphism ϕ of the groupoid $\mathcal{G}$, i.e. a map $\phi : (U, G) \to (U', G')$ that preserves the groupoid operations. This can be turned into a transformation ζ_ϕ on $\mathbb{X}$ by

$$\zeta_\phi A : g \mapsto A(\phi(g)).$$

This transformation is continuous, as is easily verified by considering the action on a net A_n converging to $A \in \mathbb{X}$. Moreover we see that ζ_ϕ leaves $d\mu_o$ invariant, since $(p_\gamma)_* d\mu_o = \otimes^{|\gamma|} d\mu_H = (p_{\phi(\gamma)})_* d\mu_o$. Thus extending the action of ζ^*_ϕ to $L^2(\mathbb{X}, d\mu_o)$ defines a unitary operator (due to equation B.15) as:

$$U_\phi \psi : A \mapsto (\zeta^*_\phi \psi)(A). \tag{B.16}$$

To construct the second set of transformations, consider a map $\Lambda : U \to H$. We can use this map to define a transformation ζ_Λ on $\mathbb{X}$ through:

$$\zeta_\Lambda A : g \mapsto \Lambda^{-1}(s(g)) A(g) \Lambda(r(g)).$$

Clearly $\zeta_\Lambda A(g_1 \circ g_2) = \zeta_\Lambda A(g_1) \zeta_\Lambda A(g_2)$. Moreover, using a net A_n converging to $A \in \mathbb{X}$ we see that ζ_Λ is continuous due to the continuity of the product in H. The translation invariance of the Haar measure implies that $(p_\gamma)_* d\mu_o(\zeta_\Lambda A) = \otimes^{|\gamma|} d\mu_H = (p_\gamma)_* d\mu_o(A)$, so ζ_Λ leaves $d\mu_o$ invariant. We thus have the assocaited unitary operator on $L^2(\mathbb{X}, d\mu_o)$ defined through:

$$U_\Lambda \psi : A \mapsto (\zeta^*_\Lambda \psi)(A). \tag{B.17}$$

These two sets of transformations allow the unitary implementation of diffeomorphisms and gauge transformations in Loop Quantum Gravity.

The procedure described here can be used to construct an "ordinary" background-dependent quantum field theory, by taking a suitable set of modes $\mathcal{M} = \{f_n\}_{n=1}^\infty$ and considering the single groupoid defined through $G = U = \mathcal{M}$ $r(f_n) = f_n, s(f_n) = f_n, e(f_n) = f_n, f_n^{-1} = f_n$ and $f_n \circ f_n = f_n$. Ordinary Klein-Gordon field theory can be constructed as above as the QFT of groupoid morphisms into $\mathbb{R}$, $\mathbb{C}$ respectively.

B.3 Loop Quantum Gravity

The construction of standard Loop Quantum Gravity uses Ashtekar variables and introduces the holonomies $h_e(A)$ of the connection along piecewise analytic curves e as well as the fluxes $E_f(S)$ of the conjugated electric field through piecewise analytic surfaces S as fundamental variables:

$$\begin{array}{rcl} h_e(A) & = & \mathcal{P}\{\exp(\int_e \tau^* A)\} \\ E_f(S) & = & \int_S \sigma^* f_i E^i, \end{array} \tag{B.18}$$

where τ and σ denote the embedding of e resp. S into Σ and $f : S \to \mathcal{L}(SU(2))$. The transformation properties of these observables under spatial diffeomorphisms ϕ are rather simple:

$$\phi \triangleright h_e(A) = h_{\phi(e)}(A) \text{ and } \phi \triangleright E_f(S) = E_{\phi^* f}(\phi(S)).$$

To be able to construct Loop Quantum gravity along the programme outlined in the previous section, we notice that a classical Ashtekar connection A defines a groupoid morphism from the path groupoid $\mathcal{P}(\Sigma)$ into the gauge group $SU(2)$ by assigning the parallel transport $A : e \mapsto h_e(A)$. Let us describe the path groupoid in a little more detail: The unit set is Σ and it turns out handy to work with a groupoid set that consists precisely of piecewise analytic paths modulus zero paths. A path is a piecewise analytic directed curve $c : [0,1] \to \Sigma$ modulus orientation-preserving reparametrizations. A zero path is a closed (sub)path that does not encircle any area. The object inclusion map is given by $e : x \in \Sigma \mapsto [t \mapsto x]$, the source- and range- maps are $s([c]) = c(0)$, $r([c]) = c(1)$ and the composition law is given by the concatenation of paths, while the inverse map is given by reversing the direction of the path.

Using the holonomy- and flux-variables as elementary variables, whose Poisson brackets are implemented as commutators, we obtain that the Poisson-bracket of any two holonomies vanishes. To calculate the Poisson-bracket of a holonomy $h_e(A)$ and a flux $E_f(S)$ we introduce a decomposition of e into pieces e_i, which are either completely inside S, completely outside S or have one boundary point on S, so $h_e(A) = h_{e_1}(A)h_{e_2}(A)...h_{e_n}(A)$. The Poisson-bracket of a holonomy h_{e_i} with $E_f(S)$ is then

$$\begin{aligned} \{h_{e_i}(A), E_f(S)\} &= \tfrac{\kappa(e,S)}{2}\begin{cases} h_{e_i}(A)\tau^i f_i(e(0)) & \text{if } S \cap e_i = e(0) \\ -f_i(e(1))\tau^i h_{e_i}(A) & \text{if } S \cap e_i = e(1), \end{cases} \\ \text{where } \kappa(e,S) &= \begin{cases} + & e \text{ above } S \\ 0 & e \text{ inside or outside } S \\ - & e \text{ beneath } S. \end{cases} \end{aligned} \tag{B.19}$$

The Jacobi-identity and equation B.19 imply that the Poisson-bracket between two fluxes fails to vanish. A detailed calculation [46] using three-dimensional regularizations of the fluxes reveals that the Poisson-bracket of fluxes satisfy a Lie-algebra structure, that includes one- and zero-dimensional quasi-surfaces. We see by inspecting equation B.19 that the action of the fluxes on the holonomies is precisely the derivative of a finite weighted decomposition function on the morphisms from the path groupoid to the gauge group. If we exponentiate this action, then we obtain Fleischacks Weyl-algebra for Loop Quantum Gravity and apply the construction precisely as in the previous section. For this exposition of Loop Quantum Gravity, we will however follow the standard approach, used in most of the literature:

B.3.1 Kinematics

Let us begin by introducing the elementary configuration variables called cylindrical functions. A cylindrical function Φ of the connection A is a function that can be constructed as follows. Given a smooth function $\phi : SU(2)^N \to \mathbb{C}$ and set of piecewise analytic paths $\gamma = (e_1, ..., e_N)$, we define a cylindrical function through

$$\Phi_\gamma(A) = \phi(h_{e_1}(A), ..., h_{e_N}(A)).$$

The space of cylindrical functions on γ will be called Cyl_γ. Notice that the same function $\Phi(A)$ can be constructed as a function on any graph γ' that contains a decomposition of all edges of γ. This defines an equivalence relation

$\sim$ between cylindrical functions that depend indistinguishably on A. The space of cylindrical functions is then

$$Cyl := (\cup_\gamma Cyl_\gamma)/\sim .$$

There is a natural gauge-invariant inner product on each Cyl_γ, given by the lattice inner product on γ:

$$\langle \Phi^1_\gamma, \Phi^2_\gamma \rangle := \int d\mu_H(g_1)...d\mu_H(g_N)\overline{\phi^1(g_1,...,g_N)}\phi^2(g_1,...,g_N), \tag{B.20}$$

where $d\mu_H$ denotes the unique normalized Haar measure on $SU(2)$. The normalization and translation-invariance of the Haar measure implies that this inner product respects the cylindrical equivalence classes. This allows for the definition of an inner product between any two cylindrical functions $\Phi^1_{\gamma^1}, \Phi^2_{\gamma^2}$, because there always exists a class of graphs γ^3 that contains a decomposition of all edges of γ^1 as well as for all of γ^2. Then equation B.20 can be used on γ^3, because there are functions ϕ^1, ϕ^2 with $\Phi^i(A) = \Phi^i_{\gamma^3}(A)$, so

$$\langle \Phi^1, \Phi^2 \rangle := \langle \Phi^1_{\gamma^3}, \Phi^2_{\gamma^3} \rangle \tag{B.21}$$

defines a Hermitian inner product, which is independent of the particular representative γ^3 due to the normalization and translation-invariance of the Haar measure.

