GLOBAL EXISTENCE AND UNIQUENESS RESULTS FOR NEMATIC LIQUID CRYSTAL AND MAGNETOVISCOELASTIC FLOWS

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Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades
an der Fakultät für Mathematik und Informatik
der Julius-Maximilians-Universität Würzburg

vorgelegt von

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Würzburg, August 2021

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Abstract

Liquid crystals and polymeric fluids are found in many technical applications with liquid crystal displays probably being the most prominent one. Ferromagnetic materials are well established in industrial and everyday use, e.g. as magnets in generators, transformers and hard drive disks. Among ferromagnetic materials, we find a subclass which undergoes deformations if an external magnetic field is applied. This effect is exploited in actuators, magnetoelastic sensors, and new fluid materials have been produced which retain their induced magnetization during the flow.

A central issue consists of a proper modelling for those materials. Several models exist regarding liquid crystals and liquid crystal flows, but up to now, none of them has provided a full insight into all observed effects. On materials encompassing magnetic, elastic and perhaps even fluid dynamic effects, the mathematical literature seems sparse in terms of models. To some extent, one can unify the modeling of nematic liquid crystals and magnetoviscoelastic materials employing a so-called energetic variational approach. Using the least action principle from theoretical physics, the actual task reduces to finding appropriate energies describing the observed behavior. The procedure leads to systems of evolutionary partial differential equations, which are analyzed in this work.

From the mathematical point of view, fundamental questions on existence, uniqueness and stability of solutions remain unsolved. Concerning the Ericksen-Leslie system modelling nematic liquid crystal flows, an approximation to this model is given by the so-called Ginzburg-Landau approximation. Solutions to the latter are intended to approximately represent solutions to the Ericksen-Leslie system. Indeed, we verify this presumption in two spatial dimensions. More precisely, it is shown that weak solutions of the Ginzburg-Landau approximation converge to solutions of the Ericksen-Leslie system in the energy space for all positive times of evolution. In order to do so, theory for the Euler equations invented by DiPerna and Majda on weak compactness and concentration measures is used.

The second part of the work deals with a system of partial differential equations modelling magnetoviscoelastic fluids. We provide a well-posedness result in two spatial dimensions for large energies and large times. Along the verification of that conclusion, existing theory on the Ericksen-Leslie system and the harmonic map flow is deployed and suitably extended.

Zusammenfassung

Flüssigkristalle und polymere Flüssigkeiten finden sich in vielen technischen Anwendungen, wobei die Liquid Crystal Displays (kurz LCDs) wahrscheinlich die bekanntesten sind. Ebenso haben viele ferromagnetische Materialien Gebrauch in der Technologie gefunden, zum Beispiel als Generatoren, Transformatoren und Hard Drive Disks. Bei einigen ferromagnetischen Materialien führt die äußere Anwendung eines Magnetfeldes zu Verformungen. Dieser Effekt wird z. B. in Aktoren ausgenutzt und es wurden neue Flüssigkeiten gefunden, welche ihre eingangs induzierte Magnetisierung beibehalten.

Bis heute besteht ein Problem darin, derartige Materialien korrekt zu modellieren. Für Flüssigkristalle und Flüssigkristallströmungen existieren mehrere Modelle, aber bisher hat keines von ihnen einen vollständigen Einblick in alle beobachteten Effekte liefern können. Zu Materialien, welche magnetischen, elastischen und vielleicht sogar fluiddynamischen Effekten unterliegen, ist die Literatur bezüglich der Modellierung auf mathematischer Seite eher spärlich. Bis zu einem gewissen Grad kann man die Modellierung von Flüssigkristallen und magnetoviskoelastischen Materialien durch einen Variationsansatz für das Wirkungsfunktional vereinheitlichen. Verwendet man das Prinzip der kleinsten Wirkung aus der theoretischen Physik, reduziert sich die eigentliche Aufgabe darauf, geeignete Energien zu finden, um das beobachtete Verhalten zu beschreiben. Das Verfahren führt zu Systemen zeitabhängiger partieller Differentialgleichungen, welche in dieser Arbeit betrachtet werden.

Aus mathematischer Sicht bleiben grundsätzliche Fragen zu Existenz, Eindeutigkeit und Stabilität von Lösungen offen. Bezüglich des Ericksen-Leslie-Modells für nematische Flüssigkristalle ist eine Approximation dieses Modells durch die sogenannte Ginzburg-Landau-Näherung gegeben. In dieser Arbeit wird bewiesen, dass Lösungen des letzteren Modells gegen Lösungen des erstgenannten in zwei Raumdimensionen konvergieren. Präzise ausgedrückt wird gezeigt, dass schwache Lösungen des Ginzburg-Landau-Systems auf beliebig großen Zeitintervallen gegen Lösungen des Ericksen-Leslie-Systems konvergieren unter der Annahme, dass die Energie des physikalischen Systems beschränkt ist. Dazu wird die von DiPerna und Majda entwickelte Theorie für die Euler-Gleichungen zu Konzentrationen unter schwacher Konvergenz verwendet.

Der zweite Teil der Arbeit beschäftigt sich mit einem System partieller Differentialgleichungen zur Modellierung magnetoviskoelastischer Flüssigkeiten. Wir zeigen, dass in zwei Raumdimensionen in gewissem Sinne ein wohlgestelltes Problem für beliebig große Energien und Zeiten vorliegt. Für den Beweis dieses Resultats verwenden und erweitern wir die bestehende Theorie zum Ericksen-Leslie-System und zum Wärmefluss harmonischer Abbildungen.

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Chapter 1

Introduction

Complex fluids appear numerously in nature and technology. They are characterized in particular by their non-Newtonian laws of evolution which let their behaviour differ drastically from Newtonian fluids. Among these complex fluids, we find blood, ketchup, lava, polymer, liquid crystals, ferrofluids etc. In a Newtonian fluid, the overall forces acting on the constitutive molecules are homogeneous and linear. The microstructure of a complex fluid behaves non-linearly, i.e. in an anisotropic manner. Complex fluids can almost be thought of as being in between a fluid and a solid state. Such intermediate states are also called mesophases. The modeling and analysis of complex media requires specific adaption to the phenomena observed during their investigation.

In this work, we focus on two materials of this big assemblage. First, we consider nematic liquid crystals, probably best known for their application in liquid crystal displays (LCDs). The second one are magnetoviscoelastic fluids, which exhibit several features: Due to their ferromagnetic properties, they react to changes of applied magnetic fields. In particular, these variations affect the motion of the fluid and compete with the deformation of the material (hence the term "viscoelastic" in the name). The coupling of magnetic and elastic effects is used in actuators and recently, new fluids, constructed with droplets, were found in [67] which retain the magnetization after the applied magnetic field is deactivated.

We are interested in the mathematical side and investigate existing models for the latter materials. This task consists of the fundamental questions of existence, uniqueness and, to some extent, stability issues of the underlying laws of motions. Therefore, the work comes within the subject area of partial differential equations and, more precisely, the branch of parabolic evolution equations.

In order to do so, we take a brief look at the modeling of liquid crystals and magnetoviscoelastic fluids in Sections 1.1 and 1.2. Section 1.3 provides an overview on the mathematical literature of both materials and introduces the definite systems of equations, which are treated in this work. Finally, Section 1.4 outlines the structure of the arguments elaborated in the thesis.

1.1 A glimpse on the modeling of nematic liquid crystals

When heated, liquid crystals may pass several temperature-dependent intermediate states [77]. Depending on the specific material, one might observe a smectic phase, a cholesteric phase and a nematic phase before entering the isotropic fluid phase. Since the temperature represents the degree of movement which molecules on average occupy, the lower the temperature is, the more structure in liquid crystal phases will occur.

The nematic liquid crystal mesophase is mainly characterized by molecules which exhibit a local alignment along a privileged direction (see Figure 1.1). But different to other phases, the molecules' distribution does not show long-range order. Commonly, this preferred direction is called the anisotropic axis. For the majority of nematic liquid crystals, one such axis is observed, hence they are termed uniaxial. However, biaxial nematic liquid crystals with a second priviledged direction are found as well.

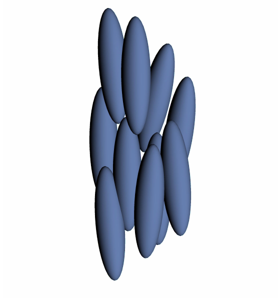

Figure 1.1: Local alignment (nematic phase)

Besides the local alignment along an axis, smectic liquid crystals undergo long-range orders. The molecules orient themselves in layers or planes because of the average lower energy assigned to the constitutive particles. The smectic mesophase is further categorized into smectic A, smectic C and smectic C*, reflecting the different positioning of the layers.

In the cholesteric mesophase, alignment similar to the nematic one is observed as well. The difference lies in the long-range helical structure. The constitutive molecules distribute themselves into two-dimensional nematic-like layers where the orientation in each layer changes gradually in the third space dimension. The orientation thus forms a spiral along the third component, resulting in the helical structure.

In the following, the modeling regarding liquid crystals is outlined. Several well-established systems of evolutionary equations exist: The Ericksen-Leslie model, the Landau-de-Gennes model along with the time-dependent counterpart, the Beris-Edwards system, and the Doi-Onsager model. In the following, the Ericksen-Leslie model is considered, introduced by Ericksen and Leslie in [26] and [53].

The Ericksen-Leslie model has its foundations especially in the works of Oseen [70] and Frank [32]. Before considering the evolution in time, liquid crystals at rest should satisfy some energy minimization principle. Like in elasticity theory or micromagnetics, the stationary state should in principle be described by a minimizing configuration of an energy functional.

As explained above, nematic liquid crystal particles prefer to align (locally) in a certain direction. This behavior is attributed to the rod-like structure of the particles. It stands to reason to assign a vector d, called director, to each particle. In the continuum limit, every point in space or some subdomain U is occupied by the nematic liquid crystal. Hence, a vector field $d : U \to \mathbb{R}^3$ is introduced which reflects the orientation $d(x)$ of a particle placed in position $x \in U$. Such a vector field is also called director field.

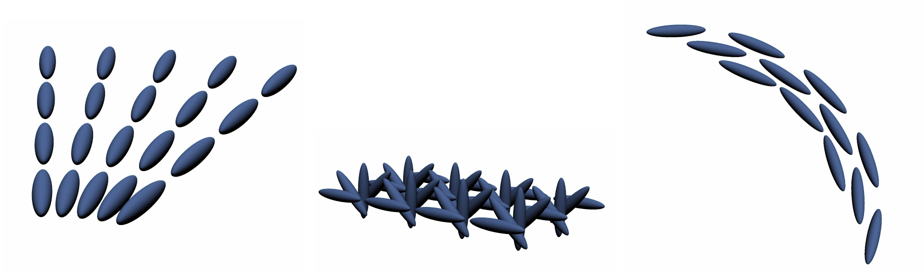

Figure 1.2: Depiction of splay, twist and bend strains

The majority of the fluids considered in this work consist of molecules which cannot be compressed or elongated. This relation translates into a unitary condition on d, i.e. $|d(x)| = 1$ for all $x \in U$.

Every configuration, represented by a director field, is assigned an energy value. The tendency of alignment of the orientations indicates that large gradients of the director field d must lead to configurations with large energies. Therefore, gradients of d must be penalized by the energy terms. In this regard, Oseen [70] and Frank [32] proposed their well-known Oseen-Frank energy density

$$\begin{aligned} W_{\mathrm{OF}}(d, \nabla d) = \frac{k_1}{2}(\operatorname{div} d)^2 + \frac{k_2}{2}|d \times (\nabla \times d)|^2 + \frac{k_3}{2}|d \cdot (\nabla \times d)|^2 \\ + \frac{1}{2}(k_2 + k_4)\left[\operatorname{tr}(\nabla d)^2 - (\operatorname{div} d)^2\right], \end{aligned} \tag{1.1.1}$$

where ground states of $d \mapsto \int_U W_{\mathrm{OF}}(d, \nabla d)$ are intended to replicate nematic liquid crystal configurations observed in reality. The parameters $k_i, i = 1, 2, 3, 4$ weigh different phenomena which are observed in distinct configurations: The first three terms represent splay, twist and bend curvature strains (see Figure 1.2), respectively, while the fourth term describes the saddle-splay term.

Proceeding to an evolutionary model, a time variable $t \in [0, \infty)$ is taken into account as well. The nematic liquid crystal state corresponds to higher temperatures and thus is closer to states of isotropic fluid. A second quantity is introduced, the velocity field $v : U \times [0, \infty) \to \mathbb{R}^3$. The velocity field $v(x, t)$ at a space point $x \in U$ and time $t \in [0, \infty)$ reflects the value and direction of velocity the particle at this position moves with. In general, the fluid's mass is distributed inhomogeneously among space leading to the density function $\rho : U \times [0, +\infty) \to \mathbb{R}_0^+$. At position x and time t, the density of the fluid, the mass per volume element, is described by the quantity ρ. In total, the three constitutive variables ρ, u and d are taken into account.

The aim is the derivation of equations relating those quantities and quantifying their evolution. As a first approach, one may rely on standard principles known from classical mechanics: The conservation of mass, of momentum and of angular momentum. Moreover, it is observed that the nematic liquid crystal flow tends to dissipate to a final state at rest as the evolution proceeds. Therefore, internal friction needs to be taken into account as well. In sum, the works of Ericksen and Leslie (see [26] and [53]) established the Ericksen-Leslie system which in its full generality reads

$$\partial_t \rho + v \cdot \nabla \rho = 0, \qquad \text{(conservation of mass)} \tag{1.1.2}$$

$$\operatorname{div} v = 0, \qquad \text{(incompressibility condition)} \tag{1.1.3}$$

$$\rho \dot{v} = \rho G_1 + \operatorname{div} \hat{\sigma}, \qquad \text{(conservation of momentum)} \tag{1.1.4}$$

$$\rho_1 \dot{\omega} = \rho_1 G_2 + \hat{g} + \operatorname{div} \pi. \qquad \text{(conservation of angular momentum)} \tag{1.1.5}$$

In this work, we refer to the above system as the general Ericksen-Leslie system. An alternative derivation of the Ericksen-Leslie model should be mentionend. Assigning a kinetic energy and an internal energy (here the Oseen-Frank energy) to the fluid as well as a dissipation rate modeling internal friction, one can employ a so-called energetic variational approach and the maximum dissipation principle. This is carried out in [83], which yields the general Ericksen-Leslie system (1.1.2)–(1.1.5) as well.

In the general Ericksen-Leslie system above, we use the following notation for $i, j = 1, 2, 3$ where we sum over repeated indices:

- The superposed dot $\dot{}$ depicts the material derivative $\dot{f} = \partial_t f + v \cdot \nabla f$,
- A is the rate of strain tensor $A = \frac{1}{2}(\nabla v + \nabla v^\top)$,
- $\Omega = \frac{1}{2}(\nabla v - \nabla v^\top)$, the skew-symmetric part of the strain rate,
- ω is the material derivative of d, i.e. $\omega = \dot{d}$,
- N is the corotational time flux of the director motion $N = \omega - \Omega d$,
- $\hat{g}$ is the intrinsic force associated to d given by

$$\hat{g}_i = \gamma d_i - \beta_j \partial_j d_i - \rho \frac{\partial W_{\mathrm{OF}}}{\partial d_i} + g_i$$

  with $\beta = (\beta_i)_i$ and γ being functions serving as Lagrangian multipliers to $|d| = 1$,
- g is the kinematic transport of d given by $g_i = \lambda_1 N_i + \lambda_2 d_j A_{ji}$
- W_{OF} is the Oseen-Frank energy (density)
- G_1 and G_2 are external forces,
- π is the director stress with

$$\pi_{ij} = \beta_i d_j + \rho \frac{\partial W_{\mathrm{OF}}}{\partial(\partial_i d_j)},$$

- $\hat{\sigma}$ is the stress tensor with

$$\hat{\sigma}_{ij} = -p\delta_{ij} - \rho \frac{\partial W_{\mathrm{OF}}}{\partial(\partial_i d_k)} \partial_j d_k + \sigma_{ij},$$

- σ is the viscous stress tensor with

$$\sigma_{ij} = \mu_1 d_k A_{kp} d_p d_i d_j + \mu_2 N_i d_j + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 A_{ik} d_k d_j + \mu_6 d_i A_{jk} d_k$$

- $\lambda_i, i = 1, 2$ are material constants which reflect the shape of the molecules,
- $\mu_i, i = 1, ..., 6$ are the Leslie coefficients.

1.2 The modeling of magnetoviscoelastic fluids

The second type of complex fluids we address in this work are magnetoviscoelastic fluids. The model here considered has recent origins (see [7, 31]) and we briefly present the governing equations in this paragraph. The term "magnetoviscoelastic fluid" comprises three notions: Fluid motion, (ferro-)magnetism and viscoelasticity. The combination of fluid motion and ferromagnetism shares mathematical similarities with the Ericksen-Leslie model. Therefore we comment on the third feature, elasticity and viscoelasticity.

The description of elasticity mainly falls into the mathematical area of Calculus of Variations (see, e.g., [17]). Therein, the utilization of Lagrangian coordinates represents a difference to fluid mechanics and micromagnetics. An elastic material, for example rubber, beams or strings, occupies a domain $U_0 \subset \mathbb{R}^3$. By application of stress to the body, the material undergoes a deformation which is represented by a mapping $y : U_0 \to\ y(U_0) = U$ in Lagragian coordinates. Elasticity means that removal of the stress results in reversion of the process, i.e., the material attains its original configuration U_0. The observed deformation (or deformed configuration) is intended to be a minimizing state of an energy functional. The material properties which determine the deformed state are encoded in an elastic energy density W. First and foremost, the density W usually depends on the deformation gradient $\widetilde{F} = \nabla y$.

Taking into consideration fluid motion as well, a subset of complex fluids are so-called viscoelastic fluids. Examples are ketchup, polymers and tooth paste. Viscoelastic fluids show two different behaviors at once: Elasticity, i.e., reattainment of original shape if exterior stresses are removed, and a constant motion reacting to stresses with dissipation to a state at rest. Seeming contradictory, the (visible) elastic behavior happens on comparatively short time scale and the dissipation on long time scales.

Since fluid mechanics is usually phrased in Eulerian coordinates on a domain U, the formulation of elasticity is as well transformed to Eulerian coordinates (see, e.g., [66]) in order to manage the description effectively. Representing the spatial coordinates in Lagrangian coordinates by $X \in U_0$, the spatial coordinates in Eulerian coordinates by $x \in U$ and the time by t, the transformation formula reads

$$F(x(X,t),t) := \widetilde{F}(X,t)$$

where F is the deformation gradient in Eulerian coordinates. If $t = 0$, no motion of the fluid has taken place so far, therefore $x(X, 0) = X$ must hold. Since x describes the position of a volume element of the material at position X and time t, the evolution is completely determined by the time derivative of $x(X,t)$, the velocity. The velocity is

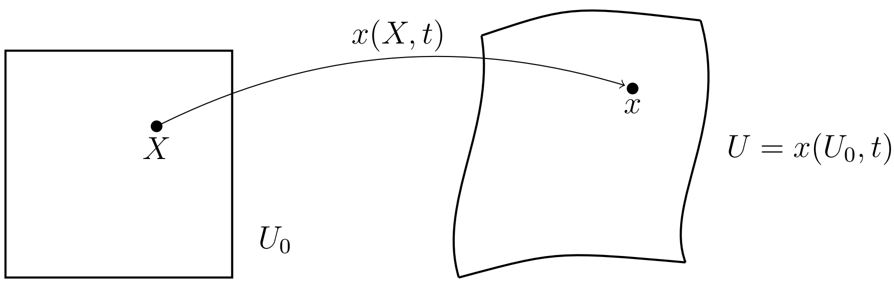

Figure 1.3: Flow map x

denoted by $u : U \times [0, +\infty) \to \mathbb{R}^3$ and hence, the coordinate change (or pullback of u) is given by

$$\partial_t x(X, t) = u(x(X, t), t).$$

Usually, the mapping $x : U_0 \times [0, +\infty) \to \mathbb{R}^3$ is called the flow map (see Figure 1.3).

Eventually, calling into play the magnetic component, ferromagnetic effects originate from the quantum mechanic spins of atoms in the materials. The spins are considered to have a fixed magnitude with variable degree of orientation. On the macroscopic level, a vector field $M : U \to \mathbb{R}^3$ is assigned to the domain U occupied by the ferromagnetic material. Because of the fixed magnitude, one sets $|M(x)| = 1$ for every $x \in U$. Without the application of a magnetic field to a magnetic specimen, the spins do not possess a specific orientation. If the magnetic field is activated, the spins react which results in the attempt to align parallel (or anti-parallel) to the magnetic field. Ferromagnetic materials are then characterized by the fact that they retain this magnetization after the magnetic field is deactivated again.

Once more, the distribution of M is intended to be modeled by a minimum configuration for a given energy functional. Here, we enter the regime of micromagnetics (see, e.g., [12] or [49] for an overview). The free energy of a magnetic material should mirror several phenomena. Similar to nematic liquid crystals, variations of the magnetization field M are penalized. Further effects must be taken into consideration: Due to possible crystalline structures in the material, there may exist preferred directions for the magnetization. This is reflected by an anisotropy energy term $\psi = \psi(M)$. What qualitively distincts the description of ferromagnets from nematic liquid crystals is the appearance of long-range interactions. For ferromagnetic materials (for example NiFe, see [45]), the formation of domains with uniform magnetization is observed. This circumstance is mainly explained by the stray field energy. Determined by Maxwell's equation

$$\operatorname{div}(M + H(M)) = 0,$$

it resembles the tendency to resolve the magnetic field induced by the magnetization M outside of the domain. Eventually, it is necessary to also take into account the application of external magnetic fields H_{ext}, resulting in the so-called Zeeman energy. In sum, the micromagnetic energy reads

$$E(M) = \int_U \langle \nabla M, A \nabla M \rangle + \psi(M) - \frac{\mu_0}{2} H(M) \cdot M - \mu_0 H_{\text{ext}} \cdot M \, \mathrm{d}x.$$

Here, A depicts a positive definite fourth-order tensor and μ_0 the permeability constant.

The term"magnetoviscoelastic" indicates that non-trivial interactions between the different described regimes can take place. For example, magnetostriction describes the relatively large deformation specific ferromagnetic materials undergo when an external magnetic field is applied. Looking for an evolutionary model of such materials, in [31, p. 35] the author employed an energetic variational approach (with maximum dissipation principle). Taking into account the information above, the resulting system reads

$$\begin{aligned}
\dot{u} &= \operatorname{div} T + \mu_0 \nabla^\top (H(M) + H_{\text{ext}}) M, \\
\operatorname{div} u &= 0, \\
\dot{F} &= \nabla u F, \\
\dot{M} &= -M \times H_{\text{eff}} - M \times (M \times H_{\text{eff}})
\end{aligned} \tag{1.2.1}$$

with the following notations:

- T is the stress tensor with

$$T_{ij} = -p\delta_{ij} - \nu\left(\partial_j u_i + \partial_i u_j\right) - \frac{\partial W(F)}{\partial F_{ik}} F_{jk} - \partial_i M_k \partial_j M_k,$$

- $\nu > 0$ the kinematic viscosity,
- $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the elastic energy density,
- H_{eff} is the effective magnetic field given by

$$(H_{\text{eff}})_i = \partial_j(A_{ijkl}\partial_k M_l) + \mu_0(H(M) + H_{\text{ext}})_i - \frac{\partial \psi(M)}{\partial M_i}.$$

We like to point out that $(1.2.1)_4$ represents a variant of the Landau-Lifshitz-Gilbert (in the following LLG) equation. The LLG equation was first derived by Landau and Lifshitz in [51] without dissipation. A unique feature of the LLG equation is the first term on the right-hand side. It describes some form of convection resulting from spin-spin interactions on the micromagnetic scale. Therefore, the LLG equation does not represent a gradient flow.

Both systems, the general Ericksen-Leslie system and (1.2.1), are (in principle) achieved by setting an energy functional and applying the principle of least action. Hence, solutions to the respective equations will be stationary points of the action functional subject to the imposed laws of dissipaton. Moreover, regardless of the assumption that our fluid is occupying a two or three-dimensional domain, the director field d and the magnetization M always attain values in $\mathbb{R}^3$. If one wants to state both systems in dimension two, one needs to extend all quantities arising to three dimensions and interpret them as functions being constant in the third spatial variable.

1.3 Well-posedness theory for liquid crystal flows and magnetoviscoelastic flows

In the following, the mathematical background of the Ericksen-Leslie model and the system for magnetoviscoelastic fluids is outlined. Ericksen and Leslie established the general Ericksen-Leslie model in [26] and [53]. The respective equations are obtained by application of the least action and maximal dissipation principle which is, e.g., elaborated in [83]. As the Ericksen-Leslie system is very difficult to analyze, let alone set up a mathematical well-posedness theory, Lin suggested in [57] a simplified system of equations: Setting $G_1 = 0, G_2 = 0, \rho = 1$ with parameters $k_1 = k_2 = k_3 = 1, k_4 = 0$ and $\mu_1 = \mu_2 = \mu_3 = \mu_5 = \mu_6 = 0, \mu_4 = 1$, furthermore ignoring the motion due to the corotational part Ω and the second order material derivatives (i.e. $\dot{\omega} \to \omega$) in (1.1.2)–(1.1.5), the Ericksen-Leslie system of PDEs reduces to

$$\begin{aligned}
&\partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p + \operatorname{div}(\nabla d \odot \nabla d) = 0,\\
&\operatorname{div} v = 0, \qquad\qquad (1.3.1)\\
&\partial_t d + (v\cdot\nabla)d = \Delta d + |\nabla d|^2 d
\end{aligned}$$

with $(\nabla d \odot \nabla d)_{ij} = \partial_i d_k \partial_j d_k$ for $i, j = 1, 2, 3$. The relation $|d| = 1$ still holds true and the term $|\nabla d|^2 d$ on the right-hand side of the third equation in (1.3.1) serves as a Lagrangian multiplier subject to the unitary constraint. In this case, the Oseen-Frank energy (1.1.1) reduces to the Dirichlet energy

$$E_{\mathrm{OF}} = \int_U \frac{|\nabla d|^2}{2}\, \mathrm{d}x.$$

Despite this reduction, this model preserves all essential nonlinear terms (stress tensor in the momentum equation, nonlinear Lagrangian multiplier in the director equation) associated with the full Ericksen-Leslie system, while neglecting other terms of comparable or lower order. Without further explanations, equation (1.3.1) can be considered in any spatial dimension $n \in \mathbb{N}$. Being a parabolic system of differential equations, one imposes initial conditions $v_0(x) = v(0, x), d_0(x) = d(x, 0)$ for $x \in U$ and boundary conditions $u(x, t) = 0, d(x, t) = d_0(x)$ for $(x, t) \in \partial U \times (0, +\infty)$ to have a chance of unique solvability in some function space.

Even for the simplified version of the Ericksen-Leslie sytem, an existence theory for (weak) solutions is mainly available in two dimensions only. The first existence results for global-in-time existence, i.e., for all times $t > 0$, and initial data with arbitrarily large energy stem from [58] and [41]. The uniqueness of solutions has been proven in [63] under additional regularity assumptions also in higher dimensions. Extensions of these works are, for instance, made in [42] and [43] which allow to include further terms of the original Ericksen-Leslie system (1.1.2)–(1.1.5). The method used in the aforementioned works relies on the similarity of (1.3.1) to the harmonic map heat flow into spheres

$$\partial_t d = \Delta d + |\nabla d|^2 d, \qquad |d| = 1, \tag{1.3.2}$$

a system that is extensively analysed in the literature. In two spatial dimensions, Struwe proves in [78] the global-in-time existence of weak solutions to (1.3.2), which are regular besides a finite number of space-time points. The sharpness of this achievement is highlighted by the existence of blowing-up solutions in finite time to (1.3.2), first found in [14]. Similarly, singular solutions to the simplified Ericksen-Leslie system are constructed in [44, 50].

In higher dimensions, an analogue of Struwe's method does not exist and results for existence and uniqueness to (1.3.1) are only known to hold true for small initial data[1] [40] or for short times [30]. Because of the difficulties handling the analysis for (1.3.1), which is due to the combination of the Navier-Stokes equations and the geometric constraint $d \in \mathbb{S}^2$, a penalization approach for the unitary constraint is considered. Substituting the requirement $|d| = 1$ by an additional term, the energy of the system reads

$$\int_U \frac{|v|^2}{2} + \frac{|\nabla d|^2}{2} + \frac{(1 - |d|^2)^2}{4\varepsilon^2}\, \mathrm{d}x$$

for $\varepsilon > 0$. Hence, one obtains the so-called Ginzburg-Landau penalization

$$\begin{aligned}
&\partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon - \Delta v_\varepsilon + \nabla p_\varepsilon + \operatorname{div}(\nabla d_\varepsilon \odot \nabla d_\varepsilon) = 0, \\
&\operatorname{div} v_\varepsilon = 0, \\
&\partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon = \Delta d_\varepsilon + \frac{(1 - |d_\varepsilon|^2) d_\varepsilon}{\varepsilon^2}.
\end{aligned} \tag{1.3.3}$$

[1] in the critical space L^3_{uloc}

If $\varepsilon > 0$ is chosen close to zero, then, it heuristically holds $|d_\varepsilon| \approx 1$ due to the penalizing term and the energy. Well-posedness of (1.3.3) is first covered in [59] by proving global existence and uniqueness of weak and classical solutions in two dimensions. Results in three dimensions are obtained in [59], too. Therein, existence and uniqueness of classical solutions is shown under a smallness condition as well as existence of weak solutions. Extensions have been made, e.g., in [30, 60] by including further terms of the general Ericksen-Leslie system (1.1.2)–(1.1.5). The idea of using a penalization again originates from results on the harmonic map heat flow (1.3.2). In [15] and [16], Chen and Struwe showed the existence of weak and partially regular solutions in any spatial dimension to (1.3.2) by using an analogous penalized system as approximation and carrying out the limit $\varepsilon \to 0^+$. Hence, one might be tempted to perform the similar arguments for the Ginzburg-Landau approximation to obtain global solutions to (1.3.1). However, the latter issue turned out to be very difficult to implement and little is known in this case. In dimensions two and three, the convergence as $\varepsilon \to 0$ is shown on a short time interval in [30, 41] and the probably biggest achievement for global (necessarily weak) solutions consists of [64], where the additional assumption on d attaining values just in the upper half-sphere $\mathbb{S}^2_+$ is needed.

The first result of this thesis aims at completing the aforementioned limit passage, when the fluid is in two spatial dimensions and as long as the initial energy of the system is finite (see the author's article [48]). A more detailed version of the proof is presented in Chapter 3, Theorem 3.1.2. The idea of the proof to Theorem (3.1.2) originates from the works of DiPerna and Majda [23] on the existence of vortex-sheet solutions to the Euler equations. Although of different type, the Euler equations and the Ericksen-Leslie system share similarities in terms of compactness in two dimensions. This is outlined in Section 3.2. The result of [48], Theorem 3.1.2 in this thesis, is generalized to arbitrary target manifolds by Du, Huang and Wang in [24].

The system (1.2.1) has more recent origins. In full generality, it is derived in [31] by the least action principle and maximum dissipation principle. To some extent, the magnetoviscoelastic system shares many similarities with the simplified Ericksen-Leslie system. In particular, the quanities v and d in (1.3.1) correspond to u and M in (1.2.1). The additional equation $(1.2.1)_3$ models the evolution of the deformation tensor and is of hyperbolic type and therefore does not contain any dissipational terms. This fact complicates the mathematical existence theory heavily and one may introduce an artificial dissipation leading to a system variant of (1.2.1),

$$\begin{aligned}
&\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = \operatorname{div}(W'(F)F^\top - \nabla M \odot \nabla M) + \mu_0 \nabla^\top (H(M) + H_{\text{ext}})M,\\
&\operatorname{div} u = 0,\\
&\partial_t F + (u \cdot \nabla)F - \nabla u F = \kappa\Delta F,\\
&\partial_t M + (u \cdot \nabla)M = -M \times H_{\text{eff}} - M \times (M \times H_{\text{eff}})
\end{aligned} \tag{1.3.4}$$

for some $\kappa > 0$. The global existence of solutions to (1.3.4) with small initial data is verified in [31] omitting the external magnetic field, the stray field term and the anisotropic energy. This result is extended in [7] by inclusion of the external field H_{ext} to the system. Both results are obtained in two dimensions and assume W to be a strongly convex function. Hence, we point out several issues accompanied with the system:

- The problem of an existence theory without artificial dissipation, i.e., $\kappa = 0$, even for simple elastic energy densities like $W(F) = \frac{|F|^2}{2}$.
- The question whether weaker than strong convexity assumptions on W are sufficient to prove an existence theory.
- The inclusion of all relevant energy terms, especially the anisotropic energy ψ and the stray field energy $H(M)$.
- The existence theory in three spatial dimensions. Considering the relation with (1.3.1), one cannot hope for better results than those for the Ericksen-Leslie system.

The last point also suggests introduction of a penalized variant of (4) again similar to the Ginzburg-Landau approximation with a parameter $\varepsilon > 0$. Here, uniqueness and weak-strong uniquenss results are established in [74]. With respect to the first point, Zhao obtains results on local-in-time existence (in two and three dimensions), weak-strong uniqueness and blow-up criterions of classical solutions in [84, 85] for $W(F) = \frac{|F|^2}{2}, \varepsilon > 0$ and including H_{ext}.

With respect to the second point, an extension to convex energies W and the inclusion of the stray field in two dimensions is perfomed in joint work of the author with M. Kalousek and A. Schlömerkemper in [46] for strong solutions local-in-time and small weak solutions global-in-time without the penalization. Furthermore, Struwe-like solutions (cf. Section 4.1), i.e., global weak solutions for arbitrary large initial data are obtained in joint work of the author with F. De Anna and A. Schlömerkemper in [18]. Stated in two dimensions, the latter results correspond to the thin-film regime in micromagnetics. A mathematical overview of the different regimes of micromagnetics is provided by [21]. In the two-dimensional thin-film limit, the non-local stray field energy reduces to a local term comparable to the anisotropic energy (see [35]).

Finally, it should be mentioned that in order to let $\kappa = 0$, one might define another type of weak solutions, the so-called dissipative solutions. Here, a measure compatible with the energy law is introduced such that (1.2.1) is balanced with the measure and the balanced version holds true in the sense of distributions. The existence of such a solution is proven in [47] in two and three dimensions including the magnetic external field.

Chapter 4 extends the results of [18], and the existence and uniqueness of a global weak solution to (1.3.4) is established with the following features: All energetic relevant terms, the external magnetic field, the stray and anisotropic energy, are included. The elastic energy density W is allowed to be non-convex. The weak solutions constructed exist for any given positive time, any initial data in the energy space, and are regular with the exception of finitely many points in time.

1.4 Outline of the thesis

The subsequent chapters deal with the proof of the convergence of Ginzburg-Landau approximations and the well-posedness of the system for magnetoviscoelastic fluids. To begin with, Chapter 2 states basic facts on Sobolev spaces and measures. The first part of Chapter 3 contains the proof of Theorem 3.1.2: The convergence of solutions of (1.3.3) to solutions of (1.3.1). Since ideas from the analysis of the two-dimensional Euler equations are used, Section 3.2 contains a sketch of the results obtained by DiPerna and Majda in [23] and a comparison between Euler and Ericksen-Leslie equations.

Chapter 4 establishes Theorem 4.4.1. In order to motivate this result, the method of Struwe in [78] is reviewed in Section 4.1. During the proof of Theorem 4.4.1, some subresults are encountered as well: The uniqueness of solutions in Section 4.2 and the existence of a local-in-time strong solution in Section 4.3 to (1.3.4). Chapter 5 is devoted to a conclusion of this thesis.

Chapter 2

Notation, Sobolev spaces on $\mathbb{T}^2$ and measures

We briefly introduce and recall notions and results used throughout the upcoming arguments.

To begin with, we denote by $C > 0$ generic constants throughout the work where the value of C might change from one inequality to the next. Unless stated otherwise, the constant only depends on external parameters. Sometimes we also use the short notation $A \lesssim B$ to denote $A \leq CB$. Further, we employ the summation convention, i.e., we sum over repeated indices.

Partial derivatives (classical or weak) with respect to spatial variables x_i are denoted by ∂_i or ∂_{x_i} and ∇f denotes the gradient of a function f. A derivative of f with respect to time is indicated by either $\partial_t f$, $\frac{\mathrm{d}}{\mathrm{d}t} f$ or f_t.

Most of the analysis in Chapters 3 and 4 takes place on the two-dimensional torus $\mathbb{T}^2$ in spatial variables. Hence we refer to [72], in particular Chapter 1 and the appendix, for the notions of weak differentiability, measurability and Bochner measurability and the well-definedness of the sets in this chapter unless stated otherwise.

The torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ is a closed manifold where one identifies left-hand with right-hand side boundary as well as lower with upper boundary. Equivalently, we can say that we consider functions $f : \mathbb{R}^2 \to \mathbb{R}^m$ which are 2π-periodic in every of the two spatial directions. Most of the time, we use functions defined on a space time domain $\mathbb{T}^2 \times [0, T]$ for some $T > 0$ which lie in some combination of Lebesgue and Sobolev spaces. Letting U be a bounded domain in $\mathbb{R}^n$ or $\mathbb{T}^2$, we set for a number $k \in \mathbb{N}_0, p \in [1, \infty]$

$$W^{k,p}(U) := \Bigg\{ f : U \to \mathbb{R}^m : f \quad k - \text{times weakly differentiable} \\ \text{and } \|f\|^p_{W^{k,p}} = \sum_{|\alpha| \leq k} \int_U |\partial_\alpha f| < \infty \Bigg\}$$

For a Banach space X, we define the time-dependent counterpart of spatial Sobolev spaces by

$$W^{k,p}(0,T;X) := \Big\{ f : (0,T) \to X : f \quad k-\text{times weakly differentiable, } \partial_t^j f$$
$$\text{is Bochner-measurable for all } 0 \le j \le k$$
$$\text{and } \|f\|^p_{W_t^{k,p}X} = \sum_{0\le j\le k} \int_0^T \left\|\partial_t^j f(t)\right\|_X^p \,\mathrm{d}t < \infty \Big\}$$

for $k \in \mathbb{N}_0, p \in [1,\infty]$. If $k = 0$, then we deal with the usual Lebesgue spaces and write $L^p(U) := W^{0,p}(U)$ as well as $L^p(0,T;X) := W^{0,p}(0,T;X)$. If $p = 2$, we often write $H^k(U) := W^{k,2}(U)$. Note that we surpress the target space $\mathbb{R}^m$ in the notation. If we additionally assume $\int_{\mathbb{T}^2} f = 0$ for $f \in W^{k,p}(\mathbb{T}^2)$, we write $f \in \dot{W}^{k,p}(\mathbb{T}^2)$.

Continuous functions $f : D \to X$ on a domain D are denoted by $C(D;X)$ or $C(D)$ if $X = \mathbb{R}^m$ with corresponding norm $\|f\|_C = \sup_{x\in D}\|f(x)\|_X$. In the case of $D = (0,T)$ and X being a Sobolev space we write $\|f\|_C = \|f\|_{C_tX_x}$. If X is a general Banach space, then the dual space of X is written as X^* with the dual pairing denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. For Sobolev spaces, we sometimes write $W^{-k,p'}(U) := (W_0^{k,p}(U))^*$ where $p' = p/(p-1)$ for $1 \le p \le \infty$ and the subscript $_0$ denotes the closure of $C_0^\infty(U)$ in the strong topology.

