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On the Order of Growth of Lerch Zeta Functions

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Abstract: We extend Bourgain's bound for the order of growth of the Riemann zeta function on the critical line to Lerch zeta functions. More precisely, we prove $L(\lambda, \alpha, 1/2 + it) \ll t^{13/84+\epsilon}$ as $t \to \infty$. For both, the Riemann zeta function as well as for the more general Lerch zeta function, it is conjectured that the right-hand side can be replaced by t^{ϵ} (which is the so-called Lindelöf hypothesis). The growth of an analytic function is closely related to the distribution of its zeros.

Keywords: Lerch zeta function; Hurwitz zeta function; (approximate) functional equation; order of growth; exponent pairs

MSC: 11M35

1. Introduction and Statement of the Main Result

Zeta and L-functions play a central role in analytic number theory and allied disciplines. The Riemann hypothesis, for example, states that all complex zeros of the Riemann zeta function ζ lie on the critical line $1/2 + i\mathbb{R}$. This claim is one of the six yet unsolved Millennium Problems and its solution would have a great impact on the distribution of prime numbers, in particular with applications to cryptography [1]. Questions concerning the zeros of an analytic function are related to the order of growth (by Jensen's formula). The Lerch zeta function L, defined in the subsequent paragraph, is a natural generalization of the Riemann zeta function; however, the location of the zeros is rather different from the expected one for ζ: the analogue of the Riemann hypothesis for L is false (except for very special parameters) [2,3]. On the other hand, the order of growth of L seems to be comparable to the one of ζ, as we shall motivate in this note. It is of great importance to understand how the order of growth and the precise location of the zeros are related.

Let $s = \sigma + it$ denote a complex variable. For parameters $0 < \alpha \le 1$ and $\lambda \in \mathbb{R}$, the Lerch zeta function is defined as an analytic function of s by the general Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m \ge 0} \frac{\mathrm{e}\,(\lambda m)}{(m+\alpha)^s},$$

where $\mathrm{e}\,(z) := \exp(2\pi i z)$. The series converges absolutely for $\sigma > 1$ and defines an analytic function in every compact subset of this half-plane. This function admits an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ if $\lambda \in \mathbb{Z}$. Moreover, it satisfies the functional equation

$$L(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Big(\mathrm{e}\,(\tfrac{s}{4} - \alpha\lambda^-)L(-\alpha, \lambda^-, s) + \mathrm{e}\,(-\tfrac{s}{4} + \alpha\lambda^+)L(\alpha, \lambda^+, s)\Big), \quad (1)$$

where $\lambda^+ := 1 - \{\lambda\}$ and

$$\lambda^- := \begin{cases} 1 & \text{if } \lambda \in \mathbb{Z}, \\ \{\lambda\} & \text{otherwise} \end{cases}$$

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with $\{\lambda\}$ being the fractional part $\lambda - \lfloor \lambda \rfloor$ of λ. Note that for s from the upper half-plane, the second term on the right dominates while for s from the lower half-plane, it is the first term. The point symmetry behind this functional equation is of great importance for the analytic behaviour; see [4], for example. Another reference for this and further properties of the Lerch zeta function is [5].

For $\lambda \in \mathbb{Z}$ the Lerch zeta function reduces to the Hurwitz zeta function $\zeta(s,\alpha) = L(1,\alpha,s)$ which itself is equal to the Riemann zeta function $\zeta(s) = \zeta(s,1)$ in the case of $\alpha = 1$; the so-called periodic zeta function arises as $F(s,\lambda) = \mathrm{e}\,(-\lambda)L(\lambda,1,s)$. These zeta functions are well-known objects in analytic number theory; the standard reference is [6].

