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A sharp Bernstein–type inequality and application to the Carleson embedding theorem with matrix weights

Daniela Kraus[1] · **Annika Moucha**[1] · **Oliver Roth**[1]

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Abstract

We prove a sharp Bernstein-type inequality for complex polynomials which are positive and satisfy a polynomial growth condition on the positive real axis. This leads to an improved upper estimate in the recent work of Culiuc and Treil (Int. Math. Res. Not. 2019: 3301–3312, 2019) on the weighted martingale Carleson embedding theorem with matrix weights. In the scalar case this new upper bound is optimal.

Keywords Carleson embedding theorem · Bernstein-type inequality

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1 Result

Lemma 1.1 *Let n be a positive integer and $p : \mathbb{C} \to \mathbb{C}$ a polynomial such that $p(s) \geq 0$ for all $s \geq 0$ and*

$$|p(s)| \leq s^{-1}(1+s)^n \qquad \text{for all } s > 0\,. \tag{1.1}$$

Then

$$|p(0)| \leq n^2\,, \tag{1.2}$$

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✉ Oliver Roth
roth@mathematik.uni-wuerzburg.de

Daniela Kraus
dakraus@mathematik.uni-wuerzburg.de

Annika Moucha
annika.moucha@mathematik.uni-wuerzburg.de

1 Department of Mathematics, University of Würzburg, Emil Fischer Strasse 40, 97074 Würzburg, Germany

with equality if

$$p(s) = p_n(s) := \frac{1}{2}\frac{(s+1)^n}{s}\left(1 - T_n\left(\frac{1-s}{1+s}\right)\right). \tag{1.3}$$

Here, $T_n(x) = \cos(n \arccos x)$ *is the* n*-th Chebyshev polynomial of the first kind.*

The source of motivation for Lemma 1.1 has been the recent work of Culiuc and Treil [1] on the Carleson embedding theorem with matrix weights. In fact, Lemma 2.2 in [1], which they attribute to F. Nazarov and M. Sodin, provides the (weaker) estimate

$$|p(0)| \le e^2 n^2 \tag{1.4}$$

for any polynomial $p : \mathbb{C} \to \mathbb{C}$ satisfying (1.1). Developing a sophisticated Bellman function technique and making use of estimate (1.4), Culiuc and Treil [1] proved the following result ([1, Theorem 1.2]). We refer to [1] for the relevant terminology and notation.

Theorem A (Carleson embedding theorem for matrix weights) *Let* W *be a* $d \times d$ *matrix–valued measure and let* A_I, $I \in \mathcal{D}$ *be a sequence of positive semidefinite* $d \times d$ *matrices. Then the following are equivalent:*

(i) $\sum_{I \in \mathcal{D}} \left|\left| A_I^{1/2} \langle W^{1/2} f \rangle_I \right|\right|^2 |I| \le A ||f||_{L^2}^2$.

(ii) $\sum_{I \in \mathcal{D}} \left|\left| A_I^{1/2} \langle W f \rangle_I \right|\right|^2 |I| \le A ||f||_{L^2}^2$.

(iii) $\frac{1}{|I_0|} \sum_{I \in \mathcal{D}, I \subset I_0} \langle W \rangle_I A_I \langle W \rangle_I |I| \le B \langle W \rangle_{I_0}$ for all $I_0 \in \mathcal{D}$.

Moreover, the best constants A and B satisfy $B \le A \le CB$, where $C = C(d) = 4e^2 d^2$.

In fact, the proof of Theorem A in [1] requires the estimate (1.4) only for polynomials $p : \mathbb{C} \to \mathbb{C}$ with degree $n = 2d$, which satisfy (1.1) and are *real and positive on the positive real axis*. Therefore Lemma 1.1 implies that one can take

$$C(d) = 4d^2$$

instead of $C(d) = 4e^2 d^2$ in Theorem A. In the scalar case ($d = 1$) this new upper bound produces the upper estimate $A \le 4B$, which is known to be optimal [4, Theorem 3.3].

Remark 1 The method we use for the proof of Lemma 1.1 can also be used to improve the bound (1.4) given by [1, Lemma 2.2], which holds for any polynomial $p : \mathbb{C} \to \mathbb{C}$ satisfying (1.1). This leads to

$$|p(0)| \le 2n^2 - n, \tag{1.5}$$

see the next section for the proof. The estimate (1.5) is presumably not best possible.

2 Proofs

The idea is to view both estimates, (1.2) and (1.5), as Bernstein–type estimates. Recall that for a polynomial h of degree N the classical Bernstein inequality says that

$$\max_{|z|=1} |h'(z)| \leq N \cdot \max_{|z|=1} |h(z)| .$$

Proof of Lemma 1.1 By assumption, $p : \mathbb{C} \to \mathbb{C}$ is a polynomial satisfying (1.1) and $p(s) \geq 0$ for all $s \geq 0$. Then $q(z) := zp(z)$ is polynomial of degree at most n with $q(0) = 0$, $p(0) = q'(0)$, and $q(s) \geq 0$ for all $s \geq 0$. We define the auxiliary function

$$f(z) := \frac{(1+z)^{2n}}{(4z)^n} q\left(-\left(\frac{1-z}{1+z}\right)^2\right) = \sum_{k=-n}^{n} a_k z^k ,$$

a Laurent polynomial of degree $\leq n$. It is not difficult to see that the growth condition (1.1) for p implies the uniform bound

$$|f(z)| \leq 1 \quad \text{for all } |z| = 1 .$$

We also note that

$$p(0) = q'(0) = -2f''(1) ,$$

so our task is to find the best upper bound for $|f''(1)|$.

