Received: 15 August 2020 | Accepted: 3 November 2020

DOI: 10.1002/pamm.202000342

Ensemble reachability of homogenous parameter-depedent systems

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In this paper we consider the class $(\theta A, B)$ of parameter-dependent linear systems given by matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. This class is of interest for several applications and the frequently met task for such systems is to steer the origin toward a given target family $f(\theta)$ by using an input that is independent from the parameter. This paper provides a collection of necessary and sufficient conditions for ensemble reachability for these systems.

1 Introduction

The focus of this paper is on parameter-dependent linear systems which are homogenous, i.e.

$$\tfrac{\partial}{\partial t} x(t,\theta) = \theta A x(t,\theta) + Bu(t), \qquad x(0,\theta) = 0 \in \mathbb{C}^n \tag{1}$$

where θ is a parameter varying over a compact and contractible set $\mathbf{P} \subset \mathbb{C}$ with empty interior and matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. Driven by applications, e.g. quantumm systems, the class of homogenous parameter-dependent systems has reached attraction in the literature, cf. [1–3] and the references therein. The essential issue is to control the entire family of systems by a parameter-independent control input which is broadcast to all members of the family and the significance is the input function u is not allowed to depend on the parameter. This assumption makes the problem infinite-dimensional, as the states are defined by functions of the parameter, whereas the input space is finite-dimensional. The question we are addressing is to steer the origin towards a given family $\{f(\theta) \,|\, \theta \in \mathbf{P}\}$ of terminal states. As pointed out in [4] exact reachability is never possible in this setting. Therefore, we consider the following notion of approximate reachability.

Definition 1.1 We call (1) (or the pair $(\theta A, B)$) uniformly ensemble reachable if for every $\varepsilon > 0$ and $f \in C(\mathbf{P}, \mathbb{C}^n)$ there are $T > 0$ and an input $u \in L^1([0,T], \mathbb{R}^m)$ such that the solution $\varphi(\cdot,\theta,u)$ to (1) satisfies $\sup_{\theta\in\mathbf{P}} \|\varphi(T,\theta,u) - f(\theta)\| < \varepsilon$.

Note that, although we used (1) to introduce the class of systems, the results are also valid for discrete-time systems. As there is also research treating general matricies $A(\theta)$ and $B(\theta)$, the goal of this paper is to provide an exposition of conditions on A and B that are necessary and sufficient for uniform ensemble reachability.

2 Ensemble reachability for homogenous parameter-dependent systems

We start the presentation by recalling known results and afterwards we contribute new conditions for uniform ensemble reachability of this class of systems. The recap is carried out in the subsequent propostion, cf. [1, 2].

Proposition 2.1 *Let $\mathbf{P}$ be compact, contractible with empty interior.*

(a) *If $\theta A, B)$ is uniformly ensemble reachable, then (A, B) is controllable, A is invertible and if $0 \in \mathbf{P}$ it follows that* $\operatorname{rank} B = n$.

(b) *If $0 \in \mathbf{P}$, then $(\theta A, B)$ is uniformly ensemble reachable if and only if* $\operatorname{rank} A = \operatorname{rank} B = n$.

(c) *Then, $(\theta A, B)$ is uniformly ensemble reachable if (A, B) is controllable, A is invertible and diagonalizable such that* $\{\theta\lambda_k \,|\, \theta \in \mathbf{P}\} \cap \{\theta\lambda_l \,|\, \theta \in \mathbf{P}\} = \emptyset$ *for all $k \neq l \in \{1, ..., r\}$.*

In particular, if the parameter set contains the origin, any single-input pair $(\theta A, b)$ is never uniformly ensemble reachable (unless $n = 1$). By [5, Proposition 3] it follows that a scalar pair $\theta a, b)$ is uniformly ensemble reachable if and only if $a \neq 0$ and $b \neq 0$, regardless of whether $0 \in \mathbf{P}$ or not. Based on the latter statement we assume from now on that A is invertible. Moreover, let $\lambda_1, ..., \lambda_r$ denote the distinct non-zero eigenvalues of A. Hence, there is an invertible matrix T such that

$$T^{-1}AT = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} \qquad T^{-1}B = \begin{pmatrix} B_1 \\ \vdots \\ B_r \end{pmatrix},$$

where J_k is the Jordan matrix consisting of all Jordan blocks to the eigenvalue λ_k. Let n_k denote the size of the Jordan matrices J_k, $k = 1, ..., r$. Using this notation we provide the following new necessary condition.

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Theorem 2.2 *Let* $\mathbf{P}$ *be compact, contractible with empty interior and* $0 \notin \mathbf{P}$*. Suppose the pair* $(\theta A, B)$ *is uniformly ensemble reachable. Then,*

(a) $\operatorname{rank} B \geq \max_{k=1,\dots,r} n_k$.