In this note, we study the order of growth of $L(\lambda,\alpha,\frac{1}{2}+it)$ in the t-aspect (i.e., when $|t| \to \infty$). Taking into account the boundedness of $L(\lambda,\alpha,s)$ for $\sigma > 1$ in combination with the functional equation and a convexity argument, more precisely the Phragmén–Lindelöf principle (see [7]), it follows that

$$L(\lambda,\alpha,\tfrac{1}{2}+it) \ll t^{1/4+\epsilon} \tag{2}$$

as $t \to \infty$, where $\epsilon > 0$ is here and elsewhere arbitrary. The case $t \to -\infty$ can be treated by conjugation and we therefore assume $t > 0$ in the sequel and consider bounds as the one above always with respect to $t \to \infty$. The so-called subconvexity problem asks for an exponent strictly less than $\frac{1}{4}$. Our main result provides such a smaller exponent.

Theorem 1. *As $t \to \infty$,*

$$L(\lambda,\alpha,\tfrac{1}{2}+it) \ll t^{13/84+\epsilon}, \tag{3}$$

where the implicit constant here and in the sequel may depend on the parameters. Moreover,

$$\begin{aligned} L(\lambda,\alpha,\tfrac{1}{2}+it) - \alpha^{-1/2-it} - \mathrm{e}\left(\frac{-1}{2\pi}t\log\frac{t}{2\pi e} + \frac{1}{8} - \alpha\lambda\right) \\ \times\left(\lambda^{-1/2+it} - \exp(-\pi t)\,\mathrm{e}\left(\frac{1}{4}+\alpha\right)(1-\{\lambda\})^{-1/2+it}\right) \\ \ll \quad t^{13/84+\epsilon}, \end{aligned} \tag{4}$$

with an absolute implicit constant, independent of $\alpha, \lambda \in (0,1]$.

The slightly more complicated uniform bound (4) results from the first constant α^{-s} in the Dirichlet series representation of $L(\lambda,\alpha,s)$ and further similar terms coming from a Dirichlet series on the right side of the functional Equation (1), respectively, a useful approximation (6).

Note that this result extends the so far best bound for the Riemann zeta function due to Bourgain [8], that is

$$\zeta(\tfrac{1}{2}+it) \ll t^{13/84+\epsilon}. \tag{5}$$

In view of $\frac{13}{84} = 0.15476\ldots$, this bound is a significant improvement upon (2) and previous estimates for the Riemann zeta function ζ. Theorem 1 improves the so far best bound for Lerch zeta functions due to Garunkštis [2], which was relying on the previously best bound for ζ due to Huxley [9]; the crucial exponent then was $\frac{32}{205} = 0.15609\ldots$. Progress in this setting, unfortunately, comes in tiny improvements upon the exponent. Huxley worked with double exponential sums and Garunkštis combined this with the classical methods of van der Corput and Weyl.

Our reasoning is quite different and relies on Bourgain's exponent pair and the subsequent approximate functional equation: for $0 < \sigma \leq 1, t \geq 1$,

$$
\begin{aligned}
L(\lambda,\alpha,s) = &\sum_{0\leq n\leq \lfloor \sqrt{t/(2\pi)-\alpha}\rfloor} \frac{\mathrm{e}\,(\lambda n)}{(n+\alpha)^s} \\
&+\left(\frac{t}{2\pi}\right)^{1/2-\sigma} \mathrm{e}\left(\frac{-1}{2\pi} t\log\frac{t}{2\pi e}+\frac{1}{8}-\alpha\lambda^-\right) \\
&\times\left(\sum_{0\leq n\leq \lfloor \sqrt{t/(2\pi)}\rfloor} \frac{\mathrm{e}\,(-\alpha n)}{(n+\lambda)^s} - \frac{\exp(-\pi t)\,\mathrm{e}\,(\sigma/2+\alpha)}{(\lambda^+)^{1-s}}\right) \\
&+O(t^{-\sigma/2}+t^{\sigma/2-1});
\end{aligned}
\tag{6}
$$

this rather complicated but remarkably useful approximation can be found (in slightly different shape) in [6,10,11]. Note that originality there is a second Dirichlet polynomial with λ^+ (as in the case of the functional Equation (1)), which is missing here except for its first term, which, however, is necessary for the uniformity in the parameters; the remaining terms of this Dirichlet polynomial contribute for $t > 1$ to the error term. A similar approximation was derived already by Meulenbeld in his dissertation [12] but his work seems to be almost forgotten.