The subscript $_{\text{loc}}$, e.g., $X_{\text{loc}}(U)$, for a function space X, the respective property, integrability, continuity or differentiability holds true locally, i.e., for every compact set $K \subset U$. Moreover, the subscript $_{\text{div}}$, e.g., X_{div}, denotes that $\operatorname{div} f = 0$ holds true classically or in the sense of distributions for all functions $f \in X_{\text{div}}$.

If $p = 2$, the notion of Sobolev spaces can be consistently to above extended to $k = s \in \mathbb{R}$ by Fourier expansion

$$f(x) = \sum_{k\in\mathbb{Z}^2} \hat{f}_k e^{ik\cdot x}, \qquad x \in \mathbb{T}^2$$

with coefficients $\hat{u}_k = (2\pi)^2 \int_{\mathbb{T}^2} e^{-ik\cdot x}u(x)$.

The probably most important applied theorems are different Sobolev embedding theorems and interpolation inequalities. More specific ones are stated when needed in Chapters 3 and 4, but the most common ones in two dimensions are given here:

Theorem 2.0.1 ([72], Theorem 1.7)**.** *Let $f : \mathbb{T}^2 \to \mathbb{R}^m$:*

- *If $1 \le p < 2$, then $W^{1,p}(\mathbb{T}^2) \subset L^q(\mathbb{T}^2)$ continuously with $q = 2p/(2-p)$. Furthermore, if $\int_{\mathbb{T}^2} f = 0$ we have*
$$\|f\|_{L^q} \le C\,\|\nabla f\|_{L^p}$$
for some constant $C > 0$.

- *If $p = 2$, then $W^{1,2}(\mathbb{T}^2) \subset L^q(\mathbb{T}^2)$ continuously for all $q < \infty$. Furthermore, if $\int_{\mathbb{T}^2} f = 0$ we have*
$$\|f\|_{L^q} \le C\,\|\nabla f\|_{L^p}$$
for some constant $C > 0$, which depends on q.

- *If $p > 2$ then $W^{1,p}(\mathbb{T}^2) \subset C^\alpha(\mathbb{T}^2)$ continuously where $\alpha = 1 - 2/p$.*

The by far most used interpolation inequality in two dimensions is

Theorem 2.0.2 ([11], p.313, Ladyzhenskaya's inequality)**.** *There exists a constant $C > 0$ such that*

$$\|f\|_{L^4}^4 \le C\left(\|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^4\right)$$

for all $f \in W^{1,2}(\mathbb{T}^2)$. If $\int_{\mathbb{T}^2} f = 0$, it is

$$\|f\|_{L^4}^4 \le C \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2 .$$

Furthermore, standard results on elliptic systems are needed.

Theorem 2.0.3. *Let $1 < p < \infty$ and $f \in \dot{L}^p(\mathbb{T}^2)$. Furthermore, let $u \in \dot{W}^{2,p}(\mathbb{T}^2)$ be a weak solution to*

$$-\Delta u = f.$$

Then there exists a constant $C = C(\mathbb{T}^2) > 0$ such that

$$\|\partial_i \partial_j u\|_{L^p} \le C \|f\|_{L^p}, \qquad i, j = 1, 2$$

for all $f \in \dot{L}^p(\mathbb{T}^2)$.

Proof. Since $W^{1,2} \hookrightarrow L^q$ for all $q < \infty$ in two dimensions, we have $\|\nabla u\|_{L^2} \le C \|f\|_{L^p}$ by Riesz' Representation Theorem. Let $\{V_k\}$ be a finite open covering with $1 \le k \le L \in \mathbb{N}$ of $\mathbb{T}^2$ and $\{\eta_k\}$ be a partition of unity subordinated to $\{V_k\}$, i.e., $\eta_k \in C_0^\infty(V_i)$ for each k, $0 \le \eta_k \le 1$ and $\sum_k \eta_k \equiv 1$. Then it holds

$$-\Delta(\eta_k u) = \eta_k f - 2\nabla\eta_k \cdot \nabla u - \Delta\eta_k u$$

for each $1 \le k \le L$. If $1 < p \le 2$ then the estimate below is trivial. Otherwise, we use the interpolation inequality $\|\nabla u\|_{L^p} \le C \|\nabla^2 u\|_{L^p}^{\alpha} \|\nabla u\|_{L^2}^{1-\alpha}$ for $\alpha = \frac{p-2}{2(p-1)}$ (see [11, p. 314]). Applying Theorem 2.0.4 gives

$$\begin{aligned}
\left\|\nabla^2 u\right\|_{L^p} &\le \sum_k \left\|\nabla^2(\eta_k u)\right\|_{L^p} \le \sum_k C_k \left(\|f\|_{L^p} + \|\nabla u\|_{L^p} + \|u\|_{L^p}\right) \\
&\le \sum_k C_k \left(\|f\|_{L^p} + \left\|\nabla^2 u\right\|_{L^p}^{\alpha} \|\nabla u\|_{L^2}^{1-\alpha}\right) \\
&\le \delta \sum_k C_k \left\|\nabla^2 u\right\|_{L^p} + \sum_k C_{k,\delta} \left(\|f\|_{L^p} + \|\nabla u\|_{L^2}\right)
\end{aligned}$$

by Young's inequality. Realizing that the sum is finite, we choose δ suffiently small which yields the assertion. □

Theorem 2.0.4 ([34], Corollary 9.10)**.** *Let $U \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$ and $f \in L^p(U)$. Furthermore, let $u \in W_0^{2,p}(U)$ be a solution to*

$$-\Delta u = f \qquad \text{on } U.$$

Then there exists a constant $C > 0$ such that

$$\|\partial_i \partial_j u\|_{L^p} \le C \|f\|_{L^p}, \qquad i, j = 1, ..., n$$

for all $f \in L^p(U)$.

We remark that the constant C in the theorem above does not depend on the size of U by scaling observations.

Regarding time-dependent spaces, a frequently used result below is Gronwall's inequality:

Theorem 2.0.5 ([72], Lemma A.25). *Let $\eta : [0,T] \to [0,\infty)$ be a continuous function satisfying*

$$\eta(t) \leq a(t) + \int_0^t \phi(s)\eta(s)\,\mathrm{d}s,$$

where $a, \phi : [0,T] \to [0,\infty)$ are integrable functions and a is increasing. Then

$$\eta(t) \leq a(t) \exp\left(\int_0^t \phi(s)\,\mathrm{d}s\right) \qquad \text{for all } t \in [0,T].$$

In particular, the result implies an explicit bound on the size of η on the time interval $[0,T]$. Often, continuity-in-time with respect to a strong topology of X is not satisfied. Therefore we say that a function $f : [0,T] \to X$ is weakly continuous if

$$t \mapsto \langle g, f(t) \rangle_{X^* \times X}$$

is continuous for every $g \in X^*$ and write $f \in C_w([0,T]; X)$.

Theorem 2.0.6 ([33]). *Let $1 < q < \infty, U \subset \mathbb{R}^n$ be a domain and $T > 0$. Furthermore, let $f \in L^\infty(0,T; L^q(U))$ with $\langle \partial_t f, \eta \rangle_{(W^{k,p})^* \times W^{k,p}} \in L^1(0,T)$ for some $k \in \mathbb{N}_0, 1 \leq p \leq \infty$ and any $\eta \in C_0^\infty(U)$. Then there exists $g \in C_w([0,T]; L^q(U))$ such that for a.a. $t \in [0,T]$ the equality $f(\cdot,t) = g(\cdot,t)$ holds true.*

At last, we recall results on weak compactness of measures (see [4]). A signed Borel measure μ defined on a set $\Omega \subset \mathbb{R}^n$ is called (inner) regular if

$$\inf\Big\{|\mu|(U \setminus K) : K \subset E \subset U,\, K \text{ closed in } \Omega \text{ and } U \text{ open in } \Omega\Big\} = 0$$

holds true for all Borel measurable sets $E \subset \Omega$. Such a measure is often called a Radon measure and we also call vector-valued set functions for which the above conditions hold true in any component a Radon measure and denote the set by $\mathcal{M}(\Omega)$. The norm on $\mathcal{M}(\Omega)$ is given by the total variation

$$\|\mu\|_{\mathcal{M}} = \sup\Big\{\sum_{i=1}^k |\mu(E_i)| : k \in \mathbb{N},\, E_i \text{ Borel set in } \Omega\Big\}$$

and equipped with this norm, the space $\mathcal{M}(\Omega)$ is a Banach space. It holds Riesz' Representation Theorem:

Theorem 2.0.7. *If $\Omega \subset \mathbb{R}^n$ is compact, it is*

$$\mathcal{M}(\Omega) \cong (C(\Omega))^*$$

valid, i.e., every bounded functional f on the space of continuous functions on Ω can be represented by a Radon measure μ_f such that

$$f(\phi) = \int_\Omega \phi\,\mathrm{d}\mu_f$$

and the map $f \mapsto \mu_f$ is an isomorphism.

Since $C(\Omega)$ is separable by Weierstrass' Theorem, we have the following consequence:

Theorem 2.0.8. *For every sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ with $\|\mu_n\|_{\mathcal{M}} \leq C$ for some $C > 0$ for all $n \in \mathbb{N}$, there exists a subsequence of $(\mu_n)_{n\in\mathbb{N}}$ and a limit point $\mu \in \mathcal{M}(\Omega)$ such that*

$$\mu_{n_i} \quad \rightharpoonup^* \quad \mu \qquad \text{as measures.}$$

In other words,

$$\lim_{i\to\infty} \int_\Omega \phi \; \mathrm{d}\mu_{n_i} = \int_\Omega \phi \; \mathrm{d}\mu$$

for every $\phi \in C(\Omega)$.

As $L^1(\Omega) \subset \mathcal{M}(\Omega)$, every bounded sequence in $L^1(\Omega)$ possesses a weak limit point in $\mathcal{M}(\Omega)$ as measures.

Chapter 3

Weak convergence of the Ginzburg-Landau approximation to liquid crystal flows on $\mathbb{T}^2$

In this chapter, we treat the singular limit problem of the Ginzburg-Landau approximation on the two-dimensional torus $\mathbb{T}^2$. More precisely, we prove that weak solutions of the Ginzburg-Landau system (1.3.3) converge to weak solutions of the simplified Ericksen-Leslie system (1.3.1) as the penalization parameter ε tends to 0^+. This result is stated in Theorem 3.1.2. The closing remarks of Section 3.1 provide the idea of a second proof to Theorem 3.1.2, which is closely related to the behavior of (approximated) harmonic maps on surfaces.
Furthermore, we compare this result to the existence problem of vortex sheet solutions for the two-dimensional Euler equations initiated by DiPerna and Majda in [23], where the inspiration for the proof of Theorem 3.1.2 originates from (see Section 3.2). The method, sometimes called concentration-cancellation, does not work in the framework of time-dependent inviscid fluid flows (see the discussion in Section 3.2.3), but surprisingly does work for liquid crystal flows.

3.1 Convergence of the Ginzburg-Landau approximation

Throughout this section, we mainly follow the author's article [48]. At first, we settle notions to the corresponding problem. As spatial domain, we consider the two-dimensional torus $\mathbb{T}^2$ while commenting on a general bounded smooth domain $\Omega \subset \mathbb{R}^2$ later (see Remark 3.1.9). Let $T > 0$ be a fixed, but possibly large, time. Given a parameter $\varepsilon > 0$, the system of the Ginzburg-Landau approximation reads

$$\partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon + \nabla p_\varepsilon - \Delta v_\varepsilon = -\mathrm{div}(\nabla d_\varepsilon \odot \nabla d_\varepsilon), \tag{3.1.1}$$
$$\mathrm{div}\, v_\varepsilon = 0, \tag{3.1.2}$$
$$\partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon = \Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1 - |d_\varepsilon|^2) d_\varepsilon \tag{3.1.3}$$

for a velocity field $v_\varepsilon : \mathbb{T}^2 \times (0,T) \to \mathbb{R}^2$, a pressure function $p_\varepsilon : \mathbb{T}^2 \times (0,T) \to \mathbb{R}$ and a director field $d_\varepsilon : \mathbb{T}^2 \times (0,T) \to \mathbb{R}^3$. Being of evolutionary type, System (3.1.1)–(3.1.3) is accompanied by initial conditions

$$v_\varepsilon(\cdot,0) = v_0, \quad d_\varepsilon(\cdot,0) = d_0, \tag{3.1.4}$$

for which the constraints $\operatorname{div} v_0 = 0$ and $|d_0| \equiv 1$ hold true. Note that v_0 and d_0 do not depend on ε (but see Remark 3.1.5). Altogether, (3.1.1)–(3.1.4) forms a (parabolic) initial value problem for which we define the following notion of weak solutions:

Definition 3.1.1. *Let $\varepsilon > 0$. A pair of functions*

$$\begin{aligned} v_\varepsilon &\in L^\infty\Big(0,T;\dot{L}^2_{\mathrm{div}}(\mathbb{T}^2)\Big) \cap L^2\Big(0,T;\dot{W}^{1,2}_{\mathrm{div}}(\mathbb{T}^2)\Big) \cap W^{1,2}\Big(0,T;(\dot{W}^{1,2}_{\mathrm{div}}(\mathbb{T}^2))^*\Big), \\ d_\varepsilon &\in L^\infty\Big(0,T;W^{1,2}(\mathbb{T}^2)\Big) \cap L^2\Big(0,T;W^{2,2}(\mathbb{T}^2)\Big) \cap W^{1,2}\Big(0,T;L^2(\mathbb{T}^2)\Big) \end{aligned}$$

is called a weak solution to the initial value problem (3.1.1)–(3.1.4) *if*

$$\begin{aligned} &\int_0^s \int_{\mathbb{T}^2} -v_\varepsilon \cdot \partial_t \phi - v_\varepsilon \otimes v_\varepsilon : \nabla\phi + \nabla v_\varepsilon : \nabla \phi - \nabla d_\varepsilon \odot \nabla d_\varepsilon : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t, \\ &\qquad = \int_{\mathbb{T}^2} v_0 \cdot \phi(0)\,\mathrm{d}x - \int_{\mathbb{T}^2} v(s)\cdot\phi(s)\,\mathrm{d}x, \\ &\partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon - \Delta d_\varepsilon - \frac{(1-|d_\varepsilon|^2)d_\varepsilon}{\varepsilon^2} = 0 \end{aligned}$$

holds true for all $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2 \times [0,T])$ and almost every $s \in [0,T]$ as well as pointwise a.e. on $\mathbb{T}^2 \times (0,T)$, respectively. Furthermore, $(v_\varepsilon, d_\varepsilon)$ attains the initial data $(v_0,d_0) \in L^2_{\mathrm{div}}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2)$ in the weak sense, i.e.

$$\int_{\mathbb{T}^2} v_\varepsilon(t)\cdot\psi\,\mathrm{d}x \to \int_{\mathbb{T}^2} v_0 \cdot \psi\,\mathrm{d}x, \qquad \int_{\mathbb{T}^2} \nabla d_\varepsilon(t) : \zeta\,\mathrm{d}x \to \int_{\mathbb{T}^2} \nabla d_0 : \zeta\,\mathrm{d}x$$

for all $\psi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2)$ and $\zeta \in C^\infty(\mathbb{T}^2)$ as $t \to 0^+$.

In Definition 3.1.1, the pressure p_ε is omitted. This limns a standard procedure when dealing with weak formulations of incompressible fluid dynamic problems since

$$\int_{\mathbb{T}^2} \nabla p_\varepsilon \cdot \phi = 0$$

for test functions ϕ with $\operatorname{div}\phi = 0$. As mentioned in the introduction, the Ginzburg-Landau approximation is derived by a variational approach (see [83]) and therefore comes with an energy law. In two dimensions, the associated energy is scale-invariant suggesting that the initial data lying in the energy space give rise to a solution. We have the following theorem that immediately follows from [59]:

Theorem 3.1.1. *Let $\varepsilon > 0, v_0 \in \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2)$ and $d_0 \in W^{1,2}(\mathbb{T}^2)$ with $|d_0| = 1$ a.e. Then there exists a unique weak solution to the initial value problem* (3.1.1)–(3.1.4).

The above theorem is proven in [59] for a bounded smooth domain Ω, accompanied by boundary conditions, via a multilevel Galerkin approximation. However, it carries over to $\Omega = \mathbb{T}^2$ since the absence of boundaries simplifies the technicalities.
The regularity properties of these weak solutions originate from the underlying energy conservation law. Formally, we have

$$\int_{\mathbb{T}^2} |v_\varepsilon(t)|^2 + |\nabla d_\varepsilon(t)|^2 + \frac{1}{2\varepsilon^2}(1-|d_\varepsilon(t)|^2)^2 + 2\int_0^t \int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 + \left|\Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1-|d_\varepsilon|^2)d_\varepsilon\right|^2 \\ = 2E_0$$

for a weak solution to (3.1.1)–(3.1.4) for any $0 \le t \le T$, where $E_0 = \frac{1}{2}\int_{\mathbb{T}^2} |v_0|^2 + |\nabla d_0|^2$ holds, the initial energy. Since all quantities on the left-hand side are non-negative, they are bounded as is the right-hand side. This justifies the regularity assumptions in Definition 3.1.1.
We turn our attention to the limiting problem $\varepsilon \to 0^+$. The energy law above provides, in particular, a bound on the quantity

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^2} \frac{(1-|d_\varepsilon(t)|^2)^2}{4\varepsilon^2}$$

independently of ε. Letting ε tend to 0, the absolute value $|d_\varepsilon|$ is then forced to converge to the constant value 1. The heuristical conclusion is that the limiting problem consists of the Euler-Lagrange equations for the energy

$$\frac{1}{2}\int_{\mathbb{T}^2} |v|^2 + |\nabla d|^2, \tag{3.1.5}$$

where d takes values in the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : |y| = 1\}$. In other words, we consider the simplified Ericksen-Leslie system (1.3.1),

$$\partial_t v + (v\cdot\nabla)v + \nabla p - \Delta v = -\operatorname{div}(\nabla d \odot \nabla d), \tag{3.1.6}$$
$$\operatorname{div} v = 0, \tag{3.1.7}$$
$$\partial_t d + (v\cdot\nabla)d = \Delta d + |\nabla d|^2 d, \qquad |d| \equiv 1, \tag{3.1.8}$$

again with velocity field $v : \mathbb{T}^2 \times (0,T) \to \mathbb{R}^2$, pressure function $p : \mathbb{T}^2 \times (0,T) \to \mathbb{R}$ and director field $d : \mathbb{T}^2 \times (0,T) \to \mathbb{S}^2$. More precisely, redoing the energetic variational approach with the energy (3.1.5), the stationary points of the least action functional are solutions to (3.1.6)–(3.1.8) (see [83]). Comparing (3.1.1)–(3.1.3) and (3.1.6)–(3.1.8), the only difference is the Lagrange multiplier $|\nabla d|^2 d$ in the d-equation subject to the unitary constraint in the Ericksen-Leslie system whereas the Ginzburg-Landau system contains the penalization term $\frac{(1-|d_\varepsilon|^2)d_\varepsilon}{\varepsilon^2}$. Still the expected regularity of weak solutions to both systems differs.

Definition 3.1.2. *A pair* (v,d) *with*

$$v \in L^\infty\Big(0,T; \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2)\Big) \ \cap\ L^2\Big(0,T; \dot{W}^{1,2}_{\mathrm{div}}(\mathbb{T}^2)\Big),$$
$$d \in L^\infty\Big(0,T; W^{1,2}(\mathbb{T}^2)\Big) \ \cap\ W^{1,2}\Big(0,T; L^{4/3}(\mathbb{T}^2)\Big)$$

is a weak solution to (3.1.6)–(3.1.8) *subject to the initial conditions*

$$v(x,0) = v_0(x),\ \operatorname{div} v_0 = 0 \quad \text{on } \mathbb{T}^2 \times \{0\}, \tag{3.1.9}$$

$$d(x,0) = d_0(x),\ |d_0| \equiv 1 \quad \text{on } \mathbb{T}^2 \times \{0\}, \tag{3.1.10}$$

if it satisfies

$$\begin{aligned}
&\int_0^s \int_{\mathbb{T}^2} -v \cdot \partial_t \phi - v \otimes v : \nabla \phi + \nabla v : \nabla \phi - \nabla d \odot \nabla d : \nabla \phi \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad = \int_{\mathbb{T}^2} v_0 \cdot \phi \, \mathrm{d}x - \int_{\mathbb{T}^2} v(s) \cdot \phi \, \mathrm{d}x, \\
&\int_0^s \int_{\mathbb{T}^2} \partial_t d \cdot \psi + (v \cdot \nabla) d \cdot \psi + \nabla d : \nabla \psi - |\nabla d|^2 d \cdot \psi \, \mathrm{d}x \, \mathrm{d}t = 0
\end{aligned}$$

for all $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2 \times [0,T])$, *all* $\psi \in C^\infty(\mathbb{T}^2 \times [0,T])$ *and for almost every* $s \in [0,T]$. *Furthermore,* (v,d) *attains the initial data* $(v_0, d_0) \in L^2_{\mathrm{div}}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2)$ *in the weak sense, i.e.*

$$\int_{\mathbb{T}^2} v(t) \cdot \xi \, \mathrm{d}x \to \int_{\mathbb{T}^2} v_0 \cdot \xi \, \mathrm{d}x, \qquad \int_{\mathbb{T}^2} \nabla d(t) : \zeta \, \mathrm{d}x \to \int_{\mathbb{T}^2} \nabla d_0 : \zeta \, \mathrm{d}x$$

for all $\xi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2)$ *and* $\zeta \in C^\infty(\mathbb{T}^2)$ *as* $t \to 0^+$.

Note that the main difference between weak solutions $(v_\varepsilon, d_\varepsilon)$ to (3.1.1)–(3.1.3) and (v,d) of (3.1.6)–(3.1.8) is the loss of $L^2(0,T;W^{2,2})$-regularity of d. As a consequence, the equation for the director field (3.1.8) is also phrased in a distributional form rather than pointwise in $\mathbb{T}^2 \times [0,T]$. This loss of regularity poses the main obstacle when considering the limit of solutions $(v_\varepsilon, d_\varepsilon)$ for $\varepsilon \to 0^+$. Nevertheless, the following main result holds true:

Theorem 3.1.2 ([48], Theorem 2.1 and 2.2)**.** *Let* $(v_0, d_0) \in \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2)$ *with* $|d_0| = 1$ *a.e. and* $(v_\varepsilon, d_\varepsilon)_{0<\varepsilon\le 1}$ *be the family of unique weak solutions to the initial value problem* (3.1.1)–(3.1.4) *with initial data* $(v_\varepsilon(0), d_\varepsilon(0)) = (v_0, d_0)$. *Then there exists a subsequence* $(\varepsilon_j)_j$ *with* $\lim_{j\to\infty} \varepsilon_j = 0^+$ *such that*

$$(v_{\varepsilon_j}, d_{\varepsilon_j}) \rightharpoonup^* (v,d) \qquad \text{in } L^\infty(0,T;L^2_{\mathrm{div}}(\mathbb{T}^2)) \times L^\infty(0,T;W^{1,2}(\mathbb{T}^2))$$

as well as pointwise a.e. on $\mathbb{T}^2 \times [0,T]$ *with* (v,d) *being a weak solution to the initial value problem in the sense of Definition 3.1.2. Moreover, the weak solution* (v,d) *satisfies the energy inequality*

$$\int_{\mathbb{T}^2} |v|^2(t) + |\nabla d|^2(t) \, \mathrm{d}x + 2 \int_0^t \int_{\mathbb{T}^2} |\nabla v|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2 \, \mathrm{d}x \, \mathrm{d}t \le \int_{\mathbb{T}^2} |v_0|^2 + |\nabla d_0|^2 \, \mathrm{d}x$$

for almost all $t \in [0,T]$.

The striking fact is that although we only have weak convergence in $L^\infty(0,T;L^2)$ available for the gradient ∇d, we can still *circumvent* the problems with nonlinearities in the Ericksen-Leslie system. The main issue is represented by the limiting behavior of the stress tensor

$$\operatorname{div}(\nabla d_\varepsilon \odot \nabla d_\varepsilon)$$

in the momentum equation of (3.1.1). In general, we only have

$$\nabla d_\varepsilon \odot \nabla d_\varepsilon \quad \rightharpoonup^* \quad \nabla d \odot \nabla d + M \qquad \text{as measures}$$

with a Radon measure $M \in L^\infty(0,T;\mathcal{M}(\mathbb{T}^2))$. The strategy is *not* to show $M = 0$ but that the weak formulation of (3.1.6)–(3.1.8) is satisfied by (v,d). In turn, this implies consequences on the structure of M which we consider in Remark 3.1.7 and Remark 3.1.8. Some further remarks on Theorem 3.1.2 are in order:

Remark 3.1.3 (Failure of strong convergence)**.** *The convergence statement $d_\varepsilon \rightharpoonup^* d$ in $L^\infty(0,T;W^{1,2}(\mathbb{T}^2))$ cannot be improved to strong convergence in $C([0,T];W^{1,2}(\mathbb{T}^2))$ in general. Indeed, if this was the case, the Dirichlet energy density $\frac{1}{2}|\nabla d|^2$ would be equi-integrable in time. In particular, for every $\delta > 0$, we could find a radius $R > 0$ small enough such that*

$$\sup_{x_0\in\mathbb{T}^2, 0\le t\le T} \frac{1}{2}\int_{B_R(x_0)} |\nabla d|^2 \le \delta.$$

The theory of Struwe-like solutions for system (3.1.6)–(3.1.8) *(see [41, 58]) then implies the smoothness of solutions as long as this condition is satisfied for a specific $\delta = \delta_0 > 0$. But in general, there exist solutions which become singular in finite time (see [44, 50]).*

Remark 3.1.4 ([48], Uniqueness)**.** *As for the harmonic map heat flow, we do not know whether the solution (v,d) constructed in Theorem 3.1.2 is a solution in the sense of Struwe in [78]. In particular, the energy is not known to be nonincreasing so far. For the harmonic map heat flow, Bertsch et al. [8] and Topping [80] proved the existence of infinitely many weak solutions with conserved but increasing energy at certain time steps. The same behavior may be possible for weak solutions (v,d) here.*

Remark 3.1.5 ([48], Stability with respect to initial data)**.** *Theorem 3.1.2 remains true for a sequence of initial data*

$$(v_0^\varepsilon, d_0^\varepsilon)_\varepsilon \to (v_0, d_0)$$

strongly in $\dot{L}^2_{\mathrm{div}}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2)$. However, if only weak convergence is given, a result like Theorem 3.1.2 may not be available in general since various oscillation and concentration effects may occur.

We briefly sketch the proof of Theorem 3.1.2 and the ingredients entering in Sections 3.1.1–3.1.4. First, we establish the energy law and derive *a-priori* bounds for solutions $(v_\varepsilon, d_\varepsilon)$. Despite standard arguments, we observe that the gradient flow structure of the d_ε-equation (3.1.3) interacts with the solenoidality condition of the test functions in the momentum equation (see Lemma 3.1.3). This is a bit subtle but necessary in the overall argument. More involved, we recover arguments from the theory of (time-independent) approximated harmonic maps which guarantee strong convergence of ∇d_ε at least on a large set of $\mathbb{T}^2$. The final crucial point is to benefit from this fact in the time-dependent problem: We fix a time t and obtain that

$$\nabla d_\varepsilon(t) \odot \nabla d_\varepsilon(t) \quad \rightharpoonup^* \quad \nabla d(t) \odot \nabla d(t) + M(t) \qquad \text{as measures}$$

up to a subsequence where the support of M is finite. The idea is to cut out the small part where M does not vanish and then recover the weak formulation in the momentum equation. Termed “concentration-cancellation”, this technique was used in [23] with respect to the stationary Euler equations. The crucial point consists of respecting the solenoidality of the test functions for the momentum equations during the cut-off procedure.

3.1.1 Energy law and *a-priori* estimates

In order to prove Theorem 3.1.2, we begin by establishing (mostly standard) *a-priori* estimates for weak solutions $(v_\varepsilon, d_\varepsilon)$ to (3.1.1)–(3.1.4). As for most systems arising from physics, this is done by employing the energy law associated to the system and, secondly, using the structure of the equations to derive duality estimates for time derivatives. We have the following

Proposition 3.1.1. *Let $(v_\varepsilon, d_\varepsilon)$ be a weak solution to (3.1.1)–(3.1.4) for $\varepsilon > 0$ with initial data $(v_0, d_0) \in \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2)$ and $|d_0| = 1$ a.e. Then the following energy law holds true:*

$$\begin{aligned}\int_{\mathbb{T}^2} |v_\varepsilon(t)|^2 + |\nabla d_\varepsilon(t)|^2 + \frac{1}{2\varepsilon^2}(1-|d_\varepsilon(t)|^2)^2 + 2\int_0^t\int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 + \left|\Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1-|d_\varepsilon|^2)d_\varepsilon\right|^2 \\ = \int_{\mathbb{T}^2} |v_0|^2 + |\nabla d_0|^2 =: 2E_0,\end{aligned} \tag{3.1.11}$$

for almost all $t \in [0,T]$.

Proof. Recalling the given regularity for weak solutions by Definition 3.1.1, we see by Ladyzhenskaya's inequality that $v_\varepsilon, \nabla d_\varepsilon \in L^4(0,T;L^4(\mathbb{T}^2))$ holds true. In view of the momentum equation (3.1.1), this implies that $\partial_t v_\varepsilon \in L^2(0,T;(\dot{W}^{1,2}(\mathbb{T}^2))^*)$ holds true. By a density argument, i.e., approximating the function $(x,s) \mapsto u(x,s)\chi_{[0,t]}$, we can therefore test the momentum equation (3.1.1) by $u\chi_{(0,t)}$ and obtain

$$\int_{\mathbb{T}^2} \frac{|v_\varepsilon(t)|^2}{2} + \int_0^t\int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 = \int_{\mathbb{T}^2} \frac{|v_0|^2}{2} + \int_0^t\int_{\mathbb{T}^2} \nabla d_\varepsilon \odot \nabla d_\varepsilon : \nabla v_\varepsilon.$$

Here we used that

$$\int_{\mathbb{T}^2} v_\varepsilon \otimes v_\varepsilon : \nabla v_\varepsilon = \int_{\mathbb{T}^2} v_\varepsilon^i v_\varepsilon^j \partial_j v_\varepsilon^i = \int_{\mathbb{T}^2} v_\varepsilon^j \frac{\partial_j}{2}|v_\varepsilon|^2 \overset{\mathrm{div}\, v_\varepsilon = 0}{=} 0.$$

For the director equation (3.1.3), we use test functions approximating $\left(-\Delta d_\varepsilon - \frac{(1-|d_\varepsilon|^2)d_\varepsilon}{\varepsilon^2}\right)\chi_{(0,t)}$ such that

$$\begin{aligned}\int_{\mathbb{T}^2} \frac{|\nabla d_\varepsilon(t)|^2}{2} + \frac{(1-|d_\varepsilon(t)|^2)^2}{4\varepsilon^2} + \int_0^t\int_{\mathbb{T}^2} \left|\Delta d_\varepsilon + \frac{(1-|d_\varepsilon|^2)d_\varepsilon}{\varepsilon^2}\right|^2 \\ = \int_{\mathbb{T}^2} \frac{|\nabla d_0|^2}{2} + \int_0^t\int_{\mathbb{T}^2} (v_\varepsilon \cdot \nabla)d_\varepsilon \cdot \Delta d_\varepsilon.\end{aligned}$$

Again, we have $\int_{\mathbb{T}^2}(v_\varepsilon \cdot \nabla)d_\varepsilon \cdot \frac{(1-|d_\varepsilon|^2)d_\varepsilon}{\varepsilon^2} = 0$ since $\mathrm{div}\, v_\varepsilon = 0$. Finally, we use integration by parts to see

$$\begin{aligned}\int_{\mathbb{T}^2} (v_\varepsilon \cdot \nabla)d_\varepsilon \cdot \Delta d_\varepsilon &= \int_{\mathbb{T}^2} v_\varepsilon^j \partial_j d_\varepsilon^i \partial_k^2 d_\varepsilon^i = \int_{\mathbb{T}^2} v_\varepsilon^j \left[\partial_k(\partial_j d_\varepsilon^i \partial_k d_\varepsilon^i) - \partial_j\partial_k d_\varepsilon^i \partial_k d_\varepsilon^i\right] \\ &= -\int_{\mathbb{T}^2} \partial_k v_\varepsilon^j \partial_j d_\varepsilon^i \partial_k d_\varepsilon^i - \int_{\mathbb{T}^2} v_\varepsilon \cdot \frac{\nabla}{2}|\nabla d_\varepsilon|^2 = -\int_{\mathbb{T}^2} \nabla v_\varepsilon : \nabla d_\varepsilon \odot \nabla d_\varepsilon.\end{aligned}$$

The sum of both equalities yields the energy law. ∎

Note that we benifited from the fact that $|d_0| \equiv 1$ almost everywhere on the right-hand side of (3.1.11). As $\varepsilon \to 0^+$, it can already be seen that $|d_\varepsilon| \to 1$ in some L^p-space. However, the gradient structure of (3.1.3) gives more information about d_ε, namely the boundedness by 1 via a maximum principle (similar calculations can be found in [3, 64]):

Lemma 3.1.1 ([48], Lemma 1). *Let $(v_\varepsilon, d_\varepsilon)$ be a weak solution to* (3.1.1)–(3.1.4). *Then d_ε satisfies*

$$|d_\varepsilon(x,t)| \leq 1$$

for almost every $(x,t) \in \mathbb{T}^2 \times [0,T]$.

Proof. For $k \in \mathbb{N}$ we define the auxiliary function $h_\varepsilon^k : \mathbb{T}^2 \times [0,T] \to \mathbb{R}$ by

$$h_\varepsilon^k(x,t) = \begin{cases} k^2 - 1 & \text{for } k < |d_\varepsilon(x,t)|, \\ |d_\varepsilon(x,t)|^2 - 1 & \text{for } 1 < |d_\varepsilon(x,t)| \leq k, \\ 0 & \text{for } |d_\varepsilon(x,t)| \leq 1. \end{cases}$$

Notice that h_ε^k is weakly differentiable by the chain rule for Lipschitz and Sobolev functions. Then, by (3.1.3), we have

$$\begin{aligned} \partial_t h_\varepsilon^k + v_\varepsilon \cdot \nabla h_\varepsilon^k &= \Delta h_\varepsilon^k - 2\chi_{\{1<|d_\varepsilon|\leq k\}} \left(|\nabla d_\varepsilon|^2 + \frac{1}{\varepsilon^2}(|d_\varepsilon|^2 - 1)|d_\varepsilon|^2 \right) \\ &\leq \Delta h_\varepsilon^k \end{aligned}$$

in the weak sense. Next, we multiply the differential inequality by h_ε^k, and an integration by parts yields (due to the periodicity of $\mathbb{T}^2$ and $|d_0| \equiv 1$)

$$\frac{1}{2}\int_{\mathbb{T}^2} |h_\varepsilon^k(t)|^2 + \int_0^t \int_{\mathbb{T}^2} |\nabla h_\varepsilon^k|^2 \leq - \int_0^t \underbrace{\int_{\mathbb{T}^2} v_\varepsilon \cdot \frac{\nabla}{2} |h_\varepsilon^k|^2}_{=0} = 0.$$

This can only be true if $h_\varepsilon^k = 0$ a.e. on $\mathbb{T}^2 \times [0,T]$ and therefore the assertion follows. □

Collecting the information of the energy law and Lemma 3.1.1, we deduce the following *a-priori* bounds where $C > 0$ is independent of $\varepsilon > 0$:

$$\|v_\varepsilon\|_{L_t^\infty \dot{L}_x^2} \leq C, \tag{3.1.12}$$

$$\|v_\varepsilon\|_{L_t^2 \dot{W}_x^{1,2}} \leq C, \tag{3.1.13}$$

$$\left\|1 - |d_\varepsilon|^2\right\|_{L_t^\infty L_x^2} \leq C\varepsilon, \tag{3.1.14}$$

$$\|d_\varepsilon\|_{L_{t,x}^\infty} \leq 1, \tag{3.1.15}$$

$$\|\nabla d_\varepsilon\|_{L_t^\infty L_x^2} \leq C, \tag{3.1.16}$$

$$\|\partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon\|_{L_t^2 L_x^2} = \left\| \Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1 - |d_\varepsilon|^2) d_\varepsilon \right\|_{L_t^2 L_x^2} \leq C. \tag{3.1.17}$$

Ladyzhenskaya's inequality (Lemma 2.0.2) also implies

$$\|v_\varepsilon\|_{L_t^4 L_x^4} \leq C. \tag{3.1.18}$$

In order to obtain the later needed strong convergence results, we make use of the generalized Aubin-Lions Lemma.

Lemma 3.1.2 ([73], Lemma 7.7)**.** *Let V_1, V_2, V_3 be Banach spaces, where V_1 is separable, reflexive and compactly embedded into V_2 as well as $V_2 \subset V_3$ continuously. Then the embedding*

$$L^p(0,T;V_1) \cap W^{1,q}(0,T;V_3) \subset L^p(0,T;V_2)$$

is compact for $1 < p < +\infty$ and $1 \leq q \leq \infty$.

We thus look for estimates on the time-derivatives $(\partial_t v_\varepsilon, \partial_t d_\varepsilon)$. A comparison of the above list of estimates and (3.1.3) shows that

$$\|\partial_t d_\varepsilon\|_{L_t^2 L_x^{\frac{4}{3}}} \leq C$$

holds true uniformly in $\varepsilon > 0$. For the time-derivative of the velocity field $\partial_t v_\varepsilon$, we could deduce in similar fashion a bound in $L^2(0,T;(C^1_{\mathrm{div}}(\mathbb{T}^2))^*)$. However, it is not sufficient for our purposes.

Lemma 3.1.3. *For a weak solution $(v_\varepsilon, d_\varepsilon)$ of (3.1.1)–(3.1.4), it holds*

$$\|\partial_t v_\varepsilon\|_{L_t^2 X_s^*} \leq C \tag{3.1.19}$$

for $X_s := W^{1,s}_{\mathrm{div}}(\mathbb{T}^2)$ for any $s > 2$ independently of $0 < \varepsilon \leq 1$.

