Mathematical Programs
with
Vanishing Constraints

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Dissertation
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with
Vanishing Constraints

Dissertation zur Erlangung
des naturwissenschaftlichen Doktorgrades
der Julius-Maximilians-Universität Würzburg

vorgelegt von

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aus

Northeim

"... And out of the confusion
Where the river meets the sea
Something new would arrive
Something better would arrive..."

(G.M. Sumner)

Acknowledgements

The doctoral thesis at hand is the result of my research during the time as a Ph.D. student at the Department of Mathematics at the University of Würzburg. In a long-term project like this one, there are, of course, several ups and downs, and hence one is lucky to have a solid scientific and social environment. Therefore, I would like to take this opportunity to thank certain people who have helped me create this dissertation, directly or indirectly.

First of all, I would like to express my deep gratitude to my supervisor Christian Kanzow for his superb scientific guidance through the past three years. He provided me with a very interesting and challenging research project, generously sharing his time and discussing new ideas. Moreover, I could substantially benefit from his great experience in the field of mathematical optimization which guided me carefully on the path which eventually led to this thesis. In addition to that, I would like to thank him for several joint publications which, clearly, had a big impact on the material presented here. Furthermore, through him, I got to know and collaborate with a couple of prominent researchers in my field of work like, for example, Jiří V. Outrata and Wolfgang Achtziger, where the latter also deserves my appreciation for agreeing to co-referee this dissertation.

Apart from my supervisor there are some more persons to whom I owe my gratitude since this whole project could hardly have been realized this way if it had not been for their support.

At this, first, I would like to thank my colleague Florian Möller for helping me with all kinds of technical questions, very amusing discussions about strength training and, after all, for being a really nice guy.

A very special thanks goes to my closest friends: Cedric Essi, Christian Struck, and Neela Struck to whom I feel deeply indebted for *caring through all the lows and also sharing the highs*.

Last but not least, I would like to thank my parents for generous financial support during my studies and my whole family (also including my sister Imke) for unrestrained emotional backup.

Contents

1. Introduction

From the ancient roots of mathematics to the latest streams of modern mathematical research, there has always been a fruitful interplay between pure mathematical theory on the one hand and the large field of applications in physics, chemistry, biology, engineering, or economics on the other. At this, all kinds of mutual influences can be observed. In many cases it happens naturally that an applicational problem leads to the genesis of a whole new discpline within mathematics; *calculus* or *algebra* are very prominent examples of that. In turn, the converse direction, in which a whole theory has been established without an immediate benefit outside of mathematics, finding enormous practical application decades later, is observed just as well.

Under the label of *applied mathematics* all mathematical disciplines are subsumed, which are concerned with the theoretical background and the computational solution of problems from all fields of applications and constantly recurring inner mathematical tasks.

In particular, the disciplines of *mathematical optimization* and *nonlinear programming*, respectively, being subdis ciplines of applied mathematics, deal with various kinds of minimization (or maximization) tasks, in which an objective function has to be minimized subject to functional or abstract constraints, from the most general to very special cases. In this thesis, however, a special class of optimization problems which can be used as a unified framework for problems from *topology optimization*, cf. Section 1.1, is investigated in depth. For these purposes consider the optimization problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& H_i(x) \geq 0 \quad \forall i = 1, \ldots, l, \\
& G_i(x)H_i(x) \leq 0 \quad \forall i = 1, \ldots, l,
\end{aligned}
\tag{1.1}
$$

with continuously differentiable functions $f, g_i, h_j, G_i, H_i : \mathbb{R}^n \to \mathbb{R}$. This type of problem is called *mathematical program with vanishing constraints*, MPVC for short. On the one hand, this terminology is due to the fact that the implicit sign constraint $G_i(x) \leq 0$ *vanishes* as soon as $H_i(x) = 0$. On the other hand, an MPVC is closely related to another type of optimization problem called *mathematical program with equilibrium constraints*, MPEC for short, see Section 1.2 for further details. In problem (1.1) the constraints $g(x) \leq 0$ and $h(x) = 0$ are supposed to be standard constraints, whereas the characteristic constraints $H_i(x) \geq 0$ and $G_i(x)H_i(x) \leq 0$ for $i = 1, \ldots, l$ are troublesome for reasons broadly explained in the sequel.

An MPVC is a very interesting type of problem for various reasons. First of all, it has a large field of applications in truss topology design, see Section 1.1 and is thus, in particular, interesting from an engineering point of view. Moreover, due to the fact that the characteristic constraints may be

reformulated by the aid of

$$H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq 0 \iff H_i(x) \geq 0, \ G_i(x) \leq 0 \quad \text{if} \quad H_i(x) > 0, \qquad (1.2)$$

a combinatorial structure being imposed on the constraints G and H comes out, which is responsible for many difficulties, which are typical for these kinds of problems as was coined by Scholtes in [59]: An MPVC is a nonconvex problem, even if all constraint functions g, h, G, H are convex, due to the product term $G_i(x)H_i(x) \leq 0$ for $i = 1, \ldots, l$. Furthermore, in most interesting and relevant cases, see Chapter 4, the standard constraint qualifications like the *linear independence*, the *Mangasarian-Fromovitz* or even the *Abadie constraint qualification* are violated. Hence, the well-known *Karush-Kuhn-Tucker conditions* cannot be viewed as first optimality conditions offhand. For these reasons, in turn, standard NLP solvers are very likely to fail for MPVCs, and so the challenge of designing more appropriate tools for their numerical solution arises naturally.

To get a first impression of what may happen when trying to analyze or solve an MPVC and in order to illustrate the above mentioned difficulties we take a look at a small academic example. For $a \in \mathbb{R}$ consider the MPVC

$$
\begin{aligned}
\min \quad & (x_1 - a)^2 + x_2^2 \\
\text{s.t.} \quad & x_2 \geq 0, \\
& x_1 x_2 \leq 0.
\end{aligned}
\qquad (1.3)
$$
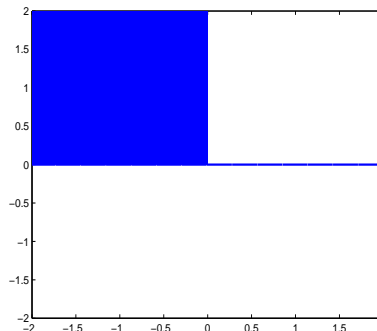
with its unique solution $x(a) = (a, 0)$. What we see is that its feasible set is nonconvex and contains some lower-dimensional areas which are particularly undesirable if the solution is located there and one tries to apply a feasible descent method to find it. On the other hand, the feasible set is at least locally convex for all feasible points except for the point $x^* = (0, 0)$. At this point both the explicit constraint $H(x) := x_2 \geq 0$ and the implicit restriction $G(x) := x_1 \leq 0$ are active, a critical situation which is responsible for many problems in the context of MPVCs. Moreover, the linear independence constraint qualification is violated at $x(a)$ for all $a \in \mathbb{R}$, and for all $a \geq 0$ even the Mangasarian-Fromovitz constraint qualification is violated at $x(a)$.

Before MPVCs have been treated systematically, there have appeared a couple of papers in the engineering literature, see, e.g., [2], [7], [13], or [33], in which particular cases of our general setup are considered.

Since MPVCs, in their general form, are quite a new class of optimization problems, very few works have only been published (or submitted) on this subject. At this, the first formal treatment has been done by Achtziger and Kanzow in [3], where the class of MPVCs was formally introduced and motivated. Subsequent to this work, there were published a couple of collaborate papers by Kanzow and the author of this thesis, see [26], [27], and [28], surveying constraint

qualifications and optimalitiy conditions for MPVCs. Needless to say, these papers are in large parts the basis for this thesis. There are two more works waiting for publication, [4] and [31], containing numerical approaches for the solution of MPVCs, where the first reference presents broad numerical results, and the second also provides stability theory and second-order conditions complementing those from [28]. The latest work in the field of MPVCs is [29] in which exact penalty results for MPVCs are investigated.

1.1. Applications of MPVCs

In order to display the relevance of programs in the fashion of (1.1) this section deals with a special problem from *topology optimization* which leads to an MPVC.

In general, topology optimization is concerned with the mathematical modelling of the engineering problem of distributing a given amount of material in a design domain subject to load and support conditions, such that the reuslting structure is in a certain sense optimally chosen. Contrary to traditional *shape design*, not only the total weight or volume of the resulting structure may be the objective of optimization, but rather the actual behaviour of the structure under load in terms of deformation energy is integrated in the optimization process.

For the more interested reader we recommend the excellent textbook [7], which has become a standard reference in this field.

The following example is taken from [3] and appears by courtesy of Wolfgang Achtziger and Christian Kanzow.

Example 1.1.1 In this example we want to find the optimal desgin for a truss structure using the so-called *ground structure approach* established in [15]. For these purposes, consider a given set M of *potential bars* defined by the coordinates of their end nodes (in $\mathbb{R}^2$ or $\mathbb{R}^3$). Moreover, for each potential bar, material parameters are given (Young's modulus E_i, relative moment of inertia s_i, stress bounds $\sigma_i^t > 0$ and $\sigma_i^c < 0$ for tension and compression, respectively). These parameters are used to formulate constraints to prevent structural failure if the calculated bar is actually realized. This, however, is the case if the calculated cross-sectional area a_i is positive. Eventually, boundary conditions (i.e., fixed nodal coordinates) and external loads (i.e., loads applying at some of the nodes) are given. Such a scenario is called a *ground structure*. The problem (*optimal truss topology design problem*) is to find cross-sectional areas a_i^* for each potential bar such that failure of the whole structure is prevented, the external load is carried by the structure, and a suitable objective function is minimal. The latter is usually the total weight of the structure or its deformation energy (*compliance*).

In order to obtain a good resulting structure after optimization, the ground structure should be 'rich' enough, i.e., it should consist of many potential bars. Figure 1.1 illustrates a ground structure in 2D in a standard design scenario. The structure (yet to be designed) is fixed to the left (indicated by a wall). On the right hand side, the given external load applies (vertical arrow) which must be carried by the structure. We have discretized a 2D rectangular design area by 15×9 nodal points. All nodal points are pair-wise connected by potential bars. After the deletion of long potential bars which are overlapped by shorter ones, we end up with 5614 potential bars. Some of these
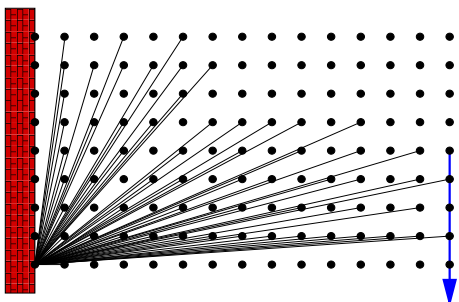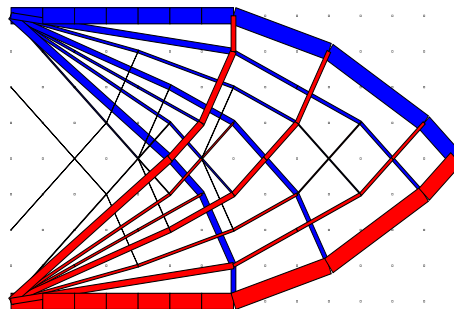
Figure 1.1.: Ground structure

Figure 1.2.: Optimal truss structure

potential bars are depicted in Figure 1.1 by black lines.

Of course, in view of a practical realization of the calculated structure after optimization, one hopes that the optimal design a^* will make use of only a few of the potential bars, i.e., $a_i^* > 0$ for a small number of indices i only, whereas most of the (many) optimal cross-sectional areas a_i^* are zero. Figure 1.2 shows the optimized structure based on the ground structure indicated in Figure 1.1. Indeed, most of the potential bars are not realized as real bars. Such a behaviour is typical in applied truss topology optimization problems.

The main difficulty in formulating (and solving) the problem lies in the fact that, generally speaking, constraints on structural failure can be formulated in a well-defined way only if there is some material giving mechanical response. As explained before, however, most potential bars will possess a zero cross-section at the optimizer. Hence, one option is the formulation of the problem as a problem with vanishing constraints. A simple formulation of the truss design problem with constraints on stresses and on local buckling takes the following form

$$
\begin{aligned}
\min \quad & f(a, u) \\
\text{s.t.} \quad & g(a, u) \leq 0, \\
& K(a)u = f^{ext}, \\
& a_i \geq 0 && \forall i = 1, \ldots, M, \\
& \sigma_i^c \leq \sigma_i(a, u) \leq \sigma_i^t && \text{if } a_i > 0 \quad \forall i = 1, \ldots, M, \\
& f_i^{int}(a, u) \geq f_i^{buck}(a) && \text{if } a_i > 0 \quad \forall i = 1, \ldots, M.
\end{aligned} \tag{1.4}
$$

At this, $a \in \mathbb{R}^M$, $a \geq 0$, is the vector of cross-sectional areas of the potential bars, and $u \in \mathbb{R}^d$ denotes the vector of nodal displacements of the structure under load, where d is the so-called degree of freedom of the structure, i.e., the number of free nodal displacement coordinates. The state variable u serves as an auxiliary variable. The objective function f often expresses structural weight or compliance but can also be any other measure evaluating a given design a and a corresponding state u. The nonlinear system of equations $K(a)u = f^{ext}$ symbolizes force equilibrium of given external loads $f^{ext} \in \mathbb{R}^d$ and internal forces along the bars expressed via Hooke's law in terms of displacements and cross-sections. The matrix $K(a) \in \mathbb{R}^{d \times d}$ is the global stiffness matrix corresponding to the structure a. This matrix is always symmetric and positive semidefinite. The

constraint $g(a, u) \leq 0$ is a resource constraint, like on the total volume of the structure if f denotes compliance or on the compliance of the structure if f denotes volume or weight. If $a_i > 0$, then $\sigma_i(a, u) \in \mathbb{R}$ is the stress along the i-th bar. Similarly, if $a_i > 0$, $f_i^{int}(a, u) \in \mathbb{R}$ denotes the internal force along the i-th bar, and $f_i^{buck}(a)$ corresponds to the permitted Euler buckling force. (We assume here that the geometry of the bar cross-section is given, e.g., as a circle or a square. Hence, the moment of inertia is a scaling of the cross-section, and the buckling force solely depends on a_i). Then the constraints on stresses and on local buckling make sense only if $a_i > 0$. Therefore, they must vanish from the problem if $a_i = 0$. Fortunately, the functions σ_i, f_i^{int}, and f_i^{buck} possess continuous extensions for $a_i \downarrow 0$, and thus may be defined also for $a_i = 0$, without any direct physical meaning, though. This, in view of (1.2), allows a reformulation of the problem in the form (1.1). In this situation, the definitions $H_i(a, u) := a_i$ for all $i = 1, \ldots, M$ will do the job.

We would like to close this section by referring the reader to [3] and Part II of this thesis for more applications of MPVCs in the 'real world '.

1.2. Comparison with MPECs

As was already suggested above, there is another class of optimization problems to which MPVCs are closely related and these are *mathematical programs with equilibrium constraints*, MPECs for short. An MPEC is a program of the following fashion

$$
\begin{aligned}
\min \quad & \tilde{f}(z) \\
\text{s.t.} \quad & \tilde{g}_i(z) \leq 0 \quad \forall i = 1, \ldots, m, \\
& \tilde{h}_j(z) = 0 \quad \forall j = 1, \ldots, p, \\
& \tilde{G}_i(z) \geq 0 \quad \forall i = 1, \ldots, l, \\
& \tilde{H}_i(z) \geq 0 \quad \forall i = 1, \ldots, l, \\
& \tilde{G}_i(z)\tilde{H}_i(z) = 0 \quad \forall i = 1, \ldots, l.
\end{aligned}
\tag{1.5}
$$

This kind of problem was already thoroughly investigated in numerous publications, where we would like to refer the reader particularly to the two monographs [37] and [44] containing comprehensive material on this subject.

Like the MPVC, an MPEC is a highly difficult problem, since it is also a representative of the class of nonconvex problems in the sense of [59], due to combinatorial structures on the characteristic constraints. As will turn out in many places of this thesis, an MPEC is even more difficult than an MPVC in many respects. For example, see [11], an MPEC violates the linear independence and the Mangasarian-Fromovitz constraint qualificiation at every feasible point, which is even worse than for MPVCs, as can be seen later.

In principle, an MPVC may be reformulated as an MPEC by introducing *slack variables*. In fact, the MPVC (1.1) is equivalent to the below MPEC in the variables $z := (x, s)$, where $s \in \mathbb{R}^l$ is the

slack variable:

$$\begin{aligned}
\min_{x,s} \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& G_i(x) - s_i \leq 0 \quad \forall i = 1, \ldots, l, \\
& H_i(x) \geq 0 \quad \forall i = 1, \ldots, l, \\
& s_i \geq 0 \quad \forall i = 1, \ldots, l, \\
& H_i(x)s_i = 0 \quad \forall i = 1, \ldots, l.
\end{aligned} \tag{1.6}$$

The precise relation between (1.1) and (1.6) is stated in the following elementary result, see [3].

Proposition 1.2.1 *(a) If x^* is a local minimizer of (1.1), then $z^* := (x^*, s^*)$ is a local minimizer of (1.6), where s^* denotes any vector with components*

$$s_i^* \begin{cases} = 0 & \text{if} \quad H_i(x^*) > 0, \\ \geq \max\{G_i(x^*), 0\} & \text{if} \quad H_i(x^*) = 0. \end{cases}$$

(b) If $z^* = (x^*, s^*)$ is a local minimizer of (1.6), then x^* is a local minimizer of (1.1).

Note that, due to Proposition 1.2.1, the following strategy for the solution of an MPVC could be applied: Reformulate the MPVC (1.1) as an MPEC in the fashion of (1.6) and apply one of the numerous solvers from the MPEC machinery.

This procedure, however, is not recommendable for various reasons: First of all, as was already suggested above, it has turned out in many situations of MPVC research, cf. [3] or [4], for example, and it will also show in this thesis, that an MPEC is even more difficult to tackle than an MPVC. Moreover, the reformulation (1.6) increases the dimension of the problem compared to (1.1). Furthermore, (1.6) involves some nonuniqueness as to the slack variables, a more serious drawback when solving it by some appropriate method.

Summing up what has been argued thus far, we have seen in Section 1.1 that an MPVC is a highly relevant problem from the viewpoint of applications. Furthermore, it was coined that it is too difficult to simply apply NLP methods for its solution. In addition to that, in Section 1.2 it was pointed out that also the reformulation of an MPVC as an MPEC is not an appropriate strategy.

Thus, the subject of this thesis, which is the theoretical investigation of MPVCs and the design of appropriate numerical solution methods, is a desirable goal.

The organization of this thesis is as follows: In the main it is divided into two major parts. Part I is concerned with the investigation of theoretical background material for MPVCs including constraint qualifications (standard and MPVC-tailored) and their associated optimality conditions. In particular, the special role of the so-called *Guignard constraint qualification* is adressed. Moreover, the notion of *M-stationarity*, an optimality concept weaker than the standard *KKT conditions*, is focussed and surveyed in depth, using, in particular, the *limiting normal cone*, see, e.g., [38], as a major tool. At this, many of the proofs are inspired by analogous considerations in the MPEC

field, as can be found in, e.g., [16], [17], [18], or [63], where the latter was also a rich source for the second-order optimality results presented in Chapter 7. Rounding off the background material, an exact penalty result for MPVCs is provided in Section 8, where also an alternative proof for M-stationarity is given.

In Part II numerical algorithms for the solution of MPVCs are established, including extensive convergence analysis and numerical applications. The first procedure is a smoothing-regularization algorithm, which was in a similar way already investigated for MPECs in [21]. For the convergence theory, *Clarke's generalized gradient* in the sense of [14] comes into play. The second one is a pure relaxation approach comparable to the one surveyed in [58] for MPECs.

Notation

In large parts most of the notation that is employed has become standard. For a brief overview we refer the reader to the end of this thesis. Nevertheless, we will now explain in more detail some of the more universal symbols which are used in many chapters.

The space of the real numbers is denoted by $\mathbb{R}$, where $\mathbb{R}_+$ and $\mathbb{R}_-$ are the nonnegative and nonpositive real numbers, respectively.

For an arbitrary set S, its n−fold cartesian product is indicated by S^n, that is, we have

$$S^n = \underbrace{S \times \cdots \times S}_{n-\text{times}}.$$

In particular, $\mathbb{R}^n$ labels the n-dimensional real vector space, where $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ describe its nonnegative and nonpositive orthant, respectively.

A vector $x \in \mathbb{R}^n$ is always understood to be a column vector, its transpose is given by x^T. Its components are denoted by x_i, which in particular justifies the notation $x = (x_i)_{i=1,\ldots,n}$. For a vector $x \in \mathbb{R}^n$ and a vector $y \in \mathbb{R}^m$ we simplify notation by

$$(x, y) := (x^T, y^T)^T.$$

Analogously, a matrix $A \in \mathbb{R}^{m \times n}$ consisting of m rows and n columns can be defined via its entries by $A := (a_{ij})_{i=1,\ldots,m,\ j=1,\ldots,n}$. Again, A^T denotes its transpose.

In general, $f : \mathbb{R}^n \to \mathbb{R}^m$ describes a function that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. In case of differentiability $f'(x)$ denotes its Jacobian at x. In addition to that, if $m = 1$, $\nabla f(x)$ denotes the gradient of f at x which is assumed to be a column vector. Moreover, for a twice differentiable function f, $\nabla^2 f(x)$ indicates the Hessian of f at x, that is we have

$$\nabla^2 f(x) = \Big(\frac{\partial}{\partial x_i \partial x_j} f(x)\Big)_{i,j=1,\ldots,n}.$$

For a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we may also partially apply the ∇-operator and we set

$$\nabla_x f(x, y) := \Big(\frac{\partial}{\partial x_i} f(x, y)\Big)_{i=1,\ldots,n} \quad \text{and} \quad \nabla_y f(x, y) := \Big(\frac{\partial}{\partial y_j} f(x, y)\Big)_{j=1,\ldots,m}.$$

The same principle will be applied to the subdifferential operators ∂, ∂^{Cl} etc.

Analogously, for the ∇^2-operator and a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ we put

$$\nabla^2_{xx} f(x, y) := \Big(\frac{\partial}{\partial x_i \partial x_j} f(x, y) \Big)_{i, j = 1, \ldots, n} \quad \text{and} \quad \nabla^2_{xy} f(x, y) := \Big(\frac{\partial}{\partial x_i \partial y_j} f(x, y) \Big)_{i = 1, \ldots, n, \; j = 1, \ldots, m}$$

Moreover, in case that there exists a vector $c \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that $f(x) = c^T x + b$ for all $x \in \mathbb{R}^n$ we call the function f *affine linear*, or simply *affine*.

The notion of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which maps elements from $\mathbb{R}^n$ to elements in $\mathbb{R}^m$ is extended to the concept of a *multifunction* or *set-valued map*. This is expressed by $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, which describes the fact that the multifunction Φ maps vectors from $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$. The graph of this multifunction is given by

$$\mathrm{gph}\,\Phi := \{(x, y) \in \mathbb{R}^{n+m} \mid y \in \Phi(x)\}.$$

We use $\| \cdot \|$ for an arbitrary l_p-norm in $\mathbb{R}^n$, that is, for $x \in \mathbb{R}^n$ we put

$$\|x\| := \|x\|_p := \begin{cases} (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} & \text{if} \quad p \in [1, \infty), \\ \sum_{i=1}^n |x_i| & \text{if} \quad p = \infty. \end{cases}$$

If a particular l_p-norm is used, this will always be noted in advance.

For $x \in \mathbb{R}^n$ and $r > 0$ we will denote the open ball with radius r around x by $B_r(x)$, i.e.,

$$B_r(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < r\}.$$

Additionally, we put

$$\mathbb{B} := \{x \in \mathbb{R}^n \mid \|x\| \le 1\},$$

i.e., $\mathbb{B}$ is the closed unit ball around the origin. For an arbitrary set $\emptyset \ne C \subseteq \mathbb{R}^n$ the function $d_C : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$d_C(x) := \inf_{y \in C} \|x - y\|,$$

denotes the distance of the vector $x \in \mathbb{R}^n$ to the set C measured in the respective norm $\| \cdot \|$. Moreover, for a closed set $C \ne \emptyset$ we define the multifunction $\mathrm{Proj}_C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\mathrm{Proj}_C(x) := \{y \in C \mid \|x - y\| = d_C(x)\}.$$

$\mathrm{Proj}_C(x)$ is then called the projection of x onto C.

Sequences in $\mathbb{R}^n$ are denoted by $\{a^k\} \subseteq \mathbb{R}^n$. In order to describe convergence to a limit point $a \in \mathbb{R}^n$ we write $a^k \to a$ or $\lim_{k \to \infty} a^k = a$. Moreover, we compactly write $\{a^k\} \to a$ for a sequence $\{a^k\} \subseteq \mathbb{R}^n$ with $a^k \to a$. For a sequence $\{a^k\} \subseteq \mathbb{R}$ we use $a^k \downarrow a$ to describe the case that $a^k \to a$ and $a^k > a$ for all $k \in \mathbb{N}$. Analogously, $a^k \uparrow a$ has to be interpreted.

Part I.

Theoretical Results

2. Concepts and results from nonlinear programming

In this chapter we briefly recall some basic notions from nonlinear programming, which are frequently employed in the subsequent analysis or that are used to motivate some of the new concepts. For these purposes, consider the general nonlinear programming problem of the form

$$
\begin{aligned}
\min \quad & \mathcal{F}(x) \\
\text{s.t.} \quad & \mathcal{G}_i(x) \leq 0 \quad \forall i = 1, \ldots, r, \\
& \mathcal{H}_j(x) = 0 \quad \forall j = 1, \ldots, s,
\end{aligned}
\tag{2.1}
$$

where $\mathcal{F}, \mathcal{G}_i, \mathcal{H}_j : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable functions. Excellent textbooks including exhaustive treatment of these kinds of problems are, e.g., [5], [22] and [43].

For further analysis we use the following definition which has become a useful standard abbreviation. If x^* is feasible for (2.1) we put

$$
I_{\mathcal{G}}(x^*) := \{ i \mid \mathcal{G}_i(x^*) = 0 \},
\tag{2.2}
$$

which is actually the set of indices for which $\mathcal{G}_i$ is active at x^*. Furthermore, we denote the feasible set of (2.1) by $\mathcal{X}$.

2.1. KKT conditions and constraint qualifications

2.1.1. The Karush-Kuhn-Tucker conditions

A fundamental, perhaps the most important result in the field of nonlinear programming is the following theorem initially proven by William Karush in his master's thesis [32] and then independently in a collaboration by Harold W. Kuhn and Albert W. Tucker in [35]. This led to calling it *Karush-Kuhn-Tucker conditions*, KKT conditions for short. Actually, it provides a necessary optimality criterion for (2.1) in case that one of the so-called *constraint qualifications*, CQs for short, holds at the point of question. We will broadly discuss some of the most prominent constraint qualifications for a standard optimization problem like (2.1) and their relationships after this result.

Theorem 2.1.1 (KKT conditions) *Let x^* be a local minimizer of (2.1) satisfying a constraint qualification. Then there exist vectors $\alpha \in \mathbb{R}^r$ and $\beta \in \mathbb{R}^s$ such that*

$$
0 = \nabla \mathcal{F}(x^*) + \sum_{i=1}^{r} \alpha_i \nabla \mathcal{G}_i(x^*) + \sum_{j=1}^{s} \beta_j \nabla \mathcal{H}_j(x^*)
\tag{2.3}
$$

and

$$\begin{aligned}
\mathcal{G}_i(x^*) \leq 0, \quad \alpha_i \geq 0, \quad \alpha_i \mathcal{G}_i(x^*) = 0 \quad &\forall i = 1, \ldots, r, \\
\mathcal{H}_j(x^*) = 0, \qquad\qquad\qquad\qquad\quad &\forall j = 1, \ldots, s.
\end{aligned} \tag{2.4}$$

In the course of increasing popularity of the subgradient calculus, a large number of generalizations of Theorem 2.1.1 have arisen, employing a nonsmooth calculus as provided in, e.g., [14], [42] and [61], where, roughly speaking, the gradients in (2.3) are replaced by the respective subgradient and the equality becomes an inclusion.

2.1.2. Constraint qualifications

Obviously, constraint qualifications play a key role in the formulation of the above theorem. A constraint qualification, in general, is a property of the feasible set represented by the constraint functions, which guarantees that the KKT conditions are in fact necessary optimality conditions. Quite a lot of different CQs have been established by different authors and shown to yield KKT conditions. A very exhaustive survey on this subject is given in [48]. For a less comprehensive but still very good overview one may also confer [6].

Three of the most common ones are the *linear independence constraint qualification (LICQ)*, the *Mangasarian-Fromovitz constraint qualification (MFCQ)* and the *Abadie constraint qualification (ACQ)*. LICQ is defined as follows and goes back to [36].

Definition 2.1.2 (LICQ) *Let x^* be feasible for* (2.1). *Then LICQ is said to hold if the gradients*

$$\begin{aligned}
\nabla \mathcal{G}_i(x^*) \quad &(i \in I_{\mathcal{G}}(x^*)), \\
\nabla \mathcal{H}_j(x^*) \quad &(j = 1, \ldots, s)
\end{aligned} \tag{2.5}$$

are linearly independent.

In turn, MFCQ obviously is due to Mangasarian and Fromovitz in [41].

Definition 2.1.3 (MFCQ) *Let x^* be feasible for* (2.1). *Then MFCQ is said to hold if the gradients*

$$\nabla \mathcal{H}_j(x^*) \quad (j = 1, \ldots, s)$$

are linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that

$$\begin{aligned}
\nabla \mathcal{G}_i(x^*)^T d < 0 \quad &(i \in I_{\mathcal{G}}(x^*)), \\
\nabla \mathcal{H}_j(x^*)^T d = 0 \quad &(j = 1, \ldots, s).
\end{aligned} \tag{2.6}$$

In order to define ACQ we need to introduce two cones which are standard tools in optimization theory. Let x^* be feasible for (2.1) then the following set

$$\mathcal{T}(x^*; X) := \{d \in \mathbb{R}^n \mid \exists \{x^k\} \subseteq X, \{t_k\} \downarrow 0 : x^k \to x^* \text{ and } \frac{x^k - x^*}{t_k} \to d\} \tag{2.7}$$

is called the *tangent cone* of the set X at the point x^*. Sometimes this cone is also referred to as *Bouligand tangent cone* or *contingent cone*. Note that the tangent cone is in fact a cone. Moreover,

note that, in particular, for the tangent cone of the feasible set of the MPVC (1.1) at a feasible point x^* we will compactly write $\mathcal{T}(x^*)$.

Now, we call the following set

$$
\mathcal{L}(x^*) = \{d \in \mathbb{R}^n \mid \nabla \mathcal{G}_i(x^*)^T d \le 0 \quad (i \in I_{\mathcal{G}}(x^*)), \\
\nabla \mathcal{H}_j(x^*)^T d = 0 \quad (j = 1, \ldots, s)\} \tag{2.8}
$$

the *linearized cone* of (2.1) at x^*, where the dependence on $\mathcal{X}$ which is reflected by the defining constraints $\mathcal{G}$ and $\mathcal{H}$, is suppressed in the notation, since it will always be clear from the context which constraint set the cone refers to. We are now in a position to state ACQ as initially done in [1].

Definition 2.1.4 (ACQ) *Let x^* be feasible for* (2.1). *Then ACQ is supposed to hold if*

$$
\mathcal{T}(x^*, \mathcal{X}) = \mathcal{L}(x^*).
$$

Note that one always has the inclusion $\mathcal{T}(x^*, \mathcal{X}) \subseteq \mathcal{L}(x^*)$, hence verifying ACQ reduces to the converse inclusion. Moreover, mind that ACQ always holds if all constraint functions are affine linear.

Another CQ which did not receive too much attention until it found application in the MPEC field, see, e.g., [17], is the *Guignard constraint qualification (GCQ)*, introduced by M. Guignard in [23]. In its definition the notion of the *dual cone* occurs which is explained below.

Definition 2.1.5 *Let $C \subseteq \mathbb{R}^n$ be a nonempty set. Then*

(a) $C^* := \{v \in \mathbb{R}^n \mid v^T d \ge 0 \ \ \forall d \in C\}$ *is the* dual cone *of C.*

(b) $C^\circ := \{v \in \mathbb{R}^n \mid v^T d \le 0 \ \ \forall d \in C\}$ *is the* polar cone *of C.*

Note that $v \in C^*$ if and only if $-v \in C^\circ$, hence C° is the negative of C^*. Furthermore, mind that the dual and the polar cone of a set is always closed and convex. Moreover, for two sets $A \subseteq B (\subseteq \mathbb{R}^n)$, apparently, one obtains the converse inclusions $B^* \subseteq A^*$ and $B^\circ \subseteq A^\circ$, respectively.

Definition 2.1.6 (GCQ) *Let x^* be feasible for* (2.1). *Then GCQ is said to hold if*

$$
\mathcal{T}(x^*, \mathcal{X})^* = \mathcal{L}(x^*)^*.
$$

At this, note that, due to what was argued above, the inclusion $\mathcal{L}(x^*)^* \subseteq \mathcal{T}(x^*, \mathcal{X})^*$ always holds. Evenually, mind that GCQ could have been equivalently defined by the use of the polar instead of the dual cone.

As can be seen in the above mentioned references [6] and [48], for example, the following simple relation holds for the four CQs that we have introduced thus far:

$$
\text{LICQ} \quad \Longrightarrow \quad \text{MFCQ} \quad \Longrightarrow \quad \text{ACQ} \quad \Longrightarrow \quad \text{GCQ} \,. \tag{2.9}
$$

The converse directions do not hold in general, cf. [48] for counterexamples.

In this chain of implications the first implication is easily verified, and the third follows immediately from the definitions. It takes more work to prove the second implication. This reflects the fact that there is quite a gap between LICQ and MFCQ on the one hand and ACQ and GCQ on the other hand in terms of strength and nature of the respective condition. First of all, cf. [48] and [6], there is a number of CQs lying between MFCQ and ACQ. And moreover, ACQ and GCQ are cone-based CQs, whereas LICQ and MFCQ are directly defined via the constraint functions.

ACQ and GCQ are typically held to be pretty weak conditions, in particular GCQ is in a sense, cf. [23] and [48], the weakest constraint qualification to yield KKT conditions at a local minimizer. Thus, they typically have good chances to hold. On the other hand, they are pretty hard to verify, in particular, since the tangent cone is involved. In turn, LICQ and MFCQ are rather strong assumptions, but may be verified pretty easily. This, in particular, makes them more appealing from a numerical viewpoint.

2.1.3. B-stationarity

At places, see Section 9.4, e.g., we will employ the notion of *B-stationarity* which is defined below.

Definition 2.1.7 (B-stationarity) *Let x^* be feasible for* (2.1). *Then x^* is called a Bouligand-stationary or B-stationary point of* (2.1) *if*

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in \mathcal{T}(x^*, \mathcal{X}). \tag{2.10}$$

Note that (2.10) is equivalent to saying that $\nabla f(x^*) \in \mathcal{T}(x^*, \mathcal{X})^*$.

The following result is well known in optimization and it states that B-stationarity is a necessary optimality condition for the nonlinear program (2.1), holding without any assumptions.

Proposition 2.1.8 *Let x^* be a local minimizer of* (2.1). *Then x^* is a B-stationary point of* (2.1).

B-stationarity is linked to the KKT-conditions in the following fashion.

Proposition 2.1.9 *Let x^* be feasible for* (2.1) *such that GCQ holds. Then x^* is B-stationary if and only if it is a KKT point.*

2.2. The convex case

As a reminder we briefly recall the notion of a convex set and a convex function. To this end, consider the following definitions.

Definition 2.2.1 *Let $C \subseteq \mathbb{R}^n$ be a nonempty set. Then C is called convex if for all $\lambda \in [0, 1]$ we have*

$$\lambda x + (1 - \lambda)y \in C \quad \forall x, y \in C.$$

Definition 2.2.2 *Let $C \in \mathbb{R}^n$ be convex and $f : C \to \mathbb{R}$. Then f is said to be*

(a) *convex on C if for all $\lambda \in [0, 1]$ it holds that*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in C.$$

(b) *strictly convex on C if for all $\lambda \in (0, 1)$ it holds that*

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in C \text{ with } x \neq y.$$

In addition to that, we say that f is (strictly) convex if it is (strictly) convex on the whole $\mathbb{R}^n$.

For differentiable functions there is a well-known characterization of convexity which is stated below.

Lemma 2.2.3 *Let $C \subseteq \mathbb{R}^n$ be convex and $f : C \to \mathbb{R}$. Then*

(a) *f is convex on C if and only if*

$$f(x) - f(y) \geq \nabla f(y)(x - y) \quad \forall x, y \in C.$$

(b) *f is strictly convex on C if and only if*

$$f(x) - f(y) > \nabla f(y)(x - y) \quad \forall x, y \in C \text{ with } x \neq y.$$

We are now in a position to be concerned with the actual subject of this section which is the following type of optimization problem

$$\begin{aligned}
\min \quad & \mathcal{F}(x) \\
\text{s.t.} \quad & \mathcal{G}_i(x) \leq 0 \quad \forall i = 1, \ldots, r, \\
& \mathcal{H}_j(x) = 0 \quad \forall j = 1, \ldots, s,
\end{aligned} \tag{2.11}$$

where the functions $\mathcal{F}, \mathcal{G}_i$ ($i = 1 \ldots, r$) are convex and the functions $\mathcal{H}_j$ ($j = 1, \ldots, s$) are affine linear. This type of problem is typically held to be pretty well-posed, in particular because its feasible region is a convex set and hence the following well-known result, see [5], e.g., applies.

Theorem 2.2.4 *Let $S \subseteq \mathbb{R}^n$ be nonempty and convex and let $f : S \to \mathbb{R}$ be convex on S. Consider the problem*

$$\min f(x) \quad \text{s.t.} \quad x \in S, \tag{2.12}$$

and suppose that x^* is a local minimizer of (2.12). Then the following holds true:

(a) *x^* is a global minimizer of (2.12).*

(b) *If either x^* is a strict local minimizer, or if f is strictly convex, then x^* is the unique global minimizer of (2.12).*

For the remainder we refer to programs in the fashion of (2.12) as *convex programs*. In particular, (2.11) is a convex program.

Moreover, in addition to the above result it is known that the KKT conditions from (2.3) and (2.4) are sufficient optimality conditions for (2.11), that is we have:

Theorem 2.2.5 *Let x^* be a KKT point of (2.11). Then x^* is a minimizer of (2.11).*

A prominent constraint qualifcation in the field of convex programming is the so-called *Slater condition* or *Slater constraint qualification (SCQ)*, which is due to M. Slater, see [60], but can be found in any comprehensive textbook like [5], [6] or [22].

Definition 2.2.6 (SCQ) *The convex program (2.12) satsifies the Slater constraint qualification (SCQ) if there exists a vector $\hat{x} \in \mathbb{R}^n$ such that*

$$\mathcal{G}_i(\hat{x}) < 0 \ (i = 1, \ldots, r), \qquad \mathcal{H}_j(\hat{x}) = 0 \ (j = 1, \ldots, s).$$

The following result relates SCQ with the standard CQs from Section 2.1.2 by discovering its sufficiency for ACQ.

Theorem 2.2.7 *Let SCQ be satisfied for the convex program (2.12). Then ACQ holds at every feasible point.*

2.3. Second-order optimality conditions

This section deals with *second-order optimality conditions* for nonlinear programs in the fashion of (2.1). We present both necessary and sufficient conditions, but we focus on the latter. For the remainder of this section we assume all functions in (2.1) to be twice continuously differentiable.

Second-order sufficient optimality conditions have initially arisen in the context of stability and sensitivity analysis of perturbed optimization problems, see [34] or [52], e.g., and are now part of any comprehensive textbook on optimization, see [5], [22] or [43].

Considering the standard nonlinear program (2.1), the basic tool for the formulation of second-order conditions is the associated function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{L}(x, \alpha, \beta) \quad &:= \quad \mathcal{F}(x) + \alpha^T \mathcal{G}(x) + \beta^T \mathcal{H}(x) \qquad &(2.13) \\ &= \quad \mathcal{F}(x) + \sum_{i=1}^{r} \alpha_i \mathcal{G}_i(x) + \sum_{j=1}^{s} \beta_j \mathcal{H}_j(x), \qquad &(2.14) \end{aligned}$$

which is called the *Lagrangian (function)* of (2.1). By the aid of the Lagrangian one may, for example, rewrite the KKT conditions from Theorem 2.1.1 as follows: A feasible point x^* of (2.1) is a KKT point if and only if there exist multipliers α, β such that

$$\nabla_x \mathcal{L}(x^*, \alpha, \beta) = 0, \ \alpha \geq 0, \ \alpha^T \mathcal{G}(x^*) = 0.$$

Second-order optimality conditions in optimization are always stated in the sense that the Hessian of the Lagrangian has (sufficient conditions) or is shown to have (necessary conditions) certain definiteness properties on a particular *critical cone*. The cones that will play this role for our purposes are given as follows. Suppose that (x^*, α, β) is a KKT point of (2.1). Then recall that

$$I_{\mathcal{G}}(x^*) = \{i \mid \mathcal{G}_i(x^*) = 0\},$$

and put

$$
\begin{aligned}
I_{\mathcal{G}}^+(x^*) &:= \{i \in I_{\mathcal{G}}(x^*) \mid \alpha_i > 0\}, \\
I_{\mathcal{G}}^0(x^*) &:= \{i \in I_{\mathcal{G}}(x^*) \mid \alpha_i = 0\}.
\end{aligned}
$$

Then we define
$$
\mathcal{K}(x^*) := \{d \in \mathbb{R}^n \mid \quad \nabla \mathcal{G}_i(x^*)^T d = 0 \quad (i \in I_{\mathcal{G}}^+(x^*)), \\
\nabla \mathcal{G}_i(x^*)^T d \le 0 \quad (i \in I_{\mathcal{G}}^0(x^*)), \\
\nabla H_j(x^*)^T d = 0 \quad (j = 1, \ldots, s)\},
$$
(2.15)

and
$$
\mathcal{K}^s(x^*) := \{d \in \mathbb{R}^n \mid \quad \nabla \mathcal{G}_i(x^*)^T d = 0 \quad (i : \alpha_i > 0)), \\
\nabla H_j(x^*)^T d = 0 \quad (j = 1, \ldots, s)\}.
$$
(2.16)

Mind, however, that the latter cones depend also on the multipliers, which are unique in the case that LICQ holds at x^*. Moreover, note that, apparently, with the linearized cone $\mathcal{L}(x^*)$ of (2.1) at x^*, one has $\mathcal{K}(x^*) \subseteq \mathcal{K}^s(x^*) \subseteq \mathcal{L}(x^*)$. Furthermore, $\mathcal{K}(x^*) = \mathcal{K}^s(x^*)$ holds, for example, under the following condition.