2. Proof of the Theorem

In analytic number theory, one often has to deal with exponential sums of the form

$$S := \sum_{M<n\leq M+u} \mathrm{e}\,(f(n))$$

where f is a real function, usually differentiable with $W \ll |f'(x)| \ll W$ for $M \leq x \leq 2M$ (which, in the sequel, we also write as $f'(x) \asymp W$), and $1 < u \leq M$. Every pair of non-negative real numbers μ, λ satisfying

$$S \ll W^\mu M^\lambda$$

and $\mu \leq 1/2 \leq \lambda \leq 1$, is said to be an *exponent pair*. The trivial estimate yields the exponent pair $(0,1)$. A slightly more advanced exponent pair is given by $(\frac{1}{2}, \frac{1}{2})$ for sufficiently smooth functions f. The much deeper exponent pair

$$(\mu,\lambda) = \left(\tfrac{13}{84}+\epsilon, \tfrac{55}{84}+\epsilon\right) \tag{7}$$

has been found by Bourgain [8], and it is related to the bound (5).

One difference from the case of the Riemann zeta function and Bourgain's work is that in the case of the Lerch zeta function, by (6), we have to deal with more general Dirichlet polynomials, namely,

$$\sum_{1\leq n\leq X} \mathrm{e}\,(\lambda n)(n+\alpha)^{-s} \tag{8}$$

rather than the simpler $\sum_n n^{-s}$. We show first that without loss of generality, we may assume λ to be rational. To see that, we use Diophantine approximations for a given irrational λ arising from the theory of continued fractions. Note that we have subtracted the first summand α^{-s} (for $n = 0$) here, which has no effect on the desired estimate (3) but needs to be considered for the uniform bound (4).

In fact, if λ is irrational, the convergents p/q of the continued fraction expansion of λ satisfy

$$\left|\lambda - \frac{p}{q}\right| < \frac{1}{q^2} \tag{9}$$

(see, e.g., [13]). We have

$$\mathrm{e}\,(\lambda n) - \mathrm{e}\,(\tfrac{pn}{q}) = 2\pi i \int_{pn/q}^{\lambda n} \mathrm{e}\,(x)\,\mathrm{d}\,x,$$

respectively,

$$|\mathrm{e}\,(\lambda n) - \mathrm{e}\,(\tfrac{pn}{q})| \leq 2\pi n \left|\lambda - \frac{p}{q}\right|,$$

It thus follows from (9) that

$$\left|\sum_{n\leq X} \frac{\mathrm{e}\,(\lambda n)}{(n+\alpha)^s} - \sum_{0\leq n\leq X} \frac{\mathrm{e}\,(\frac{pn}{q})}{(n+\alpha)^s}\right| \leq \sum_{n\leq X} \frac{|\mathrm{e}\,(\lambda n) - \mathrm{e}\,(\frac{pn}{q})|}{(n+\alpha)^\sigma} \ll \frac{1}{q^2}\sum_{n\leq X} n^{1-\sigma} \ll \frac{X^{2-\sigma}}{q^2}.$$

Since the denominators q can be chosen arbitrarily large, independent of all other parameters, we can approximate an arbitrary Dirichlet polynomial of the form (8) by one with a rational parameter with respect to λ, hence the general case follows from the rational one.