Since a measure on an infinite dimensional space is conveniently defined as consistent family of cylindrical measures, let us define a measure $d\mu_o(A)$ on the space of connections by defining the push-forward $d\mu_\gamma(A)$ onto each Cyl_α to be the product measure of Haar measures on the holonomies on each edge of γ. The normalization and translation invariance of the Haar measure ensure the cylindrical consistency of the family $d\mu_\gamma$, thus defining $d\mu_o$ unambiguously. Equation B.21 then reduces to

$$\langle Cyl^1, Cyl^2 \rangle := \int d\mu_o(A)\overline{Cyl^1(A)}Cyl^2(A).$$

The Hilbert-space completion of Cyl in this inner product defines the Hilbert space $\mathcal{H} = L^2(\mathcal{A}, d\mu_o)$. The occurring quantum configuration space $\mathcal{A}$ is precisely the space of groupoid morphisms form the groupoid of piecewise analytic paths to the gauge group $SU(2)$, given the weakest topology such that all cylindrical functions are continuous.

The algebra of elementary quantum operators contains the fluxes as well. The cylindrical functions are represented on $\mathcal{H}$ as multiplication operators:

$$\pi(Cyl)\Phi : A \mapsto Cyl(A)\Phi(A),$$

whereas the fluxes are represented through the action of their respective Hamilton vector-fields:

$$\pi(E_f(S))\Phi : A \mapsto i\{E_f(S), \Phi\}(A),$$

which implements the elementary commutator:

$$[\pi(E_f(S)), \pi(Cyl)] = i\{E_f(S), Cyl\}$$

as we desired.

Defining a normal-ordering : ... : of products of these elementary operators by ordering all configuration operators to the left and all flux operators to the right, allows us to identify the vacuum state:

$$\omega(: Cyl\, E_1 ... E_n :) := \int d\mu_o(A) Cyl(A).$$

The connection representation, that we just constructed, arises as the GNS-representation of the $*$-algebra generated by the commutative algebra of cylindrical functions and i-times the Poisson action of the fluxes as self-adjoint elements.

Let us now construct a convenient Hilbert-basis for $\mathcal{H}$: Let us first recall the Peter-Weyl theorem, which states that the matrix elements $u(g)$ of the irreducible representations of a compact group G furnish a Hilbert-basis for $L^2(G, d\mu_H)$, which are normalized, when divided by the square root of the dimension of the representation. The matrix-elements of a Lie-group are represented by the eigenvalues of the Casimir operators (labeling the representation) and by the eigenvalues of a maximal commuting set of left-/right- invariant vector fields (labeling the matrix element), so for $SU(2)$, there is a nonnegative half-integer j for the representation and two labels $n, m = -j, -j+1, ..., +j$ giving basis $\phi^j_{nm} = u^j_{nm}(g)/\sqrt{2j+1}$.

A gauge-variant spin network function SNF is a special cylindrical function on a graph γ that can be written as a product of normalized nontrivial representation matrix elements:

$$SNF_\gamma(A) = \prod_{e \in \gamma} \phi^{j_e}_{m_e n_e}|_{j_e \neq 0}.$$

It follows that all gauge-variant spin network functions are orthogonal and normalized in $\mathcal{H}$ and that together with the trivial spin network function $A \mapsto 1$ they are dense in $\mathcal{H}$. For the construction of the gauge-invariant Hilbert-space it is however useful to consider a different spin network decomposition: Let us consider the Hilbert space completion $\mathcal{H}_\gamma$ of Cyl_γ and let us fix a group element $g(v) \in SU(2)$ for each vertex $v \in \gamma$ and consider the gauge transformations U_Λ:

$$U_\Lambda Cyl_\gamma : A \mapsto Cyl_\gamma(g^{-1}(i(e_1))h_{e_1}(A)g(f(e_1)), ..., g^{-1}(i(e_N))h_{e_N}(A)g(f(e_N))).$$

This operation is a unitary representation of G^M in $\mathcal{H}$ due to the translation invariance of the Haar measure, so one can write $\mathcal{H}_\gamma$ as a direct sum over irreducible representations of the edges (labeled by half-integers $\vec{j}$) and vertices (labeled by half-integers $\vec{l}$) under U_Λ, hence:

$$\mathcal{H}_\gamma = \oplus_{\vec{j},\vec{l}} \mathcal{H}_{\gamma,\vec{j},\vec{l}}. \qquad \text{(B.22)}$$

Notice that each $\mathcal{H}_{\gamma,\vec{j},\vec{l}}$ is finite-dimensional. To implement a decomposition of $\mathcal{H}$, let us introduce the auxiliary Hilbert space $\mathcal{H}'_\gamma$ as the closure of the span of all $\mathcal{H}_{\gamma,\vec{j},\vec{l}}$, where no representation is trivial. Then $\mathcal{H}$ can be decomposed into finite dimensional Hilbert-spaces:

$$\mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma, \tag{B.23}$$

where the span of the trivial spin network function $A \mapsto 1$ is the summand that corresponds to the trivial graph.

B.3.2 Kinematic Constraints

Classical general relativity (in terms of Ashtekar variables) is invariant under all fibre-bundle morphisms; this group is a semidirect product of the local gauge transformations and spatial diffeomrophisms. Let us now investigate the action of these transformations on holonomies of the Ashtekar connection along piecewise analytic curves. A gauge transfromation is labeled by a map $\Lambda : \Sigma \to SU(2)$ and its classical action on a holonomy $h_e(A)$ is:

$$\Lambda \triangleright h_e(A) = \Lambda(i(e)) h_e(A) \Lambda^{-1}(f(e)), \tag{B.24}$$

where $i(e), f(e)$ denote the initial and final point of e. This allows us to implement the action of gauge transformations on cylindrical functions as a pull-back under this transformation:

$$U_\Lambda \phi(h_{e_1}(A), ..., h_{e_N}(A)) := \phi(\Lambda(i(e_1))h_{e_1}(A)\Lambda^{-1}(f(e_1)), ..., \Lambda(i(e_N))h_{e_N}(A)\Lambda^{-1}(f(e_N))), \tag{B.25}$$

which turns out to be unitary in $\mathcal{H}$ due to the translation invariance of the Haar measure. Cylindrical consistency is provided by the trivial action on the interior of decompositions and the trivial action of a pull-back under a variable that the cylindrical function is independent of.