Definition 2.3.1 (SCS) *Let (x^*, α, β) be a KKT point of* (2.1). *Then we say that strict complementarity slackness (SCS) holds if*

$$\alpha_i + \mathcal{G}_i(x^*) \ne 0 \quad \forall i = 1, \ldots, r.$$

The notion of strict complementarity slackness has been successfully employed in many situations of optimization theory. For example, SCS yields differentiability of most of the prominent *NCP-functions*, like the *Fischer-Burmeister function*, see [20], or the *min-function* as used in, e.g., [45]. Thus, in the presence of SCS, the KKT conditions can be rewritten as a differentiable system of equations.

Eventually, we may now state the second-order sufficient conditions that we need in the sequel.

Definition 2.3.2 *Let (x^*, α, β) be a KKT point of* (2.1). *Then we say that*

(a) second-order sufficient condition (SOSC) is satisfied if

$$d^T \nabla_{xx}^2 \mathcal{L}(x^*, \alpha, \beta)d > 0 \quad \forall d \in \mathcal{K}(x^*) \setminus \{0\},$$

(b) strong second-order condition (SSOSC) is satisfied if

$$d^T \nabla_{xx}^2 \mathcal{L}(x^*, \alpha, \beta)d > 0 \quad \forall d \in \mathcal{K}^s(x^*) \setminus \{0\}.$$

Note that, SOSC and SSOSC coincide under SCS.

The following result is well known in optimization and can be found in, e.g., [5].

Theorem 2.3.3 *Let* (x^*, α, β) *be a KKT point of* (2.1) *satisfying SOSC. Then* x^* *is a strict local minimizer of* (2.1).

Obviously, since SSOSC implies SOSC, we get the following corollary.

Corollary 2.3.4 *Let* (x^*, α, β) *be a KKT point of* (2.1) *satisfying SSOSC. Then* x^* *is a strict local minimizer of* (2.1).

For completeness' sake and since it motivates the MPVC-tailored results in Chapter 7, we also provide a prominent second-order necessary result for (2.1), which can also be found in [5], for example.

Theorem 2.3.5 *Let* x^* *be a local minimizer of* (2.1) *satisfying LICQ. Furthermore, let* (α, β) *be the associated (unique) multipliers such that* (x^*, α, β) *is a KKT point of* (2.1). *Then it holds that*

$$d^T \nabla_{xx}^2 \mathcal{L}(x^*, \alpha, \beta) d \geq 0 \quad \forall d \in \mathcal{K}(x^*).$$

3. Tools for MPVC analysis

This chapter is supposed to provide some concepts and abbreviations which have turned out to be extremely helpful for the analysis of MPVCs.

For the remainder we decide to denote the feasible set of the MPVC (1.1) by X and we put

$$\theta_i(x) := G_i(x)H_i(x) \quad \forall i = 1, \dots, l. \tag{3.1}$$

A first crucial tool is the following list of index sets. For these purposes, let $x^* \in X$. Then we put

$$
\begin{aligned}
J &:= \{1, \dots, p\}, \\
I_g &:= \{i \mid g_i(x^*) = 0\}, \\
I_+ &:= \{i \mid H_i(x^*) > 0\}, \\
I_0 &:= \{i \mid H_i(x^*) = 0\}.
\end{aligned}
\tag{3.2}
$$

Furthermore, we divide the index set I_+ into the following subsets:

$$
\begin{aligned}
I_{+0} &:= \{i \mid H_i(x^*) > 0, G_i(x^*) = 0\}, \\
I_{+-} &:= \{i \mid H_i(x^*) > 0, G_i(x^*) < 0\}.
\end{aligned}
\tag{3.3}
$$

Similarly, we partition the set I_0 in the following fashion:

$$
\begin{aligned}
I_{0+} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) > 0\}, \\
I_{00} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) = 0\}, \\
I_{0-} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) < 0\}.
\end{aligned}
\tag{3.4}
$$

Note that the first subscript indicates the sign of $H_i(x^*)$, whereas the second subscript stands for the sign of $G_i(x^*)$. Mind, however, that the above index sets substantially depend on the chosen point x^*, but for our purposes it will always be clear from the context which point they refer to. Moreover, note that a very special role will be played by the bi-active set I_{00}, as was already foreshadowed in the introduction.

The gradient of the function θ_i from (3.1) at a feasible point $x^* \in X$ may be expressed with the above index sets as

$$
\nabla\theta_i(x^*) = \begin{cases}
G_i(x^*)\nabla H_i(x^*) & \text{if } i \in I_{0-} \cup I_{0+}, \\
0 & \text{if } i \in I_{00}, \\
G_i(x^*)\nabla H_i(x^*) + H_i(x^*)\nabla G_i(x^*) & \text{if } i \in I_{+-}, \\
H_i(x^*)\nabla G_i(x^*) & \text{if } i \in I_{+0}.
\end{cases}
\tag{3.5}
$$

3.1. Some MPVC-derived problems

At places we will make use of some auxiliary problems that are derived directly from the MPVC. For these purposes, let x^* be feasible for (1.1). Then $\mathcal{P}(I_{00})$ denotes the set of all partitions of the index set I_{00}. Now, let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be an arbitrary partition of the index set I_{00} into two subsets, that is $\beta_1 \cup \beta_2 = I_{00}$ and $\beta_1 \cap \beta_2 = \emptyset$. Then $NLP_*(\beta_1, \beta_2)$ describes the nonlinear program

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& H_i(x) = 0 \quad \forall i \in I_{0+}, \\
& H_i(x) \geq 0 \quad \forall i \in I_{0-}, \\
& G_i(x) \leq 0 \quad \forall i \in I_{+0}, \\
& H_i(x) \geq 0 \quad \forall i \in \beta_1, \\
& G_i(x) \leq 0 \quad \forall i \in \beta_1, \\
& H_i(x) = 0 \quad \forall i \in \beta_2, \\
& H_i(x) \geq 0 \quad \forall i \in I_+, \\
& G_i(x) \leq 0 \quad \forall i \in I_{+-} \cup I_{0-}.
\end{aligned}
\tag{3.6}
$$

Note that $NLP_*(\beta_1, \beta_2)$ does not contain any product constraints and thus, does not show a combinatorial aspect.

This program will turn out to be an appropriate tool of proof for an intrinsic characterization of the tangent and the *MPVC-linearized cone* which is still to be introduced in the subsequent section. Thus, it is reasonable to already envision the linearized cone of this program, which is then given, cf. (2.8), by

$$
\begin{aligned}
\mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*) = \{ d \in \mathbb{R}^n \mid \ & \nabla g_i(x^*)^T d \leq 0 \quad (i \in I_g), \\
& \nabla h_j(x^*)^T d = 0 \quad (j = 1, \ldots, p), \\
& \nabla H_i(x^*)^T d = 0 \quad (i \in I_{0+}), \\
& \nabla H_i(x^*)^T d \geq 0 \quad (i \in I_{0-}), \\
& \nabla G_i(x^*)^T d \leq 0 \quad (i \in I_{+0}), \\
& \nabla H_i(x^*)^T d \geq 0 \quad (i \in \beta_1), \\
& \nabla G_i(x^*)^T d \leq 0 \quad (i \in \beta_1), \\
& \nabla H_i(x^*)^T d = 0 \quad (i \in \beta_2) \}.
\end{aligned}
\tag{3.7}
$$

Another useful problem is the so-called *tightened nonlinear program*, $TNLP(x^*)$ for short, which is defined by

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& H_i(x) = 0 \quad \forall i \in I_{0+} \cup I_{00}, \\
& H_i(x) \geq 0 \quad \forall i \in I_{0-} \cup I_+, \\
& G_i(x) \leq 0 \quad \forall i = 1, \ldots, l
\end{aligned}
\tag{3.8}
$$

The reason why it is called *tightened* is that its feasible set is obviously contained in X. (Another tightened nonlinear program in the context of MPECs was used in [56] in order to define MPEC-

tailored constraint qualifications.) The $TNLP(x^*)$ will serve to investigate relations of some of the MPVC-tailored constraint qualifications very concisely, see Chapter 5.

3.2. Representations of the standard cones and the MPVC-linearized cone

By the aid of the index sets from (3.2)-(3.4) it is possible to find a very handy representation for the linearized cone, cf. (2.8), at a feasible point of an MPVC.

Lemma 3.2.1 *Let $x^* \in X$ be a feasible point for* (1.1). *Then the linearized cone at x^* is given by*

$$
\begin{aligned}
\mathcal{L}(x^*) = \{ d \in \mathbb{R}^n \mid \ & \nabla g_i(x^*)^T d \leq 0 && (i \in I_g), \\
& \nabla h_j(x^*)^T d = 0 && (j = 1, \ldots, p), \\
& \nabla H_i(x^*)^T d = 0 && (i \in I_{0+}), \\
& \nabla H_i(x^*)^T d \geq 0 && (i \in I_{00} \cup I_{0-}), \\
& \nabla G_i(x^*)^T d \leq 0 && (i \in I_{+0}) \}.
\end{aligned}
\tag{3.9}
$$

Proof. Let θ_i for $i = 1, \ldots, l$ denote the function from (3.1). Then, using the definition of the index sets from (3.2)-(3.4), it follows from its definition, see (2.8), that the linearized cone of the program (1.1) at x^* is given by

$$
\begin{aligned}
\mathcal{L}(x^*) = \{ d \in \mathbb{R}^n \mid \ & \nabla g_i(x^*)^T d \leq 0 && (i \in I_g), \\
& \nabla h_j(x^*)^T d = 0 && (j = 1, \ldots, p), \\
& \nabla H_i(x^*)^T d \geq 0 && (i \in I_0), \\
& \nabla \theta_i(x^*)^T d \leq 0 && (i \in I_0 \cup I_{+0}) \}.
\end{aligned}
$$

Now, using the expression of the gradient $\nabla \theta_i(x^*)$ for $i \in I_0 \cup I_{+0}$ as given in (3.5), it follows that

$$
\begin{aligned}
\nabla \theta_i(x^*)^T d \leq 0 \quad &\Leftrightarrow \quad \nabla H_i(x^*)^T d \leq 0 \ \forall i \in I_{0+}, \\
\nabla \theta_i(x^*)^T d \leq 0 \quad &\Leftrightarrow \quad 0 \leq 0 \ \forall i \in I_{00}, \\
\nabla \theta_i(x^*)^T d \leq 0 \quad &\Leftrightarrow \quad \nabla H_i(x^*)^T d \geq 0 \ \forall i \in I_{0-}, \\
\nabla \theta_i(x^*)^T d \leq 0 \quad &\Leftrightarrow \quad \nabla G_i(x^*)^T d \leq 0 \ \forall i \in I_{+0}.
\end{aligned}
$$

The first equivalence, together with $\nabla H_i(x^*)^T d \geq 0$ for all $i \in I_0$, gives $\nabla H_i(x^*)^T d = 0$ for all $i \in I_{0+}$, whereas the second and third equivalences do not provide any new information. Putting together all these pieces of information, we immediately get the desired representation of the linearized cone. $\qquad \square$

Another cone, which was initially employed in [26], is

$$
\begin{aligned}
\mathcal{L}_{MPVC}(x^*) := \{d \in \mathbb{R}^n \mid \ &\nabla g_i(x^*)^T d \leq 0 && (i \in I_g), \\
&\nabla h_j(x^*)^T d = 0 && (j = 1, \ldots, p), \\
&\nabla H_i(x^*)^T d = 0 && (i \in I_{0+}), \\
&\nabla H_i(x^*)^T d \geq 0 && (i \in I_{00} \cup I_{0-}), \\
&\nabla G_i(x^*)^T d \leq 0 && (i \in I_{+0}), \\
&(\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \leq 0 && (i \in I_{00})\}.
\end{aligned}
\tag{3.10}
$$

We will call $\mathcal{L}_{MPVC}(x^*)$ the *MPVC-linearized cone* since it takes into account the special structure of the MPVC. Note that it is, in general, a nonconvex cone, and that the only difference between $\mathcal{L}_{MPVC}(x^*)$ and the linearized cone $\mathcal{L}(x^*)$ is that we add a quadratic term in the last line of (3.10), cf. Lemma 3.2.1. In particular, we always have the inclusion $\mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*)$.

Recalling the program $NLP_*(\beta_1, \beta_2)$ from (3.6) we are now in a position to state a result which provides a very fruitful characterization of both the MPVC-linearized cone and the tangent cone of the MPVC (1.1).

Lemma 3.2.2 *Let x^* be feasible for* (1.1). *Then the following statements hold:*

(a) $\mathcal{T}(x^*) = \displaystyle\bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*).$

(b) $\mathcal{L}_{MPVC}(x^*) = \displaystyle\bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*).$

Proof. (a) $'\subseteq'$: Let $d \in \mathcal{T}(x^*)$. Then there exist sequences $\{x^k\} \subseteq X$ and $\{t_k\} \subseteq \mathbb{R}$ with $t_k \downarrow 0$ such that $\frac{x^k - x^*}{t_k} \to d$. Thus, it suffices to show that there exists a partition $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(I_{00})$ and an infinite set $K \subseteq \mathbb{N}$ such that x^k is feasible for $NLP_*(\hat{\beta}_1, \hat{\beta}_2)$ for all $k \in K$. Since x^k is feasible for (1.1) and all functions are at least continuous, we have $g_i(x^k) \leq 0$ $(i = 1, \ldots, m)$, $h_j(x^k) = 0$ $(j = 1, \ldots, p)$, $H_i(x^k) \geq 0$ $(i \in I_{0-})$, $H_i(x^k) \geq 0$ $(i \in I_+)$ and $G_i(x^k) \leq 0$ $(i \in I_{+-} \cup I_{0-})$ for all $k \in \mathbb{N}$ sufficiently large. For $i \in I_{0+}$ we have $G_i(x^k) > 0$ for k sufficiently large, again by continuity. Therefore, we obtain $H_i(x^k) = 0$ for all $i \in I_{0+}$ and all k sufficiently large, as x^k is feasible for (1.1). Using a similar argument, we also obtain $G_i(x^k) \leq 0$ for all $i \in I_{+0}$ for k sufficiently large. Now put

$$
\beta_{1,k} := \{i \in I_{00} \mid G_i(x^k) \leq 0\} \quad \text{and} \quad \beta_{2,k} := \{i \in I_{00} \mid G_i(x^k) > 0\}
$$

for all $k \in \mathbb{N}$. Since $\mathcal{P}(I_{00})$ contains only a finite number of partitions, we can find a particular partition $(\hat{\beta}_1, \hat{\beta}_2)$ and an infinite set $K \subseteq \mathbb{N}$ such that $(\beta_{1,k}, \beta_{2,k}) = (\hat{\beta}_1, \hat{\beta}_2)$ for all $k \in K$. Then $(\hat{\beta}_1, \hat{\beta}_2)$ and K have the desired properties.

$'\supseteq'$: For all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ one can easily see by the definition of the respective programs that any feasible point of $NLP_*(\beta_1, \beta_2)$ is also feasible for (1.1). Hence, we obtain $\mathcal{T}(x^*) \supseteq \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*)$ for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, which implies the claimed inclusion.

(b) $'\subseteq'$: Let $d \in \mathcal{L}_{MPVC}(x^*)$. Recalling the representations of the corresponding linearized cones, see (3.10) and (3.7), respectively, we only need to show that there exists a partition $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$

such that $\nabla G_i(x^*)^T d \leq 0$ ($i \in \beta_1$) and $\nabla H_i(x^*)^T d = 0$ ($i \in \beta_2$) holds, since all other restrictions are trivially satisfied. To this end, put

$$\beta_1 := \{i \in I_{00} \mid \nabla G_i(x^*)^T d \leq 0\}, \quad \beta_2 := \{i \in I_{00} \mid \nabla G_i(x^*)^T d > 0\}.$$

Since we have $(\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \leq 0$ and $\nabla H_i(x^*)^T d \geq 0$ for all $i \in I_{00}$ by assumption, we can conclude from the above definitions that $\nabla H_i(x^*)^T d = 0$ holds for all $i \in \beta_2$ which proves the first inclusion.

$' \supseteq '$: This inclusion follows immediately from the definitions of the corresponding cones. $\qquad\square$

The previous Lemma may be viewed as the counterpart of corresponding results known from the MPEC literature, see, e.g., [37, 47, 17].

An immediate consequence of Lemma 3.2.2 is the following corrollary.

Corollary 3.2.3 *Let x^* be feasible for* (1.1). *Then we have* $\mathcal{T}(x^*) \subseteq \mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*)$.

Proof. Since the tangent cone is always a subset of the corresponding linearized cone, we clearly have $\mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) \subseteq \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$ for all $(\beta_1,\beta_2) \in \mathcal{P}(I_{00})$. Invoking Lemma 3.2.2, we therefore obtain

$$\mathcal{T}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) \subseteq \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*) = \mathcal{L}_{MPVC}(x^*),$$

which proves first inclusion. The second inclusion follows immediately from the definition of the respective cones. $\qquad\square$

4. Standard CQs in the context of MPVCs

In this chapter we investigate how appropriate standard constraint qualifications such as LICQ, MFCQ, ACQ and GCQ are for MPVC analysis.

At this, it is argued that both LICQ and MFCQ must be held to be too restrictive for MPVCs. Moreover, ACQ, too, will be shown to be a very strong assumption for MPVCs and hence is violated in many cases. Only GCQ will turn out to be a reasonable assumption for the MPVC.

The following Section 4.1 is based on material investigated in [3], whereas Section 4.2 and 4.3 go back to [26].

4.1. Violation of LICQ and MFCQ

The first result reveals that standard LICQ, see Definition 2.1.2, is always violated for an MPVC under pretty mild assumptions. Recall for the subsequent analysis that we have set $\theta_i := G_i H_i$ for $i = 1, \ldots, l$.

Lemma 4.1.1 *Let x^* be feasible for* (1.1) *such that $I_0 \neq \emptyset$. Then LICQ is violated at x^*.*

Proof. Let $j \in I_0$. Then $\nabla \theta_j(x^*) = G_j(x^*) \nabla H_j(x^*)$, that is, $\nabla \theta_j(x^*)$ is a multiple of $\nabla H_j(x^*)$, and since both the H_j- and θ_j-constraint are active at x^*, LICQ is violated. $\qquad \square$

The following lemma shows that under slightly stronger assumptions MFCQ, cf. Definition 2.1.3 does not hold for an MPVC either.

Lemma 4.1.2 *Let x^* be feasible for* (1.1) *such that $I_{00} \cup I_{0+} \neq \emptyset$. Then MFCQ is violated at x^*.*

Proof. Let $j \in I_{00} \cup I_{0+}$. If $j \in I_{00}$ then $\nabla \theta_j(x^*) = 0$ and thus, $\nabla \theta_j(x^*)^T d = 0$ for all $d \in \mathbb{R}^n$, and hence MFCQ is violated. In turn, for $j \in I_{0+}$ it holds that $\nabla \theta_j(x^*) = G_j(x^*) \nabla H_j(x^*)$. Thus, if for some $d \in \mathbb{R}^n$ we have $\nabla H_j(x^*)^T d > 0$ this yields $\nabla \theta_j(x^*)^T d > 0$, which shows that MFCQ is not fulfilled in this case either. $\qquad \square$

The previous two results were taken from [3], with slightly different proofs though, where it is also argued that the assumption $I_{00} \cup I_{0+} \neq \emptyset$ is quite reasonable for MPVCs and satisfied for a big class of applications from truss topology optimization. Thus, one must come to the conclusion that both the LICQ and MFCQ are too strong assumptions for MPVCs.

Note that for MPECs the situation is even worse, that is, LICQ and MFCQ are always violated at any feasible point, see [11].

4.2. Necessary conditions for ACQ

We will now discuss the Abadie constraint qualification, see Definition 2.1.4, in the context of MPVCs.

The Abadie constraint qualification requires that the tangent cone $\mathcal{T}(x^*)$ is equal to the linearized cone $\mathcal{L}(x^*)$. Hence a necessary condition for the ACQ to be satisfied is that $\mathcal{T}(x^*)$ is a polyhedral convex cone. The aim is now to provide several characterizations of this necessary condition. To this end, we first state the following assumption.

(A1) ACQ is satisfied for all nonlinear programs $NLP_*(\beta_1, \beta_2)$, $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, where x^* denotes a given feasible point of the MPVC.

This assumption is held to be fairly weak, and a sufficient condition is the LICQ-type assumption to be formally introduced in Section 5, which is also shown to imply GCQ, see Theorem 4.3.2.

Using (A1), we are able to state the following result that may be viewed as a counterpart of [47, Proposition 3] (note, however, that part of its proof is different).

Proposition 4.2.1 *Let $x^* \in X$ be a feasible point of the MPVC from* (1.1) *such that assumption (A1) holds. Then the following statements are equivalent:*

(a) $\mathcal{T}(x^*)$ is polyhedral.

(b) $\mathcal{T}(x^*)$ is convex.

(c) For all $d^1, d^2 \in \mathcal{T}(x^*)$ and all $i \in I_{00}$, we have $(\nabla G_i(x^*)^T d^1)(\nabla H_i(x^*)^T d^2) \leq 0$.

(d) There exists a partition $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ such that $\mathcal{T}(x^*) = \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$.

Proof. (a) $\Longrightarrow$ (b): This is obvious.

(b) $\Longrightarrow$ (c): Let $d^1, d^2 \in \mathcal{T}(x^*)$ and $i \in I_{00}$ be arbitrarily given. Define $d(\lambda) := \lambda d^1 + (1 - \lambda)d^2$ for $\lambda \in (0, 1)$. Due to (b), we have $d(\lambda) \in \mathcal{T}(x^*)$ for all $\lambda \in (0, 1)$. Because of (A1) and Lemma 3.2.2, however, we have $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$. This implies $d(\lambda) \in \mathcal{L}_{MPVC}(x^*)$ for all $\lambda \in (0, 1)$. In particular, we therefore have

$$(\nabla G_i(x^*)^T d(\lambda))(\nabla H_i(x^*)^T d(\lambda)) \leq 0.$$

Using the definition of $d(\lambda)$, this can be rewritten as

$$
\begin{aligned}
0 \geq \ & \lambda^2 [(\nabla G_i(x^*)^T d^1)(\nabla H_i(x^*)^T d^1)] \\
& + (1 - \lambda)^2 [(\nabla G_i(x^*)^T d^2)(\nabla H_i(x^*)^T d^2)] \\
& + \lambda(1 - \lambda)[(\nabla G_i(x^*)^T d^1)(\nabla H_i(x^*)^T d^2) + (\nabla G_i(x^*)^T d^2)(\nabla H_i(x^*)^T d^1)].
\end{aligned}
\tag{4.1}
$$

Now suppose that $(\nabla G_i(x^*)^T d^1)(\nabla H_i(x^*)^T d^2) > 0$ (the case with d^1, d^2 being exchanged can be treated in a similar way). Since $d^2 \in \mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$ and $i \in I_{00}$, we have $\nabla H_i(x^*)^T d^2 \geq 0$. This therefore implies $\nabla G_i(x^*)^T d^1 > 0$ and $\nabla H_i(x^*)^T d^2 > 0$. Again exploiting the fact that d^1, d^2 belong to the cone $\mathcal{L}_{MPVC}(x^*)$, we obtain $\nabla G_i(x^*)^T d^2 \leq 0$ and $\nabla H_i(x^*)^T d^1 = 0$. Taking this into account, dividing (4.1) by $1 - \lambda$, and then letting $\lambda \uparrow 1$, we get the contradiction $(\nabla G_i(x^*)^T d^1)(\nabla H_i(x^*)^T d^2) \leq 0$ from (4.1).

(c) $\Longrightarrow$ (d): Let (c) hold and recall that $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$. Furthermore, mind that the cone $\mathcal{L}_{MPVC}(x^*)$ is defined by the following set of equations and inequalities:

$$
\begin{aligned}
\nabla g_i(x^*)^T d &\leq 0 & (i &\in I_g), \\
\nabla h_j(x^*)^T d &= 0 & (j &= 1, \ldots, p), \\
\nabla H_i(x^*)^T d &= 0 & (i &\in I_{0+}), \\
\nabla H_i(x^*)^T d &\geq 0 & (i &\in I_{00} \cup I_{0-}), \\
\nabla G_i(x^*)^T d &\leq 0 & (i &\in I_{+0}), \\
(\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) &\leq 0 & (i &\in I_{00}).
\end{aligned}
\tag{4.2}
$$

Now let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be a particular partition defined as follows: β_1 contains all the indices $i \in I_{00}$ such that there is a vector $d = d^i$ which satisfies the system (4.2) and such that, in addition, it holds that $\nabla H_i(x^*)^T d > 0$, i.e., this inequality is satisfied strictly. Then put $\beta_2 := I_{00} \setminus \beta_1$. Thus, for all $i \in \beta_2$ and all vectors d satisfying the system (4.2), we necessarily have $\nabla H_i(x^*)^T d = 0$. We now claim that $(\mathcal{T}(x^*) =) \mathcal{L}_{MPVC}(x^*) = \mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)$ ($= \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*)$ in view of (A1)). Comparing the definitions of the two cones $\mathcal{L}_{MPVC}(x^*)$ and $\mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)$, we only have to verify that $\nabla H_i(x^*)^T d = 0$ for all $i \in \beta_2$ and $\nabla G_i(x^*)^T d \leq 0$ for all $i \in \beta_1$. The former is true in view of our previous comments, and the latter follows from the definition of β_1 which says that, for any $i \in \beta_1$, we can find a particular vector $\tilde{d}$ satisfying the whole system (4.2) such that, in addition, $\nabla H_i(x^*)^T \tilde{d} > 0$. Assumption (c) then implies the desired inequality $\nabla G_i(x^*)^T d \leq 0$.

(d) $\Longrightarrow$ (a): This follows immediately from Assumption (A1). $\qquad \square$

At this point, we would like to point out that the statements (a)–(d) from Proposition 4.2.1 are only necessary but not sufficient conditions for ACQ. In fact, it is known, see [1] for a simple standard optimization example, that the tangent cone might be polyhedral without being equal to the corresponding linearized cone.

For MPVCs, however, the situation is even more complicated since Lemma 3.2.2 tells us that the tangent cone $\mathcal{T}(x^*)$ is typically the union of finitely many cones. Consequently, the tangent cone $\mathcal{T}(x^*)$ is usually nonconvex, i.e., the Abadie constraint qualification does not hold.

4.3. Sufficient conditions for GCQ

Our aim is to provide conditions which are reasonable for MPVCs but still sufficient for GCQ. Since it is well known, see, e.g., [23] or Chapter 2, that GCQ implies KKT conditions as a nec-

essary optimality criterion at a local minimizer of a standard optimization problem, we hereby obtain constraint qualifications to imply KKT conditions of the MPVC, and which have a much better chance to be satisfied opposite to standard constraint qualifications like LICQ, MFCQ or ACQ, see the discussion above.

The major goal of this section is to show that GCQ holds at a feasible point of an MPVC under the presence of an LICQ-type constraint qualification which occured first in the context of MPVC analysis in [3, Corollary 2] and will be formally introduced in Chapter 5.

For these purposes consider the following auxiliary result, where again the problem $NLP_*(\beta_1, \beta_2)$ from (3.1) comes into play.

Lemma 4.3.1 *Let x^* be feasible for the MPVC* (1.1) *such that the gradients*

$$
\begin{aligned}
\nabla h_j(x^*) && (j = 1, \ldots, p), \\
\nabla g_i(x^*) && (i \in I_g), \\
\nabla H_i(x^*) && (i \in I_0), \\
\nabla G_i(x^*) && (i \in I_{00} \cup I_{+0})
\end{aligned}
$$

are linearly independent. Then standard LICQ holds at x^* for all programs $NLP_*(\beta_1, \beta_2)$ and all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$.

Proof. Let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be given. In view of the definition of $NLP_*(\beta_1, \beta_2)$ in (3.6), we have to show that the gradients

$$
\begin{aligned}
\nabla h_j(x^*) && (j = 1, \ldots, p), \\
\nabla g_i(x^*) && (i \in I_g), \\
\nabla H_i(x^*) && (i \in I_0), \\
\nabla G_i(x^*) && (i \in \beta_1 \cup I_{+0})
\end{aligned}
$$

are linearly independent. Since we have $\beta_1 \subseteq I_{00}$, this is trivially satisfied, because of the assumed LICQ-type condition. $\qquad \square$

The latter result enables us to prove the above mentioned suffiency result for GCQ.

Theorem 4.3.2 *Let x^* be feasible for the MPVC* (1.1) *such that the assumptions of Lemma 4.3.1 hold. Then GCQ is satisfied at x^*.*

Proof. In view of Definition 2.1.6 and the well-known inclusion $\mathcal{L}(x^*)^* \subseteq \mathcal{T}(x^*)^*$, we only need to prove that the converse inclusion $\mathcal{T}(x^*)^* \subseteq \mathcal{L}(x^*)^*$ holds. To this end, first recall that we have

$$
\mathcal{T}(x^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*)
$$

in view of Lemma 3.2.2 (a). Invoking [6, Theorem 3.1.9] therefore yields

$$
\mathcal{T}(x^*)^* = \bigcap_{(\beta_1, \beta_2) \in \mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*)^*. \tag{4.3}
$$

Since MPVC-LICQ holds at x^* for (1.1), we know by Lemma 4.3.1 that LICQ and thus ACQ are satisfied at x^* for $NLP_*(\beta_1, \beta_2)$ and for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Hence, we have $\mathcal{T}_{NLP_*(\beta_1, \beta_2)}(x^*) = \mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)$ for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Recalling the representation of $\mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)$ from (3.7) and using [6, Theorem 3.2.2], we obtain

$$
\mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)^* =
$$
$$
\{v \in \mathbb{R}^n \mid v = -\sum_{i \in I_g} \mu_i^g \nabla g_i(x^*) - \sum_{j=1,\dots,p} \mu_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \mu_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0} \cup \beta_1} \mu_i^G \nabla G_i(x^*)
$$
$$
\text{with} \quad \mu_i^g \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-} \cup \beta_1), \quad \mu_i^G \geq 0 \ (i \in I_{+0} \cup \beta_1)\}.
$$

In a similar way, we obtain

$$
\mathcal{L}(x^*)^* =
$$
$$
\{v \in \mathbb{R}^n \mid v = -\sum_{i \in I_g} \mu_i^g \nabla g_i(x^*) - \sum_{j=1,\dots,p} \mu_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \mu_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0}} \mu_i^G \nabla G_i(x^*)
$$
$$
\text{with} \quad \mu_i^g \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-} \cup I_{00}), \quad \mu_i^G \geq 0 \ (i \in I_{+0})\}.
$$

Now let $v \in \mathcal{T}(x^*)^* = \bigcap_{(\beta_1, \beta_2) \in \mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)^*$. Moreover, choose $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ arbitrarily and put $(\tilde{\beta}_1, \tilde{\beta}_2) := (\beta_2, \beta_1)$. Using the above representation of $\mathcal{L}_{NLP_*(\beta_1, \beta_2)}(x^*)^*$, it follows that there exists a vector $\mu = (\mu^g, \mu^h, \mu^H, \mu^G)$ with

$$
\mu_i^g \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-} \cup \beta_1), \quad \mu_i^G \geq 0 \ (i \in I_{+0} \cup \beta_1) \tag{4.4}
$$

such that

$$
v = -\sum_{i \in I_g} \mu_i^g \nabla g_i(x^*) - \sum_{j=1,\dots,p} \mu_j^h \nabla h_j(x^*) + \sum_{i \in \beta_1 \cup \beta_2 \cup I_{0-} \cup I_{0+}} \mu_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0} \cup \beta_1} \mu_i^G \nabla G_i(x^*). \tag{4.5}
$$

However, since v also belongs to $\mathcal{L}_{NLP_*(\tilde{\beta}_1, \tilde{\beta}_2)}(x^*)^*$, we obtain in a similar way the existence of a certain vector $\tilde{\mu} = (\tilde{\mu}^g, \tilde{\mu}^h, \tilde{\mu}^H, \tilde{\mu}^G)$ satisfying

$$
\tilde{\mu}_i^g \geq 0 \ (i \in I_g), \quad \tilde{\mu}_i^H \geq 0 \ (i \in I_{0-} \cup \tilde{\beta}_1), \quad \tilde{\mu}_i^G \geq 0 \ (i \in I_{+0} \cup \tilde{\beta}_1)
$$

such that

$$
v = -\sum_{i \in I_g} \tilde{\mu}_i^g \nabla g_i(x^*) - \sum_{j=1,\dots,p} \tilde{\mu}_j^h \nabla h_j(x^*) + \sum_{i \in \tilde{\beta}_1 \cup \tilde{\beta}_2 \cup I_{0-} \cup I_{0+}} \tilde{\mu}_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0} \cup \tilde{\beta}_1} \tilde{\mu}_i^G \nabla G_i(x^*). \tag{4.6}
$$

Subtracting the two representations (4.5) and (4.6) of v from each other, we obtain

$$
0 = -\sum_{i \in I_g} (\mu_i^g - \tilde{\mu}_i^g) \nabla g_i(x^*) - \sum_{j=1,\dots,p} (\mu_j^h - \tilde{\mu}_j^h) \nabla h_j(x^*) + \sum_{i \in I_{0-} \cup I_{0+}} (\mu_i^H - \tilde{\mu}_i^H) \nabla H_i(x^*)
$$
$$
+ \sum_{i \in \beta_1 (= \tilde{\beta}_2)} (\mu_i^H - \tilde{\mu}_i^H) \nabla H_i(x^*) + \sum_{i \in \beta_2 (= \tilde{\beta}_1)} (\mu_i^H - \tilde{\mu}_i^H) \nabla H_i(x^*) - \sum_{i \in \beta_1} \mu_i^G \nabla G_i(x^*)
$$

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$$+ \sum_{i \in \beta_2(=\tilde{\beta}_1)} \tilde{\mu}_i^G \nabla G_i(x^*) - \sum_{i \in I_{+0}} (\mu_i^G - \tilde{\mu}_i^G) \nabla G_i(x^*).$$

Since MPVC-LICQ holds at x^*, all gradients occuring in the previous formula are linearly independent. Consequently, all coefficients are zero. In particular, we obtain $\mu_i^H = \tilde{\mu}_i^H \geq 0$ ($i \in \beta_2$) and $\mu_i^G = 0$ ($i \in \beta_1$). Taking this into account and using (4.5), (4.4), we obtain the representation

$$v = -\sum_{i \in I_g} \mu_i^g \nabla g_i(x^*) - \sum_{j=1,\dots,p} \mu_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \mu_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0}} \mu_i^G \nabla G_i(x^*)$$

with

$$\mu_i^g \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-} \cup I_{00}), \quad \mu_i^G \geq 0 \ (i \in I_{+0}).$$

This shows that v belongs to $\mathcal{L}(x^*)^*$, cf. the above representation of this dual cone. □

5. MPVC-tailored constraint qualifications

In Chapter 4 we found out that standard constraint qualifications such as LICQ and MFCQ are in most interesting situations not satisfied for MPVCs, see Section 4.1. Also ACQ, cf. Section 4.2, was shown to be a rather strong assumption in this context. Only GCQ, see Section 4.3, has a good chance to hold under some reasonable conditions. In view of these difficulties, this chapter is dedicated to introducing some new, MPVC-tailored constraint qualifications. At this, we are guided on the one hand by the standard CQs and on the other hand by some specialized tools like the MPVC-linearized cone and the assumptions of Lemma 4.3.1 which led to promising results like, e.g., Lemma 3.2.2 and Theorem 4.3.2, respectively.

5.1. MPVC-counterparts of standard CQs

In this section we establish MPVC-counterparts of LICQ, MFCQ, ACQ and GCQ as defined in Section 2.1.2.

We commence with the definition of an MPVC-tailored variant of LICQ, which is motivated, in particular, by Theorem 4.3.2 and will also play a very important role in convergence analysis of the numerical algorithms to be investigated in Part II.

Definition 5.1.1 *We say that MPVC-LICQ is satisfied at a feasible point x^* of* (1.1) *if the gradients*

$$
\begin{aligned}
&\nabla h_j(x^*) && (j = 1, \ldots, p), \\
&\nabla g_i(x^*) && (i \in I_g), \\
&\nabla H_i(x^*) && (i \in I_0), \\
&\nabla G_i(x^*) && (i \in I_{00} \cup I_{+0})
\end{aligned}
$$

are linearly independent.

A very useful observation is stated in the following lemma, which reveals that MPVC-LICQ is in fact (standard) LICQ of the *tightened nonlinear program $TNLP(x^*)$* which was established in Section 3.1, see (3.8).

Lemma 5.1.2 *Let x^* be feasible for* (1.1). *Then MPVC-LICQ is satisfied at x^* if and only if LICQ holds at x^* for $TNLP(x^*)$.*

Proof. The proof follows immediately from the definitions of LICQ (Definition 2.1.2), MPVC-LICQ (Definition 5.1.1) and $TNLP(x^*)$, see (3.8). □

In view of the above result, it is very natural to define an MPVC analogon of MFCQ in the following fashion.

Definition 5.1.3 *Let x^* be feasible for* (1.1). *Then we say that MPVC-MFCQ is satisfied at x^* if MFCQ is satisfied at x^* for $TNLP(x^*)$.*

An immediate consequence is the following lemma.

Lemma 5.1.4 *Let x^* be feasible for* (1.1) *such that MPVC-LICQ holds at x^*. Then MPVC-MFCQ is satisfied at x^*.*

Proof. The proof follows immediately from Lemma 5.1.2, the definition of MPVC-MFCQ and the fact that LICQ implies MFCQ in the standard case, see Section 2.1.2. □

At places we will need an explicit characterization of MPVC-MFCQ. For these purposes, note that MPVC-MFCQ holds at a point $x^* \in X$ if and only if the gradients

$$\nabla h_j(x^*) \, (j = 1, \ldots, p) \quad \text{and} \quad \nabla H_i(x^*) \, (i \in I_{0+} \cup I_{00}) \tag{5.1}$$

are linearly independent, and there exists a vector d such that

$$\begin{aligned}
\nabla g_i(x^*)^T d &< 0 \quad \forall i \in I_g, \\
\nabla H_i(x^*)^T d &> 0 \quad \forall i \in I_{0-}, \\
\nabla G_i(x^*)^T d &< 0 \quad \forall i \in I_{+0} \cup I_{00}, \\
\nabla h_j(x^*)^T d &= 0 \quad \forall j = 1, \ldots, p, \\
\nabla H_i(x^*)^T d &= 0 \quad \forall i \in I_{0+} \cup I_{00}.
\end{aligned} \tag{5.2}$$

The converse direction of the above lemma does not hold true in general as can be seen in the below example, whose feasible region beautifully displays the possible ill-posedness of an MPVC and will also be frequently referred to later on.

Example 5.1.5 Consider the MPVC

$$\begin{aligned}
\min \quad & f(x) := x_1 + x_2^2 \\
\text{s.t} \quad & g_1(x) := x_2 - x_1 \leq 0, \\
& H_1(x) := x_1^3 - x_2 \geq 0, \\
& G_1(x)H_1(x) := -x_1(x_1^3 - x_2) \leq 0.
\end{aligned} \tag{5.3}$$

Its feasible set can be seen in Figure 5.1 It is immediately clear that $x^* = \binom{0}{0}$ is a local minimizer for (5.3). We have $I_g = \{1\}$ as well as $I_{00} = \{1\}$. Furthermore, $\nabla H_1(x^*) = \binom{0}{-1}$, $\nabla G_1(x^*) = \binom{-1}{0}$ and $\nabla g_1(x^*) = \binom{-1}{1}$ are obviously linearly dependent and thus, MPVC-LICQ is violated. In turn, MPVC-MFCQ is satisfied, since if we choose $d := \binom{1}{0}$, then $\nabla H_1(x^*)^T d = 0$, $\nabla G_1(x^*)^T d = -1 < 0$ and $\nabla g_1(x^*)^T d = -1 < 0$.
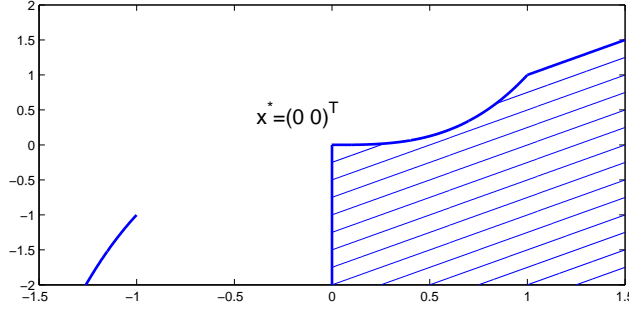
Figure 5.1.: Feasible set of (5.3)

.