Next, we consider $\lambda = \frac{a}{b}$, where, without loss of generality, $1 \leq a \leq b$ are coprime. We have

$$\begin{aligned} L(\tfrac{a}{b},\alpha,s) &= \sum_{0\leq c<b} \mathrm{e}\,(\tfrac{ac}{b}) \sum_{\substack{n\geq 0\\ n\equiv c \bmod b}} (n+\alpha)^{-s} \\ &= \sum_{0\leq c<b} \mathrm{e}\,(\tfrac{ac}{b}) \sum_{m\geq 0} (bm+c+\alpha)^{-s} \\ &= b^{-s} \sum_{0\leq c<b} \mathrm{e}\,(\tfrac{ac}{b}) L(1,\tfrac{\alpha+c}{b},s). \end{aligned} \tag{10}$$

Hence, we may assume without loss of generality that $\lambda = 1$, which is the case of Hurwitz zeta functions $\zeta(s,\alpha) = L(1,\alpha,s)$.

Therefore, in place of the Dirichlet polynomials of the form (8), we may consider a sum of the form

$$E(s,\alpha,X) := \sum_{n\leq X} (n+\alpha)^{\sigma+it}. \tag{11}$$

We rewrite this sum as

$$E(s,\alpha,X) = \sum_{1\leq j\leq \log X/\log 2} \sum_{2^{-j}X<n\leq 2^{1-j}X} (n+\alpha)^{\sigma+it} + O(1).$$

By partial summation, we get for any inner sum

$$\begin{aligned} \sum_{Y<n\leq 2Y} (n+\alpha)^{\sigma+it} &= \sum_{n\leq 2Y} (n+\alpha)^{it} \cdot (2Y)^\sigma - \sum_{n\leq Y} (n+\alpha)^{it} \cdot Y^\sigma \\ &\quad -\sigma \int_Y^{2Y} \sum_{n\leq u} (n+\alpha)^{it} \cdot u^{\sigma-1}\,\mathrm{d}\,u \\ &\ll Y^\sigma \max_{Y\leq u\leq 2Y} \left|\sum_{Y<n\leq u} (n+\alpha)^{it}\right|, \end{aligned}$$

where $Y = 2^{-j}X$. Hence,

$$E(s,\alpha,X) \ll \sum_{1\leq j\leq \log X/\log 2} (2^{-j}X)^\sigma \max_{2^{-j}X\leq u\leq 2^{1-j}X} \left|\sum_{2^{-j}X<n\leq u} (n+\alpha)^{it}\right|. \tag{12}$$

In order to make use of Bourgain's exponent pair (7), we verify that the function f related to (11), i.e.,

$$f(x) := \frac{t}{2\pi}\log(x+\alpha),$$

is infinitely often differentiable for $x \geq 0$ with derivatives $f^{(k)}$ satisfying

$$(-1)^{k+1} f^{(k)}(x) = \frac{k!}{2\pi} \cdot \frac{t}{(x+\alpha)^k} \asymp \frac{t}{Y^k}, \tag{13}$$

for $k = 1, 2, \ldots$ and $Y \leq x \leq 2Y$ (with the same choice of Y as above) and independent of $\alpha \in (0, 1]$; the implicit constant may depend on k. The case $\alpha = 1$ is related to the Riemann zeta function, for the more general f the same bounds hold and (7) can be applied. For a thorough introduction to the theory of exponent pairs, we refer to [14,15].

Under these mild assumptions, we have, for every suitable exponential pair (μ, λ),

$$\sum_{Y<n\leq u} \mathrm{e}\,(f(n)) \ll (t/Y)^\mu Y^\lambda = t^\mu Y^{\lambda-\mu}.$$

Substituting this in (12) leads to

$$E(s,\alpha,X) \ll t^\mu X^{\sigma+\lambda-\mu} \cdot \log X, \tag{14}$$

where the log-factor arises from our dyadic dissection in (12).