Proof. First, note that for $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2)$ one has

$$\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2} (\nabla d_\varepsilon)^\top (1-|d_\varepsilon|^2) d_\varepsilon \cdot \phi = -\frac{1}{4\varepsilon^2}\int_{\mathbb{T}^2} \nabla(1-|d_\varepsilon|^2)^2 \cdot \phi = 0.$$

Secondly, we employ the identity $\mathrm{div}(\nabla d_\varepsilon \odot \nabla d_\varepsilon) = \nabla \frac{|\nabla d_\varepsilon|^2}{2} + (\nabla d_\varepsilon)^\top \Delta d_\varepsilon$. Definition 3.1.1 implies that there exists $\partial_t v_\varepsilon \in L^2(0,T;(\dot{W}^{1,2}_{\mathrm{div}}(\mathbb{T}^2))^*)$ such that

$$\int_0^s \langle \partial_t v_\varepsilon, \phi\rangle_{(\dot{W}^{1,2})^* \times \dot{W}^{1,2}} + \int_0^s \int_{\mathbb{T}^2} -v_\varepsilon \otimes v_\varepsilon : \nabla\phi + \nabla v_\varepsilon : \nabla \phi - \nabla d_\varepsilon \odot \nabla d_\varepsilon : \nabla \phi = 0$$

for every $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2 \times [0,T])$ and a.e. $s \in [0,T]$. Therefore we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2\times[0,T]} \langle \partial_t v_\varepsilon, \phi\rangle\right| \\
&\leq \left|\int_{\mathbb{T}^2\times[0,T]} v_\varepsilon \otimes v_\varepsilon : \nabla\phi\right| + \left|\int_{\mathbb{T}^2\times[0,T]} \nabla v_\varepsilon : \nabla\phi\right| \\
&\quad + \left|\int_{\mathbb{T}^2\times[0,T]} (\nabla d_\varepsilon)^\top \left(\Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1-|d_\varepsilon|^2)d_\varepsilon\right)\cdot\phi\right| \\
&\leq \left(\|v_\varepsilon\|^2_{L_t^4 L_x^4} + \|v_\varepsilon\|_{L_t^2 \dot{W}_x^{1,2}} + \|\nabla d_\varepsilon\|_{L_t^\infty L_x^2} \cdot \left\|\Delta d_\varepsilon + \frac{1}{\varepsilon^2}(1-|d_\varepsilon|^2)d_\varepsilon\right\|_{L_t^2 L_x^2}\right) \times \\
&\quad \times \left(\|\phi\|_{L_t^2 W^{1,2}_{\mathrm{div}}} + \|\phi\|_{L_t^2 C_{\mathrm{div}}}\right) \\
&\leq C\left(\|\phi\|_{L_t^2 W^{1,2}_{\mathrm{div}}} + \|\phi\|_{L_t^2 C_{\mathrm{div}}}\right)
\end{aligned}$$

by the *a-priori* estimates (3.1.13), (3.1.16), (3.1.17) and (3.1.18). The assertion follows since $X_s = W^{1,s}_{\mathrm{div}}(\mathbb{T}^2) \subset W^{1,2}_{\mathrm{div}}(\mathbb{T}^2) \cap C_{\mathrm{div}}(\mathbb{T}^2)$ is true for any $s > 2$. □

We point out that we benefited from the interaction of solenoidal test functions and the gradient flow structure of the system. As a consequence of the above estimates, we can choose a subsequence $(\varepsilon_i)_{i\in\mathbb{N}} \subset (0,1]$ with $\lim_{i\to\infty} \varepsilon_i = 0^+$ such that

$$\begin{aligned}
v_{\varepsilon_i} &\to v && \text{in } L^2(\mathbb{T}^2 \times [0,T]) \text{ and a.e.,} && (3.1.20)\\
\nabla v_{\varepsilon_i} &\rightharpoonup \nabla v && \text{in } L^2(\mathbb{T}^2 \times [0,T]), && (3.1.21)\\
\partial_t v_{\varepsilon_i} &\rightharpoonup \partial_t v && \text{in } L^2(0,T;X_s^*) \text{ for } s>2, && (3.1.22)\\
d_{\varepsilon_i} &\to d && \text{in } L^p(\mathbb{T}^2 \times [0,T]) && (3.1.23)\\
& && \text{for any } p \in (1,\infty) \text{ and a.e.,} &&\\
|d_{\varepsilon_i}|^2 &\to 1 && \text{in } L^\infty(0,T;L^1(\mathbb{T}^2)) \text{ and a.e.,} && (3.1.24)\\
\nabla d_{\varepsilon_i} &\rightharpoonup^* \nabla d && \text{in } L^\infty(0,T;L^2(\mathbb{T}^2)), && (3.1.25)\\
\partial_t d_{\varepsilon_i} + (v_{\varepsilon_i}\cdot\nabla) d_{\varepsilon_i} &\rightharpoonup \partial_t d + (v\cdot\nabla)d && \text{in } L^2(\mathbb{T}^2 \times [0,T]). && (3.1.26)
\end{aligned}$$

Additionally, we can choose the subsequence such that

$$\begin{aligned}
v_{\varepsilon_i}(\cdot,t) &\to v(\cdot,t) && \text{in } L^2(\mathbb{T}^2) \text{ for a.a. } t\in[0,T], && (3.1.27)\\
d_{\varepsilon_i}(\cdot,t) &\to d(\cdot,t) && \text{in } L^2(\mathbb{T}^2) \text{ for a.a. } t\in[0,T]. && (3.1.28)
\end{aligned}$$

3.1.2 Regularity estimates for fixed times t

For the time being, we follow the idea of fixing a time $t \in [0,T]$ and investigating limiting and regularity questions of solutions to (3.1.1)–(3.1.4). Formally, with $d_\varepsilon(t) = u_\varepsilon$, Equation (3.1.3) becomes

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon = \tau_\varepsilon \tag{3.1.29}$$

on $\mathbb{T}^2$ for some $\tau_\varepsilon \rightharpoonup \tau$ in $L^2(\mathbb{T}^2)$ and $u_\varepsilon \in W^{2,2}(\mathbb{T}^2)$ being a strong solution for $\varepsilon > 0$. The integrability of $(\tau_\varepsilon)_\varepsilon$ stems from the energy equality (3.1.11). As $\varepsilon \to 0^+$, we have a singular limit problem and we presume that $(u_\varepsilon)_\varepsilon$ converges to a so-called approximated harmonic map $u : \mathbb{T}^2 \to \mathbb{S}^2$, i.e.

$$\Delta u + |\nabla u|^2 u = \tau - (\tau\cdot u)u$$

in some sense. In general, strong convergence of $(u_\varepsilon)_\varepsilon$ in $W^{1,2}$ cannot be expected because of a (geometrical) phenomenom called bubbling (see [71, 82]). For our next purposes, we need to keep in mind that the appearence of Dirac measures in the energy (density)

$$e_\varepsilon(u_\varepsilon) := \frac{|\nabla u_\varepsilon|}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \tag{3.1.30}$$

may occur as $\varepsilon \to 0^+$.
However, the size of support of such defect measures in the energy density can actually be bounded. For this we use the fundamental idea of partial regularity for PDEs: If a certain scale-invariant intrinsic quantity (often times a rescaled form of the energy) is small on a subset of the domain, then solutions to the PDE must be smooth on this subset.

Actually, for (3.1.29), the energy itself is invariant under scaling. Equation (3.1.11) then motivates the assumption

$$\sup_{0<\varepsilon\leq 1}\int_{\mathbb{T}^2} e_\varepsilon(u_\varepsilon) \leq E_0. \tag{3.1.31}$$

Inspired by [61, 64], we show the following

Theorem 3.1.3 ([48], Theorem 3)**.** *Suppose that $(u_\varepsilon)_\varepsilon$ is a sequence of strong solutions to* (3.1.29) *with $0<\varepsilon\leq 1$ satisfying* (3.1.31). *Further, let $\tau_\varepsilon \rightharpoonup \tau$ in $L^2(\mathbb{T}^2)$ for $\varepsilon \to 0^+$ and $|u_\varepsilon| \leq 1$ for $\varepsilon > 0$. Then there exists an $\varepsilon_0 > 0$ such that if for $x_0 \in \mathbb{T}^2$ the condition*

$$\sup_{0<\varepsilon\leq 1}\int_{B_{r_1}(x_0)} e_\varepsilon(u_\varepsilon) \leq \varepsilon_0^2 \tag{3.1.32}$$

holds true for some $\operatorname{diam}\mathbb{T}^2 \geq r_1 > 0$*, there exists a subsequence with*

$$u_\varepsilon \to u \text{ strongly in } W^{1,2}(B_{r_1/4}(x_0))$$

for $\varepsilon \to 0^+$.

Proof. The proof follows [48] and is divided into four steps. Let $x_0 = 0$ without loss of generality.
Step 1: We show that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\frac{|x-y|}{\varepsilon} + \left(C\frac{|x-y|}{\varepsilon}\right)^{1/2} \quad \text{on } \mathbb{T}^2.$$

To do so, we use a splitting ansatz $u_\varepsilon = (u_\varepsilon - v_\varepsilon) + v_\varepsilon$, where v_ε solves

$$\Delta v_\varepsilon = \tau_\varepsilon$$

on $\mathbb{T}^2$ with $\int_{\mathbb{T}^2} v_\varepsilon = 0$. Theorem 2.0.3 gives $\|v_\varepsilon\|_{\dot{W}^{2,2}} \leq C\|\tau_\varepsilon\|_{L^2}$ and Morrey's embedding in two dimensions, $W^{2,2} \hookrightarrow C^{1/2}$, implies

$$|v_\varepsilon(x) - v_\varepsilon(y)| \leq C^{1/2}|x-y|^{1/2},$$

since $(\tau_\varepsilon)_\varepsilon$ is bounded in L^2. Moreover, we have $\|v_\varepsilon\|_{L^\infty} \leq C$. Secondly, it is

$$\Delta(u_\varepsilon - v_\varepsilon) = \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon.$$

Using a regularity result for the Poisson equation, [9, Lemma A.2.], we have

$$\begin{aligned}
\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty}^2 &\leq C\|u_\varepsilon - v_\varepsilon\|_{L^\infty}\frac{\|(1-|u_\varepsilon|^2)u_\varepsilon\|_{L^\infty}}{\varepsilon^2} \leq \frac{C}{\varepsilon^2}(\|u_\varepsilon\|_{L^\infty} + \|v_\varepsilon\|_{L^\infty}) \\
&\leq \frac{C}{\varepsilon^2}(1+\|\tau_\varepsilon\|_{L^2})
\end{aligned}$$

because $|u_\varepsilon| \leq 1$ holds true. Hence, we have

$$|u_\varepsilon(x) - v_\varepsilon(x) - (u_\varepsilon(y) - v_\varepsilon(y))| \leq \frac{C}{\varepsilon}|x-y|$$

Since $\varepsilon \le 1$, this shows the assertion.
Step 2: We use the Hölder continuity to show that $|u_\varepsilon(x)| \ge \frac{1}{2}$ on B_{r_1}. On the contrary, assume there existed some $x_1 \in B_{r_1}$ with $|u_\varepsilon(x_1)| < 1/2$. Due to the Hölder estimate from Step 1, we have, for $x \in B_{\varepsilon\theta_0}(x_1)$, that

$$|u_\varepsilon(x)| \le \frac{3}{4}$$

provided $0 < \theta_0 < \min\left\{\frac{1}{64C}, r_1\right\}$. Therefore it follows

$$\int_{B_{\theta_0\varepsilon}(x_1)} \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \ge \left(\frac{7}{16}\right)^2 \frac{\theta_0^2\varepsilon^2\pi}{4\varepsilon^2} = \left(\frac{7}{32}\right)^2 \theta_0^2\pi$$

which contradicts the assumption that

$$\int_{B_{\theta_0\varepsilon}(x_1)} \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \le \int_{B_{r_1}(0)} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \varepsilon_0^2$$

for a chosen sufficiently small $\varepsilon_0 > 0$.
Step 3: We use $|u_\varepsilon| \ge \frac{1}{2}$ on B_{r_1} to engage the polar decomposition

$$u_\varepsilon = |u_\varepsilon|\frac{u_\varepsilon}{|u_\varepsilon|} =: \rho_\varepsilon\psi_\varepsilon.$$

Notice that $|\psi_\varepsilon| \equiv 1$ as well as

$$|\nabla\psi_\varepsilon| + |\nabla\rho_\varepsilon| \lesssim |\nabla u_\varepsilon| \lesssim |\nabla\psi_\varepsilon| + |\nabla\rho_\varepsilon|.$$

Multiplying (3.1.29) by ψ_ε and applying the multiplication operator $\frac{1}{\rho_\varepsilon}((\cdot) - (\psi_\varepsilon \cdot (\cdot))\psi_\varepsilon)$ to (3.1.29), we obtain the system of equations

$$\Delta\rho_\varepsilon + \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) - \rho_\varepsilon|\nabla\psi_\varepsilon|^2 = \tau_\varepsilon\psi_\varepsilon =: g_\varepsilon \tag{3.1.33}$$

$$\Delta\psi_\varepsilon = -|\nabla\psi_\varepsilon|^2\psi_\varepsilon - \frac{2}{\rho_\varepsilon}\nabla\psi_\varepsilon\nabla\rho_\varepsilon + \frac{1}{\rho_\varepsilon}(\tau_\varepsilon - (\tau_\varepsilon\psi_\varepsilon)\psi_\varepsilon) =: f_\varepsilon \tag{3.1.34}$$

on B_{r_1}, respectively. Considering the second equation, we take a cut-off function $\eta \in C_0^\infty(B_{r_1})$ such that $\eta \equiv 1$ on $B_{r_1/2}$ with $\|\nabla\eta\|_{L^\infty} \lesssim 1/r_1$ and $\|\nabla^2\eta\|_{L^\infty} \lesssim 1/r_1^2$. Then the function $\eta\psi_\varepsilon$ satisfies

$$\Delta(\eta\psi_\varepsilon) = \eta f_\varepsilon + 2\nabla\eta\cdot\nabla\psi_\varepsilon + \Delta\eta\psi_\varepsilon =: \tilde{f}_\varepsilon.$$

Hence, we again employ estimates from the theory of elliptic equations [34, Corollary 9.10] by

$$\begin{aligned}\left\|\nabla^2(\eta\psi_\varepsilon)\right\|_{L^{\frac{4}{3}}} &\le C\left\|\tilde{f}_\varepsilon\right\|_{L^{\frac{4}{3}}} \le C\left(\|\eta f_\varepsilon\|_{L^{\frac{4}{3}}} + \frac{\|\nabla\psi_\varepsilon\|_{L^2}}{r_1^{1/3}} + \frac{\|\psi_\varepsilon\|_{L^\infty}}{r_1^{1/2}}\right)\\ &\le C(\|\nabla\psi_\varepsilon\|_{L^2} + \|\nabla\rho_\varepsilon\|_{L^2})\|\eta\nabla\psi_\varepsilon\|_{L^4} + C\|\tau_\varepsilon\|_{L^2} + \frac{C}{r_1^{1/2}}.\end{aligned}$$

We point out that the value of the constant $C > 0$ in the previous inequality is independent of r_1. Observe that $\eta \nabla \psi_\varepsilon = \nabla(\eta\psi_\varepsilon) - \nabla\eta \otimes \psi_\varepsilon$ is valid. The Sobolev imbedding (for this version in $W_0^{1,4/3}$, see [28, Chapter 5, Theorem 3]) on B_{r_1} gives

$$\|\nabla(\eta\psi_\varepsilon)\|_{L^4(B_{r_1})} \lesssim \left\|\nabla^2(\eta\psi_\varepsilon)\right\|_{L^{\frac{4}{3}}(B_{r_1})}.$$

Again, the constant is independent of r_1, which can be seen by a scaling argument. Combining both above inequalities, we use the assumption $\|\nabla u_\varepsilon\|_{L^2} \le \sqrt{2}\varepsilon_0$ for small enough $\varepsilon_0 > 0$ to absorb the first term on the right-hand side of the elliptic inequality and get

$$\|\nabla(\eta\psi_\varepsilon)\|_{L^4} \lesssim \|\tau_\varepsilon\|_{L^2} + r_1^{-1/2}.$$

Thus $(\nabla\psi_\varepsilon)_\varepsilon$ is uniformly bounded in $L^4(B_{r_1/2}) \cap W^{1,\frac{4}{3}}(B_{r_1/2})$ and admits a strongly convergent subsequence in $W^{1,2}(B_{r_1/2})$.

Multiplying (3.1.33) by $1-\rho_\varepsilon$ and integrating by parts over some B_{r_2} with $0 < r_2 \le r_1/2$, we obtain

$$\begin{aligned}
&\int_{B_{r_2}} |\nabla\rho_\varepsilon|^2 + \int_{B_{r_2}} \frac{1}{\varepsilon^2}(1-\rho_\varepsilon^2)\rho_\varepsilon(1-\rho_\varepsilon) \\
&\quad = \int_{\partial B_{r_2}} (1-\rho_\varepsilon)\frac{\partial(1-\rho_\varepsilon)}{\partial r} + \int_{B_{r_2}} \tau_\varepsilon\psi_\varepsilon(1-\rho_\varepsilon) + \int_{B_{r_2}} \rho_\varepsilon(1-\rho_\varepsilon)|\nabla\psi_\varepsilon|^2 \\
&\quad \lesssim \int_{\partial B_{r_2}} (1-\rho_\varepsilon)\left|\frac{\partial\rho_\varepsilon}{\partial r}\right| + \left(\|\tau_\varepsilon\|_{L^2(B_{r_2})} + \|\tau_\varepsilon\|^2_{L^2(B_{r_2})} + 1\right)\|1-\rho_\varepsilon\|_{L^2(B_{r_2})}.
\end{aligned} \tag{3.1.35}$$

By Cavalieri's principle and the mean value theorem we see for some $r_2 \in [r_1/4, r_1/2]$ that

$$\int_{\partial B_{r_2}} (1-\rho_\varepsilon)\left|\frac{\partial\rho_\varepsilon}{\partial r}\right| \le \frac{C}{r_1}\int_{B_{r_2}} (1-\rho_\varepsilon)\left|\frac{\partial\rho_\varepsilon}{\partial r}\right|$$

holds true. Returning to inequality (3.1.35) we have

$$\int_{B_{r_2}} |\nabla\rho_\varepsilon|^2 \lesssim (\|\nabla\rho_\varepsilon\|_{L^2(B_{r_2})} + 1)\|1-\rho_\varepsilon\|_{L^2(B_{r_2})} \lesssim \varepsilon,$$

which implies $\rho_\varepsilon \to 1$ strongly in $W^{1,2}(B_{r_1/4})$.

Step 4: Summarizing the information above, we have in particular

$$\begin{aligned}
\psi_\varepsilon \to \psi &\quad \text{in } W^{1,2}(B_{r_1/4}, \mathbb{R}^3), \\
\rho_\varepsilon \to \rho \equiv 1 &\quad \text{in } W^{1,2}(B_{r_1/4})
\end{aligned}$$

as well as pointwise a.e. This eventually yields

$$u_\varepsilon = \rho_\varepsilon\psi_\varepsilon \to \rho\psi = u$$

in $L^2(B_{r_1/4})$ and since $\rho_\varepsilon = |u_\varepsilon| \le 1$, we have

$$\nabla u_\varepsilon = \psi_\varepsilon \otimes \nabla\rho_\varepsilon + \rho_\varepsilon\nabla\psi_\varepsilon \quad \to \quad \psi \otimes \nabla\rho + \rho\nabla\psi = \nabla u$$

in $L^2(B_{r_1/4})$ due to the generalized dominated convergence theorem. □

3.1.3 The concentration set Σ

Theorem 3.1.3 required a smallness assumption (3.1.32) on the local energy of the sequence $(u_\varepsilon)_\varepsilon$. Of course in general, such an assumption is not satisfied. Take for example a sequence of Dirac measures $(\delta_n)_{n\in\mathbb{N}}$ on $[0,1]$ where δ_n is supported on the point $nr \bmod 1$ for an irrational number r. Clearly, the sequence does not satisfy a condition like (3.1.32), i.e.

$$\int_I \mathrm{d}\delta_n \leq \varepsilon$$

for $\varepsilon < 1$ for an open interval I in $[0,1]$ since there are infinitely many δ_n supported in any open interval of $[0,1]$ due to the irrationality of r. Hence, there is also no weak limit. On the other hand, it is easy to select a subsequence such that δ_{n_k} converges weakly to some δ_{x_0} for some $x_0 \in [0,1]$. Even more, the subsequence strongly converges to 0 outside of any open interval containing x_0 and trivially satisfies a smallness condition such as the one stated above.
Our task is therefore to determine the properties of the set where strong convergence of $(u_{\varepsilon_k})_k = (d_{\varepsilon_k}(t))_k$ is available at least for a subsequence. We see that strong convergence fails in finitely many (isolated) points. In order to do so we define, for a sequence $(u_\varepsilon)_\varepsilon$ solving (3.1.29), the set of singular points by

$$\Sigma := \bigcap_{0<r} \left\{ x_0 \in \mathbb{T}^2 : \liminf_{k\to\infty} \int_{B_r(x_0)} e_\varepsilon(u_\varepsilon) > \varepsilon_0^2 \right\}$$

where ε_0 is given by Theorem 3.1.3. Reading the above definition we realize that the set Σ contains exactly the points in $\mathbb{T}^2$ which will never allow assumption (3.1.32) being verified even for a subsequence, how small $r > 0$ might be chosen. But due to (3.1.31), Σ must be small.

Lemma 3.1.4 ([48], Lemma 2). *There exists a constant $K = K(E_0) > 0$, where E_0 is taken from* (3.1.31)*, such that*

$$\#\Sigma \leq K$$

holds true.

Proof. Choose a finite subset $A_N := \{x_l\}_{1\leq l\leq N} \subset \Sigma$ for $N \in \mathbb{N}$ with mutually disjoint balls $\{B_{r_l}(x_l)\}_l$. Since A_N is finite, there is a $k_0 \in \mathbb{N}$ such that

$$\varepsilon_0^2 < \int_{B_{r_l}(x_l)} e_\varepsilon(u_\varepsilon)$$

for all $k \geq k_0$ by construction of Σ. Thus we have

$$\#A_N = N < \frac{1}{\varepsilon_0^2} \sum_{l=1}^{N} \int_{B_{r_l}(x_l)} e_\varepsilon(u_\varepsilon) \leq \frac{E_0}{\varepsilon_0^2}.$$

due to the energy estimate (3.1.11). By the arbitrariness of A_N, the set Σ consists of at most $K := \left\lceil \frac{E_0}{\varepsilon_0^2} \right\rceil$ points. $\square$

As a consequence of the previous lemma, we find a subsequence of $(u_\varepsilon)_\varepsilon$ strongly converging on $\mathbb{T}^2\backslash\Sigma$.

Lemma 3.1.5 ([48], Lemma 3)**.** *Let $(u_\varepsilon)_\varepsilon$ be as in Theorem 3.1.3. Then there exists a subsequence such that*

$$\nabla u_\varepsilon \to \nabla u_\varepsilon$$

in $L^2_{\mathrm{loc}}(\mathbb{T}^2\backslash\Sigma)$.

Proof. Let $\{z_j\}_{j\in\mathbb{N}}$ be the set of rational points in $\mathbb{T}^2\backslash\Sigma$ and define

$$r_j := \sup\left\{ r > 0 : \liminf_{k\to\infty} \int_{B_r(z_j)} e_\varepsilon(u_\varepsilon) \le \varepsilon_0^2 \right\}. \tag{3.1.36}$$

In general the radii r_j might be too large to satisfy

$$\int_{B_{r_j}(z_j)} e_\varepsilon(u_\varepsilon) \le \varepsilon_0^2,$$

which is why we consider $\frac{4}{5}r_j$. In view of Theorem 3.1.3 we want to prove $\bigcup_{j\in\mathbb{N}} B_{r_j/5}(z_j) = \mathbb{T}^2\backslash\Sigma$.
Let $z \in \mathbb{T}^2\backslash\Sigma$ with $r_z > 0$ be such that

$$\liminf_{k\to\infty} \int_{B_{r_z}(z)} e_\varepsilon(u_\varepsilon) \le \varepsilon_0^2.$$

By density, we choose a z_{j_0} such that $|z_{j_0} - z| < \frac{r_z}{6}$. Thus we have $r_{j_0} \ge \frac{5}{6}r_z$ from (3.1.36) and therefore $|z_{j_0} - z| < \frac{r_{j_0}}{5}$.

Since the covering is countable we use a diagonal argument, Theorem 3.1.3 and Lemma 3.1.4 to extract a subsequence which fulfills the assertion. □

All in all, we improved the convergence of $(u_\varepsilon)_\varepsilon$ (up to a subsequence) from weakly in $W^{1,2}(\mathbb{T}^2)$ to

$$|\nabla u_\varepsilon|^2 \quad \rightharpoonup^* \quad |\nabla u|^2 + \sum_{i=1}^{K} a_i \delta_{x_i}$$

in measures. The defect measure of the energy density (without the penalization term) is therefore only caused by concentration. Oscillations are ruled out in total. The cause for this behavior is the information that the sequence $(u_\varepsilon)_\varepsilon$ stems from the solution of a critical elliptic problem, namely (3.1.29).

3.1.4 Limit passage as $\varepsilon \to 0^+$ and proof of Theorem 3.1.2

Using the previous results we now prove Theorem 3.1.2, i.e., we show that weak solutions of (3.1.1)–(3.1.3) converge to weak solutions of (3.1.6)–(3.1.8). In comparison to [48], we split the argument into different parts and first focus on the easier of the two main equations, the one for the director field d_ε (3.1.3).

Lemma 3.1.6. *Let $(v_\varepsilon, d_\varepsilon)_\varepsilon$ be a sequence of weak solutions to* (3.1.1)–(3.1.4) *satisfying* (3.1.20)–(3.1.28)*. Then (v, d) satisfies* (3.1.8) *weakly and $d(t) \rightharpoonup d_0$ in $W^{1,2}(\mathbb{T}^2)$.*

Proof. The proof imitates the idea of [3, 15]. For $a, b, c : \mathbb{T}^2 \to \mathbb{R}^3$ being weakly differentiable, we set $(a \wedge \nabla b) : \nabla c = \sum_j (a \wedge \partial_{x_j} b) \cdot \partial_{x_j} c$ where $\wedge$ denotes the vector product on $\mathbb{R}^3$. Let $\xi \in L^\infty(0, T; W^{1,2}(\mathbb{T}^2)) \cap L^\infty(\mathbb{T}^2 \times [0, T])$. Using a density argument, we can test (3.1.3) by $d_\varepsilon \wedge \xi$. The identity $d_\varepsilon \wedge \Delta d_\varepsilon = \operatorname{div}(d_\varepsilon \wedge \nabla d_\varepsilon)$ yields

$$\int_0^s \int_{\mathbb{T}^2} \big(d_\varepsilon \wedge (\partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon)\big) \cdot \xi + \int_0^s \int_{\mathbb{T}^2} (d_\varepsilon \wedge \nabla d_\varepsilon) : \nabla \xi = 0$$

for almost all $s \in [0, T]$. From the convergence statements (3.1.20), (3.1.23)– (3.1.26) and the bound of the maximum principle, $|d_\varepsilon| \leq 1$ a.e. for all $\varepsilon > 0$, we conclude that the limit of the weak formulation is

$$\int_0^s \int_{\mathbb{T}^2} \big(d \wedge (\partial_t d + (v \cdot \nabla) d)\big) \cdot \xi + \int_0^s \int_{\mathbb{T}^2} (d \wedge \nabla d) : \nabla \xi = 0.$$

Here we have $|d| \equiv 1$ a.e. Therefore all derivatives (in particular the first term involving $\partial_t d$ and $\partial_{x_j} d$ for $j = 1, 2$) are perpendicular to d a.e. Using this fact, setting $\xi = d \wedge \Phi$ with $\Phi \in C^\infty(\mathbb{T}^2 \times [0, T])$ and employing another density argument to test the above equation by ξ, we obtain

$$\int_0^s \int_{\mathbb{T}^2} (\partial_t d + (v \cdot \nabla) d) \cdot \Phi + \int_0^s \int_{\mathbb{T}^2} \nabla d : \nabla \Phi - \int_0^T \int_{\mathbb{T}^2} |\nabla d|^2 d \cdot \Phi = 0.$$

Here we exploited the Lagrange identity $(a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$ for the wedge-product. This shows that the limit (v, d) satisfies the director equation (3.1.8). $\square$

The whole trick of the above proof consists of canceling the penalizing term $\frac{(1-|d_\varepsilon|^2) d_\varepsilon}{\varepsilon^2}$ by using the properties of the vector product. In the end, the symmetry of $\mathbb{S}^2$ plays the key role here to get rid of the question if

$$\frac{(1 - |d_\varepsilon|^2) d_\varepsilon}{\varepsilon^2} \quad \to \quad |\nabla d|^2 d$$

holds true in some sense.
Such a method is not applicable in the momentum equation (3.1.6) and the stress tensor $\operatorname{div}(\nabla d_\varepsilon \odot \nabla d_\varepsilon)$ for two simple reasons: It is an equation governing the evolution of v_ε, and the test functions we are using in the weak formulation are solenoidal. The latter fact complicates other arguments, like cut-offs of test functions, as well. This is where we use the main idea of the proof: the concentration-cancellation technique from [23].
In order to do so we state following

Lemma 3.1.7. *For every $f \in \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2)$ there exists $g \in \dot{W}^{1,2}(\mathbb{T}^2)$ with*

$$f = \nabla^\perp g, \qquad \nabla^\perp = (-\partial_2, \partial_1)^\top.$$

Proof. We write f as Fourier expansion $f = \sum_{k \in \mathbb{Z}^2} \hat{f}_k e^{ik\cdot(\cdot)}$ with $\hat{f}_0 = 0$ since $\int_{\mathbb{T}^2} f = 0$. The solenoidality of f implies $k \cdot \hat{f}_k = 0$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$, which in turn implies $\hat{f}_k = (-k_2, k_1)^\top \lambda_k$ for some $\lambda_k \in \mathbb{C}$ and all $k \in \mathbb{Z}^2 \setminus \{0\}$. Then $g = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \lambda_k e^{ik(\cdot)}$ is the desired g since

$$\|\nabla g\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^2 |\lambda_k|^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{f}_k|^2 < \infty$$

holds true. $\square$

Remark 3.1.6. *Such a statement holds true for higher-order Sobolev spaces $W^{m,2}(\mathbb{T}^2)$ by an analogous proof.*

We briefly sketch our strategy for the remainder of the proof of Theorem 3.1.2: The goal is to verify an alternative weak version of (3.1.6) for a.e. time $t \in [0,T]$. We do so by looking at a "good" set of full measure of times $t \in [0,T]$ for which the quantities of the energy law (3.1.11) are bounded independently of ε. This allows us to view the sequence as solutions to (3.1.29) and to apply Theorem 3.1.3 and Lemma 3.1.5. Finally, we use the aforementioned concentration-cancellation procedure to cut out possible defect measures and verify in the end the weak formulation of (3.1.6) for general solenoidal test functions.

Proof of Theorem 3.1.2. We proceed similarly to [48]. Regarding Lemma 3.1.6, the remaining part is to show that (v,d) fulfills the momentum equation (3.1.6) in a weak sense. We set $\tau_\varepsilon := \partial_t d_\varepsilon + (v_\varepsilon \cdot \nabla) d_\varepsilon$ and $\tau := \partial_t d + (v \cdot \nabla) d$. Due to (3.1.20)–(3.1.28), the set

$$A := \left\{ t \in [0,T] : \liminf_{\varepsilon \to 0^+} \left(\|\partial_t v_\varepsilon(t)\|_{X_r^*} + \|\nabla v_\varepsilon(t)\|_{L^2} + \|\nabla d_\varepsilon(t)\|_{L^2} + \|\tau_\varepsilon(t)\|_{L^2} \right) < \infty \right\}$$

has full measure by Fatou's lemma, i.e., $|A| = T$ (recall $X_r = W^{1,r}_{\mathrm{div}}(\mathbb{T}^2)$ for $r > 2$). Without loss of generality, let A be such that $(v_\varepsilon, d_\varepsilon)(t) \to (v,d)(t)$ as in (3.1.27) and (3.1.28) for every $t \in A$. Fix $t \in A$. Now there exists a subsequence for which

$$\left(\partial_t v_{\varepsilon_j}, \nabla v_{\varepsilon_j}, \nabla d_{\varepsilon_j}, \tau_{\varepsilon_j}\right)(t) \rightharpoonup (\partial_t v, \nabla v, \nabla d, \tau)(t),$$

where we identified the limit in t by the strong convergence of $(v_{\varepsilon_j}(t), d_{\varepsilon_j}(t))_{j\in\mathbb{N}}$ in L^2. Since this is true for any subsequence, the full sequence $((\partial_t v_\varepsilon, \nabla v_\varepsilon, \nabla d_\varepsilon, \tau_\varepsilon)(t))_\varepsilon$ converges weakly.

Next, we take a test function $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2)$. Since ϕ is solenoidal, it is

$$\phi = \nabla^\perp \eta = (-\partial_2, \partial_1)^\top \eta$$

for some $\eta \in C^\infty(\mathbb{T}^2)$ (see Remark 3.1.6) if $\int_{\mathbb{T}^2} \phi = 0$. Similar to the argument in Lemma 3.1.3, we use the weak formulation of Definition 3.1.1 to deduce the existence of $\partial_t v_\varepsilon \in L^2(0,T;(\dot W^{1,2}_{\mathrm{div}}(\mathbb{T}^2))^*)$ such that

$$\begin{aligned} \int_{\mathbb{T}^2} \langle \partial_t v_\varepsilon(t), \phi \rangle_{(\dot W^{1,2}_{\mathrm{div}})^* \times \dot W^{1,2}_{\mathrm{div}}} + \int_{\mathbb{T}^2} v_\varepsilon(t) \otimes v_\varepsilon(t) : \nabla\phi + \int_{\mathbb{T}^2} \nabla v_\varepsilon(t) : \nabla \phi \\ - \int_{\mathbb{T}^2} \nabla d_\varepsilon(t) \odot \nabla d_\varepsilon(t) : \begin{pmatrix} -\partial_1\partial_2\eta & -\partial_2^2\eta \\ \partial_1^2\eta & \partial_1\partial_2\eta \end{pmatrix} = 0 \end{aligned} \tag{3.1.37}$$

is fulfilled w.l.o.g. at time $t \in A \setminus \{0\}$ by ϕ. Also, we have

$$\tau_\varepsilon(t) = \Delta d_\varepsilon(t) + \frac{(1 - |d_\varepsilon(t)|^2) d_\varepsilon(t)}{\varepsilon^2}$$

which is, taking into account the bound on τ_ε above, (3.1.29) for $d_\varepsilon(t) = u_\varepsilon$. Thus, by Lemma 3.1.5, there exists a subsequence $(v_\varepsilon, d_\varepsilon)_\varepsilon$, which in general depends on t, such that

$$\nabla d_\varepsilon(t) \to \nabla d(t)$$

in $L^2_{\mathrm{loc}}(\mathbb{T}^2\backslash\Sigma_t)$, where Σ_t is finite according to Lemma 3.1.4.

By density, it suffices to show the weak formulation (3.1.37) in the limit for all functions $\eta(x) = e^{ik\cdot x}$ with $k \in \mathbb{Z}^2$ where the case $k = (0,0)$ is trivial. First note that the only problematic terms are the ones related to $\partial_t v$ and $\nabla d \odot \nabla d$. However, choosing a smooth cut–off function ψ which vanishes in a neighborhood of Σ_t, we may pass to the limit with the test function $\nabla^\perp\eta(x) = \nabla^\perp\left(e^{ik\cdot x}\psi(x)\right)$, i.e.

$$\begin{aligned} \left\langle \partial_t v(t), \nabla^\perp\eta\right\rangle_{X_r^*,X_r} + \int_{\mathbb{T}^2} v(t)\otimes v(t) : \nabla\nabla^\perp\eta + \int_{\mathbb{T}^2} \nabla v(t) : \nabla\nabla^\perp\eta \\ - \int_{\mathbb{T}^2} \nabla d(t)\odot\nabla d(t) : \nabla\nabla^\perp\eta = 0. \end{aligned} \tag{3.1.38}$$

It remains to “fill the holes” and we do so by considering every point in Σ_t separately. To this end, observe that the equations (3.1.6) and (3.1.1) are covariant under rotations. To be more specific, let $0 = x_0 \in \Sigma_t$, without loss of generality, be the invariant point of the rotation. Taking a test function $\nabla^\perp\beta$ with $\operatorname{supp}\beta \subset B_r$, $r > 0$ small enough, the rotation of coordinates $Qy = x$ for $Q^\top = Q^{-1}$ yields

$$\int_{B_r} (\nabla d_\varepsilon \odot \nabla d_\varepsilon)(x,t) : \nabla\nabla^\perp\beta(x,t)\;\mathrm{d}x = \int_{B_r} (\nabla_y d_\varepsilon \odot \nabla_y d_\varepsilon)(Qy,t) : \nabla_y\nabla_y^\perp\beta(Qy,t)\;\mathrm{d}y \tag{3.1.39}$$

by a change of variables. Similar identities hold for all other terms and we have

$$\int f_n(x)\phi(x)\;\mathrm{d}x \to \int f(x)\phi(x)\;\mathrm{d}x \qquad \text{iff} \qquad \int f_n(Qy)\phi(Qy)\;\mathrm{d}y \to \int f(Qy)\phi(Qy)\;\mathrm{d}y.$$

For $r > 0$ sufficiently small we know by Lemma 3.1.5 that $(\nabla d_\varepsilon \odot \nabla d_\varepsilon)(\cdot,t)$ only concentrates in $x_0 = 0 \in B_r$ and so does $(\nabla_y d_\varepsilon \odot \nabla_y d_\varepsilon)(Q(\cdot),t)$ by the same token. Thanks to the rotational covariance, it is enough to consider test functions $h(x) = h(x_1)$ in a neighborhood of the concentration point $x_0 = 0$. We can do so by choosing $h(x) = h(x\cdot v)$ for some $v \in \mathbb{R}^2\setminus\{(0,0)\}$ and Q such that $Q^\top v$ is a multiple of e_1 in (3.1.39). To cut off the concentration point, define h_n for $n \in \mathbb{N}$ large enough by the elliptic ODE

$$h_n'' = (1 - \mathbb{1}_{(-1/n,1/n)})h'', \qquad h_n(-r) = h(-r),\; h_n(r) = h(r).$$

We properly localize the function h_n by considering $\eta_n(x_1,x_2) = h_n(x_1)\chi(x_1,x_2)$ with χ being smooth, $\chi \equiv 1$ on $B_{r/2}$ and zero outside of B_r (the whole construction is illustrated in Figure 3.1) . We set $\eta = h\chi$ respectively and note that

$$\nabla^2\eta_n = \nabla^2(h_n\cdot\chi) \to \nabla^2(h\cdot\chi) = \nabla^2\eta \qquad \text{almost everywhere on } \mathbb{T}^2$$

and by dominated convergence in any $L^p(\mathbb{T}^2)$, $1 \le p < \infty$; therefore $\eta_n \to \eta$ in $W^{2,p}(\mathbb{T}^2)$ (in particular $\nabla^\perp\eta_n \to \nabla^\perp\eta$ in $X_s = W^{1,s}_{\mathrm{div}}(\mathbb{T}^2,\mathbb{R}^2)$ for $2 < s < \infty$).