A spatial diffeomorphism $\phi : \Sigma \to \Sigma$ acts classically on a holonomy $h_e(A)$ of the Ashtekar connection as:

$$\phi \triangleright h_e(A) = h_{\phi(e)}(A). \tag{B.26}$$

Again, one can implement the action on a cylindrical function Cyl as a pull-back under the classical action on holonomies:

$$U_\phi \psi(h_{e_1}(A), ..., h_{e_N}(A)) = \psi(h_{\phi(e_1)}(A), ..., h_{\phi(e_N)}(A)). \tag{B.27}$$

This action is unitary due to the invariance of $d\mu_o(A)$ under diffeomorphisms, as can be checked directly

$$\begin{array}{rcl} \int d\mu_o(A) U_\phi Cyl_\gamma(A) & = & \int d\mu_o(A) Cyl_{\phi(\gamma)}(A) \\ & = & \int d\mu_H(g_1)...d\mu_H(g_N)\phi(g_1, ..., g_N) \\ & = & \int d\mu_o(A) Cyl_\gamma(A). \end{array}$$

The strategy is now to solve the kinematic constraints using the group averaging procedure. It turns however out that it is rather simple to solve the Gauss constraint directly, which is what we want to do here:

As we saw in the previous section, there is a decomposition of $\mathcal{H} = \oplus_{\gamma,\vec{j},\vec{l}}\mathcal{H}_{\gamma,\vec{j},\vec{l}}$. Since a gauge transformation acts on the vertices of a graph, we see that the gauge-invariant states are precisely those, that lie in the summands with $\vec{l} = 0$, thus the gauge invariant Hilbert space is:

$$\mathcal{H}_{Gauss} = \oplus_{\gamma,\vec{j}}\mathcal{H}_{\gamma,\vec{j},0}. \tag{B.28}$$

These states are precisely the product states of traces over holonomies of closed loops, where the spins on each edge on γ are symmetrized giving the spin j_k representation for the k-th edge. It follows that the vertices furnish gauge-invariant inter-twiners between the adjacent j_k-representations. The gauge-invariant spin network states are thus of the form:

$$T_\gamma = \left(\prod_{e\in\gamma}\rho^{j_e}(h_e(A))_{m_e,n_e}\right) M_{m_{e_1}n_{e_1},\dots,m_{e_N}n_{e_N}}, \tag{B.29}$$

where M is a direct product of gauge-invariant inter-twiners.

Let us now solve the diffeomorphism constraint with the group-averaging procedure. Using the observation that $\mathcal{H} = \oplus_\gamma\mathcal{H}_\gamma$, we can split the group averaging: Let I_γ be the subgroup of the diffeomorphisms that maps γ onto itself and let T_γ be the subgroup of the diffeomorphism group that acts trivially on γ, then the group of graph symmetries $S_\gamma := I_\gamma/T_\gamma$ is a finite group. This allows us to define the operator $\hat{P}_\gamma$ on $\mathcal{H}_\gamma$ defined as:

$$\hat{P}_\gamma Cyl_\gamma := \frac{1}{|S_\gamma|}\sum_{\phi\in S_\gamma} Cyl_{\phi(\gamma)}. \tag{B.30}$$

We are now able to define the anti-linear rigging map $\eta(Cyl_\gamma)$ through:

$$\eta(Cyl_\gamma) : \Psi \mapsto \sum_{\phi\in Diff/I_\gamma}\langle U_\phi\hat{P}_\gamma Cyl_\gamma, \Psi\rangle, \tag{B.31}$$

which is well defined and finite despite the over-countability of the diffeomorphisms in $\phi \in Diff/I_\gamma$, because the only contributing summand in this sum is the one that maps γ onto the graph that underlies Ψ. The rules for group averaging then imply that the diffeomorphism invariant Hilbert space is made up of the completion of the image of the cylindrical functions under the rigging map η, i.e. $\eta : Cyl \subset \mathcal{H} \to \mathcal{H}_{diff.}$. This Hilbert space carries the implied inner product $\langle .,.\rangle_{diff.}$:

$$\langle\eta(\phi), \eta(\psi)\rangle_{diff.} := \eta(\phi)[\psi], \tag{B.32}$$

which is Hermitian by construction and well defined since it is independent of the particular representative ϕ used to define $\eta(\phi)$. For diffeomorphism-invariant operators O we have that:

$$\eta(\phi)[O\psi] =: \langle O^*\eta(\phi), \eta(\psi)\rangle_{diff.}$$

is independent of the representative of ψ used to define $\eta(\psi)$, which yields the involution in the algebra of diffeomorphism invariant operators.

B.3.3 Dynamics

In contrast to the mathematically well defined kinematics and implementation of the kinematic constraints of Loop Quantum Gravity, the dynamics (i.e. the imposition of the set of scalar constraints) is still unsatisfactory. There is however a proposal by Thiemann, that anomaly-freely implements the set of scalar constraints, which we want to present here. This idea rests on a number of observations that we have to explain first:

First of all, one needs to take care of the square root of the determinant of E, which is non-polynomial in the flux operator. This can be taken care of by realizing that

$$e^i_a = \frac{\sqrt{\iota}}{2}\eta_{abc}\epsilon^{ijk}\frac{E^b_j E^c_k}{\sqrt{|E|}} = \frac{2}{\kappa\iota}\{A^i_a, V\}, \tag{B.33}$$

so one can express the troublesome non-polynomial expression in the momentum variables as a Poisson-bracket, which will be implemented as i-times the commutator in the quantum theory. The same trick can be used to implement the intrinsic curvature K as a Poisson-bracket expressed using $\bar{K} := \iota^{-3/2}\{C^{Eucl.}(1), V\}$:

$$K^i_a = \frac{1}{\kappa\iota}\left\{A^i_a, \bar{K}\right\}. \tag{B.34}$$

Second, one can split the scalar constraint into two summands: $C(N) = C^{Eucl.} - 2(\iota^2+1)T(N)$, where $C^{Eucl.}$ is the constraint of the Euclidean theory:

$$C^{Eucl.}(N) = \frac{2}{\kappa^2\iota}\int_\Sigma d^3x N(x)\eta^{abc}Tr\left(F_{ab}(x)\{A_c(x), V\}\right), \tag{B.35}$$

whereas the second summand can be expressed as:

$$T(N) = -\frac{2}{\kappa^4\iota^3}\int_\Sigma d^3x N(x) Tr\left(\{A_a(x),\bar{K}\}\{A_b(x),\bar{K}\}\{A_c(x),V\}\right). \tag{B.36}$$

Third, for any small line segment s of coordinate length ϵ and for the boundary curve β of any coordinate square of area ϵ^2, we have the approximations:

$$\begin{array}{rcl} \{\int_s A, V\} & = & -(h_s(A))^{-1}\{h_s(A), V\} + O(\epsilon) \\ \{\int_s A, \bar{K}\} & = & -(h_s(A))^{-1}\{h_s(A), \bar{K}\} + O(\epsilon) \\ \int_q F & = & \frac{1}{2}\left(h_{\beta^{-1}}(A) - h_\beta(A)\right) + O(\epsilon^2), \end{array} \tag{B.37}$$

which allow us to regularize the expressions B.35, B.36 in terms of the regulator ϵ using a partition P_ϵ of Σ into coordinate cells of coordinate size approximately ϵ^3. [2] This allows us to use a Riemann sum approximation for the expressions

[2]The simplest such partition is a partition of $\mathbb{R}^3$ into regular coordinate cubes of size ϵ^3, but any partition that will for a general open region R of coordinate volume $V_c(R)$ yield that the number of cells in R is approximately $V_c(R)/\epsilon^3$ as $\epsilon \to 0$ is admissible.