In order to define MPVC counterparts of ACQ and MFCQ, we recall Corollary 3.2.3, which tells us that at any point $x^* \in X$ we have $\mathcal{T}(x^*) \subseteq \mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*)$. In Section 4.2 it was already coined that ACQ, that is $\mathcal{T}(x^*) = \mathcal{L}(x^*)$, is a very strong assumption for MPVCs, due to the fact that $\mathcal{L}(x^*)$ is in general a polyhedral convex cone, whereas $\mathcal{T}(x^*)$ is, most often, not. In view of this difficulty, Corollary 3.2.3 suggests to replace the linearized cone $\mathcal{L}(x^*)$ by the MPVC-linearized cone $\mathcal{L}_{MPVC}(x^*)$. This leads to the following MPVC counterparts of ACQ and GCQ.

Definition 5.1.6 *Let $x^* \in X$ be feasible for* (1.1). *Then we say that*

(a) *MPVC-ACQ holds at x^* if $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$.*

(b) *MPVC-GCQ holds at x^* if $\mathcal{T}(x^*)^* = \mathcal{L}_{MPVC}(x^*)^*$.*

An immediate consequence of the above definitions and Corollary 3.2.3 is the following result.

Proposition 5.1.7 *Let x^* be feasible for* (1.1). *Then the following holds true:*

(i) *If MPVC-ACQ holds at x^* then MPVC-GCQ is satisfied at x^*.*

(ii) *If ACQ holds at x^* then MPVC-ACQ is satisfied at x^*.*

(iii) *If GCQ holds at x^* then MPVC-GCQ is satisfied at x^*.*

Note that the converse implications of the above Proposition do not hold in general. This is displayed by the following two examples, where in the first one we have an MPVC which satisfies MPVC-ACQ (and hence MPVC-GCQ) but GCQ (and thus ACQ) is violated. Thus, the reversion of neither Proposition 5.1.7 (ii) nor (iii) hold in general.

Example 5.1.8 Consider the MPVC from Example 5.1.5 with its minimizer $x^* = (0,0)^T$. Then a quick calculation shows that $\mathcal{T}(x^*) = \{d \in \mathbb{R}^2 \mid d_1 \geq 0, \ d_2 \leq 0\}$ and hence, $\mathcal{T}(x^*)^* = \{v \in \mathbb{R}^2 \mid v_1 \leq 0, \ v_2 \geq 0\}$. Furthermore, we have $\mathcal{L}(x^*) = \{d \in \mathbb{R}^2 \mid d_1 \geq d_2, \ d_2 \leq 0\}$ and thus, $\mathcal{L}(x^*)^* = \{v \in \mathbb{R}^2 \mid v_1 + v_2 \geq 0, \ v_1 \leq 0\}$. In particular, this yields that GCQ is violated. Moreover, $\mathcal{L}_{MPVC}(x^*) = \{d \in \mathcal{L}(x^*) \mid d_1 d_2 \leq 0\} = \mathcal{T}(x^*)$ and hence MPVC-ACQ is fulfilled.

The second example shows that MPVC-GCQ has a chance to hold even though MPVC-ACQ does not, and so MPVC-GCQ happens to be strictly weaker than MPVC-GCQ, that is, the reversion of Proposition 5.1.7 (i) does not hold in general.

Example 5.1.9 Consider the optimization problem

$$
\begin{aligned}
\min \quad & f(x) := x_1^2 + x_2^2 \\
\text{s.t.} \quad & g_1(x) := -x_2 \leq 0, \\
& H_1(x) := x_2 - x_1^3 \geq 0, \\
& G_1(x)H_1(x) := x_1^3(x_2 - x_1^3) \leq 0.
\end{aligned}
\tag{5.4}
$$

Its unique solution is $x^* := (0,0)^T$. One can easily see by geometric arguments or by Lemma 3.2.2 that $\mathcal{T}(x^*) = \{d \in \mathbb{R}^2 \mid d_2 \geq 0, \ d_1 d_2 \leq 0\}$. One can also compute that $\mathcal{L}_{MPVC}(x^*) = \{d \in \mathbb{R}^2 \mid d_2 \geq 0\}$. Thus, MPVC-ACQ is obviously violated, whereas MPVC-GCQ holds, since we have $\mathcal{T}(x^*)^* = \{v \in \mathbb{R}^2 \mid v_1 = 0, \ v_2 \geq 0\} = \mathcal{L}_{MPVC}(x^*)^*$.

Proposition 5.1.7 (i) together with Lemma 5.1.4 almost yields the corresponding chain of implications to (2.9) for the MPVC counterparts. The only gap that has not been filled yet is the implication between MPVC-MFCQ and MPVC-ACQ. For these purposes, like in the standard case, some work is needed. For these purposes, a first sufficiency result for MPVC-ACQ is given below. At this, again, the auxiliary program $NLP_*(\beta_1, \beta_2)$ from (3.6) comes into play. Note that the assumptions in the below lemma are exactly assumption $(A1)$ from Section 4.2.

Lemma 5.1.10 *Let x^* be feasible for* (1.1). *If, for all partitions $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, the Abadie constraint qualification holds for $NLP_*(\beta_1, \beta_2)$, then MPVC-ACQ holds for* (1.1).

Proof. Using our assumption and Lemma 3.2.2, we obtain

$$
\mathcal{T}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*) = \mathcal{L}_{MPVC}(x^*),
$$

which gives the assertion. $\qquad\square$

A very nice and immediate consequence of this lemma is that MPVC-ACQ holds at any feasible point for the MPVC (1.1) as soon as all constraint functions are affine linear.

Theorem 5.1.11 *Let x^* be feasible for* (1.1) *and assume that all functions g_i, h_j, G_i, and H_i are affine linear. Then MPVC-ACQ holds at x^*.*

Proof. Since all constraints of $NLP_*(\beta_1, \beta_2)$ are affine linear for any $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, it follows from a well-known result in optimization, see also Section 2.1.2, that ACQ holds for each $NLP_*(\beta_1, \beta_2)$, $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Lemma 5.1.10 therefore gives the desired result. $\qquad\square$

In order to clarify the relationship between MPVC-MFCQ and MPVC-ACQ, we need the following auxiliary result.

Lemma 5.1.12 *Let x^* be feasible for* (1.1) *such that MPVC-MFCQ is satisfied. Then, for any* $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, *MFCQ holds at x^* for NLP$_*(\beta_1, \beta_2)$.*

Proof. Let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be given arbitrarily. We have to show that the gradients

$$
\begin{aligned}
\nabla h_j(x^*) \quad &\forall j = 1, \ldots, p, \\
\nabla H_i(x^*) \quad &\forall i \in I_{0+} \cup \beta_2
\end{aligned}
\tag{5.5}
$$

are linearly independent, and that there exists a vector $\tilde{d}$ such that

$$
\begin{aligned}
\nabla g_i(x^*)^T \tilde{d} < 0 \quad &\forall i \in I_g, \\
\nabla H_i(x^*)^T \tilde{d} > 0 \quad &\forall i \in I_{0-} \cup \beta_1, \\
\nabla G_i(x^*)^T \tilde{d} < 0 \quad &\forall i \in I_{+0} \cup \beta_1, \\
\nabla h_j(x^*)^T \tilde{d} = 0 \quad &\forall j = 1, \ldots, p, \\
\nabla H_i(x^*)^T \tilde{d} = 0 \quad &\forall i \in I_{0+} \cup \beta_2.
\end{aligned}
\tag{5.6}
$$

The linear independence of (5.5) is trivially satisfied, as we have $\beta_2 \subseteq I_{00}$ and MPVC-MFCQ holds, cf. (5.1).

Since the occurring gradients are linearly independent, the linear system

$$
\begin{pmatrix}
\nabla h_j(x^*)^T & (j = 1, \ldots, p) \\
\nabla H_i(x^*)^T & (i \in I_{0+} \cup \beta_2) \\
\nabla H_i(x^*)^T & (i \in \beta_1)
\end{pmatrix}
d =
\begin{pmatrix}
0 \\
0 \\
e
\end{pmatrix}
$$

has a solution $\hat{d}$, where $e \in \mathbb{R}^{|\beta_1|}$ denotes the vector of all ones. Now, choose d such that (5.2) is satisfied, and put

$$
d(\delta) := d + \delta \hat{d}.
$$

Then, for all $\delta > 0$, we have

$$
\begin{aligned}
\nabla h_j(x^*)^T d(\delta) = 0 \quad &\forall j = 1, \ldots, p, \\
\nabla H_i(x^*)^T d(\delta) = 0 \quad &\forall i \in I_{0+} \cup \beta_2, \\
\nabla H_i(x^*)^T d(\delta) > 0 \quad &\forall i \in \beta_1.
\end{aligned}
$$

Furthermore, for $\delta > 0$ sufficiently small, we have

$$
\begin{aligned}
\nabla g_i(x^*)^T d(\delta) < 0 \quad &\forall i \in I_g, \\
\nabla H_i(x^*)^T d(\delta) > 0 \quad &\forall i \in I_{0-}, \\
\nabla G_i(x^*)^T d(\delta) < 0 \quad &\forall i \in \beta_1 \cup I_{+0}.
\end{aligned}
$$

This concludes the proof. $\qquad\qquad\square$

The next theorem states that MPVC-MFCQ is a sufficient condition for MPVC-ACQ.

Theorem 5.1.13 *Let x^* be feasible for* (1.1) *such that MPVC-MFCQ holds. Then MPVC-ACQ is satisfied.*

Proof. Lemma 5.1.12 shows that standard MFCQ holds for every program $NLP_*(\beta_1, \beta_2)$ with $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Hence standard ACQ holds for each program $NLP_*(\beta_1, \beta_2)$. The statement therefore follows from Lemma 5.1.10. □

Eventually, we have furnished proof for the following chain of implication, which is the MPVC analogon to (2.9):

$$\text{MPVC-LICQ} \implies \text{MPVC-MFCQ} \implies \text{MPVC-ACQ} \implies \text{MPVC-GCQ}. \tag{5.7}$$

5.2. More MPVC-tailored constraint qualifications

The goal of this section is to provide further MPVC-tailored constraint qualifications and to investigate their relationships. The analysis follows results presented in [27] and is motivated by similar considerations for MPECs in [63] and bilevel programs in [64], for example, see also the treatment for standard optimization problems in [40] and elsewhere.

In order to state these constraint qualifications, we first recall the definition of two well-known cones from, e.g., [5]. Given a (feasible) set $\mathcal{X} \subseteq \mathbb{R}^n$ and a point $x \in \mathcal{X}$, we call

$$\mathcal{A}(x, \mathcal{X}) := \{d \in \mathbb{R}^n \mid \exists \delta > 0, \exists \alpha : \mathbb{R} \to \mathbb{R}^n : \ \alpha(\tau) \in \mathcal{X} \ \forall \tau \in (0, \delta),$$
$$\alpha(0) = x, \ \lim_{\tau \downarrow 0} \frac{\alpha(\tau) - \alpha(0)}{\tau} = d\} \tag{5.8}$$

the *cone of attainable directions* of $\mathcal{X}$ at x, and

$$\mathcal{F}(x, \mathcal{X}) := \{d \in \mathbb{R}^n \setminus \{0\} \mid \exists \delta > 0 : \ x + \tau d \in \mathcal{X} \ \forall \tau \in (0, \delta)\} \tag{5.9}$$

the *cone of feasible directions* of $\mathcal{X}$ at x. For the MPVC (1.1) we suppress the feasible set X in the notation and thus, for x^* in X the following chain of inclusions

$$cl(\mathcal{F}(x^*)) \subseteq cl(\mathcal{A}(x^*)) \subseteq \mathcal{T}(x^*) \subseteq \mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*) \tag{5.10}$$

holds, cf. [5, Lemma 5.2.1] and Lemma 3.2.2. Now, the standard *Zangwill constraint qualification* (ZCQ for short) is said to hold at x if $\mathcal{L}(x) \subseteq cl(\mathcal{F}(x, \mathcal{X}))$, and the standard *Kuhn-Tucker constraint qualification* (KTCQ for short) is satisfied at x if $\mathcal{L}(x) \subseteq cl(\mathcal{A}(x, \mathcal{X}))$. Using (5.10), we immediately see that

$$\text{ZCQ} \implies \text{KTCQ} \implies \text{ACQ}. \tag{5.11}$$

Since ACQ is already too strong for MPVCs, we therefore cannot expect ZCQ or KTCQ to hold for our program (1.1). However, similar to the definition of MPVC-ACQ and MPVC-GCQ, we obtain MPVC-tailored variants of these constraint qualifications by using the MPVC-linearized cone instead of the linearized cone itself.

Definition 5.2.1 *Let x^* be feasible for* (1.1)*. Then*

(a) the MPVC-ZCQ *holds at x^* if $\mathcal{L}_{MPVC}(x^*) \subseteq cl(\mathcal{F}(x^*))$.*

(b) the MPVC-KTCQ *holds at* x^* *if* $\mathcal{L}_{MPVC}(x^*) \subseteq cl(\mathcal{A}(x^*))$.

An immediate consequence of the above definition and (5.10) are the implications

$$\text{MPVC-ZCQ} \implies \text{MPVC-KTCQ} \implies \text{MPVC-ACQ},$$

which are the counterparts of (5.11). Moreover, standard ZCQ (standard KTCQ) implies MPVC-ZCQ (MPVC-KTCQ).

In classical optimization, the case of a convex program, where all equality constraints are supposed to be (affine) linear and all the inequality constraints (as well as the objective function) are supposed to be convex, is often considered, cf. Section 2.2. Very popular constraint qualifications to be used in this context are the Slater-type constraint qualifications (SCQ for short), see Definition 2.2.6.

Since the G_iH_i-restrictions in (1.1), being a product of two nonconstant functions, are very likely to be nonconvex, these standard Slater-type constraint qualifications will rather often fail to hold in the case of an MPVC. Thus, it is our goal to find suitable variants for MPVCs. To this end, let us introduce the following terminology.

Definition 5.2.2 *The program* (1.1) *is called* MPVC-convex *if the functions* h_j, G_i, H_i *are affine linear and all components* g_i *are convex.*

The next definition states the MPVC-tailored versions of two Slater-type constraint qualifications.

Definition 5.2.3 *Let the program* (1.1) *be MPVC-convex. Then this program is said to satisfy*

(a) weak MPVC-SCQ *or* MPVC-WSCQ *at a feasible point* x^* *if there exists a vector* $\hat{x}$ *such that*

$$\begin{aligned}
g_i(\hat{x}) &< 0 && \forall i \in I_g, \\
h_j(\hat{x}) &= 0 && \forall j = 1, \ldots, p, \\
G_i(\hat{x}) &\leq 0 && \forall i \in I_{+0} \cup I_{00}, \\
H_i(\hat{x}) &= 0 && \forall i \in I_{0+} \cup I_{00}, \\
H_i(\hat{x}) &\geq 0 && \forall i \in I_{0-}.
\end{aligned} \tag{5.12}$$

(b) MPVC-SCQ *if there exists a vector* $\hat{x}$ *such that*

$$\begin{aligned}
g_i(\hat{x}) &< 0 && \forall i = 1, \ldots, m, \\
h_j(\hat{x}) &= 0 && \forall j = 1, \ldots, p, \\
G_i(\hat{x}) &\leq 0 && \forall i = 1, \ldots, l, \\
H_i(\hat{x}) &= 0 && \forall i = 1, \ldots, l.
\end{aligned}$$

Note that MPVC-SCQ obviously implies MPVC-WSCQ, whereas MPVC-SCQ has the advantage that it can be checked without knowledge of the feasible point x^*. With these definitions, we are now in a position to state the next theorem which tells us that MPVC-WSCQ implies MPVC-ZCQ and thus, in view of our previous results, we also see that MPVC-WSCQ and MPVC-SCQ are sufficient conditions for MPVC-ACQ.

Theorem 5.2.4 *Let x^* be feasible for the MPVC-convex program such that MPVC-WSCQ is satisfied. Then MPVC-ZCQ holds at x^*.*

Proof.   Let $d \in \mathcal{L}_{MPVC}(x^*)$. We need to show that there is a sequence $d^k \in \mathcal{F}(x^*)$ such that d^k converges to d. To this end, choose $\hat{x}$ satisfying (5.12), a positive sequence $\{t_k\} \downarrow 0$, and put $d^k := d + t_k \hat{d} := d + t_k(\hat{x} - x^*)$. Then d^k obviously converges to d.

Now, let k be fixed for the time being. In order to see that d^k is an element of $\mathcal{F}(x^*)$, we need to prove that $x^* + \tau d^k$ is feasible for (1.1) for all $\tau > 0$ sufficiently small.

First of all, note that, since the functions g_i $(i = 1, \ldots, l)$ are convex, we have, invoking Lemma 2.2.3,

$$\nabla g_i(x^*)^T \hat{d} = \nabla g_i(x^*)^T(\hat{x} - x^*) \leq g_i(\hat{x}) - g_i(x^*) < 0 \quad \forall i \in I_g. \tag{5.13}$$

Furthermore, we also have

$$\nabla g_i(x^*)^T d \leq 0 \quad \forall i \in I_g, \tag{5.14}$$

since d is an element of $\mathcal{L}_{MPVC}(x^*)$. Together, (5.13) and (5.14) imply

$$\nabla g_i(x^*)^T d^k < 0 \quad \forall i \in I_g.$$

Invoking Taylor's formula, it follows that, for all $\tau > 0$ sufficiently small, we have

$$g_i(x^* + \tau d^k) = g_i(x^*) + \tau \nabla g_i(x^*)^T d^k + o(\tau) = \tau \nabla g_i(x^*)^T d^k + o(\tau) < 0 \quad \forall i \in I_g. \tag{5.15}$$

By continuity, we also have $g_i(x^* + \tau d^k) < 0$ for all $i \notin I_g$ and all $\tau > 0$ sufficiently small, which together with (5.15) yields

$$g_i(x^* + \tau d^k) \leq 0 \quad \forall i = 1, \ldots, l, \tag{5.16}$$

for all $\tau > 0$ sufficiently small. In order to check the remaining constraints, we put $u := \tau t_k$ and note that $u > 0$ becomes arbitrarily small for $\tau \to 0$. The definition of u implies $x^* + \tau d^k = (1-u)x^* + u\hat{x} + \tau d$. Invoking the linearity of the respective functions and exploiting the fact that $d \in \mathcal{L}_{MPVC}(x^*)$, we thus obtain, for $\tau > 0$ sufficiently small,

$$\begin{aligned}
h_j(x^* + \tau d^k) &= h_j((1-u)x^* + u\hat{x}) + \tau \underbrace{\nabla h_j(x^*)^T d}_{=0} \\
&= (1-u)\underbrace{h_j(x^*)}_{=0} + u\underbrace{h_j(\hat{x})}_{=0} = 0 \quad \forall j = 1, \ldots, p.
\end{aligned} \tag{5.17}$$

Similarly, we can compute that, for $\tau > 0$ sufficiently small, we have

$$\begin{aligned}
H_i(x^* + \tau d^k) &= H_i((1-u)x^* + u\hat{x}) + \tau \nabla H_i(x^*)^T d \\
&= (1-u)H_i(x^*) + uH_i(\hat{x}) + \tau \nabla H_i(x^*)^T d \begin{cases} > 0, & \text{if } i \in I_+, \\ = 0, & \text{if } i \in I_{0+}, \\ \geq 0, & \text{if } i \in I_{0-} \cup I_{00}, \end{cases}
\end{aligned} \tag{5.18}$$

which, in particular, implies

$$H_i(x^* + \tau d^k) \geq 0 \quad \forall i = 1, \ldots, l. \tag{5.19}$$

Furthermore, for $\tau > 0$ sufficiently small, we also have

$$G_i(x^* + \tau d^k) = (1 - u)G_i(x^*) + uG_i(\hat{x}) + \tau \nabla G_i(x^*)^T d \begin{cases} < 0, & \text{if } i \in I_{+-} \cup I_{0-}, \\ > 0, & \text{if } i \in I_{0+}, \\ \leq 0, & \text{if } i \in I_{+0}. \end{cases} \qquad (5.20)$$

Together, we obtain $G_i(x^* + \tau d_k)H_i(x^* + \tau d_k) \leq 0$ for all $i \in \{1, \ldots, l\} \setminus I_{00}$ and for all $\tau > 0$ sufficiently small. Thus, it remains to check the G_iH_i-restriction for $i \in I_{00}$. First, let $i \in I_{00}$ such that $\nabla G_i(x^*)^T d > 0$. Since we have $d \in \mathcal{L}_{MPVC}(x^*)$, this implies $\nabla H_i(x^*)^T d = 0$ and thus $H_i(x^* + \tau d^k) = 0$, in view of (5.18), that is we have $G_i(x^* + \tau d_k)H_i(x^* + \tau d_k) = 0$. Second, let $i \in I_{00}$ such that $\nabla G_i(x^*)^T d \leq 0$. Then we have $G_i(x^* + \tau d^k) \leq 0$ in view of (5.20), and thus $G_i(x^* + \tau d_k)H_i(x^* + \tau d_k) \leq 0$, which concludes the proof. $\qquad\square$

The below figure summarizes the major results which were actually shown in Section 5.1, 5.2 and 4.3, fixing all CQs relevant for MPVCs and their relationships.

At this, MPVC-affine refers to the situation from Theorem 5.1.11 where all mappings g_i, h_j, G_i, H_i are affine linear.
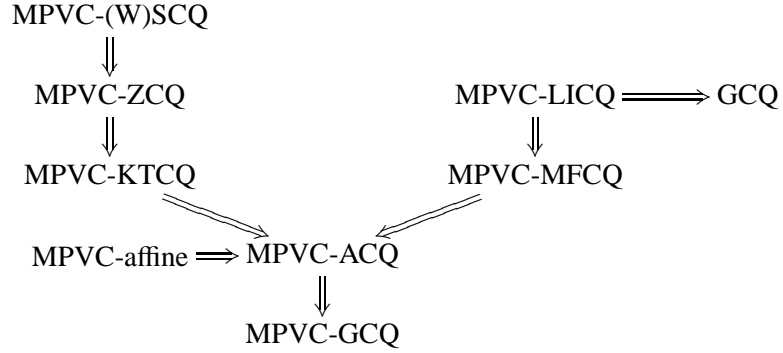
6. First-order optimality conditions for MPVCs

In this chapter we investigate first-order optimality conditions for MPVCs. First, we present necessary optimality criteria, where we mainly focus on two concepts: The first one is *strong stationarity* , which will be seen to be equivalent to the KKT conditions, cf. Section 2.1.1. The second one is *M-stationarity*, a weaker condition, holding under milder assumptions, in particular under all MPVC-tailored constraints from Chapter 5. For completeness' sake we also establish the notion of *weak stationarity*, since this one, being a very weak assumption though, sometimes occurs in the context of convergence analysis of various numerical algorithms for the solution of MPVCs.

Secondly, a first-order sufficient optimality result is proven for a special, convex-type MPVC.

6.1. First-order necessary optimality conditions

6.1.1. Strong stationarity

This whole section is concerned with a stationarity condition for MPVCs which is called *strong stationarity*. Its definition is given below. When the notion of strong stationarity appeared first in [3], it was derived directly from the KKT conditions of the MPVC.

Definition 6.1.1 (Strong stationarity) *Let x^* be feasible for* (1.1). *Then we say that x^* is strongly stationary if there exist Lagrange multipliers $(\lambda, \mu, \eta^G, \eta^H) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}^l$ such that*

$$0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(x^*) \qquad (6.1)$$

and

$$
\begin{aligned}
&h_j(x^*) = 0 \quad \forall j = 1, \ldots, p, \\
&\lambda_i \geq 0, \; g_i(x^*) \leq 0, \; \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \ldots, m, \\
&\eta_i^H = 0 \, (i \in I_+), \; \eta_i^H \geq 0 \, (i \in I_{00} \cup I_{0-}), \; \eta_i^H \textit{ free } (i \in I_{0+}), \\
&\eta_i^G = 0 \, (i \in I_0 \cup I_{+-}), \; \eta_i^G \geq 0 \, (i \in I_{+0}).
\end{aligned}
\qquad (6.2)
$$

Note that in the above situation, we will call both x^* and $(x^*, \lambda, \mu, \eta^G, \eta^H)$ a strongly stationary point of the MPVC.

As mentioned before, strong stationarity was originally derived from the KKT conditions of the MPVC (1.1). In fact, a feasible point x^* of (1.1) is strongly stationary if and only it is a KKT point. This is confirmed by the below result.

Proposition 6.1.2 *Let x^* be feasible for the MPVC* (1.1)*. Then the following assertions hold true.*

(a) *If* $(x^*, \lambda, \mu, \rho, \nu)$ *is a KKT point of* (1.1)*, then* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *with*

$$\eta_i^G := \nu_i H_i(x^*), \quad \eta_i^H := \rho_i - \nu_i G_i(x^*) \quad \forall i = 1, \ldots, l,$$

is a strongly stationary point of (1.1)*.*

(b) *If* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *is strongly stationary of* (1.1) *then* $(x^*, \lambda, \mu, \rho, \nu)$ *with*

$$\nu_i \begin{cases} = \frac{\eta_i^G}{H_i(x^*)} & \text{if } i \in I_{+0}, \\ = 0 & \text{if } i \in I_{0+}, \\ \geq \max\{0, -\frac{\eta_i^H}{G_i(x^*)}\} & \text{if } i \in I_{0+}, \\ \in [0, -\frac{\eta_i^H}{G_i(x^*)}] & \text{if } i \in I_{0-}, \\ \geq 0 & \text{if } i \in I_{00}. \end{cases}$$

and

$$\rho_i := \eta_i^H + \nu_i G_i(x^*) \quad \forall i = 1, \ldots, l,$$

is a KKT point of (1.1)*.*

In particular, x^* is a KKT point of (1.1) *if and only if it is a strongly stationary point of* (1.1)*.*

Proof. See [3]. □

Due to its equivalence to the KKT conditions, it is immediately clear that strong stationarity is a necessary optimality criterion for the MPVC under all constraint qualifications that imply GCQ since one has:

Proposition 6.1.3 *Let $x^* \in X$ be a local minimizer of* (1.1) *such that GCQ is satisfied. Then x^* is a strongly stationary point for* (1.1)*.*

Proof. The proof follows immediately from the fact that a local minimizer satisfying GCQ is a KKT point, see Section 2.1, and, by Proposition 6.1.2, every KKT point is also strongly stationary. □

An immediate consequence is the below result.

Corollary 6.1.4 *Let $x^* \in X$ be a local minimizer of* (1.1) *such that MPVC-LICQ is satisfied at x^*. Then x^* is a strongly stationary point for* (1.1) *with unique multipliers* $(\lambda, \mu, \eta^G, \eta^H)$ *such that* (6.1) *and* (6.2) *hold.*

Proof. The fact that x^* is strongly stationary is due to Proposition 6.1.3, because MPVC-LICQ implies GCQ, see Theorem 4.3.2. The uniqueness follows immediately from the linear indepedence of the gradients occuring in MPVC-LICQ. □

6.1.2. M-stationarity

In Chapter 4 it was argued that all standard constraint qualifications but GCQ must be held too strong for MPVCs. This was the major reason for establishing more applicable constraint qualifications in Chapter 5. These CQs, however, are in general weaker than their standard counterparts. In particular, except for MPVC-LICQ, these MPVC-tailored CQs do not imply GCQ and thus, strong stationarity cannot be expected to be a necessary optimality condition under these assumptions.

Due to this misery it had to be investigated which type of necessary optimality criterion may hold under MPVC-GCQ and hence under all other MPVC-tailored CQs.

Our technique of proof is motivated by the corresponding analysis carried out in [18] for MPECs, and is heavily based on the so-called *limiting normal cone*.

Definition 6.1.5 *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed set, and let $a \in C$. Then*

(a) *the* Fréchet normal cone *to C at a is defined by $\hat{N}(a, C) := (\mathcal{T}_C(a))^\circ$, i.e., the Fréchet normal cone is the polar of the tangent cone.*

(b) *the* limiting normal cone *to C at a is defined by*

$$N(a, C) := \{ \lim_{k \to \infty} w^k \mid \exists \{a^k\} \subseteq C : a^k \to a, \ w^k \in \hat{N}(a^k, C) \}. \tag{6.3}$$

The Fréchet normal cone is sometimes also called the *regular normal cone*, most notably in [54], whereas the limiting normal cone comes with a number of different names, including *normal cone*, *basic normal cone*, and *Mordukhovich normal cone* due to the many contributions of Mordukhovich in this area, see, in particular, [38, 39] for an extensive treatment and many applications of this cone. In case of a convex set C, both the Fréchet normal cone and the limiting normal cone coincide with the standard normal cone from convex analysis, cf. [53].

For the remainder, we put

$$q := |I_{00}|.$$

The following result calculates both the Fréchet and the limiting normal cone of a particular set that will play an essential role in the analysis of MPVCs.

Lemma 6.1.6 *Let the set*

$$C := \{(v, \rho) \in \mathbb{R}^q \times \mathbb{R}^q \mid \rho_i \geq 0, \rho_i v_i \leq 0 \ \forall i = 1, \ldots, q\}$$

be given. Then the following statements hold:

(a) $\hat{N}((0, 0), C) = \{(u, v) \mid u = 0, v \leq 0\}.$

(b) $N((0, 0), C) = \{(u, v) \mid u_i \geq 0, u_i v_i = 0 \ \forall i = 1, \ldots, q\}.$

Proof. Reordering the elements of the set C in a suitable way, we see that C can be expressed as a Cartesian product $C_1 \times \cdots \times C_q$ with closed sets $C_i := \{(\nu_i, \rho_i) \in \mathbb{R}^2 \mid \rho_i \geq 0, \rho_i \nu_i \leq 0\}$. Invoking [54, Proposition 6.41], it follows that we simply have to calculate the Fréchet and the limiting normal cones of the set $M := \{(\nu, \rho) \in \mathbb{R}^2 \mid \rho \geq 0, \rho\nu \leq 0\}$ at $(0, 0) \in \mathbb{R}^2$.

(a) Because of the above remark, it suffices to show that $\hat{N}((0, 0), M) = \{0\} \times \mathbb{R}_-$. It is easy to see, however, that $\mathcal{T}((0, 0), M) = M$ holds. Thus, the Fréchet normal cone is given by $\hat{N}((0, 0), M) = M^\circ = \{(c, d) \in \mathbb{R}^2 \mid c = 0, \ d \leq 0\} = \{0\} \times \mathbb{R}_-$, which proves assertion (a).

(b) It suffices to show that $N((0, 0), M) = \{(r, s) \in \mathbb{R}^2 \mid r \geq 0, rs = 0\}$ holds.

$' \subseteq '$: In view of the definition of the limiting normal cone in (6.3), we first need to figure out how the Fréchet normal cone of M at an arbitrary point $(\nu, \rho) \in M$ looks like. To this end, we consider five cases:

1) $\nu < 0, \rho > 0$: This implies $\mathcal{T}((\nu, \rho), M) = \mathbb{R}^2$. Hence $\hat{N}((\nu, \rho), M) = \{0\} \times \{0\} =: A_1$.

2) $\nu = 0, \rho > 0$: This implies $\mathcal{T}((\nu, \rho), M) = \mathbb{R}_- \times \mathbb{R}$. Hence $\hat{N}((\nu, \rho), M) = \mathbb{R}_+ \times \{0\} =: A_2$.

3) $\nu < 0, \rho = 0$: This implies $\mathcal{T}((\nu, \rho), M) = \mathbb{R} \times \mathbb{R}_+$. Hence $\hat{N}((\nu, \rho), M) = \{0\} \times \mathbb{R}_- =: A_3$.

4) $\nu > 0, \rho = 0$: This implies $\mathcal{T}((\nu, \rho), M) = \mathbb{R} \times \{0\}$. Hence $\hat{N}((\nu, \rho), M) = \{0\} \times \mathbb{R} =: A_4$.

5) $\nu = \rho = 0$: This implies $\mathcal{T}((\nu, \rho), M) = M$. Hence $\hat{N}((\nu, \rho), M) = \{0\} \times \mathbb{R}_- = A_3$.

Now let $w \in N((0, 0), M)$. Then there is a sequence $\{w^k\} \to w$ such that $w^k \in \hat{N}((\nu_k, \rho_k), M)$ for all $k \in \mathbb{N}$ and some sequence $\{(\nu_k, \rho_k)\} \subseteq M$ converging to $(0, 0)$. Then it follows from the above five cases that all w^k belong to the set $A_1 \cup A_2 \cup A_3 \cup A_4 = A_2 \cup A_4 = \mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R} = \{(r, s) \in \mathbb{R}^2 \mid r \geq 0, rs = 0\}$. Since this set is closed, the limiting element w also belongs to this set. This gives the desired inclusion.

$' \supseteq '$: Let $(a, b) \in \{(r, s) \in \mathbb{R}^2 \mid r \geq 0, rs = 0\}$. First, we consider the case $a > 0$ (hence $b = 0$). In order to prove $(a, b) \in N((0, 0), M)$, we define the sequence $\{(u_k, v_k)\} \subseteq M$ by putting $u_k := 0$ and selecting v_k such that we have $v_k \downarrow 0$. Then we are in the above second case for all $k \in \mathbb{N}$. Consequently, we have $(a_k, b_k) := (a, 0) \in \hat{N}((u_k, v_k), M)$ for all $k \in \mathbb{N}$ which proves the desired inclusion. Next, consider the case $a = 0$ (and b arbitrary). Then let $\{(u_k, v_k)\} \subseteq M$ be any sequence with $u_k \downarrow 0$ and $v_k = 0$ for all $k \in \mathbb{N}$. Then the above fourth case shows that $\hat{N}((u_k, v_k), M) = \{0\} \times \mathbb{R}$. Defining $(a_k, b_k) := (0, b)$ for all $k \in \mathbb{R}$, it therefore follows that $(a_k, b_k) \in \hat{N}((u_k, v_k), M)$ for all $k \in \mathbb{N}$, and this gives the desired inclusion also in this case. □

Now let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the following sets:

$$
\mathcal{D}_1 := \{(d, v, \rho) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q \mid
\begin{array}{ll}
\nabla g_i(x^*)^T d \le 0 & (i \in I_g), \\
\nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\
\nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\
\nabla H_i(x^*)^T d \ge 0 & (i \in I_{0-}), \\
\nabla G_i(x^*)^T d \le 0 & (i \in I_{+0}), \\
\nabla G_i(x^*)^T d - v_i = 0 & (i \in I_{00}), \\
\nabla H_i(x^*)^T d - \rho_i = 0 & (i \in I_{00})\}.
\end{array}
\tag{6.4}
$$

and

$$
\mathcal{D}_2 := \{(d, v, \rho) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q \mid \rho_i \ge 0, \ v_i \rho_i \le 0 \ \forall i = 1, \ldots, q\}.
\tag{6.5}
$$

These two sets will be crucial for the proof of our upcoming main result.

Lemma 6.1.7 *Let the multifunction* $\Phi : \mathbb{R}^{n+2q} \rightrightarrows \mathbb{R}^{n+2q}$ *be given by*

$$
\Phi(v) := \{w \in \mathcal{D}_1 \mid v + w \in \mathcal{D}_2\}.
\tag{6.6}
$$

Then Φ *is a polyhedral multifunction, e.g.,* gphΦ *is the union of finitely many convex sets.*

Proof. Since the graph of Φ may be expressed as

$$
\text{gph}\Phi = \{(d^v, v^v, \rho^v, d^w, v^w, \rho^w) \mid
\begin{array}{ll}
\nabla g_i(x^*)^T d^w \le 0 & (i \in I_g), \\
\nabla h_j(x^*)^T d^w = 0 & (j = 1, \ldots, p), \\
\nabla H_i(x^*)^T d^w = 0 & (i \in I_{0+}), \\
\nabla H_i(x^*)^T d^w \ge 0 & (i \in I_{0-}), \\
\nabla G_i(x^*)^T d^w \le 0 & (i \in I_{+0}), \\
\nabla G_i(x^*)^T d^w - v_i^w = 0 & (i \in I_{00}), \\
\nabla H_i(x^*)^T d^w - \rho_i^w = 0 & (i \in I_{00}), \\
\rho^v + \rho^w \ge 0, \\
(\rho_i^v + \rho_i^w)(v_i^v + v_i^w) \le 0 & (i = 1, \ldots, q)\}
\end{array}
$$

$$
= \bigcup_{(\alpha_1, \alpha_2) \in \mathcal{P}(\{1, \ldots, q\})} \{(d^v, v^v, \rho^v, d^w, v^w, \rho^w) \mid
\begin{array}{ll}
\nabla g_i(x^*)^T d^w \le 0 & (i \in I_g), \\
\nabla h_j(x^*)^T d^w = 0 & (j = 1, \ldots, p), \\
\nabla H_i(x^*)^T d^w = 0 & (i \in I_{0+}), \\
\nabla H_i(x^*)^T d^w \ge 0 & (i \in I_{0-}), \\
\nabla G_i(x^*)^T d^w \le 0 & (i \in I_{+0}), \\
\nabla G_i(x^*)^T d^w - v_i^w = 0 & (i \in I_{00}), \\
\nabla H_i(x^*)^T d^w - \rho_i^w = 0 & (i \in I_{00}), \\
\rho_{\alpha_1}^v + \rho_{\alpha_1}^w \ge 0, \\
\rho_{\alpha_2}^v + \rho_{\alpha_2}^w = 0, \\
v_{\alpha_1}^v + v_{\alpha_1}^w \le 0\},
\end{array}
$$

gphΦ is the union of finitely many polyhedral convex sets. Hence the assertion follows. □

The previous results allow us to state the following main result of this section.

Theorem 6.1.8 *Let x^* be a local minimizer of* (1.1) *such that MPVC-GCQ holds. Then there exist multipliers* $(\lambda, \mu, \eta^G, \eta^H)$ *such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(x^*) = 0 \qquad (6.7)$$

and

$$
\begin{aligned}
&\lambda_i \geq 0, \;\; g_i(x^*) \leq 0, \;\; \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \ldots, m, \\
&\eta_i^H = 0 \,(i \in I_+), \;\; \eta_i^H \geq 0 \,(i \in I_{0-}), \;\; \eta_i^H \text{ free } (i \in I_{0+}), \\
&\eta_i^G = 0 \,(i \in I_{+-} \cup I_{0-} \cup I_{0+}), \;\; \eta_i^G \geq 0 \,(i \in I_{+0} \cup I_{00}), \\
&\eta_i^H \eta_i^G = 0 \,(i \in I_{00}).
\end{aligned}
\qquad (6.8)
$$

Proof. Since x^* is a local minimizer of (1.1), standard results from optimization imply that $\nabla f(x^*)^T d \geq 0$ for all $d \in \mathcal{T}(x^*)$, see, e.g., Section 2.1.3. Since MPVC-GCQ holds at x^*, it therefore follows that $\nabla f(x^*) \in \mathcal{T}(x^*)^* = \mathcal{L}_{MPVC}(x^*)^*$. Consequently, we have $\nabla f(x^*)^T d \geq 0$ for all $d \in \mathcal{L}_{MPVC}(x^*)$. This is equivalent to $d^* = 0$ being a minimizer of

$$\min_{d} \nabla f(x^*)^T d \quad \text{s.t.} \quad d \in \mathcal{L}_{MPVC}(x^*). \qquad (6.9)$$

Now, $d^* = 0$ being a minimizer of (6.9) is equivalent to $(d^*, v^*, \rho^*) := (0, 0, 0)$ being a minimizer of

$$\min_{d,v,\rho} \nabla f(x^*)^T d \quad \text{s.t.} \quad (d, v, \rho) \in \mathcal{D} := \mathcal{D}_1 \cap \mathcal{D}_2 \qquad (6.10)$$

with $\mathcal{D}_1$ and $\mathcal{D}_2$ as defined in (6.4) and (6.5), respectively. Once more, since $(0, 0, 0)$ is a minimizer of (6.10), we have $(\nabla f(x^*)^T, 0, 0)^T w \geq 0$ for all $w \in \mathcal{T}((0, 0, 0), \mathcal{D})$, where $\mathcal{T}((0, 0, 0), \mathcal{D})$ denotes the tangent cone of $\mathcal{D}$ at the origin. Using [54, Proposition 6.5], this implies

$$(-\nabla f(x^*)^T, 0, 0)^T \in \mathcal{T}((0, 0, 0), \mathcal{D})^\circ = \hat{N}((0, 0, 0), \mathcal{D}) \subseteq N((0, 0, 0), \mathcal{D}). \qquad (6.11)$$

Since Φ, as defined in (6.6), is a polyhedral multifunction by Lemma 6.1.7, [52, Proposition 1] may be invoked to show that Φ is locally upper Lipschitz at every point $v \in \mathbb{R}^{n+2q}$. In particular, it is therefore calm at every $(v, w) \in \text{gph}\Phi$ in the sense of [25] (see also Definition 8.2.3 for a definition of calmness of a multifunction). Invoking [25, Corollary 4.2], we see that (6.11) implies

$$(-\nabla f(x^*)^T, 0, 0)^T \in N((0, 0, 0), \mathcal{D}_1) + N((0, 0, 0), \mathcal{D}_2).$$

Since $\mathcal{D}_1$ is polyhedral convex, the limiting normal cone of $\mathcal{D}_1$ is equal to the standard normal cone from convex analysis, and standard results on the representation of this normal cone (see,

e.g., [6, 18]) yield the existence of certain vectors $\lambda, \mu, \mu^H, \mu^G$ such that

$$
\begin{pmatrix} -\nabla f(x^*) \\ 0 \\ 0 \end{pmatrix} \in \sum_{i \in I_g} \lambda_i \begin{pmatrix} \nabla g_i(x^*) \\ 0 \\ 0 \end{pmatrix} + \sum_{j=1}^{p} \mu_j \begin{pmatrix} \nabla h_j(x^*) \\ 0 \\ 0 \end{pmatrix}
$$
$$
- \sum_{i \in I_{0+} \cup I_{0-}} \mu_i^H \begin{pmatrix} \nabla H_i(x^*) \\ 0 \\ 0 \end{pmatrix} + \sum_{i \in I_{+0}} \mu_i^G \begin{pmatrix} \nabla G_i(x^*) \\ 0 \\ 0 \end{pmatrix} \tag{6.12}
$$
$$
- \sum_{i \in I_{00}} \mu_i^H \begin{pmatrix} \nabla H_i(x^*) \\ 0 \\ -e^i \end{pmatrix} + \sum_{i \in I_{00}} \mu_i^G \begin{pmatrix} \nabla G_i(x^*) \\ -e^i \\ 0 \end{pmatrix}
$$
$$
+ N((0,0,0), \mathcal{D}_2)
$$

with

$$
\lambda_i \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-}), \quad \mu_i^G \geq 0 \ (i \in I_{+0}), \tag{6.13}
$$

where e^i denotes the compatible unit vector in $\mathbb{R}^q$.