Choosing $\phi = \nabla^\perp\eta_n$ in (3.1.37), we are able to pass to the limit in ε since $\nabla\nabla^\perp\eta_n$ vanishes around the concentration point. The limit then reads

$$\begin{aligned} \left\langle \partial_t v(t), \nabla^\perp\eta_n\right\rangle_{X_r^*,X_r} + \int_{B_r} v(t)\otimes v(t) : \nabla\nabla^\perp\eta_n + \int_{B_r} \nabla v(t) : \nabla\nabla^\perp\eta_n \\ - \int_{B_r\setminus B_{r/2}} \nabla d(t)\odot\nabla d(t) : \nabla\nabla^\perp\eta_n - \int_{B_{r/2}} [\nabla d\odot\nabla d]_{2,1}h_n'' = 0. \end{aligned} \tag{3.1.40}$$

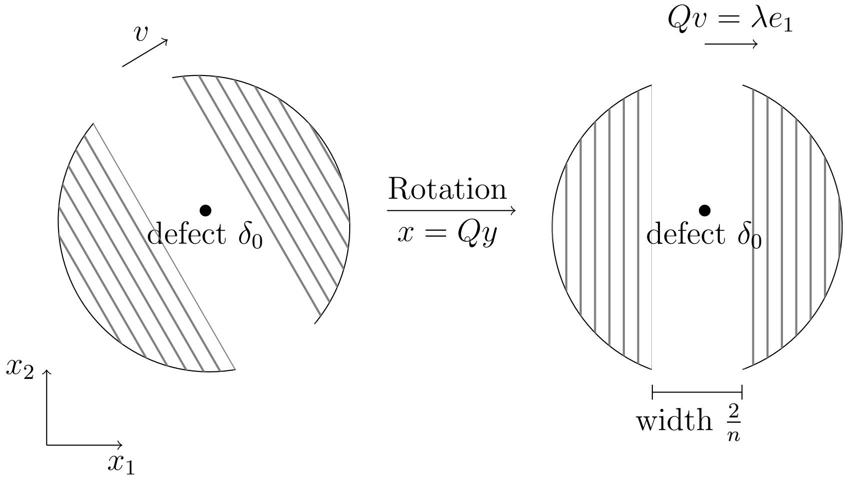

Figure 3.1: Cut-off of plane waves

with $[A]_{ij} = a_{ij}$ for $A = (a_{ij})_{ij} \in \mathbb{R}^{M\times N}$. Notice that we need the regularity of $\partial_t v \in L^2(0,T;X_r^*)$ here. As $n \to \infty$ we are able to replace η_n by η in the second, third and fourth term due to $v \otimes v \in L^2(\mathbb{T}^2, \mathbb{R}^2), \nabla v \in L^2(\mathbb{T}^2, \mathbb{R}^{2\times 2}), \nabla d \odot \nabla d \in L^1(\mathbb{T}^2 \backslash B_{r/2}, \mathbb{R}^{2\times 2})$ from (3.1.20)–(3.1.25). For the first term we have

$$\left\langle \partial_t v(t), \nabla^\perp \eta_n \right\rangle_{X_s^*, X_s} \quad \to \quad \left\langle \partial_t v(t), \nabla^\perp \eta \right\rangle_{X_s^*, X_s} \tag{3.1.41}$$

since $\partial_t v(t) \in X_s^*$ and $\nabla^\perp \eta_n \to \nabla^\perp \eta$ in X_s. In order to use Lebesgue's dominated convergence theorem for the last term of (3.1.40), we observe that

$$\begin{array}{rcll} [\nabla d \odot \nabla d]_{2,1} h_n'' & \to & [\nabla d \odot \nabla d]_{2,1} h'' & \text{a.e.} \\ \left| [\nabla d \odot \nabla d]_{2,1} h_n'' \right| & \leq & \left| [\nabla d \odot \nabla d]_{2,1} h'' \right| & \in L^1(B_{r/2}) \end{array}$$

is valid. This and $[\nabla \nabla^\perp \eta]_{2,1} = h''$ on $B_{r/2}$ yield (3.1.38) for $\eta = h\chi$.

By (3.1.39), we deduce that the weak formulation is also satisfied for test functions of the form $\nabla^\perp \eta(x) = \nabla^\perp \left(e^{ik\cdot x} \chi(x) \right)$, $k \in \dot{\mathbb{Z}}^2$, where χ is a proper chosen cut-off function around a concentration point. Combining this with (3.1.38) and using the density (for example in the $W^{3,2}(\mathbb{T}^2)$-topology, see also Remark 3.1.6) of $\{e^{ik\cdot(\cdot)} : k \in \mathbb{Z}^2\}$ in the space of test functions, we eventually obtain that (v, d) satisfies the weak formulation

$$\begin{aligned} \langle \partial_t v(t), \phi \rangle_{X_r^*, X_r} + \int_{\mathbb{T}^2} v(t) \otimes v(t) : \nabla \phi + \int_{\mathbb{T}^2} \nabla v(t) : \nabla \phi \\ - \int_{\mathbb{T}^2} \nabla d(t) \odot \nabla d(t) : \nabla \phi = 0 \end{aligned} \tag{3.1.42}$$

for every $\phi \in C^\infty_{\text{div}}(\mathbb{T}^2)$ and $t \in A$. As t was arbitrary and A has full measure, (3.1.42) holds for a.a. $t \in (0, T]$.

In order to deal with the time dependence we multiply (3.1.42) by $\zeta(t)$ with $\zeta \in C^\infty([0,T])$ and integrate over $[0, s]$ with $0 < s \leq T$. The density of $C^\infty_{\text{div}}(\mathbb{T}^2) \otimes C^\infty([0,T])$ in

$C^\infty_{\text{div}}(\mathbb{T}^2 \times [0,T])$ yields

$$\int_0^s \langle \partial_t v, \phi \rangle_{X_r^*, X_r} + \int_0^s \int_{\mathbb{T}^2} v \otimes v : \nabla\phi + \int_0^s \int_{\mathbb{T}^2} \nabla v : \nabla \phi \\ - \int_0^s \int_{\mathbb{T}^2} \nabla d \odot \nabla d : \nabla \phi = 0$$

for all $\phi \in C^\infty_{\text{div}}(\mathbb{T}^2 \times [0,T])$ and $\phi(T) = 0$. Furthermore, we know from

$$\int_0^s \langle \partial_t v_\varepsilon, \phi \rangle_{(\dot{W}^{1,2})^* \times \dot{W}^{1,2}} = \int_{\mathbb{T}^2} v_\varepsilon(s) \cdot \phi(s) - \int_{\mathbb{T}^2} v_0 \cdot \phi(0) - \int_0^s \int_{\mathbb{T}^2} v_\varepsilon \cdot \partial_t \phi$$

for every $\varepsilon > 0$ that we have

$$\int_0^s \langle \partial_t v, \phi \rangle_{X_r^*, X_r} = \int_{\mathbb{T}^2} v(s) \cdot \phi(s) - \int_{\mathbb{T}^2} v_0 \cdot \phi(0) - \int_0^s \int_{\mathbb{T}^2} v \cdot \partial_t \phi.$$

according to (3.1.20), (3.1.22), (3.1.27) and (3.1.28) for a.e. $s \in [0,T]$. From (3.1.20)–(3.1.26) we gain an improvement of regularity, i.e. $(v, \nabla d)$ lie in $C_w([0,T]; L^2(\mathbb{T}^2))$. In particular, the solution (v,d) attains the initial data (v_0, d_0). Hence (3.1.6)–(3.1.10) possesses a weak solution in the sense of Definition 3.1.2. The energy inequality follows from (3.1.11) and (3.1.20)–(3.1.28) as well as the lower semicontinuity of the norms with respect to weak convergence. □

In addition to the arguments in [48], we give some remarks on the proof above:

Remark 3.1.7. *As mentioned after the statement of Theorem 3.1.2, we did not achieve the weak formulation through an improved convergence result on approximative solutions* $(v_\varepsilon, d_\varepsilon)$. *The defect measure* $M \in L^\infty(0,T; \mathcal{M}(\mathbb{T}^2))$ *defined by*

$$\nabla d_\varepsilon \odot \nabla d_\varepsilon \quad \rightharpoonup^* \quad \nabla d \odot \nabla d + M$$

in the sense of Radon measures does not appear in the weak formulation of (3.1.6). *On the other hand, we have*

$$\int_0^s \int_{\mathbb{T}^2} -v \cdot \partial_t \phi - v \otimes v : \nabla \phi + \nabla v : \nabla \phi - (\nabla d \odot \nabla d + M) : \nabla \phi = 0$$

by standard limit passage in (3.1.1) *for any divergence-free test function* ϕ *and a.e.* $s \in [0,T]$. *Subtracting the above equation from the achieved weak formulation in Theorem 3.1.2, we obtain*

$$\int_0^s \int_{\mathbb{T}^2} M : \nabla \nabla^\perp \eta = 0$$

for all $\phi = \nabla^\perp \eta$ *and* $\eta \in C^\infty(\mathbb{T}^2)$. *For a fixed time* $t \in [0,T]$ *(a.e. in* $[0,T]$*), there exists a subsequence of* $(\nabla d_\varepsilon \odot \nabla d_\varepsilon)_\varepsilon$ *such that*

$$M(t) = \sum_{i=1}^K \begin{pmatrix} a_i & b_i \\ b_i & c_i \end{pmatrix} \delta_{x_i}$$

for points $x_i \in \mathbb{T}^2, i = 1, ..., K$ *by Lemma 3.1.5 and if we localize* η*, we have up to translation* $M(t) = A(t)\delta_0, A \in \mathbb{R}^{2\times 2}$ *with* A *being symmetric. Using test function* $\eta(x_1, x_2) = x_j^2 \chi(x_1, x_2), j = 1, 2, \chi \in C_0^\infty(B_r), r > 0$ *small and* $\chi = 1$ *on* $B_{r/2}$*, the identity*

$$\int_{\mathbb{T}^2} M(t) : \nabla\nabla^\perp \eta = \int_{\mathbb{T}^2} \begin{pmatrix} a & b \\ b & c \end{pmatrix} : \begin{pmatrix} -\partial_1\partial_2\eta & -\partial_2\partial_2\eta \\ \partial_1\partial_1\eta & \partial_1\partial_2\eta \end{pmatrix} \mathrm{d}\delta_0$$

implies

$$b = 0.$$

In other words, we have

$$\partial_1 d_\varepsilon(t) \cdot \partial_2 d_\varepsilon(t) \quad \to \quad \partial_1 d(t) \cdot \partial_2 d(t)$$

in the sense of distributions. By a local rotation by 45 *degrees, the statement* $|\partial_1 d_\varepsilon(t)|^2 - |\partial_2 d_\varepsilon(t)|^2 \quad \to \quad |\partial_1 d(t)|^2 - |\partial_2 d(t)|^2$ *follows as well with the conclusion* $a = c$.

Remark 3.1.8. *The above remark acually shows that the defect measure* $M(t)$ *must be a gradient up to a time-dependent subsequence, i.e.*

$$\operatorname{div} M(t) = \sum_{i=1}^{K} a_i(t) \nabla \delta_{x_i(t)}$$

in the sense of distributions for bounded measurable functions $a_i : [0, T] \to \mathbb{R}$ *and* $i = 1, ..., K$ *with* $K > 0$ *being the constant in Lemma 3.1.4. However, the dependence of* $a_i, x_i : [0, T] \to \mathbb{R} \times \mathbb{T}^2$ *is not quantifiable so far. We expect that the mapping* $t \mapsto x_i(t)$ *defines rectifiable-in-time curves.*

Remark 3.1.9 (Boundary conditions)**.** *Theorem 3.1.2 is phrased in the periodic domain* $\mathbb{T}^2$*. However, if we impose boundary conditions on a smooth bounded domain* Ω*, the observations on the defect measure and the local structure of the cut-off argument actually allow us to verify an analogous version of Theorem 3.1.2 on* Ω*. Indeed, the restriction to plane waves as test functions on* $\mathbb{T}^2$ *is not necessary. Similarly to Remark 3.1.7, we first verify the weak formulation for test function* $\eta = x_i^2\chi, i = 1, 2$*. The conclusion is that the defect consists of a gradient which is not seen by general divergence-free test functions in* $C^\infty_{0,\mathrm{div}}(\Omega)$*. The energy density* $|\nabla d_\varepsilon|^2$ *might concentrate on the boundary, but since functions in* $C_0^\infty(\Omega)$ *vanish close to the boundary, no contribution is seen in the weak formulation.*

Remark 3.1.10. *In Remark 3.1.7 we note that*

$$\partial_1 d_\varepsilon(t) \cdot \partial_2 d_\varepsilon(t) \to \partial_1 d(t) \cdot \partial_2 d(t), \qquad |\partial_1 d_\varepsilon(t)|^2 - |\partial_2 d_\varepsilon(t)|^2 \to |\partial_1 d(t)|^2 - |\partial_2 d(t)|^2$$

in the sense of distributions by usage of the momentum equation (3.1.6)*. But one cannot expect that the coupling to an additional equation should increase the behavior of solutions to the harmonic map heat flow. Actually, the above statements hold true for sequences of (approximate) harmonic maps themselves, which is related to the (coefficients of the) Hopf differential*

$$\mathcal{H} = |\partial_1 d|^2 - |\partial_2 d|^2 - 2i\partial_1 d \cdot \partial_2 d.$$

For the theory of harmonic maps on two-dimensional surfaces, see e.g. [38]. In particular, in [38, p. 134], the connection of the Hopf differential to Euler equations and the result of [29] is noted. Moreover, [24], an analogue of Theorem 3.1.2 is shown via observations on the Hopf differential for $\varepsilon > 0$.

3.2 A comparison to the two-dimensional incompressible Euler equations

In this section, we sketch a selected part of the mathematical theory of the two-dimensional Euler equations from fluid mechanics. We do so since we use an idea of the work of DiPerna and Majda [23], the concentration-cancellation method, as central ingredient to prove Theorem 3.1.2. The situation of inviscid fluid dynamics is compared to the above result on nematic liquid crystal flows. The intention is to outline the theory of so-called vortex sheet solutions to Euler equations without treating many of the technicalities which appear in there. Rather, an intuition is given on what is happening in the argument. The first and main results in this direction were settled in the late 1980's to early 1990's with the introduction of generalized Young measures (DiPerna-Majda measures) in [22], the investigation of concentration in measures in [23] and the existence result of vortex-sheet solutions with non-negative vorticity in [20]. In this presentation, we mainly follow [69].

The Euler equations are the standard model to describe incompressible fluid flows in the large Reynolds number limit. They read

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= 0, \\ \operatorname{div} v &= 0 \end{aligned} \tag{3.2.1}$$

for a velocity field $v : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ and $p : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ being the pressure. The system originates from Newton's second law and the conservation of mass and does not include internal friction of the fluid. Therefore, the system is not of parabolic type but rather hyperbolic (in some situations). Solutions to (3.2.1) will at best propagate the regularity fed into the system by v_0 but not smooth out over time. Although highly important in the topic of fluid mechanics, we ignore the effects resulting from boundaries and consider the whole space $\mathbb{R}^2$.

Given an initial state $v(x, 0) = v_0(x)$ with $v_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $\operatorname{div} v_0 = 0$, the solution of the initial value problem will formally preserve the kinetic energy

$$\int_{\mathbb{R}^2} \frac{|v(t)|^2}{2} = \int_{\mathbb{R}^2} \frac{|v_0|^2}{2} \tag{3.2.2}$$

for all times $t \geq 0$. Some other quantities such as the barycenter of mass are conserved as well (see [69, Chapter 1]). One of the difficult questions regarding Euler and Navier-Stokes equations is whether, or how, turbulence is modeled by them or how turbulent motion can be deduced from them. We will not enter in the discussion of turbulence itself as it has no fully consistent definition. But clearly, one speaks of the fluid being in the turbulent regime if the fluid can be observed swirling on various length scales with a seemingly random pattern.

From the mathematical point of view, this behavior leads to a fundamental quantity in (Newtonian) fluid mechanics: the vorticity. It describes the local spinning of the fluid and is given by the curl of the velocity,

$$\omega := \operatorname{curl} v = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Hence, $\omega : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ is a scalar quantity in two dimensions. Of course, it is then important to describe the evolution of vorticity along time and we derive the vorticity

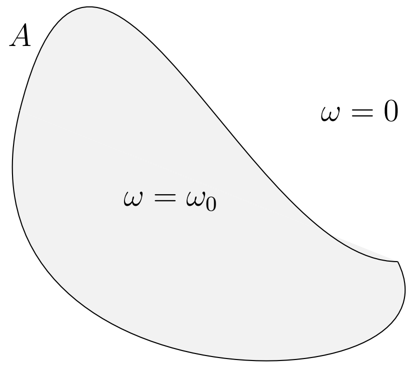

Figure 3.2: Vortex patch $\omega = \omega_0 \chi_A$

equation by applying the curl to (3.2.1),

$$\partial_t \omega + (v \cdot \nabla)\omega = 0. \tag{3.2.3}$$

The velocity v is not independent of the vorticity and can be recovered by the Biot-Savart law

$$v = K * \omega, \qquad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \tag{3.2.4}$$

for $x \in \mathbb{R}^2 \setminus \{0\}$ (see [69, Chapter 2]). Equation (3.2.3) is essentially a transport equation which depicts a seemingly simpler structure than the Euler equations itself. The difficulty of the subject becomes apparent with the non-local and nonlinear coupling of ω with itself through the Biot-Savart law. Nevertheless, with v being solenoidal, transport equations in general and (3.2.3) in particular preserve any L^p-norm over time. Given an initial vorticity function $\omega(x, 0) = \omega_0$ with $\omega_0 : \mathbb{R}^2 \to \mathbb{R}$, we formally have

$$\int_{\mathbb{R}^2} |\omega(t)|^p = \int_{\mathbb{R}^2} |\omega_0|^p \tag{3.2.5}$$

for a solution to (3.2.3) for any $1 \le p \le \infty$. Choosing different values of p, one wants to model different situations. For example, for $p = \infty$, the interest lies in so-called vortex patch solutions where ω_0 is given by (a multiple of) an indicator function of a smooth domain, see Figure 3.2. One important point then consists of the description of time-evolution of the boundary of the domain A. However, we focus on the other endpoint case $p = 1$. Actually, we look for even more singular data, namely measures. We require

$$\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap W^{-1,2}_{\text{loc}}(\mathbb{R}^2).$$

Such an initial datum is called a vortex sheet data (see [69, Chapter 9]).

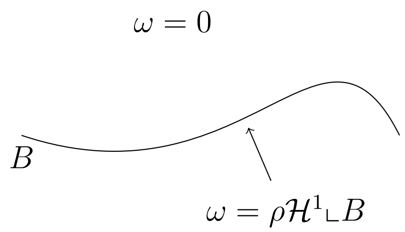

Figure 3.3: Vortex sheet

The actual motivation is the following: For very singular flows of inviscid fluids (jets, wakes), the vorticity is thought to concentrate on a one-dimensional curve, namely a weighted one-dimensional Hausdorff measure supported on a curve in $\mathbb{R}^2$ (see Figure 3.3). This regularity[1] is propagated by (3.2.2) and (3.2.5) and therefore bounded in time.

In view of the constraint $\operatorname{div} v = 0$, the velocity can be represented by a stream function $\psi : \mathbb{R}^2 \to \mathbb{R}$ through $v = \nabla^\perp \psi$ (this is the Helmholtz decomposition). Together with the definition of the velocity, we have

$$-\Delta\psi = \omega. \tag{3.2.6}$$

For vortex sheet initial data, we have a Poisson equation with right-hand side in L^1 or the space of Radon measures respectively. Notice the similarity of ψ to d in the director equation of the Ericksen-Leslie system (3.1.8) and (3.1.29) where the most singular term $|\nabla d|^2 d$ is also bounded just in L^1. Therefore, one might be interested in transferring arguments known for vortex-sheet solutions to the Ericksen-Leslie system. Since the proof of Theorem 3.1.2 is at the core more concerned with the (quasi-)stationary system (3.1.29), we are led to investigate time-independent solutions to (3.2.1).

3.2.1 Stationary solutions and concentration effects

In constrast to the Ericksen-Leslie system (3.1.6)–(3.1.8), it is easier to find non-trivial solutions to the Euler equations. If we ignore the dependence on time in (3.2.1) and add an external force $f : \mathbb{R}^2 \to \mathbb{R}^2$, we arrive at the stationary Euler equations

$$\operatorname{div}(v \otimes v) + \nabla p = f, \qquad \operatorname{div} v = 0 \tag{3.2.7}$$

and the corresponding stationary vorticity equation

$$\operatorname{div}(v\omega) = \operatorname{curl} f. \tag{3.2.8}$$

Since we are interested in singular effects, as concentration of measures depicts one, we introduce weak solutions:

Definition 3.2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. A velocity field $v \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ with $\operatorname{div} v = 0$ is called a weak solution to* (3.2.7) *if*

$$-\int_{\mathbb{R}^2} v \otimes v : \nabla\phi = \int_{\mathbb{R}^2} f \cdot \phi$$

holds true for every $\phi \in C^\infty_{0,\mathrm{div}}(\mathbb{R}^2)$.

[1]The velocity v lies in L^2 if and only if $\int_{\mathbb{R}^2} \omega = 0$ because of the Biot-Savart law. Otherwise v is at least locally in L^2 which still is bounded in time by a Gronwall inequality, see [69, Chapter 3].

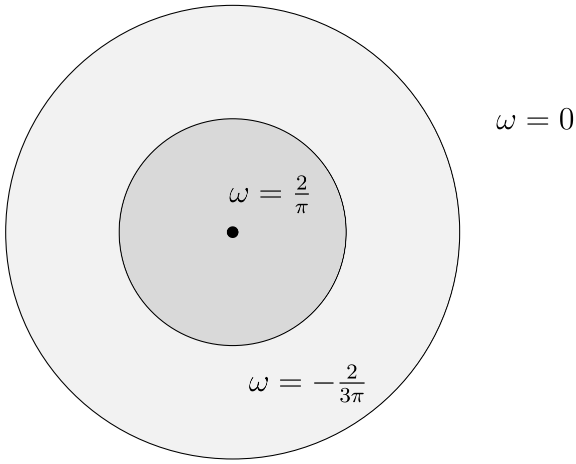

Figure 3.4: Phantom vortex

Once again, we get rid of the pressure function p by choosing a special class of solenoidal test functions for (3.2.7). For some forces f, there might not exist solutions to the above equations. On the other hand, letting $f = 0$, every smooth radial vorticity function $\omega(x) = \omega(r), r = |x|$ is a solution to (3.2.8). In order to see this, note that the Biot-Savart law for a radial vorticity reads ([69, Chapter 2])

$$v(x) = \frac{x^\perp}{r^2} \int_0^r s\omega(s)\,\mathrm{d}s. \tag{3.2.9}$$

Then we have

$$\operatorname{div}(v\omega)(x) = \left(\frac{1}{r^2}\int_0^r s\omega(s)\,\mathrm{d}s\right) x^\perp \cdot \omega'(x)\frac{x}{r} = 0$$

for every $x \in \mathbb{R}^2$. As a model solution and first example, we consider such radial eddies and in particular phantom vorticies. The name “phantom” stems from vanishing total vorticity condition, i.e., $\int_{\mathbb{R}^2} \omega = 0$. Let us take (see Figure 3.4)

$$\omega_1(x) = \begin{cases} \frac{2}{\pi}, & \text{for } |x| \le \frac{1}{2}, \\ -\frac{2}{3\pi}, & \text{for } \frac{1}{2} < |x| \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.10}$$

Then ω_1 is supported on $B = \{x \in \mathbb{R}^2 |\, |x| \le 1\}$, has vanishing mean value and the total variation is $\|\omega_1\|_{L^1} = \int_B |\omega_1| = 1$. Because of the radial symmetry, it is a (weak) solution to (3.2.8) and furthermore, the associated velocity field v_1 via Biot-Savart law also vanishes outside B by construction.

Our focus now lies on concentration phenomena. In order to do so, we rescale the vortex solution ω_1 by introducing

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon^2}\omega_1\left(\frac{x}{\varepsilon}\right)$$

for $\varepsilon > 0$. The L^1-norm is invariant under this scaling law, i.e., $\int_{\mathbb{R}^2} |\omega_\varepsilon| = 1$ for $\varepsilon > 0$. Further, the L^2-norm of the associated velocity $v_\varepsilon = K * \omega_\varepsilon = \frac{1}{\varepsilon}v_1(\frac{\cdot}{\varepsilon})$ since[2] $\int_{\mathbb{R}^2} \omega_\varepsilon = 0$.

[2] This is not true if $\int_{\mathbb{R}^2} \omega_1 \neq 0$.

By the computations below, the set $\{\omega_\varepsilon\}_{\varepsilon>0}$ is also bounded in $W^{-1,2}(\mathbb{R}^2)$ and therefore in the vortex sheet space $\mathcal{M} \cap W^{-1,2}_{\mathrm{loc}}$ from above. We have (compare to [69, p. 412])

$$\begin{aligned} \omega_\varepsilon &\rightharpoonup^* 0 \qquad \text{as Radon measures,} \\ v_\varepsilon &\rightharpoonup 0 \qquad \text{in } L^2(\mathbb{R}^2), \end{aligned}$$

but no strong convergence in either cases and we notice that the limit $\omega, v = 0$ is a solution to (3.2.7). The nonlinear terms of the convection in (3.2.7) are $(v_\varepsilon)_i(v_\varepsilon)_j$ for $i, j = 1, 2$ which are bounded in L^1 as $\varepsilon \to 0^+$. Taking a test function $\phi \in C_0(\mathbb{R}^2)$ we compute

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} \phi(x)(v_\varepsilon)_i(x)(v_\varepsilon)_j(x) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} \phi(x)\frac{1}{\varepsilon^2}(v_1)_i\left(\frac{x}{\varepsilon}\right)(v_1)_j\left(\frac{x}{\varepsilon}\right) \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} \phi(\varepsilon x)(v_1)_i(x)(v_1)_j(x) = \phi(0)\int_{\mathbb{R}^2}(v_1)_i(x)(v_1)_j(x). \end{aligned}$$

Using (3.2.9), we have

$$\begin{aligned} \int_{\mathbb{R}^2}(v_1)_i(x)(v_1)_j(x) &= \int_{\mathbb{R}^2} \frac{(x^\perp)_i(x^\perp)_j}{r^4}\left(\int_0^r s\omega_1(s)\,\mathrm{d}s\right)^2 \mathrm{d}x \\ &= \int_0^\infty \int_0^{2\pi} \frac{(e_\theta)_i(e_\theta)_j}{r}\left(\int_0^r s\omega_1(s)\,\mathrm{d}s\right)^2 \mathrm{d}\theta\,\mathrm{d}r = C\delta_{ij} \end{aligned}$$

for some $C > 0$ with e_θ representing the angular unit vector and δ_{ij} being the Kronecker-delta. In total, it is

$$v_\varepsilon \otimes v_\varepsilon \rightharpoonup^* C\begin{pmatrix}\delta_0 & 0 \\ 0 & \delta_0\end{pmatrix},$$

so the convection term converges in measures to a multiple of the Dirac measure supported in the origin. In particular, it is $|v_\varepsilon|^2 \rightharpoonup^* 2C\delta_0$. Remarkably, the solutions v_ε converge to a solution, although the nonlinear convection term fails to converge to its limit expression. This fact is due to the structure of the defect measure and the weak formulation. Taking a test function $\phi \in C^\infty_{0,\mathrm{div}}(\mathbb{R}^2)$, we have

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^2} \nabla\phi : \mathrm{d}v_\varepsilon \otimes v_\varepsilon = C\int_{\mathbb{R}^2} \nabla\phi : \mathrm{d}\begin{pmatrix}\delta_0 & 0 \\ 0 & \delta_0\end{pmatrix} = C(\partial_1\phi_1(0) + \partial_2\phi_2(0)) = 0.$$

The concentration of measure is not seen by the equation and we speak of concentration-cancellation. This prototypical example illustrates what actually also happens in the proof of Theorem 3.1.2 for liquid crystal flows.

We turn our attention to a more elaborate example of solutions which is due to Greengard and Thomann in [36]. It shows that the construction of a sequence of stationary solutions to (3.2.7) is possible such that the associated kinetic energy density concentrates on the unit square Q of $\mathbb{R}^2$ with the Lebesgue measure itself supported on Q as defect measure. The sequence of solutions again converges weakly to zero, hence to a solution of (3.2.7). In fact, it shows that every Radon measure can be constructed by the defect measure of solutions sequence to (3.2.7).

Theorem 3.2.1 ([36]). *There exists a sequence $(v_n)_n$ of weak solutions to (3.2.7) such that*

$$\begin{aligned} v_n &\rightharpoonup 0 && \text{in } L^2(\mathbb{R}^2), \\ \omega_n &\rightharpoonup^* 0 && \text{as Radon measures}, \\ \int_{\mathbb{R}^2} |\nabla v_n| &\to \infty, \\ |v_n|^2 &\rightharpoonup^* a\mathcal{L}^2 \llcorner Q && \text{as Radon measures} \end{aligned}$$

with $\mathcal{L}^2$ being the two-dimensional Lebesgue measure, $a > 0$ and $Q = [0,1]^2$.

Proof. We sketch the proof following [69, Chapter 11]. Suppose we have two phantom vorticies, i.e., radial vorticity functions ω_1, ω_2 with vanishing mean, and both with compact but disjoint supports B_1 and B_2. Then the corresponding velocities , v_1 and v_2, also vanish outside of B_1 and B_2 respectively by (3.2.9). The consequence is that although the Euler equation is nonlinear, the sum $v_1 + v_2$ is still a solution of (3.2.7).

Next we consider a dyadic lattice Λ^n on Q. Then the sum

$$\sum_{k,l} \frac{1}{4^n} \delta_{k,l},$$

with $\delta_{k,l}$ being a Dirac mass supported on the (k,l)-th node of Λ^n, converges as Radon measures to the Lebegue measure on the square Q. But we have seen above that a sequence of phantom vortices leads to a Dirac measure as defect, therefore the idea is to place a small enough phantom vortex $\omega_{k,l}$ on every node of Λ^n. The sum $\sum_{k,l} \omega_{k,l}$ is again a solution if the supports do not intersect by our above observation. In order to satisfy these constraints one needs to introduce two scales. The vortices $\omega_{k,l}$ we look for are translated copies of (see [69, p. 419])

$$\omega_n(x) = \begin{cases} \omega_n^+, & \text{for } |x| \leq \delta_n, \\ 0, & \text{for } \delta_n < |x| \leq \delta_n^{1/2} \text{ or } |x| > R_n, \\ \omega_n^-, & \text{for } \delta_n^{1/2} < |x| \leq R_n \end{cases}$$

for the choices (see Figure 3.5)

$$\delta_n = \exp\left(-\frac{4^{m+1}}{2\pi}\right), \quad R_n = \left(\frac{5}{4}\delta_n\right)^{1/2}, \quad \omega_n^+ = \frac{1}{4^n \delta_n^2}$$

and ω_n^- such that $\int_{\mathbb{R}^2} \omega_n = 0$. The crucial point is the appearance of two different scales. The inner, positive part of ω concentrates with quadratical speed compared to the outer, negative part supported on an annulus. With the specific parameters chosen, it ensures the summability of $\omega_n = \sum_{k,l} \omega_{k,l}$ as well as the corresponding bounds for the velocity v_n which is proven in [69, p. 419] □

Recall the meaning of the stream function $\nabla^\perp \psi = v$. Then the essentials of Theorem 3.2.1 consist of the fact that solutions of

$$\Delta \psi_n = \omega_n, \qquad \nabla^\perp \psi_n \otimes \nabla^\perp \psi_n + \nabla p_n = 0$$

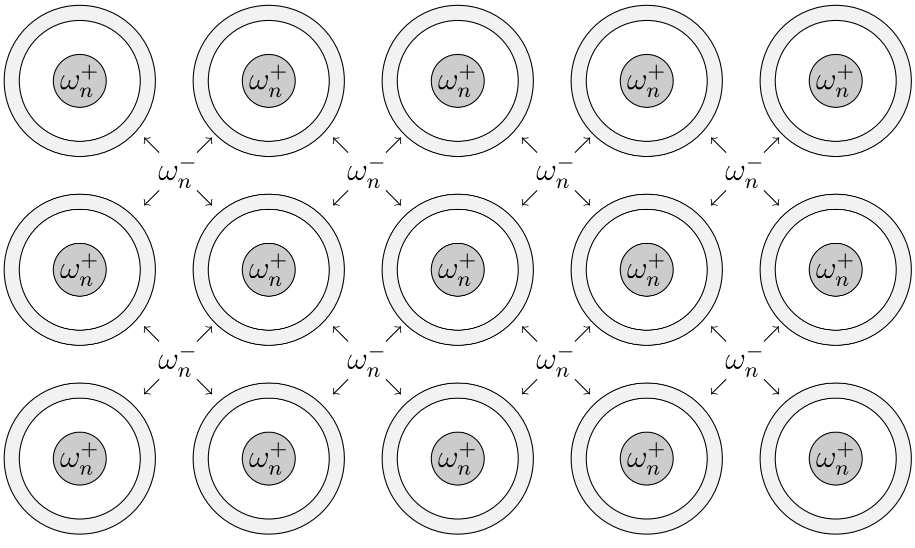

Figure 3.5: Greengard-Thomann example

can develop the worst possible defect measures in the energy density $\frac{|\nabla \psi_n|^2}{2}$ but still, in the limit, satisfy the limit equation. None of these examples are accidental as we will see in the following section.

In comparison to approximated harmonic maps (3.1.29), the Ericksen-Leslie equations or the solutions to

$$-\Delta u_n = |\nabla u_n|^2 u_n + \tau_n$$

with bounded energy and tension field $\int |\nabla u_n|^2 + |\tau_n|^2 \leq C$, we observe a similar behavior in the proof of Theorem 3.1.2. However, the defect measure concentrates on a smaller (at most finite) set since a form of elliptic ε-regularity is available. Example (3.2.10) shows (by multiplication with a small constant) on the other hand that concentration effects can always occur in (3.2.7), no matter how small the local energy is chosen.

3.2.2 The reduced defect measure and weak stability of stationary Euler equations

We turn our attention to a sequence of solutions of (3.2.7). In the previous section, we have seen convergence results to a limit solution for special cases despite the failure of the convergence of the nonlinear terms in (3.2.7). Such a behavior is true in general and first proven in [23]:

Theorem 3.2.2 ([69], Theorem 12.2)**.** *Let $f_n \rightharpoonup f$ in $L^1(\mathbb{R}^2)$, the velocity fields $v_n \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ satisfy*

$$\operatorname{div}(v_n \otimes v_n) + \nabla p_n = f_n, \qquad \operatorname{div} v_n = 0$$

in the weak sense with $\int_K |v_n|^2, \int_K |\omega_n| = \int_K |\operatorname{curl} v_n|$ being bounded for every $K \subset\subset \mathbb{R}^2$. Then the sequence $(v_n)_n$ possesses a limit point $v \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ which satisfies (3.2.7) *in the weak sense.*

The proof (we follow a more streamlined version from [27, 69]) involves technical arguments from capacity theory at some stages which we will only sketch.

First, we introduce another notion of defect measure, the so-called reduced defect measure

$$\theta(E) := \limsup_{n\to\infty} \int_E |v_n - v|^2$$

for any Borel set $E \subset \mathbb{R}^2$. To be precise, the set function θ only depicts a premeasure since it is not countably subadditive but finitely subadditive (a counterexample is provided by the sequence from the Greengard-Thomann example from the previous section, see [69, p. 421]). The defect measure we consider so far is the weak*-defect measure σ defined by

$$|v_n|^2 \,\mathrm{d}x - |v|^2 \,\mathrm{d}x \rightharpoonup^* \sigma$$

as Radon measures. As we see below the reduced defect measure can behave much better than σ.

We also recall the notion of Hausdorff measure. For a given Borel set $E \subset \mathbb{R}^2$, the Hausdorff premeasure of order γ for a given $r > 0$ is defined by

$$H_r^\gamma(E) := \inf \left\{ \sum_{r_i \le r} r_i^\gamma \,\middle|\, E \subset \bigcup_{i=1}^\infty B_{r_i}(x_i), x_i \in \mathbb{R}^2 \text{ for every } i \in \mathbb{N} \right\}.$$

Taking the (well-defined) limit $r \to 0^+$ leads us to the Hausdorff measure of order γ,

$$H^\gamma(E) := \lim_{r\to 0} H_r^\gamma(E),$$

and we say that E has Hausdorff dimension $d \ge 0$ if $d = \inf\{\gamma |\, H^\gamma(E) = 0\}$. Morevoer, we settle the meaning of concentration in terms of the reduced defect measure (see [69, p. 410]):

Definition 3.2.2. *We say that θ concentrates inside a set with Hausdorff dimension p, if, given $\delta, r > 0$ there exists a family of sets $\{F_r\}$ such that*

- $\theta(F_r) = 0$,
- $H_r^{p+\delta}(F_r^c) \le C$ *with a constant $C > 0$ independent of r and δ.*

Remark 3.2.3. *The assertion $\theta(E) = 0$ is equivalent to the strong convergence $v_n \to v$ in $L^2(E)$. In general, this is not the case for σ restricted to E (a counterexample is provided by $n\chi_{(0,1/n)}$ on $E = (0, 1)$). If E is closed , then we have $\sigma(E) \ge \theta(E)$ (see [69, p. 411]).*

For a given sequence of solutions to (3.2.7), the weak*-defect measure may concentrate on a set of Hausdorff-dimension two as illustrated in the Greengard-Thomann example. However, in terms of the reduced defect measure, concentration happens on a set of dimension zero in the sense of Definition 3.2.2. This fact is not accidental.

Theorem 3.2.3 ([69], Theorem 12.1)**.** *Let $(v_n)_n$ be sequence such that*

- $\operatorname{div} v_n = 0$,
- $\operatorname{curl} v_n = \omega_n$,

- $\int_K |v_n|^2 + \int_K |\omega_n| \leq C(K)$ *for every compact set* $K \subset \mathbb{R}^2$.

Then $\theta(\cdot) = \limsup_{n\to\infty} \int_{(\cdot)} |v_n - v|^2$ *concentrates inside a set of Hausdorff dimension zero in the sense of Definition 3.2.2.*

Proof. A detailed proof can be found in [27, 69]. We roughly sketch the idea:
By our assumptions, we introduce a stream function to the problem as in (3.2.6),

$$-\Delta\psi_n = \omega_n,$$

where the right-hand side is bounded in L^1 and $\nabla^\perp\psi_n = v_n$ in L^2 locally. As n tends to infinity, the sequence $(\omega_n)_n$ might concentrate only in a countable number of points to a Dirac measure which as a set has Hausdorff dimension 0. Aside from this set, the limit measure $\mu = \overline{|\omega_n|}$ must be continuous. To be more precise, for a given $0 < p < 2$, we consider the set

$$E_p = \left\{ x \in K : \sup_{n\in\mathbb{N}} \sup_{0<r} \frac{1}{r^p} \int_{B_r(x)} |\omega_n| < \infty \right\}$$

for a compact set $K \subset \mathbb{R}^2$. A covering argument shows that $H^p((E_p)^c)$ is finite for every $0 < p < 2$ because of the bound on the total variation of $(\omega_n)_n$. On the other hand, the embeddings for Riesz potentials I_α (as the Biot-Savart kernel is one of order $\alpha = 1$) from [1] show

$$\|I_1 f\|_{L^q_{\mathrm{loc}}} \leq C\|f\|_{L^{1,p}}$$

for $n = 2, q < 2 + \frac{p}{1-p}$. Here, the space $L^{1,p}$, the Morrey space, contains all functions satisfying

$$\sup_{x_0\in\mathbb{R}^2, r>0} r^{-p} \int_{B_r(x_0)} |f| \leq C.$$

Consequently, we have strong convergence $v_n \to v$ in $L^2(E_p)$ since the sequence converges pointwise and is bounded in some L^q-norm with $q > 2$. Since the complementary set $(E_p)^c$ satisfies the bound $H((E_p)^c) \leq c_p$, the concentration set $C \approx \bigcap_{0<p<2}(E_p)^c$ must satisfy $H^\gamma(C) = 0$ for every $\gamma > 0$. □

The technicalities of the argument are substantially more complicated. For example, the notion of p-capacity enters in arguments related to the size and properties of E_p and strong convergence of $(v_n)_n$. Note that it is not necessary that v_n solves the stationary Euler equations.

The final argument in the proof of Theorem 3.2.2 consists of the aforementioned concentration-cancellation arguments. Let us note a technicality: Since Theorem 3.2.2 is phrased in terms of local Lebesgue spaces, we need to restrict ourselves to approximating sequences $(v_n)_n$ with $\operatorname{supp} v_n \subset B_R$ for some $R > 0$ for almost all $n \in \mathbb{N}$. It does not pose any restriction on our situation which is proven in the appendix of [23].