B.35, B.36, so in the limit $\epsilon \to 0$ one has the regulated expressions:

$$\begin{array}{rcl} C^{Eucl.}_\epsilon(N) & = & \sum_{c\in P_\epsilon} N(c) C^{Eucl.}_c \\ C^{Eucl.}_c & = & \frac{1}{\kappa^2\iota}\sum_{iJ} C^{iJ} Tr\left((\rho(h_{\beta_i}) - \rho(h_{\beta_i^{-1}})\rho(h_{s_J^{-1}}))\{\rho(h_{s_j}, V)\}\right) \\ T_\epsilon(N) & = & \sum_{c\in P_\epsilon} N(c) T_c \\ T_c & = & \frac{1}{\kappa^2\iota^3}\sum_{IJK} T^{IJK} \\ & & Tr\left(\rho(h_{s_I^{-1}})\{\rho(h_{s_I}), \bar{K}\}\rho(h_{s_j^{-1}})\{\rho(h_{s_j}), \bar{K}\}\rho(h_{s_K^{-1}})\{\rho(h_{s_K}), V\}\right) \\ \bar{K} & = & \iota^{-3/2}\{C^{Eucl.}(1), V\}, \end{array} \tag{B.38}$$

where s_K and β_i are coordinate line segments into the K-direction and loops around coordinates squares in $j \times k$-direction and C^{iJ} and T^{IJK} are constants depending on the representation ρ used, yield suitably regularized expressions. The freedom in the choice of the regulator is then used to find a family of partitions P_ϵ together with line segments s_K and loops β_i such that (1) the family of scalar constraints is cylindrically consistent (2) it transforms covariantly under diffeomorphisms and (3) it leaves the domain of cylindrical functions invariant. Then replacing $i\hbar\{.,.\}$ with the respective commutator gives a densely defined set of scalar constraints, which define the Hermitian set of constraints. We will not quote a particular family of partitions, but rather focus on the last two key observations that are necessary for Thiemanns construction:

Fourth, due to the decomposition of the Hilbert space of Loop Quantum Gravity into $\mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma$, we can adapt the construction of the regulator to the graph γ and define one regulator P^γ_ϵ for each graph. Cylindrical consistency gives restrictions on these regulators that can be resolved easily. Diffeomorphism covariance on the other hand poses the restriction that if γ and γ' are diffeomorphic then their regulators have to be diffeomorphic. Amazingly, one is able to construct such families of regulators e.g. [44]. The resulting regulated scalar constraints generally act only on the vertices of γ for $t(\gamma) > \epsilon > 0$ for some $t(\gamma)$ and add line-segments s_k and loops β_i that contain only one vertex of γ.

Fifth, one has to remove the regulator. This is possible on the diffeomorphism invariant Hilbert space, due to the observation that

$$\eta(Cyl^1_{\gamma_1})[\hat{C}_\epsilon(N) Cyl^2_{\gamma_2}] = \langle C^*_\epsilon(N)\eta(Cyl^1_{\gamma_1}), \eta(Cyl^2_{\gamma_2})\rangle_{diff}, \tag{B.39}$$

so as soon as the resulting graph of $\hat{C}(N)$ acting on $Cyl^2_{\gamma_2}$ does not change its diffeomorphism class as ϵ, one has already obtained the limit $\epsilon \to 0$. Meaning that the regulator ϵ can be removed trivially on the diffeomorphism invariant Hilbert space, which is the domain for Thiemanns set of scalar constraints.

B.4 Loop Quantum Cosmology

This section serves to introduce standard Loop Quantum Cosmology.

B.4.1 Classical Symmetry Reduction

We already introduced the classical symmetry reduction used to construct Bianchi cosmologies in chapter 2.1.2. Let us now consider Bianchi I cosmology in terms of Ashtekar variables:

This means we assume a base-manifold $\Sigma = \mathbb{R}^3$, fix a global chart (U, ϕ), and a symmetry group $G = \mathbb{R}^3$ acting transitively on Σ, generated by the three vector fields ∂_i (we will always use the global chart (U, ϕ) for all vector fields). To further reduce the symmetry, let us assume local rotational symmetry, generated by $\epsilon_{3jk} x_j \partial_k$ as well as complete rotational symmetry generated by all three $\epsilon_{ijk} x_j \partial_k$.

Since Loop Quantum Gravity is constructed as a theory of connections, we have to classify G-symmetric connections. Following [16] we decompose a G-symmetric connection on a fibre bundle over Σ into a connection on a reduced bundle over Σ/G plus a G-multiplet of scalars on Σ/G. We first use their classification of G-symmetric principal fibre bundles and then consider connections thereon:

Connections are by definition invariant under vertical bundle morphisms, i.e. gauge transformations. Moreover, let G act on a principal fibre bundle $P(\Sigma, H, \pi)$ as a group of bundle morphisms, such that all G orbits are isomorphic. Σ/G is reductive and I is the isotropy group of a point, such that $\Sigma = (\Sigma/G) \times (G/I)$, then Σ can be considered an orbit bundle over Σ/G. Notice that each point in $p \in P$ defines a morphism $\rho : I \to H$ by:

$$\rho_p : i \mapsto \alpha_i(p),$$

where α commutes with the right action in the fibre, such that

$$\rho_{ph} = Ad_{h^{-1}} \rho_p.$$

If we now fix one particular ρ, we can construct a symmetric subbundle

$$P_{sym}(\Sigma/H, C_H(\rho(I)), \pi|_{sym})$$

over Σ/H with the reduced structure group given by the centralizer of the image of I under ρ, which is isomorphic to any other subbundle constructed using a ρ in the same conjugacy class. This means that the G-symmetric fibre bundles are completely classified by the reduced bundle P_{sym} and the conjugacy class $[\rho]_{conj}$ of the map ρ.

Having the symmetric bundles classified by the data $(P_{sym}, [\rho]_{conj})$, we need to consider a symmetric connection ω on P, which by restriction defines a connection ω_{sym} on P_{sym} but it also defines a linear map $L_p : \mathcal{L}(G) \to \mathcal{L}(H)$ through:

$$L_p : V \mapsto \omega_p(V).$$

The image of the orthogonal complement of $\mathcal{L}(I)$, i.e. $\mathcal{L}(I_\perp)$, under the map L is horizontal in the full bundle, so it contains information about the connection, but not tangential to any direction in Σ/G. Since a connection is invariant under vertical bundle morphisms, we obtain that not any linear map L can arise, but only those that satisfy:

$$L_p(Ad_i V) = Ad_{\rho(i)}(L_p(V)),$$

for all $i \in I$, which we can take as the definition of the transformation law of the "Higgs-field" L under gauge transformations. Using the invariance of the Maurer-Cartan form on G as well as the embedding $e : G/I \to G$, we can decompose any symmetric connection into the part parallel to Σ/G, i.e. the symmetric connection on P_{sym}, and the part parallel to the G-orbits in Σ as:

$$\omega = \omega_{sym} + L \circ i^* \theta_{MC}, \tag{B.40}$$

where θ_{MC} denotes the Maurer-Cartan form on G and ω_{sym} is the connection on P_{sym}.

Following [13], we can apply this classification to Binachi I cosmologies in terms of the Ashtekar connection. The assumed base manifold $\Sigma = \mathbb{R}^3$, and the symmetry group $G = \mathbb{R}^3$ acting on Σ as translations. Since the left-invariant 1-forms on G are dx^i, we can express $\theta_{MC} = g_i dx^i$. The Isotropy group of a point is trivial, so there is only the identity embedding $i : \mathbb{R}^3 \to \mathbb{R}^3$. The orbit space Σ/G consists of one orbit x_o only, so the the reduced fibre bundle is a fibre bundle over the trivial space.The linear map becomes a matrix over this point, meaning that we can express the symmetric connection as:

$$\omega = L \circ \theta_{MC} = A^I_a \tau_I dx^a,$$

where τ^I denotes the I-th generator of the gauge group $SU(2)$ and A is a constant matrix. The transformation constraint (of the Higgs field under gauge transformations or of the linear map under vertical bundle morphisms) is trivial, so it is satisfied by all matrices A. Moreover, using the dual basis X_i to $\omega^i = dx^i$, we can give the symmetric Ashtekar variables (A, E) for Bianchi I cosmologies:

$$A = A^I_a \tau_I dx^a \qquad E = E^a_I \tau^I X_a. \tag{B.41}$$

Let us now impose the local rotational constraint around the 3-axis: The isotropy-group of a point is $U(1)$, so $\rho_n : U(1) \to SU(2)$ can be chosen to be $\rho_n(e^{aY}) = e^{a\tau_3}$, where Y denotes the generator of $U(1)$. The only integer n for which the Higgs constraint is satisfied is $n = 1$ and the matrices A turn out to be of the form:

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{B.42}$$

where a, b, c are real parameters.