Using [54, Proposition 6.41] and Lemma 6.1.6, we get the following explicit representation of the remaining normal cone:

$$
N((0,0,0), \mathcal{D}_2) = N(0, \mathbb{R}^n) \times N((0,0), \{(v, \rho) \mid \rho_i \geq 0, \rho_i v_i \leq 0 \ \forall i = 1, \dots, q\})
$$
$$
= \{0\}^n \times \{(u, v) \mid u_i \geq 0, u_i v_i = 0 \ \forall i = 1, \dots, q\}.
$$

Applying the above equality to (6.12) yields

$$
\mu_i^G \geq 0 \ \wedge \ \mu_i^G \mu_i^H = 0 \quad \forall i \in I_{00}. \tag{6.14}
$$

Putting $\lambda_i := 0$ for $i \notin I_g$, $\eta_i^H := 0$ for $i \in I_+$, $\eta_i^G := 0$ for $i \in I_{0+} \cup I_{0-} \cup I_{+-}$, $\eta_i^G := \mu_i^G$ and $\eta_i^H := \mu_i^H$ for all other indices, we see from (6.13), (6.14) and (6.12) that (6.7) and (6.8) are satisfied. $\quad\square$

Motivated by a corresponding terminology for MPECs (where it was introduced in [58]) and based on the fact that the optimality conditions (6.7), (6.8) from Theorem 6.1.8 were derived using the Mordukhovich normal cone, we call them the *M-stationarity conditions* of an MPVC. They are slightly weaker than the strong stationarity conditions (6.1), (6.2) from Definition 6.1.1. In fact, in the latter we have $\eta_i^H \geq 0$ and $\eta_i^G = 0$ for all $i \in I_{00}$, whereas now we only have $\eta_i^G \geq 0$ and $\eta_i^H \eta_i^G = 0$ for all $i \in I_{00}$. In particular, M- and strong stationarity coincide as soon as $I_{00} = \emptyset$.
For the sake of completeness we give a formal definition of M-stationarity below.

Definition 6.1.9 *Let x^* be feasible for (1.1). Then we say that x^* is M-stationary if there exist multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that*

$$
\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(x^*) = 0 \tag{6.15}
$$

and

$$h_j(x^*) = 0 \quad \forall j = 1, \ldots, p,$$

$$\lambda_i \geq 0, \quad g_i(x^*) \leq 0, \quad \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \ldots, m,$$

$$\eta_i^H = 0 \, (i \in I_+), \quad \eta_i^H \geq 0 \, (i \in I_{0-}), \quad \eta_i^H \, free \, (i \in I_{0+}), \tag{6.16}$$

$$\eta_i^G = 0 \, (i \in I_{0-} \cup I_{0+} \cup I_{+-}), \quad \eta_i^G \geq 0 \, (i \in I_{+0} \cup I_{00}),$$

$$\eta_i^G \eta_i^H = 0 \quad (i \in I_{00}).$$

As MPVC-GCQ is the weakest among the MPVC-tailored constraints, M-stationarity becomes a necessary optimality condition in the presence of any of these CQs.

Corollary 6.1.10 *Let x^* be a local minimizer of* (1.1) *such that either MPVC-ACQ, -MFCQ, - KTCQ, -ZCQ or -(W)SCQ is satisfied. Then x^* is M-stationary.*

Proof. The proof follows from Theorem 6.1.8 and the fact that all the assumed CQs imply MPVC-GCQ. □

The following example considers an MPVC with a local minimizer being M- but not strongly stationary. Thus, this example nicely illustrates the situation described in the introduction of this section in which strong stationarity is a too restrictive tool for necessary optimality results. Hence, M-stationarity and all the MPVC-tailored CQs that go with it are appropriate and relevant devices for MPVC analysis.

Example 6.1.11 Consider the MPVC from Example 5.1.5 with its minimizer $x^* = (0, 0)^T$. Apparently, since MPVC-MFCQ holds at x^* as was argued in Example 5.1.5, due to Corollary 6.1.10, x^* is at least M-stationary. If the point x^* was strongly stationary, the equation

$$\begin{aligned} 0 &= \nabla f(x^*) + \eta^G \nabla G_1(x^*) - \eta^H \nabla H_1(x^*) + \lambda \nabla g_1(x^*) \\ &= \binom{1}{0} + \eta^G \binom{-1}{0} - \eta^H \binom{0}{-1} + \lambda \binom{-1}{1} \end{aligned} \tag{6.17}$$

would yield $0 \leq \eta^H = -\lambda \leq 0$ an thus, $\eta^H = \lambda = 0$, which, in turn, implies $\eta^G = 1 > 0$, showing that x^* is M-stationary but not strongly stationary and in particular not a KKT point.

6.1.3. Weak stationarity

At places, mainly in Part II, a stationary condition still weaker than M-stationarity occurs in the context of MPVCs. It was originally employed and formally introduced in [31], where it was already coined to be very mild, justifying its name *weak stationarity*.

Definition 6.1.12 *Let x^* be feasible for the MPVC* (1.1). *Then x^* is called* weakly stationary *if there exist multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that*

$$0 = \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j=1}^p \mu_j \nabla h_j(x^*) - \sum_{i=1}^l \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^l \eta_i^G \nabla G_i(x^*) \tag{6.18}$$

and

$$h_j(x^*) = 0 \quad \forall j = 1, \ldots, p,$$

$$\lambda_i \geq 0, \quad g_i(x^*) \leq 0, \quad \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \ldots, m,$$

$$\eta_i^H = 0 \, (i \in I_+), \quad \eta_i^H \geq 0 \, (i \in I_{0-}), \quad \eta_i^H \text{ free } (i \in I_{0+} \cup I_{00}),$$

$$\eta_i^G = 0 \, (i \in I_{+-} \cup I_{0-} \cup I_{0+}), \quad \eta_i^G \geq 0 \, (i \in I_{+0} \cup I_{00}).$$

In order to conclude the section on first-order necessary optimality conditions we state a result which sums up the relations of strong, M- and weak stationarity and emphasizes their differences.

Proposition 6.1.13 *Let* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *be a weakly stationary for* (1.1)*. Then the following holds true.*

(a) If in addition we have

$$\eta_i^G \eta_i^H = 0 \quad \forall i \in I_{00}$$

then $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *is M-stationary.*

(b) If furthermore we assume that

$$\eta_i^H \geq 0, \ \eta_i^G = 0 \quad \forall i \in I_{00}$$

then $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *is strongly stationary.*

In particular one has the following chain of implications:

$$\text{strong stationarity} \ \Rightarrow \ \text{M-stationarity} \ \Rightarrow \ \text{weak stationarity}. \tag{6.19}$$

6.2. A first-order sufficient optimality condition

We know from the discussion of the previous section that both strong and M-stationarity are first-order necessary optimality conditions for MPVCs in the presence of suitable constraint qualifications. In the case of a standard nonlinear program, the usual KKT conditions are also known to be sufficient optimality conditions under certain convexity assumptions, see Theorem 2.2.5. In our case, however, this result cannot be applied since the product term $G_i(x)H_i(x)$ usually does not satisfy any convexity requirements. Nevertheless, we will see in this section that M- and strong stationarity are also sufficient optimality conditions for our nonconvex MPVC problem, provided that the mappings g_i, h_j, G_i, H_i satisfy some convexity assumptions (but not necessarily the products $G_i H_i$ themselves). Our analysis here is motivated by a related result from [63] in the context of MPECs and was originally published by Kanzow and the author of this thesis in [28].

In order to state the desired result, we first recall some well-known terms concerning certain convexity properties of real-valued functions, see, for example, [5, 40].

Definition 6.2.1 *Let* $S \subseteq \mathbb{R}^n$ *be a nonempty convex set and let* $f : S \rightarrow \mathbb{R}$*. Then* f *is called* quasiconvex *if, for each* $x, y \in S$*, the following inequality holds:*

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\} \quad \forall \lambda \in (0, 1).$$

Definition 6.2.2 *Let $S \subseteq \mathbb{R}^n$ be a nonempty open set and let $f : S \to \mathbb{R}$ be a differentiable function. Then f is called* pseudoconvex *if, for each $x, y \in S$, the following implication holds:*

$$\nabla f(x)^T (y - x) \geq 0 \implies f(y) \geq f(x).$$

Now, let x^* be an M-stationary point of the MPVC (1.1) with corresponding multipliers $\lambda, \mu, \eta^G, \eta^H$. Then we define the following index sets:

$$
\begin{aligned}
J^+ &:= \{j \in J \mid \mu_j > 0\}, \\
J^- &:= \{j \in J \mid \mu_j < 0\}, \\
I_{00}^+ &:= \{i \in I_{00} \mid \eta_i^H > 0\}, \\
I_{00}^- &:= \{i \in I_{00} \mid \eta_i^H < 0\}, \\
I_{0-}^+ &:= \{i \in I_{0-} \mid \eta_i^H > 0\}, \\
I_{0+}^+ &:= \{i \in I_{0+} \mid \eta_i^H > 0\}, \\
I_{0+}^- &:= \{i \in I_{0+} \mid \eta_i^H < 0\}, \\
I_{+0}^{0+} &:= \{i \in I_{+0} \mid \eta_i^H = 0, \ \eta_i^G > 0\} = \{i \in I_{+0} \mid \eta_i^G > 0\}, \\
I_{00}^{0+} &:= \{i \in I_{00} \mid \eta_i^H = 0, \ \eta_i^G > 0\} = \{i \in I_{00} \mid \eta_i^G > 0\}.
\end{aligned}
\tag{6.20}
$$

Note that, for a strongly stationary point, the two index sets I_{00}^- and I_{00}^{0+} are empty.

Using these index sets and definitions, we are able to state the main result of this section.

Theorem 6.2.3 *Let x^* be an M-stationary point of the MPVC* (1.1). *Suppose that f is pseudoconvex at x^* and that g_i $(i \in I_g), h_j$ $(j \in J^+), -h_j$ $(j \in J^-), G_i$ $(i \in I_{+0}^{0+}), H_i$ $(i \in I_{0+}^-), -H_i$ $(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{0-}^+)$ are quasiconvex. Then the following statements hold:*

(a) If $I_{00}^- \cup I_{00}^{0+} = \emptyset$ then x^* is a local minimizer of (1.1).

(b) If $I_{0+}^- \cup I_{00}^- \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$ then x^* is a global minimizer of (1.1).

Proof. Since x^* is an M-stationary point of (1.1) there exist multipliers $\lambda, \mu, \eta^G, \eta^H$ such that

$$\nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) - \sum_{i \in I_0} \eta_i^H \nabla H_i(x^*) + \sum_{i \in I_{+0} \cup I_{00}} \eta_i^G \nabla G_i(x^*) = 0 \tag{6.21}$$

with

$$
\begin{aligned}
\lambda_i &\geq 0 \quad \forall i \in I_g, \quad &\eta_i^H &\geq 0 \quad \forall i \in I_{0-} \\
\eta_i^G &\geq 0 \quad \forall i \in I_{00} \cup I_{+0}, \quad &\eta_i^H \eta_i^G &= 0 \quad \forall i \in I_{00}.
\end{aligned}
\tag{6.22}
$$

Now let x be any feasible point of (1.1). For $i \in I_g$, we then have $g_i(x) \leq 0 = g_i(x^*)$. Thus, by the quasiconvexity of g_i $(i \in I_g)$, we obtain

$$g_i(x^* + t(x - x^*)) = g_i((1 - t)x^* + tx) \leq \max\{g_i(x), g_i(x^*)\} = 0 = g_i(x^*)$$

for all $t \in (0, 1)$, which implies

$$\nabla g_i(x^*)^T (x - x^*) = g_i'(x^*; x - x^*) = \lim_{t \downarrow 0} \frac{g_i(x^* + t(x - x^*)) - g_i(x^*)}{t} \leq 0 \quad \forall i \in I_g.$$

In view of (6.22), we therefore have

$$\lambda_i \nabla g_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_g. \tag{6.23}$$

By similar arguments, we also obtain

$$\nabla h_j(x^*)^T(x - x^*) \leq 0 \quad \forall j \in J^+, \quad \text{and} \quad -\nabla h_j(x^*)^T(x - x^*) \leq 0 \quad \forall j \in J^-,$$

which gives

$$\mu_j \nabla h_j(x^*)^T(x - x^*) \leq 0 \quad \forall j \in J, \tag{6.24}$$

taking the definitions of J^+ and J^- into account.

Again, since x is feasible for (1.1), we particularly have $-H_i(x) \leq 0$ for all $i = 1, \ldots, l$. Thus, by the quasiconvexity of $-H_i$ for $i \in I_{0+}^+ \cup I_{00}^+ \cup I_{0-}^+$, we obtain with the above arguments $-\nabla H_i(x^*)^T(x - x^*) \leq 0$ and thus, in view of the definition of the occurring index sets, we have

$$-\eta_i^H \nabla H_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_{0+}^+ \cup I_{00}^+ \cup I_{0-}^+. \tag{6.25}$$

We now verify statement (b) first. To this end, let $I_{0+}^- \cup I_{00}^- \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$. Then it is clear from (6.22), (6.25), and the definition of the index sets that we even have

$$-\eta_i^H \nabla H_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_0, \quad \eta_i^G \nabla G_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_{00} \cup I_{+0}, \tag{6.26}$$

where the second inequality is an equality due to the fact that $\eta_i^G = 0$ for all (remaining) indices $i \in I_{00} \cup I_{+0}$. Then (6.23), (6.24), (6.26) together with (6.21) imply

$$
\begin{aligned}
-\nabla f(x^*)^T(x - x^*) &= \Big(\sum_{i \in I_g} \lambda_i \nabla g_i(x^*)^T + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) - \sum_{i \in I_0} \eta_i^H \nabla H_i(x^*) + \ldots \\
&\quad \cdots + \sum_{i \in I_{+0} \cup I_{00}} \eta_i^G \nabla G_i(x^*) \Big)^T (x - x^*) \leq 0.
\end{aligned}
$$

Hence we have $\nabla f(x^*)^T(x - x^*) \geq 0$, which implies $f(x) \geq f(x^*)$, as f is pseudoconvex by assumption. Since x is an arbitrary feasible point of (1.1), x^* is a global minimizer of (1.1) in the case that $I_{0+}^- \cup I_{00}^- \cup I_{+0}^{0+} \cup I_{00}^{0+} = \emptyset$ holds, which proves assertion (b).

To verify statement (a), we only need to show, in view of the above arguments, that for any feasible x sufficiently close to x^*, we have

$$-\eta_i^H \nabla H_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_{0+}^- \tag{6.27}$$

and

$$\eta_i^G \nabla G_i(x^*)^T(x - x^*) \leq 0 \quad \forall i \in I_{+0}^{0+}, \tag{6.28}$$

since then we see that (6.23), (6.24) and (6.26) are satisfied, and thus, by analogous reasoning as above, we obtain $f(x) \geq f(x^*)$ for all feasible x sufficiently close to x^*.

First let $i \in I_{0+}^-$. By continuity, it follows that $G_i(x) > 0$ and thus $H_i(x) = 0$ for any $x \in X$ sufficiently close to x^*. Invoking the quasiconvexity of H_i ($i \in I_{0+}^-$), this implies $\nabla H_i(x^*)^T(x - x^*) \leq 0$, and since we have $\eta_i^H < 0$ ($i \in I_{0+}^-$), (6.27) follows immediately.

Second, let $i \in I_{+0}^{0+}$. By continuity, it follows that $H_i(x) > 0$ and thus $G_i(x) \le 0$ for any $x \in X$ sufficiently close to x^*. Invoking the quasiconvexity of G_i ($i \in I_{+0}^{0+}$), this implies $\nabla G_i(x^*)^T(x - x^*) \le 0$, which gives (6.28), since we have $\eta_i^G > 0$ ($i \in I_{+0}^{0+}$). $\qquad\square$

We next state a simple consequence of Theorem 6.2.3 where the M-stationarity of x^* is replaced by the strong stationarity assumption.

Corollary 6.2.4 *Let x^* be a strongly stationary point of the MPVC* (1.1). *Suppose that f is pseudoconvex at x^* and that g_i ($i \in I_g$), h_j ($j \in J^+$), $-h_j$ ($j \in J^-$), G_i ($i \in I_{+0}^{0+}$), H_i ($i \in I_{0+}^-$), $-H_i$ ($i \in I_{0+}^+ \cup I_{00}^+ \cup I_{0-}^+$) are quasiconvex. Then the following statements hold:*

(a) x^* is a local minimizer of (1.1).

(b) If $I_{0+}^- \cup I_{+0}^{0+} = \emptyset$ then x^* is a global minimizer of (1.1).

Proof. Since the assumptions of Theorem 6.2.3 are satisfied and strong stationarity implies that $I_{00}^- \cup I_{00}^{0+} = \emptyset$, (a) and (b) follow immediately from Theorem 6.2.3 (a) and (b), respectively. $\qquad\square$

If we sharpen the assumptions to an MPVC-convex setup, see Definition 5.2.2, we obtain the following handy result.

Corollary 6.2.5 *Let the program* (1.1) *be MPVC-convex such that f is convex. Furthermore, let x^* be a strongly stationary point of* (1.1). *Then the following statements hold:*

(a) x^* is a local minimizer of (1.1).

(b) If $I_{0+}^- \cup I_{+0}^{0+} = \emptyset$, then x^* is a global minimizer of (1.1).

Proof. Follows immediately from Corollary 6.2.4, since convex functions are both pseudo- and quasiconvex. $\qquad\square$

We would like to point out that we find the above result somehow remarkable: The MPVC-convex program, though being equipped with convex and affine linear functions g_i, h_j, H_i, G_i, must yet be assumed to be a nonconvex program, due to the $G_i H_i$-constraints. Nevertheless, Corollary 6.2.5 tells us that the strong stationarity conditions (and thus the KKT conditions themselves) are sufficient optimality conditions. That means, we have shown the KKT conditions to be a sufficient optimality criterion for a class of usually nonconvex programs.

At this point it might be useful to go through a simple example of an MPVC in order to illustrate some of the above introduced concepts and results.

Example 6.2.6 For $a, b \in \mathbb{R}$ consider the following two-dimensional MPVC:
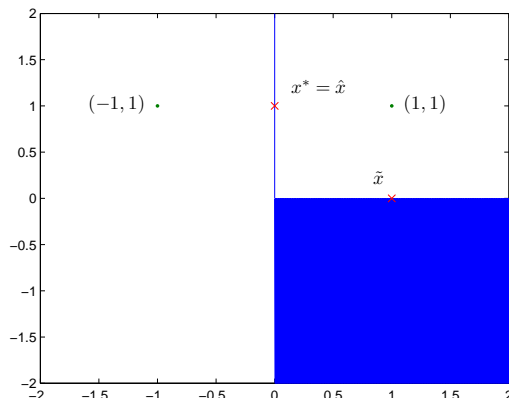
Figure 6.1.: Feasible set of (6.29)

$$
\begin{array}{rcll}
\min & f(x) & := & (x_1 - a)^2 + (x_2 - b)^2 \\
\text{s.t.} & H(x) & := & x_1 \geq 0, \\
& G(x)H(x) & := & x_2 x_1 \leq 0.
\end{array}
\tag{6.29}
$$

Its feasible set and also some relevant points for the upcoming discussion are given in Figure 6.1. Geometrically speaking, in (6.29), one is searching for the projection of (a, b) onto the feasible set.

First of all, we see that the gradients $\nabla H(x) = (1, 0)^T$ and $\nabla G(x) = (0, 1)^T$ are linearly independent for all $x \in \mathbb{R}^2$, hence, MPVC-LICQ, see Definition 5.1.1, is satisfied at any feasible point. Therefore, strong stationarity is a necessary optimality condition.

Furthermore, the function f is convex and the functions G, H are linear. Thus, the program is MPVC-convex (but still nonconvex!). By Corollary 6.2.5, we then know that strong stationarity is a sufficient condition for a local minimizer and, under some additional condition concerning certain index sets, even for a global minimizer. Together, the above considerations yield that a feasible point of (6.29) is a local minimizer if and only if it is a strongly stationary point. We will verify this by considering the above MPVC for two different choices of (a, b) and calculating the respective strongly stationary points.

For all choices (a, b), the strong stationarity conditions of (6.29) read

$$
0 = \begin{pmatrix} 2x_1 - 2a \\ 2x_2 - 2b \end{pmatrix} - \eta^H \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\tag{6.30}
$$

with

$$
\eta^H \begin{cases} = 0, & \text{if } x_1 > 0, \\ \geq 0, & \text{if } x_1 = 0, x_2 \leq 0, \\ \text{free}, & \text{if } x_1 = 0, x_2 > 0, \end{cases}
\qquad
\eta^G \begin{cases} \geq 0, & \text{if } x_1 > 0, x_2 = 0, \\ = 0, & \text{else.} \end{cases}
\tag{6.31}
$$

For the choice $(a, b) := (1, 1)$, it is quickly calculated that there are two strongly stationary points.

The first one is $\hat{x} := (0, 1)^T$ with associated multipliers $\hat{\eta}^G := 0$, $\hat{\eta}^H := -2$. The second point is $\tilde{x} := (1, 0)^T$, where the corresponding multipliers are given by $\tilde{\eta}^G := 2$, $\tilde{\eta}^H := 0$. These are the only local minimzers of (6.29), as was argued above, for the special choice $(a, b) := (1, 1)$. In fact, they are even global minimizers as can be seen easily by geometric arguments, even though the sufficient condition from Corollary 6.2.5 (b) is not satisfied, illustrating that this is only a *sufficient* criterion.

The next choice is $(a, b) := (-1, 1)$, where we can compute only one strongly stationary point $x^* := (0, 1)^T$ with multipliers given by $\eta^G := 0, \eta^H := 2$. In particular, we then have $I_{0+}^- \cup I_{+0}^{0+} = \emptyset$, so that, in this case, we can invoke Corollary 6.2.5 (b) to ensure that this is not only a local, but a global minimizer of (6.29).

7. Second-order optimality conditions for MPVCs

The goal of this chapter is to provide (necessary and sufficient) second-order optimality conditions for MPVCs. The analysis is motivated by general results from optimization or, more specialized, from the MPEC field and was part of the publication [28] by Kanzow and the author of this work.

In order to state second-order optimality results for nonlinear programs, a suitable cone, usually a subset of the linearized cone, is needed, on which the Hessian of the Lagrangian is or is shown to be positive (semi-)definite, see Section 2.3. For our purposes, in order to obtain MPVC-tailored results we will substitute the standard Lagrangian for the following function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ by

$$L(x, \lambda, \mu, \eta^G, \eta^H) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j \in J} \mu_j h_j(x) - \sum_{i=1}^{l} \eta_i^H H_i(x) + \sum_{i=1}^{l} \eta_i^G G_i(x) \qquad (7.1)$$

and call this function the *MPVC-Lagrangian*. For example, a feasible point x^* of (1.1) is strongly stationary (or M-stationary) if and only if there exist multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that

$$\nabla_x L(x^*, \lambda, \mu, \eta^G, \eta^H) = 0$$

and $(\lambda, \mu, \eta^G, \eta^H)$ satisfies (6.2) (or (6.16)). The critical cone which will play the above mentioned role in our context is defined below as a subset of the MPVC-linearized cone, which was introduced in Section 3.2. Given a feasible point x^* of (1.1), the MPVC-linearized cone is, according to (3.10), given by

$$\begin{aligned}
\mathcal{L}_{MPVC}(x^*) = \{d \in \mathbb{R}^n \mid\ & \nabla g_i(x^*)^T d \leq 0 & (i \in I_g), \\
& \nabla h_j(x^*)^T d = 0 & (j \in J), \\
& \nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\
& \nabla H_i(x^*)^T d \geq 0 & (i \in I_{00} \cup I_{0-}), \\
& \nabla G_i(x^*)^T d \leq 0 & (i \in I_{+0}), \\
& (\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \leq 0 & (i \in I_{00})\}.
\end{aligned} \qquad (7.2)$$

In many situations of MPVC-analysis, see Section 5, the MPVC-linearized cone has been succesfully used instead of the usual linearized cone. Thus, it is not surprising that it occurs in the context of second-order optimality conditions for MPVCs, too.

For the definition of the above mentioned subset of the MPVC-linearized cone, we assume that we have a strongly stationary point $(x^*, \lambda, \mu, \eta^G, \eta^H)$ of (1.1). Then we define $C(x^*)$ by

$$\begin{aligned}
C(x^*) := \{d \in \mathcal{L}_{MPVC}(x^*) \mid\ & \nabla g_i(x^*)^T d = 0 & (i \in I_g^+), \\
& \nabla H_i(x^*)^T d = 0 & (i \in I_{00}^+ \cup I_{0-}^+), \\
& \nabla G_i(x^*)^T d = 0 & (i \in I_{+0}^{0+})\},
\end{aligned} \qquad (7.3)$$

that is, in fact, we have (taking into account that $I_{00}^- = \emptyset$ at a strongly stationary point)

$$
\begin{aligned}
C(x^*) = \{d \in \mathbb{R}^n \mid \; & \nabla g_i(x^*)^T d \le 0 & (i \in I_g^0), \\
& \nabla g_i(x^*)^T d = 0 & (i \in I_g^+), \\
& \nabla h_j(x^*)^T d = 0 & (j \in J), \\
& \nabla H_i(x^*)^T d \ge 0 & (i \in I_{00}^0 \cup I_{0-}^0), \\
& \nabla H_i(x^*)^T d = 0 & (i \in I_{0+} \cup I_{00}^+ \cup I_{0-}^+), \\
& \nabla G_i(x^*)^T d \le 0 & (i \in I_{+0}^{00}), \\
& \nabla G_i(x^*)^T d = 0 & (i \in I_{+0}^{0+}), \\
& (\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \le 0 & (i \in I_{00})\},
\end{aligned}
\tag{7.4}
$$

where we put

$$
\begin{aligned}
I_g^+ &:= \{i \in I_g \mid \lambda_i > 0\}, \\
I_g^0 &:= \{i \in I_g \mid \lambda_i = 0\}, \\
I_{00}^+ &:= \{i \in I_{00} \mid \eta_i^H > 0\}, \\
I_{00}^0 &:= \{i \in I_{00} \mid \eta_i^H = 0\}, \\
I_{0-}^+ &:= \{i \in I_{0-} \mid \eta_i^H > 0\}, \\
I_{0-}^0 &:= \{i \in I_{0-} \mid \eta_i^H = 0\}, \\
I_{+0}^{00} &:= \{i \in I_{+0} \mid \eta_i^G = 0\}, \\
I_{+0}^{0+} &:= \{i \in I_{+0} \mid \eta_i^G > 0\}
\end{aligned}
\tag{7.5}
$$

in accordance with (6.20).

The definition of these index sets may, again, appeal a bit complicated and make the proof of our theorems somewhat technical, but on the other hand we prove pretty strong results, showing that we can use the same cone $C(x^*)$ for both the necessary and the sufficient second-order condition.

Note that for the whole chapter, all functions occuring in (1.1) are assumed to be at least twice continuously differentiable.

7.1. A second-order necessary condition

In this section a second-order necessary condition for the MPVC (1.1) is established.

The following lemma is a direct preparation for the upcoming theorem on second-order necessary optimality conditions. Its technique of proof goes back to similar considerations in the context of standard nonlinear programs, see [22], for example. Note, however, that we cannot simply apply these standard results since, e.g., the usual LICQ assumption typically does not hold for MPVCs, see Section 4.1. Instead of this we employ MPVC-LICQ as given in Definition 5.1.1.

Lemma 7.1.1 *Let x^* be a strongly stationary point of* (1.1) *such that MPVC-LICQ holds. Furthermore, let $d \in C(x^*)$. Then there exists an $\varepsilon > 0$ and a twice continuously differentiable curve $x : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $x(0) = x^*$, $x'(0) = d$, $x(t) \in X$ for $t \in [0, \varepsilon)$ and such that, in addition,*

we have

$$
\begin{array}{rcll}
g_i(x(t)) & = & 0 & (i \in I_g^+), \\
h_j(x(t)) & = & 0 & (j \in J), \\
H_i(x(t)) & = & 0 & (i \in I_{00}^+ \cup I_{0-}^+ \cup I_{0+}), \\
G_i(x(t)) & = & 0 & (i \in I_{+0}^{0+}).
\end{array}
\tag{7.6}
$$

Proof. Let $d \in C(x^*)$ and let $(\lambda, \mu, \eta^G, \eta^H)$ be the (unique) multipliers such that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point. We define some further subsets (depending on x^* and the particular vector d chosen from $C(x^*)$) of the index sets which were defined previously:

$$
\begin{array}{rcl}
I_{g,=}^0 & := & \{i \in I_g^0 \mid \nabla g_i(x^*)^T d = 0\}, \\
I_{g,<}^0 & := & \{i \in I_g^0 \mid \nabla g_i(x^*)^T d < 0\}, \\
I_{00,=}^0 & := & \{i \in I_{00}^0 \mid \nabla H_i(x^*)^T d = 0\}, \\
I_{00,>}^0 & := & \{i \in I_{00}^0 \mid \nabla H_i(x^*)^T d > 0\}, \\
I_{0-,=}^0 & := & \{i \in I_{0-}^0 \mid \nabla H_i(x^*)^T d = 0\}, \\
I_{0-,>}^0 & := & \{i \in I_{0-}^0 \mid \nabla H_i(x^*)^T d > 0\}, \\
I_{+0,*=}^{00} & := & \{i \in I_{+0}^{00} \mid \nabla G_i(x^*)^T d = 0\}, \\
I_{+0,*<}^{00} & := & \{i \in I_{+0}^{00} \mid \nabla G_i(x^*)^T d < 0\}, \\
I_{00,>=}^0 & := & \{i \in I_{00}^0 \mid \nabla H_i(x^*)^T d > 0, \ \nabla G_i(x^*)^T d = 0\}, \\
I_{00,><}^0 & := & \{i \in I_{00}^0 \mid \nabla H_i(x^*)^T d > 0, \ \nabla G_i(x^*)^T d < 0\}.
\end{array}
\tag{7.7}
$$

Then we define the mapping $z : \mathbb{R}^n \to \mathbb{R}^q$, where $q := |I_g^+ \cup I_{g,=}^0| + |J| + |I_{0+} \cup I_{00}^+ \cup I_{0-}^+ \cup I_{00,=}^0 \cup I_{0-,=}^0| + |I_{+0}^{0+} \cup I_{+0,*=}^{00} \cup I_{00,>=}^0|$, by

$$
z(x) :=
\left(
\begin{array}{ll}
g_i(x) & (i \in I_g^+ \cup I_{g,=}^0) \\
h_j(x) & (j \in J) \\
H_i(x) & (i \in I_{0+} \cup I_{00}^+ \cup I_{0-}^+ \cup I_{00,=}^0 \cup I_{0-,=}^0) \\
G_i(x) & (I_{+0}^{0+} \cup I_{+0,*=}^{00} \cup I_{00,>=}^0)
\end{array}
\right),
\tag{7.8}
$$

and denote the j-th component function of z by z_j. Furthermore, let $\bar{H} : \mathbb{R}^{q+1} \to \mathbb{R}^q$ be the mapping defined by

$$
\bar{H}_j(y, t) := z_j(x^* + td + z'(x^*)^T y) \quad \forall j = 1, \dots, q.
$$

The system $\bar{H}(y, t) = 0$ has a solution $(y^*, t^*) := (0, 0)$, and the partial Jacobian

$$
\bar{H}_y(0, 0) = z'(x^*) z'(x^*)^T \in \mathbb{R}^{q \times q}
$$

is nonsingular since the matrix $z'(x^*)$ has full rank q due to the MPVC-LICQ assumption. Thus, invoking the implicit function theorem and using the twice continuous differentiability of all mappings involved in the definition of z, there exists an $\varepsilon > 0$ and a twice continuously differentiable

curve $y : (-\varepsilon, \varepsilon) \to \mathbb{R}^q$ such that $y(0) = 0$ and $\bar{H}(y(t), t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Moreover, its derivative is given by

$$y'(t) = -(\bar{H}_y(y(t), t))^{-1} \bar{H}_t(y(t), t) \quad \forall t \in (-\varepsilon, \varepsilon).$$

In particular, this implies

$$y'(0) = -(\bar{H}_y(0, 0))^{-1} \bar{H}_t(0, 0) = -(\bar{H}_y(0, 0))^{-1} \underbrace{z'(x^*)d}_{=0} = 0,$$

due to the properties of d. Now define

$$x(t) := x^* + td + z'(x^*)^T y(t).$$

Then $x(\cdot)$ is twice continuously differentiable on $(-\varepsilon, \varepsilon)$, and we obviously have $x(0) = x^*$ and $x'(0) = d$. Hence, we still need to show that $x(t) \in X$ and that $x(\cdot)$ satisfies (7.6) for all t sufficiently close to 0.

For these purposes, first note that $\bar{H}_j(y(t), t) = 0$ implies $z_j(x(t)) = 0$ and thus we obtain

$$
\begin{array}{llll}
g_i(x(t)) & = & 0 & (i \in I_g^+ \cup I_{g,=}^0), \\
h_j(x(t)) & = & 0 & (j \in J), \\
H_i(x(t)) & = & 0 & (i \in I_{0+} \cup I_{00}^+ \cup I_{0-}^+ \cup I_{00,=}^0 \cup I_{0-,=}^0), \\
G_i(x(t)) & = & 0 & (i \in I_{+0}^{0+} \cup I_{+0,*=}^{00} \cup I_{00,>=}^0),
\end{array}
\tag{7.9}
$$

so that (7.6) and the feasibility of $x(t)$ for the above occuring index sets is garantueed for all $t \in (-\varepsilon, \varepsilon)$.

By simple continuity arguments, one can also verify that we have $g_i(x(t)) < 0$ ($i \notin I_g$), $G_i(x(t)) < 0$ ($i \in I_{0-} \cup I_{+-}$) and $H_i(x(t)) > 0$ ($i \in I_+$) for all t sufficiently close to 0. Thus, taking into account the definition of $C(x^*)$, it remains to show that

$$
\begin{array}{llll}
g_i(x(t)) & \leq & 0 & (i \in I_{g,<}^0), \\
H_i(x(t)) & \geq & 0 & (i \in I_{00,>}^0 \cup I_{0-,>}^0),
\end{array}
\tag{7.10}
$$

and that

$$G_i(x(t))H_i(x(t)) \leq 0 \quad (i \in I_{00,><}^0 \cup I_{0-,>}^0 \cup I_{+0,*<}^{00}) \tag{7.11}$$

for $t > 0$ sufficiently small.

In order to verify (7.10), let $i \in I_{g,<}^0$. Then we have $\nabla g_i(x^*)^T d < 0$ by definition. This implies $\nabla g_i(x(\tau))^T x'(\tau) < 0$ for all $|\tau|$ sufficiently small. From the mean value theorem, we obtain a $\tau_t \in (0, t)$ such that $g_i(x(t)) = g_i(x(0)) + \nabla g_i(x(\tau_t))^T x'(\tau_t)(t - 0) = t\nabla g_i(x(\tau_t))^T x'(\tau_t) < 0$ for all $t > 0$ sufficiently small, which proves the first statement of (7.10).

In order to prove the second statement, let $i \in I_{00,>}^0 \cup I_{0-,>}^0$. Then it follows, by definition, that $\nabla H_i(x^*)^T d > 0$, and thus by continuity, it holds that $\nabla H_i((x(t))^T x'(t) > 0$ for all t sufficiently close to 0. Since we have $H_i(x(0)) = H_i(x^*) = 0$, this implies $H_i(x(t)) > 0$ for all $t > 0$ sufficiently small, using the above arguments.

To verify (7.11), first let $i \in I^0_{0-,>}$. Then we have $G_i(x(t)) < 0$ by continuity, and with the above reasoning we get $H_i(x(t)) > 0$ for $t > 0$ sufficiently small, so that $G_i(x(t))H_i(x(t)) \leq 0$ holds in this case.

Now, let $i \in I^0_{00,><}$. Then, by definition, we have $\nabla H_i(x^*)^T d > 0$ and $\nabla G_i(x^*)^T d < 0$. Then, with analogous reasoning as above, it follows that $H_i(x(t)) > 0$ and $G_i(x(t)) < 0$ for $t > 0$ sufficiently small, which gives (7.11) in this case.

Finally, let $i \in I^{00}_{+0,*<}$. Then we have $H_i(x(t)) > 0$ for $|t|$ sufficiently small. And since we have $\nabla G_i(x^*)^T d < 0$, we obtain $G_i(x(t)) < 0$ for all $t > 0$ sufficiently small, which eventually proves (7.11). $\qquad\square$

The proof of the following theorem exploits the existence of the curve $x(\cdot)$ from the above lemma.

Theorem 7.1.2 *Let x^* be a local minimizer of (1.1) such that MPVC-LICQ holds. Then we have*

$$d^T \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) d \geq 0 \quad \forall d \in C(x^*),$$

where $\lambda, \mu, \eta^G, \eta^H$ are the (unique) multipliers corresponding to (the strongly stationary) point x^* of (1.1).

Proof. First recall from Corollary 6.1.4 that MPVC-LICQ ensures the existence of (unique) multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point.

Let $d \in C(x^*)$. Using the curve $x(\cdot)$ (and $\varepsilon > 0$) from Lemma 7.1.1, we are in a position to define the function $\phi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$\phi(t) := L(x(t), \lambda, \mu, \eta^G, \eta^H),$$

where L denotes the MPVC-Lagrangian from (7.1). Then ϕ is twice continuously differentiable with

$$\phi'(t) = x'(t)^T \nabla_x L(x(t), \lambda, \mu, \eta^G, \eta^H)$$

and

$$\phi''(t) = x''(t)^T \nabla_x L(x(t), \lambda, \mu, \eta^G, \eta^H) + x'(t)^T \nabla^2_{xx} L(x(t), \lambda, \mu, \eta^G, \eta^H) x'(t).$$

Using Lemma 7.1.1, we therefore obtain

$$\phi'(0) = d^T \nabla_x L(x^*, \lambda, \mu, \eta^G, \eta^H) = 0$$

and

$$\phi''(0) = d^T \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) d,$$

since we have $\nabla_x L(x^*, \lambda, \mu, \eta^G, \eta^H) = 0$, as $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of (1.1). Now, suppose that $\phi''(0) = d^T \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) d < 0$. By continuity, we thus have $\phi''(t) < 0$ for t sufficiently close to 0. Invoking Taylor's formula, we obtain

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2}\phi''(\xi_t)$$

for all $t \in (-\varepsilon, \varepsilon)$ and a suitable point ξ_t depending on t. Since we have $\phi'(0) = 0$ and $\phi''(\xi_t) < 0$ for t sufficiently close to 0, we thus have $\phi(t) < \phi(0)$ for these $t \in (-\varepsilon, \varepsilon)$. Since $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of (1.1), we have

$$\phi(0) = f(x^*) + \sum_{i \in I_g} \lambda_i g_i(x^*) + \sum_{j \in J} \mu_j h_j(x^*) + \sum_{i \in I_{+0}} \eta_i^G G_i(x^*) - \sum_{i \in I_0} \eta_i^H H_i(x^*) = f(x^*)$$

and, in view of (7.6) and the feasibility of $x(t)$ for $t > 0$ sufficiently small, we also have

$$\phi(t) = f(x(t)) + \sum_{i \in I_g} \lambda_i g_i(x(t)) + \sum_{j \in J} \mu_j h_j(x(t)) + \sum_{i \in I_{+0}} \eta_i^G G_i(x(t)) - \sum_{i \in I_0} \eta_i^H H_i(x(t)) = f(x(t)),$$

which yields $f(x(t)) < f(x^*)$ for all $t > 0$ sufficiently small, in contradiction to x^* being a local minimizer of (1.1). $\qquad\square$

7.2. A second-order sufficient condition

In this section we state a second-order sufficiency condition. Note, again, that this result makes use of the same set $C(x^*)$ as the second-order necessary condition from Theorem 7.1.2.