For the weak formulation of (3.2.7), take a test function $\phi \in C^\infty_{0,\mathrm{div}}(\mathbb{R}^2)$ which can be written as $\nabla^\perp\eta$ for $\eta \in C^\infty_0(\mathbb{R}^2)$ by Helmholtz decomposition (compare to Remark 3.1.6). Then (3.2.7) reads

$$\int_{\mathbb{R}^2} v_n \otimes v_n : \nabla\nabla^\perp\eta = -\int_{\mathbb{R}^2} f_n \cdot \nabla^\perp\eta. \tag{3.2.11}$$

Similarly to (3.1.39), one has

$$\int_{\mathbb{R}^2} \nabla_y \nabla_y^\perp \eta(Qy) : (Q^\top v_n \otimes Q^\top v_n)(Qy)\,\mathrm{d}y = \int_{\mathbb{R}^2} \nabla_x \nabla_x^\perp \eta(x) : (v \otimes v)(x)\,\mathrm{d}x \tag{3.2.12}$$

for any $Q \in SO(2)$, see [69, p. 457]. Next, by Fourier expansion $\eta(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ik\cdot x}\hat{\eta}(k)\,\mathrm{d}k$, it is enough to consider a smooth plane wave function $x \mapsto \tilde{h}(k\cdot x)$ as test function (note that the integral expressions in the weak formulation make sense). Regarding (3.2.12), we can choose Q such that $\tilde{h}(k\cdot x) = h(x_1)$ and (3.2.11) reads

$$-\int_{\mathbb{R}^2} h''(v_n)_1(v_n)_2 = \int_{\mathbb{R}^2} (f_n)_2 h'.$$

By Theorem 3.2.3 we know that the reduced defect measure of $(v_n)_n$ is concentrated inside of a set of Hausdorff dimension zero. The same holds true for the rotated versions $Q^\top v_n(Q\cdot)$ (see [69, Lemma 12.3]).

Finally, we implement a proper cutoff procedure for the test functions. By the definition of the reduced defect measure, there exists a family $\{F_r\} \subset 2^{\mathbb{R}^2}$ of closed sets such that

$$\lim_{n\to\infty} \int_{F_r} |v_n - v|^2 = 0$$

with a constant $C > 0$ (independent of r) such that for any $r > 0$ it holds

$$H_r^p(F_r^c) \leq C(p)$$

for any $p > 0$. In particular, take $p = \frac{1}{2}$. Then, for $r > 0$, there exists a covering $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}, r_i \leq r$ with $F_r^c \subset \bigcup_{i\in\mathbb{N}} B_{r_i}(x_i)$ and $\sum_{i=1}^\infty r_i^{\frac{1}{2}} \leq 2C$ by the definition of Hausdorff measure. Let $P_1 : \mathbb{R}^2 \to \mathbb{R}$ denote the projection onto the x_1-axis. Then we have

$$\mathcal{L}^1(P_1 F_r^c) \leq \sum_{i=1}^\infty \mathcal{L}^1(P_1 B_{r_i}(x_i)) = 2\sum_{i=1}^\infty r_i \leq 2r^{\frac{1}{2}} \sum_{i=1}^\infty r_i^{\frac{1}{2}} \leq 4C \cdot r^{\frac{1}{2}},$$

i.e., the one-dimensional Lebesgue measure of $P_1 F_r^c$ tends to zero as $r \to 0$. Thus we define $h_r : \mathbb{R} \to \mathbb{R}$ by

$$h_r''(x_1) = (1 - \chi_{P_1 F_r^c}(x))h''(x)$$

(with boundary conditions leaving h_r' bounded as $r \to 0$). It holds $h_r \to h$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R})$ and $(h_r'', h_r')_r$ remain bounded by a uniform constant as $r \to 0$. Inserting h_r into the weak formulation above we have

$$-\int_{\mathbb{R}^2} h_r''(v_n)_1(v_n)_2 = \int_{\mathbb{R}^2} (f_n)_2 h_r' \qquad \to \qquad -\int_{\mathbb{R}^2} h_r'' v_1 v_2 = \int_{\mathbb{R}^2} f_2 h_r'$$

as $n \to \infty$ by the assumptions of Theorem 3.2.2 and the fact that $v_n \to v$ in L^2 on the support of the functions h_r''. The convergence $h_r'' \to h''$ holds true a.e. on $\mathbb{R}$ and by dominated convergence (it is $|h_r'' v_1 v_2| \leq |h'' v_1 v_2|$), we finally conclude

$$-\int_{\mathbb{R}^2} h'' v_1 v_2 = \int_{\mathbb{R}^2} f_2 h'$$

for any plane wave h. Regarding the above arguments, we deduce that the weak formulation of (3.2.7) is satisfied for every test function proving Theorem 3.2.2.

We recognize that the concentration-cancellation method becomes more technical in the regime of Euler equations than in Section 3.1. Clearly, this is related to the more elaborate notion of reduced defect measure. However, the method above does work for all concentration sets of Hausdorff dimension less than one (in the sense of Definition 3.2.2). The (perhaps critical) case of the concentration set being of reduced Hausdorff dimension in space-time (!) is dealt with in [2, 86].

3.2.3 Comparison of Euler equations and the Ericksen-Leslie system

We give a final comparison of the Ericksen-Leslie system (3.1.6)–(3.1.8) and the Euler equations (3.2.1) in two dimensions. The conclusions drawn below are summarized in Table 3.1. First, both are systems of evolutionary equations with the Ericksen-Leslie system of parabolic type and the Euler equations of (partially) hyperbolic type. Therefore, solutions (v, d) of (3.1.6)–(3.1.8) smooth out after initial time modulo singularities whereas (3.2.1) (at best) preserves initial regularity, e.g., the L^p-norm of the vorticity.

We specifically turn our attention to the behavior of the velocity field v_E (the subscript $_E$ stands for Euler) in (3.2.1) and ∇d in (3.1.6)–(3.1.8). Along (approximate) solution sequences of their respective equations, both quantities might concentrate some part of their energy in some measure. By Theorems 3.1.2 and 3.2.2 we know that solutions are attained in the limit of the sequences for the Ericksen-Leslie system and the stationary Euler equations. This fact is mainly due to concentration-cancellation, i.e., the potential concentration of the limit measure $\overline{|v_E|^2}$ and $\overline{|\nabla d|^2}$ is not seen in the weak formulation of the equations. While the support of the concentrated measure (in the sense of Radon measures) is only (locally) finite in space (Lemma 3.1.5) for liquid crystal flows, we need to introduce another notion, the reduced defect measure, in order to quantify the sufficient smallness of concentration support with respect to inviscid fluid flows (see Theorem 3.2.3).

Considering the space-time defect, one would expect rectifiable lines as defect measure in the liquid crystal case. More precisely, the defect should be supported on a set of finite two-dimensional parabolic Hausdorff measure. At least, this is the case for the harmonic map heat flow (see [62, Chapters 7 and 8]). For the Euler equations, not too much is known about the reduced defect measure. Seemingly, the best known result [68] states that the space-time reduced Hausdorff measure concentrates inside a set of dimension $2+\varepsilon$ for any $\varepsilon > 0$. Therefore one might wonder if the method of the proof of Theorem 3.1.2 is useful to show the analogous result for the time-dependent Euler equations. In order to do so, one would fix times $t > 0$, use a similar concentration-cancellation argument as in Theorem 3.2.3 and verify a weak formulation for almost every $t \in (0, T)$. However, this scheme depends on the regularity of the time-derivative $\partial_t v$ which is proven to lie in $(W^{1,r}_{\mathrm{div}})^*$ for any $r > 2$ in the liquid crystal case (see Lemma 3.1.3). We use the gradient flow structure of the Ericksen-Leslie system to obtain (3.1.19), but the Euler equations do not provide such an internal structure (at least it is not observed up to now). So far, the best known estimate for general solutions of (3.2.1) is $\partial_t v_E \in (C^1_{\mathrm{div}})^*$ which directly follows from the energy estimate. Under such bad regularity assumptions, the limit procedure in (3.1.41) would not be possible. Indeed, in [75], it is shown that the

Table 3.1: Comparison of Euler and Ericksen-Leslie system

$n = 2$	Euler equations v_E	Ericksen-Leslie system ∇d
weak compactness of stationary solutions	$\surd$	$(\surd)$
weak compactness of time-dependent solutions	not so far	$\surd$
concentration sets Σ (time t fixed)	$H(\Sigma) = 2$ (weak* defect), $H(\Sigma) \approx 0$ (reduced defect)	Σ (locally) finite
regularity of $\partial_t v$	$(C^1_{\mathrm{div}})^*$	$(W^{1,r}_{\mathrm{div}})^*, \quad r > 2$

method works if the measure of an external force $f \in (C^1_{\mathrm{div}})^*$ acting on first derivatives of a test function is continuous which is in general not the case for distributions in $(C^1_{\mathrm{div}})^*$.

Chapter 4

Well-posedness theory of global solutions to magnetoviscoelastic flows

In the following, we address the well-posedness of a system of equations modeling the flow of magnetoviscoelastic fluids. In detail, we review the uniqueness, short-time existence of strong solutions and global-in-time of weak solutions to system (1.3.4). All results are proven in two spatial dimensions which, from the modeling point of view, is thought to represent the thin film regime in micromagnetics.

We recall the definition and meaning of the quantities arising in (1.3.4). The fluid mechanics point of view prompts to consider Eulerian coordinates and therefore the first state quantity consists of a velocity field $u : \mathbb{T}^2 \times [0,T] \to \mathbb{R}^2$. Since nontrivial deformations are intended to be pictured as well, a deformation gradient (in Eulerian coordinates) is introduced as matrix-valued field $F : \mathbb{T}^2 \times [0,T] \to \mathbb{R}^{2\times 2}$. Further, we restrict ourselves to incompressible materials. As third ingredient, the fluid is built by (ferro-)magnetic particles which requires the introduction of a magnetization field $M : \mathbb{T}^2 \times [0,T] \to \mathbb{S}^2$ undergoing the laws of micromagnetics.

Setting up the corresponding energy functional and applying a variational approach, we are given the following system of magnetoviscoelastic fluids:

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = -\operatorname{div}(\nabla M \odot \nabla M - W'(F)F^\top) + \mu_0 \nabla^\top H_{\text{ext}} M, \tag{4.0.1}$$

$$\operatorname{div} u = 0, \tag{4.0.2}$$

$$\partial_t F + u\cdot\nabla F - \nabla u F = \kappa\Delta F, \tag{4.0.3}$$

$$\operatorname{div} F^\top = 0, \quad |M|^2 = 1, \tag{4.0.4}$$

$$\partial_t M + (u\cdot\nabla)M = -M\wedge H_{\text{eff}} - M\wedge M\wedge H_{\text{eff}}, \tag{4.0.5}$$

on $\mathbb{T}^2 \times (0,T)$ with the effective magnetic field

$$H_{\text{eff}} = \Delta M + \mu_0 H_{\text{ext}} - \psi'(M),$$

and initial conditions

$$(u(\cdot,0), F(\cdot,0), M(\cdot,0)) = (u_0, F_0, M_0), \qquad \text{on } \mathbb{T}^2, \tag{4.0.6}$$

$$\operatorname{div} u_0 = 0, \ \operatorname{div} F_0^\top = 0, \ |M_0| = 1, \qquad \text{on } \mathbb{T}^2. \tag{4.0.7}$$

The derivatives of $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and $\psi : \mathbb{R}^3 \to \mathbb{R}$ are denoted by $W'(F)$ and $\psi'(M)$ with $(W'(F))_{ij} = \frac{\partial W(F)}{\partial F_{ij}}$, $i, j = 1, 2$ and $(\psi'(M)) = \frac{\partial \psi(M)}{\partial M_i}$, $i = 1, 2, 3$. To explain the occuring quantities, $H_{\text{ext}} : \mathbb{T}^2 \times [0, T] \to \mathbb{R}^3$ represents the application of an external magnetic field to the magnetic fluid. The elastic energy density $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ depicts the internal elastic reaction of the fluid to deformation and $\psi : \mathbb{R}^3 \to \mathbb{R}$ is a combined expression for the anisotropic energy as well as the stray field energy in micromagnetics. We mention here that in the two-dimensional thin film regime, the use of a local energy term for the stray field energy is feasible, see [35].

For the remainder of the chapter, we make the following assumptions:

- The external magnetic field H_{ext} is a function in $W^{1,\infty}(\mathbb{T}^2 \times [0, T])$, i.e.,

$$\sup_{(x,t)\in\mathbb{T}^2\times[0,T]} \Big(|H_{\text{ext}}(x, t)| + |\partial_t H_{\text{ext}}(x, t)| + |\nabla H_{\text{ext}}(x, t)| \Big) < \infty. \tag{4.0.8}$$

- The elastic energy density W is a twice differentiable function on $\mathbb{R}^{2\times 2}$ and there exist constants $\chi, C_1, C_2 > 0$ such that

$$|W'(A) - \chi A| \le C_1 \qquad \text{for all } A \in \mathbb{R}^{2\times 2} \tag{4.0.9}$$

  and

$$|W''(A)| \le C_2 \qquad \text{for all } A \in \mathbb{R}^{2\times 2}. \tag{4.0.10}$$

- The combined magnetic anisotropic and stray field energy density ψ is a non-negative, even polynomial of degree at most eight in M, i.e.

$$\psi(M) = \sum_{|\alpha|\le 8} a_\alpha M^\alpha, \quad \psi(M) = \psi(-M), \quad \psi(M) \ge 0 \tag{4.0.11}$$

  for all $M \in \mathbb{R}^3$.

The requirement (4.0.8) depicts the worst expectable regularity when inducing an external magnetic field. For example, an H_{ext} with saw-tooth profile in time is covered by these assumptions. Regarding the elastic energy density, we point out that W is not assumed to be convex in any way in contrast to previous works [7, 18, 46]. From the physical point of view, energy densities like the double-well potentials (see Figure 4.1), where the minimizers include the set $SO(2)$, are covered by (4.0.9) and (4.0.10). The assumption on ψ to be a polynomial of eighth degree is of physical nature as well. In [45], different types of anisotropy energies are described with maximal degree of order eight. Moreover, the stray field energy consists of a non-negative, even quadratic polynomial as well if we consider two spatial dimensions, see [35]. The non-negativity of ψ makes sense, since ψ is intended to model an energy term which, by labeling, is non-negative.

Before we enter mathematical arguments, we summarize the content of the following sections: Section 4.1 will introduce a special notion of weak solutions which we call Struwe-like solutions. Proving the existence of Struwe-like solutions is the main motivation for investigating uniqueness of (4.0.1)–(4.0.5) in Section 4.2 and the existence of strong solutions locally in time in Section 4.3. The final part of investigation of Struwe-like solutions forms Section 4.4 where the unique existence of such a solution is proven for any initial data in the energy space. Section 4.5 contains a localized energy law of (4.0.1)–(4.0.5).

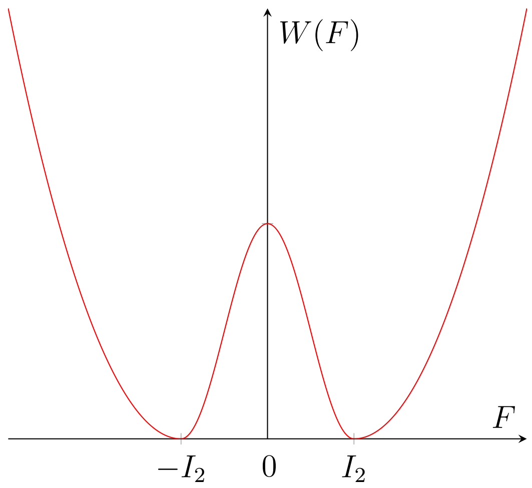

Figure 4.1: Double-well potential

4.1 Struwe's solutions

Let us introduce and motivate the notion of a Struwe-like solution of (4.0.1)–(4.0.5). We already mentioned in the introduction that system (4.0.1)–(4.0.5) shares similarities with the Ericksen-Leslie system (3.1.6)–(3.1.8) regarding the analysis. Here we specifically address the main analogy, the unitary constraint $|d(x,t)| = |M(x,t)| = 1$ for any space-time point (x,t) in the domain of the respective functions. Therefore, we need to handle this geometric constraint and may equivalently say the image of d and M lies in $\mathbb{S}^2$. The modeling of magnetic and liquid crystal materials advices to us the Dirichlet energies

$$\int \frac{|\nabla d|^2}{2} \quad \text{and} \quad \int \frac{|\nabla M|^2}{2}$$

in the variational approach. Ignoring the velocity, deformation gradient and convective forces in the LLG equation (4.0.5), we are left with the harmonic map heat flow

$$\partial_t v - \Delta v = |\nabla v|^2 v, \qquad v(0) = v_0 \tag{4.1.1}$$

for $d = M = v$ and $v : D \times [0,T] \to \mathbb{S}^2$ on a domain $D \subset \mathbb{R}^2$.

The origin of (4.1.1) is connected to a different problem in geometry. Instead of $\mathbb{R}^2$ (or open subsets) and $\mathbb{S}^2$, let us consider general closed Riemannian manifolds (M,g) as spatial domain and (N,h) as target space of v. The analogy of (4.1.1) reads

$$v_t - \Delta_g v = A(v)(\nabla v, \nabla v), \qquad v(0) = v_0 \tag{4.1.2}$$

for $v : M \times [0,T] \to N$ and with A being the second fundamental form of N in the normal direction ν of N. Equation (4.1.2) was first considered in [25] with the intention to solve the following problem: Given a map v_0 in some homotopy class, is there a harmonic map in the same class? In order to answer this task positively, one needs, e.g., to give a homotopy from v_0 to such a harmonic map and the idea was roughly the following: Using v_0 as initial condition, the heat flow (4.1.2) evolves in time with a limit as $t \to +\infty$. This limit is characterized by $\partial_t v = 0$, hence

$$-\Delta_g v = A(v)(\nabla v, \nabla v)$$

and therefore is a critical point to the functional

$$\int_M |dv|^2 \, \mathrm{d}M$$

among all $v \in W^{1,2}(M, N)$. Hence, if the solution to (4.1.2) is smooth, such a homotopy is found. In this regard, it was obtained in [25] that solutions (4.1.2) remain smooth if the sectional curvature of N is non-positive (see also [62, Theorem 5.3.1]).

On the other hand, the sphere $\mathbb{S}^2$ is a surface of positive curvature everywhere. Still, one might hope for global regularity in (4.1.2) for any target manifold N on two-dimensional surfaces M since the Dirichlet energy is critical. This was disproven in [14] by obtainment of a local smooth solution to (4.1.1) which blows up in finite time. It shows that a previous result of Struwe from 1985 is sharp:

Theorem 4.1.1. *[78, Theorem 4.2] Let (M, g) and (N, h) be closed Riemannian manifolds with* $\dim M = 2$. *Then for any $v_0 \in W^{1,2}(M; N)$ there exists a unique solution $v : M \times [0, \infty) \to N$ of* (4.1.2) *which is smooth except of finitely many points $(x_l, t_l) \in M \times (0, \infty)$, $1 \leq l \leq L$ for some integer $L \in \mathbb{N}$.*

The problem of singularity formation represents a geometric issue. Considering such an (x_l, t_l), a phenomenom called bubbling happens. Some amount of energy concentrates in this point as t approaches t_l. Rescaling of the flows shows that the amount of energy concentrated in the singular point corresponds to the energy of finitely many harmonic maps $\omega_i : \mathbb{S}^2 \to N$, called bubbles. In some sense, those bubbles separate and flow can smoothly be continued after time t_l. These facts are proven, e.g., in [62, 71, 78, 82]. Here, the analytic side of the proof of Theorem 4.1.1 is reviewed.

Roughly speaking, a solution to a (nonlinear) parabolic PDE is smooth if a certain coercive critical quantity is small. The reason is that the typically superlinear terms become even smaller than the linear ones in this case. It pushes the PDE almost back to the linear regime where smoothness of solutions is valid. Hence, a proof of (partial) regularity usually possesses the following basic ingredients:

- A smallness assumption on such a critical quantity,
- an interpolation/embedding inequality which compares a quantity to its derivatives and
- a local inequality inherent to the PDE considered which makes the application of the smallness assumption possible.

For (4.1.2), a global energy law holds true,

$$\int_M |\nabla v(t)|^2 + 2 \int_0^t \int_M |\partial_t v|^2 \leq \int_M |\nabla v_0|^2 = 2E_0 \tag{4.1.3}$$

for all $t \in (0, \infty)$ and since $\dim M = 2$, the energy is critical. Luckily, there also exists a sensible local energy inequality as well,

$$\int_{B_R(x_0)} |\nabla v(t_2)|^2 \leq \int_{B_{2R}(x_0)} |\nabla v(t_1)|^2 + c \frac{t_2 - t_1}{R} E_0 \tag{4.1.4}$$

for all $0 \leq t_1 < t_2$, $\operatorname{diam} M/2 > R > 0$ and $x_0 \in M$. As interpolation inequality in [78], a (localized) version of Ladyzhenskaya's inequality is used,

$$\int_{t_1}^{t_2}\int_M |\nabla v|^4 \leq C \sup_{(x_0,t)\in M\times(t_1,t_2)} \int_{B_R(x_0)} |\nabla v|^2 \times \left(\int_{t_1}^{t_2}\int_M |\nabla^2 v|^2 + \frac{1}{R^2}\int_{t_1}^{t_2}\int_M |\nabla v|^2\right) \tag{4.1.5}$$

for $R > 0$. The last inequality is very similar to the energy inquality and states

$$\int_{t_1}^{t_2}\int_M |\nabla^2 v|^2 \leq \int_{t_1}^{t_2}\int_M |\nabla v|^4. \tag{4.1.6}$$

The argument then goes as follows: Suppose the critical quantity is small, i.e.

$$\sup_{(x_0,t)\in M\times(t_1,t_2)} \int_{B_R(x_0)} |\nabla v|^2 \leq \varepsilon << 1.$$

Then the combination of (4.1.5), (4.1.6) and (4.1.3) yields

$$\int_{t_1}^{t_2}\int_M |\nabla^2 v|^2 \leq C\varepsilon\left(\int_{t_1}^{t_2}\int_M |\nabla^2 v|^2 + \frac{t_2-t_1}{R^2}E_0\right)$$

and therefore we have

$$\int_{t_1}^{t_2}\int_M |\nabla^2 v|^2 \leq C\frac{t_2-t_1}{R^2}E_0.$$

If the $W^{2,2}$-norm of v is finite, a bootstrap argument shows that v is actually smooth (see [78]. The question is if the smallness assumption can be justified since $t \mapsto \int_{B_R(x_0)} |\nabla v(t)|^2$ might oscillate heavily. Indeed, this is doable by (4.1.4), at least for some small time $[t_1, t_2]$. The local energy of $v(t_2)$ is controlled by the local energy of $v(t_1)$ and some contribution dependent on the difference $t_2 - t_1$. However, one might need to choose a very small R to conclude the smallness which in turn limits the length of $[t_1, t_2]$. But on a small time interval to the right of any given time $t^* \in [0, \infty)$, the condition is always guaranteed.

In contrast, the converse statement must be the following: If v becomes non-smooth in t^*, then

$$\sup_{(x_0,t)\in M\times(t_1,t^*)} \int_{B_R(x_0)} |\nabla v(t)|^2 \geq \varepsilon$$

must hold true. How often is it possible to satisfy this property? Since there is an L^2-bound on $\partial_t v$, the mapping $t \mapsto \nabla v(t)$ is at least weakly continuous on $[0, \infty)$. With respect to the lower semicontinuity of the norm under weak convergence, we have

$$\liminf_{t\nearrow t^*} \int_M |\nabla v(t)|^2 - \int_M |\nabla v(t^*)|^2 = a > 0.$$

The strict inequality holds because the convergence is strictly weak (otherwise, by equi-integrability, the smallness condition can be satisfied). Further, the number a needs to have at least the size of ε, since otherwise, again the smallness assumption is satisfied. Consequently,

$$\int_M |\nabla v(t^*)|^2 \leq 2E_0 - \varepsilon.$$

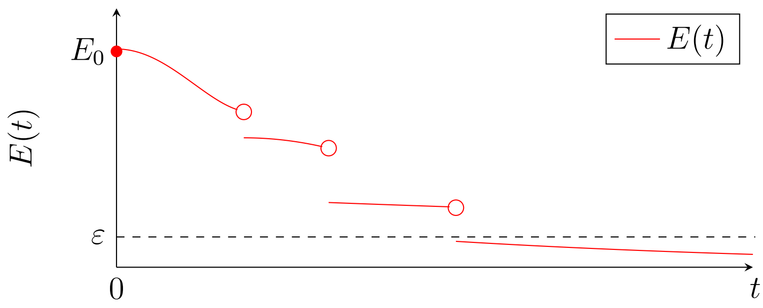

Figure 4.2: Singularities and loss of energy

With $v(t^*)$ being the initial data, one can restart the flow (4.1.2) and obtain another local solution. If it becomes singular, another decrease of energy of at least size ε occurs. However, only finitely many drops may occur since otherwise, the energy becomes negative which is a contradiction. Therefore also only finitely many singularities may occur during the flow, see Figure 4.2.

Despite considerable subtleties, we mimic these argument in the following to derive the existence of a Struwe-like solution, which is the most regular solution available, to (4.0.1)–(4.0.5). To begin with, we state the definition of a Struwe-like solution.

Definition 4.1.1 (Struwe-like solutions)**.** *A triple* $(u,\, F,\, M)$ *is called a Struwe-like solution to system* (4.0.1)–(4.0.6) *in* $\mathbb{T}^2 \times (0,T)$ *with initial data* $(u_0,\, F_0,\, M_0) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ *satisfying* $\operatorname{div} u_0 = 0, \operatorname{div} F_0^\top = 0$ *and* $|M_0| = 1$ *if it fulfills*

$$\begin{aligned}
u &\in L^\infty(0,T,\, L^2(\mathbb{T}^2)) \cap L^2(0,T;\, H^1(\mathbb{T}^2)) && \textit{with} && \partial_t u \ \in L^2(T_i,\widetilde{T};\, H^{-1}(\mathbb{T}^2)),\\
F &\in L^\infty(0,T,\, L^2(\mathbb{T}^2)) \cap L^2(0,T;\, H^1(\mathbb{T}^2)) && \textit{with} && \partial_t F \ \in L^2(0,T;\, H^{-1}(\mathbb{T}^2)),\\
M &\in L^\infty(0,T,\, H^1(\mathbb{T}^2)) \cap L^2(T_i,\widetilde{T};\, H^2(\mathbb{T}^2)) && \textit{with} && \partial_t M \in L^2(T_i,\widetilde{T};\, L^2(\mathbb{T}^2)),
\end{aligned} \tag{4.1.7}$$

for any $\widetilde{T} \in [T_i, T_{i+1})$ *and for a suitable ordered finite family of times* $\{T_1, \ldots, T_N\}$*, with* $T_0 = 0$ *and* $T_N \leq T$ *. Furthermore, the following identities are satisfied:*

$$\begin{aligned}
&\int_{\mathbb{T}^2} u(t,\, x) \cdot \nabla\phi(x)\, \mathrm{d}x \,=\, 0 \qquad \textit{for a.e.} \quad t \in (0,T),\\
&\int_{\mathbb{T}^2} F(t,\, x)^\top \cdot \nabla\Xi(x)\, \mathrm{d}x \,=\, 0 \qquad \textit{for a.e.} \quad t \in (0,T),\\
&|M(t,x)| = 1 \qquad \textit{for a.e.}\ (t,x) \in (0,T) \times \mathbb{T}^2
\end{aligned}$$

for any functions ϕ, Ξ *in* $\dot{H}^1(\mathbb{T}^2)$*, as well as*

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^2} &- u \cdot \partial_t\varphi - u \otimes u : \nabla\varphi + \nu\nabla u : \nabla\varphi\\
&= \int_{\mathbb{T}^2} u_0(x) \cdot \varphi(0,x)\\
&+ \int_0^T \int_{\mathbb{T}^2} \nabla M \odot \nabla M : \nabla\varphi - W'(F)F^\top : \nabla\varphi + \mu_0 \nabla H_{\text{ext}} M \cdot \varphi
\end{aligned}$$

for any smooth vector function $\varphi \in C^\infty(\mathbb{T}^2 \times [0,T])$*, with* $\operatorname{div}\varphi = 0$*. Similarly, for any matrix function* $\Xi \in C^\infty(\mathbb{T}^2 \times [0,T])$*,*

$$\int_0^T \int_{\mathbb{T}^2} -F \cdot \partial_t \Xi - u \otimes F : \nabla \Xi - \sum_{i,j,k=1}^{2} u_i F_{jk} \partial_j \Xi_{ik} + \kappa \nabla F : \nabla \Xi = -\int_{\mathbb{T}^2} F_0 \cdot \partial_t \Xi.$$

Finally, the following identity must be satisfied pointwise almost everywhere in $\mathbb{T}^2 \times (0,T)$*,*

$$\partial_t M + (u \cdot \nabla) M = -M \wedge H_{\text{eff}} - M \wedge (M \wedge H_{\text{eff}}),$$

together with the initial condition $(u(t,\cdot), F(t,\cdot), \nabla M(t,\cdot)) \rightharpoonup (u_0, F_0, M_0)$ *as* $t \to 0^+$ *in* $L^2(\mathbb{T}^2)$*.*
If $\{T_1, \ldots, T_N\}$ *is the minimum set for which the above relations holds true, then we say that* $\{T_1, \ldots, T_N\}$ *is the set of singular times of the Struwe-like solution.*

The goal of the remaining sections is the confirmation of existence and uniqueness of such a solution in Theorem 4.4.1. We follow here to some extend [18] which is inspired by [41, 58] which, in turn, is based on the original paper of Struwe [78] sketched above. The final arguments leading to the unique existence of a Struwe-like solution are carried out in Section 4.4. Along the way, we investigate the uniqueness properties and existence of strong solutions to (4.0.1)–(4.0.5) for two reasons. First, these results are of independent interest. In particular, the proof of uniqueness of solutions, cf. Theorem 4.2.1, is relatively short compared to [18] and also differs from the predecessor version for the Ericksen-Leslie system in [41] and [63]. Secondly, both results form an integral part of the verification of Theorem 4.4.1.

4.2 Uniqueness of magnetoviscoelastic flows

We carry out the uniqueness of weak solutions for (4.0.1)–(4.0.6) as long as the solution belongs to the energy space and $\nabla^2 M$ is bounded in $L^2(\mathbb{T}^2 \times (0,T))$. Regarding the scaling properties, the situation is very similar to the two-dimensional Navier-Stokes equations (see [72, p. 85]). On the other hand, the technicalities are considerably more complicated if one tries to show uniqueness through the energy estimate in L^2, see [18, Section 5] and [41, 63]. Therefore, we consider a sort of energy approach for anti-derivatives of $(u, F, \nabla M)$ which is easily accessible on the torus. A similar argument is carried out in [54] for the general Ericksen-Leslie system.

Theorem 4.2.1. *Let* $T > 0$ *and let* $(u^i, F^i, M^i), i = 1,2$ *be weak solutions to* (4.0.1)–(4.0.5) *with initial conditions*

$$u_0^1 = u_0^2 \in L^2(\mathbb{T}^2), \quad F_0^1 = F_0^2 \in L^2(\mathbb{R}^2), \quad M_0^1 = M_0^2 \in W^{1,2}(\mathbb{T}^2)$$

for $t = 0$ *and* $\operatorname{div} u_0^i = 0, \operatorname{div} {F_0^i}^\top = 0, |M_0^i| = 1$ *for* $i = 1,2$*. Furthermore, the solutions and* H_{ext} *satisfy the regularity properties*

$$\begin{aligned} (u^i, F^i, \nabla M^i) &\in L^\infty(0,T; L^2(\mathbb{T}^2)) \cap L^2(0,T; W^{1,2}(\mathbb{T}^2)), \qquad \text{for } i = 1,2, \\ H_{\text{ext}} &\in L^2(0,T; W^{1,2}(\mathbb{T}^2)). \end{aligned}$$

Then $(u^1, F^1, M^1) = (u^2, F^2, M^2)$ *on* $\mathbb{T}^2 \times [0,T]$*.*

Proof. For a quantity $a = u, F, M$ we note the difference by $a := a^1 - a^2$. Further, let $(u^i) = \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} u^i$ and $(M^i) = \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} M^i$, the mean-values of u^i and M^i, and $F^i = (F_1^i, F_2^i)$ with $F_1^i, F_2^i : \mathbb{T}^2 \times [0,T] \to \mathbb{R}^2$ being the first and the second column vectors of F^i for $i = 1, 2$. Then $\operatorname{div} {F^i}^\top$ implies $\operatorname{div} F_j^i$ for $i, j = 1, 2$. Because of the solenoidality, we have $u^i - (u^i) = \nabla^\perp \eta^i$ and $F^i = (F_1^i, F_2^i) = (\nabla^\perp \chi_1^i, \nabla^\perp \chi_2^i)$ for $i = 1, 2$ by Lemma 3.1.7 with $(\eta^i, \chi_1^i, \chi_2^i) \in \dot{W}^{1,2}(\mathbb{T}^2)$ and the standard Poincaré inequality yields

$$\left\|(\eta^i, \chi_1^i, \chi_2^i)\right\|_{L^2(\mathbb{T}^2)} \le C \left\|(u^i, F^i)\right\|_{L^2(\mathbb{T}^2)}.$$

By taking the average of (4.0.1), the evolution of (u^i) follows the equation

$$\partial_t (u^i) = \frac{\mu_0}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} \nabla^\top H_{\mathrm{ext}} M^i \tag{4.2.1}$$

(all remaining terms vanish due to Gauß' Theorem). Therefore we conclude by the assumptions made on the regularity that we have

$$(u^i, F^i, \nabla M^i) \in L^4(0, T; L^4(\mathbb{T}^2))$$

due to Ladyzhenskaya's inequality and the assumptions on H_{ext}. Now for the solutions (u^i, F^i, M^i) of (4.0.1)–(4.0.6), we take the difference and arrive at the weak formulations of

$$\begin{aligned}
&\partial_t u + (u^1 \cdot \nabla) u + (u \cdot \nabla) u^2 - \nu \Delta u + \nabla p \\
&= -\operatorname{div}\Big(\nabla M^1 \odot \nabla M + \nabla M \odot \nabla M^2 - W'(F^1) F^\top - (W'(F^1) - W'(F^2)) {F^2}^\top\Big) \\
&\quad + \mu_0 \nabla^\top H_{\mathrm{ext}} M, && (4.2.2) \\
&\operatorname{div} u = \operatorname{div} u^1 = \operatorname{div} u^2 = 0, && (4.2.3) \\
&\partial_t F + (u^1 \cdot \nabla) F + (u \cdot \nabla) F^2 - \nabla u^1 F - \nabla u F^2 = \kappa \Delta F, && (4.2.4) \\
&\operatorname{div} F^\top = \operatorname{div} {F^1}^\top = \operatorname{div} {F^2}^\top = 0, \quad |M^1|^2 = |M^2|^2 = 1, && (4.2.5) \\
&\partial_t M + (u^1 \cdot \nabla) M + (u \cdot \nabla) M^2 \\
&= -M^1 \wedge H_{\mathrm{eff}}^1 + M^2 \wedge H_{\mathrm{eff}}^2 - M^1 \wedge M^1 \wedge H_{\mathrm{eff}}^1 + M^2 \wedge M^2 \wedge H_{\mathrm{eff}}^2, && (4.2.6)
\end{aligned}$$

on $\mathbb{T}^2 \times (0, T)$ with the effective magnetic field

$$H_{\mathrm{eff}}^i = \Delta M^i + \mu_0 H_{\mathrm{ext}} - \psi'(M^i), \qquad i = 1, 2$$

and initial conditions

$$\begin{aligned}
&(u(0), F(0), M(0)) = (0, 0, 0) \qquad \text{on } \mathbb{T}^2, && (4.2.7) \\
&\operatorname{div} u_0 = 0, \ \operatorname{div} F_0^\top = 0, \qquad \text{on } \mathbb{T}^2. && (4.2.8)
\end{aligned}$$

The idea is to test the equations with

$$\left(\nabla^\perp (-\Delta)^{-1} \eta, \nabla^\perp (-\Delta)^{-1} \chi_1, \nabla^\perp (-\Delta)^{-1} \chi_2, M\right).$$

We mention the following point: The operator $(-\Delta)^{-1}$ means the solution operator of the Poisson equation, see Theorem 2.0.3. Further we assumed that (u^i, F^i, M^i) are weak

solutions as in Definition 4.1.1 with no singular time appearing. As test functions, we initially allow for smooth functions but by a density argument and the integrability properties assumed, we are able to make the above choice for the test functions.
Calculations to (4.2.2)*:*
We test (4.2.2) by $\nabla^\perp(-\Delta)^{-1}\eta$ and execute first only the spatial estimates which hold true at a.e. time $t \in (0,T)$. It is

$$
\begin{aligned}
-\int_{\mathbb{T}^2} u\cdot\nabla^\perp(-\Delta)^{-1}\eta_t = -\int_{\mathbb{T}^2}\nabla^\perp\eta\cdot\nabla^\perp(-\Delta)^{-1}\eta_t &= \int \eta(-\Delta)(-\Delta)^{-1}\eta_t \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{\eta^2}{2}.
\end{aligned}
$$

Since $\nabla(u) = 0$, we have for the convective term

$$
\begin{aligned}
&\int_{\mathbb{T}^2}(u^1\cdot\nabla)u\cdot\nabla^\perp(-\Delta)^{-1}\eta = \int_{\mathbb{T}^2}(u^1\cdot\nabla)(u-(u))\cdot\nabla^\perp(-\Delta)^{-1}\eta \\
&= \int_{\mathbb{T}^2}\operatorname{div}\Big(u^1\otimes(u-(u))\Big)\cdot\nabla^\perp(-\Delta)^{-1}\eta = -\int_{\mathbb{T}^2}u^1\otimes(u-(u)) : \nabla\nabla^\perp(-\Delta)^{-1}\eta \\
&\le \left\|u^1\right\|_{L^4}\|u-(u)\|_{L^2}\left\|\nabla\nabla^\perp(-\Delta)^{-1}\eta\right\|_{L^4} \le C\left\|u^1\right\|_{L^4}\left\|\nabla^\perp\eta\right\|_{L^2}\|\eta\|_{L^4} \\
&\le \delta\|\nabla\eta\|_{L^2}^2 + C_\delta\left\|u^1\right\|_{L^4}^2\|\eta\|_{L^4}^2 \le \delta\|\nabla\eta\|_{L^2}^2 + C_\delta\left\|u^1\right\|_{L^4}^2\|\eta\|_{L^2}\|\nabla\eta\|_{L^2} \\
&\le 2\delta\|\nabla\eta\|_{L^2}^2 + C_\delta\left\|u^1\right\|_{L^4}^4\|\eta\|_{L^2}^2.
\end{aligned}
$$

Here we used that $\|u-(u)\|_{L^2} = \left\|\nabla^\perp\eta\right\|_{L^2} = \|\nabla\eta\|_{L^2}$, Ladyzhenskaya's inequality (Theorem 2.0.2) and the estimate on the operator $(-\Delta)^{-1}$ on $L^p, 1<p<\infty$ in Theorem 2.0.3. Furthermore, it is

$$
\begin{aligned}
\int_{\mathbb{T}^2}(u\cdot\nabla)u^2\cdot\nabla^\perp(-\Delta)^{-1}\eta &= \int_{\mathbb{T}^2}\Big((u-(u))\cdot\nabla\Big)u^2\cdot\nabla^\perp(-\Delta)^{-1}\eta \\
&\quad + \int_{\mathbb{T}^2}((u)\cdot\nabla)u^2\cdot\nabla^\perp(-\Delta)^{-1}\eta
\end{aligned}
$$

and similarly to above, we conclude the estimate

$$
\int_{\mathbb{T}^2}\Big((u-(u))\cdot\nabla\Big)u^2\cdot\nabla^\perp(-\Delta)^{-1}\eta \le 2\delta\|\nabla\eta\|_{L^2}^2 + \left\|u^2\right\|_{L^4}^4\|\eta\|_{L^2}^2.
$$