Imposing complete rotational symmetry, we obtain the isotropy group of a point to be $SU(2)$, which can be embedded into the gauge group by the identity embedding. The solutions to the Higgs constraint are of the form:

$$A = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{B.43}$$

for a real parameter c.

Let us form now on assume complete rotational invariance, so we can construct a chart and a gauge such that the Ashtekar varibales take the form:

$$A(x) = \tilde{c}\tau_I \delta^I_a dx^a \qquad E = \tilde{p}\tau^I \delta^a_I X_a.$$

To be able to convert the gravitational Poisson bracket into a finite Poisson bracket for these remaining degrees of freedom, we consider these in a cell of volume V_o and use the densities: $c := V_o^{\frac{1}{3}}\tilde{c}$ and $p := 8\pi G\iota V_o^{\frac{2}{3}}\tilde{p}$, so the gravitational Poisson bracket induces the Poisson bracket between p and c. The classical Hamilton constraint (with constant laps) takes in these variables the form

$$-\frac{6}{\iota^2}c^2 sgn(p)\sqrt{|p|} + C_{matt} = 0.$$

Notice that the physical triad and cotriad have to be scaled with V_o.

B.4.2 Kinematics

Standard Loop Quantum Cosmology uses the holonomies along straight lines as the fundamental configuration variables, which are in the global chart (U, ϕ) of the form:

$$e = \{e^a(0) + l\dot{e}^a t : 0 \leq t \leq 1\},$$

and the holonomies of the symmetric connection along these straight lines can be calculated directly:

$$h_e = \cos(\frac{lc}{2}) + 2\dot{e}^a \delta^I_a \tau_I \sin(\frac{lc}{2}). \tag{B.44}$$

The algebra of configuration variables is generated by the matrix elements of these holonomies, so it consists of the span of the exponentials $e^{\frac{i}{2}lc}$, and a cylindrical function is thus:

$$Cyl(A) = \sum_k \xi_k \exp(\frac{i}{2}l_k c). \tag{B.45}$$

The smallest C^*-algebra that contains all these cylindrical functions is the algebra of almost periodic functions on $\mathbb{R}$, and its spectrum is the Bohr-compactification $\bar{\mathbb{R}}_B$ of $\mathbb{R}$. Let us now consider p as the flux through a unit square and calculate its Poisson bracket with cylindrical functions by calculating the Poisson action of fluxes through a unit square on the cylindrical functions:

$$\{Cyl(A), p\} = \frac{8\pi\iota G}{6}\sum_k \frac{i}{2}l_k \xi_k \exp(\frac{i}{2}l_k c), \tag{B.46}$$

which we now want to represent on a Hilbert space. Since each $e^{\frac{i}{2}lc}$ can be obtained as a matrix element of a holonomy over along edge of parameter length l, we can induce a scalar product for these functions by $\langle e^{\frac{i}{2}l_1 c}, e^{\frac{i}{2}l_2 c}\rangle :=$

$\langle (h_{e(l_1)})_{11}, (h_{e(l_2)})_{11}\rangle = \delta^{Kron.}_{l1,l_2}$, which is precisely the scalar product of the Hilbert space $L^2(\bar{\mathbb{R}}_B, d\mu_H)$, which has the representation:

$$\langle e^{\frac{i}{2}l_1 c}, e^{\frac{i}{2}l_2 c}\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dc\overline{e^{\frac{i}{2}l_1 c}}e^{\frac{i}{2}l_2 c} = \delta^{Kron.}_{l_1,l_2}. \tag{B.47}$$

This nonseparable Hilbert space has the Hilbert basis $\langle c|\mu\rangle := e^{\frac{i}{2}\mu c}$ and the cylindrical functions are represented thereon as multiplication operators. The fluxes through unit squares on the other hand are represented by $-i\hbar$ times the Poisson action B.46:

$$\begin{array}{rcl} \langle c, \pi(Cyl)\mu\rangle & = & Cyl(A)e^{\frac{i}{2}\mu c} \\ \langle c, \pi(p)\mu\rangle & = & \frac{8\pi\mu\iota l^2_{Pl.}}{6}e^{\frac{i}{2}\mu c}, \end{array} \tag{B.48}$$

where $l_{Pl.}$ denotes the Planck length. This shows that the states $|\mu\rangle$ are eigenstates of the triad operator, meaning that they measure the physical size of the cell. Homogeneity and isotropy lets us find a simple operator for the cell volume:

$$V|\mu\rangle = \left(\frac{8\pi\iota|\mu|}{6}\right)^{\frac{3}{2}} l^3_{Pl.}|\mu\rangle =: V_\mu|\mu\rangle. \tag{B.49}$$

For the purpose of conatructing a dynamics analogous to Thiemanns dynamics in the full theory, one needs inverse powers of the flux operators, which we do using the Poisson bracket $\{c, V^{\frac{1}{3}}\} = sgn(p)|p|^{-\frac{1}{2}}$. Replacing the $i\hbar$ times the Poisson bracket with the respective commutator yields:

$$\frac{sgn(p)}{\sqrt{|p|}}|\mu\rangle = \frac{6}{8\pi\iota l^2_{Pl.}}\left(V^{\frac{1}{3}}_{\mu+1} - V^{\frac{1}{3}}_{\mu-1}\right)|\mu\rangle. \tag{B.50}$$

B.4.3 Dynamics

The classical Hamilton constraint can be constructed for constant lapse, which is precisely the constraint that is implemented in standard Loop Quantum Cosmology. Using the observation that the full scalar constraint reduces to $C = -\frac{V_o}{\sqrt{8\pi G\iota^2}}F^{ij}_{ab}\frac{E^{ai}E^{bj}}{\sqrt{det(E)}}$, we proceed as in the Thiemann quantization of the scalar constraint and express the curvature part as the holonomy around a closed square loop $\Box(l_o)$ of side length l_o, i.e. $F^i_{ab}\tau_i \sim \frac{1}{l_o^2 V_o^{\frac{2}{3}}}\left(h_{\Box(l_o)} - 1\right)$, where $h_{\Box(l_o)} = h_{e_i(l_o)}h_{e_j(l_o)}h^{-1}_{e_i(l_o)}h^{-1}_{e_j(l_o)}$ with the edges $e_i(l_o)$ starting at a fixed point and extending for background length l_o into direction i. Using the construction of equation B.50, we obtain the regularized classical expression for the scalar constraint as:

$$C(l_o) = -\frac{4}{8\pi G\iota^3 l_o^3}\sum_{ijk} Tr\left(h_{e_i(l_o)}h_{e_j(l_o)}h^{-1}_{e_i(l_o)}h^{-1}_{e_j(l_o)}h_{e_k(l_o)}\left\{h^{-1}_{e_k(l_o)}, V\right\}\right). \tag{B.51}$$

Replacing $i\hbar$times the Poisson bracket with the commutator and the classical quantities with the respective elementary operators yields the constraint operator:

$$C_{l_o}|\mu\rangle = \frac{3}{8\pi G l_o^3 \iota^3 l_{Pl.}^2} (V_{\mu+l_o} - V_{\mu-l_o}) \left(|\mu + 4l_o\rangle - 2|\mu\rangle + |\mu + 4l_o\rangle\right). \quad \text{(B.52)}$$

This is the pure gravitational constraint, a matter cosmological model will have additional degrees of freedom (let us assume that $|\nu\rangle$ form a dense set in the matter Hilbert space), so a state in a matter model will have the form $|\psi\rangle = \sum_k a(\mu_k, \nu_k)|\mu_k\rangle \otimes \nu_k\rangle$. Denoting the matter part of the scalar constraint by $C_{matt.}$ yields the full constraint:

$$\begin{aligned} &(V_{\mu+5l_o} - V_{\mu+3l_o})a(\mu + 4l_o, \nu) - 2(V_{\mu+l_o} - V_{\mu-l_o})a(\mu, \nu) + \\ &(V_{\mu-3l_o} - V_{\mu-5l_o})a(\mu - 4l_o, \nu) \;=\; -\tfrac{8\pi G \iota^3 l_o^3 l_{Pl.}^2}{3} \hat{C}_{matt.}(\mu) a(\mu, \nu) \end{aligned} \quad \text{(B.53)}$$

Physical states have to satisfy the full constraint, so $a(\mu, \nu)$ has to satisfy this equation. Notice, that the regulator can not be removed from standard Loop Quantum Cosmology, due to the fixing of the diffeomorphisms at the classical level.