Theorem 7.2.1 *Let $(x^*, \lambda, \mu, \eta^G, \eta^H)$ be a strongly stationary point of the MPVC* (1.1) *such that*

$$d^T \nabla_{xx}^2 L(x^*, \lambda, \mu, \eta^G, \eta^H) d > 0 \quad \forall d \in C(x^*) \setminus \{0\}. \tag{7.12}$$

Then x^* is a strict local minimizer of (1.1).

Proof. Assume that x^* is not a strict local minimizer of (1.1). Then there exists a sequence $\{x^k\} \subseteq X$ tending to x^* with $f(x^k) \leq f(x^*)$ for all k. Now, put $t_k := \|x^k - x^*\|$. Then we have $t_k \downarrow 0$. Furthermore, we define the sequence $\{d^k\} \subseteq \mathbb{R}^n$ by $d^k := \frac{x^k - x^*}{t_k}$. Since we have $\|d^k\| = 1$ for all $k \in \mathbb{N}$, we can assume, without loss of generality, that $\{d^k\}$ has a limit $d \in \mathbb{R}^n \setminus \{0\}$. Furthermore, by construction, we see that d lies in the tangent cone $\mathcal{T}(x^*)$ of (1.1) and thus, invoking Corollary 2.5 from [26], we particularly have $d \in \mathcal{L}_{MPVC}(x^*)$. Hence, we have

$$\begin{array}{rcll} \nabla g_i(x^*)^T d & \leq & 0 & (i \in I_g), \\ \nabla h_j(x^*)^T d & = & 0 & (j \in J), \\ \nabla H_i(x^*)^T d & = & 0 & (i \in I_{0+}), \\ \nabla H_i(x^*)^T d & \geq & 0 & (i \in I_{00} \cup I_{0-}), \\ \nabla G_i(x^*)^T d & \leq & 0 & (i \in I_{+0}), \end{array} \tag{7.13}$$

as well as

$$(\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) \leq 0 \quad (i \in I_{00}). \tag{7.14}$$

Furthermore, since we have $f(x^k) \leq f(x^*)$ for all k by assumption, the mean value theorem yields a vector ξ^k on the connecting line between x^k and x^* such that $\nabla f(\xi^k)^T (x^k - x^*) \leq 0$ for all k. Dividing by $\|x^k - x^*\|$ and passing to the limit thus implies

$$\nabla f(x^*)^T d \leq 0. \tag{7.15}$$

Now, we consider two different cases, which both lead to a contradiction.

First, consider the case that equality holds in (7.13) for all indices $i \in I_g^+ \cup I_{0-}^+ \cup I_{00}^+ \cup I_{+0}^{0+}$. Then we have $d \in C(x^*)$. Since x^k is feasible for (1.1) for all k and we have $x^k \to x^*$, the following statements hold for all k sufficiently large:

$$
\begin{aligned}
\lambda_i \underbrace{g_i(x^k)}_{\leq 0} &\leq 0 \quad (i \in I_g), \\
\mu_j \underbrace{h_j(x^k)}_{=0} &= 0 \quad (j \in J), \\
\eta_i^H \underbrace{H_i(x^k)}_{=0} &= 0 \quad (i \in I_{0+}), \\
-\eta_i^H \underbrace{H_i(x^k)}_{\geq 0} &\leq 0 \quad (i \in I_{0-} \cup I_{00}), \\
\eta_i^G \underbrace{G_i(x^k)}_{\leq 0} &\leq 0 \quad (i \in I_{+0}),
\end{aligned}
\tag{7.16}
$$

where we use continuity arguments as well the fact that we have $G_i(x^k) H_i(x^k) \leq 0$ for all $i = 1, \dots, l$ and all k, for the third and fifth statement. Invoking (7.16) and the properties of the multipliers $(\lambda, \mu, \eta^G, \eta^H)$, we obtain

$$
\begin{aligned}
f(x^*) &\geq f(x^k) \\
&\geq f(x^k) + \sum_{i \in I_g} \lambda_i g_i(x^k) + \sum_{j \in J} \mu_j h_j(x^k) + \sum_{i \in I_{+0}} \eta_i^G G_i(x^k) - \sum_{i \in I_0} \eta_i^H H_i(x^k) \\
&= l(x^k),
\end{aligned}
\tag{7.17}
$$

where we put $l(x) := L(x, \lambda, \mu, \eta^G, \eta^H)$. Applying Taylor's formula to (7.17) yields a vector ξ^k on the connecting line between x^* and x^k such that

$$
\begin{aligned}
f(x^*) &\geq l(x^k) \\
&= \underbrace{l(x^*)}_{=f(x^*)} + \underbrace{\nabla l(x^*)^T}_{=\nabla_x L(x^*, \lambda, \mu, \eta^G, \eta^H)=0} (x^k - x^*) + \tfrac{1}{2}(x^k - x^*)^T \nabla^2 l(\xi^k)(x^k - x^*) \\
&= f(x^*) + \tfrac{1}{2}(x^k - x^*)^T \nabla_{xx}^2 L(\xi^k, \lambda, \mu, \eta^G, \eta^H)(x^k - x^*),
\end{aligned}
\tag{7.18}
$$

also exploiting the fact that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of (1.1). Dividing by $\|x^* - x^k\|^2$ and letting $k \to \infty$ gives

$$
d^T \nabla_{xx}^2 L(x^*, \lambda, \mu, \eta^G, \eta^H) d \leq 0,
\tag{7.19}
$$

which contradicts assumption (7.12) of our theorem, because we have $0 \neq d \in C(x^*)$.

Second, consider the opposite case, that is, assume that there is an index $i \in I_g^+ \cup I_{0-}^+ \cup I_{00}^+ \cup I_{+0}^{0+}$ such that a strict inequality holds in (7.13). We only consider the case that there exists an index $i \in I_g^+$ such that $\nabla g_i(x^*)^T d < 0$, since the other cases can be treated in the same way. Now, let

$s \in I_g^+$ such that $\nabla g_s(x^*)^T d < 0$. Then it follows from (7.13) and (7.15) that

$$
\begin{aligned}
0 \;\geq\; & \nabla f(x^*)^T d \\
= \;& -\Big(\sum_{i \in I_g} \lambda_i \nabla g_i(x^*)^T d + \sum_{j \in J} \mu_j \nabla h_j(x^*)^T d + \sum_{i \in I_{+0}} \eta_i^G \nabla G_i(x^*)^T d - \sum_{i \in I_0} \eta_i^H \nabla H_i(x^*)^T d \Big) \\
\geq \;& -\sum_{i \in I_g^+} \lambda_i \nabla g_i(x^*)^T d \\
\geq \;& -\lambda_s \nabla g_s(x^*)^T d > 0,
\end{aligned}
$$

which yields the desired contradiction also in this case. $\qquad\square$

Closing this section, we would like to point out that for Example 6.2.6 the conclusion of Theorem 7.1.2 as well as the assumptions of Theorem 7.2.1 are obviously satisfied, since the Hessian of the MPVC-Lagrangian is a positive multiple of the identity at any feasible point and thus in particular positive definite on the whole $\mathbb{R}^n$.

8. An exact penalty result for MPVCs

In this chapter an exact penalty function for the MPVC (1.1) is constructed. On the basis of this, M-stationarity is recovered as a necessary optimality condition for a local minimizer.

The material presented here goes back to current results from [29].

8.1. The concept of exact penalization

The notion of *penalization* is as old as the whole discipline of mathematical optimization. At this, the ultimate goal is to transform a constrained into an unconstrained optimization problem in the following fashion: Consider a mathematical program of the form

$$\min f(x) \quad \text{s.t.} \quad F(x) \in \Lambda, \tag{8.1}$$

with functions $f : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^m$ and a nonempty closed set $\Lambda \subseteq \mathbb{R}^m$. Now, suppose we have a function $\psi : \mathbb{R}^n \to \mathbb{R}_+$ such that $\psi(x) = 0$ if and only if $F(x) \in \Lambda$. Herewith define the function $P : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$P(x; \alpha) := f(x) + \alpha \psi(x). \tag{8.2}$$

Then P is called a *penalty function* for (8.1) and $\alpha > 0$ is a *penalty parameter*. The idea of penalization now consists in considering a sequence of parameterized unconstrained problems

$$\min_{x \in \mathbb{R}^n} P(x; \alpha) \qquad\qquad U(\alpha)$$

for some penalty parameter $\alpha > 0$. By letting $\alpha \to \infty$, infeasiblity is more and more *penalized* and thus, one hopes that for a certain finite $\bar{\alpha} > 0$, the minimizers of (8.1) can be detected via the minimizers of (the hopefully easier to solve problem) $U(\alpha)$ for $\alpha > \bar{\alpha}$. The crucial concept in this context is the notion of an *exact penalty function* given in the below definition.

Definition 8.1.1 *Let x^* be a local minimizer of* (8.1) *and let $P : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be a penalty function for* (8.1). *Then P is called exact at x^* if there exists a finite penalty parameter $\bar{\alpha} > 0$ such that x^* is a local minimzer of $U(\alpha)$ for all $\alpha > \bar{\alpha}$.*

8.2. A generalized mathematical program

In this section, we consider a general mathematical program of the form

$$\min f(x) \quad \text{s.t.} \quad F(x) \in \Lambda, \tag{8.3}$$

with locally Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^m$ and a nonempty closed set $\Lambda \subseteq \mathbb{R}^m$. This type of problem was already fruitfully employed in many situations, e.g. in the field of MPECs in [19] .

As soon as one tries to investigate exact penalty results for a class of optimization problems, the very closely linked concept of *calmness* of the respective problem, cf. [9, 10, 14], arises naturally for reasons explained below.

In order to define calmness for our general optimization problem (8.3), consider the associated family of perturbed problems

$$\min f(x) \quad \text{s.t.} \quad F(x) + p \in \Lambda, \qquad\qquad \Pi(p)$$

for some parameter $p \in \mathbb{R}^m$. Note that, obviously, it holds that (8.3) and $\Pi(0)$ are the same problems. The following definition of calmness is due to Burke, see [9, Def. 1.1].

Definition 8.2.1 *Let x^* be feasible for $\Pi(0)$. Then the problem is called* calm *at x^* if there exist constants $\bar{\alpha} > 0$ and $\varepsilon > 0$ such that for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying $x \in B_\varepsilon(x^*)$ and $F(x) + p \in \Lambda$, one has*

$$f(x) + \bar{\alpha}\|p\| \geq f(x^*).$$

In this context $\bar{\alpha}$ and ε are called the *modulus* and the *radius* of calmness for $\Pi(0)$ at x^*. Note that the original definition by Clarke, see [14, Def. 6.4.1], also involves that $p \in B_\varepsilon(0)$. Actually, these definitions coincide as soon as the function F is continuous, as was coined in [9, Prop. 2.1], which is in particular fulfilled in our setup.

When Clarke established the notion of calmness as a tool for sensitivity analysis of parameterized optimization problems, he already was aware of its close connection to the concept of exact penalization. He showed that calmness is a sufficient condition for exact penalization. The full relation, however, is due to Burke, see [9, Th. 1.1], and is restated in the following result.

Proposition 8.2.2 *Let x^* be feasible for $\Pi(0)$. Then $\Pi(0)$ is calm at x^* with modulus $\bar{\alpha}$ and radius ε if and only if x^* is a minimum of*

$$P(x; \alpha) := f(x) + \alpha d_\Lambda(F(x)) \qquad\qquad (8.4)$$

over $B_\varepsilon(x^*)$ for all $\alpha \geq \bar{\alpha}$.

Proof. See [9, Th. 1.1]. □

In the course of rising popularity of the calculus of multifunctions and their applications to optimization problems, another calmness concept has been established and successfully employed in the context of mathematical programming. The following definition of calmness of a multifunction can be found, e.g., in [54].

Definition 8.2.3 *Let $\Phi : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be a multifunction with a closed graph and $(u, v) \in \mathrm{gph}\Phi$. Then we say that Φ is* calm *at (u, v) if there exist neighbourhoods U of u, V of v and a modulus $L \geq 0$ such that*

$$\Phi(u') \cap V \subseteq \Phi(u) + L\|u - u'\|\mathbb{B} \quad \forall u' \in U. \qquad\qquad (8.5)$$

The application to our mathematical programming setup from (8.3) and $\Pi(p)$ follows by virtue of the so-called *perturbation map*, a multifunction $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$M(p) := \{x \in \mathbb{R}^n \mid F(x) + p \in \Lambda\}. \tag{8.6}$$

By means of the perturbation map, the feasible set of $\Pi(p)$ is then given by $M(p)$, in particular, one has $F^{-1}(\Lambda) = M(0)$.

Part of the gain from the notion of calmness of multifunctions for optimization is revealed by the following two results. In the first result, we see that calmness of the perturbation map at a particular point is in fact equivalent to the existence of local error bounds, see [46].

Proposition 8.2.4 *Let $x^* \in M(0)$ be feasible for* (8.3). *Then the following statements are equivalent.*

(1) M is calm at $(0, x^*)$.

(2) There exists a neighbourhood U of x^* and a constant $\rho > 0$ such that

$$d_{F^{-1}(\Lambda)}(x) \le \rho d_\Lambda(F(x)) \quad \forall x \in U. \tag{8.7}$$

Proof. See [24, Corollary 1]. □

The second result shows that, roughly speaking, calmness of the perturbation map (Definition 8.2.3) yields calmness of the unperturbed problem $\Pi(0)$ (Definition 8.2.1).

Proposition 8.2.5 *Let $x^* \in M(0)$ be a local minimizer of* (8.3) *such that M is calm at $(0, x^*)$. Then $\Pi(0)$ is calm at x^*.*

Proof. By assumption, M is calm at $(0, x^*)$ and hence, due to Proposition 8.2.4, there exist constants $\tilde{\varepsilon}, \rho > 0$ such that

$$d_{F^{-1}(\Lambda)}(x) \le \rho d_\Lambda(F(x)) \quad \forall x \in B_{\tilde{\varepsilon}}(x^*).$$

Now, choose $\hat{\varepsilon} \in (0, \tilde{\varepsilon}]$ such that f attains a minimum over $B_{\hat{\varepsilon}}(x^*) \cap F^{-1}(\Lambda)$ at x^*. Then put $\varepsilon := \frac{\hat{\varepsilon}}{2}$ and choose $x \in B_\varepsilon(x^*)$ arbitrarily. Moreover, let

$$x_0 \in \mathrm{Proj}_{F^{-1}(\Lambda)}(x).$$

In particular, this implies $x_0 \in B_{\hat{\varepsilon}}(x^*)$. Together, one obtains

$$
\begin{aligned}
f(x^*) &\le f(x_0) \\
&\le f(x) + L\|x - x_0\| \\
&= f(x) + Ld_{F^{-1}(\Lambda)}(x) \\
&\le f(x) + \rho L d_\Lambda(F(x)),
\end{aligned}
\tag{8.8}
$$

where $L > 0$ denotes the local Lipschitz constant of f around x^*. If, now, we put $\bar{\alpha} := \rho L$ and mind that, for $p \in \mathbb{R}^m$, we have $d_\Lambda(F(x)) \le \|p\|$ whenever $F(x) + p \in \Lambda$, we apparently get the

desired calmness of $\Pi(0)$. □

An immediate consequence is the following corollary.

Corollary 8.2.6 *Let $x^* \in M(0)$ be such that M is calm at $(0, x^*)$. Then the penalty function from* (8.4) *is exact at x^*.*

Proof. The proof follows immediately from Prop. 8.2.5 and 8.2.2. □

In the sequel of this section we will provide sufficient conditions for the calmness of the multifunction M at $(0, x^*)$ for some $x^* \in M(0)$. Thus, we automatically obtain sufficient conditions for the function $P(x; \alpha) = f(x) + \alpha d_\Lambda(F(x))$ to be exact at x^*. From now on we will assume the functions f and F to be continuously differentiable. Then we can define the following generalization of the Mangasarian-Fromovitz constraint qualification, see [19].

Definition 8.2.7 *Let x^* be feasible for* (8.3). *We say that the* generalized Mangasarian-Fromovitz constraint qualification *(GMFCQ) holds at x^* if the following implication holds:*

$$\left. \begin{array}{r} F'(x^*)^T \lambda = 0 \\ \lambda \in N(F(x^*), \Lambda) \end{array} \right\} \Longrightarrow \lambda = 0. \tag{8.9}$$

Note that, if $\Lambda = \mathbb{R}^m_-$, (8.9) reduces to standard MFCQ.

The notion of $GMFCQ$ leads to the following result.

Proposition 8.2.8 *Let $x^* \in M(0)$ be feasible for* (8.3) *such that GMFCQ is satisfied. Then the perturbation map M is calm at $(0, x^*)$.*

Proof. See the proof of [19, Corollary 2.4]. □

The following corollary follows immediately.

Corollary 8.2.9 *Let $x^* \in M(0)$ be feasible for* (8.3) *such that GMFCQ is satisfied. Then the penalty function from* (8.4) *is exact at x^*.*

8.3. Deriving an exact penalty function for MPVCs

In order to derive an exact penalty function for the MPVC (1.1), we are guided by the results from Section 8.2, in particular Corollary 8.2.9. The path that we follow starts with a reformulation of the MPVC in the fashion of (8.3). Afterwards we will provide sufficient conditions for the GMFCQ to hold for the rewritten MPVC, which eventually yields an exact penalty function. Note, however, that the question whether GMFCQ holds or not, substantially depends on the chosen representation of the feasible set.

For the sake of reformulating the MPVC, consider the characteristic set

$$C := \{(a, b) \in \mathbb{R}^2 \mid b \geq 0, \ ab \leq 0\}, \tag{8.10}$$

and put

$$\Lambda_{VC} := \mathbb{R}_-^m \times \{0\}^p \times C^l. \tag{8.11}$$

Furthermore, define the map $F^{VC} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{2l}$ by

$$F^{VC}(x) := \begin{pmatrix} g_i(x) & (i = 1 \ldots, l) \\ h_j(x) & (j = 1, \ldots, p) \\ \binom{G_i(x)}{H_i(x)} & (i = 1, \ldots, l) \end{pmatrix}. \tag{8.12}$$

By means of these definitions, we are able to write the MPVC (1.1) as the following program

$$\min f(x) \quad \text{s.t.} \quad F^{VC}(x) \in \Lambda_{VC}. \tag{8.13}$$

The perturbation map for (8.13) is consequently given by

$$M^{VC}(p) := \{x \in \mathbb{R}^n \mid F^{VC}(x) + p \in \Lambda_{VC}\}.$$

In order to find conditions to yield GMFCQ for (8.13), we need the following auxiliary result, which is concerned with calculating the limiting normal cone of the characteristic set C from (8.10).

Lemma 8.3.1 *Let* $(a, b) \in C$*. Then it holds that*

$$N((a, b), C) = \begin{cases} \{0\} \times \{0\} & \text{if} \quad b > 0, \ a < 0, \\ \mathbb{R}_+ \times \{0\} & \text{if} \quad b > 0, \ a = 0, \\ \{0\} \times \mathbb{R}_- & \text{if} \quad b = 0, \ a < 0, \\ \{0\} \times \mathbb{R} & \text{if} \quad b = 0, \ a > 0, \\ \{(u, v) \mid u \geq 0, \ uv = 0\} & \text{if} \quad a = b = 0. \end{cases} \tag{8.14}$$

Proof. See the proof of Lemma 6.1.6. $\qquad\square$

By the aid of the above Lemma, we are now able to prove a first sufficiency result for GMFCQ in the MPVC setup.

Theorem 8.3.2 *Let* $x^* \in M(0)$ *be feasible for* (1.1) *and assume that for all* $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ *the following two conditions are satisfied:*

(i) *There exists a vector* $d \in \mathbb{R}^n$ *such that*

$$\begin{aligned} \nabla g_i(x^*)^T d &> 0 & (i \in I_g), \\ \nabla h_j(x^*)^T &= 0 & (j = 1, \ldots, p), \\ \nabla G_i(x^*)^T d &> 0 & (i \in I_{+0} \cup \beta_2), \\ \nabla H_i(x^*)^T d &< 0 & (i \in I_{0-}), \\ \nabla H_i(x^*)^T d &= 0 & (i \in I_{0+} \cup \beta_1). \end{aligned} \tag{8.15}$$

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(ii) The gradients $\nabla h_j(x^*)$ $(j = 1\ldots, p)$ and $\nabla H_i(x^*)$ $(i \in I_{0+} \cup \beta_1)$ are linearly independent.

Then GMFCQ holds for (8.13).

Proof. Observe first that due to [54, Proposition 6.41] we have

$$N(F^{VC}(x^*), \Lambda_{VC}) = \bigtimes_{i=1}^{m} N(g_i(x^*), \mathbb{R}_-) \times \bigtimes_{j=1}^{p} N(h_j(x^*), \{0\}) \times \bigtimes_{i=1}^{l} N((G_i(x^*), H_i(x^*)), C)$$

$$= \bigtimes_{i=1}^{m} \left\{ \begin{array}{ll} \mathbb{R}_+ & (i \in I_g) \\ \{0\} & (i \notin I_g) \end{array} \right. \times \mathbb{R}^p \times \bigtimes_{i=1}^{l} N((G_i(x^*), H_i(x^*)), C).$$

Hence, by means of Lemma 8.3.1 it follows that GMFCQ amounts to the condition

$$\left. \begin{array}{l} 0 = \sum_{i=1}^{m} \lambda_i^g \nabla g_i(x^*) + \sum_{j=1}^{p} \lambda_j^h \nabla h_j(x^*) + \sum_{i=1}^{l} \lambda_i^G \nabla G_i(x^*) + \sum_{i=1}^{l} \lambda_i^H \nabla H_i(x^*) \\ \lambda_i^g = 0 \ (i \notin I_g), \quad \lambda_i \geq 0 \ (i \in I_g), \\ \lambda_i^G = 0 \ (i \in I_{+-} \cup I_{0+} \cup I_{0-}), \quad \lambda_i^G \geq 0 \ (i \in I_{+0} \cup I_{00}), \\ \lambda_i^H = 0 \ (i \in I_+), \quad \lambda_i^H \leq 0 \ (i \in I_{0-}), \\ \lambda_i^G \lambda_i^H = 0 \ (i \in I_{00}), \end{array} \right\} \implies \begin{array}{l} \lambda^g = 0, \ \lambda^h = 0, \\ \lambda^G = \lambda^H = 0. \end{array}$$

This is equivalent to

$$\left. \begin{array}{l} 0 = \sum_{i \in I_g} \lambda_i^g \nabla g_i(x^*) + \sum_{j=1}^{p} \lambda_j^h \nabla h_j(x^*) + \sum_{i \in I_{+0} \cup I_{00}} \lambda_i^G \nabla G_i(x^*) + \sum_{i \in I_0} \lambda_i^H \nabla H_i(x^*) \\ \lambda_i^g \geq 0 \ (i \in I_g), \\ \lambda_i^G \geq 0 \ (i \in I_{+0} \cup I_{00}), \\ \lambda_i^H \leq 0 \ (i \in I_{0-}), \\ \lambda_i^G \lambda_i^H = 0 \ (i \in I_{00}), \end{array} \right\} \implies \begin{array}{l} \lambda_i^g = 0 \ (i \in I_g), \\ \lambda_j^h = 0 \ (j = 1, \ldots, p), \\ \lambda_i^G = 0 \ (i \in I_{+0} \cup I_{00}), \\ \lambda_i^H = 0 \ (i \in I_0). \end{array}$$

This, eventually, is equivalent to the following condition: For all partitions $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, the implication

$$\left. \begin{array}{l} 0 = \sum_{i \in I_g} \lambda_i^g \nabla g_i(x^*) + \sum_{j=1}^{p} \lambda_j^h \nabla h_j(x^*) + \sum_{i \in I_{+0} \cup \beta_2} \lambda_i^G \nabla G_i(x^*) + \sum_{i \in I_{0-} \cup I_{0+} \cup \beta_1} \lambda_i^H \nabla H_i(x^*) \\ \lambda_i^g \geq 0 \ (i \in I_g), \\ \lambda_i^G \geq 0 \ (i \in I_{+0} \cup \beta_2), \\ \lambda_i^H \leq 0 \ (i \in I_{0-}), \end{array} \right\} \implies \begin{array}{l} \lambda_i^g = 0 \ (i \in I_g), \\ \lambda_j^h = 0 \ (j = 1, \ldots, p), \\ \lambda_i^G = 0 \ (i \in I_{+0} \cup \beta_2) \\ \lambda_i^H = 0 \ (i \in I_{0-} \cup I_{0+} \cup \beta_1) \end{array}$$

$$(8.16)$$

holds true. Invoking *Motzkin's Theorem of the alternative*, cf. [40], for example, we see that the implication (8.16) is, in case that $I_g \cup I_{0-} \cup I_{+0} \cup \beta_2 \neq \emptyset$, equivalent to condition (i). In turn, if $I_g \cup I_{0-} \cup I_{+0} \cup \beta_2 = \emptyset$, (8.16) reduces to the linear independence of the gradients $\nabla h_j(x^*)$ $(j = 1, \ldots, p)$, $\nabla H_i(x^*)$ $(i \in I_{0+} \cup \beta_1)$, which is condition (ii). $\qquad \square$

The following result, which is an immediate consequence of Theorem 8.3.2, will state that *MPVC-MFCQ*, see Section 5.1, is a sufficient condition for calmness of the perturbation map M^{VC}.

Corollary 8.3.3 *Let x^* be feasible for (1.1) such that MPVC-MFCQ holds at x^*. Then M^{VC} is calm at $(0, x^*)$.*

Proof. MPVC-MFCQ obviously implies condition (i) and (ii) from Theorem 8.3.2 and hence, GMFCQ holds. Due to Proposition 8.2.8, GMFCQ implies calmness of M^{VC} at $(0, x^*)$. □

Putting all pieces of information together, we can state a satisfactory exact penalty result for the MPVC.

Theorem 8.3.4 *Let x^* be feasible for (1.1) such that MPVC-MFCQ holds at x^*. Then the function*

$$P^{VC}(x, \alpha) := f(x) + \alpha d_{\Lambda_{VC}}(F^{VC}(x)) \tag{8.17}$$

is exact at x^*.

In order to find an explicit representation of the penalty function from (8.17), the following elementary result is crucial.

Lemma 8.3.5 *Let C be given by (8.10). Then for $(a, b) \in C$ we have*

$$d_C(a, b) = \max\{0, -b, \min\{a, b\}\} = \begin{cases} \min\{a, b\}, & if \quad a, b \geq 0, \\ 0, & if \quad a \leq 0, \ b \geq 0, \\ -b, & if \quad b \leq 0. \end{cases}$$

Note, that the latter result provides an explicit representation which is totally independent of the chosen l_p-norm to induce the distance function.

Corollary 8.3.6 *Let $x \in \mathbb{R}^n$. Then we have*

$$d_{\Lambda_{VC}}(F^{VC}(x)) = \left\| \begin{pmatrix} \max\{g_i(x), 0\} & (i = 1, \ldots, m) \\ |h_j(x)| & (j = 1 \ldots, p) \\ \max\{0, -H_i(x), \min\{G_i(x), H_i(x)\}\} & (i = 1, \ldots, l) \end{pmatrix} \right\|.$$

8.4. The limiting subdifferential

In this section we will briefly introduce the so-called *limiting subdifferential* for lower semicontinuous (lsc) functions, since we use it to derive M-stationarity in the following section. The limiting subdifferential is closely linked to the limiting normal cone, see Definition 6.1.5, and is investigated in depth in [38, 39] or [54].

In order to define it, the notion of the *Fréchet subdifferential* is needed. Note that the latter is sometimes also called the regular subdifferential, cf. [54].

Definition 8.4.1 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be lsc and $f(x)$ finite.*

(a) *The set*

$$\hat{\partial}f(x) := \left\{ s \in \mathbb{R}^n \,\middle|\, \liminf_{y \to x} \frac{f(y) - f(x) - s^T(y - x)}{\|y - x\|} \geq 0 \right\}$$

is called the Fréchet subdifferential *of f at x.*

(b) *The set*

$$\partial f(x) := \left\{ \lim_{k \to \infty} s^k \,\middle|\, \exists\, x^k \underset{f}{\to} x, \ s^k \in \hat{\partial}f(x^k) \right\}$$

is called the limiting subdifferential *of f at x.*

8.5. An alternative proof for M-stationarity

We consider again the penalty function P^{VC} from (8.17). Under certain assumptions (like MPVC-MFCQ, cf. Theorem 8.3.4), this penalty function is exact, hence a local minimum of the MPVC is also a local minimizer of $P^{VC}(\cdot, \alpha)$ for some $\alpha > 0$. This implies that $0 \in \partial_x P(x^*, \alpha)$, and this condition can be used in order to derive optimality conditions for the MPVC itself. However, it is not clear in advance what type of optimality result we can expect to get from this condition. At least, since, on the one hand, MPVC-MFCQ gives exactness of the penalty function P^{VC}, but, on the other hand, is not enough in order to yield strong stationarity at a local minimizer x^* of (1.1), it is not possible to derive strong stationarity from the condition $0 \in \partial_x P(x^*, \alpha)$. The best we can expect to get is therefore M-stationarity, and this is precisely the aim of this section.

Hence, suppose that x^* is a local minimizer of $P^{VC}(\cdot, \alpha)$ for some $\alpha > 0$, such that $0 \in \partial_x P(x^*, \alpha)$. In view of the definition of P^{VC} in (8.17) we are, for obvious reasons, particularly interested in the limiting subdifferential of the distance function d_C from Lemma 8.3.5. To this end, we define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi(a, b) := d_C(a, b). \tag{8.18}$$

Then the limiting subdifferential of ϕ at points from the set C is given in the below lemma.

Lemma 8.5.1 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by (8.18) and let $(a, b) \in C$. Then we have*

$$\partial\phi(a, b) = \begin{cases} \left\{\binom{0}{0}\right\} & if \quad b > 0, a < 0, \\ \mathrm{conv}\left\{\binom{0}{0}, \binom{1}{0}\right\} & if \quad b > 0, a = 0, \\ \mathrm{conv}\left\{\binom{0}{-1}, \binom{0}{1}\right\} & if \quad b = 0, a > 0, \\ \mathrm{conv}\left\{\binom{0}{-1}, \binom{0}{0}\right\} & if \quad b = 0, a < 0, \\ \mathrm{conv}\left\{\binom{0}{1}, \binom{0}{-1}\right\} \cup \mathrm{conv}\left\{\binom{0}{0}, \binom{1}{0}\right\} & if \quad a = b = 0. \end{cases}$$

Proof. Due to the fact that $\phi(a, b) = d_C(a, b)$ for all $(a, b) \in \mathbb{R}^2$, where d_C can be induced by any l_p-norm in $\mathbb{R}^2$, especially by the Euclidean norm, we may invoke [54, Example 8.53], which yields that

$$\partial\phi(a, b) = N((a, b), C) \cap \mathbb{B} \quad \forall (a, b) \in C. \tag{8.19}$$

The representation of the limiting normal cone from Lemma 8.3.1 together with (8.19) eventually gives the desired result. □

The following main result of this section reveals that exactness of the penalty function P^{VC} from (8.4) at a local minimizer of the MPVC yields M-stationarity as an optimality condition.

Theorem 8.5.2 *Let x^* be a local minimizer of the MPVC (1.1) such that P^{VC} is exact at x^*. Then M-stationarity holds at x^*.*

Proof. Due to the fact that P^{VC} is exact at the local minimizer x^* of (1.1), there exists a penalty paramter $\alpha > 0$ such that x^* is also a local minimizer of $P^{VC}(\cdot, \alpha)$. In particular, we thus have $0 \in \partial_x P^{VC}(x^*, \alpha)$. Now, recall that by Corollary 8.3.6 we have

$$P^{VC}(x, \alpha) = f(x) + \alpha \left\| \begin{pmatrix} \max\{g_i(x), 0\} & (i = 1, \ldots, m) \\ |h_j(x)| & (j = 1 \ldots, p) \\ \phi(G_i(x), H_i(x)) & (i = 1, \ldots, l) \end{pmatrix} \right\|.$$

Due to the fact that P^{VC} is exact for an arbitrary l_p-norm if and only if it is exact when using the l_1-norm, we restrict ourselves to this case, since we may apply well-known sum rules for the limiting subdifferential then. Thus, consider the case that

$$P^{VC}(x, \alpha) = f(x) + \alpha + \sum_{i=1}^{m} g_i(x) + \alpha \sum_{j=1}^{p} h_j(x) + \alpha \sum_{i=1}^{l} \phi(G_i(x^*), H_i(x^*)).$$

Invoking [54, Exercise 10.10] we hence obtain

$$0 \in \partial_x P^{VC}(x^*, \alpha) \subseteq \{\nabla f(x^*)\} + \alpha \sum_{i=1}^{m} \partial(\max\{g_i(x), 0\}) + \alpha \sum_{j=1}^{p} \partial(|h_j(x)|) + \alpha \sum_{i=1}^{l} \partial(\phi(G_i(x^*), H_i(x^*)),$$

and therefore, due to [8, p. 151], there exist vectors $\lambda_i \in \partial \max\{g_i(x^*), 0\}$ for $i = 1 \ldots, m$, $\mu_i \in \partial|h_j(x^*)|$ for $j = 1, \ldots, p$ and $(\rho_i, v_i) \in \partial \phi(G_i(x^*), H_i(x^*))$ for $i = 1, \ldots, l$ such that

$$0 = \nabla f(x^*) + \alpha \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \alpha \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) + \alpha \sum_{i=1}^{l} (\rho_i \nabla G_i(x^*) + v_i \nabla H_i(x^*)). \tag{8.20}$$

Now, put
$$\eta_i^G := \alpha \rho_i, \quad \eta_i^H := -\alpha v_i \quad \forall i = 1, \ldots, l.$$

Then (8.20), Lemma 8.5.1 and the well-known formulas for the limiting subdifferential of the max- and the absolute value function imply that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is an M-stationary point of (1.1). □

The above result allows us to regard exactness of the penalty function P^{VC} as an MPVC-tailored constraint qualification.

Combining the previous result with the sufficiency condition for the exactness of P^{VC} from Section 8.3, we can immediately show that MPVC-MFCQ yields M-stationarity at a local minimizer of (1.1), which is already well known, cf. Corollary 6.1.10.

Corollary 8.5.3 *Let x^* be a local minimizer of* (1.1) *such that MPVC-MFCQ holds. Then x^* is an M-stationary point.*

Proof. The proof follows immediately from Theorem 8.3.4 and Theorem 8.5.2. □

Part II.

Numerical Approaches

9. A smoothing-regularization approach

In this chapter a numerical algorithm for the solution of the MPVC (1.1) is investigated which is on the basis of a pretty simple idea:

The characteristic constraints $H_i(x) \geq 0$, $G_i(x)H_i(x) \leq 0$ for $i = 1, \dots, l$ are substituted for set a of (in-)equalities $\varphi(G_i(x), H_i(x)) = 0$ (or $\varphi(G_i(x), H_i(x)) \leq 0$) for $i = 1, \dots, l$ with a locally Lipschitz (not necessarily smooth) function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ satisfying the condition

$$\varphi(a, b) = 0 \iff b \geq 0, \ ab \leq 0, \tag{9.1}$$

such that the resulting program

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \dots, p, \\
& \varphi(G_i(x), H_i(x)) = 0 \quad \forall i = 1, \dots, l.
\end{aligned}
\tag{9.2}
$$

is equivalent to (1.1). This program is then embedded in a sequence of regularized and smooth problems $NLP(t)$, for a smoothing and regularization parameter $t > 0$, which are hopefully easier to solve than the original MPVC, and such that $NLP(0)$ coincides with (9.2).

Due to the fact that φ will be chosen nonsmooth, for reasons explained below, the analysis of the behaviour of the smoothed problems $NLP(t)$ for $t \to 0$ involves nonsmooth calculus. Since, for our purposes, we have chosen to employ *Clarke's generalized gradient* as established in [14], we briefly recall some of the basic concepts that we will use in the sequel.

9.1. Clarke's generalized gradient

We commence by giving a definition of the *Bouligand subdifferential* of a locally Lipschitz, real-valued function, which is our key to the generalized gradient in the sense of Clarke. For these purposes, recall that by a theorem of Rademacher, see [49], a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere, in the sense that the set of nondifferentiable points is a null set for the *Lebesgue measure*.

Definition 9.1.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and let D_f be the set*

$$D_f := \{x \in \mathbb{R}^n \mid f \text{ is differentiable at } x\}$$

of all differentiable points of f. Then for $x \in \mathbb{R}^n$ the following set

$$\partial^B f(x) := \{g \in \mathbb{R}^n \mid \exists \{x^k\} \subseteq D_f : x^k \to x \ \wedge \ \nabla f(x^k) \to g\}$$

is called the Bouligand subdifferential of f at x.

By means of the Bouligand subdifferential there exists a very handy characterization of Clarke's generalized gradient in the finite-dimensional case, cf. [14, Theorem 2.5.1], which we use as a definition.

Definition 9.1.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and $x \in \mathbb{R}^n$. Then the following set*

$$\partial^{Cl} f(x) := conv\{\partial^B f(x)\}$$

is called Clarke's generalized gradient of f at x.

Note, however, that Clarke's generalized gradient was originally introduced via the notion of generalized directional derivatives.

Some basic properties of Clarke's generalized gradient are subsumed in the following result.

Proposition 9.1.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and $x \in \mathbb{R}^n$. Then the generalized gradient (of Clarke) $\partial^{Cl} f(x)$ of f at x is nonempty, convex and compact.*

Proof. See [14, Th. 2.1.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

9.2. Reformulation of the vanishing constraints

In this section, we present a reformulation of the vanishing constraints, as was suggested above, using a suitable function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ satisfying the condition

$$\varphi(a,b) = 0 \iff b \geq 0, \ ab \leq 0. \tag{9.3}$$

As soon as we have a function with this property, we can reformulate the original problem (1.1) in the fashion of (9.2)

Before we present a particular function φ with the property (9.3), we first motivate why we use a nonsmooth mapping φ. To this end, we need the following preliminary result.

Lemma 9.2.1 *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function satisfying (9.3). Then $\nabla\varphi(a,b) = 0$ holds for all $(a,b) \in \mathbb{R}^2$ with $a \leq 0, b \geq 0$.*

Proof. First let $b > 0$ (and $a \leq 0$). Then we obtain for all $h < 0$ sufficiently small that

$$(a+h)b \leq 0 \implies \varphi(a+h,b) = 0 \implies \frac{\partial\varphi(a,b)}{\partial a} = \lim_{h\uparrow 0} \frac{\varphi(a+h,b) - \varphi(a,b)}{h} = 0,$$

and

$$a(b+h) \leq 0 \implies \varphi(a,b+h) = 0 \implies \frac{\partial\varphi(a,b)}{\partial b} = \lim_{h\uparrow 0} \frac{\varphi(a,b+h) - \varphi(a,b)}{h} = 0.$$

Next consider the case $b = 0$ (and $a \leq 0$). Then it follows for all $h > 0$ sufficiently small that

$$(a+h)b = 0 \leq 0 \implies \varphi(a+h,b) = 0 \implies \frac{\partial\varphi(a,b)}{\partial a} = \lim_{h\downarrow 0} \frac{\varphi(a+h,b) - \varphi(a,b)}{h} = 0,$$

and

$$a(b + h) \leq 0 \implies \varphi(a, b + h) = 0 \implies \frac{\partial \varphi(a, b)}{\partial b} = \lim_{h \downarrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = 0.$$

Since φ is assumed to be differentiable, we obtain $\nabla \varphi(a, b) = 0$ in either case. $\qquad \square$

An immediate consequence of Lemma 9.2.1 is the following result.

Proposition 9.2.2 *Let the reformulated problem (9.2) be defined with a differentiable function φ satisfying (9.3), and let x^* be any feasible point for (9.2) such that $I_{0-} \cup I_{00} \cup I_+ \neq \emptyset$ holds. Then MFCQ is not satisfied at x^*.*

Proof. Let $r_i(x) := \varphi(G_i(x), H_i(x))$. Using the chain rule, we obtain

$$\nabla r_i(x^*) = (\nabla G_i(x^*), \nabla H_i(x^*)) \nabla \varphi(G_i(x^*), H_i(x^*)).$$

Since x^* is feasible and there exists an index $i \notin I_{0+}$ by assumption, we obtain from Lemma 9.2.1 that $\nabla \varphi(G_i(x^*), H_i(x^*)) = 0$. This implies $\nabla r_i(x^*) = 0$, hence MFCQ cannot hold. $\qquad \square$

Since the assumptions in Proposition 9.2.2 are fairly weak, it must be supposed that in case of a smooth reformulation of (1.1), MFCQ and thus LICQ do mostly not hold at any feasible point. In particular, these constraint qualifications then do not hold at a solution of (1.1). This observation motivates the use of nonsmooth reformulations of (1.1). The function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(a, b) := \max\{ab, 0\} - \min\{b, 0\} \tag{9.4}$$

will turn out to be a useful choice. Some of its properties are stated in the following result.

Lemma 9.2.3 *The function φ from (9.4) has the following properties:*

(a) φ satisfies (9.3).

(b) φ is locally Lipschitz and nonnegative.

(c) The set of differentiable points of φ is given by

$$D_\varphi = \{(a, b)^T \in \mathbb{R}^2 \mid a \neq 0 \text{ and } b \neq 0\}.$$

In fact, φ is continuously differentiable at these points.