Using (4.2.1), we realize that

$$
\begin{aligned}
&\int_{\mathbb{T}^2}((u)\cdot\nabla)u^2\cdot\nabla^\perp(-\Delta)^{-1}\eta = -\int_{\mathbb{T}^2}u^2\otimes(u) : \nabla\nabla^\perp(-\Delta)\eta \\
&\le |u|\left\|u^2\right\|_{L^2}\left\|\nabla\nabla^\perp(-\Delta)^{-1}\eta\right\|_{L^2} \le C\left\|u^2\right\|_{L^2}\|\eta\|_{L^2}\int_{\mathbb{T}^2}|\nabla H_{\mathrm{ext}}||M| \\
&\le C\left\|u^2\right\|_{L^2}\|\eta\|_{L^2}\|\nabla H_{\mathrm{ext}}\|_{L^2}\|M\|_{L^2} \le C\left\|u^2\right\|_{L^2}\|\nabla H_{\mathrm{ext}}\|_{L^2}\Big(\|\eta\|_{L^2}^2 + \|M\|_{L^2}^2\Big)
\end{aligned}
$$

holds true. For the dissipational part, we have

$$\nu \int_{\mathbb{T}^2} \nabla u : \nabla\nabla^\perp(-\Delta)^{-1}\eta = \nu \int_{\mathbb{T}^2} \nabla\nabla^\perp \eta : \nabla\nabla^\perp(-\Delta)^{-1}\eta$$

$$= \nu \int_{\mathbb{T}^2} \nabla^\perp\eta \cdot (-\Delta)\nabla^\perp(-\Delta)^{-1}\eta = \nu \int_{\mathbb{T}^2} |\nabla^\perp\eta|^2 \tag{4.2.9}$$

$$= \nu \int_{\mathbb{T}^2} |\nabla\eta|^2. \tag{4.2.10}$$

The estimates involving ∇M read

$$\begin{aligned}
&\int_{\mathbb{T}^2} \operatorname{div}(\nabla M^1 \odot \nabla M) \cdot \nabla^\perp(-\Delta)^{-1}\eta \leq \left\|\nabla M^1\right\|_{L^4} \|\nabla M\|_{L^2} \left\|\nabla\nabla^\perp(-\Delta)^{-1}\eta\right\|_{L^4} \\
&\leq C \left\|\nabla M^1\right\|_{L^4} \|\nabla M\|_{L^2} \|\eta\|_{L^4} \leq \delta \|\nabla M\|_{L^2}^2 + C_\delta \left\|\nabla M^1\right\|_{L^4}^2 \|\eta\|_{L^4}^2 \\
&\leq \delta \|\nabla M\|_{L^2}^2 + C_\delta \left\|\nabla M^1\right\|_{L^4}^2 \|\eta\|_{L^2} \|\nabla\eta\|_{L^2} \\
&\leq \delta\Big(\|\nabla M\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2 \Big) + C_\delta \left\|\nabla M^1\right\|_{L^4}^4 \|\eta\|_{L^2}^2
\end{aligned}$$

and analogously

$$\int_{\mathbb{T}^2} \operatorname{div}(\nabla M \odot \nabla M^2) \cdot \nabla^\perp(-\Delta)^{-1}\eta \leq \delta\Big(\|\nabla M\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2 \Big) + C_\delta \left\|\nabla M^2\right\|_{L^4}^4 \|\eta\|_{L^2}^2 .$$

Concerning the terms resulting from the elastic energy density, we estimate

$$\int_{\mathbb{T}^2} W'(F^1)F^\top : \nabla\nabla^\top(-\Delta)^{-1}\eta \leq C \int_{\mathbb{T}^2} (1 + |F^1|)|F||\nabla\nabla^\top(-\Delta)^{-1}\eta|$$

where we used assumption (4.0.9). Recalling the definition $F = (\nabla^\perp\chi_1, \nabla^\perp\chi_2)$, we proceed with

$$\begin{aligned}
&\leq C(1 + \left\|F^1\right\|_{L^4}) \left\|\nabla^\perp(\chi_1,\chi_2)\right\|_{L^2} \left\|\nabla\nabla^\top(-\Delta)^{-1}\eta\right\|_{L^4} \\
&\leq C(1 + \left\|F^1\right\|_{L^4}) \|\nabla(\chi_1,\chi_2)\|_{L^2} \|\eta\|_{L^4} \leq \delta \|\nabla(\chi_1,\chi_2)\|_{L^2}^2 + C_\delta \left\|F^1\right\|_{L^4}^2 \|\eta\|_{L^4}^2 \\
&\leq \delta \|\nabla(\chi_1,\chi_2)\|_{L^2}^2 + C_\delta \left\|F^1\right\|_{L^4}^2 \|\eta\|_{L^2} \|\nabla\eta\|_{L^2} \\
&\leq \delta\Big(\|\nabla(\chi_1,\chi_2)\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2 \Big) + C_\delta \left\|F^1\right\|_{L^4}^4 \|\eta\|_{L^2}^2 .
\end{aligned}$$

In the second term, we use the Lipschitz continuity of W' implied by (4.0.10) and obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2} \Big(W'(F^1) - W'(F^2)\Big)(F^2)^\top : \nabla\nabla^\perp(-\Delta)^{-1}\eta \\
&\quad \leq C \int_{\mathbb{T}^2} |F||F^2||\nabla\nabla^\perp(-\Delta)^{-1}\eta|.
\end{aligned}$$

Similar to the above estimate, we get

$$\begin{aligned}
&\int_{\mathbb{T}^2} \Big(W'(F^1) - W'(F^2)\Big)(F^2)^\top : \nabla\nabla^\perp(-\Delta)^{-1}\eta \\
&\quad \leq \delta(\|\nabla(\chi_1,\chi_2)\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2) + C_\delta \left\|F^2\right\|_{L^4}^4 \|\eta\|_{L^2}^2 .
\end{aligned}$$

Regarding the term involving H_{ext}, it is

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla^\top H_{\text{ext}} M \cdot \nabla^\perp(-\Delta)^{-1}\eta &= -\int_{\mathbb{T}^2} (H_{\text{ext}} \cdot \nabla) M \cdot \nabla^\perp(-\Delta)^{-1}\eta \\ &\le \|\nabla M\|_{L^2} \|H_{\text{ext}}\|_{L^4} \left\|\nabla^\perp(-\Delta)^{-1}\eta\right\|_{L^4}.\end{aligned}$$

Now, we apply the Sobolev embedding to $\nabla^\perp(-\Delta)^{-1}\eta$, i.e.

$$\left\|\nabla^\perp(-\Delta)^{-1}\eta\right\|_{L^4} \le C\|\eta\|_{L^2}$$

and proceed with

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla^\top H_{\text{ext}} M \cdot \nabla^\perp(-\Delta)^{-1}\eta &\le C\|\nabla M\|_{L^2} \|H_{\text{ext}}\|_{L^4} \|\eta\|_{L^2} \\ &\le \delta \|\nabla M\|_{L^2}^2 + C_\delta \|H_{\text{ext}}\|_{L^4}^2 \|\eta\|_{L^2}^2.\end{aligned}$$

Calculations to (4.2.4)*:*
Recall that $F^i = (F_1^i, F_2^i) = (\nabla^\perp \chi_1^i, \nabla^\perp \chi_2^i)$ holds true where the upper indices stand for different solutions to (4.2.2)–(4.2.6), while the lower indices denote the different columns of the matrix F^i. As for F^i, the equation (4.2.4) can be read componentwise in columns, i.e.

$$\partial_t F_j + (u^1 \cdot \nabla) F_j + (u \cdot \nabla) F_j^2 - \nabla u^1 F_j - \nabla u F_j^2 = \kappa \Delta F_j, \tag{4.2.11}$$

for $j = 1, 2$. Because of the div – curl-structure in (4.2.4) and the solenoidality of u^i and F^i, we have (componentwise)

$$\begin{aligned}(u^1 \cdot \nabla) F_j - \nabla u^1 F_j &= \operatorname{div}(u^1 \otimes F_j - F_j \otimes u^1), \\ (u \cdot \nabla) F_j^2 - \nabla u F_j^2 &= \operatorname{div}(u \otimes F_j^2 - F_j^2 \otimes u)\end{aligned}$$

for $j = 1, 2$. We test (4.2.11) by $\nabla^\perp(-\Delta)^{-1}\chi_j$, $j = 1, 2$ and perform the spatial estimates for a.e. $t \in [0, T]$. This yields the following estimates:

$$\begin{aligned}-\int_{\mathbb{T}^2} F_j \cdot \nabla^\perp(-\Delta)^{-1}(\chi_j)_t = -\int_{\mathbb{T}^2} \nabla^\perp\chi_j \cdot \nabla^\perp(-\Delta)^{-1}(\chi_j)_t &= \int_{\mathbb{T}^2} \chi_j(-\Delta)(-\Delta)^{-1}(\chi_j)_t \\ = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} \frac{|\chi_j|^2}{2}\end{aligned}$$

and

$$\kappa \int_{\mathbb{T}^2} \nabla F_j \cdot \nabla\nabla^\perp(-\Delta)^{-1}\chi_j = \kappa \int_{\mathbb{T}^2} |\nabla \chi_j|^2$$

similar to the calculations in (4.2.9). Analogously to the momentum equation, the convective terms are mainly dealt with usage of Ladyzhenskaya's inequality, i.e.

$$\begin{aligned}\int_{\mathbb{T}^2} \operatorname{div}(u^1 \otimes \nabla^\perp\chi_j - \nabla^\perp\chi_j \otimes u^1) \cdot \nabla^\perp(-\Delta)^{-1}\chi_j &\le \|\nabla\chi_j\|_{L^2} \left\|u^1\right\|_{L^4} \left\|\nabla\nabla^\perp(-\Delta)^{-1}\chi_j\right\|_{L^4} \\ \le \delta \|\nabla\chi_j\|_{L^2}^2 + C_\delta \left\|u^1\right\|_{L^4}^2 \|\nabla\chi_j\|_{L^2} \|\chi_j\|_{L^2} &\le 2\delta \|\nabla\chi_j\|_{L^2}^2 + C_\delta \left\|u^1\right\|_{L^4}^4 \|\chi_j\|_{L^2}^2.\end{aligned}$$

Terms including u must again be handled by the decomposition $u = u - (u) + (u)$. Since $\operatorname{div} F_j^2 = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} \operatorname{div}(u \otimes F_j^2 - F_j^2 \otimes u) \cdot \nabla^\perp (-\Delta)^{-1} \chi_j \\
&\quad = \int_{\mathbb{T}^2} \operatorname{div}((u-(u)) \otimes F_j^2 - F_j^2 \otimes (u-(u))) \cdot \nabla^\perp (-\Delta)^{-1} \chi_j \\
&\quad\quad - \int_{\mathbb{T}^2} \operatorname{div}(F_j^2 \otimes (u)) \cdot \nabla^\perp (-\Delta)^{-1} \chi_j
\end{aligned}$$

and in the same manner as above, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2} \operatorname{div}((u-(u)) \otimes F_j^2 - F_j^2 \otimes (u-(u))) \cdot \nabla^\perp (-\Delta)^{-1} \chi_j \\
&\quad \le \delta(\|\nabla \eta\|_{L^2}^2 + \|\nabla \chi_j\|_{L^2}^2) + C_\delta \left\|F_j^2\right\|_{L^4}^4 \|\chi_j\|_{L^2}^2 .
\end{aligned}$$

The second term above is estimated with the help of (4.2.1):

$$\begin{aligned}
&\int_{\mathbb{T}^2} \operatorname{div}(F_j^2 \otimes (u)) \cdot \nabla^\perp (-\Delta)^{-1} \chi_j = -\int F_j^2 \otimes (u) : \nabla \nabla^\perp (-\Delta)^{-1} \chi_j \\
&\quad \le |(u)| \left\|F_j^2\right\|_{L^2} \left\|\nabla \nabla^\perp (-\Delta)^{-1} \chi_j\right\|_{L^2} \le C \|\nabla H_{\mathrm{ext}}\|_{L^2} \|M\|_{L^2} \left\|F_j^2\right\|_{L^2} \|\chi_j\|_{L^2} \\
&\quad \le C \|\nabla H_{\mathrm{ext}}\|_{L^2} \left\|F_j^2\right\|_{L^2} \left(\|M\|_{L^2}^2 + \|\chi_j\|_{L^2}^2 \right).
\end{aligned}$$

Calculations to (4.2.6)*:*
Equation (4.2.6) is tested by M and for a.e. $t \in [0,T]$ we obtain

$$\int_{\mathbb{T}^2} \partial_t M \cdot M = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^2} \frac{|M|^2}{2}.$$

The first term of the convective part vanishes due to $\operatorname{div} u^1 = 0$, i.e.

$$\int_{\mathbb{T}^2} (u^1 \cdot \nabla) M \cdot M = 0$$

and since $(M) \int_{\mathbb{T}^2} (u \cdot \nabla) M^2 = 0$ holds true, we have for the second one

$$\begin{aligned}
&\int_{\mathbb{T}^2} (u \cdot \nabla) M^2 \cdot M = \int_{\mathbb{T}^2} (u \cdot \nabla) M^2 \cdot (M - (M)) \le \|u\|_{L^2} \left\|\nabla M^2\right\|_{L^4} \|M - (M)\|_{L^4} \\
&\quad \le C \Big(\|u - (u)\|_{L^2} + C|(u)| \Big) \left\|\nabla M^2\right\|_{L^4} \|M - (M)\|_{L^2}^{1/2} \|\nabla M\|_{L^2}^{1/2} \\
&\quad \le \delta \|\nabla \eta\|_{L^2}^2 + C_\delta |(u)|^2 + C_\delta \left\|\nabla M^2\right\|_{L^4}^2 \|\nabla M\|_{L^2} \|M - (M)\|_{L^2} \\
&\quad \le \delta(\|\nabla \eta\|_{L^2}^2 + \|\nabla M\|_{L^2}^2) + C_\delta (1 + \|\nabla H_{\mathrm{ext}}\|_{L^2}^2 + \left\|\nabla M^2\right\|_{L^4}^4)(\|M\|_{L^2}^2 + |(M)|^2).
\end{aligned}$$

In the next terms, some cancellations appear as well due to the wedge product. It is

$$\int_{\mathbb{T}^2} M \wedge H_{\mathrm{ext}} \cdot M = 0$$

and using (4.0.11), in particular the Lipschitz continuity of ψ' on $\mathbb{S}^2$, we have

$$\int_{\mathbb{T}^2} \Big(M \wedge \psi'(M^1) + M^2 \wedge (\psi(M^1) - \psi(M^2) \Big) \cdot M \le C \int_{\mathbb{T}^2} |M^2||M^1 - M^2||M| \\ \le C \left\| M \right\|_{L^2}^2 .$$

Let $\operatorname{div}(a \wedge \nabla b) := \sum_j \partial_j (a \wedge \partial_j b)$. Then it follows

$$\begin{aligned} &\int_{\mathbb{T}^2} (-M^1 \wedge \Delta M^1 + M^2 \wedge \Delta M^2) \cdot M = \int_{\mathbb{T}^2} -\operatorname{div}(M \wedge \nabla M^1 + M^2 \wedge \nabla M) \cdot M \\ &= \int_{\mathbb{T}^2} M \wedge \nabla M^1 : \nabla M \\ &\le C \left\| \nabla M^1 \right\|_{L^4} \left\| \nabla M \right\|_{L^2} \Big(\left\| M - (M) \right\|_{L^4} + |(M)| \Big) \\ &\le C \left\| \nabla M^1 \right\|_{L^4} \left\| \nabla M \right\|_{L^2} \Big(\left\| M - (M) \right\|_{L^2}^{1/2} \left\| \nabla M \right\|_{L^2}^{1/2} + |(M)| \Big) \\ &\le \delta \left\| \nabla M \right\|_{L^2}^2 + C_\delta \left\| \nabla M^1 \right\|_{L^4}^2 |(M)|^2 + C_\delta \left\| \nabla M^1 \right\|_{L^4}^2 \left\| \nabla M \right\|_{L^2} \left\| M - (M) \right\|_{L^2} \\ &\le 2\delta \left\| \nabla M \right\|_{L^2}^2 + C_\delta \Big(\left\| \nabla M^1 \right\|_{L^4}^2 + \left\| \nabla M^1 \right\|_{L^4}^4 \Big) |(M)|^2 + C_\delta \left\| \nabla M^1 \right\|_{L^4}^4 \left\| M \right\|_{L^2}^2 . \end{aligned}$$

Of course, we have a dissipational part

$$\int_{\mathbb{T}^2} -\Delta M \cdot M = \int_{\mathbb{T}^2} |\nabla M|^2.$$

When treating the highest-order term

$$|\nabla M^1|^2 M^1 - |\nabla M^2|^2 M^2 = (\nabla M : \nabla M^1) M^1 + (\nabla M^2 : \nabla M) M^1 + |\nabla M^2|^2 M,$$

the first two terms can be estimated as follows:

$$\begin{aligned} &\int_{\mathbb{T}^2} (\nabla M : \nabla M^1) M^1 \cdot M \le \left\| \nabla M \right\|_{L^2} \left\| \nabla M^1 \right\|_{L^4} \left\| M^1 \right\|_{L^\infty} \left\| M \right\|_{L^4} \\ &\quad \le 2\delta \left\| \nabla M \right\|_{L^2}^2 + C_\delta \Big(\left\| \nabla M^1 \right\|_{L^4}^2 + \left\| \nabla M^1 \right\|_{L^4}^4 \Big) (\left\| M \right\|_{L^2}^2 + |(M)|^2), \\ &\int_{\mathbb{T}^2} (\nabla M^2 : \nabla M) M^1 \cdot M \\ &\quad \le 2\delta \left\| \nabla M \right\|_{L^2}^2 + C_\delta \Big(\left\| \nabla M^2 \right\|_{L^4}^2 + \left\| \nabla M^2 \right\|_{L^4}^4 \Big) (\left\| M \right\|_{L^2}^2 + |(M)|^2) \end{aligned}$$

as above where we used $|M^1| = 1$. For the last term, we have (we denote the square of M by $[M]^2$, not to be mistaken with M^2)

$$\begin{aligned} &\int_{\mathbb{T}^2} |\nabla M^2|^2 [M]^2 \le \left\| \nabla M^2 \right\|_{L^4}^2 \left\| M \right\|_{L^4}^2 \le C \left\| \nabla M^2 \right\|_{L^4}^2 \Big(\left\| M - (M) \right\|_{L^4}^2 + |(M)|^2 \Big) \\ &\quad \le \delta \left\| \nabla M \right\|_{L^2}^2 + C \left\| \nabla M^2 \right\|_{L^4}^4 (\left\| M \right\|_{L^2}^2 + |(M)|^2) \end{aligned}$$

The remaining terms are mostly standard:

$$\begin{aligned}
&\int_{\mathbb{T}^2} (M^1 \cdot H_{\text{ext}})M^1 - (M^2 \cdot H_{\text{ext}})M^2 \leq 2\,\|H_{\text{ext}}\|_{L^2}\,\|M\|_{L^4}^2 \\
&\quad \leq \delta\,\|\nabla M\|_{L^2}^2 + C\,\|H_{\text{ext}}\|_{L^2}^2\,(\|M\|_{L^2}^2 + |(M)|^2), \\
&\int_{\mathbb{T}^2} (M^1 \wedge M^1 \wedge \psi'(M^1) - M^2 \wedge M^2 \wedge \psi'(M^2)) \cdot M \leq C\,\|M\|_{L^2}^2 .
\end{aligned}$$

We summarize all inequalities and for sufficiently small $\delta > 0$, we absorb all involved terms on the right-hand side to the left-hand side dissipational terms. This yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{T}^2}\left[|\eta|^2 + |(\chi_1,\chi_2)|^2 + |M|^2\right] + |(M)|^2\right\} + \int_{\mathbb{T}^2}\left[\nu|\nabla\eta|^2 + \kappa|\nabla(\chi_1,\chi_2)|^2 + |\nabla M|^2\right] \\
&\quad \leq C\left(1 + \left\|(u^1,u^2,F^1,F^2,\nabla M^1,\nabla M^2)\right\|_{L^4}^4 + \|H_{\text{ext}}\|_{W^{1,2}}^2\right) \\
&\qquad \times \left(\int_{\mathbb{T}^2}\left[|\eta|^2 + |(\chi_1,\chi_2)|^2 + |M|^2\right] + |(M)|^2\right)
\end{aligned}$$

on $[0,T]$ and since the prefactor on the right-hand side is in $L^1(0,T)$ by assumption, Gronwall's inequality gives the desired assertion. □

Remark 4.2.1. *Despite the lengthy calculations, the principles of the above proof are quite simple. It actually corresponds to using the $W^{-1,2}$-norm for the energy quantities $(u, F, \nabla M)$ instead of the L^2-norm. If one attempts to show uniqueness to the harmonic map heat flow* (4.1.1)*, testing by M suffices rather than using ΔM as test function (see [62, p. 137]). But u and F do not undergo the same scaling law as M whereas η and χ do. In this regard, it seems more natural to consider the above proof for uniqueness.*

4.3 Local existence of strong solutions to magneto-viscoelastic flows

In this section, we investigate the local existence theory of strong solutions to (4.0.1)–(4.0.6). In [46], a local well-posedness result for strong solutions subject to boundary conditions on a bounded domain is given. We consider solutions on the torus under more general assumptions on ψ and W and therefore extend the result obtained in [46].

Lemma 4.3.1 ([46], Theorem 2.4)**.** *Let W, ψ and H_{ext} satisfy the assumptions* (4.0.8)–(4.0.11)*. Then for given initial data $v_0 \in W^{1,2}_{\text{div}}(\mathbb{T}^2)$, $F_0 \in W^{1,2}(\mathbb{T}^2)$ and $M \in W^{2,2}(\mathbb{T}^2)$ with $\operatorname{div} F_0^\top = 0$ and $|M_0| \equiv 1$ there exists a $T > 0$ and a unique strong solution $(v, p, F, M) : \mathbb{T}^2 \times [0,T] \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{2\times 2} \times \mathbb{S}^2$ of* (4.0.1)–(4.0.6) *such that*

$$\begin{aligned}
(v, F, M) &\in C\Big([0,T]; W^{1,2}(\mathbb{T}^2) \times W^{1,2}(\mathbb{T}^2) \times W^{2,2}(\mathbb{T}^2)\Big) \cap \\
&\qquad L^2\Big(0,T; W^{2,2}(\mathbb{T}^2) \times W^{2,2}(\mathbb{T}^2) \times W^{3,2}(\mathbb{T}^2)\Big), \\
p &\in L^2(0,T; W^{1,2}(\mathbb{T}^2)).
\end{aligned}$$

The proof is based on the Galerkin method, or more precisely, on a two-level approximation of the system (4.0.1)–(4.0.5). We mainly rely on the arguments and (sub-)results of [7] and [46] and highlight the different arguments needed for our extension.

The first of the two approximation steps consists of solving (4.0.3)–(4.0.5) for a given, smooth enough velocity $u = v$. To begin with, we introduce the basis of the function spaces involved. By $\{\xi_i\}_{i=1}^{\infty}$ we denote an orthogonal basis of $W^{1,2}_{\mathrm{div}}(\mathbb{T}^2)$ that is orthonormal in $L^2_{\mathrm{div}}(\mathbb{T}^2)$ consisting of eigenfunctions of the Stokes operator on $\mathbb{T}^2$. The details can be found in [72, Chapter 2]. For $n \in \mathbb{N}_0$, we set

$$H^n = \mathrm{span}\{\xi_0, \xi_1, \xi_2, \dots, \xi_n\}$$

and denote the projection of L^2 onto H^n by P_{H_n}. Following [7], we introduce, for $t_0 \in (0, T]$ and $L = \|v_0\|_{L^2} + 1$,

$$\begin{aligned} V_n(t_0) = \Big\{ v(t,x) = \sum_{i=1}^{n} g_n^i(t)\xi_i(x) : \mathbb{T}^2 \times [0, t_0) \to \mathbb{R}^2 \Big| \\ \sup_{t\in[0,t_0)} \sum_{i=1}^{n} |g_n^i(t)|^2 \le L^2, g_n^i(0) = \int_{\mathbb{T}^2} v_0(x)\xi_i(x)\,\mathrm{d}x \Big\}, \end{aligned}$$

where the g_n^i are Lipschitz solutions of the corresponding ordinary differential equations, the projection of (4.0.1) on H^n (see [7, Definition 4.1] for the details). For any given $v \in V^n(t_0)$, we find solutions of (4.0.3)–(4.0.5) with $u = v$. Therefore, the result on the first approximation level is given as follows:

Lemma 4.3.2 ([46]). *Let the assumptions of Theorem 4.3.1 be satisfied and let $v \in V_n(t_0)$. Then there is $t^* \in (0, t_0)$ depending on n, L, t_0, F_0, M_0 and H_{ext} such that we can find a unique pair (F, M) possessing the regularity*

$$\begin{aligned} &F \in L^\infty\Big(0, t^*; W^{1,2}(\mathbb{T}^2)\Big) \cap L^2\Big(0, t^*; W^{2,2}(\mathbb{T}^2)\Big),\ \partial_t F \in L^2\Big(0, t^*; L^2(\mathbb{T}^2)\Big), \\ &M \in W^{1,\infty}\Big(0, t^*; L^2(\mathbb{T}^2)\Big) \cap W^{1,2}\Big(0, t^*; W^{1,2}(\mathbb{T}^2)\Big) \cap L^2\Big(0, t^*; W^{3,2}(\mathbb{T}^2)\Big) \end{aligned} \tag{4.3.1}$$

satisfying (4.0.3) *and* (4.0.5) *a.e. in* $(0, t^*) \times \mathbb{T}^2$, *i.e.*

$$\begin{aligned} &\partial_t F + (v \cdot \nabla)F = \nabla v F + \kappa \Delta F, \\ &\partial_t M + (v \cdot \nabla)M = |\nabla M|^2 M + \Delta M - M \wedge H_{\mathrm{eff}} - M \wedge (M \wedge (H_{\mathrm{eff}} - \Delta M)) \end{aligned} \tag{4.3.2}$$

together with the initial conditions from (4.0.6). *Moreover,* $\mathrm{div}\, F^\top = 0$ *and* $|M| = 1$ *a.e. in* $(0, t^*) \times \mathbb{T}^2$.

Remark 4.3.1. *Lemma 4.3.2 is mainly proven in [7, Section 5] with the extension, essentially the regularity statement* $(4.3.1)_1$*, in [46, Lemma 3.1]. It relies on a combination of a Galerkin approximation, standard energy estimates for higher order derivatives of* (F, M) *and Schauder's fixed point Theorem. Since many of these estimates reoccur hereafter, we will not prove the previous lemma but mention the minor technical differences of Lemma 4.3.2 with respect to the version of [46].*

At first, since the spatial domain is $\mathbb{T}^2$*, no boundary conditions occur, which simplifies the situation. Furthermore, we consider a non-convex elastic energy density* W*. This* *poses no obstacle because the (strong) convexity is not used in [7] to conclude the first approximation step (cf. [7, equation (87)]). At last, the anisotropic and stray field term ψ is not considered in [7] and as special quadratic polynomial in [46]. We consider ψ to be an even polynomial of eighth order here. However, in view of the regularity proven for M in* (4.3.1)*, a (non-negative) function $\psi = \psi(M)$ depicts a lower-order term. The non-negativity assures the compatibility with the energy law, see* (4.4.3)*.*

With the existence of a Galerkin approximation $v_n \in V_n(t^*)$ and a corresponding pair (F_n, M_n) at hand, we proceed with the second step. More precisely, we know of the existence of a Galerkin approximation $v_n \in V_n(t^*)$ for some $t^* \in (0,T)$ and fixed $n \in \mathbb{N}$ satisfying

$$\int_{\mathbb{T}^2} \partial_t v_n \cdot \xi + (v_n \cdot \nabla) v_n \cdot \xi - \left(\nabla M_n^\top \nabla M_n - W'(F_n)(F_n)^\top - \nabla v_n\right) \cdot \nabla \xi \\ -(\nabla H_{\text{ext}}^\top M_n) \cdot \xi = 0 \qquad \text{in } (0,t^*) \text{ for all } \xi \in H^n \tag{4.3.3}$$

and a corresponding pair (F_n, M_n) enjoying the regularity (4.3.1) and satisfying

$$\langle \partial_t F_n, \Xi \rangle + \int_{\mathbb{T}^2} (v_n \cdot \nabla) F_n \cdot \Xi - \int_{\mathbb{T}^2} \nabla v_n F_n \cdot \Xi + \kappa \int_{\mathbb{T}^2} \nabla F_n \cdot \nabla \Xi = 0 \\ \text{for all } \Xi \in W^{1,2}(\mathbb{T}^2) \text{ a.e. in } (0,t^*), \tag{4.3.4}$$

$$\partial_t M_n + (v_n \cdot \nabla) M_n - \Delta M_n + M_n \wedge H_{\text{eff}} - |\nabla M_n|^2 M_n \\ + (M_n \cdot H_{\text{res}}) M_n - H_{\text{res}} = 0 \quad \text{a.e. in } (0,t^*) \times \mathbb{T}^2 \tag{4.3.5}$$

with $H_{\text{res}} = H_{\text{ext}} - \psi'(M_n)$. Additionally, the pair (F_n, M_n) fulfills the constraint

$$\operatorname{div} F^\top = 0, \ \ |M_n| = 1 \text{ a.e. in } (0,t^*) \times \mathbb{T}^2 \tag{4.3.6}$$

and

$$v_n(0) = P_{H^n} v_0, \quad F(0) = F_0, \quad M(0) = M_0.$$

Note that all weak formulations stated above actually hold pointwise a.e. on $\mathbb{T}^2 \times (0,t^*)$ due to the regularity given by Lemma 4.3.2, in particular

$$\partial_t F_n + (v_n \cdot \nabla) F_n - \nabla v_n F_n - \kappa \Delta F_n = 0 \quad \text{a.e. in } (0,t^*) \times \mathbb{T}^2. \tag{4.3.7}$$

Next, we look for *a-priori* estimates on (v_n, F_n, M_n). In order to circumvent the non-convexity of W, we define a substitute energy denoted by

$$E(t) = \frac{1}{2}\left(\|v_n(t)\|_{L^2}^2 + \chi \|F_n(t)\|_{L^2}^2 + \|\nabla M_n(t)\|_{L^2}^2 + 2\|\psi(M_n(t))\|_{L^1}\right)$$

and the corresponding dissipation

$$D(t) = \int_0^t \nu \|\nabla v_n\|_{L^2}^2 + \kappa\chi \|\nabla F_n\|_{L^2}^2 + \|M_n \wedge M_n \wedge H_{\text{eff}}\|_{L^2}^2$$

with χ being the constant in (4.0.9). Hence, referring to Lemma 4.4.2, we have

$$E(t) + D(t) \le K(E(0), H_{\text{ext}}, t^*) \tag{4.3.8}$$

for all $t \in [0,t^*)$ which represents one global energy estimate. We need to remark that the constant on the right-hand side remains finite as long as t^* is finite. However, since we are interested in strong solutions, we look for additional estimates on the derivatives of (v_n, F_n, M_n). At this point we deviate from [46] and verify the following Lemma:

Lemma 4.3.3. *Let (v_n, F_n, M_n) satisfy (4.3.3)–(4.3.7) in the interval $[0, t^*)$. Then the following inequality holds true:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v_n\|_{L^2}^2 + \|\nabla F_n\|_{L^2}^2 + \|\Delta M_n\|_{L^2}^2\right) + \left(\nu\|\Delta v_n\|_{L^2}^2 + \kappa\|\Delta F_n\|_{L^2}^2 + \|\nabla\Delta M_n\|_{L^2}^2\right)$$
$$\leq C\left(1 + \|v_n\|_{L^2}^2 + \|F_n\|_{L^2}^2 + \|\nabla M_n\|_{L^2}^2\right)\left(1 + \|\nabla v_n\|_{L^2}^2 + \|\nabla F_n\|_{L^2}^2 + \|\Delta M_n\|_{L^2}^2\right)^2, \tag{4.3.9}$$

for a suitable constant $C > 0$ which only depends on $\mathbb{T}^2, H_{\mathrm{ext}}, W$ and ψ.

Proof. For notational convenience, we forgo the subscripts $(\cdot)_n$ in the following argument. We first test the momentum equation (4.3.3) by $-\Delta v$. Hence recalling that $\operatorname{div} v = 0$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|_{L^2}^2 + \nu\|\Delta v\|_{L^2}^2 = \underbrace{\int_{\mathbb{T}^2}(v\cdot\nabla)v\cdot\Delta v - \mu_0\nabla^\top H_{\mathrm{ext}}M\cdot\Delta v - \operatorname{div}(W'(F)F^\top)\cdot\Delta v}_{\sum_{i=1}^3 I_i}$$
$$+\int_{\mathbb{T}^2}(\Delta v\cdot\nabla)M\cdot\Delta M.$$

Next, we multiply the deformation tensor equation of (4.3.7) by $-\Delta F$ and we integrate both in time and space, to gather

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla F\|_{L^2}^2 + \kappa\|\Delta F\|_{L^2}^2 = \int_{\mathbb{T}^2}(v\cdot\nabla)F : \Delta F - \nabla v F : \Delta F = \sum_{i=4}^5 I_i.$$

Additionally, we apply the gradient to (4.3.5), multiply by $\nabla\Delta M$ and integrate the result in $\mathbb{T}^2$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta M\|_{L^2}^2 + \|\nabla\Delta M\|_{L^2}^2 = -\int_{\mathbb{T}^2}(\Delta v\cdot\nabla)M\cdot\Delta M + \sum_{i=6}^{15} I_i,$$

where

$$\begin{aligned}
I_6 &= -2\sum_{k=1}^2\int_{\mathbb{T}^2}(\partial_k v\cdot\nabla)\partial_k M\cdot\Delta M,\\
I_7 &= -\int_{\mathbb{T}^2}(v\cdot\nabla)\Delta M\cdot\Delta M,\\
I_8 &= -\int_{\mathbb{T}^2}2\left((\nabla^2 M\nabla M)\otimes M\right)\cdot\nabla\Delta M,\\
I_9 &= -\int_{\mathbb{T}^2}|\nabla M|^2\nabla M\cdot\nabla\Delta M,\\
I_{10} &= \int_{\mathbb{T}^2}(\nabla M\wedge(\Delta M + H_{\mathrm{ext}}))\cdot\nabla\Delta M,\\
I_{11} &= \int_{\mathbb{T}^2}(M\wedge\nabla H_{\mathrm{ext}})\cdot\nabla\Delta M,
\end{aligned}$$

$$
\begin{aligned}
I_{12} &= \int_{\mathbb{T}^2} (M \cdot H_{\text{ext}}) \cdot (\nabla M \cdot \nabla \Delta M),\\
I_{13} &= \int_{\mathbb{T}^2} \left((\nabla \Delta M)^\top M\right) \cdot \left((\nabla M)^\top H_{\text{ext}} + (\nabla H_{\text{ext}})^\top M\right),\\
I_{14} &= \int_{\mathbb{T}^2} \nabla H_{\text{ext}} \cdot \nabla \Delta M,\\
I_{15} &= -\int_{\mathbb{T}^2} \nabla(M \wedge \psi'(M) + M \wedge M \wedge \psi'(M)) \cdot \nabla \Delta M.
\end{aligned}
$$

We proceed estimating the terms I_i by Hölder's, Ladyzhenskaya's, Young's and interpolation inequalities. Recall $(v) = \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} v$. We obtain that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}^2} ((v-(v))\cdot\nabla)v\cdot\Delta v \le \|v-(v)\|_{L^4}\|\nabla v\|_{L^4}\|\Delta v\|_{L^2}\\
&\lesssim \delta\|\Delta v\|_{L^2}^2 + \|v-(v)\|_{L^2}\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2} \lesssim \delta\|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2\|\nabla v\|_{L^2}^4\\
I_2 &\lesssim \delta\|\Delta v\|_{L^2}^2 + \|\nabla H_{\text{ext}}\|_{L^2}^2\\
I_3 &\lesssim \|F\|_{L^4}\|\nabla F\|_{L^4}\|\Delta v\|_{L^2} \lesssim \delta\|\Delta v\|_{L^2}^2 + \|F\|_{L^2}\|\nabla F\|_{L^2}^2\|\Delta F\|_{L^2}\\
&\lesssim \delta\left(\|\Delta v\|_{L^2}^2 + \|\Delta F\|_{L^2}^2\right) + \|F\|_{L^2}^2\|\nabla F\|_{L^2}^4,
\end{aligned}
$$

where, in the last inequality, we have used that $|W''(F)| \le C_2$ and $|W'(F)| \le C_1(1+\chi|F|)$ as assumed in (4.0.9)–(4.0.10). Furthermore, we get by Ladyzhenskaya's and Young's inequality

$$
\begin{aligned}
I_4 &= \int_{\mathbb{T}^2} ((v-(v))\cdot\nabla)F\cdot\Delta F \lesssim \delta\|\Delta F\|_{L^2}^2 + \|v\|_{L^2}^2\left(\|\nabla v\|_{L^2}^4 + \|\nabla F\|_{L^2}^4\right)\\
I_5 &\le \int_{\mathbb{T}^2} |\nabla v||F||\Delta F| \lesssim \delta\|\Delta F\|_{L^2}^2 + \|\nabla v\|_{L^4}^2\|F\|_{L^4}^2\\
&\lesssim \delta\|\Delta F\|_{L^2}^2 + \|\nabla v\|_{L^2}\|\Delta v\|_{L^2}\|F\|_{L^2}\|\nabla F\|_{L^2}\\
&\lesssim \delta\left(\|\Delta v\|_{L^2}^2 + \|\Delta F\|_{L^2}^2\right) + \|F\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla F\|_{L^2}^2\\
I_6 &\le 2\int_{\mathbb{T}^2} |\nabla v||\nabla^2 M||\Delta M| \lesssim \|\nabla v\|_{L^2}\|\Delta M\|_{L^4}^2 \lesssim \delta\|\nabla\Delta M\|_{L^2}^2 + \left(\|\nabla v\|_{L^2}^4 + \|\Delta M\|_{L^2}^4\right).
\end{aligned}
$$