Appendix C

Loop Quantum Geometry

This appendix serves as an overview over Loop Quantum Geometry. Technical reasons suggest the use of the stratified analytic category, which is described in section C.1. Thereafter we review the standard theory of Loop Quantum Geometry (sections C.2,C.3 and C.4). This appendix introduces the "standard" version of Loop Quantum Geometry andS serves as the background for the construction of the altered version of Loop Quantum Geometry constructed in chapter 8.

C.1 Stratified Analytic Diffeomorphisms

This thesis uses a subgroup of the stratified analytic homeomorphisms as an extension of the diffeomorphism group of Loop Quantum Gravity. We are following Fleischhack [23], who himself was following Hardt [25]. The reason for considering this class of mappings is the desire to conserve the nice properties of analytic curves, surfaces and maps while still being able to work locally, i.e. not having a global structure dictated by local properties of a curve, surface or mapping.

Definition 56 *Let $\mathbb{X}$ be a differential manifold with differentiability class p= (n, ∞ or ω) and let U be a subset of $\mathbb{X}$.*
Then $\mathcal{M}$ is called a **stratification**, *iff it is a locally finite, disjoint decomposition of $\mathbb{X}$ into connected embedded C^p manifolds $\mathbb{X}_i \subset \mathbb{X}$, such that*

$$\mathbb{X}_i \cap \partial X_j \neq \emptyset \to \mathbb{X}_i \subseteq \partial X_j \text{ and } dim(\mathbb{X}_i) < dim(\mathbb{X}_j).$$

The elements $\mathbb{X}_i$ of the decomposition are called **strata**.
$\mathcal{M}$ is called a stratification of U, iff U is the union of some elements of $\mathcal{M}$.

Having a notion of stratification, we can define stratified analytic curves as finite 1-dimensional submanifolds $c \subset \mathbb{X}$ that are can be composed from a finite number of elements of an analytic stratification $\mathcal{M}$ of $\mathbb{X}$. A stratified analytic surface is a 2-dimensional submanifold $S \subset \mathbb{X}$ that can be composed from a finite number of elements of an analytic stratification $\mathcal{M}$ of $\mathbb{X}$.

Moreover, we use the the notion of a stratification to define stratified maps and particularly stratified C^p-diffeomorphisms:

Definition 57 *Let f be a continuous map from a C^p-manifold $\mathbb{X}$ to a C^p-manifold $\mathbb{Y}$ (p=n, ∞ or ω). Then f is called*

- *a* **stratified map***, iff there is a pair of stratifications $\mathcal{M}, \mathcal{N}$ of $\mathbb{X}, \mathbb{Y}$ respectively and for each stratum X_i there exists an open neighborhood U_i and a C^p-map $f_i : \mathbb{X}_i \subseteq U_i \to \mathbb{X}$ satisfying $\overline{\mathbb{X}}_i \subseteq U_i$, $f_i|_{\mathbb{X}_i} = f|_{\mathbb{X}_i}$, $f_i(\mathbb{X}_i) \in \mathcal{N}$ and $rank(f|_{X_i}) = dim(f(\mathbb{X}_i))$.*

- *a* **stratified diffeomorphism** *iff $f|_{\mathbb{X}_i}$ is injective and the restriction of each f_i to the respective U_i is a C^p-diffeomorphism.*

C.2 Area Operators

We will follow [39] to construct an area operator for each closed 2-dimensional submanifold surface that is embedded by $\tau : S \to \Sigma$. Let us re-express the classical area functional $A(S) = \int_S d^2x \sqrt{|\tau^* h|}$ in terms of the fundamental flux operators $P^i_S(E) = E^f(S)$ where $f = \tau^i = const.$:

$$A(S) := 8\pi\iota G \lim_{\leftarrow \mathcal{P}(S)} \sum_{S_k \in \mathcal{P}(S)} \sqrt{\eta_{ij} P^i_{S_k}(E) P^j_{S_k}(E)}, \tag{C.1}$$

where $\mathcal{P}(S)$ denotes a projective family of partitions of S and the projective limit is taken w.r.t. the partial ordering of partitions given by $\mathcal{P}_1 \geq \mathcal{P}_2$ iff each element of P_2 is a composition of elements of elements of $\mathcal{P}_1$. Where we have to use a family of partitions that satisfies a density condition, i.e. for each open subset $U \subset S$ and each $N \in \mathbb{N}$ there is $\mathcal{P}^U_N$ in this family, such that the number of cells inside U is greater or equal to N.

The strategy is to define this operator on the set of spin network functions as an essentially self-adjoint operator and to use its unique Hermitian extension as the respective area operator. Let us adapt the family of partitions $\mathcal{P}(S)$ to the graph γ of a spin network function SNF_γ as follows:
There exists a partition $\mathcal{P}^\gamma_o$ such that each transversal puncture of γ through S is in a separate cell. Then for each $n + 1 \in \mathbb{N}$ let there be a partition $\mathcal{P}^\gamma_{n+1}$ such that each cell of $\mathcal{P}^\gamma_n$ is the composition of 3×3 cells of $\mathcal{P}^\gamma_{n+1}$ with the constraint that if the cell $\mathcal{P}^\gamma_n$ contains a transversal puncture of γ, then this puncture lies in the central cell of the 3×3 decomposition of this cell. Since the area functional does not depend on the particular family if partitions, we can define one for each graph γ:

$$A(S) := 8\pi\iota G \lim_{n\to\infty} \sum_{S_k \in \mathcal{P}^\gamma_n(S)} \sqrt{\eta_{ij} P^i_{S_k}(E) P^j_{S_k}(E)}. \tag{C.2}$$

Replacing the classical flux variables with the quantum flux operators yields an area operator for each spin network function SNF_γ. Notice that each cell S_k contains at most one transversal intersection point with the graph γ. Let us consider the equivalent graph γ' that contains a vertex at each intersection point, so the operator $\eta_{ij}\hat{P}^i_{S_k}\hat{P}^j_{S_k}$ acts on $SNF_{\gamma'}$ precisely the same way as the vertex Laplace operator $\Delta_{v,S,\gamma'} = -\eta^{ij}(\hat{J}^{v,S,up}_i - \hat{J}^{v,S,down}_i)(\hat{J}^{v,S,up}_j - \hat{J}^{v,S,down}_j)$, where v denotes the vertex at the transversal intersection point; the J-operators act only as left-invariant vector fields on the components of the spin network function, and $\eta_{ij}\hat{P}^i_{S_k}\hat{P}^j_{S_k}$ acts trivially on $SNF_{\gamma'}$, if S_k does not contain a transversal intersection point of γ'. This means that each element $\sum_{S_k \in \mathcal{P}^\gamma_n(S)} \sqrt{\eta_{ij}\hat{P}^i_{S_k}\hat{P}^j_{S_k}}$ acts on $SNF_{\gamma'}$ in precisely the same way, meaning that the limit is attained already for $n = 0$. The operator is cylindrically consistent, as can be verified by using different representatives of the same cylindrical function.