(b) $\operatorname{rank} B_k = n_k$.

(c) *if* $r = 2$ *it holds* $\{\theta\lambda_1 \mid \theta \in \mathbf{P}\} \cap \{\theta\lambda_2 \mid \theta \in \mathbf{P}\} = \emptyset$.

In other words, the rank of the input matrix is necessarily greater or equal to the size of the largest Jordan matrix of A. Before we provide the proof, we have to recap some notation. The continuous mappings $\Gamma_1, \dots, \Gamma_r$ given by $\Gamma_k \colon \theta \mapsto \theta\lambda_k$ have the following properties:

$$\Gamma_k(\theta) \neq \Gamma_l(\theta) \quad \forall\, k \neq l \quad \text{and} \quad \bigcup_{i=1}^{r} \Gamma_i(\theta) = \sigma(\theta A),$$

where $\sigma(\theta A)$ denotes the spectrum of θA. In the sense of [5], the mappings $\Gamma_1, \dots, \Gamma_r$ are called a pointwise disjoint eigenvalue selection. Also it is easy to see that the pair $(\theta A, B)$ is uniformly ensemble reachable if and only if $(\theta T^{-1}AT, T^{-1}B)$ is uniformly ensemble reachable.

Proof. To see (a), let $n_r = \max_{k=1,\dots,r} n_k$. Then, there is change of coordinates S such that

$$S^{-1}AS = \begin{pmatrix} M & N \\ 0 & J_r \end{pmatrix} \qquad S^{-1}B = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_2 \\ \tilde{B}_3 & \tilde{B}_4 \end{pmatrix}$$

for some matrices M, N and $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ and $\tilde{B}_4$. In particular, the pair $(\theta J_r, \begin{pmatrix} \tilde{B}_3 & \tilde{B}_4 \end{pmatrix})$ is uniformly ensemble reachable and from [5, Proposition 6] we conclude that $n_r = \operatorname{rank} \begin{pmatrix} \tilde{B}_3 & \tilde{B}_4 \end{pmatrix} \leq \operatorname{rank} B$.

To see (b), note that the mappings $\Gamma_1, \dots, \Gamma_r$ define a pointwise disjoint continuous eigenvalue selection. Then, the application of Theorem 2 in [5] yields that each pair $(\theta J_k, B_k)$ is uniformly ensemble reachable. The assertion then follows from [5, Proposition 6].

To see (c), we note that due to space constraints we examplarily treat the pair

$$\left(\theta \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & 0 \end{pmatrix} \right), \quad \alpha \neq 0.$$

If uniform ensemble reachability holds, it follows that single-input system

$$\tfrac{\partial}{\partial t} z(t,\theta) = \begin{pmatrix} \theta\lambda_1 & \theta \\ 0 & \theta\lambda_2 \end{pmatrix} z(t,\theta) + \begin{pmatrix} 1 \\ \alpha \end{pmatrix} u(t)$$

is uniformly ensemble reachable. Thus, by [5, Theorem 3 (c)], one has $\{\theta_1\lambda_1, \theta_1\lambda_2\} \cap \{\theta_2\lambda_1, \theta_2\lambda_2\} = \emptyset$ for all $\theta_1 \neq \theta_2$. This shows assertion (c) and the proof is complete. □

We close this paper with the following sufficient conditions.

Theorem 2.3 *Let* $\mathbf{P}$ *be compact, contractible with empty interior and* $0 \notin \mathbf{P}$ *and* $\operatorname{rank} B = m < n$*. Then,* $(\theta A, B)$ *is uniformly ensemble reachable if* $\operatorname{rank} B_k = n_k$ *and*

$$\{\theta\lambda_k \mid \theta \in \mathbf{P}\} \cap \{\theta\lambda_l \mid \theta \in \mathbf{P}\} = \emptyset \quad \forall\, k \neq l. \tag{2}$$

Proof. Since the eigenvalues $\lambda_1, \dots, \lambda_r$ are non-zero, the eigenvalue functions $\lambda_k(\theta) := \theta\lambda_k$ are injective. Since $\operatorname{rank} B_k = n_k$, we can apply Proposition 5 in [5] and conclude that every pair $\theta J_k, B_k)$ is uniformly ensemble reachable. By condition (2), toghether with [5, Theorem 1] it follows that $(\theta T^{-1}AT, T^{-1}B)$ is uniformly ensemble reachable. This shows the assertion. □

We conjecture that the condition (2) is also necessary for the case of more than two distinct eigenvalues.

Acknowledgements This research was funded by the German Research Foundation (DFG) under grant SCHO 1780/1-1.
Open access funding enabled and organized by Projekt DEAL.

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