For an application of this estimate to the Hurwitz zeta function $\zeta(s,\alpha) = L(1,\alpha,s)$, we rewrite (6) as

$$\begin{aligned} \zeta(s,\alpha) \quad = \quad & \alpha^{-s} + \sum_{1\leq n\leq X'} (n+\alpha)^{-s} - \left(\frac{t}{2\pi}\right)^{1/2-\sigma} \\ & \times \mathrm{e}\left(\frac{-1}{2\pi} t \log \frac{t}{2\pi e} + \frac{1}{8} - \alpha\right) \sum_{0\leq n\leq X} \frac{\mathrm{e}\,(-\alpha n)}{(n+1)^s} \\ & + O(t^{-1/4}), \end{aligned} \tag{15}$$

where

$$X' := \left\lfloor \sqrt{\frac{t}{2\pi}} - \alpha \right\rfloor \sim X := \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor \asymp t^{1/2},$$

and the difference with respect to α has no effect for the following. The second sum on the right of (15) has a numerator $\mathrm{e}\,(-\alpha n)$. If α is irrational, we can use a good rational approximation $\frac{p}{q}$ to α in order to replace the sum in question by $\sum_n \mathrm{e}\,(-pn/q) n^{-s}$ as we did in the beginning of this section. If $\alpha = \frac{a}{b}$ is rational, then we observe

$$\sum_{0\leq n\leq X} \frac{\mathrm{e}\,(-\frac{an}{b})}{(n+1)^s} = b^{-s} \sum_{0\leq c<b} \mathrm{e}\,(\tfrac{ac}{b}) E\left(s, \tfrac{c+1}{b}, \frac{X-c}{b}\right),$$

similarly to (10). Hence, substituting (14) in (15) leads to

$$\begin{aligned} \zeta(\sigma+it,\alpha) \quad \ll \quad & 1 + \sum_{\substack{j\leq \log X/\log 2 \\ N=2^{-j}\lfloor\sqrt{t/2\pi}\rfloor}} \left(N^{-\sigma} \cdot t^\mu N^{\lambda-\mu} + N^{\sigma-1} t^{1/2-\sigma} \cdot t^\mu N^{\lambda-\mu}\right) \\ \ll \quad & t^{(\mu+\lambda-\sigma)/2} \log t, \end{aligned}$$

valid for $\sigma \geq 1/2$ and $\lambda - \mu \geq \sigma$. Inserting (7) (for which (13) is necessary, see [8,14]), yields

$$\zeta(\sigma+it,\alpha) \ll t^{(17/21-\sigma)/2+\epsilon}.$$

In view of (10), the same estimate holds for Lerch zeta functions with rational parameter $\lambda = \frac{a}{b}$; the special case $\sigma = 1/2$ gives:

$$L(\tfrac{a}{b}, \alpha, \tfrac{1}{2} + it) \ll t^{13/84+\epsilon}.$$

This implies (3). For the uniform estimate (4), one just has to subtract the terms corresponding to the summands with $n = 0$ in the Dirichlet polynomials appearing in (6). This concludes the proof of Theorem 1.

3. Concluding Remarks

Of course, Theorem 1 implies the same bound for Dirichlet L-functions $L(s, \chi)$. This follows immediately from the representation

$$L(s, \chi) = q^{-s} \sum_{1 \leq a < q} \chi(a) \zeta(s, \tfrac{a}{q}),$$

where χ is supposed to be a character modulo q.

Our proof suggests that every improvement of Bourgain's estimate (5) for the Riemann zeta function by use of a new exponent pair should lead to the same improvement of (3) for the Lerch zeta function, and Garunkštis' paper [2] is showing the same for any improvement coming from double exponential sums. This indicates that Lerch zeta functions (and Dirichlet L-functions as well) should in general have the same order of growth as the Riemann zeta function.

A much stronger estimate than in (5) and Theorem 1 is expected to hold. The yet unproved Lindelöf hypothesis claims that

$$\zeta(\tfrac{1}{2} + it) \ll t^{\epsilon}.$$

The counterpart of this open conjecture for Lerch zeta functions seems meaningful (though the Riemann hypothesis fails for Lerch zeta functions in general). An effect of the Lindelöf hypothesis for the Riemann zeta function would be that there are *not too many* zeros to the right of the critical line [16]. This also applies for Lerch zeta functions; see [17] for a discussion of a Lindelöf hypothesis for the Lerch zeta function.

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