An integration by parts and $\operatorname{div} v = 0$ yields that $I_7 = 0$. Similarly as before we obtain that

$$
\begin{aligned}
I_8 &\le \int_{\mathbb{T}^2} |\nabla M||\nabla^2 M||\nabla\Delta M|^2 \lesssim \delta\|\nabla\Delta M\|_{L^2}^2 + \|\nabla M\|_{L^2}^2\|\Delta M\|_{L^2}^4\\
I_9 &\le \int_{\mathbb{T}^2} |\nabla M|^3|\nabla\Delta M| \le \|\nabla M\|_{L^6}^3\|\nabla\Delta M\|_{L^2} \lesssim \delta\|\nabla\Delta M\|_{L^2}^2 + \|\nabla M\|_{L^2}^2\|\Delta M\|_{L^2}^4,
\end{aligned}
$$

where we employed an interpolation of L^6 between H^1 and L^2, namely the inequality $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}^{2/3}\|f\|_{L^2}^{1/3}$ for average-free functions f (see [11, p. 313]). Recalling the assumptions on H_{ext} and ψ, (4.0.8) and (4.0.11), we derive similarly to above that

$$\begin{aligned}
I_{10} &\leq \delta \|\nabla\Delta M\|_{L^2}^2 + (\|H_{\text{ext}}\|_{L^4}^2 + \|\Delta M\|_{L^4}^2) \|\nabla M\|_{L^4}^2 \\
&\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + (\|H_{\text{ext}}\|_{W^{1,2}}^2 + \|\nabla\Delta M\|_{L^2} \|\Delta M\|_{L^2}) \|\nabla M\|_{L^2} \|\Delta M\|_{L^2} \\
&\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + (1 + \|\nabla M\|_{L^2}^2) \|\Delta M\|_{L^2}^4 + 1 \\
I_{11} &\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + 1 \\
I_{12} &\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + \|\nabla M\|_{L^2}^2 \left(\|\Delta M\|_{L^2}^4 + 1\right) \\
I_{13} &\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + \|\nabla M\|_{L^2}^2 \|\Delta M\|_{L^2}^4 + 1 \\
I_{14} &\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + 1 \\
I_{15} &\lesssim \delta \|\nabla\Delta M\|_{L^2}^2 + \|\nabla M\|_{L^2}^2 .
\end{aligned}$$

After summarizing all inequalities, we use the energy estimate (4.3.8) and choosing δ sufficiently small, we absorb all terms involving δ to the right-hand side. Eventually, we deduce that (4.3.9) holds true. $\square$

Inequality (4.3.8) yields a bound on the energy $E(t)$ of (v_n, F_n, M_n). Regarding (4.3.9), a comparison to the initial value problem (see Lemma 4.5.2)

$$\begin{aligned}
z_n'(t) &= c\left(1 + z_n^2(t)\right), \\
z_n(0) &= \|v_0\|_{W^{1,2}}^2 + \|F_0\|_{W^{1,2}}^2 + \|\nabla M_0\|_{W^{1,2}}^2
\end{aligned}$$

shows that there exists a $T^* > 0$ (w.l.o.g. $T^* \leq t^*$) such that

$$\sup_{t\in[0,T^*]} \Bigg(\|\nabla v_n(t)\|_{L^2}^2 + \|\nabla F_n(t)\|_{L^2}^2 + \|\Delta M_n(t)\|_{L^2}^2 + \int_0^t \|\Delta v_n(s)\|_{L^2}^2 + \|\Delta F_n(s)\|_{L^2}^2 + \|\nabla\Delta M_n(s)\|_{L^2}^2 \,\mathrm{d}s \Bigg) \leq C$$

for a constant $C > 0$ for all $n \in \mathbb{N}$ with c depending on the bounds of the $W^{1,2}$-norms of $(u_0, F_0, \nabla M_0)$, the norm of H_{ext} and the constants in the assumptions on W and ψ. Therefore we conclude

$$\begin{aligned}
\|v_n\|_{L^\infty(0,T^*;W^{1,2}(\mathbb{T}^2))} &\leq C, \\
\|v_n\|_{L^2(0,T^*;W^{2,2}(\mathbb{T}^2))} &\leq C, \\
\|F_n\|_{L^\infty(0,T^*;W^{1,2}(\mathbb{T}^2))} &\leq C, \\
\|F_n\|_{L^2(0,T^*;W^{2,2}(\mathbb{T}^2))} &\leq C, \\
\|M_n\|_{L^\infty(0,T^*;W^{2,2}(\mathbb{T}^2))} &\leq C, \\
\|M_n\|_{L^2(0,T^*;W^{3,2}(\mathbb{T}^2))} &\leq C.
\end{aligned}$$

Hence, there exists a (not explicitly relabeled) subsequence $\{(v_n, F_n, M_n)\}_{n=1}^\infty$ such that

$$\begin{aligned}
&v_n \rightharpoonup^* v \ \text{ in } L^\infty(0,T^*;W^{1,2}(\mathbb{T}^2)), && v_n \rightharpoonup v \ \text{ in } L^2(0,T^*;W^{2,2}(\mathbb{T}^2)), \\
&F_n \rightharpoonup^* F \ \text{ in } L^\infty(0,T^*;W^{1,2}(\mathbb{T}^2)), && F_n \rightharpoonup F \ \text{ in } L^2(0,T^*;W^{2,2}(\mathbb{T}^2)), \\
&M_n \rightharpoonup^* M \ \text{ in } L^\infty(0,T^*;W^{2,2}(\mathbb{T}^2)), && M_n \rightharpoonup M \ \text{ in } L^2(0,T^*;W^{3,2}(\mathbb{T}^2)).
\end{aligned}$$

We note that the above estimates and the estimates on the time derivatives of v_n, F_n, M_n and ∇M_n provide strong convergences that are necessary to verify that (v, F, M) satisfies the weak formulation of (4.0.1)–(4.0.5). This is done by deducing estimates on $(\partial_t v_n, \partial_t F_n, \partial_t M_n)$ by duality and the use of the Aubin-Lions lemma 3.1.2, see Steps 3 and 5 of the proof of [7, Theorem 3.2]. Thanks to the regularity of the limit functions v, F, M, we can integrate by parts in space and time to get

$$\begin{aligned}
&\int_0^{T^*}\int_{\mathbb{T}^2} (\partial_t v + (v\cdot\nabla)v + \operatorname{div}\left(\nabla M^\top \nabla M - W'(F)F^\top - \nabla v\right)\\
&\qquad\qquad - (\nabla H_{\text{ext}}^\top M))\cdot\phi \,\mathrm{d}x\,\mathrm{d}t = 0,\\
&\int_0^{T^*}\int_{\mathbb{T}^2} (\partial_t F + (v\cdot\nabla)F - (\nabla v F) - \kappa\Delta F)\cdot\xi\,\mathrm{d}x\,\mathrm{d}t = 0,\\
&\int_0^{T^*}\int_{\mathbb{T}^2} (\partial_t M + (v\cdot\nabla M) + M\times H_{\text{eff}} - |\nabla M|^2 M\\
&\qquad\qquad - \Delta M + M\times(M\times H_{\text{res}}))\cdot\theta\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}\tag{4.3.10}$$

for all $\phi \in L^2(0,T;W^{1,2}_{\text{div}}(\mathbb{T}^2))$, $\xi \in L^2(0,T;W^{1,2}(\mathbb{T}^2))$ and $\theta \in L^2(0,T;L^2(\mathbb{T}^2))$. Obviously, from $(4.3.10)_{2,3}$ it follows that equations (4.0.3) and (4.0.5) are satisfied a.e. in $\mathbb{T}^2\times(0,T^*)$.

The existence of an associated pressure follows similar to [46] from (4.3.10). Obviously, the regularity of v, F, M and H_{ext} implies that

$$\begin{aligned}
G(s) :=& \partial_t v(s) + (v(s)\cdot\nabla)v(s)\\
&+ \operatorname{div}\left(\nabla M(s)\odot\nabla M(s) - W'(F(s))(F(s))^\top - \nabla v(s)\right)\\
&- (\nabla H_{\text{ext}}^\top(s)M(s)) \in L^2(\mathbb{T}^2)
\end{aligned}\tag{4.3.11}$$

for a.a. $s\in[0,T^*]$. Moreover, $(4.3.10)_1$ and the Helmholtz-Weyl decomposition (see [72, Section 2.1]) imply that $G(s) = \nabla\tilde{p}(s)$ for some $\tilde{p}(s)\in W^{1,2}(\mathbb{T}^2)$ a.e. in $[0,T^*]$. Consequently, we have $\|\nabla\tilde{p}\|_{L^2(0,T^*;L^2(\mathbb{T}^2))} = \|G\|_{L^2(0,T^*;L^2(\mathbb{T}^2))}$. Defining $p = \tilde{p} - \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2}\tilde{p}$, we obtain

$$\|p\|_{L^2(0,T^*;L^2(\mathbb{T}^2))} \le C\|\nabla p\|_{L^2(0,T^*;L^2(\mathbb{T}^2))}$$

by the Poincaré inequality. Therefore we have shown the existence of an associated pressure and from (4.3.11) we conclude that (4.0.1) is fulfilled a.e. in $\mathbb{T}^2\times(0,T^*)$. □

Since it is of interest for the following chapter, we restate Lemma 4.3.3 for strong solutions of (4.0.1)–(4.0.5).

Lemma 4.3.4. *Let (u, F, M) satisfy (4.0.1)–(4.0.5) in the interval $[0,T)$. Then the following inequality holds true:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla F\|_{L^2}^2 + \|\Delta M\|_{L^2}^2\right) + \left(\nu\|\Delta u\|_{L^2}^2 + \kappa\|\Delta F\|_{L^2}^2 + \|\nabla\Delta M\|_{L^2}^2\right)\\
&\quad\le C\left(1 + \|u\|_{L^2}^2 + \|F\|_{L^2}^2 + \|\nabla M\|_{L^2}^2\right)\left(1 + \|\nabla u\|_{L^2}^2 + \|\nabla F\|_{L^2}^2 + \|\Delta M\|_{L^2}^2\right)^2,
\end{aligned}\tag{4.3.12}$$

for a suitable constant C. In particular, the loss of regularity of the solution at the time T is characterized by

$$\lim_{t\nearrow T}\int_0^t \|\Delta M\|_{L^2}^2(s)\,\mathrm{d}s = +\infty.$$

Proof. The calculations leading to (4.3.12) are exactly those for the proof of Lemma 4.3.3. Now, as long as the quantity

$$\int_0^t \|\Delta M(s)\|_{L^2}^2 \, \mathrm{d}s$$

remains finite, the quantity

$$\int_0^t \|\nabla v(s)\|_{L^2}^2 + \|\nabla F(s)\|_{L^2}^2 + \|\Delta M(s)\|_{L^2}^2 \, \mathrm{d}s$$

remains finite as well by (4.3.8). Hence we can use Gronwall's inequality in (4.3.12) and conclude the boundedness of the right-hand side on (4.3.12) on $(0, t)$ which shows that the solution remains strong on $(0, t)$. □

4.4 Existence of global weak solutions to magneto-viscoelastic flows

This section is devoted to the proof of existence and uniqueness of Struwe-like solutions. This means, we give an extension to [18, Theorem 2.1] with the difference being the assumptions on W and ψ, cf. (4.0.9)–(4.0.11). We use the previous results in order to show

Theorem 4.4.1 ([18]). *Let $T > 0$ and $(u_0, F_0, M_0) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ be given initial data satisfying $\operatorname{div} u_0 = 0$ and $\operatorname{div} F_0^\top = 0$ in the sense of distributions. Assume that $|M_0| \equiv 1$ almost everywhere in $\mathbb{T}^2$ and H_{ext}, W and ψ satisfy (4.0.8)–(4.0.11). Then there exists a unique global Struwe-like solution to (4.0.1)–(4.0.6) as defined in Definition 4.1.1. Furthermore, there are two constants $\varepsilon_0 > 0$ and $R_0 > 0$ such that for any singular time T_i, there is at least one singular point $y_i \in \mathbb{T}^2$, characterized by the condition*

$$\limsup_{t \nearrow T_i} \int_{B_R(y_i)} |\nabla M(t, x)|^2 \, \mathrm{d}x \geq \varepsilon_0,$$

for any $R > 0$ with $R \leq R_0$.

Remark 4.4.1. *The above statement requires a minor adjustment we avoided for the sake of presentation: If T itself is a time in which a singularity of M occurs, then we slightly increase its value to ensure that M is smooth in T.*

Regarding the proof of 4.4.1, we roughly follow the argument of [18]. However, the part of uniqueness is completely different from [18] and elementary, see Section 4.2. We introduce the following simplified notation of the function spaces in which our solutions belong to

$$H(a,b) := \Big\{ v : [a,b] \times \mathbb{T}^2 \to \mathbb{R}^2 : v \text{ measurable and}$$
$$\text{ess sup}_{a \le t \le b} \int_{\mathbb{T}^2} |v|^2(t) + \int_a^b \int_{\mathbb{T}^2} |\nabla v|^2 < \infty \Big\},$$
$$K(a,b) := \Big\{ G : [a,b] \times \mathbb{T}^2 \to \mathbb{R}^{2\times 2} : G \text{ measurable and}$$
$$\text{ess sup}_{a \le t \le b} \int_{\mathbb{T}^2} |G|^2(t) + \int_a^b \int_{\mathbb{T}^2} |\nabla G|^2 < \infty \Big\}$$

and

$$V(a,b) := \Big\{ M : [a,b] \times \mathbb{T}^2 \to \mathbb{S}^2 : M \text{ measurable and}$$
$$\text{ess sup}_{a \le t \le b} \int_{\mathbb{T}^2} |\nabla M|^2(t) + \int_a^b \int_{\mathbb{T}^2} |\nabla^2 M|^2 + |\partial_t M|^2 < \infty \Big\}.$$

We start by presenting a brief overview of the strategy we perform throughout this section. This strategy is very similar to the one for solutions of the harmonic map heat flow (4.1.1) of [78] outlined in Section 4.1 and the simplified Ericksen-Leslie system in [41, 58].

1. As a first step, we use the existence of strong solutions (cf. Section 4.3) that are defined locally-in-time. More precisely we prove the existence of a suitable time $T^* \in (0,T)$, for which a weak solution of (4.0.1)–(4.0.6) exists within the following functional framework:
$$u,\ F \in C\Big([0,T^*]; L^2(\mathbb{T}^2)\Big) \ \cap \ L^2\Big(0,T^*; H^1(\mathbb{T}^2)\Big),$$
$$M \in C\Big([0,T^*]; H^1(\mathbb{T}^2)\Big) \cap L^2\Big(0,T^*; H^2(\mathbb{T}^2)\Big).$$
In particular, no singularity occurs within the time interval $(0,T^*)$.

2. We secondly extend the aforementioned solution until a first singularity occurs at a time $T_1 > 0$ in $[T^*, T)$. In addition, we show that at time T_1 the system "loses" a fixed amount of energy.

3. We finally extend our solution on the right-hand side of the singularity and we recursively perform the previous steps in $[T^*, T]$. We conclude the recursive procedure showing that no accumulation point is reached by the set of singular times. Hence our solution can be smoothly extended to the entire interval $[0,T]$, with the exception of a finite amount of times in which a singularity can occur.

Figure 4.3 illustrates the difference with respect to a closed physical system. Energy is "produced" due to the appearance of an external magnetic field H_{ext}.

The first part of the above strategy is developed in a rather standard manner: We begin by regularizing the initial data leading to a sequence $(u_0^m,\ F_0^m,\ M_0^m)_{m\in\mathbb{N}}$ in $H^1(\mathbb{T}^2)\times H^1(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ such that

$$u_0^m \to u_0 \quad \text{in} \quad L^2(\mathbb{T}^2), \qquad F_0^m \to F_0 \quad \text{in} \quad L^2(\mathbb{T}^2), \qquad M_0^m \to M_0 \quad \text{in} \quad H^1(\mathbb{T}^2)$$

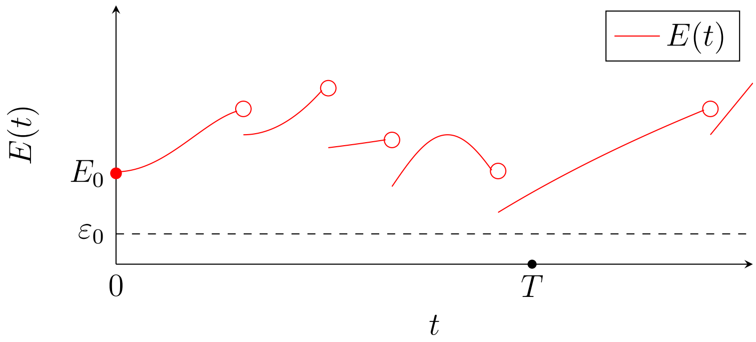

Figure 4.3: Finite number of singularities and energy “creation”

with $\operatorname{div} u_0^m = 0$, $\operatorname{div} F_0^{m\top} = 0$. Here, the important point is that M_0^m is chosen such that $|M_0^m| = 1$ holds true a.e. on $\mathbb{T}^2$. This (differential geometric) fact is proven in [76, p. 267]. Hence, we apply Theorem 4.3.1 which implies that there exists a time $T^m > 0$ and a unique local-in-time strong solution (u^m, F^m, M^m) of (4.0.1)–(4.0.6) with

$$\begin{aligned}(u^m,\, F^m) &\in C([0,T^m], H^1(\mathbb{T}^2)) \cap L^2(0,T^m; H^2(\mathbb{T}^2)),\\ M^m &\in C([0,T^m], H^2(\mathbb{T}^2)) \cap L^2(0,T^m; H^3(\mathbb{T}^2)).\end{aligned}$$

We proceed to prove that there exists a time $T^* \in (0,T)$ that provides a lower bound for the family of the lifespans $T^m > 0$ of the approximate solutions, i.e., $T^* < T^m$, for any $m \in \mathbb{N}$, up to a subsequence. To this end, we perform suitable *a-priori* estimates below, whose calculations are justified since $(u^m, F^m, M^m)_{m\in\mathbb{N}}$ is a strong solution.
Particularly in this part, we follow the strategy used in [41, 58] when treating the well-posedness of the Ericksen-Leslie system for liquid crystals. To do so, we first consider the energy law of (4.0.1)–(4.0.6). Actually, we employ several variants and substitutes of energy inequalities since the appearance of H_{ext}, the non-convexity of W and the dissipational term concerning F destroy any chance of energy conservation in the system. The following energy identity holds true.

Lemma 4.4.1. *Suppose* $(u, F, M) \in H(0,\tilde{T}) \times K(0,\tilde{T}) \times V(0,\tilde{T})$ *is a weak solution to* (4.0.1)–(4.0.7). *Then we have*

$$\begin{aligned}\int_{\mathbb{T}^2} &|u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2 + 2\psi(M(t)) - 2\mu_0 M(t)H_{\text{ext}}(t)\\ &\qquad + 2\int_0^t\int_{\mathbb{T}^2} \nu|\nabla u| + \chi\kappa|\nabla F|^2 + |H_{\text{eff}}|^2 - |M\cdot H_{\text{eff}}|^2\\ &\quad = \int_{\mathbb{T}^2} |u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2 + 2\psi(M_0) - 2\mu_0 M_0\cdot H_{\text{ext}}(0)\\ &\qquad + 2\int_0^t\int_{\mathbb{T}^2} (W'(F) - \chi F) : \nabla u F - 2\mu_0 M\cdot\partial_t H_{\text{ext}}\end{aligned} \tag{4.4.1}$$

for almost all $t \in [0,\tilde{T}]$.

Proof. By a density argument, we multiply (4.0.1) by u, (4.0.3) by χF, where χ is given by (4.0.9), and (4.0.5) by $H_{\text{eff}} = \Delta M + H_{\text{ext}} - \psi'(M)$ and integrate over $\mathbb{T}^2 \times (0,t)$. Using integration by parts and $\operatorname{div} u = 0$, we derive from the first equation

$$\int_{\mathbb{T}^2} \frac{|u(t)|^2}{2} - \frac{|u_0|^2}{2} + \int_0^t \int_{\mathbb{T}^2} \nu |\nabla u|^2 - \nabla M \odot \nabla M : \nabla u + W'(F)F^T : \nabla u - \mu_0 \nabla^\top H_{\mathrm{ext}} M \cdot u = 0.$$

The second equation yields

$$\int_{\mathbb{T}^2} \chi \frac{|F(t)|^2}{2} - \chi \frac{|F_0|^2}{2} + \int_0^t \int_{\mathbb{T}^2} \chi\kappa |\nabla F|^2 - \chi \nabla u F : F = 0.$$

For the third equation, we have

$$\int_0^t \int_{\mathbb{T}^2} (\partial_t M + (u \cdot \nabla) M) \cdot (\Delta M + \mu_0 H_{\mathrm{ext}} - \psi'(M)) = \int_0^t \int_{\mathbb{T}^2} |H_{\mathrm{eff}}|^2 - |M \cdot H_{\mathrm{eff}}|^2$$

where we used $M \wedge H_{\mathrm{ext}} \cdot H_{\mathrm{ext}} = 0$ and $-M \wedge (M \wedge H_{\mathrm{eff}}) = H_{\mathrm{eff}} - (H_{\mathrm{eff}} \cdot M)M$ since $|M| = 1$ a.e. On the left-hand side, note that

$$\int_0^t \int_{\mathbb{T}^2} \partial_t M \cdot \Delta M = \int_{\mathbb{T}^2} \frac{|\nabla M_0|^2}{2} - \frac{|\nabla M(t)|^2}{2},$$
$$\int_0^t \int_{\mathbb{T}^2} (u \cdot \nabla) M \cdot H_{\mathrm{ext}} = - \int_0^t \int_{\mathbb{T}^2} \nabla^\top H_{\mathrm{ext}} M \cdot u$$

by $\operatorname{div} u = 0$. We also have

$$- \int_0^t \int_{\mathbb{T}^2} \partial_t M \cdot \psi'(M) = \int_{\mathbb{T}^2} \psi(M_0) - \psi(M(t)),$$
$$\int_0^t \int_{\mathbb{T}^2} (u \cdot \nabla) M \cdot \Delta M = - \int_0^t \int_{\mathbb{T}^2} \nabla M \cdot \nabla M : \nabla u$$

again by $\operatorname{div} u = 0$ and $\operatorname{div}(\nabla M \odot \nabla M) = (\nabla M)^\top \Delta M + \frac{\nabla}{2} |\nabla M|^2$. The solenoidality of u is also used in

$$\int_0^t \int_{\mathbb{T}^2} (u \cdot \nabla) M \cdot \psi'(M) = 0.$$

Finally, note that

$$\int_0^t \int_{\mathbb{T}^2} \partial_t M \cdot 2\mu_0 H_{\mathrm{ext}} = 2\mu_0 \int_{\mathbb{T}^2} (M(t) \cdot H_{\mathrm{ext}} - M_0 \cdot H_{\mathrm{ext}}) - \int_0^t \int_{\mathbb{T}^2} 2\mu_0 M \cdot \partial_t H_{\mathrm{ext}}$$

holds true, which, by summation over the three identities, yields (4.4.1). □

The "real" energy of (4.0.1)–(4.0.5) rather involves $W(F(t))$ instead of $\chi |F(t)|^2$. However, the above form is more useful since it circumvents coercivity problems arising from the term

$$\kappa \nabla F : W''(F) \nabla F$$

which, due to non-convexity of W, might attain negative values. Let us define some expressions depending on the initial data and the external magnetic field describing suitable behaviors of the energy of the system,

$$\begin{aligned}
2E_0 &:= \int_{\mathbb{T}^2} |u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2 + 2\psi(M_0),\\
2E(t) &= \int_{\mathbb{T}^2} |u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2 + 2\psi(M(t)),\\
D(t) &= \int_0^t \int_{\mathbb{T}^2} \nu|\nabla u|^2 + \chi\kappa|\nabla F|^2 - |H_{\mathrm{eff}}|^2 + |M\cdot H_{\mathrm{eff}}|^2
\end{aligned}$$

for $t \geq 0$.

Lemma 4.4.2. *Suppose* $(u, F, M) \in H(0, \widetilde{T}) \times K(0, \widetilde{T}) \times V(0, \widetilde{T})$ *is a weak solution to* (4.0.1)–(4.0.7)*. Then we have*

$$2E(t) + D(t) \leq 2E(0) + 4\mu_0 \|H_{\mathrm{ext}}\|_{L^\infty} + t\cdot 2\mu_0 \|\partial_t H_{\mathrm{ext}}\|_{L^\infty} + C\int_0^t E(s)\,\mathrm{d}s \tag{4.4.2}$$

for a.e. $t \in [0, \widetilde{T}]$*. In particular, we have*

$$E(t) + D(t) \leq K(E_0, H_{\mathrm{ext}}, \widetilde{T}) \tag{4.4.3}$$

for a.e. $t \in [0, \widetilde{T}]$ *where the constant* $K(E_0, H_{\mathrm{ext}}, \widetilde{T}) > 0$ *depends on* E_0*, the* L^∞*-norm of* H_{ext} *and* $\partial_t H_{\mathrm{ext}}$ *and* $\widetilde{T} > 0$*. Moreover, the following inequality holds true for all* $0 \leq t_1 \leq t_2 \leq \widetilde{T}$*:*

$$2E(t_2) + D(t_2) - D(t_1) \leq 2E(t_1) + C(H_{\mathrm{ext}}, \widetilde{T})K(E_0, H_{\mathrm{ext}}, \widetilde{T})\sqrt{t_2 - t_1}. \tag{4.4.4}$$

For the remainder of this chapter, we denote by

$$K(E_0, H_{\mathrm{ext}}, \widetilde{T})$$

the constant derived in the arguments leading to (4.4.3).

Proof. We use the growth condition (4.0.9) in (4.4.1) in order to estimate

$$\int_0^t \int_{\mathbb{T}^2} (W'(F) - \chi F) : \nabla u F \leq \int_0^t \int_{\mathbb{T}^2} \frac{\nu}{2}|\nabla u|^2 + C\int_0^t \int_{\mathbb{T}^2} |F|^2$$

and standard estimates regarding H_{ext} with $|M| = 1$ yield (4.4.2). By a Gronwall argument, we obtain (4.4.3) from (4.4.2). With this information at hand, we proceed analogously to the proof of (4.4.1), integrate over $\mathbb{T}^2 \times (t_1, t_2)$ but quit to integrate by parts in the term $\int_0^t \int_{\mathbb{T}^2} \partial_t M \cdot H_{\mathrm{ext}}$. The result is

$$\begin{aligned}
&2E(t_2) + 2(D(t_2) - D(t_1))\\
&\quad = 2E(t_1) + 2\mu_0 \int_{t_1}^{t_2}\int_{\mathbb{T}^2} \partial_t M\cdot H_{\mathrm{ext}} + \int_{t_1}^{t_2} (W'(F) - \chi F) : \nabla u F
\end{aligned} \tag{4.4.5}$$

Next, we derive the following estimate from (4.0.5) and (4.4.3):

$$\|\partial_t M\|_{L^2(0,\widetilde{T};L^{\frac{4}{3}})} \lesssim \|(u\cdot\nabla)M\|_{L^4(0,\widetilde{T};L^{\frac{4}{3}})} + \|-M\wedge(M\wedge H_{\mathrm{eff}})\|_{L^2(0,\widetilde{T};L^2)} \lesssim K(E_0,H_{\mathrm{ext}},\widetilde{T}),$$

Once again, we absorb the L^2-norm of ∇u on the right-hand side of (4.4.5) into the dissipation. Eventually, we use this information in

$$\begin{aligned}2\mu_0\int_{t_1}^{t_2}\int_{\mathbb{T}^2}\partial_t M\cdot H_{\mathrm{ext}} + C\int_{t_1}^{t_2}\int_{\mathbb{T}^2}|F|^2 &\le C\,\|H_{\mathrm{ext}}\|_{L^\infty}\sqrt{t_2-t_1} + K(E_0,H_{\mathrm{ext}},\widetilde{T})(t_2-t_1)\\ &\le C(H_{\mathrm{ext}},\widetilde{T})K(E_0,H_{\mathrm{ext}},\widetilde{T})\sqrt{t_2-t_1}.\end{aligned}$$

□

In contrast to Struwe-like solutions for harmonic map flow, we need to deal with the exterior force in terms of the external magnetic field H_{ext}. In particular, additional energy is fed into the system and the energy functional might increase in time. This implies that estimate (4.4.3) is too weak to rule out the possibility of infinitely many singularities appearing in time. Hence estimate (4.4.4) plays a major role in showing that only finitely many singularities of the solution appear at most. To this end, the dependence of $K(E_0,H_{\mathrm{ext}},T)$ just on the data and T will become crucial.
We recall that $(u^m,\ F^m,\ M^m)_{m\in\mathbb{N}}$ stands for a sequence of approximate solutions, whose existence is given by Lemma 4.3.1. Without loss of generality, we can assume that T^m is the lifespan of each approximate solution. Regarding (4.0.1)–(4.0.5), the limit passage of some of the nonlinear terms involving M is made possible by the control of the second gradient of M. To this end, the following lemma shows that the loss of smoothness of our approximate solutions is characterized by the blow-up of the L^2-norm of $\nabla^2 M$.

Lemma 4.4.3 (Blow-up criterion)**.** *Suppose that (u^m,F^m,M^m) is a strong solution to* (4.0.1)–(4.0.6)*. Then the following inequality holds true:*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u^m\|_{L^2}^2+\|\nabla F^m\|_{L^2}^2+\|\Delta M^m\|_{L^2}^2\right)(t)+\left(\nu\,\|\Delta u^m\|_{L^2}^2+\kappa\,\|\Delta F^m\|_{L^2}^2+\|\nabla\Delta M^m\|_{L^2}^2\right)\\ &\quad\le C\left(1+\|u^m\|_{L^2}^2+\|F^m\|_{L^2}^2+\|\nabla M^m\|_{L^2}^2\right)\left(1+\|\nabla u^m\|_{L^2}^2+\|\nabla F^m\|_{L^2}^2+\|\Delta M^m\|_{L^2}^2\right)^2,\end{aligned}\tag{4.4.6}$$

for a suitable constant C that does not depend on the index m. In particular, the loss of regularity of the solution at the lifespan T^m is characterized by

$$\lim_{t\nearrow T^m}\int_0^t\|\Delta M^m\|_{L^2}^2(s)\,\mathrm{d}s=+\infty.$$

Proof. This follows from Lemma 4.3.4 in Section 4.3. □

Because of Lemma 4.4.3, the overall goal is to gain a bound on the critical L^2-norm of ΔM^m. The following semi-localized version of Ladyzhenskaya's inequality forms the corner stone in this argument:

Lemma 4.4.4 ([78], Lemma 3.2). *There exists a constant C_1 such that for any $T > 0$, $f \in H(0,T)$ and any $R > 0$,*

$$\int_{[0,T]\times\mathbb{T}^2} |f|^4\phi \le C_1 \left(\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{T}^2} \int_{B_R(x)} |f(t)|^2 \right) \left(\int_{[0,T]\times\mathbb{T}^2} |\nabla f|^2\phi + R^{-2} \int_{[0,T]\times\mathbb{T}^2} |f|^2\phi \right)$$

holds true for every $\phi \in C_0^\infty(B_R(x))$ with $\phi(y) = \widetilde{\phi}(|y-x|)$ and $\widetilde{\phi}$ nonincreasing.

A partition of unity argument (see [78]) hence entails the following corollary.

Corollary 4.4.2 ([78], Lemma 3.1). *There exists a constant C_1 such that for any $f \in H(0,T)$ and any $R > 0$,*

$$\int_{[0,T]\times\mathbb{T}^2} |f|^4 \le C_1 \left(\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{T}^2} \int_{B_R(x)} |f(t)|^2 \right) \left(\int_{[0,T]\times\mathbb{T}^2} |\nabla f|^2 + R^{-2} \int_{[0,T]\times\mathbb{T}^2} |f|^2 \right)$$

holds true.

Corollary 4.4.2 unlocks a criterion for a bound on $\|\Delta M\|_{L^2}$ in terms of a local smallness condition of ∇M, as expressed in the following lemma.

Lemma 4.4.5. *Let $\widetilde{T} > 0$ be a general positive time and $(u, F, M) \in H(0,\widetilde{T}) \times K(0,\widetilde{T}) \times V(0,\widetilde{T})$ be a weak solution of (4.0.1)–(4.0.7). Then there exists a constant $\varepsilon_1 > 0$ such that if*

$$\operatorname*{ess\,sup}_{0\le t\le \widetilde{T}, x\in\mathbb{T}^2} \int_{B_R(x)} |\nabla M(t)|^2 < \varepsilon_1$$

for a suitable $R > 0$, then the following estimate holds true:

$$\int_{[0,\widetilde{T}]\times\mathbb{T}^2} |\nabla u|^2 + |\nabla F|^2 + |\Delta M|^2 \le C \left[(1 + \widetilde{T}R^{-2}) K(E_0, H_{\text{ext}}, \widetilde{T}) + \|H_{\text{ext}}\|^2_{L^2L^2} + \widetilde{T} \right].$$

Proof. We first remark that the following identity concerning the dissipation of the magnetization field holds true:

$$\begin{aligned} H_{\text{eff}}^2 - (M\cdot H_{\text{eff}})^2 &= |\Delta M|^2 - |\nabla M|^4 + \mu_0^2 H_{\text{ext}}^2 - \mu_0 (M\cdot H_{\text{ext}})^2 \\ &\quad + 2\mu_0 [\Delta M \cdot H_{\text{ext}} - (M\cdot\Delta M)(M\cdot H_{\text{ext}})] \\ &\quad + \psi'(M)^2 - (\psi'(M)\cdot M)^2 - 2\Delta M\cdot\psi'(M) - 2\mu_0 H_{\text{ext}}\cdot\psi'(M) \\ &\quad + 2(\Delta M\cdot M)\cdot(M\cdot\psi'(M)) + 2\mu_0 (H_{\text{ext}}\cdot M)(\psi'(M)\cdot M) \\ &\ge \frac{1}{2}|\Delta M|^2 - C|\nabla M|^4 - C\mu_0^2 |H_{\text{ext}}|^2 - C\sigma^2 \end{aligned}$$

with $\sigma := \sup_{M\in\mathbb{S}^2} |\psi'(M)|$. Thus, we combine the above inequality with (4.4.3) to gather

$$\begin{aligned} &\int_{[0,\widetilde{T}]\times\mathbb{T}^2} \nu|\nabla u|^2 + \chi\kappa|\nabla F|^2 + |\nabla^2 M|^2 \\ &\qquad \le 2\int_{[0,\widetilde{T}]\times\mathbb{T}^2} \nu|\nabla u|^2 + \chi\kappa|\nabla F|^2 + 2\Big(H_{\text{eff}}^2 - (M\cdot H_{\text{eff}})^2\Big) \\ &\qquad\quad + 2|\nabla M|^4 + C\Big(|H_{\text{ext}}|^2 + \sigma^2\Big) \\ &\qquad \le 2K(E_0, H_{\text{ext}}, \widetilde{T}) + \int_{[0,\widetilde{T}]\times\mathbb{T}^2} C\Big(|H_{\text{ext}}|^2 + \sigma^2\Big) + 2|\nabla M|^4. \end{aligned}$$

Now we use Corollary 4.4.2 for $f = \nabla M$, set $\varepsilon_1 = \frac{1}{4C_1}$ and employ (4.4.3) to arrive at the assertion. $\square$

Lemma 4.4.5 becomes useful if one is able to uniformly control the exchange energy $|\nabla M(t)|^2$ on any suitable ball $B_R(x)$ of fixed radius R, i.e., if $\{\nabla M(t)\}_{t\geq 0}$ is equi-integrable. In general, this is not the case since the best control we can achieve from weak solutions is the following energy estimate,

$$\text{ess sup}_{t\in[0,\widetilde{T}]} \int_{\mathbb{T}^2} |\nabla M(t)|^2 \leq C,$$

which only yields a bound in the entire domain $\mathbb{T}^2$. Nevertheless, to better understand the main feature presented by this criterion, we first analyze how the local energy term evolves in time.

Lemma 4.4.6. *Let $\widetilde{T} > 0$ be a general positive time and $(u, F, M) \in H(0, \widetilde{T}) \times K(0, \widetilde{T}) \times V(0, \widetilde{T})$ be a solution to (4.0.1)–(4.0.6) with initial condition $(u_0, F_0, M_0) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$. Then there exist constants $\varepsilon_1 > 0$ and $R > 0$ such that if*

$$\operatorname*{ess\,sup}_{0\leq t\leq \widetilde{T}, x\in\mathbb{T}^2} \int_{B_{2R}(x)} |\nabla M(t)|^2 < \varepsilon_1$$

then for any $t \in (0, \widetilde{T})$, for any $x_0 \in \mathbb{T}^2$ and for any $R > 0$,

$$\begin{aligned}
&\int_{B_R(x_0)} \Big(|u|^2(t) + \chi|F|^2(t) + |\nabla M|^2(t)\Big) \\
&\qquad \leq \Big[\int_{B_{2R}(x_0)} \Big(|u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2\Big)\Big] \\
&\qquad\quad + C_1(1+t)R^2 + C_2 t\Big(1 + \frac{1}{R^2}\Big) + C_3 \frac{t^{\frac{1}{3}}}{R^{\frac{2}{3}}} \Big(1 + t\Big(1 + \frac{1}{R^2}\Big)\Big)^{\frac{2}{3}},
\end{aligned}$$

where C_1, C_2 and C_3 are three positive constants, that depend only on $\|H_{\text{ext}}\|_{W^{1,\infty}}$, E_0 and $K(E_0, H_{\text{ext}}, \widetilde{T})$.

Proof. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ be a cut-off function with $\phi \equiv 1$ on $B_R(x_0)$ and $|\nabla\phi| \lesssim \frac{1}{R}$, $|\nabla^2\phi| \lesssim \frac{1}{R^2}$ for all $R \leq R_0$. Using the local energy inequality provided by Lemma 4.5.1 of the next section, i.e.

$$\begin{aligned}
&\int_{\mathbb{T}^2} \Big(|u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2\Big)\phi^2 \\
&\quad + \int_0^t \int_{\mathbb{T}^2} \Big(\nu|\nabla u|^2 + \kappa\chi|\nabla F|^2 + |\Delta M + |\nabla M|^2 M|^2\Big)\phi^2 \\
&\leq \int_{\mathbb{T}^2} \Big(|u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2\Big)\phi^2 + C\Big\{ \int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2 + |p|\Big)|u||\phi||\nabla\phi| \\
&\quad + \int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2\Big)\Big(|\nabla\phi|^2 + |\phi||\nabla^2\phi|\Big) + \int_0^t \int_{\mathbb{T}^2} |\nabla u||F||\chi F - W'(F)|\phi^2 \\
&\quad + \int_0^t \int_{\mathbb{T}^2} |u|^2\phi^2 + \int_0^t \int_{\mathbb{T}^2} \Big(|H_{\text{ext}}|^2 + |\nabla H_{\text{ext}}|^2 + |\psi'(M)|^2\Big)\phi^2\Big\},
\end{aligned}$$

for $0 \le t \le \widetilde{T}$, we have that

$$
\begin{aligned}
&\int_{B_R(x_0)} \Big(|u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2\Big) \\
&\qquad + \int_0^t \int_{\mathbb{T}^2} \Big(\nu|\nabla u|^2 + \kappa\chi|\nabla F|^2 + |\Delta M + |\nabla M|^2 M|^2\Big)\phi^2 \\
&\qquad \le \int_{B_{2R}(x_0)} \Big(|u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2\Big) \\
&\qquad + C\int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2 + |p|\Big)|u||\phi||\nabla\phi| \\
&\qquad + C\frac{t}{R^2}K(E_0, H_{\text{ext}}, \widetilde{T}) + C\int_0^t \int_{\mathbb{T}^2} |\nabla u||F||\chi F - W'(F)|\phi^2 + C_1(1+t)R^2,
\end{aligned}
\tag{4.4.7}
$$

for a constant C_1 depending on H_{ext} and ψ' and some positive constant C. We used (4.4.3) in the third term on the right-hand side. Hence, we proceed estimating any terms on the right-hand side.