We are thus able to define the action of the area operator on a spin network function SNF_γ as:

$$\hat{A}(S)\, SNF_\gamma := 4\pi\iota l^2_{Pl} \sum_{v \in S\cap\gamma \text{transv.}} \sqrt{-\Delta_{v,S,\gamma}} SNF_\gamma. \tag{C.3}$$

Using the spectral root, we can apply this operator to twice differentiable cylindrical functions immediately. Density of the essential domain and essential self-adjointness defines the unique Hermitian extension of the area operator as an unbounded Hermitian operator on $L^2(\mathbb{X}, \mu_{AL})$. This operator extends to quasi-surfaces as well: Given an open 2-dimensional surface S and divide it into two open 2-dimensional surfaces S_1, S_2 and one open 1-dimensional quasi-surface S_3 then $A(S_3) := A(S) - A(S_1) - A(S_2)$ is a nonzero operator. The same procedure applied to 1-dimensional quasi-surfaces defines the area operator for zero-dimensional quasi-surfaces.

Working with spin network functions, we can calculate the eigenvalues of the area operator: Writing $\Delta_{v,S,\gamma} = (J^{S,v,up} + J^{S,v,down})^2 - 2(J^{S,v,up})^2 - 2(J^{S,v,down})^2$, we obtain that the eigenvalues of the area operator are given by:

$$\lambda = 4\pi\iota l^2_{Pl} \sum \sqrt{2j^{up}(j^{up}+1) + 2j^{down}(j^{down}+1) - j^{up+down}(j^{up+down}+1)},$$

so the smallest nonzero value is $2\pi l^2_{Pl}\sqrt{3}$.

C.3 Volume Operators

The quantization strategy for the volume operator of an open region R is precisely the same as for the area operator of a surface: We use the classical expression for the volume functional of a region, reexpress it as a limit of cell volumes of a family of partitions of the region, which we then reexpress using the classical flux variables. Then we replace the classical flux variables with the corresponding flux operators and take he limit on spin network functions.

There is however one additional caveat: We have to ensure that the partitioning is defined background independently. Since it is much simpler to define the partition using a chart, we will have to average over a background-independent set of families of partitions to obtain a background independent result. The final operator is then the Hermitian extension of the essentially self-adjoint operator defined on spin network functions.

We follow [40] and consider the classical volume functional $V(R)$ of an open region R:

$$V(R) = \int_R \det q = (\sqrt{8\pi\iota G})^3 \int_R d^3x \sqrt{|\det P|}.$$

We assume without loss of generality that R is contained in one chart, if this is not the case the we use a partition of unity to write $R = \cup_i R_i$, where each R_i is contained in a single chart. We will again adapt the partition $\mathcal{P}(R)$ to the graph γ by starting with a cubical partition[1] $P_o^\gamma(R)$, such that each cell contains at most one vertex of γ which is supposed to be located at the coordinate center of mass of the cell, and each cell that does not contain a vertex contains at most one edge of γ. For each $n+1 \in \mathbb{N}$ we define P_{n+1}^γ as a partition that contains a $3 \times 3 \times 3$-decomposition of each cell of P_n^γ, such that a cell that contains a vertex contains it at its coordinate center of mass[2]. For each cubical cell define three surfaces S_1, S_2, S_3 as the surfaces that go through the coordinate center of mass of the cell which are subsets of the $x^i = const.$-surfaces for the ith chart coordinate.

Using the average cell volume density for the cell $c \in P_n(R)$:

$$q_c = \frac{(8\pi\iota G)^3}{3!} \epsilon^{ijk} \eta_{abc} P^i_{S_a}(E) P^j_{S_b}(E) P^k_{S_c}(E)$$

and the expression for the approximate volume of R in terms of these:

$$V_n(R) = \sum_{c \in P_n(R)} \sqrt{|q_c|}$$

we are able to identify the classical volume functional with the limit:

$$V(R) = \lim_{n\to\infty} V_n(R), \tag{C.4}$$

which we are able to quantize immediately by replacing the classical flux variables with the respective flux operators $\hat{q}_c = \frac{(8\pi\iota G)^3}{3!} \epsilon^{ijk} \eta_{abc} \hat{P}^i_{S_a} \hat{P}^j_{S_b} \hat{P}^k_{S_c}$. Hence the definition of the volume operator associated to a family $\mathcal{F}$ of partitions $P_n(R)$ is

$$\hat{V}_{\mathcal{F}}(R) \lim_{n\to\infty} \sum_{c \in P_n(R)} \sqrt{|\hat{q}_c|}. \tag{C.5}$$

[1]Each cell in the partition is assumed to be a coordinate rectangle and two cells share at most one face. We assume that the boundaries of the rectangles are subsets of $x^i = 0$ for the ith chart coordinate.

[2]We assume that the refinement of the cells is achieved by cutting the cell into subcells at coordinate thirds in each direction.

It turns out that this volume operator depends on the chart used to define the family $\mathcal{F}$ of partitions. Let us remove this dependence by averaging over a diffeomorphism invariant set of families of partitions. The final result is:

$$\hat{q}_{c,\gamma} = \frac{(8\pi\iota l_{Pl})^3}{48} \sum_{\text{all triples of edges}} \epsilon_{ijk}\kappa(e_1, e_2, e_3)\hat{J}_i^{v,e_1}\hat{J}_j^{v,e_2}\hat{J}_k^{v,e_3}, \tag{C.6}$$

where $\kappa = +1$ if the ordered triple of tangent vectors at the vertex is right handed -1 if left handed and 0 if degenerate.

A different quantization strategy yields a volume operator [41, 42] based on the cell volume density

$$\hat{q}_{c,\gamma}^{RS} = \frac{(8\pi\iota l_{Pl})^3}{48} \sum_{\text{all triples of edges}} \epsilon_{ijk}\hat{J}_i^{v,e_1}\hat{J}_j^{v,e_2}\hat{J}_k^{v,e_3}.$$

C.4 Length Operators

A length operator [43] was constructed by Thiemann using the observation that $\frac{1}{\kappa}\{A_a^i, V\} = \frac{sgn|e|}{2}e_a^i$. Using the volume operator Thiemann quantized the length functional $L(c) = \int_c dx\sqrt{\dot{c}^a(x)\dot{c}^b(x)q_{ab}(c(x))}$ in terms of commutators of holonomies with the total volume operator. We present a slight modification of [43] in this section, that will prove useful in chapter 8.

Thiemanns ansatz for the classical length functional $L(c)$ of a piecewise analytic curve c requires a family $\mathcal{F}$ of partitions P of the curve c with the density property: for open subset c_i of c and each $n \in \mathbb{N}$ there is a partition P_n^i in $\mathcal{F}$ such that the open set contains at least n pieces of the partition P_n^i. Using such a family one can rewrite the classical length functional as:

$$L_{\mathcal{F}} = \lim_{n\to\infty} \frac{1}{\kappa} \sum_{c_i \in P_n} \sqrt{2Tr(\{h_{c_i}, V\}\{h_{c_i}^{-1}, V\})}, \tag{C.7}$$

where h_{c_i} denotes the holomony along the curve $c_i \in P$. Thiemann quantizes this length functional by replacing the holomomies with holonomy operators, the volume functionals with the volume operator and $i\hbar$ times the Poisson bracket with the commutator of the respective quantities, yielding the length operator:

$$\hat{L}_{\mathcal{F}} = \lim_{n\to\infty} 2l_{Pl} \sum_{c_i \in P_n} \sqrt{-2Tr([\hat{h}_{c_i}, \hat{V}][\hat{h}_{c_i}^{-1}, \hat{V}])}. \tag{C.8}$$