(d) The gradient of φ at an arbitrary differentiable point $(a, b) \in D_\varphi$ is given by

$$\nabla \varphi(a, b)^T = \begin{cases} (b, a), & \text{if } a, b > 0, \\ (b, a - 1), & \text{if } a, b < 0, \\ (0, 0), & \text{if } a < 0, b > 0, \\ (0, -1), & \text{if } a > 0, b < 0. \end{cases}$$

(e) The generalized gradient at an arbitrary nondifferentiable point $(a, b) \notin D_\varphi$ is given by

$$
\partial^{Cl}\varphi(a, b)^T = \begin{cases}
\{(\lambda b, 0) \mid \lambda \in [0, 1]\}, & \text{if } a = 0, b > 0, \\
\{(\lambda b, -1) \mid \lambda \in [0, 1]\}, & \text{if } a = 0, b < 0, \\
\{(0, \lambda a - (1 - \lambda)) \mid \lambda \in [0, 1]\}, & \text{if } a > 0, b = 0, \\
\{(0, \lambda a - \lambda) \mid \lambda \in [0, 1]\}, & \text{if } a < 0, b = 0, \\
\{(0, -\lambda) \mid \lambda \in [0, 1]\}, & \text{if } a = 0, b = 0.
\end{cases}
$$

(f) φ is a regular function (in the sense of Clarke [14, Def. 2.3.4]).

Proof. (a) First let $\varphi(a, b) = 0$. This implies $0 \leq \max\{ab, 0\} = \min\{b, 0\} \leq 0$. Thus, we have $\max\{ab, 0\} = 0 = \min\{b, 0\}$, which implies $ab \leq 0$ and $b \geq 0$. The converse direction is obvious.

(b) The first statement is obvious, and the second one follows from the alternative representation $\varphi(a, b) = \max\{ab, 0\} + \max\{-b, 0\}$.

(c) It is easy to see that the mapping φ is (continuously) differentiable at all points $(a, b) \in D_\varphi$. Hence it remains to show that it is nondifferentiable for all $(a, b) \notin D_\varphi$. Then $a = 0$ or $b = 0$. By considering several cases separately, we show that the partial derivatives do not exist in this case, hence φ cannot be differentiable.

Case 1: $a = 0, b > 0$. Then an elementary calculation shows that

$$
\lim_{h \downarrow 0} \frac{\varphi(a + h, b) - \varphi(a, b)}{h} = \lim_{h \downarrow 0} \frac{hb}{h} = b > 0,
$$

whereas, on the other hand, we have

$$
\lim_{h \uparrow 0} \frac{\varphi(a + h, b) - \varphi(a, b)}{h} = \lim_{h \uparrow 0} \frac{0}{h} = 0.
$$

Thus, $\frac{\partial \varphi(a,b)}{\partial a}$ does not exist, and consequently φ is not differentiable at (a, b).

Case 2: $a = 0, b < 0$. Then we have

$$
\lim_{h \downarrow 0} \frac{\varphi(a + h, b) - \varphi(a, b)}{h} = 0 \quad \text{and} \quad \lim_{h \uparrow 0} \frac{\varphi(a + h, b) - \varphi(a, b)}{h} = b < 0,
$$

hence $\frac{\partial \varphi(a,b)}{\partial a}$ does not exist.

Case 3: $b = 0, a > 0$. Here, a simple calculation shows that

$$
\lim_{h \downarrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = a > 0 \quad \text{and} \quad \lim_{h \uparrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = -1,
$$

and, therefore, φ is not differentiable at (a, b).

Case 4: $b = 0, a < 0$. Then

$$
\lim_{h \downarrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = 0 \quad \text{and} \quad \lim_{h \uparrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = a - 1 < 0,
$$

showing that φ is nondifferentiable also in this case.

Case 5: $a = 0, b = 0$. Here we obtain

$$\lim_{h\downarrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = 0 \quad \text{and} \quad \lim_{h\uparrow 0} \frac{\varphi(a, b + h) - \varphi(a, b)}{h} = -1,$$

so the two one-sided directional derivatives do not coincide also in this case.

(d) This statement can be verified by a simple calculation.

(e) Let $(a, b) \notin D_\varphi$ be arbitrarily given, and recall that the generalized gradient of Clarke, see Definition 9.1.2, is given by the convex hull

$$\partial^{Cl}\varphi(a, b) := \text{conv}\{\partial^B\varphi(a, b)\} \tag{9.5}$$

of the set

$$\partial^B\varphi(a, b) := \{g \in \mathbb{R}^2 \mid \exists\{(a_k, b_k)\} \subseteq D_\varphi : (a_k, b_k) \to (a, b) \text{ and } \nabla\varphi(a_k, b_k) \to g\}. \tag{9.6}$$

In the following, let $\{(a_k, b_k)\} \subseteq D_\varphi$ be an arbitrary sequence converging to (a, b). As in the proof of part (c), we consider a number of cases separately.

Case 1: $a = 0, b > 0$. Here, we basically have two possibilities of convergence to (a, b):

- $a_k \uparrow 0$: Then (d) gives $\nabla\varphi(a_k, b_k)^T = (0, 0) \to (0, 0)$ for k sufficiently large.

- $a_k \downarrow 0$: Then (d) implies $\nabla\varphi(a_k, b_k)^T = (b_k, a_k) \to (b, 0)$ for k sufficiently large.

Hence (9.6) gives us $\partial^B\varphi(0, b)^T = \{(0, 0), (b, 0)\}$. Then (9.5) shows that the generalized gradient is given by $\partial^{Cl}\varphi(0, b)^T = \text{conv}\{\{(0, 0), (b, 0)\}\} = \{(\lambda b, 0) \mid \lambda \in [0, 1]\}$, so we obtain the desired result in this case.

Case 2: $a = 0, b < 0$. Again, there are basically the following two possibilities of convergence to (a, b):

- $a_k \uparrow 0$: Then (d) gives $\nabla\varphi(a_k, b_k)^T = (b_k, a_k - 1) \to (b, -1)$ for k sufficiently large.

- $a_k \downarrow 0$: Then (d) implies $\nabla\varphi(a_k, b_k)^T = (0, -1) \to (0, -1)$ for all k sufficiently large.

We therefore get $\partial^{Cl}\varphi(0, b)^T = \text{conv}\{\{(0, -1), (b, -1)\}\} = \{(\lambda b, -1) \mid \lambda \in [0, 1]\}$.

Case 3: $b = 0, a > 0$. Here we have the following two possibilities:

- $b_k \uparrow 0$: Then (d) gives $\nabla\varphi(a_k, b_k)^T = (0, -1) \to (0, -1)$ for k sufficiently large.

- $b_k \downarrow 0$: Then (d) implies $\nabla\varphi(a_k, b_k)^T = (b_k, a_k) \to (0, a)$ for k sufficiently large.

Consequently, we get $\partial^{Cl}\varphi(a, 0)^T = \text{conv}\{\{(0, -1), (0, a)\}\} = \{(0, \lambda a - (1 - \lambda) \mid \lambda \in [0, 1]\}$.

Case 4: $b = 0, a < 0$. Then the following possibilities occur:

- $b_k \uparrow 0$: Then (d) gives $\nabla\varphi(a_k, b_k)^T = (b_k, a_k - 1) \to (0, a - 1)$ for k sufficiently large.

- $b_k \downarrow 0$: Then (d) implies $\nabla\varphi(a_k, b_k)^T = (0, 0) \to (0, 0)$ for k sufficiently large.

Hence we have $\partial^{Cl}\varphi(a, 0)^T = \text{conv}\{\{(0, 0), (0, a - 1)\}\} = \{(0, \lambda a - \lambda) \mid \lambda \in [0, 1]\}$.

Case 5: $a = 0, b = 0$. In this case, we have to consider four possibilities:

- $a_k \uparrow 0$, $b_k \uparrow 0$: Then we obtain $\nabla\varphi(a_k, b_k)^T = (b_k, a_k - 1) \to (0, -1)$ from (d).

- $a_k \downarrow 0$, $b_k \uparrow 0$: Here we get $\nabla\varphi(a_k, b_k)^T = (0, -1) \to (0, -1)$ from (d).

- $a_k \downarrow 0$, $b_k \downarrow 0$: Then (d) implies $\nabla\varphi(a_k, b_k)^T = (b_k, a_k) \to (0, 0)$.

- $a_k \uparrow 0$, $b_k \downarrow 0$: Using (d) once again, we get $\nabla\varphi(a_k, b_k)^T = (0, 0) \to (0, 0)$.

Together, this gives $\partial^{Cl}\varphi(0, 0)^T = \text{conv}\{\{(0, 0), (0, -1)\}\} = \{(0, -\lambda) \mid \lambda \in [0, 1]\}$.

(f) Recall that $\varphi(a, b) = \max\{ab, 0\} + \max\{-b, 0\}$. As the composition of a regular function with a continuously differentiable function is regular (cf. [14, Thm. 2.3.9 (iii)]), and since positive linear combinations of regular functions are regular as well [14, Prop 2.3.6 (c)], we only need to show the regularity of the mapping $\xi \mapsto \max\{\xi, 0\}$ in view of the above representation of φ. However, this function is convex and, therefore, regular by [14, Prop. 2.3.6 (b)]. $\qquad\square$

Using Lemma 9.2.3 (a), (b), it follows that we can reformulate our MPVC from (1.1) as

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& r_i(x) \leq 0 \quad \forall i = 1 \ldots, l,
\end{aligned} \tag{9.7}$$

where

$$r_i(x) := \varphi(G_i(x), H_i(x)) \quad \forall i = 1 \ldots, l, \tag{9.8}$$

and φ denotes the particular function from (9.4).

9.3. A smoothing-regularization approach to the reformulated problem

Let φ be the function from (9.4), let $r_i(\cdot)$ be the corresponding mapping defined in (9.8), and recall that our MPVC from (1.1) is equivalent to the nonlinear program (9.7). However, the solution of this nonlinear program is still a difficult task since the mapping φ and, therefore, r_i is not differentiable everywhere. An obvious idea is therefore to approximate the nonsmooth function φ by a suitable smooth mapping. Since φ involves max-terms, there exist plenty of possibilities, see the corresponding discussion in [12], for example.

In order to simplify our subsequent analysis, we will use the particular smoothing function

$$\varphi^t(a, b) := \frac{1}{2}(ab + \sqrt{a^2b^2 + t^2} + \sqrt{b^2 + t^2} - b), \tag{9.9}$$

where $t \in \mathbb{R}$ denotes the smoothing parameter. Note that φ^t indeed reduces to φ for $t = 0$ since we have $\varphi(a, b) = \max\{ab, 0\} + \max\{-b, 0\}$. Some further properties of the smoothing function φ^t are summarized in the following result.

Lemma 9.3.1 *Let φ^t denote the smoothing function from (9.9). Then the following statements hold for all $t > 0$:*

(a) *We have $\lim_{t \to 0} \varphi^t(a, b) = \varphi(a, b)$ for all $(a, b) \in \mathbb{R}^2$.*

(b) *The gradient is given by $\nabla \varphi^t(a, b)^T = \frac{1}{2}(b + \frac{ab^2}{\sqrt{a^2 b^2 + t^2}}, \; a + \frac{a^2 b}{\sqrt{a^2 b^2 + t^2}} + \frac{b}{\sqrt{b^2 + t^2}} - 1).$*

(c) *It holds that*

$$\varphi^t(a, b) \begin{cases} = t & if \quad b = 0, \\ < t & if \quad b > 0, a \leq 0, \\ > t & if \quad b < 0, a \leq 0. \end{cases}$$

Proof. Statement (a) is obvious, and (b) follows from standard calculus rules. Hence, it remains to consider part (c). For $b = 0$, we have $\varphi^t(a, b) = \frac{1}{2}(\sqrt{t^2} + \sqrt{t^2}) = t$. For $b > 0$ and $a \leq 0$, on the other hand, we have $\sqrt{b^2 + t^2} < \sqrt{b^2 + 2bt + t^2} = b + t$ and $\sqrt{a^2 b^2 + t^2} \leq \sqrt{t^2 - 2tab + (ab)^2} = -ab + t$. Thus, we obtain $\varphi^t(a, b) = \frac{1}{2}(ab + \sqrt{a^2 b^2 + t^2} + \sqrt{b^2 + t^2} - b) < \frac{1}{2}(ab - ab + t + b + t - b) = t$. Finally, for $b < 0$ and $a \leq 0$, we have $ab \geq 0, -b > 0, \sqrt{a^2 b^2 + t^2} \geq t$ and $\sqrt{b^2 + t^2} \geq t$. Consequently, we get $\varphi^t(a, b) = \frac{1}{2}(ab + \sqrt{a^2 b^2 + t^2} + \sqrt{b^2 + t^2} - b) > \frac{1}{2}(ab + t + t - b) > t$. $\qquad \square$

Using the approximation φ^t from (9.9) of the mapping φ, we obtain the functions

$$r_i^t(x) := \varphi^t(G_i(x), H_i(x)) \quad \forall i = 1, \dots, l \tag{9.10}$$

as the corresponding approximations of the mappings r_i from (9.8). Based on Lemma 9.3.1, we get the following properties of r_i^t.

Corollary 9.3.2 *Let r_i and r_i^t be defined by (9.8) and (9.10), respectively. Then the following statements hold for all $t > 0$:*

(a) *We have $\lim_{t \to 0} r_i^t(x) = r_i(x)$ for all $x \in \mathbb{R}^n$ and all $i = 1, \dots, l$.*

(b) *The gradient of r_i^t is given by*

$$\nabla r_i^t(x) = \begin{cases} -\frac{1}{2} \nabla H_i(x) & if \quad i \in I_{00}(x), \\ \frac{1}{2}(G_i(x) - 1) \nabla H_i(x) & if \quad i \in I_{0+}(x) \cup I_{0-}(x), \\ \frac{1}{2}\left[H_i(x) \nabla G_i(x) + (\frac{H_i(x)}{\sqrt{H_i(x)^2 + t^2}} - 1) \nabla H_i(x) \right] & if \quad i \in I_{+0}(x) \cup I_{-0}(x), \\ \frac{1}{2}[a_i(x) \nabla G_i(x) + b_i(x) \nabla H_i(x)] & else, \end{cases}$$

where

$$a_i(x) := H_i(x) + \frac{G_i(x) H_i(x)^2}{\sqrt{G_i(x)^2 H_i(x)^2 + t^2}},$$

$$b_i(x) := G_i(x) + \frac{G_i(x)^2 H_i(x)}{\sqrt{G_i(x)^2 H_i(x)^2 + t^2}} + \frac{H_i(x)}{\sqrt{H_i(x)^2 + t^2}} - 1.$$

(c) For all $x \in \mathbb{R}^n$, we have

$$r_i^t(x) \begin{cases} = t & \text{if } i \in I_0(x), \\ < t & \text{if } i \in I_{+0}(x) \cup I_{+-}(x), \\ > t & \text{if } i \in I_{-0}(x) \cup I_{--}(x). \end{cases}$$

Proof. (a) This is an immediate consequence of Lemma 9.3.1 (a).

(b) This is implied by Lemma 9.3.1 (b), the definition of the corresponding index sets and the fact that

$$\nabla r_i^t(x) = D_a \varphi^t(G_i(x), H_i(x)) \nabla G_i(x) + D_b \varphi^t(G_i(x), H_i(x)) \nabla H_i(x)$$

for all $x \in \mathbb{R}^n$ and for all $i = 1, \ldots, l$, where $D_a \varphi^t(a, b)$ and $D_b \varphi^t(a, b)$ denote the partial derivatives of the mapping φ^t with respect to its first and second argument.

(c) This is an immediate consequence of Lemma 9.3.1 (c) and the definition of the corresponding index sets. $\qquad\square$

A natural idea to get a smooth counterpart of the nonsmooth reformulation (9.7) of our MPVC would now be to replace the constraints $r_i(x) \leq 0$ by $r_i^t(x) \leq 0$. However, it is easy to see that this pure smoothing approach results in a nonlinear program which may have an empty feasible set. We therefore also enlarge the feasible region by replacing the constraints $r_i(x) \leq 0$ by $r_i^t(x) \leq t$. We therefore obtain the smooth nonlinear program

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\ & h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\ & r_i^t(x) \leq t \quad \forall i = 1 \ldots, l, \end{aligned} \qquad NLP(t)$$

which for $t = 0$ is equivalent to the MPVC from (1.1). The program $NLP(t)$ was obtained from (9.7) by using a smoothing idea for the nonsmooth mapping φ and a regularization of the feasible set. We therefore call this a *smoothing-regularization approach*. The following result is important from a practical point of view since it shows that the regularization enlarges the feasible region, in particular, it therefore follows that the feasible set of the program NLP(t) is always nonempty (cf. also Ch. 9.5.1 below for an illustration). In the below result and in the remainder of this chapter let, for an arbitrary point $x \in \mathbb{R}^n$, the index sets

$$\begin{array}{ccc} I_{--}(x), & I_{-0}(x), & I_{-+}(x), \\ I_{0-}(x), & I_{00}(x), & I_{0+}(x), \\ I_{+-}(x), & I_{+0}(x), & I_{++}(x), \end{array}$$

be defined in the fashion of (3.3) and (3.4), that is, the first and second subscripts indicate the signs of $H_i(x)$ and $G_i(x)$, respectively. Analogously, we put

$$I_g(x) := \Big\{ i \in \{1, \ldots, m\} \mid g_i(x) = 0 \Big\}.$$

Proposition 9.3.3 *Let X and X(t) denote the feasible sets of MPVC and NLP(t), respectively. Then $X \subseteq X(t)$ for all $t > 0$.*

Proof. Let $t > 0$ and $x \in X$ be arbitrarily given. Then we have $g(x) \leq 0$, $h(x) = 0$, and $r_i(x) \leq 0$ for all $i = 1, \ldots, l$. Hence we need to show that $r_i^t(x) \leq t$ holds for all $i = 1, \ldots, l$.

Since x is feasible for the MPVC, we have the following partitioning of the index set $\{1, \ldots, l\}$:

$$\{1, \ldots, l\} = I_0(x) \cup I_{+0}(x) \cup I_{+-}(x).$$

Corollary 9.3.2 (c) therefore gives the desired result. □

9.4. Convergence results

The optimization problems $NLP(t)$ are ordinary smooth constrained nonlinear programs which typically do not contain any critical kinks in their feasible sets like the original MPVC. We therefore believe that the programs $NLP(t)$ can be solved by standard optimization software, at least in the sense that this software is able to find a stationary point x^t which, together with some multipliers, satisfies the usual KKT conditions of NLP(t). In this section, we now investigate the properties of sequences $\{x^t\} \subseteq \mathbb{R}^n$ for $t \downarrow 0$, where x^t is an arbitrary stationary point of NLP(t).

Before presenting our main convergence theorems, however, we need some preliminary results that will play an important role in the subsequent analysis. To this end, given $t > 0$ and a feasible point $x \in X(t)$, we first define the index set

$$M(x, t) := \{i \mid r_i^t(x) = t\}. \tag{9.11}$$

Note that this is the set of active r_i^t-constraints. Some important properties of this and some other index sets are given in the following result.

Lemma 9.4.1 *Let $x^* \in X$ be feasible for the MPVC* (1.1). *Then there is an $\varepsilon > 0$ such that, for all $x \in B_\varepsilon(x^*)$, the following statements hold:*

(a) $I_g(x) \subseteq I_g(x^*)$.

(b) $I_0(x) \subseteq I_0(x^*)$.

(c) $M(x, t) \subseteq I_{+0}(x^*) \cup I_0(x^*)$.

Proof. Obviously, it suffices to show that there is an ε for each of the three statements (a)–(c).

(a) Let $i \notin I_g(x^*)$. Then $g_i(x^*) < 0$ holds. Since g_i is continuous, there is an $\varepsilon > 0$ such that $g_i(x) < 0$ for all $x \in B_\varepsilon(x^*)$. We therefore have $i \notin I_g(x)$ for all $x \in B_\varepsilon(x^*)$.

(b) Let $i \notin I_0(x^*)$. Then $H_i(x^*) > 0$ and, therefore, by continuity of H_i, we have $H_i(x) > 0$ for all $x \in B_\varepsilon(x^*)$ with some $\varepsilon > 0$ sufficiently small. This shows that $i \notin I_0(x)$ for all $x \in B_\varepsilon(x^*)$.

(c) In view of Corollary 9.3.2 (c), we necessarily have

$$M(x, t) \subseteq I_0(x) \cup I_{++}(x) \cup I_{-+}(x)$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Hence, it suffices to show that the following inclusions hold for all $x \in B_\varepsilon(x^*)$ with some $\varepsilon > 0$ sufficiently small:

1) $I_0(x) \subseteq I_0(x^*) \cup I_{+0}(x^*)$,

2) $I_{++}(x) \subseteq I_0(x^*) \cup I_{+0}(x^*)$,

3) $I_{-+}(x) \subseteq I_0(x^*) \cup I_{+0}(x^*)$.

ad 1): This follows immediately from part (b).

ad 2): Suppose this statement does not hold. Then there is a sequence $\{\varepsilon_k\} \downarrow 0$, a sequence $\{x^k\} \subseteq \mathbb{R}^n$ with $x^k \in B_{\varepsilon_k}(x^*)$, and an index $i_k \in I_{++}(x^k)$ such that $i_k \notin I_0(x^*) \cup I_{+0}(x^*)$ for all $k \in \mathbb{N}$. Since x^* is feasible for our MPVC, we therefore have $i_k \in I_{+-}(x^*)$ for all k. In particular, it follows that

$$G_{i_k}(x^k) > 0 \qquad \text{and} \qquad G_{i_k}(x^*) < 0 \qquad (9.12)$$

for all $k \in \mathbb{N}$. Since the index set $\{1, \dots, l\}$ is finite, there is an index i_0 and an infinite subset $K \subseteq \mathbb{N}$ such that $i_k = i_0$ for all $k \in K$. We then obtain from (9.12) that $G_{i_0}(x^k) > 0$ and $G_{i_0}(x^*) < 0$ for all $k \in K$. Since $x^k \to x^*$, however, this is a contradiction to the continuity of G_{i_0}.

ad 3): This can be verified in a way similar to 2). $\qquad \square$

The main ingredient for our convergence results is given in the next proposition.

Proposition 9.4.2 *Let $\{x^k\} \subseteq \mathbb{R}^n$ and $t_k \downarrow 0$ be sequences with $x^k \to x^*$ for some feasible point $x^* \in X$ of our MPVC. Then the following statements hold for all $i \in \{1, \dots, l\}$:*

(a) $\displaystyle \lim_{k \to \infty} d_{\partial^{Cl} r_i(x^*)}(\nabla r_i^{t_k}(x^k)) = 0.$

(b) *Every accumulation point of the sequence $\{\nabla r_i^{t_k}(x^k)\}$ belongs to $\partial^{Cl} r_i(x^*)$.*

(c) *For any $i \in I_{+-}(x^*)$, we have $\displaystyle \lim_{k \to \infty} \nabla r_i^{t_k}(x^k) = 0.$*

Proof. (a) Let $\{x^k\}$ and $\{t_k\}$ be the sequences specified above, and consider an arbitrary index $i \in \{1, \dots, l\}$. We have to show that, for any given $\varepsilon > 0$, there is an index $K \in \mathbb{N}$ such that, for each $k \geq K$, we can find an element $g^k \in \partial^{Cl} r_i(x^*)$ such that $\|\nabla r_i^{t_k}(x^k) - g_k\| \leq \varepsilon$ for all $k \geq K$. To this end, we first recall that

$$\nabla r_i^t(x) = (\nabla G_i(x), \nabla H_i(x)) \nabla \varphi^t(G_i(x), H_i(x)).$$

Furthermore, taking into account that φ is a regular function in view of Lemma 9.2.3 (f), it follows from the chain rule in [14, Thm. 2.3.9 (iii)] that

$$\partial^{Cl} r_i(x^*) = (\nabla G_i(x^*), \nabla H_i(x^*)) \partial^{Cl} \varphi(G_i(x^*), H_i(x^*)), \qquad (9.13)$$

hence any element $g \in \partial^{Cl} r_i(x^*)$ is of the form $g = (\nabla G_i(x^*), \nabla H_i(x^*)) d$ for some vector $d \in \partial^{Cl} \varphi(G_i(x^*), H_i(x^*))$. Since

$$
\begin{aligned}
d_{\partial^{Cl} r_i(x^*)}(\nabla r_i^{t_k}(x^k)) &\leq \left\| (\nabla G_i(x^k), \nabla H_i(x^k)) \nabla \varphi^t(G_i(x^k), H_i(x^k)) - (\nabla G_i(x^*), \nabla H_i(x^*)) d_k \right\| \\
&\leq \left\| (\nabla G_i(x^k), \nabla H_i(x^k)) \right\| \left\| \nabla \varphi^{t_k}(G_i(x^k), H_i(x^k)) - d_k \right\|
\end{aligned}
$$

$$+\left\|(\nabla G_i(x^k), \nabla H_i(x^k)) - (\nabla G_i(x^*), \nabla H_i(x^*))\right\| \left\|d_k\right\|$$

for all $d_k \in \partial^{Cl}\varphi(G_i(x^*), H_i(x^*))$, it follows from the continuity of $(\nabla G_i, \nabla H_i)$ as well as the boundedness of the set $\partial^{Cl}\varphi(G_i(x^*), H_i(x^*))$ (cf. Prop. 9.1.3) that it suffices to show the following statement: For every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that, for each $k \geq K$, we can find an element $d_k \in \partial^{Cl}\varphi(G_i(x^*), H_i(x^*))$ such that

$$\left\|\nabla\varphi^{t_k}(G_i(x^k), H_i(x^k)) - d_k\right\| \leq \varepsilon \quad \forall k \geq K. \tag{9.14}$$

We will prove this statement by considering several cases separately. In order to simplify the notation, we will always write (a_k, b_k) for $(G_i(x^k), H_i(x^k))$, and (a, b) for $(G_i(x^*), H_i(x^*))$.

Case 1: $i \in I_{00}(x^*)$. Then we have $a_k \to 0, b_k \to 0$, and $t_k \downarrow 0$. This implies

$$\frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}, \ \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} \to 0 \tag{9.15}$$

and

$$\frac{1}{2}\Big(\frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big) \in [-1, 0] \quad \forall k \in \mathbb{N}. \tag{9.16}$$

Now let $\varepsilon > 0$ be arbitrarily given. In view of (9.15), we can find a sufficiently large $K \in \mathbb{N}$ such that the inequalities $|a_k|, |b_k|, \left|\frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}\right|, \left|\frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}}\right| \leq \frac{\varepsilon}{2}$ hold for all $k \geq K$. For all $k \geq K$, we then define

$$d_k := \Big(0, \frac{1}{2}\Big(\frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big)\Big)^T.$$

Then it follows from (9.16) and Lemma 9.2.3 (e) that

$$d_k \in \{(0, -\lambda)^T \mid \lambda \in [0, 1]\} = \partial^{Cl}\varphi(0, 0).$$

Since the gradient of φ^t is given by

$$\nabla\varphi^t(a, b)^T = \frac{1}{2}\Big(b + \frac{ab^2}{\sqrt{a^2b^2 + t^2}}, \ a + \frac{a^2b}{\sqrt{a^2b^2 + t^2}} + \frac{b}{\sqrt{b^2 + t^2}} - 1\Big), \tag{9.17}$$

cf. Lemma 9.3.1 (b), we now obtain for any $k \geq K$

$$\begin{aligned}
\left\|\nabla\varphi^{t_k}(a_k, b_k) - d_k\right\| &\leq \frac{1}{2}\Big|b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}\Big| + \frac{1}{2}\Big|a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}}\Big| \\
&\leq \frac{1}{2}\Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big) + \frac{1}{2}\Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big) \\
&= \varepsilon.
\end{aligned}$$

This proves (9.14) in the present case.

Case 2: $i \in I_{0+}(x^*)$. Then we have $a_k \to a > 0, b_k \to 0$, and $t_k \downarrow 0$. Consequently, we have

$$\frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \to 0, \tag{9.18}$$

and an elementary calculation shows that, for all $k \in \mathbb{N}$ sufficiently large, we have

$$\frac{1}{2}\Big(a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big) \in [-1, a_k]. \tag{9.19}$$

Now let $\varepsilon > 0$ be given. Using (9.18), we can find a number $K \in \mathbb{N}$ such that $\big|b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}\big| \le \varepsilon$ and $|a - a_k| \le \frac{\varepsilon}{2}$ for all $k \ge K$. Then define the vector

$$d_k^T := \begin{cases} (0, a), & \frac{1}{2}[a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1] > a, \\ \big(0, \frac{1}{2}[a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1]\big), & \text{else.} \end{cases}$$

Using (9.19) and Lemma 9.2.3 (e), we see that

$$d_k \in \{(0, \lambda a - (1 - \lambda))^T \mid \lambda \in [0, 1]\} = \partial^{Cl}\varphi(a, 0).$$

Using (9.17), we then obtain for all $k \ge K$

$$\begin{aligned}
\big\|\nabla\varphi^{t_k}(a_k, b_k) - d_k\big\| &\le \frac{1}{2}\Big|b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}\Big| + \Big|\frac{1}{2}\Big(a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big) - d_{2,k}\Big| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This proves (9.14) also in the second case.

Case 3: $i \in I_{0-}(x^*)$. Then we have $a_k \to a < 0, b_k \to 0$, and $t_k \downarrow 0$. This implies

$$\frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \to 0 \tag{9.20}$$

and, for all $k \in \mathbb{N}$ sufficiently large, we have

$$\frac{1}{2}\Big(a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big) \in [a_k - 1, 0]. \tag{9.21}$$

Let $\varepsilon > 0$ be arbitrarily given. In view of (9.20), we can find a number $K \in \mathbb{N}$ such that $\big|b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}\big| \le \varepsilon$ and $|a - a_k| \le \frac{\varepsilon}{2}$ for all $k \ge K$. For each $k \in \mathbb{N}$, let us define

$$d_k^T := \begin{cases} (0, a - 1), & \frac{1}{2}[a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1] < a - 1, \\ \big(0, \frac{1}{2}[a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1]\big), & \text{else.} \end{cases}$$

Then we obtain

$$d_k \in \{(0, \lambda a - \lambda)^T \mid \lambda \in [0, 1]\} = \partial^{Cl} \varphi(a, 0)$$

from (9.21) and Lemma 9.2.3 (e). Using (9.17), we therefore obtain

$$
\begin{aligned}
\left\| \nabla \varphi^{t_k}(a_k, b_k) - d_k \right\| &\leq \frac{1}{2} \left| b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \right| + \left| \frac{1}{2} \left(a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1 \right) - d_{2,k} \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

for all $k \geq K$. This proves (9.14) also in Case 3.

Case 4: $i \in I_{+0}(x^*)$. Then we have $a_k \to 0, b_k \to b > 0$, and $t_k \downarrow 0$. This implies

$$\frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} \to 0, \quad \frac{b_k}{\sqrt{b_k^2 + t_k^2}} \to 1 \tag{9.22}$$

and, for $k \in \mathbb{N}$ sufficiently large, we have

$$\frac{1}{2} \left(b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \right) \in [0, b_k]. \tag{9.23}$$

Let $\varepsilon > 0$. Using (9.22), we choose $K \in \mathbb{N}$ large enough such that $|b_k - b| \leq \frac{\varepsilon}{2}$ and $\left| a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1 \right| \leq \varepsilon$ for all $k \geq K$. Define

$$
d_k^T := \begin{cases} (b, 0), & \frac{1}{2} \left(b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \right) > b, \\ \left(\frac{1}{2} \left(b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \right), 0 \right), & \text{else.} \end{cases}
$$

Then we obtain from (9.23) and Lemma 9.2.3 (e)

$$d_k \in \{(\lambda b, 0) \mid \lambda \in [0, 1]\} = \partial^{Cl} \varphi(0, b),$$

for all $k \geq K$. Taking (9.17) into account again, it follows that

$$
\begin{aligned}
\left\| \nabla \varphi^{t_k}(a_k, b_k) - d_k \right\| &\leq \left| \frac{1}{2} \left(b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}} \right) - d_{1,k} \right| + \frac{1}{2} \left| a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1 \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

This proves (9.14) in Case 4.

Case 5: $i \in I_{+-}(x^*)$. In this case, (9.14) follows from (c), as the generalized gradient of φ at a point

(a, b) with $a < 0, b > 0$ only contains the zero vector, see Lemma 9.2.3 (d).

(b) This follows from part (a) since $\partial^{Cl} r_i(x^*)$ is closed.

(c) Since we have

$$\nabla r_i^{t_k}(x^k) = (\nabla G_i(x^k), \nabla H_i(x^k)) \nabla \varphi^{t_k}(G_i(x^k), H_i(x^k)),$$

we only need to show that $\nabla \varphi^{t_k}(a_k, b_k) \to 0$ for $a_k \to a < 0, \ b_k \to b > 0, \ t_k \downarrow 0$. However, it is easy to see that

$$
\begin{aligned}
\nabla \varphi^{t_k}(a_k, b_k) &= \frac{1}{2}\Big(b_k + \frac{a_k b_k^2}{\sqrt{a_k^2 b_k^2 + t_k^2}}, a_k + \frac{a_k^2 b_k}{\sqrt{a_k^2 b_k^2 + t_k^2}} + \frac{b_k}{\sqrt{b_k^2 + t_k^2}} - 1\Big) \\
&\to \frac{1}{2}\Big(b + \frac{ab^2}{|ab|}, a + \frac{a^2 b}{|ab|} + \frac{b}{|b|} - 1\Big) \\
&= \frac{1}{2}(b - b, a - a + 1 - 1) = (0, 0),
\end{aligned}
$$

and this completes the proof of part (c). $\qquad\qquad\square$

We are now in a position to prove our first main convergence result. Basically, it says that every limit point of a sequence of KKT points of NLP(t) for $t \downarrow 0$ gives a strongly stationary point of the MPVC.

Theorem 9.4.3 *Let $(x^t, \lambda^t, \mu^t, \tau^t)$ be a KKT point of NLP(t), and suppose that $(x^t, \lambda^t, \mu^t, \tau^t) \to (x^*, \lambda^*, \mu^*, \tau^*)$ holds for $t \downarrow 0$. Then there exist multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of the MPVC (1.1).*

Proof. First of all, letting $t \downarrow 0$, we obtain

$$
\begin{aligned}
g_i(x^t) \le 0 &\implies g_i(x^*) \le 0 \quad \forall i = 1, \ldots, m, \\
h_j(x^t) = 0 &\implies h_j(x^*) = 0 \quad \forall j = 1, \ldots, p, \\
r_i^t(x^t) \le t &\implies r_i(x^*) \le 0 \quad \forall i = 1, \ldots, l
\end{aligned}
$$

by continuity. Thus, x^* is at least feasible for our MPVC.

Now let $t > 0$ be sufficiently small. Then x^t is sufficiently close to x^*. Since $(x^t, \lambda^t, \mu^t, \tau^t)$ satisfies the KKT conditions of NLP(t), we therefore obtain from Lemma 9.4.1

$$0 = \nabla f(x^t) + \sum_{i=1}^{m} \lambda_i^t \nabla g_i(x^t) + \sum_{j \in J} \mu_j^t \nabla h_j(x^t) + \sum_{i=1}^{l} \tau_i^t \nabla r_i^t(x^t) \tag{9.24}$$

and

$$
\begin{aligned}
\lambda_i^t &\ge 0 \ (i \in I_g(x^t) \subseteq I_g(x^*)), \quad \lambda_i^t = 0 \ (i \notin I_g(x^t)), \\
\tau_i^t &\ge 0 \ (i \in M(x^t, t) \subseteq I_{+0}(x^*) \cup I_0(x^*)), \quad \tau_i^t = 0 \ (i \notin M(x^t, t)).
\end{aligned}
\tag{9.25}
$$

Now let r_i^* for $i \in \{1, \ldots, l\}$ be an arbitrary accumulation point of the bounded sequence $\{\nabla r_i^t(x^t)\}$, cf. Proposition 9.4.2 (a). Then Proposition 9.4.2 (b) shows that $r_i^* \in \partial^{Cl} r_i(x^*)$ for all $i = 1, \ldots, l$. Using the fact that

$$\partial^{Cl} r_i(x^*) = (\nabla G_i(x^*), \nabla H_i(x^*)) \partial^{Cl} \varphi(G_i(x^*), H_i(x^*)),$$

cf. (9.13), together with the representation of $\partial^{Cl}\varphi$ from Lemma 9.2.3 (d), (e), we obtain

$$
\begin{aligned}
r_i^* &= v_i \nabla G_i(x^*) + \omega_i \nabla H_i(x^*) & \text{with} \quad v_i = 0, \ \omega_i \in [-1, 0] \quad (i \in I_{00}(x^*)), \\
r_i^* &= v_i \nabla G_i(x^*) + \omega_i \nabla H_i(x^*) & \text{with} \quad v_i = 0, \ \omega_i \in [G_i(x^*) - 1, 0] \quad (i \in I_{0-}(x^*)), \\
r_i^* &= v_i \nabla G_i(x^*) + \omega_i \nabla H_i(x^*) & \text{with} \quad v_i = 0, \ \omega_i \in [-1, G_i(x^*)] \quad (i \in I_{0+}(x^*)), \\
r_i^* &= v_i \nabla G_i(x^*) + \omega_i \nabla H_i(x^*) & \text{with} \quad v_i \in [0, H_i(x^*)], \ \omega_i = 0 \quad (i \in I_{+0}(x^*)), \\
r_i^* &= v_i \nabla G_i(x^*) + \omega_i \nabla H_i(x^*) & \text{with} \quad v_i = 0, \ \omega_i = 0 \quad (i \in I_{+-}(x^*)).
\end{aligned}
\tag{9.26}
$$

Since the sequences $\{\nabla r_i^t(x^t)\}$ are bounded for all i, the components have a joint convergent subsequence. By passing to the limit on this subsequence, we then obtain from (9.24), (9.25), and (9.26):

$$
0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) + \sum_{i=1}^{l} \tau_i^* v_i \nabla G_i(x^*) + \sum_{i=1}^{l} \tau_i^* \omega_i \nabla H_i(x^*)
\tag{9.27}
$$

with

$$
\begin{aligned}
&\lambda_i^* \geq 0 \ (i \in I_g(x^*)), \quad \lambda_i^* = 0 \ (i \notin I_g(x^*)), \\
&\tau_i^* v_i \geq 0 \ (i \in I_{+0}(x^*)), \quad \tau_i^* v_i = 0 \ (i \notin I_{+0}(x^*)), \\
&\tau_i^* \omega_i \leq 0 \ (i \in I_{00}(x^*)) \cup I_{0-}(x^*)), \quad \tau_i^* \omega_i \ free \ (i \in I_{0+}(x^*)), \quad \tau_i^* \omega_i = 0 \ (i \in I_{+}(x^*)).
\end{aligned}
\tag{9.28}
$$

Putting

$$
\begin{aligned}
\lambda_i &:= \lambda_i^* \quad \forall i = 1, \ldots, m, \\
\mu_j &:= \mu_j^* \quad \forall j = 1, \ldots, p, \\
\eta_i^G &:= \tau_i^* v_i \quad \forall i = 1, \ldots, l, \\
\eta_i^H &:= -\tau_i^* \omega_i \quad \forall i = 1, \ldots, l,
\end{aligned}
$$

we see that the strong stationarity conditions (6.1), (6.2) follow immediately from (9.27), (9.28).

□

Note that Theorem 9.4.3 holds with basically no assumptions except for the minimum requirement that the sequence of KKT points $\{(x^t, \lambda^t, \mu^t, \tau^t)\}$ exists and attains a limit. We also point out that the limit point automatically gives a strongly stationary point of the original MPVC, whereas in corresponding results for MPECs, even under stronger assumptions, the limit points typically satisfy some first order optimality conditions that are weaker than the strong stationarity conditions for an MPEC, see the corresponding discussion at the end of this section.

Our next aim is to show that the above mentioned minimum requirements in Theorem 9.4.3 can still be weakened to reasonable assumptions. To this end, we first introduce the concept of asymptotic nondegeneracy. This definition is similar to the one used in the MPEC literature, where it was introduced in [21].

Definition 9.4.4 *Let x^* be feasible for our MPVC. Then a sequence $\{x^t\}$ of feasible points of NLP(t) converging to x^* for $t \downarrow 0$ is called* asymptotically nondegenerate, *if any accumulation point of $\{\nabla r_i^t(x^t)\}$ is different from 0 for each $i \in I_{+0}(x^*) \cup I_0(x^*)$.*

Note that asymptotic nondegeneracy is required to hold in Definition 9.4.4 only for the components i from the index sets $I_{+0}(x^*)$ and $I_0(x^*)$, but not for those belonging to $I_{+-}(x^*)$, cf. Proposition 9.4.2 (c) in this context.

The concept of asymptotic nondegeneracy will play an essential role in the proof of the following result.

Lemma 9.4.5 *Let x^* be feasible for our MPVC and suppose that the gradient vectors*

$$\nabla h_j(x^*) \ (j = 1, \ldots, p), \ \nabla g_i(x^*) \ (i \in I_g), \ \nabla G_i(x^*) \ (i \in I_{+0}), \ \nabla H_i(x^*) \ (i \in I_0) \tag{9.29}$$

are linearly independent. Furthermore, let $\{x^t\}$ be a sequence of feasible points of NLP(t) converging to x^* and being asymptotically nondegenerate. Then there exists a parameter $\bar{t} > 0$ such that standard LICQ holds for NLP(t) at x^t for all $t \in (0, \bar{t})$.