We now introduce a small parameter $\delta > 0$ that will be determined at the end of the proof. First we recall that $|W'(A) - \chi A| \le C_2$ for any $A \in \mathbb{R}^{2\times 2}$ and a positive constant C_2 by (4.0.9). Hence we remark that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{T}^2} |\nabla u||F||\chi F - W'(F)|\phi^2 \\
&\qquad \le C_2 \int_0^t \int_{\mathbb{T}^2} |\nabla u||F|\phi^2 \le \delta \int_0^t \int_{\mathbb{T}^2} |\nabla u|^2\phi^2 + \frac{C}{\delta}\int_0^t \int_{\mathbb{T}^2} \chi|F|^2\phi^2.
\end{aligned}
$$

Next, thanks to Lemma 4.4.4, we obtain that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2\Big)|u||\phi||\nabla\phi| \\
&\le \delta \int_0^t \int_{\mathbb{T}^2} \Big(|u|^4 + |F|^4 + |\nabla M|^4\Big)\phi^2 + \frac{C}{\delta}\int_0^t \int_{\mathbb{T}^2} |u|^2|\nabla\phi|^2 \\
&\le C\delta \operatorname*{ess\,sup}_{0\le s\le \widetilde{T}, x\in\mathbb{T}^2} \Big(\int_{B_R(x)} |u(s)|^2 + \chi|F(s)|^2 + |\nabla M(s)|^2\Big) \times \\
&\quad \times \Big\{ \int_0^t \int_{\mathbb{T}^2} \Big(|\nabla u|^2 + \chi|\nabla F|^2 + |\nabla^2 M|^2\Big)\phi^2 \\
&\quad + \frac{1}{R^2}\int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + \chi|F|^2 + |\nabla M|^2\Big)\phi^2 \Big\} + \frac{C}{\delta}\frac{t}{R^2}K(E_0, H_{\text{ext}}, \widetilde{T}) \\
&\le C\delta K(E_0, H_{\text{ext}}, \widetilde{T}) \left[\int_0^t \int_{\mathbb{T}^2} \Big(|\nabla u|^2 + \chi|\nabla F|^2\Big)\phi^2\right] + C\frac{\delta}{R^2} t K(E_0, H_{\text{ext}}, \widetilde{T})^2 \\
&\quad + C\delta K(E_0, H_{\text{ext}}, \widetilde{T}) \left[\int_0^t \int_{\mathbb{T}^2} |\nabla^2 M|^2\phi^2\right] + \frac{C}{\delta}\frac{t}{R^2}K(E_0, H_{\text{ext}}, \widetilde{T}),
\end{aligned}
\tag{4.4.8}
$$

where we have applied (4.4.3) as well as the assumption (4.0.9) on W. For the pressure p we first note that

$$
p = (-\Delta)^{-1}\operatorname{div}\operatorname{div}(W'(F)F^\top - \nabla M \odot \nabla M - u \otimes u) + (-\Delta)^{-1}\operatorname{div}\left(\nabla^\top H_{\text{ext}} M\right)
$$

holds true weakly and recall Theorem 2.0.3. Hence, thanks to Lemma 4.4.5 for $t = \widetilde{T}$,

$$
\begin{aligned}
\int_0^t \int_{\mathbb{T}^2} |p|^2 &\leq C \int_0^t \int_{\mathbb{T}^2} \Big(|u|^4 + |F|^4 + |\nabla M|^4 + |\nabla H_{\text{ext}}|^2 \Big) \\
&\leq C \underset{0 \leq s \leq \widetilde{T}, x \in \mathbb{T}^2}{\text{ess sup}} \Big(\int_{B_{2R}(x)} |u(s)|^2 + |F(s)|^2 + |\nabla M(s)|^2 \Big) \times \\
&\quad \times \Big(\int_0^t \int_{\mathbb{T}^2} |\nabla u|^2 + |\nabla F|^2 + |\nabla^2 M|^2 + \frac{1}{R^2} \int_0^t \int_{\mathbb{T}^2} |u|^2 + |F|^2 + |\nabla M|^2 \Big) \\
&\quad + \int_0^t \int_{\mathbb{T}^2} |\nabla H_{\text{ext}}|^2 \\
&\leq CK(E_0, H_{\text{ext}}, \widetilde{T}) \Big(\Big(1 + \frac{t}{R^2} \Big) K(E_0, H_{\text{ext}}, \widetilde{T}) + \\
&\quad + \|H_{\text{ext}}\|^2_{L_t^2 L_x^2} + (1 + \|\nabla H_{\text{ext}}\|^2_{L_t^\infty L_x^2}) t + \frac{t}{R^2} K(E_0, H_{\text{ext}}, \widetilde{T}) \Big) \\
&\leq C \Big(K(E_0, H_{\text{ext}}, \widetilde{T}) + 1 \Big)^2 \Big(1 + t \Big(1 + \frac{1}{R^2} \Big) \Big).
\end{aligned}
$$

Thus, we deduce that

$$
\begin{aligned}
\int_0^t \int_{\mathbb{T}^2} |p||u||\phi||\nabla \phi| &\leq \delta \int_0^t \int_{\mathbb{T}^2} |u|^4 \phi^2 + \frac{C}{\delta} \int_0^t \int_{B_{2R}(x_0)} |p|^{\frac{4}{3}} |\phi|^{\frac{2}{3}} |\nabla \phi|^{\frac{4}{3}} \\
&\leq C\delta \underset{0 \leq s \leq \widetilde{T}, x \in \mathbb{T}^2}{\text{ess sup}} \Big(\int_{B_{2R}(x)} |u|^2 \Big) \Big(\int_0^t \int_{\mathbb{T}^2} |\nabla u|^2 \phi^2 + \frac{1}{R^2} \int_0^t \int_{\mathbb{T}^2} |u|^2 \phi^2 \Big) \\
&\quad + \frac{C}{\delta} \int_0^t \int_{B_{2R}(x_0)} |p|^{\frac{4}{3}} |\phi|^{\frac{2}{3}} |\nabla \phi|^{\frac{4}{3}} \\
&\leq CK(E_0, H_{\text{ext}}, \widetilde{T}) \delta \int_0^t \int_{\mathbb{T}^2} \Big(|\nabla u|^2 |\phi|^2 + \frac{|u|^2}{R^2} \Big) + \frac{Ct^{\frac{1}{3}}}{\delta R^{\frac{2}{3}}} \Big(\int_0^t \int_{B_{2R}(x_0)} |p|^2 \Big)^{\frac{2}{3}} \\
&\leq C\delta \int_0^t \int_{\mathbb{T}^2} \Big(|\nabla u|^2 |\phi|^2 + \frac{|u|^2}{R^2} \Big) \\
&\quad + C \frac{t^{\frac{1}{3}}}{R^{\frac{2}{3}}} \Big(1 + t \Big(1 + \frac{1}{R^2} \Big) \Big)^{\frac{2}{3}} \Big(K(E_0, H_{\text{ext}}, \widetilde{T}) + 1 \Big)^{\frac{4}{3}}.
\end{aligned}
$$

A control on $\nabla^2 M$ in terms of ΔM is given by

$$
\int_{\mathbb{T}^2} |\nabla^2 M| \phi^2 \leq \int_{\mathbb{T}^2} |\Delta M|^2 \phi^2 + 4 \int_{\mathbb{T}^2} |\nabla^2 M| |\nabla M| |\phi| |\nabla \phi|
$$

and an application of Young's inequality shows

$$
\int_{\mathbb{T}^2} |\nabla^2 M| \phi^2 \leq C \left(\int_{\mathbb{T}^2} |\Delta M|^2 \phi^2 + \int_{\mathbb{T}^2} |\nabla M|^2 |\nabla \phi|^2 \right).
$$

From Lemma 4.4.4 it follows that

$$\begin{aligned}\int_0^t \int_{\mathbb{T}^2} |\nabla^2 M|^2 \phi^2 &\leq C \int_0^t \int_{\mathbb{T}^2} \Big(\Big(|\Delta M + |\nabla M|^2 M|^2 + |\nabla M|^4 \Big) \phi^2 + |\nabla M|^2 |\nabla \phi|^2 \Big) \\ &\leq C \Big(\varepsilon_1 \int_0^t \int_{\mathbb{T}^2} |\nabla^2 M| \phi^2 + \frac{1}{R^2} \int_0^t \int_{\mathbb{T}^2} |\nabla M|^2 (1 + \phi^2) \\ &\quad + \int_0^t \int_{\mathbb{T}^2} |\Delta M + |\nabla M|^2 M|^2 \phi^2 \Big),\end{aligned}$$

i.e.,

$$\int_0^t \int_{\mathbb{T}^2} |\nabla^2 M|^2 \phi^2 \leq C \int_0^t \int_{\mathbb{T}^2} |\Delta M + |\nabla M|^2 M|^2 + \frac{Ct}{R^2} K(E_0, H_{\text{ext}}, \widetilde{T}). \tag{4.4.9}$$

Finally, for small enough δ, we combine the previous inequalities to deduce that

$$\begin{aligned}\int_{B_R(x_0)} |u|^2(t) + \chi |F|^2(t) + |\nabla M|^2(t) &\leq \int_{B_{2R}(x_0)} \Big(|u_0|^2 + \chi |F_0|^2 + |\nabla M_0|^2 \Big) + \\ &+ C K(E_0, H_{\text{ext}}, \widetilde{T})^2 t \Big(1 + \frac{1}{R^2} \Big) \\ &+ C \frac{t^{\frac{1}{3}}}{R^{\frac{2}{3}}} \Big(1 + t \Big(1 + \frac{1}{R^2} \Big) \Big)^{\frac{2}{3}} \Big(K(E_0, H_{\text{ext}}, \widetilde{T}) + 1 \Big)^{\frac{4}{3}} + C_1 (1 + t) R^2.\end{aligned}$$

Defining the constants

$$C_2 = C K(E_0, H_{\text{ext}}, \widetilde{T})^2, \qquad C_3 = C (K(E_0, H_{\text{ext}}, \widetilde{T}) + 1)^{\frac{4}{3}},$$

we conclude the proof of the lemma. □

Remark 4.4.3. *We assert that Lemma 4.4.6 can be extended to the case of an external magnetic field H_{ext} in $H^1((0,T) \times \mathbb{T}^2)$. The choice of a more regular external field H_{ext} in $W^{1,\infty}$ has been made for the sake of clear presentation, since in this framework we can explicitly control the local energy of the system within the ball $B_{2R}(x_0)$ with respect to the radius $R > 0$ the time $t \in (0,T)$ and the initial data (u_0, F_0, M_0).*

Under the previous considerations, we are finally in the position to address the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. Let $(u_0, F_0, M_0) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ be an initial datum satisfying $|M_0| = 1$, $\operatorname{div} u_0 = 0$ and $\operatorname{div}_0 F^\top = 0$. As depicted at the beginning of this section, we consider a sequence $(u_0^m, F_0^m, M_0^m)_m \subset H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ satisfying the constraints $\operatorname{div} u_0^m, \operatorname{div} F_0^{m\top} = 0$ and $|M_0^m| = 1$ and converging strongly in $L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ to (u_0, F_0, M_0). Every triple (u_0^m, F_0^m, M_0^m) generates a strong solution (u^m, F^m, M^m) to (4.0.1)–(4.0.5) on a maximal time interval $[0, T^m)$ according to Theorem 4.3.1.

By Lemma 4.4.3, the solution (u^m, F^m, M^m) satisfies $\lim_{t \nearrow T^m} \int_0^t \|\Delta M^m\|_{L^2}^2 (s) \, \mathrm{d}s = +\infty$. In turn, Lemma 4.4.5 provides a bound on $\int_0^t \|\Delta M_m\|_{L^2}^2 (s) \, \mathrm{d}s$ if

$$\operatorname*{ess\,sup}_{0 \leq s \leq t, x \in \mathbb{T}^2} \int_{B_R(x)} |\nabla M^m(s)|^2 < \varepsilon_1$$

for some $\varepsilon_1 > 0$ and $R > 0$. Lemma 4.4.6 yields for any $t \in (0, T^m)$ the estimate

$$\begin{aligned}\int_{B_R(x_0)} &|u^m(t)|^2 + |F^m(t)|^2 + |\nabla M^m(t)|^2 \\ &\leq \int_{B_{2R}(x_0)} |u_0^m|^2 + |F_0^m|^2 + |\nabla M_0^m|^2 \\ &\quad + C_1(1+t)R^2 + C_2 t\Big(1 + \frac{1}{R^2}\Big) + C_3 \frac{t^{\frac{1}{3}}}{R^{\frac{2}{3}}} \left(1 + t\Big(1 + \frac{1}{R^2}\Big)\right)^{\frac{2}{3}}.\end{aligned} \tag{4.4.10}$$

We can choose an $R > 0$ such that

$$\int_{B_{2R}(x_0)} |u_0^m|^2 + |F_0^m|^2 + |\nabla M_0^m|^2 < \frac{\varepsilon_1}{4}, \qquad C_1(1+t)R^2 < \frac{\varepsilon_1}{4}$$

for all $t \in (0, T^m)$, $x_0 \in \mathbb{T}^2$ and $m \in \mathbb{N}$. Hence, we define

$$T^* := \min\left\{ \frac{\varepsilon_1 R^2}{4C_2(1+R^2)}, \frac{\varepsilon_1^3 R^2}{4^3 C_3^3 \Big(1 + T\Big(1 + \frac{1}{R^2}\Big)\Big)^2} \right\}.$$

Combining the above relations with (4.4.10), we obtain that

$$\operatorname*{ess\,sup}_{0 \leq s \leq \min\{T^*, T^m\}} \int_{B_R(x_0)} |\nabla M^m(s)|^2 < \varepsilon_1$$

for all $x_0 \in \mathbb{T}^2$ and $m \in \mathbb{N}$. Hence Lemma 4.4.5 yields that

$$\int_0^{\min\{T^*, T^m\}} \|\Delta M^m(s)\|_{L^2}^2 \, \mathrm{d}s < +\infty,$$

which can only be the case if $T^* < T^m$ for every $m \in \mathbb{N}$. Therefore, the strong solution (u^m, F^m, M^m) exists on $[0, T^*]$, where we highlight that T^* does not depend on m. Finally, we pass to the limit with the *a-priori* bounds given in energy inequality (4.4.3) and Lemma 4.4.5 which yields local existence of a solution with

$$\begin{aligned}(u, F, M) \in C&\Big([0, T^*]; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)\Big) \cap \\ &L^2\Big(0, T^*; H^1(\mathbb{T}^2) \times H^1(\mathbb{T}^2) \times H^2(\mathbb{T}^2)\Big)\end{aligned}$$

for given initial data $(u_0, F_0, M_0) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$.
Because of uniqueness of these solutions (cf. Theorem 4.2.1), we can extend our solution up to a singular time $T_1 \in (0, T)$, characterized by the following relation

$$\operatorname*{ess\,sup}_{0 \leq t \leq T_1, x \in \mathbb{T}^2} \int_{B_R(x)} |\nabla M(t)|^2 \geq \varepsilon_1$$

for any $R > 0$. Even more, for every $0 < \delta < T_1$ and $R > 0$, we have

$$\operatorname*{ess\,sup}_{T_1 - \delta \leq t \leq T_1, x \in \mathbb{T}^2} \int_{B_R(x)} |\nabla M(t)|^2 \geq \varepsilon_1$$

because the solution is regular on $\mathbb{T}^2 \times [0, T_1)$. This assertion in turn implies

$$\sup_{x\in\mathbb{T}^2} \limsup_{t\nearrow T_1} \int_{B_R(x)} |\nabla M(t)|^2 \geq \varepsilon_1 \tag{4.4.11}$$

for any $R > 0$. Since $(u, F, M) \in C_w([0, T_1]; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2))$, the solution (u, F, M) is well-defined at the time T_1, in particular $(u, F, M)(T_1) \in L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ with $\operatorname{div} u(T_1) = 0, \operatorname{div} F^\top(T_1) = 0, |M(T_1)| = 1$. Therefore we claim that the following loss of energy occurs at this first time singularity (see (4.4.4)):

$$2E(T_1) \leq 2E_0 + CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{T_1} - \varepsilon_1. \tag{4.4.12}$$

In other words, the external field H_{ext} and the non-convexity of W (mathematically) feeds energy into the system through $CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{T_1}$ while the singularity decreases the total energy of the fixed amount given by $\varepsilon_1 > 0$. To prove this statement we proceed by contradiction. We assume that (4.4.12) is false, then it follows by (4.4.4) that

$$\begin{aligned}
0 &\leq \underbrace{\limsup_{t\nearrow T_1} (2E(t) - 2E(T_1))}_{=:a} \\
&< \limsup_{t\nearrow T_1} \Big(2E_0 + CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{t} - 2E_0 - CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{T_1} + \varepsilon_1\Big) = \varepsilon_1.
\end{aligned}$$

Defining the local energy $2E_{R,x}(t) := \int_{B_R(x)} |u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2 + 2\psi(M(t))$ for $t \in [0, T]$, we have for any $x \in \mathbb{T}^2$

$$\begin{aligned}
\limsup_{t\nearrow T_1} \int_{B_R(x)} |\nabla M(t)|^2 &\leq \limsup_{t\nearrow T_1}\ 2E_{R,x}(t) = \limsup_{t\nearrow T_1} \Big\{E_{R,x}(t) - E_{R,x}(T_1) + E_{R,x}(T_1)\Big\} \\
&\leq a + E_{R,x}(T_1) < \varepsilon_1
\end{aligned}$$

for a sufficiently small radius $R > 0$ which is a contradiction to (4.4.11).
Since $(u(T_1), F(T_1), M(T_1))$ is defined in $L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$, we can start our entire procedure once again with this new set of initial data, extending our solution to a new time interval $[T_1, T_2]$, where a new singularity appears at time T_2:

$$\begin{aligned}
2E(T_2) &\leq 2E(T_1) + CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{T_2 - T_1} - \varepsilon_1 \\
&\leq 2E_0 + CK(E_0, H_{\text{ext}}, \widetilde{T})(\sqrt{T_1} + \sqrt{T_2 - T_1}) - 2\varepsilon_1.
\end{aligned}$$

Then, we continue this procedure by recursion, leading to a unique solution on the intervals $[T_1, T_2], [T_2, T_3]$ and so on. Next we show that, at most, a finite amount of singularities occurs before reaching the final time T of system (4.0.1)–(4.0.6). This is satisfied if we prove that there is no accumulation point for the set of any singular time T_i. To this end, we proceed by contradiction: We assume that there exists a sequence of singular times $(T_i)_{i\in\mathbb{N}}$ such that $T_i < T_{i+1} < T$ (with an abuse of notation we set $T_0 = 0$). By (4.4.12), every finite maximal time of existence comes with the loss of ε_1 for the energy, more precisely

$$\begin{aligned}
0 \leq 2E(T_n) &\leq 2E_0 + \sum_{i=1}^{n} \Big(CK(E_0, H_{\text{ext}}, \widetilde{T})\sqrt{T_i - T_{i-1}} - \varepsilon_1\Big) \\
&\leq 2E_0 - n\varepsilon_1 + CK(E_0, H_{\text{ext}}, \widetilde{T}) \sum_{i=1}^{n} \sqrt{T_i - T_{i-1}}.
\end{aligned} \tag{4.4.13}$$

Since $T_i - T_{i-1}$ converges towards 0, eventually $\varepsilon_1 > K_E\sqrt{T_i - T_{i-1}}$ and thus the right-hand side of the above inequality converges towards $-\infty$ as n goes to $+\infty$, which is a contradiction. Thus we deduce that only a finite amount of time singularities occurs before the solution becomes smooth until the final time $T > 0$. The uniqueness follows as well by a recursive application of Theorem 4.2.1 and the uniqueness of the weak limit for $(u(t), F(t), \nabla M(t))$ in the singular times $T_i, i = 1, ..., n$. This concludes the proof of Theorem 4.4.1. $\square$

4.5 Auxiliary results

In this section, we provide the proof of some technical results used in the previous sections. First, we have a localized form of the energy law for (4.0.1)–(4.0.5).

Lemma 4.5.1. *Let $\widetilde{T} > 0$ be a general positive time and let $(u, F, M) \in H(0, \widetilde{T}) \times K(0, \widetilde{T}) \times V(0, \widetilde{T})$ be a weak solution of (4.0.1)–(4.0.7) with initial condition $(u_0,\ F_0,\ M_0)$. Further, let ϕ be a smooth function on $\mathbb{T}^2$. Then the following identity holds true*

$$
\begin{aligned}
&\int_{\mathbb{T}^2} \Big(|u(t)|^2 + \chi|F(t)|^2 + |\nabla M(t)|^2\Big)\phi^2 \\
&\quad + \int_0^t \int_{\mathbb{T}^2} \Big(\nu|\nabla u|^2 + \kappa\chi|\nabla F|^2 + |\Delta M + |\nabla M|^2 M|^2\Big)\phi^2 \\
&\le \int_{\mathbb{T}^2} \Big(|u_0|^2 + \chi|F_0|^2 + |\nabla M_0|^2\Big)\phi^2 + C\Big\{ \int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2 + |p|\Big)|u||\phi||\nabla\phi| \\
&\quad + \int_0^t \int_{\mathbb{T}^2} \Big(|u|^2 + |F|^2 + |\nabla M|^2\Big)\Big(|\nabla\phi|^2 + |\phi||\nabla^2\phi|\Big) + \int_0^t \int_{\mathbb{T}^2} |\nabla u||F||\chi F - W'(F)|\phi^2 \\
&\quad + \int_0^t \int_{\mathbb{T}^2} |u|^2\phi^2 + \int_0^t \int_{\mathbb{T}^2} \Big(|H_{\text{ext}}|^2 + |\nabla H_{\text{ext}}|^2 + |\psi'(M)|^2\Big)\phi^2\Big\}
\end{aligned}
$$

for all $0 \le t \le \widetilde{T}$.

Proof. By a density argument, we multiply (4.0.1) by $u\phi^2$ and integration over $\mathbb{T}^2$, we have that

$$
\begin{aligned}
&\int_{\mathbb{T}^2} u_t \cdot u\phi^2 + (u\cdot\nabla)u \cdot u\phi^2 - \nu\Delta u \cdot u\phi^2 + \nabla p \cdot u\phi^2 \\
&\qquad\qquad = \int_{\mathbb{T}^2} \Big(- \operatorname{div}(\nabla M \odot \nabla M)\cdot u + \operatorname{div}(W'(F)F^\top) + \mu_0\nabla^\top H_{\text{ext}} M\Big)\cdot u\phi^2.
\end{aligned}
$$

We first proceed analyzing any term on the left-hand side, since the ones on the right-hand side will eventually be simplified under suitable combination with terms of the other equations. Thus

$$\begin{aligned}
\int_{\mathbb{T}^2} u_t \cdot u\phi^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} |u|^2\phi^2,\\
\int_{\mathbb{T}^2} (u\cdot\nabla)u\cdot u\phi^2 &= \int_{\mathbb{T}^2} u_j\partial_j u_i \cdot u_i\phi^2 = -\int_{\mathbb{T}^2} \frac{|u|^2}{2} u_j 2\phi\partial_j\phi \le C\int_{\mathbb{T}^2} |u|^3|\phi||\nabla\phi|,\\
-\nu\int_{\mathbb{T}^2} \Delta u\cdot u\phi^2 &= \nu\int_{\mathbb{T}^2} \partial_j u_i\partial_j u_i\phi^2 + \partial_j u_i \cdot u_i 2\phi\partial_j\phi\\
&= \nu\int_{\mathbb{T}^2} |\nabla u|^2\phi^2 - \nu\int_{\mathbb{T}^2} |u|^2(\partial_j\phi\partial_j\phi + \phi\partial_j^2\phi)\\
&= \nu\int_{\mathbb{T}^2} |\nabla u|^2\phi^2 - |u|^2(|\nabla\phi|^2 + \phi\Delta\phi)\\
&\ge \nu\int_{\mathbb{T}^2} |\nabla u|^2\phi^2 - |u|^2(|\nabla\phi|^2 + |\phi||\nabla^2\phi|),\\
\int_{\mathbb{T}^2} \nabla p\cdot u\phi^2 &= -\int_{\mathbb{T}^2} 2pu\cdot\phi\nabla\phi \le \int_{\mathbb{T}^2} 2|p||u||\phi||\nabla\phi|.
\end{aligned}$$

Furthermore, the contribution of the external magnetic field to the momentum equation is dealt with by

$$\mu_0\int_{\mathbb{T}^2} \nabla^\perp H_{\mathrm{ext}} M\cdot u\phi^2 \le C\Big(\int_{\mathbb{T}^2} |\nabla H_{\mathrm{ext}}|^2\phi^2 + \int_{\mathbb{T}^2} |u|^2\phi^2\Big).$$

Next, we multiply the equation for F in (4.0.3) by $\chi F\phi^2$, where we recall that the positive constant $\chi > 0$ is such that $|W'(A) - \chi A| \le C_2$, for any $A \in \mathbb{R}^2$. Integrating over $\mathbb{T}^2$, we gather

$$\chi\int_{\mathbb{T}^2} \Big(F_t : F\phi^2 + (u\cdot\nabla)F : F\phi^2 - \nabla uF : F\phi^2 - \kappa\Delta F : F\phi^2\Big) = 0,$$

where

$$\begin{aligned}
\int_{\mathbb{T}^2} F_t : F\phi^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} |F|^2\phi^2,\\
\int_{\mathbb{T}^2} (u\cdot\nabla)F : F\phi^2 &= -\int_{\mathbb{T}^2} |F|^2 u_j\phi\partial_j\phi \le \int_{\mathbb{T}^2} |F|^2|u||\phi||\nabla\phi|,\\
-\kappa\int_{\mathbb{T}^2} \Delta F : F\phi^2 &= \kappa\int_{\mathbb{T}^2} \Big(\nabla F : \nabla F\phi^2 + \partial_j F_{ik}F_{ik}2\phi\partial_j\phi\Big)\\
&= \kappa\int_{\mathbb{T}^2} \Big(|\nabla F|^2\phi^2 - |F|^2\Big(|\nabla\phi|^2 + \phi\Delta\phi\Big)\Big)\\
&\ge \kappa\int_{\mathbb{T}^2} \Big(|\nabla F|^2\phi^2 - |F|^2\Big(|\nabla\phi|^2 + |\phi||\nabla^2\phi|\Big)\Big).
\end{aligned}$$

Additionally, we have that

$$\begin{aligned}
&-\int_{\mathbb{T}^2} \chi\nabla uF : F\phi^2 + \operatorname{div}(W'(F)F^\top)\cdot u\phi^2 = -\int_{\mathbb{T}^2} \partial_j u_i \cdot F_{jk}\chi F_{ik}\phi^2 + \partial_j(W'(F)_{ik}F_{jk})u_i\phi^2\\
&= \int_{\mathbb{T}^2} u_i W'(F)_{ik}F_{jk}2\phi\partial_j\phi - \int_{\mathbb{T}^2} \nabla uF : (\chi F - W'(F))\phi^2\\
&\le C\Big\{\int_{\mathbb{T}^2} |u||F|^2|\phi||\nabla\phi| + \int_{\mathbb{T}^2} |u|^2\phi^2 + \int_{\mathbb{T}^2} |F|^2|\nabla\phi|^2\Big\} + \int_{\mathbb{T}^2} |\nabla u||F||\chi F - W'(F)|\phi^2.
\end{aligned}$$

Moreover, multiplying the LLG equation (4.0.5) by $-(\Delta M+|\nabla M|^2 M)\phi^2$ and integrating over $\mathbb{T}^2$, we deduce that

$$\begin{aligned}
&-\int_{\mathbb{T}^2}\Big(M_t+(u\cdot\nabla)M\Big)\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2\\
&\quad=\int_{\mathbb{T}^2} M\wedge H_{\text{eff}}\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2+\int_{\mathbb{T}^2} M\wedge(M\wedge H_{\text{eff}})\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2.
\end{aligned}$$

Developing the first term on the right-hand side, we infer

$$\begin{aligned}
\int_{\mathbb{T}^2} M\wedge H_{\text{eff}}\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2 &= \int_{\mathbb{T}^2} M\wedge\Big(\mu_0 H_{\text{ext}}-\psi'(M)\Big)\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2\\
&\leq C\int_{\mathbb{T}^2}\Big(|H_{\text{ext}}|^2+|\psi'(M)|^2\Big)\phi^2\\
&\quad+\frac14\int_{\mathbb{T}^2}|\Delta M+|\nabla M|^2M|^2\phi^2,
\end{aligned}$$

while the second term yields

$$\begin{aligned}
&\int_{\mathbb{T}^2} M\wedge(M\wedge H_{\text{eff}})\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2\\
&\quad=\int_{\mathbb{T}^2} M\wedge\Big(M\wedge(\mu_0H_{\text{ext}}-\psi'(M))\Big)\cdot(\Delta M+|\nabla M|^2M)\phi^2-\int_{\mathbb{T}^2}|\Delta M+|\nabla M|^2M|^2\phi^2\\
&\quad\leq C\int_{\mathbb{T}^2}(|H_{\text{ext}}|^2+|\psi'(M)|^2)\phi^2-\frac34|\Delta M+|\nabla M|^2M|^2\phi^2.
\end{aligned}$$

We hence remark that the following estimates holds true (recall $(\nabla M)^\top M=$ since $|M|=$ 1 a.e.)

$$\begin{aligned}
&\int_{\mathbb{T}^2}(u\cdot\nabla)M\cdot\Big(\Delta M+|\nabla M|^2M\Big)\phi^2-\operatorname{div}(\nabla M\odot\nabla M)\cdot u\phi^2\\
&\quad=\int_{\mathbb{T}^2} u_i\partial_iM_k\cdot\partial_j^2M_k\phi^2-\partial_j(\partial_iM_k\partial_jM_k)u_i\phi^2=\int_{\mathbb{T}^2}\frac12|\nabla M|^2u\cdot 2\phi\nabla\phi\\
&\quad\leq C\int_{\mathbb{T}^2}|\nabla M|^2|u||\phi||\nabla\phi|.
\end{aligned}$$

Next, we have $M_t\cdot|\nabla M|^2M=|\nabla M|^2\partial_t(|M|^2/2)=0$ and

$$\int_{\mathbb{T}^2} M_t\cdot(-\Delta M)\phi^2=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}|\nabla M|^2\phi^2+\int_{\mathbb{T}^2}\partial_tM_j\partial_kM_j2\phi\partial_k\phi.$$

Thanks to the LLG equation,

$$\begin{aligned}
\int_{\mathbb{T}^2} \partial_t M_j \partial_k M_j 2\phi \partial_k \phi &= -\int_{\mathbb{T}^2} (u \cdot \nabla M_j) \partial_k M_j 2\phi \partial_k \phi \\
&\quad - \int_{\mathbb{T}^2} \Big(M \wedge (\Delta M + \mu_0 H_{\text{ext}} - \psi'(M) \Big)_j \partial_k M_j 2\phi \partial_k \phi \\
&\quad - \int_{\mathbb{T}^2} \Big(M \wedge M \wedge \Big(\Delta M + \mu_0 H_{\text{ext}} - \psi'(M) \Big) \Big)_j \partial_k M_j 2\phi \partial_k \phi \\
&= - \int_{\mathbb{T}^2} (u \cdot \nabla M_j) \partial_k M_j 2\phi \partial_k \phi \\
&\quad - \int_{\mathbb{T}^2} (M \wedge (\Delta M + |\nabla M|^2 M + \mu_0 H_{\text{ext}} - \psi'(M))_j \partial_k M_j 2\phi \partial_k \phi \\
&\quad - \int_{\mathbb{T}^2} \Big(M \wedge M \wedge \Big(\Delta M + |\nabla M|^2 M + \mu_0 H_{\text{ext}} - \psi'(M) \Big) \Big)_j \partial_k M_j 2\phi \partial_k \phi \\
&\leq C \int_{\mathbb{T}^2} |u| |\nabla M|^2 |\phi| |\nabla \phi| + \frac{1}{4} \int_{\mathbb{T}^2} |\Delta M + |\nabla M|^2 M|^2 \phi^2 \\
&\quad + C \int_{\mathbb{T}^2} |\nabla M|^2 |\nabla \phi|^2 + C \int_{\mathbb{T}^2} (|H_{\text{ext}}|^2 + |\psi'(M)|^2) \phi^2.
\end{aligned}$$

Finally, integration over $[0, t]$ yields the assertion. □

The second result provides a local-in-time substitute of Gronwall's inequality.

Lemma 4.5.2. *Let $z : [0, t^+) \to \mathbb{R}_0^+$ solve*

$$z' = c(1 + z^2), \qquad z(0) \geq 0 \tag{4.5.1}$$

for some $c > 0$. Then if $y \geq 0$ solves

$$y' \leq c(1 + y^2), \qquad z(0) \geq y(0) \geq 0 \tag{4.5.2}$$

on $[0, t^*)$, it is $y \leq z$ on $[0, t^*)$.

Proof. Taking the difference of (4.5.1) and (4.5.2) gives

$$z' - y' \geq c(z^2 - y^2) = c(z - y)(z + y) \geq c(z - y).$$

If $z(t_0) - y(t_0) \geq 0$ for some $t_0 \in [0, t^*)$, the function $t \mapsto z(t) - y(t)$ is therefore non-negative on some interval to the right of t_0 by Taylor's formula. Hence the set $A = \{t \in [0, t^*) : z(t) - y(t) \geq 0\}$ is right-open, non-empty and closed by continuity of z and y. So $A = [0, t^*)$. □

Chapter 5

Conclusion and open problems

Finally, we relate the results of this thesis to some open problems. Starting with Theorem 3.1.2 and the Ericksen-Leslie model for liquid crystals, we positively answer the convergence of the Ginzburg-Landau approximations in two spatial dimensions. We already mentioned in Remark 3.1.10 that the deeper reason behind the validity of Theorem 3.1.2 consists of the Hopf differential. Yet, the proof suggests that the set of singularities of a solution to (3.1.6)–(3.1.8) might be finite at any time $t \geq 0$. A more elaborate formulation is the following conjecture: As $\varepsilon \to 0^+$ in the Ginzburg-Landau approximation (3.1.1)–(3.1.3), the limit point (v, d) is a regular solution to the Ericksen-Leslie system away from a set of two-dimensional parabolic Hausdorff measure. Such a behavior roughly corresponds to an evolution of spatial concentration points of the energy on a rectifiable curve and is known for the harmonic map heat flow (see [16, 62]). If true however, the obstacles to a verification of this statement are the non-localizibilty of energy in (3.1.6)–(3.1.8) up to now. Neither Struwe's monotonicity formula [79] nor a sensible form[1] of local energy inequality (as, e.g., for the Navier-Stokes equations) is available (so far).

The following speculation is closely connected: For every initial data of finite energy, we find a unique solution to the Ericksen-Leslie model, which is smooth away from finitely many space-time points. In [41] and [58], the authors show the finiteness of singular times and conjecture the former statement. Again, the lack of localizibility complicates the situation in contrast to the harmonic map heat flow where this is known (see [78]).

Likely more important are the issues in the three-dimensional theory for liquid crystals. Global existence results are known only for special cases such as the director field attaining values restricted to the upper half-sphere (see [64]) or under smallness conditions on the initial data (see [40]). If one again considers the limit of the Ginzburg-Landau approximation $\varepsilon \to 0^+$, the spatial defect measure is, in general, a rectifiable one-dimensional set for fixed time. This defect measure might evolve as motion by mean curvature and could actually appear in the momentum equation (3.1.6) or, alternatively, induce boundary conditions on the velocity at the support of defect measure. Once more, the barrier to a proof seems to be the absence of an opportunity to localize the energy.

Turning our attention to the system for magnetoviscoelastic fluids (4.0.1)–(4.0.5), we face issues in several directions: On the one hand, we have to deal with the same questions

[1] Referring to the local energy inequality in Section 4.1 or Lemma 4.5.1, the quantity $\operatorname{ess\,sup}_{(t_0-R^2,t_0)} \int_{B_R} e(t)\,\mathrm{d}x$ is no good candidate to be required small. Instead, a suitable quantity poses $R^{-2}\int_{t_0-R^2}^{t_0}\int_{B_R} e(t)\,\mathrm{d}x\,\mathrm{d}t$, cf. [79].

as above because of the similarity to the Ericksen-Leslie system in the magnetization variable. On the other hand, the mathematical theory regarding the viscoelastic subsystem involving only the velocity field and the deformation gradient is even less developed. At this point, there are two directions to proceed in. The first one is to develop a more expanded existence theory. Global existence for weak solutions is only known in special cases [65] or under restrictive (smallness) assumptions on initial data [52]. Again, the issue is the stress tensor $W'(F)F^\top$ in the momentum equation which could also undergo oscillations due to the lack of dissipation for $\kappa = 0$ in (4.0.3).

The second way to pursue is the search for more appropiate dissipation laws regarding the deformation gradient. The Laplacian operator in (4.0.3) does not harmonize with the incompressibility constraint $\det F = 1$. Other terms are proposed, e.g., in [6]. However, regarding usual questions in the calculus of variations like the different notions of convexity for elastic energies W, it is not clear how they interfere with such arguments originating in fluid mechanics.

Eventually, we mention two other possible extensions to Theorem 4.4.1: The system (4.0.1)–(4.0.5) does not encompass a coupling of F and M in the anisotropy energy for example. Such a coupling in term is, however, plausible, since F deforms the domain occupied by the magnetic fluid and therefore also the easy axis, i.e., the preferred direction for the magnetization. At last, all of the previous work featured the spatial domain $\mathbb{T}^2$, i.e., no boundary conditions are considered for a smooth bounded domain in $\mathbb{R}^2$. Remark 3.1.9 mentions how to verify Theorem 3.1.2 on a smooth bounded domain. However, it does not give insight into the phenomena of harmonic maps defined on surfaces with boundary. Here, we mention [13] for a result on bubbling near the boundary. With respect to Theorem 4.4.1, we conjecture that an analogous result can be proven on a smooth bounded domain subject to standard boundary conditions. The discussion in this case consists more of the question which boundary conditions are appropiate for the deformation gradient to hold.

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Acknowledgements

First of all I would like to express my gratitude to my advisor Prof. Anja Schlömerkemper for her continuous support and encouragement. I thank her for giving me the opportunity to write this PhD thesis, for always being friendly and having a sympathetic ear during my time in the team as well for her constructive feedback. Not to mention I appreciate a lot the help on the scientific work habits: Ideas for research topics, preparation of talks and publications.

Secondly, I would like to express my graditude to Prof. Changyou Wang from Purdue University who consented to review my PhD thesis.

Likewise, I express my graditute to Prof. Chun Liu from the Illinois Institute of Technology for consenting to review my PhD thesis.

Furthermore, I thank my collaborators and co-authors Francesco De Anna, from the University of Würzburg, and Martin Kalousek from the Institute of Mathematics at the Czech Academy of Sciences. There are many things to acknowledge: Discussions on the projects leading to this work, help when questions occured during the work and the careful proofreading of my previous publications. Furthermore, I thank Francesco De Anna again for the fruitful discussions, proofreading parts of the final thesis and provision of some of his images for my thesis.

Next, I wish to thank all current and former members of our team: Barbora Benešová, now employed at the Charles University in Prague, for her very useful advices on literature; Laura Lauerbach, now employed at the University of Kassel, for her great help in getting started and facing all the rapids in a working life at the university; Jesse Ratzkin, for his helpful answers regarding geometry, discussions and, in particular, for pointing a mistake in an early version of one of my preprints; Prof. Barbara Zwicknagl, now employed at the Humboldt University in Berlin, also for having a sympathetic ear and the willingness to answer naive questions in the beginning of a PhD student's work and Sourav Mitra for his interest in my topic and fruitful discussions.

For his support and proofreading the thesis, I thank my long-time friend Roman Strittmatter very, very much.

Also, I would like to ackowledge very much the interest of and the time working with Yuning Liu from the Shanghai Campus of the New York University.

Moreover, I acknowledge the financial support of the DFG (German Research Foundation), grant SCHL 1706/4-2, project number 3916822047.

Finally, I owe my deepest gratitude to my parents, who enabled myself to study mathematics at a university, always supported me in achieving my goals and therefore to write this PhD thesis.