To be able to take the limit in the quantum theory, we have to adapt the family of partitions again to the graph γ of the spin network function SNF_γ that $\hat{L}(c)$ acts upon. Using piecewise analyticity of both c and γ, we know that for each graph γ there is an appropriate partition P^γ, such that there are only five topological relations that a segment c_i may have with γ and that each c_i is analytic:

1. $\gamma \cap c_i = \emptyset$

2. $\gamma \cap c_i$ is an interior point of an edge of γ

3. $\gamma \cap c_i$ is contained in the interior of an edge of γ

4. $\gamma \cap c_i$ is one vertex of γ

5. $\gamma \cap c_i = c_i$ and contains both one vertex and the interior of an edge of γ

Using the observation that the volume operator acts nontrivially only at vertices and that a vertex of $\gamma \cup c_i$ is either a vertex of γ or a boundary point $p \in \partial c_i$, we can rewrite the commutator

$$[\hat{h}_{c_i}^{-1}, \hat{V}]Cyl_\gamma = \left(\sum_{v \in (V(\gamma) \cup \partial c_i)} [\hat{h}_{c_i}^{-1}, \hat{V}_v] - \sum_{v \in (V(\gamma \cap c_i) \setminus V(\gamma))} \hat{V}_v h_{c_i}^{-1} \right) Cyl_\gamma,$$

where $V(\gamma)$ denotes the set of vertices of γ and V_v denotes the volume operator acting only at vertex v and the germ of the edges adjacent to this vertex.
Case 1: The first term vanishes because there is no vertex of γ that coincides with a boundary point of c_i, the second term vanishes because $\gamma \cap c_i = \emptyset$.
Case 2: The first term vanishes because there is no vertex of γ that coincides with a boundary point of c_i, the second term contains a trivalent vertex, so the second term vanishes by gauge invariance.
Case 3: The first term vanishes because there is no vertex of γ that coincides with a boundary point of c_i, the second term vanishes because the vertices in $V(\gamma \cap c_i) = \partial c_i$ are bi-valent.
Case 4: Generally, the first term does not vanish because there is one vertex of γ that coincides with a boundary point of c_i, the second term vanishes because $V(\gamma \cap c_i) \setminus V(\gamma) = \emptyset$.
Case 5: Generally, the first term does not vanish because there is one vertex of γ that coincides with a boundary point of c_i, generally the second term does not vanish because $V(\gamma \cap c_i) \setminus V(\gamma) \neq \emptyset$.
Thus, we can use a partition $P_o^\gamma = P^\gamma$ and use a refinement procedure $P_n^\gamma \to P_{n+1}^\gamma$ that splits pieces of case 1 into pieces of case 1, pieces of case 2 into pieces of case 2, pieces of case 3 into pieces of case 1 and one endpiece of case 3, pieces of case 4 into pieces of case 1 and one endpiece of case 4 and pieces of case 5 into pieces of case 2 and one endpiece of case 5. This defines the adapted family of partitions $\mathcal{F}^\gamma = \{P_n^\gamma\}_{n=0}^\infty$. Observing for cases 3,4 and 5 that $[\hat{h}_{c_i}^{-1}, \hat{V}]Cyl_\gamma$ depends only on the germ of c_i at the endpoint and that cases 1 and 2 do not contribute, we conclude that the limit $n \to \infty$ of $L_{\mathcal{F}^\gamma}(c)$ is already attained with the first partition P_o^γ. Using this observation and the observation that the volume operator acts only nontrivially on $V(\gamma) \cup (\cup_i \partial c_i)$, we are able to write the action of the length operator on an spin network function as the operator:

$$\hat{L}(c) := 2 l_{Pl} \sum_{v \in V(gamma)} \sum_{c_i \in P_o : v \in \partial c_i} \sqrt{-2Tr([\hat{h}_{c_i}, \hat{V}][\hat{h}_{c_i}^{-1}, \hat{V}])} \tag{C.9}$$

Cylindrical consistency of the definition of this operator is easily checked by verifying that the action of the length operator on different representatives for the same cylindrical function is independent of the representative.

Observing that the length-segment-squared operator $l^2_{c_i} := -8l_{Pl}[h_{c_i}, V_v][h^{-1}_{c_i}, V_v] = 8l_{Pl}\sum_{AB} K^\dagger_{AB}K_{AB}$, because $[(h_{c_i})_{AB}, V_v]^\dagger = -[(h^{-1}_{c_i})^T_{AB}, V_v]$ by the unitarity of the $SU(2)$-holonomy matrix $(h_s)_{AB}$for any curve s. This implies that $l^2_{c_i}$ is positive semi-definite on the domain SNF_γ. To show that $l^2_{c_i}$ is self-adjoint on gauge-invariant spin network functions, we observe that the only graph-changing cases are 1, 2, 4 and 5. However $[\hat{h}^{-1}_{c_i}, \hat{V}]Cyl_\gamma$ always vanishes for case 1 and vanishes by gauge invariance for case 2, so gauge-invariance ensures for the length operator to be non-graph changing for gauge-invariant states in these cases. The resolution of cases 4 and 5 needs the gauge invariance of the length operator, which maps gauge-invariant states to gauge invariant states; which makes the argument for the non-graph changing action of $l^2_{c_i}$ more complicated[3].

Since cylindrical functions with nontrivial dependence on two different graphs are always orthogonal, we conclude that $l^2_{c_i}$ is a positive semi-definite symmetric operator on the gauge-invariant spin network functions, implying after using the spectral root for the definition of the length operator, that the length operator is a positive semi-definite real symmetric operator on the gauge-invariant spin network states. Density of the gauge-invariant spin network states in the gauge-invariant Hilbert space yields the existence of a Hermitian length operator thereon.

[3]Let us quote Thiemanns proof that if a gauge-invariant cylindrical function Cyl_γ depends nontrivially on every edge in a graph γ then $l^2_{c_i}\,Cyl_\gamma$ depends at most on γ: $l^2_{c_i}\,Cyl_\gamma$ is gauge invariant, since $l^2_{c_i}$ is gauge invariant and depends at most on $\gamma \cup c_i$. Decomposing $l^2_{c_i}\,Cyl_\gamma$ into gauge-invariant spin network functions yields:
Case 4: There is no gauge-invariant spin network function that depends nontrivially on the extra edge c_i.
Case 5: c_i coincides with the initial segment of an edge, say $e \in \gamma$. $\gamma \cup c_i$ does not contain e, but c_i and $e \backslash c_i$ and the additional bivalent vertex v between these two edges. The only gauge-invariant two-valent intertwiner is the trivial one, so the only gauge-invariant spin network functions that depend nontrivially on $c_i, e \setminus c_i$ depend on e only

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Curriculum Vitae

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List of Publications

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4. T. Koslowski: ”A Cosmological Sector in Loop Quantum Gravity”, [arXiv:0711.1098]

Erklärung

Gemäß Paragraph 5 Absatz 2 der Promotionsordnung vom 22.09.2003 der Fakultät für Physik und Astronomie der Universität Würzburg erkläre ich hiermit an Eides statt, da ich diese Dissertation selbständig und ohne Hilfe eines Promotionsberaters angefertigt und keine anderen als die im Literaturverzeichnis angegebenen Quellen und die folgenden Hilfsmittel benutzt habe: (1) das Computeralgebraprogramm Mathematica zur Formelmanipulation an den angezeigten Stellen, (2) das Textsatzsystem LaTeXzum setzen der Dissertation sowie (3) verschiedene Computergraphikprogramme zum Erstellen der Bider.

Diese Dissertation wurde von Prof. Thorsten Ohl betreut und bisher weder in gleicher, noch in anderer Form, in einem anderen Prüfungsfach oder an einer anderen Hochschule mit dem Ziel, einen akademischen Grad zu erwerben, vorgelegt.

Würzburg, den 20. Mai 2008

Tim Koslowski