Proof. We have to show that, for t sufficiently small, the vectors

$$\nabla g_i(x^t) \ (i \in I_g(x^t)), \ \nabla h_j(x^t) \ (j = 1, \ldots, p), \ \nabla r_i^t(x^t) \ (i \in M(x^t, t))$$

are linearly independent. By Lemma 9.4.1 (c), we know that $M(x^t, t) \subseteq I_{+0}(x^*) \cup I_0(x^*)$ for all t sufficiently small. By Proposition 9.4.2, we also know that for $i \in I_{+0}(x^*) \cup I_0(x^*)$ and t sufficiently small, the vector $\nabla r_i^t(x^t)$ is arbitrarily close to a vector $r_i^*(t) \in \partial^{Cl} r_i(x^*)$, which has the representation (cf. (9.26))

$$r_i^*(t) = \begin{cases} \omega_i(t) \nabla H_i(x^*), & i \in I_0(x^*), \\ \nu_i(t) \nabla G_i(x^*), & i \in I_{+0}(x^*), \end{cases} \tag{9.30}$$

with certain scalars $\omega_i(t), \nu_i(t)$ which are, for t sufficiently small, different from 0 since $\{x^t\}$ is asymptotically nondegenerate. Using (9.29) and the above argument, the vectors

$$\nabla g_i(x^*) \ (i \in I_g(x^*)), \ \nabla h_j(x^*) \ (j = 1, \ldots, p),$$
$$\nu_i(t) \nabla G_i(x^*) \ (i \in I_{+0}(x^*)), \ \omega_i(t) \nabla H_i(x^*) \ (i \in I_0(x^*))$$

are linearly independent for t sufficiently small. This implies the linear independence of

$$\nabla g_i(x^t) \ (i \in I_g(x^t)), \ \nabla h_j(x^*) \ (j = 1, \ldots, p), \ \nabla r_i^t(x^t) \ (i \in M(x^t, t))$$

since $I_g(x^t) \subseteq I_g(x^*)$ and $M(x^t, t) \subseteq I_{+0}(x^*) \cup I_0(x^*)$ for t sufficiently small. $\qquad \square$

The linear independence of the gradients in (9.29) is an assumption that was also used in [3] in a different context. It is called VC-LICQ and is weaker than MPVC-LICQ as given in Definition 5.1.1. In particular, VC-LICQ is then a weaker constraint qualification than (standard) LICQ, since LICQ already implies MPVC-LICQ.

Using Lemma 9.4.5, we are now in a position to prove our second main convergence result. To this end, recall the notion of B-stationarity from Section 2.1.3.

Theorem 9.4.6 *Let x^t be a B-stationary point of NLP(t) for all $t > 0$. Furthermore, let $x^t \to x^*$ for $t \downarrow 0$ such that $\{x^t\}$ is asymptotically nondegenerate, and suppose that the gradient vectors from (9.29) are linearly independent. Then the following statements hold:*

(a) *For t sufficiently small, there are unique multipliers $(\lambda^t, \mu^t, \tau^t)$ such that $(x^t, \lambda^t, \mu^t, \tau^t)$ is a KKT point of NLP(t).*

(b) *The sequence $\{(\lambda^t, \mu^t, \tau^t)\}$ has a convergent subsequence. Let (λ, μ, τ) be a limit point.*

(c) *There are unique multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of the MPVC.*

Proof.   (a) By Lemma 9.4.5, we know that standard LICQ holds at x^t for each t sufficiently small. Since x^t is a B-stationary point of NLP(t), it therefore follows from standard results in optimization, see Section 2.1.3, that there exist unique multipliers $(\lambda^t, \mu^t, \tau^t)$ such that $(x^t, \lambda^t, \mu^t, \tau^t)$ is a KKT point of NLP(t).

(b) Because of (a) (for t sufficiently small), there are multipliers $(\lambda^t, \mu^t, \tau^t)$ such that $(x^t, \lambda^t, \mu^t, \tau^t)$ is a KKT point of NLP(t). Using Lemma 9.4.1, we therefore obtain

$$-\nabla f(x^t) = \sum_{i \in I_g} \lambda_i^t \nabla g_i(x^t) + \sum_{j=1}^p \mu_j^t \nabla h_j(x^t) + \sum_{i \in I_0 \cup I_{+0}} \tau_i^t \nabla r_i^t(x^t).$$

In matrix-vector notation, this can be rewritten as

$$A(x^t)^T z^t = -\nabla f(x^t), \tag{9.31}$$

where

$$A(x^t) := \begin{pmatrix} \nabla g_i(x^t)^T & (i \in I_g) \\ \nabla h_j(x^t)^T & (j = 1, \ldots, p) \\ \nabla r_i^t(x^t)^T & (i \in I_0 \cup I_{+0}) \end{pmatrix},$$

and

$$z^t := \begin{pmatrix} \lambda_i^t & (i \in I_g) \\ \mu_j^t & (j = 1, \ldots, p) \\ \tau_i^t & (i \in I_0 \cup I_{+0}) \end{pmatrix}$$

is the vector containing the corresponding multipliers of the potentially active constraints. All other multipliers are 0 for t sufficiently small, in particular, they converge to 0. By Proposition 9.4.2 (a), we know that the sequences $\{\nabla r_i^t(x^t)\}$ are bounded for all i. Hence $A(x^t)$ converges on a subsequence, say, to a matrix $A(x^*)$ which, using Proposition 9.4.2 and the representation of $\partial^{Cl} r_i(x^*)$ from (9.13) (see also (9.30)), has the following structure

$$A(x^*) := \begin{pmatrix} \nabla g_i(x^*)^T & (i \in I_g) \\ \nabla h_j(x^*)^T & (j = 1, \ldots, p) \\ \omega_i \nabla H_i(x^*)^T & (i \in I_0) \\ \nu_i \nabla G_i(x^*)^T & (i \in I_{+0}) \end{pmatrix}.$$

Since $\{x^t\}$ is asymptotically nondegenerate, it follows that $\omega_i \neq 0 \, (i \in I_0)$ and $\nu_i \neq 0 \, (i \in I_{+0})$. Hence the assumed linear independence of the gradients from (9.29) shows that the matrix $A(x^*)$ has full row rank. Since $\nabla f(x^t)$ converges to $\nabla f(x^*)$, it follows that the sequence $\{z^t\}$ from (9.31)

can be chosen in such a way that it is bounded and, therefore, convergent on a suitable subsequence. Hence the multipliers of the potentially active constraints have a convergent subsequence, which together with the convergence of the multipliers of the nonactive constraints proves assertion (b).

(c) Because of (a) and (b), we are in the situation of Theorem 9.4.3 (by considering the convergent subsequence only) which gives the existence of multipliers such that the strong stationarity conditions (6.1), (6.2) hold. The uniqueness of the multipliers follows immediately from the linear independence of the gradient vectors from (9.29). □

We would like to close this section with a brief comparison of the above convergence theorem for MPVCs on the one hand and corresponding convergence results for some related methods in the MPEC field on the other hand.

In [21] a (pure) smoothing-continuation method for MPECs is presented and our approach for MPVCs is to some extend an adaption of this idea (though we cannot use a pure smoothing method). However, the convergence result [21, Th. 3.1] assumes, in our notation, MPEC-LICQ at x^* and a nondegeneracy assumption on $\{x^t\}$. Together with a second-order-type condition for x^t, the authors show that their limit point x^* is a B-stationary point (and, therefore, under MPEC-LICQ, a strongly stationary point). Note that the second-order condition is not needed in our analysis, and that we use a weaker LICQ-type assumption.

The paper [58] introduces a pure regularization approach. The assumptions in the main convergence result [58, Cor. 3.4] are very similar to those from [21, Th. 3.1]. More precisely, this paper also assumes MPEC-LICQ and a second-order condition, and replaces the nondegeneracy condition from [21] by an upper level strict complementarity (ULSC) assumption. Note that this ULSC assumption is not needed in our analysis.

The convergence result for the penalty approach in [30] is essentially the same as the one from [58], so, again, the authors need stronger assumptions than those that we require in our MPVC-setting.

Note that [58] and [30] also present convergence results under weaker assumptions, but then their limit point is no longer guaranteed to be a strongly stationary (or KKT) point of the MPEC.

Eventually, we are inclined to say that the properties of MPVCs in terms of convergence results of a numerical approach are in a sense better than the properties of MPECs, since, roughly speaking, stronger (or at least similar) results can be shown under milder assumptions. This, again, motivates to tackle the MPVC formulation of an optimization problem rather than taking the MPEC formulation of an MPVC from [3] and to apply a standard MPEC solver to this MPEC formulation.

9.5. Numerical results

In this chapter we present some numerical experiments with the proposed smoothing-regularization scheme. All numerical problems in this chapter have been attacked with the solver `Ipopt`,
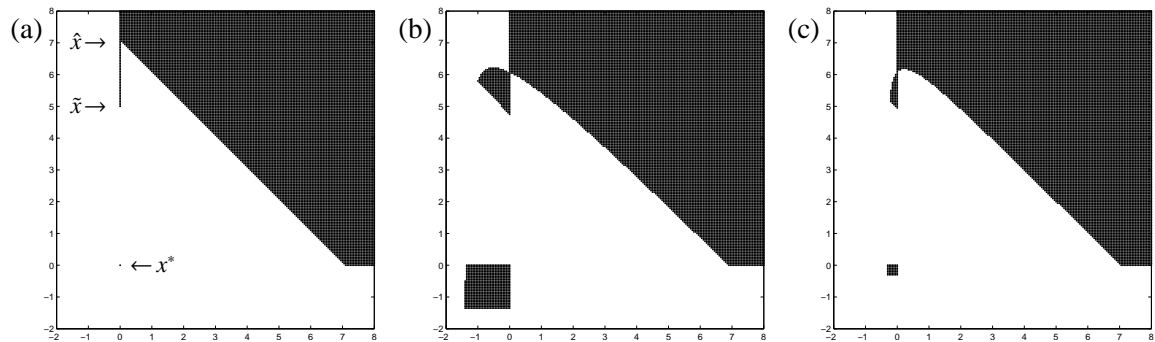
Figure 9.1.: Feasible sets: (a) Original problem, (b) problem $NLP(2)$, (c) problem $NLP(\frac{1}{2})$ (cf. Ch. 9.5.1)

Version 3.3.3, see [62], and its default settings. We say that `Ipopt` 'terminates successfully' if it terminates with the message 'Optimal solution found'.

9.5.1. Academic example

This example in two variables is known in the field of structural optimization. It arises in truss topology optimization (cf. also Ch. 9.5.2 below) where the variables $x_1, x_2 \geq 0$ represent cross-sectional areas of two different groups of truss bars and the meaning of the objective function is the weight of the structure. All the mechanical modeling (force equilibrium, boundary conditions, material law etc.) are analytically expressed in the variables x_1, x_2 (cf., e.g., [33, 13]). After this, one arrives at the following MPVC problem formulation.

$$\min_{x \in \mathbb{R}^2} \quad 4x_1 + 2x_2$$
$$\text{s.t.} \quad x_1 \geq 0,$$
$$x_2 \geq 0,$$
$$(5\sqrt{2} - x_1 - x_2)x_1 \leq 0,$$
$$(5 - x_1 - x_2)x_2 \leq 0. \tag{9.32}$$

The feasible set of this program is shown in Fig. 9.1(a). It consists of the union of an unbounded polyhedron, of an attached line segment $\{(0, x_2)^T \mid 5 \leq x_2 \leq 5\sqrt{2}\}$, and of the isolated point $\{(0,0)^T\}$. As the geometry indicates, numerical methods based on feasible descent concepts generally converge to the point $\hat{x} := (0, 5\sqrt{2})^T$ (cf. Fig. 9.1(a)). Hence, this example is a good test example for academic purposes. Moreover, in the practical application indicated above, the origin must be excluded by an additional constraint, and then the unique optimal global solution to the problem is the point $\tilde{x} := (0, 5)^T$ (see also [2]). In our test, however, we keep the point $x^* := (0,0)^T$, since it will be interesting whether our approach can find it. Clearly x^* is the global minimizer of problem (9.32), and $\tilde{x}$ is a local minimizer. It is a simple exercise to prove that these two points

are also the only strongly stationary (KKT) points of the problem, cf. also [2]. In particular, $\hat{x}$ is not a KKT point as wrongly stated in [13].

With the definitions $f(x) := 4x_1 + 2x_2$, $H_i(x) := x_i$ for $i = 1, 2$, $G_1(x) := 5\sqrt{2} - x_1 - x_2$, $G_2(x) := 5 - x_1 - x_2$, and $t > 0$ we arrive at the perturbed problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^2} \quad & f(x) \\
\text{s.t.} \quad & r_i^t(x) \le t \quad \text{for } i = 1, 2.
\end{aligned}
\tag{9.33}
$$

(cf. *NLP(t)*). The feasible set of this problem is illustrated in Fig. 9.1(b) and (c) for $t = 2$ and $t = \frac{1}{2}$, respectively. These figures also nicely illustrate the result of Proposition 9.3.3.

First we make some tests on the original problem (9.32). We select the 144 different starting points with $x_1^{\text{start}}, x_2^{\text{start}} \in \{0, 1, \dots, 10, 20\}$. Note that sign constraints are part of problem (9.32). Hence, starting points with negative entries are projected by `Ipopt` onto the nonnegative orthant in a first, pre-processing step. Therefore we restrict ourselves to starting points from the nonnegative orthant. All 144 problems are terminated successfully with iteration numbers between 20 and 65, with an average of 37.3. In 10 problems, with starting point close to x^*, the termination point was the global minimizer $x^* = (0, 0)^T$. In the other 134 problems the termination point was the local minimizer $\tilde{x} = (0, 5)^T$. Figure 9.2(a) surveys this behaviour in more detail. Each starting point x^{start} is given a mark, indicating the termination point which has been reached by using this starting point. The feasible region of problem (9.32) is indicated by lines. The black dots mark the local/global optimizers $\tilde{x}, x^*$.

We add that, surprisingly, not each solver is able to successfully terminate at a local minimizer of (9.32) starting from one of the above mentioned starting points although (9.32) is a problem in only 2 variables with 2 mildly nonlinear constraints. For example, the black-box solver `fmincon` from the `Matlab`-toolbox fails for quite some of the 144 starting points. Although `fmincon` is not a state-of-the-art solver, this tells us something about the severe ill-conditioning hidden in the MPVC problem structure.

Next we make a similar test of different starting points for problem formulation (9.33). Since sign constraints are not part of problem (9.33), we also try starting points with negative entries. We solve 289 problems where $x_1^{\text{start}}, x_2^{\text{start}} \in \{-5, \dots, 10, 20\}$, and $t := 10^{-3}$ is constant in all problems. `Ipopt` terminates successfully for all problems, requiring between 25 and 144 iterations (average: 49.7). As a surprise, the convergence behavior is different than for (9.32). In 283 of the 289 problems the termination point was $(-0.000686, -0.000655)^T \approx (0, 0)^T = x^*$ while only 6 problems terminated at $(-0.000474, 5.00032)^T \approx (0, 5)^T = \tilde{x}$. Figure 9.2(b) illustrates this behaviour. As we see, the starting points finally leading to $\approx (0, 0)^T$ are not necessarily close to $(0, 0)^T$. Obviously, the nonlinearity of the problem and the absence of sign constraints cause `Ipopt` to collect information from a larger neighbourhood of the starting point, and thus it is likely that the local minimizer $\tilde{x}$ is avoided. Another reason might be that the feasible set of (9.33) is larger than that of (9.32). More precisely, the *critical* parts of this set (the region around x^*, and the part between $\tilde{x}$ and $\hat{x}$) possess non-empty interiors which might be useful. Moreover, we stress that in all $144 + 289 = 433$ test problems one of the two local minimizers $\tilde{x}, x^*$ has been reached, and convergence to the point $\hat{x}$ did never occur. The reason for this lies in the fact that $\hat{x}$ does not satisfy the strong stationarity (KKT) conditions while `Ipopt` is based on the solution of the KKT
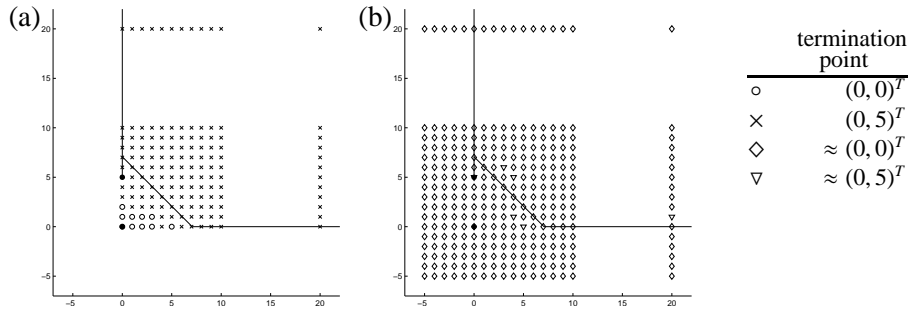
Figure 9.2.: Starting points and corresponding termination points of `Ipopt`: (a) Problem (9.32), (b) problem (9.33) with $t = 10^{-3}$

t	#it	obj. fctn.	termination point	
10^{0}	12	-4.14274	$(-0.696813,$	$-0.677747)^T$
10^{-1}	8	-0.406339	$(-0.0686829,$	$-0.0658035)^T$
10^{-2}	24	-0.0405365	$(-0.00685652,$	$-0.00655523)^T$
10^{-3}	46	-0.00405273	$(-0.000685544,$	$-0.000655277)^T$
10^{-4}	72	-0.000405336	$(-0.0000685653,$	$-0.0000655375)^T$
10^{-5}	629	-0.0000406067	$(-0.00000686863,$	$-0.00000656612)^T$
10^{-6}	639	-0.00000413668	$(-0.000000699312,$	$-0.000000669715)^T$

Table 9.1.: Results for problem (9.33) for different values of t

conditions (cf. also [2]).

Next we investigate the influence of the choice of t in problem (9.33). For these purposes we fix the starting point $x^{\text{start}} := (10, 10)^T$, and (9.33) is treated for each $t = 10^{-k}$, $k = 0, 1, \ldots, 6$. For each of these 7 problems, `Ipopt` terminated successfully close to x^*. Table 9.1 displays the main results where the column "#it" stands for the required iteration numbers. We observe that the factor 0.1 in t leads to one digit more in the precision of the calculated solution and thus also in the optimal function value.

For t^{-k} with $k > 6$ `Ipopt` does not terminate successfully within the first 3000 iterations due to numerical difficulties. Obviously, the functions r_i^t, $i = 1, 2$, are then 'numerically nonsmooth' (Note that $\varphi^t(a, b) \approx \max\{ab, 0\} + \max\{-b, 0\}$ for t close to zero; cf. Lemma 9.3.1(a)).

Finally we return to the practical background of problem (9.32), the truss design problem. To this end, we must artificially exclude the point $x^* = (0, 0)^T$. We do this by adding the linear constraint $3 - x_1 - x_2 \leq 0$ to (9.32) and to (9.33). For the latter problem again we test the smoothing-regularization approach with starting point $x^{\text{start}} := (10, 10)^T$ and $t := 10^{-k}$, $k = 0, 1, \ldots, 6$. The results are displayed in Table 9.2. As expected, we observe convergence to the desired point $\tilde{x}$ for $t \searrow 0$. Again we gain one digit for each decrease of t.

t	#it	obj. fctn.	termination point	
10^0	31	8.75287	$(-0.494657,$	$5.36575)^T$
10^{-1}	31	9.8752	$(-0.0476219,$	$5.03284)^T$
10^{-2}	64	9.98753	$(-0.00473909,$	$5.00324)^T$
10^{-3}	82	9.99875	$(-0.000473693,$	$5.00032)^T$
10^{-4}	162	9.99988	$(-0.0000473881,$	$5.00003)^T$
10^{-5}	99	9.99999	$(-0.00000475987,$	$5.00000)^T$
10^{-6}	288	10.00000	$(-0.000000496364,$	$5.00000)^T$

Table 9.2.: Results for problem (9.33) with an additional constraint excluding $(0, 0)$ for different values of t

9.5.2. Examples in truss topology optimization

In this section we focus on a practical application where vanishing constraints are a 'genuine' part of the modeling. The main task is to calculate an optimal design of a truss structure. Trusses are pin-jointed frameworks consisting of bars like, e.g., electricity masts, support constructions produced from steel bars etc. The usual mechanical modeling of a truss is solely based on geometry, i.e., bending moments at the joints are neglected (in contrast to so-called *frames*). Hence, the resulting design problem is easy to formulate. We refer to the monograph [7] and the literature therein for a profound overview on topology optimization problems, not only trusses.

The challenging part of current research in topology problems of structural optimization are (local) stress-constraints. This means, possible failure of the calculated structure due to high stresses is prevented by the inclusion of appropriate constraints. For each single bar in the truss, one stress constraint must be included to the problem (cf. also below).

In truss topology problems, the topology is (also) optimized. This means, starting with a dense grid of so-called potential bars, a large set of feasible structures is defined. Each potential bar is allowed to have a positive cross-sectional area, $a_i > 0$, or a zero cross-sectional area, $a_i = 0$. The latter means that, after optimization, this potential bar will not be realized as a real bar in the structure, and thus is skipped. In this sense, the *topology* of a truss is optimized, and, besides the optimal cross-sections $a_i^* > 0$ for bars to be realized in the final design, the optimization process itself takes care of the 'principal shape' of the structure. The user-defined grid of potential bars is called a "ground structure". It includes the definition of the boundary conditions (Dirichlet type). Typical ground structures can be seen in Fig. 9.3(a), Fig. 9.4(a), and Fig. 9.5(b) below.

The crucial difficulty in the treatment of stress constraints in a topology problem arises from the fact that stress constraints must be considered only for those bars which are present in the structure, i.e., if $a_i > 0$. Otherwise, it may happen that the 'fictitious stresses', i.e., values of the stress function for bars with $a_i = 0$, cause a restriction on the current design which is not appropriate. Note that all a_i's are variables, and thus the stress function must be defined also for the case $a_i = 0$. Of course, in reality, a non-existent bar, i.e., with $a_i = 0$, does not possess any stress. A simple workaround in modeling is to multiply the stress function of bar i with the area a_i, hence ending up in an MPVC formulation (cf. below).

In this chapter we consider planar trusses only. The only reason for this is that the visualization of 3D-structures is difficult, and good benchmark examples in 3D are hardly known. The modeling

and the structure of the optimization problem presented below, however, does not change if one switches from 2D to 3D. Finally, we mention that problems of truss topology design provide good benchmarks for the development of optimization methods for continuum structures discretized by finite elements.

Next we present the treated problem formulation. With the truss ground structure, external loads are given to be carried by the real structure. In practice, a few so-called load cases must be considered, i.e., different loads apply at different points of time. This is modeled by the consideration of different corresponding vectors u_ℓ of nodal displacements. We consider the same (elastic, isotropic) material for all bars with Young's modulus E. Our goal is to minimize the weight of the structure. Since the material is the same for all bars, we minimize its total material volume instead. Let N denote the number of potential bars in the ground structure, and for all $i = 1, \ldots, N$ let ℓ_i be the length of the potential bar and a_i the corresponding cross-sectional area (so-called *design variable*). Hence, the volume of the structure is given by the sum $\sum_{i=1}^{N} \ell_i a_i$. We use nodal displacements as auxiliary variables to express force equilibrium and stresses. Let L denote the number of load cases. Then for each $\ell \in \{1, \ldots, L\}$ the displacements of the nodal points in the structure are collected in a vector u_ℓ (so called "state variables"). We assume that the support nodes can carry arbitrarily large forces. Hence, Dirichlet boundary conditions can be modeled in a way that corresponding (fixed) displacement coordinates are simply deleted from the problem. Hence, $u_\ell \in \mathbb{R}^d$ where $d := \dim \cdot (\#\text{nodes}) - s$ denotes the so-called "(number of) degrees of freedom of the structure", dim = 2 refers to trusses in 2D, "#nodes" is the number of nodal points of the ground structure, and s is the number of support conditions in Dirichlet sense referring to fixed nodal coordinates. Finally, for simplicity of notation, the vectors u_ℓ, $\ell = 1, \ldots, L$, are collected in a single vector $u := (u_1^T, \ldots, u_L^T)^T \in \mathbb{R}^{L \cdot d}$.

With the variables (a, u) our problem can be stated as follows.

$$
\begin{aligned}
\min_{a \in \mathbb{R}^N, u \in \mathbb{R}^{L \cdot d}} \quad & \sum_{i=1}^{N} \ell_i a_i \\
\text{s.t.} \quad & K(a)u = f_\ell && \forall \ell = 1, \ldots, L, \\
& f_\ell^T u_\ell \leq c && \forall \ell = 1, \ldots, L, \\
& a_i \leq \bar{a} && \forall i = 1, \ldots, N, \\
& a_i \geq 0 && \forall i = 1, \ldots, N, \\
& (\sigma_{i\ell}(a, u)^2 - \bar{\sigma}^2)a_i \leq 0 && \forall i = 1, \ldots, N, \ \forall \ell = 1, \ldots, L.
\end{aligned}
\tag{9.34}
$$

Here the matrix $K(a)$ is the global stiffness matrix of the structure a which for trusses takes the form

$$
K(a) := \sum_{i=1}^{N} a_i \frac{E}{\ell_i} \gamma_i \gamma_i^T \in \mathbb{R}^{d \times d}
$$

with vectors $\gamma_i \in \mathbb{R}^d$. In each component corresponding to a nodal displacement coordinate of the end nodes of bar i, the vector γ_i contains the value $-\cos(\alpha)$ where α is the angle between the displacement coordinate axis and the bar axis. Hence, γ_i contains all information on the location and geometry of potential bar i in the ground structure. The vector $f_\ell \in \mathbb{R}^d$ contains the external forces (load case ℓ) applying at the nodal points, expressed in the displacement coordinate

system $\mathbb{R}^d$. The equilibrium equation $K(a)u_\ell = f_\ell$ models force equilibrium, Hooke's law, and compatibility conditions.

With a user-defined constant $c > 0$ the constraint $f_\ell^T u_\ell \leq c$ bounds the so-called *compliance* $f_\ell^T u_\ell$ of the structure, i.e., the external work caused by load f_ℓ. This energy constraint is required to make the problem well-posed. It should be noted that always $f_\ell^T u_\ell \geq 0$ holds due to the equilibrium constraints.

Moreover, we have box constraints on the cross sectional areas, $0 \leq a_i \leq \bar{a}$ for all $i = 1, \ldots, N$, where $\bar{a} > 0$ is a user-defined constant. The sign constraints $a_i \geq 0$ are part of the vanishing stress constraints, our main interest of the problem. For each i and each ℓ the function $\sigma_{i\ell}$ denotes the stress of the i-th potential bar when the structure is loaded by load case ℓ. We work with the usual displacement-based modeling of stress for bar elements and linearly-elastic material with Young's modulus E, i.e.,

$$\sigma_{i\ell}(a, u) := E \frac{\gamma_i^T u}{\ell_i} \qquad \forall i = 1, \ldots, N \; \forall \ell = 1, \ldots, L.$$

A positive stress value indicates tension of the bar while negative stress indicates compression. For simplicity, however, we use the same user-defined threshold value $\bar{\sigma} > 0$ for bars under tension and compression. Hence, stress-constraints for present bars can be formulated as the quadratic constraints

$$\sigma_{i\ell}(a, u)^2 \leq \bar{\sigma}^2 \qquad \forall i : a_i > 0 \; \forall \ell = 1, \ldots, L. \tag{9.35}$$

As already outlined above, we must find a way to formulate stress constraints also for potential bars with $a_i = 0$. This is done by simple multiplication of the inequalities in (9.35) with a_i (cf. problem (9.34)).

All in all, problem (9.34) possesses $n := N + L \cdot d$ variables, $p := L \cdot d$ equality constraints, $m := L + N$ (ordinary) inequality constraints, and, formally, $N \cdot L$ couples $(H_{i\ell}, G_{i\ell})$ corresponding to vanishing (stress) constraints where

$$H_{i\ell}(a, u) \quad := \quad a_i, \tag{9.36}$$

$$G_{i\ell}(a, u) \quad := \quad \sigma_{i\ell}(a, u)^2 - \bar{\sigma}^2 \tag{9.37}$$

for all $i = 1, \ldots, N$ and all $\ell = 1, \ldots, L$. Notice, however, that $H_{i\ell} = H_{i\ell'}$ for all ℓ, ℓ'.

With these notations we may switch to the corresponding problem $NLP(t)$ approximating problem (9.34) through our smoothing-regularization approach for $t > 0$. We arrive at

$$\begin{aligned}
\min_{a \in \mathbb{R}^N, u \in \mathbb{R}^{L \cdot d}} \quad & \sum_{i=1}^{N} \ell_i a_i \\
\text{s.t.} \quad & K(a)u_\ell = f_\ell && \forall \ell = 1, \ldots, L, \\
& f_\ell^T u_\ell \leq c && \forall \ell = 1, \ldots, L, \\
& a_i \leq \bar{a} && \forall i = 1, \ldots, N, \\
& a_i \geq 0 && \forall i = 1, \ldots, N, \\
& r_{i\ell}^t(a, u) \leq t && \forall i = 1, \ldots, N, \; \forall \ell = 1, \ldots, L,
\end{aligned} \tag{9.38}$$

where

$$r_{i\ell}^t(a, u) := \varphi^t(H_{i\ell}(a, u), G_{i\ell}(a, u)) \qquad \forall i = 1, \ldots, n \; \forall \ell = 1, \ldots, L.$$
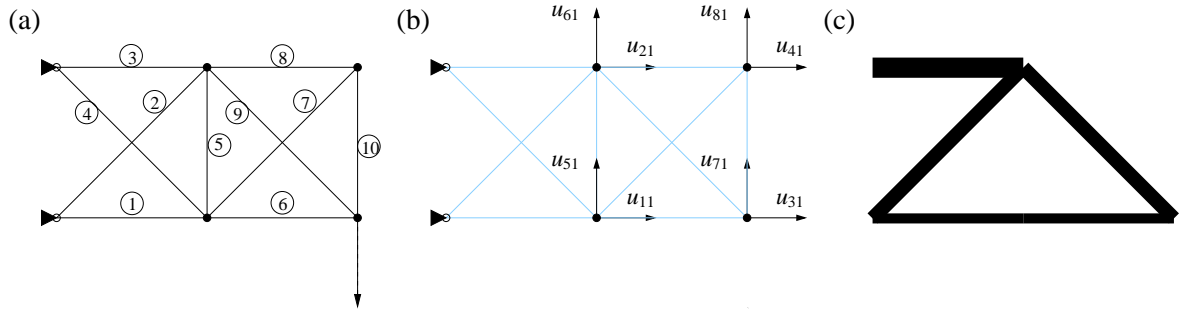
Figure 9.3.: Ten-bar truss example (cf. Ch. 9.5.2)

with φ^t from (9.9). Mind that we have left the constraints $H_{i\ell}(a, u) = a_i \geq 0$ for all i, ℓ in the program. This is to avoid negative bar areas because we want to enforce that the outcome of an optimization run can be interpreted as a meaningful structure and is manufacturable. Moreover, it turned out that the presence of these sign constraints can improve the solution process in large-scaled problems.

If not stated otherwise, we use the (infeasible) starting point $(a, u) := (0, 0) \in \mathbb{R}^N \times \mathbb{R}^{L \cdot d}$. Moreover, as a simplification in all problems below we use the setting $E := 1$ for the Young's modulus, which can be regarded as a scaling of the problem and is not essential.

Ten-bar Truss

First we consider a well-studied academic example for which we also provide the full data description and thus, interested readers may easily verify our numerical results by their own method. We consider the ground structure depicted in Fig. 9.3(a) consisting of $N = 10$ potential bars and 6 nodal points. For obvious reasons this example is called the *ten-bar truss* in the engineering literature. The numbering of the bars is depicted in Fig. 9.3(a) (numbers in circles). We consider $L = 1$ load which applies at the bottom right hand node pulling vertically to the ground with force $\|f_1\|_2 = 1$. The two left hand nodes are fixed, and hence the structure has $d = 8$ degrees of freedom for displacements,

$$u = u_1 = (u_{11}, u_{21}, u_{31}, u_{41}, u_{51}, u_{61}, u_{71}, u_{81})^T \in \mathbb{R}^8.$$

The numbering of the displacement coordinates is indicated in Fig. 9.3(b). The resulting vectors $\gamma_i \in \mathbb{R}^8$, $i = 1, \ldots, 10$, are given in Table 9.3. The i-th column of this table contains the vector γ_i, where only the non-zero entries are displayed. The j-th line of the table corresponds to displacement coordinate u_{j1} which is indicated in the last column. The size of the ground structure is 2×1, i.e., the bar lengths are $\ell_i = 1$ for $i \in \{1, 3, 5, 6, 8, 10\}$ and $\ell_i = \sqrt{2}$ for $i \in \{2, 4, 7, 9\}$.

All in all, problem (9.38) possesses 18 variables, 8 bilinear equality constraints, $1 + 2 \cdot 10 = 21$ linear inequality constraints, and 10 nonlinear inequality constraints modeling the vanishing stress constraints.

We solve three different problem instances:

$$\begin{array}{c|cccccccccc|c}
i = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \\
\hline
 & 1 & & & \frac{\sqrt{2}}{2} & & -1 & -\frac{\sqrt{2}}{2} & & & & u_{11} \\
 & & \frac{\sqrt{2}}{2} & 1 & & & & & -1 & -\frac{\sqrt{2}}{2} & & u_{21} \\
 & & & & & 1 & & & & \frac{\sqrt{2}}{2} & & u_{31} \\
 & & & & & & & \frac{\sqrt{2}}{2} & 1 & & & u_{41} \\
 & & & & -\frac{\sqrt{2}}{2} & -1 & & -\frac{\sqrt{2}}{2} & & & & u_{51} \\
 & & \frac{\sqrt{2}}{2} & & & 1 & & & & \frac{\sqrt{2}}{2} & & u_{61} \\
 & & & & & & & & & -\frac{\sqrt{2}}{2} & -1 & u_{71} \\
 & & & & & & & \frac{\sqrt{2}}{2} & & & 1 & u_{81} \\
\end{array}$$

Table 9.3.: Vectors γ_i for ten-bar truss

	Results of problem `tenbar1`					Results of problem `tenbar2`			
i	a_i^{*1}	$\sigma_{i1}(a^{*1}, u^{*1})$	j	u_{j1}^{*1}	i	a_i^{*2}	$\sigma_{i1}(a^{*2}, u^{*2})$	j	u_{j1}^{*2}
1	0.99627	−1.00374	1	−1.00374	1	0.99996	−1.00004	1	−1.00004
2	1.41048	−1.00265	2	1.00187	2	1.41418	−1.00003	2	1.00002
3	1.99626	1.00187	3	−2.00748	3	1.99996	1.00002	3	−2.00008
4	0	1.77565	4	4.87088	4	0	−1.01099	4	9.19016
5	0	1.54788	5	−4.55505	5	0	−4.02202	5	1.02194
6	0.99627	−1.00374	6	−3.00717	6	0.99996	−1.00004	6	−3.00007
7	0	0.83993	7	−8.02182	7	0	0.40163	7	−8.00022
8	0	3.86901	8	−8.74981	8	0	8.19015	8	−8.36500
9	1.41048	1.00265	$f_1^T u_1^{*1} = 10$		9	1.41418	1.00003	$f_1^T u_1^{*2} = 10$	
10	0	−0.72799	$V^* = 7.97825$		10	0	−0.36478	$V^* = 7.99978$	

Table 9.4.: Results of problems `tenbar1` and `tenbar2` (cf. Ch. 9.5.2)

First we set $c := 10$, $\bar{a} := 100$ (will not be active), $\bar{\sigma} := 1$, and $t := 10^{-2}$ and call this problem setting `tenbar1`. `Ipopt` requires 106 iterations terminating at the point (a^{*1}, u^{*1}). Table 9.4, left, shows the full data where also the stress values σ_{i1} are displayed. The structure consists of 5 bars and is shown in Fig. 9.3(c). Here we have counted the indices i with $a_i^{*1} > 0$. In practice, of course, the bars 1 and 6 would be realized as one "melted" bar without a joint.

Note that the values for u_{41}^* and u_{81}^* denote fictitious displacements because $a_7^{*1} = a_8^{*1} = a_{10}^{*1} = 0$, and thus there is no bar adjacent to the upper right hand node. Nevertheless, mind that $\sigma_{i1}(a^{*1}, u^{*1}) \neq 0$ for $i = 7, 8, 10$. These values may be considered as 'fictitious stress values'.

The stress values in Table 9.4 show that

$$\sigma_{\max}^{*1} := \max_{1 \leq i \leq N} |\sigma_{i1}(a^{*1}, u^{*1})| = 3.86901$$

while

$$\hat{\sigma}_{\max}^{*1} := \max_{1 \leq i \leq N: \, a_i^{*1} > 0} |\sigma_{i1}(a^{*1}, u^{*1})| = 1.00374 . \tag{9.39}$$

This nicely shows the effect of vanishing constraints, because by (9.36) and (9.37) we have for $i \in \{4, 5, 8\}$ that $a_i^{*1} = 0 = H_{i1}(a^{*1}, u^{*1})$ and $G_{i1}(a^{*1}, u^{*1}) > 0$. Note, however, that the vanishing
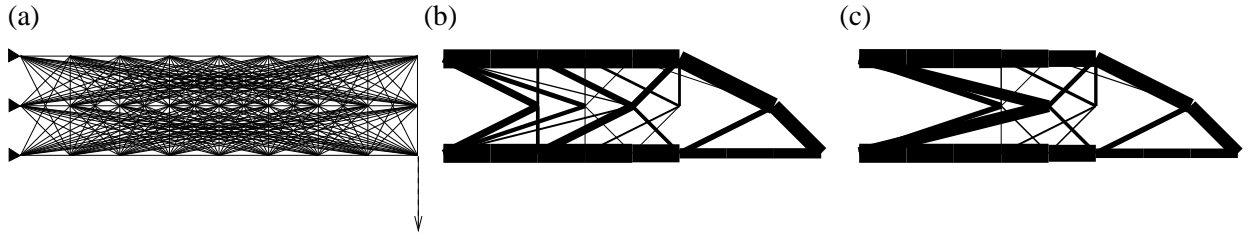
Figure 9.4.: Ground structure and results for cantilever arm example (cf. Ch. 9.5.2)

constraints are part of the original problem (9.34) while (a^{*1}, u^{*1}) is the solution of the *approximating* problem (9.38). This also explains why the stress bound $\bar{\sigma} = 1$ is slightly exceeded in the optimizer (cf. (9.39)).

We hope that the gap between $\hat{\sigma}^{*1}_{\max}$ and $\bar{\sigma}$ closes for $t \searrow 0$ (cf. Ch. 9.4). Hence, we choose $c := 10$, $\bar{a} := 100$, and $\bar{\sigma} := 1$ as before, but we put $t = 10^{-4}$. Moreover, we use (a^{*1}, u^{*1}) as a starting point. This problem setting is called `tenbar2`. `Ipopt` needs 115 iterations until successful termination at the point (a^{*2}, u^{*2}). The full data is also displayed in Table 9.4, right. Since the feasible set of the problem is smaller than for `tenbar1`, the optimal volume V^* increased from 7.97825 to 7.99978. The optimal structure, however, looks right as before ($\|a^{*1} - a^{*2}\|_\infty = 0.0037$). Again the value

$$\sigma^{*2}_{\max} := \max_{1 \le i \le N} |\sigma_{i1}(a^{*2}, u^{*2})| = 8.19015$$

is much bigger than

$$\hat{\sigma}^{*2}_{\max} := \max_{1 \le i \le N: \, a^{*2}_i > 0} |\sigma_{i1}(a^{*2}, u^{*2})| = 1.00004$$

showing the effect of vanishing constraints. Finally, now $\hat{\sigma}^{*2}_{\max} \approx \bar{\sigma}$ holds, as expected.

Cantilever Arm

This example deals with the design of a cantilever arm. Its ground structure consists of $9 \times 3 = 27$ nodal points on an 8×2 area in size. All 27 nodal points are pairwise connected while long bars overlapping shorter ones are deleted resulting in $N = 228$ potential bars. The three left hand nodes are fixed, i.e., $d = 48$. Again we consider a single load case, $L = 1$, acting at the bottom right node pulling to the ground with magnitude $\|f_1\|_2 = 1$. Ground structure, boundary conditions and load are illustrated in Fig. 9.4(a). Problem (9.38) possesses $N + d = 276$ variables, $d = 48$ bilinear equality constraints, $2 \cdot N = 556$ box constraints, and $N = 228$ nonlinear constraints.

First we treat problem (9.38) with $c := 100$, $\bar{a} := 1$, $t := 10^{-2}$, and $\bar{\sigma} := 100.0$. Here the stress bound is chosen very large (and thus will be inactive), because we want to study the effect of stress constraints on the design. After 38 iterations `Ipopt` successfully terminates at the point (a^{*1}, u^{*1}) with optimal volume $V^* = 23.1399$. Moreover, $\max_{1 \le i \le N} a^{*1}_i = \bar{a}$, and $f^T_1 u^{*1}_1 = c$. The obtained structure makes use of 38 bars (where we consider a^{*1}_i to be positive if $a^*_i \ge 0.005 \cdot \bar{a}$). This

structure is displayed in Fig. 9.4(b). From an engineering point of view, the result may well be close to a global minimizer of problem (9.34). An analysis of the stress values shows that

$$\sigma^{*1}_{\max} := \max_{1 \le i \le N} |\sigma_{i1}(a^{*1}, u^{*1})| = 2.7813 = \hat{\sigma}^{*1} := \max_{1 \le i \le N: a^{*1}_i > 0} |\sigma_{i1}(a^{*1}, u^{*1})| \ .$$

As expected, by the large choice of $\bar{\sigma}$, absolute stresses as well es absolute 'fictitious stresses' (i.e., $|\sigma_{i1}|$ for zero bars) are still small compared to $\bar{\sigma}$, and thus the difficulty of vanishing constraints is not challenging at the point (a^{*1}, u^{*1}).