In order to find such an estimate, it turns out to be essential that the auxiliary function f is *real and positive (i.e.,* ≥ 0) on $|z| = 1$. To see this just note that

$$k(z) = \frac{z}{(1+z)^2} = \frac{1}{4}\left(1 - \left(\frac{1-z}{1+z}\right)^2\right)$$

is the Koebe function, familiar from the classical theory of univalent functions, which maps the unit circle $|z| = 1$ onto the half–line $[1/4, +\infty)$. Hence, on $|z| = 1$, $f(z)$ is the product of two real and positive functions.

We are thus in a position to apply the Fejér–Riesz theorem [2] for the Laurent polynomial f. This gives us a complex polynomial P of degree $\leq n$ with no zeros in $|z| < 1$ such that

$$f(z) = P(z)\overline{P(1/\overline{z})} , \qquad z \in \mathbb{C} \setminus \{0\} .$$

Clearly, $|P(z)| \leq 1$ for all $|z| = 1$. We can therefore apply a sharpening of Bernstein's inequality due to P. Lax [3] (confirming an earlier conjecture of Erdös) which asserts that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \cdot \max_{|z|=1} |P(z)| \leq \frac{n}{2} .$$

In particular,

$$|p(0)| = |q'(0)| = 2|f''(1)| = 4|P'(1)|^2 \leq n^2 \,,$$

proving (1.2). Clearly, the polynomial $P_n(z) = (z^n - 1)/2$ has the property $|P_n'(1)| = n/2$, so $|f_n''(1)| = n^2/2$ for $f_n(z) := P_n(z)\overline{P_n(1/\overline{z})}$. It is easy to see that

$$f_n(z) = \frac{(1+z)^{2n}}{(4z)^n} q_n \left(- \left(\frac{1-z}{1+z} \right)^2 \right)$$

for a polynomial q_n of degree at most n with $q_n(0) = 0$, and it is straightforward to check that $p_n(z) := q_n(z)/z$ has the form (1.3). □

Proof of (1.5) By assumption, $p : \mathbb{C} \to \mathbb{C}$ is a polynomial satisfying (1.1). Then $q(z) := zp(z)$ is polynomial of degree at most n with $q(0) = 0$ and $p(0) = q'(0)$. We define, closely following the proof of [1, Lemma 2.2], the auxiliary function

$$g(z) := \frac{(1+z)^{2n}}{4^n} q \left(- \left(\frac{1-z}{1+z} \right)^2 \right) ,$$

a polynomial of degree $N \leq 2n$. As before, the polynomial g has the property that

$$|g(z)| \leq 1 \quad \text{for all } |z| = 1 \,.$$

Now note that

$$p(0) = -2g''(1) \,.$$

Hence, we could apply the classical Bernstein inequality twice, first for g' and then for g'', but this would result in

$$|p(0)| = 2|g''(1)| \leq 2N(N-1) \leq 4n(2n-1) \,,$$

which is not particularly good. However, as observed in [1, Proof of Lemma 2.2] we can assume without loss of generality that g has no zeros in $|z| < 1$. We can therefore apply as above the inequality of Lax which leads to

$$\max_{|z|=1} |g'(z)| \leq \frac{N}{2} \cdot \max_{|z|=1} |g(z)| \leq n \,.$$

This brings us in a position to apply Corollary 14.2.8 in [5] for the polynomial g' which has degree $\leq 2n - 1$. Hence

$$|g''(z)| + |(2n-1)g'(z) - zg''(z)| \leq n(2n-1) \,, \qquad |z| \leq 1 \,.$$

Taking $z = 1$ and noting that $g'(1) = nq(0) = 0$, gives $2|g''(1)| \leq n(2n-1)$, as required. □

3 Remarks

The polynomials p which occur in the proof of Theorem A in [1] are of the form

$$p(s) = \sum_{I \in \mathcal{D}} p_I(s)|I|\,,$$

with $p_I(s) \geq 0$ for all $s \geq 0$ and each p_I a polynomial of degree at most $2(d-1)$. The extremal polynomial p_{2d} in Lemma 1.1 has degree $2(d-1)$ and *all its* $2(d-1)$ *zeros are on the positive real axis and are double zeros*. This implies that

$$p(s) = p_{2d}(s) \quad \Longleftrightarrow \quad \forall_{I \in \mathcal{D}} \exists_{c(I) \geq 0}\, p_I|I| = c(I)p_{2d}\,.$$

Hence the extremal polynomial p_{2d} of Lemma 1.1 shows up in the proof of Theorem A only if each p_I is a multiple of p_{2d}.

After acceptance of the paper the authors found another short proof of Lemma 1.1 based on Markov's inequality [5, Theorem 15.1.4] which allows to identify *all* extremal polynomials. In fact, using the change of variables $s = (1-x)/(1+x)$ we have

$$q(x) := 1 - 2^{1-n}(1+x)^{n-1}(1-x)p\left(\frac{1-x}{1+x}\right) = 1 - 2\frac{sp(s)}{(1+s)^n}, \quad x \in (-1, 1).$$

By assumptions, q is a polynomial of degree at most n such that $q(1) = 1$, $q'(1) = p(0)$ and $|q(x)| \leq 1$ for all $x \in [-1, 1]$. By Markov's inequality, $|p(0)| = |q'(1)| \leq n^2$ with equality if and only if $q(x) = T_n(x)$. This proves (1.2) with equality if and only if $p = p_n$ as in (1.3).

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