Now we tighten the problem and change the stress bound to $\bar{\sigma} := 2.2$ The values $c = 100$, $\bar{a} = 1$, and $t = 10^{-2}$ remain untouched, and we use (a^{*1}, u^{*1}) as a starting point. `Ipopt` struggles in 294 iterations to successfully terminate at (a^{*2}, u^{*2}) with optimal volume $V^* = 23.6608$. The obtained structure consists of 37 bars and is shown in Fig. 9.4(c). Now, we have

$$\sigma^{*2}_{\max} := \max_{1 \le i \le N} |\sigma_{i1}(a^{*2}, u^{*2})| = 22.0794 \gg \hat{\sigma}^{*2} := \max_{1 \le i \le N: a^{*2}_i > 0} |\sigma_{i1}(a^{*2}, u^{*2})| = 2.2017,$$

i.e., we observe the effect of vanishing constraints! Again, the discrepancy $\hat{\sigma}^{*2} - \bar{\sigma} = 0.017$ is due to the perturbation hidden in the functions r^t_{i1} (resp. in φ^t) for $t > 0$.

Therefore, in a third step we radically decrease t to $t := 10^{-5}$ while keeping $c = 100$, $\bar{a} = 1$, and $\bar{\sigma} = 2.2$ from before. As a starting point we use (a^{*2}, u^{*2}). After 316 iterations `Ipopt` terminates successfully at (a^{*3}, u^{*3}) with $V^* = 23.6633$. The structure a^{*3} consists of 31 bars and hardly differs from a^{*2} (cf. Fig. 9.4(c)). Similarly to before we have

$$\sigma^{*3}_{\max} := \max_{1 \le i \le N} |\sigma_{i1}(a^{*3}, u^{*3})| = 21.2456 \gg \hat{\sigma}^{*3} := \max_{1 \le i \le N: a^{*3}_i > 0} |\sigma_{i1}(a^{*3}, u^{*3})| = 2.20000 = \bar{\sigma} \ .$$

Hence, again "properly vanishing constraints" are active. A closer analysis shows that (out of $N = 224$ in total) there are 24 bars (resp. indices i) satisfying the two inequalities

$$a^*_i < 0.005 = 0.005 \cdot \bar{a} \qquad \text{and} \qquad |\sigma_{i1}(a^*, u^*)| > \bar{\sigma} \ .$$

Because of $\hat{\sigma}^{*3} = \bar{\sigma}$ the calculated point (a^{*3}, u^{*3}) is feasible (and hopefully optimal) for the original problem (9.34)!

The Hook Example

In this chapter we deal with an example which has been considered also by a few other authors who are interested in stress constraints, but mainly for the case of discretized continuum structures. The covered domain has the shape of a *hook* where the top nodes are fixed. A sketch is shown in Fig. 9.5(a). We use a 7×9 nodal grid (6×6 in size) where the upper right quarter is cut out. Like in the previous example, all nodal points are pairwise connected, and bars overlapping (in length) are deleted. In this way we arrive at the ground structure shown in Fig. 9.5(b) consisting of 51 nodes and $N := 703$ potential bars. The top 4 nodes are fixed, and hence $d := 94$. We consider $L := 2$ load cases which both apply solely at the middle right hand node with $\|f_1\|_2 = 1$ and $\|f_2\|_2 = 1.5$. The forces are indicated in Fig. 9.5(a) by dashed arrows. All in all, problem (9.38) possesses
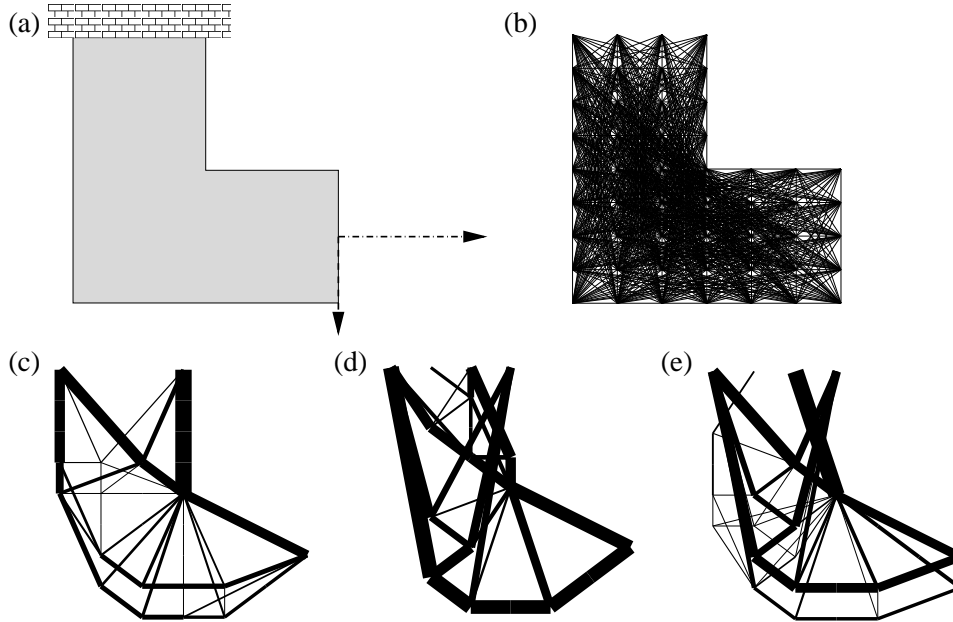
Figure 9.5.: Hook example (cf. Ch. 9.5.2): (a)and (b) Ground structure and load cases, (c) results of problem `hook1`, (d)`hook2` , (e) `hook3`

$N + L \cdots d = 891$ variables, $L \cdot d = 188$ bilinear equality constraints, $2 + 2 \cdot N = 1408$ linear constraints, and $L \cdot N = 1406$ nonlinear constraints approximating vanishing stress constraints.

We treat five problem instances. The values $c := 100$ and $\bar{a} := 100$ (always inactive) are chosen the same in all five problems. Table 9.5 shows the names of the problem instances and the rest of the input data.

instance	$\bar{\sigma}$	starting point	t
`hook1`	100	$(0, 0)$	0.01
`hook2`	3.5	$(0, 0)$	0.01
`hook2+`	3.5	result of `hook2`	0.0001
`hook3`	3.0	$(0, 0)$	0.01
`hook3+`	3.0	result of `hook3`	0.001

Table 9.5.: Problem instances for hook example (cf. Ch. 9.5.2)

Table 9.6 summarizes the results of these five problems. The columns display the number #it of iterations of `Ipopt` until successful termination, the optimal objective function value V^*, the maximal bar area $\max a_i^* = \max_{1 \leq i \leq N} a_i^*$, the number #bars of bars with $a_i^* > 0$ (where, similarly to above, a_i^* is regarded to be positive if $a_i^* > 0.005 \cdot \max_j a_i^*$), the maximal absolute stress in

problem	#it	V^*	max a_i^*	#bars	$\hat{\sigma}_1^*$	max $|\sigma_{i1}^*|$	$\hat{\sigma}_2^*$	max $|\sigma_{i2}^*|$	Fig. 9.5
hook1	66	9.6702	0.5098	49	4.3961	4.3961	4.5968	4.6183	(c)
hook2	1824	12.9125	0.3716	31	3.3794	11.7310	3.5259	17.1068	(d)
hook2+	703	12.9159	0.3715	31	3.3722	11.3081	3.5003	17.0362	as in (d)
hook3	1575	13.7870	0.5042	49	3.0906	19.3449	3.0775	18.3761	(e)
hook3+	645	13.8305	0.5050	46	3.0944	20.8425	3.0990	15.4809	as in (e)

Table 9.6.: Results of problem instances for hook example

present bars w.r.t. load case $\ell = 1, 2$,

$$\hat{\sigma}_\ell^* := \max_{1 \leq i \leq N:\, a_i^* > 0} |\sigma_{i\ell}(a^*, u^*)|,$$

and the maximal absolute stress w.r.t. load case $\ell = 1, 2$ including also fictitious stresses,

$$\max |\sigma_{i\ell}^*| := \max_{1 \leq i \leq N} |\sigma_{i\ell}(a^*, u^*)| \, .$$

The last column refers to the subfigure in Fig. 9.5 where the solution structure a^* of each problem, respectively, is displayed.

For each of the five problems, $\max |\sigma_{i\ell}^*| > \hat{\sigma}_\ell^*$, $\ell = 1, 2$. Hence, for each of the five problem instances we observe the effect of "properly vanishing constraints", even for both load cases. The plots of the optimal structures in Fig. 9.5 show nicely that the decrease of $\bar{\sigma}$ from 100 (i.e., inactive stress constraints; Fig. 9.5(c)) to 3.5 forces the structure to invest much more material into the bottom arch (Fig. 9.5(d)). When $\bar{\sigma}$ is further reduced to 3.0 then the stress in this (compressive) arch becomes too big, and hence the arch is again split into two arches (Fig. 9.5(e)).

Finally, we observe that the decrease of t in problem hook2+ (resp. in hook3+) did not help to substantially decrease the approximation gaps $\hat{\sigma}_\ell^* - \bar{\sigma}$, $\ell = 1, 2$, when compared to hook2 (resp. hook3). It seems that the problem is too large scaled such that a gap reduction is possible without spending efforts on the adjustment of accuracy parameters, maximum iteration numbers etc. of Ipopt.

10. A relaxation approach

In this chapter, like in the previous one, a numerical approach for the solution of the MPVC (1.1) is investigated. At this, the main idea is to consider parametric nonlinear programs $NLP(t)$ of the form

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m, \\
& h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\
& H_i(x) \geq 0 \quad \forall i = 1, \ldots, l, \\
& G_i(x)H_i(x) \leq t \quad \forall i = 1, \ldots, l.
\end{aligned}
\qquad NLP(t)
$$

Apparently, if we denote the feasible set of $NLP(t)$ for $t > 0$ by $X(t)$ we get the analogous result to Proposition 9.3.3.

Proposition 10.0.1 *Consider the parametric problem $NLP(t)$ from above. Then we have $X \subseteq X(t)$ for all $t > 0$. Moreover, it holds that $X(0) = X$.*

In view of the definition of $NLP(t)$ and the above result we call this a *relaxation approach.* This type of scheme was initially introduced in the field of MPECs in [58], see also [55] for a more refined analysis. For MPVCs this scheme was also analyzed in [31]. Some of our results resemble those from the latter reference, with different proofs though, and some material is new, see also the discussion following Theorem 10.2.10.

10.1. Preliminaries

Like in the previous chapter, some auxiliary results are needed in order to establish the desired convergence theory.

Now, for $t > 0$ let $x \in X(t)$ be any feasible point of $NLP(t)$. Then we analogously define the index sets

$$
\begin{aligned}
I_g(x) \quad &:= \quad \{i \,\big|\, g_i(x) = 0\}, \\
I_0(x) \quad &:= \quad \{i \,\big|\, H_i(x) = 0\}, \\
M(x, t) \quad &:= \quad \{i \,\big|\, G_i(x)H_i(x) = t\}.
\end{aligned}
$$

Throughout, the regarded feasible point of $NLP(t)$ will always be given, when defining the latter index sets.

There are some trivial inclusions which hold for some of the above defined index sets. These are stated in the following lemma.

Lemma 10.1.1 *Let x^* be feasible for $NLP(0)$. Then there exists an $\varepsilon > 0$ such that, for all $t > 0$ and all $x \in B_\varepsilon(x^*) \cap X(t)$, we have*

(a) $I_g(x) \subseteq I_g$.

(b) $I_0(x) \subseteq I_0$.

(c) $M(x,t) \subseteq I_{00} \cup I_{+0} \cup I_{0+}$.

Proof. We verify the statements separately:

(a) Let $i \notin I_g$, that is $g_i(x^*) < 0$. By continuity, we thus have $g_i(x) < 0$ for all x sufficiently close to x^*, hence $i \notin I_g(x)$.

(b) Let $i \notin I_0$, that is we have $H_i(x^*) > 0$, and by the above arguments it follows immediately that for all x sufficiently close to x^* we have $i \notin I_0(x)$.

(c) Let $i \notin I_{00} \cup I_{+0} \cup I_{0+}$, that is we have $i \in I_{+-} \cup I_{0-}$. Thus, for all $x \in X(t)$ sufficiently close to x^*, we obtain $G_i(x) < 0$ and thus we have $G_i(x)H_i(x) \leq 0 < t$ for all $t > 0$. This implies $i \notin M(x,t)$ for all $x \in X(t)$ sufficiently close to x^* and all $t > 0$. $\qquad\square$

For numerical methods, usually the satisfaction of constraint qualifications like MFCQ, most often LICQ, must be assumed at a limit point in order to prove convergence. As was already mentioned at many places, cf., e.g., Chapter 4, MPVCs have the unpleasant property to violate these assumptions in many interesting cases. Thus, one had to make up more specialized constraint qualifications, see Section 5, that are more reasonable in the context of MPVCs, but which still ensure the desired properties.

The next lemma states that assuming MPVC-LICQ, see Definition 5.1.1, at a feasible point of (1.1) guarantuees the existence of a neighbourhood such that standard LICQ holds for $NLP(t)$ for all $t > 0$ at all points in that neighbourhood which are feasible for $NLP(t)$.

Lemma 10.1.2 *Let $x^* \in X$ such that MPVC-LICQ holds at x^*. Then there exists an $\varepsilon > 0$ such that LICQ holds for $NLP(t)$ for all $t > 0$ and for all $x \in B_\varepsilon(x^*) \cap X(t)$.*

Proof. Let $t > 0$ and choose $\hat{\varepsilon} > 0$ small enough such that the assertions of Lemma 10.1.1 hold. Then let $x \in B_{\hat{\varepsilon}}(x^*) \cap X(t)$. Thus, we obtain

$$
\begin{aligned}
I_g(x) &\subseteq I_g, \\
I_0(x) &\subseteq I_0, \\
M(x,t) &\subseteq I_{0+} \cup I_{00} \cup I_{+0}.
\end{aligned}
$$

Since we obviously have $M(x,t) \cap I_0(x) = \emptyset$ for all $x \in X(t)$, the above inclusions and the MPVC-LICQ assumption yield that the following gradients are linearly independent for all $x \in B_{\hat{\varepsilon}}(x^*) \cap$

$X(t)$:

$$\begin{aligned}
\nabla g_i(x^*) \quad & (i \in I_g(x)), \\
\nabla h_j(x^*) \quad & (j \in J), \\
\nabla H_i(x^*) \quad & (i \in I_0(x)), \\
\underbrace{G_i(x^*)}_{>0} \nabla H_i(x^*) \quad & (i \in M(x,t) \cap I_{0+}), \\
\underbrace{H_i(x^*)}_{>0} \nabla G_i(x^*) \quad & (i \in M(x,t) \cap I_{+0}), \\
\nabla H_i(x^*) \quad & (i \in M(x,t) \cap I_{00}), \\
\nabla G_i(x^*) \quad & (i \in M(x,t) \cap I_{00}).
\end{aligned}$$

Hence, there exists an $\varepsilon > 0$ such that the vectors

$$\begin{aligned}
\nabla g_i(x) \quad & (i \in I_g(x)), \\
\nabla h_j(x) \quad & (j \in J), \\
\nabla H_i(x) \quad & (i \in I_0(x)), \\
G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) \quad & (i \in M(x,t) \cap I_{0+}), \\
G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) \quad & (i \in M(x,t) \cap I_{+0}), \\
\nabla H_i(x) \quad & (i \in M(x,t) \cap I_{00}), \\
\nabla G_i(x) \quad & (i \in M(x,t) \cap I_{00})
\end{aligned} \tag{10.1}$$

are linearly independent for all $x \in B_\varepsilon(x^*) \cap X(t)$. Now, let $x \in B_\varepsilon(x^*) \cap X(t)$. Then the equation

$$\begin{aligned}
0 \;=\;& \sum_{i \in I_g(x)} \alpha_i \nabla g_i(x) + \sum_{j \in J} \beta_j \nabla h_j(x) + \sum_{i \in I_0(x)} \gamma_i \nabla H_i(x) + \sum_{i \in M(x,t)} \delta_i (G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x)) \\
=\;& \sum_{i \in I_g(x)} \alpha_i \nabla g_i(x) + \sum_{j \in J} \beta_j \nabla h_j(x) + \sum_{i \in I_0(x)} \gamma_i \nabla H_i(x) + \sum_{i \in M(x,t) \cap (I_{0+} \cup I_{+0})} \delta_i (G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x)) \\
& + \sum_{i \in M(x,t) \cap I_{00}} (\delta_i G_i(x)) \nabla H_i(x) + \sum_{i \in M(x,t) \cap I_{00}} (\delta_i H_i(x)) \nabla G_i(x)
\end{aligned}$$

yields that, due to the linear independence of (10.1) and the fact that $G_i(x), H_i(x) \neq 0$ ($i \in M(x,t)$), all numbers $\alpha_i, \beta_i, \delta_i, \gamma_i$ are zero. This, in turn, implies that the vectors

$$\begin{aligned}
\nabla g_i(x) \quad & (i \in I_g(x)), \\
\nabla h_j(x) \quad & (j \in J), \\
\nabla H_i(x) \quad & (i \in I_0(x)), \\
G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) \quad & (i \in M(x,t))
\end{aligned}$$

are linearly independent, that is, LICQ holds for $NLP(t)$ at x. $\qquad \square$

10.2. Convergence Results

The following theorem can be viewed as the main convergence result of this chapter. It follows an idea from [58], where the whole approach is executed for MPECs. At this, the behaviour of

a sequence of KKT points $\{x^t, \lambda^t, \mu^t, \rho^t, \nu^t\}_{t>0}$ of $NLP(t)$ is investigated, where the convergence of $\{x^t\}_{t>0}$ is still assumed. Analogous to [58], we analyze which conditions are needed to gain a weakly or strongly stationary point as a limit. In addition to [58], we also provide a characteristic condition for M-stationarity and we establish an explicit rule for constructing the MPVC multipliers from the KKT multipliers of the relaxed problems in a fashion that is useful for algorithmical purposes.

Theorem 10.2.1 *Let x^* be feasible for (1.1) such that MPVC-LICQ is satisfied, and let $(x^t, \lambda^t, \mu^t, \rho^t, \nu^t)$ be a KKT point of $NLP(t)$ for all $t > 0$ with $x^t \to x^*$ as $t \downarrow 0$. Then the following assertions hold true:*

(a) If we put

$$
\begin{aligned}
\eta_i^{G,t} &:= \nu_i^t H_i(x^t) && (i = 1, \ldots, l), \\
\eta_i^{H,t} &:= \rho_i^t - \nu_i^t G_i(x^t) && (i = 1, \ldots, l),
\end{aligned}
\tag{10.2}
$$

then the multipliers $(\lambda^t, \mu^t, \eta^{G,t}, \eta^{H,t})$ converge to unique MPVC-multipliers $(\lambda, \mu, \eta^G, \eta^H)$ such that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a weakly stationary point of (1.1).

(b) The point $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is M-stationary if and only if

$$
\lim_{t \to 0} (\nu_i^t)^2 t = 0 \quad (i \in I_{00} \cap M(x^t, t) \; \forall t > 0 \text{ sufficiently small}).
$$

(c) The point $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is strongly stationary if and only if

$$
\lim_{t \to 0} G_i(x^t)\nu_i^t = \lim_{t \to 0} H_i(x^t)\nu_i^t = 0 \quad (i \in I_{00} \cap M(x^t, t) \; \forall t > 0 \text{ sufficiently small}).
\tag{10.3}
$$

Proof. (a) Let us define the multipliers $\eta_i^{H,t}$ and $\eta_i^{G,t}$ as proposed in (10.2). Then, using the implications

$$
i \in I_0(x^t) \Longrightarrow i \notin M(x^t, t) \Longrightarrow \nu_i^t = 0 \Longrightarrow \eta_i^{H,t} = \rho_i^t,
\tag{10.4}
$$

$$
i \in M(x^t, t) \Longrightarrow i \notin I_0(x^t) \Longrightarrow \rho_i^t = 0
\tag{10.5}
$$

and employing Lemma 10.1.1, the KKT conditions for $NLP(t)$ yield

$$
\begin{aligned}
-\nabla f(x^t) &= \sum_{i \in I_g(x^t)} \lambda_i^t \nabla g_i(x^t) + \sum_{j \in J} \mu_j^t \nabla h_j(x^t) - \sum_{i \in I_0(x^t)} \rho_i^t \nabla H_i(x^t) + \sum_{i \in M(x^t,t)} \nu_i^t \nabla \theta_i(x^t) \\
&= \sum_{i \in I_g(x^t)} \lambda_i^t \nabla g_i(x^t) + \sum_{j \in J} \mu^t \nabla h_j(x^t) \\
&\quad + \sum_{i \in M(x^t,t) \cap I_{+0}} \eta_i^{G,t} \left(\nabla G_i(x^t) + \frac{G_i(x^t)}{H_i(x^t)} \nabla H_i(x^t) \right) \\
&\quad - \sum_{i \in M(x^t,t) \cap I_{0+}} \eta_i^{H,t} \left(\nabla H_i(x^t) + \frac{H_i(x^t)}{G_i(x^t)} \nabla G_i(x^t) \right) \\
&\quad - \sum_{i \in M(x^t,t) \cap I_{00}} \eta_i^{H,t} \nabla H_i(x^t) + \sum_{i \in M(x^t,t) \cap I_{00}} \eta_i^{G,t} \nabla G_i(x^t) - \sum_{i \in I_0(x^t)} \eta_i^{H,t} \nabla H_i(x^t).
\end{aligned}
\tag{10.6}
$$

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If, now, we define the matrix $A(x^t) \in \mathbb{R}^{(|I_g|+|J|+|I_0|+|I_{+0}|+|I_{00}|) \times n}$ by

$$
A(x^t) := \begin{pmatrix}
\nabla g_i(x^t)^T & (i \in I_g) \\
\nabla h_j(x^t)^T & (j \in J) \\
-\nabla H_i(x^t)^T & (i \in I_{00} \cup I_{0-} \cup (I_{0+} \setminus M(x^t,t))) \\
-(\nabla H_i(x^t)^T + \frac{H_i(x^t)}{G_i(x^t)} \nabla G_i(x^t)^T) & (i \in M(x^t,t) \cap I_{0+}) \\
\nabla G_i(x^t)^T & (i \in I_{00} \cup (I_{+0} \setminus M(x^t,t))) \\
\nabla G_i(x^t)^T + \frac{G_i(x^t)}{H_i(x^t)} \nabla H_i(x^t)^T & (i \in M(x^t,t) \cap I_{+0})
\end{pmatrix}
$$

and the vector $z^t \in \mathbb{R}^{|I_g|+|J|+|I_0|+|I_{+0}|+|I_{00}|}$ by

$$
z^t := \begin{pmatrix}
\lambda_i^t & (i \in I_g) \\
\mu_j^t & (j \in J) \\
\eta_i^{H,t} & (i \in I_0) \\
\eta_i^{G,t} & (i \in I_{00} \cup I_{+0})
\end{pmatrix}
$$

for all $t > 0$ sufficiently small, then (10.6) can be written as $A(x^t)^T z^t = -\nabla f(x^t)$ for all $t > 0$ sufficiently small. Here, we have used the fact that

$$
\begin{aligned}
\lambda_i^t &= 0 \quad \forall i \in I_g \setminus I_g(x^t), \\
\eta_i^{H,t} &= 0 \quad \forall i \in (I_{00} \cup I_{0-} \cup (I_{0+} \setminus M(x^t,t))) \setminus ((M(x^t,t) \cap I_{00}) \cup I_0(x^t)), \\
\eta_i^{G,t} &= 0 \quad \forall i \in (I_{00} \cup (I_{+0} \setminus M(x^t,t))) \setminus (M(x^t,t) \cap I_{00})
\end{aligned}
$$

which can be verified by similar considerations as in (10.4), (10.5). Now, since the matrix $A(x^t)$ converges to the matrix

$$
A(x^*) := \begin{pmatrix}
\nabla g_i(x^*)^T & (i \in I_g) \\
\nabla h_j(x^*)^T & (j \in J) \\
-\nabla H_i(x^*)^T & (i \in I_0) \\
\nabla G_i(x^*)^T & (i \in I_{00} \cup I_{+0})
\end{pmatrix},
$$

which has full rank by the MPVC-LICQ assumption and as $\nabla f(x^t) \to \nabla f(x^*)$ for $t \downarrow 0$, it follows that z^t, too, converges for $t \downarrow 0$, that is, the multipliers λ_i^t for $i \in I_g$, μ_j^t for $j \in J$, $\eta_i^{H,t}$ for $i \in I_0$ and $\eta_i^{G,t}$ for $i \in I_{00} \cup I_{+0}$ are convergent. For $i \notin I_g$ and t sufficiently small we have $\lambda_i^t = 0$ and hence, $\lim_{t\to 0} \lambda_i^t = 0$. Similarly, for $i \in I_{+-}$ and t sufficiently small we have $\eta_i^{G,t} = \eta_i^{H,t} = 0$, thus $\lim_{t\to 0} \eta_i^{G,t} = \lim_{t\to 0} \eta_i^{H,t} = 0$. Now, for $i \in I_{+0}$, it was argued above that $\eta_i^{G,t} = \nu_i^t H_i(x^t)$ is convergent. But, as $\lim_{t\to 0} H_i(x^t) = H_i(x^*) > 0$ the multiplier ν_i^t is bounded and thus, $\lim_{t\to 0} \eta_i^{H,t} = \lim_{t\to 0} \nu_i^t G_i(x^t) = 0$. Finally, for $i \in I_{0+} \cup I_{0-}$ we have $\lim_{t\to 0} \eta_i^{G,t} = 0$. To verify this statement, it suffices to show that $\{\nu_i^t\}$ is bounded for all indices $i \in I_{0+} \cup I_{0-}$. Suppose there is such an index with $\{\nu_i^t\}$ being unbounded. If $i \in I_{0-}$, it follows that $\rho_i^t - \nu_i^t G_i(x^t)$ is unbounded, contradicting the fact that $\eta_i^{H,t}$ is convergent. On the other hand, if $i \in I_{0+}$, it follows that also ρ_i^t is unbounded, hence both ν_i^t and ρ_i^t are positive for sufficiently small $t > 0$, implying $\theta_i(x^t) = t$ and $H_i(x^t) = 0$, a contradiction.

Together, these considerations yield that the whole sequence of multipliers $(\lambda^t, \mu^t, \eta^{G,t}, \eta^{H,t})$ is convergent to a limit point which we denote by $(\lambda, \mu, \eta^G, \eta^H)$. Obviously, these multipliers satisfy

$$0 = \nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{j \in J} \mu_j \nabla h_j(x^*) + \sum_{i \in I_{00} \cup I_{+0}} \eta_i^G \nabla G_i(x^*) - \sum_{i \in I_0} \eta_i^H \nabla H_i(x^*),$$

as well as

$$\lambda_i \quad \begin{cases} \geq 0, & \text{if } i \in I_g, \\ = 0, & \text{else,} \end{cases}$$

$$\eta_i^G = \begin{cases} \lim_{t \to 0} v_i^t H_i(x^t) \geq 0, & \text{if } i \in M(x^t, t) \cap (I_{00} \cup I_{+0}) \ \forall t > 0 \text{ suff. small,} \\ 0, & \text{else,} \end{cases} \tag{10.7}$$

$$\eta_i^H = \begin{cases} \lim_{t \to 0} \rho_i^t \geq 0, & \text{if } i \in I_0(x^t) \ \forall t > 0 \text{ suff. small,} \\ -\lim_{t \to 0} v_i^t G_i(x^t) \leq 0, & \text{if } i \in M(x^t, t) \cap (I_{00} \cup I_{0+}) \ \forall t > 0 \text{ suff. small,} \\ 0 & \text{else.} \end{cases}$$

In particular, $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a weakly stationary point of (1.1) then, which proves a), since the uniqueness is due to the MPVC-LICQ assumption.

(b), (c): This follows immediately from the proof of (a). □

Note that the characteristic conditions (10.3) for strong stationarity hold especially for the case of bounded multipliers v_i^t ($i \in I_{00}$). This boundedness condition is satisfied, in particular, if these multipliers are convergent. We therefore obtain the following consequence of Theorem 10.2.1.

Corollary 10.2.2 *Let x^* be feasible for* (1.1) *such that MPVC-LICQ is satisfied. Furthermore let $(x^t, \lambda^t, \mu^t, \rho^t, v^t)$ be a KKT point of NLP(t) for all $t > 0$ and let $\eta^{G,t}$ and $\eta^{H,t}$ be defined as in* (10.2). *Then every limit point of the sequence $\{(x^t, \lambda^t, \mu^t, \eta^{G,t}, \eta^{H,t})\}_{t>0}$ for $t \to 0$ is a strongly stationary point of* (1.1).

In fact, the last theorem is also valid if one deletes the MPVC-LICQ assumption, but we could not have stated it as a corollary of Theorem 10.2.1 then.

We would like to point out here that there is no result in the fashion of Theorem 10.2.1 and Corollary 10.2.2 in [31], though this work contains a couple of convergence results, but these all differ substantially in both, assumptions and assertions, from the latter two results.

The following example illustrates that the boundedness of the multipliers, albeit a sufficient condition to get a strongly stationary limit point, is not necessary.

Example 10.2.3 Consider the MPVC equipped with the following functions:

$$\begin{aligned} f(x) &:= -(\tfrac{2}{3}x_1^3 + \tfrac{2}{3}x_2^3 + x_1 x_3 + x_2 x_4), \\ H(x) &:= x_1^2 + x_3, \\ G(x) &:= x_2^2 + x_4. \end{aligned}$$

Since we have

$$\nabla H(x) := \begin{pmatrix} 2x_1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \nabla G(x) := \begin{pmatrix} 0 \\ 2x_2 \\ 0 \\ 1 \end{pmatrix},$$

MPVC-LICQ holds at any feasible point of the MPVC, in particular at $x^* := (0,0,0,0)^T$. Now, put $\rho := 0$ and $\nu := 1$. Then we have

$$0 = \underbrace{\nabla f(x^*)}_{=0} - \underbrace{\rho}_{=0} \nabla H(x^*) + \nu \underbrace{\nabla \theta(x^*)}_{=0},$$

and

$$\rho \geq 0, \rho H(x^*) = 0, \qquad \nu \geq 0, \nu \theta(x^*) = 0.$$

This implies that (x^*, ρ, ν) is a KKT point of the MPVC, that is, x^* is a strongly stationary point (note, however, that the multiplier $\nu = 1$ can be replaced by any nonnegative number).

Now, consider the sequence $\{x^t\}_{t>0}$ defined by $x^t := (\sqrt[4]{t}, \sqrt[4]{t}, 0, 0)^T$ for all $t > 0$. Then, obvioulsy, the sequence $\{x^t\}_{t>0}$ converges to x^* as $t \downarrow 0$ and x^t is feasible for $NLP(t)$ for all $t > 0$. Furthermore, if we put $\rho^t := 0$ and $\nu^t := \frac{1}{\sqrt[4]{t}}$, we obtain

$$\nabla f(x^t) - \underbrace{\rho^t \nabla H(x^t)}_{=0} + \nu^t \nabla \theta(x^t) = - \begin{pmatrix} 2\sqrt{t} \\ 2\sqrt{t} \\ \sqrt[4]{t} \\ \sqrt[4]{t} \end{pmatrix} + \begin{pmatrix} 2\sqrt{t} \\ 2\sqrt{t} \\ \sqrt[4]{t} \\ \sqrt[4]{t} \end{pmatrix} = 0$$

and

$$\rho^t \geq 0, \rho^t H(x^t) = 0, \qquad \nu^t \geq 0, \nu^t(\theta(x^t) - t) = 0,$$

for all $t > 0$. This means that (x^t, ρ^t, ν^t) is a KKT point of $NLP(t)$ for all $t > 0$, and we also have $\nu^t \to \infty$ as $t \downarrow 0$. Moreover, note that both the condition $\lim_{t \to 0}(\nu^t)^2 t = \lim_{t \to 0} \sqrt{t} = 0$ for M-stationarity holds and the conditions $\lim_{t \to 0} \nu^t G(x^t) = \lim_{t \to 0} \nu^t H(x^t) = \lim_{t \to 0} \sqrt[4]{t} = 0$ are also satisfied.

In the remainder of this section, we want to provide sufficient conditions such that the following statements hold:

- There exists a sequence of KKT points of $NLP(t)$.

- The corresponding sequence $\{x^t\}$ converges.

- Every limit point of a sequence of KKT points gives a strongly stationary point of the MPVC, i.e., the characteristic conditions (10.3) are satisfied.

To this end, we first introduce the following strict complementarity notions for MPVCs, cf. also the concept of strict complementarity in the standard case in Chapter 2.

Definition 10.2.4 *Let* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *be a strongly stationary point of* (1.1).

(a) The upper level strict complementarity condition (ULSCC) *is said to hold if*

$$\eta_i^G > 0 \ (i \in I_{+0}), \quad \eta_i^H > 0 \ (i \in I_{00} \cup I_{0-}).$$

(b) The strong upper level strict complementarity condition (SULSCC) *is said to hold if ULSCC holds and in addition*
$$\eta_i^H \neq 0 \ (i \in I_{0+}).$$

Note that ULSCC was introduced in [31]. It also has its counterpart in the MPEC setting, cf. [58], e.g. The stronger concept SULSCC holds, in particular, if ULSCC is satisfied and $I_{0+} = \emptyset$. The SULSCC condition allows us to state the following consequence of Theorem 10.2.1 which will be used in the proof of our second main result, Theorem 10.2.10 below.

Corollary 10.2.5 *Let the assumptions of Theorem 10.2.1 hold such that strong stationarity and, in addition, SULSCC holds for* $(x^*, \lambda, \mu, \eta^G, \eta^H)$. *Then we have the following two equivalences:*

(a) $\eta_i^G > 0$ *or* $\eta_i^H < 0 \Longleftrightarrow \theta_i(x^t) = t$ *for all* $t > 0$ *sufficiently small.*

(b) $\eta_i^H > 0 \Longleftrightarrow H_i(x^t) = 0$ *for all* $t > 0$ *sufficiently small.*

Proof. (a) '$\Longrightarrow$:' Let first $\eta_i^H < 0$. In view of (10.7), this immediately implies $i \in M(x^t, t)$ for all $t > 0$ sufficiently small, i.e., $\theta_i(x^t) = t$ for all these t. The same argument also shows that $\eta_i^G > 0$ gives $i \in M(x^t, t)$ for all $t > 0$ sufficiently small.
'$\Longleftarrow$:' Let $\theta_i(x^t) = t$ for $t > 0$ sufficiently small. Due to Lemma 10.1.1 this yields that

$$i \in M(x^t, t) \cap (I_{00} \cup I_{+0} \cup I_{0+})$$

for all $t > 0$ sufficiently small. Let us first suppose that $i \in M(x^t, t) \cap I_{00}$ for all $t > 0$ sufficiently small. Then, by the SULSCC assumption, $\eta_i^H > 0$. Hence (10.7) implies $\eta_i^{H,t} = \rho_i^t > 0$ for all $t > 0$ sufficiently small. On the other hand, since $i \in M(x^t, t)$ and, therefore, $i \notin I_0(x^t)$, we have $\rho_i^t = 0$ for all $t > 0$ sufficiently small. This contradiction shows that this case cannot occur. Now, let $i \in M(x^t, t) \cap I_{0+}$ for all $t > 0$ sufficiently small. Then SULSCC shows that $\eta_i^H \neq 0$. However, $\eta_i^H > 0$ gives a contradiction as in the case discussed before. Hence we necessarily have $\eta_i^H < 0$. Finally, let $i \in M(x^t, t) \cap I_{+0}$ for all $t > 0$ sufficiently small. Then we immediately obtain $\eta_i^G > 0$ from SULSCC.

(b) '$\Longrightarrow$:' Let $\eta_i^H > 0$. Then (10.7) implies that $\eta_i^{H,t} = \rho_i^t > 0$ for all $t > 0$ sufficiently small. Thus $i \in I_0(x^t)$ for all $t > 0$ sufficiently small, i.e., $H_i(x^t) = 0$ for all these t.
'$\Longleftarrow$:' Let $H_i(x^t) = 0$ for all $t > 0$ sufficiently small. Then $i \in I_0(x^t)$ and, therefore, $\eta_i^H \geq 0$ in view of (10.7). By SULSCC, we necessarily obtain $\eta_i^H > 0$, as desired. $\qquad\square$

It is immediately clear from the previous proof that in the above result, SULSCC is only needed for the '$\Longleftarrow$'-directions.

We continue with providing a condition to ensure that the multipliers v_i^t ($i \in I_{00}$) are equal to zero for $t > 0$ sufficiently small. In particular, this yields boundedness and thus strong stationarity in Theorem 10.2.1. To this end, we need the following technical results.

Lemma 10.2.6 *Let $x^* \in X$ be given and consider an arbitrary index $i \in I_{00}$. Then there exists a neighbourhood U_i of x^* and a positive constant c_i such that for any $x \in U_i$ with $H_i(x) \geq 0, G_i(x)H_i(x) \geq t$ and all $t > 0$ sufficiently small, we have $\|x - x^*\| \geq c_i \sqrt{t}$.*

Proof. Suppose for contradiction that there exists a sequence $\{t_k\} \downarrow 0$ and a sequence $\{x^k\} \rightarrow x^*$ with $G_i(x^k)H_i(x^k) \geq t_k$, $H_i(x^k) \geq 0$ and

$$\frac{\sqrt{t_k}}{\|x^k - x^*\|} \rightarrow \infty. \tag{10.8}$$

By taking a subsequence if necessary, we either have $G_i(x^k) \geq \sqrt{t_k}$ or $H_i(x^k) \geq \sqrt{t_k}$ for all k. If $G_i(x^k) \geq \sqrt{t_k}$ for all k then, with a positive constant l_i satisfying $l_i \geq \|\nabla G_i(x^*)\|$, it follows that

$$
\begin{aligned}
\sqrt{t_k} \leq G_i(x^k) &= G_i(x^k) - G_i(x^*) \\
&= \nabla G_i(x^*)(x^k - x^*) + o(\|x^k - x^*\|) \\
&\leq 2l_i\|x^k - x^*\|,
\end{aligned}
$$

for k sufficiently large, in contradiction to (10.8). In case that $H_i(x^k) \geq \sqrt{t_k}$ for all k, we obtain the same contradiction which eventually proves the assertion. $\qquad\square$

Lemma 10.2.7 *Let $x^* \in X$ be given, and let $\{x^t\}_{t>0}$ be a sequence satisfying $\|x^t - x^*\| = O(t)$ and $H_i(x^t) \geq 0$ for all $i \in I_{00}$. Then we have $G_i(x^t)H_i(x^t) < t$ for all $i \in I_{00}$ and $t > 0$ sufficiently small.*

Proof. Due to Lemma 10.2.6 there exists a neighbourhood U of x^* and a positive constant $\hat{c}$ such that if $t \in (0, \hat{t}]$, $x \in U$ and $H_i(x) \geq 0, G_i(x)H_i(x) \geq t$ for $i \in I_{00}$, we have $\|x - x^*\| \geq c\sqrt{t}$. Since $\|x^t - x^*\| = O(t)$, we have for small $t > 0$ that $x^t \in U$, and $\|x^t - x^*\| < c\sqrt{t}$. Hence, due to the fact that $H(x^t) \geq 0$, we must have $G_i(x^t)H_i(x^t) < t$ for all $i \in I_{00}$ and $t > 0$ sufficiently small. $\qquad\square$

The above lemmas imply the following result.

Proposition 10.2.8 *Let x^* be a feasible point of the MPVC (1.1), and let $\{x^t\}_{t>0}$ be a sequence of KKT points of NLP(t) with $\|x^t - x^*\| = O(t)$. Then the associated multipliers of x^t are bounded.*

Proof. Since the assumptions of Lemma 10.2.7 are satisfied, we obtain $G_i(x^t)H_i(x^t) < 0$ for all $i \in I_{00}$ and $t > 0$ sufficiently small. In particular this implies that the corresponding multipliers v_i^t ($i \in I_{00}$) vanish for all $t > 0$ sufficiently small. The proof of Theorem 10.2.1 shows that all other

multipliers are bounded, too, so that the assertion follows. □

We would like to finish this section with a stability-type result. In stability analysis some second-order-type conditions arise naturally, cf. [50, 51, 34] for classical references. The condition needed in our context is given in the next definition, where we recall that the function L denotes the MPVC-Lagrangian from (7.1).

Definition 10.2.9 *Let* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *be a strongly stationary point of* (1.1). *Then we say that the* MPVC strong second-order sufficient condition (MPVC-SSOSC) *holds if*

$$d^T \nabla_{xx}^2 L(x^*, \lambda, \mu, \eta^G, \eta^H) d > 0$$

for all $d \in C(x^*)$, *where*

$$
C(x^*) := \{d \in \mathbb{R}^n \mid \begin{array}{ll} \nabla g_i(x^*)^T d & = 0 \quad (i : \lambda_i > 0), \\ \nabla h_j(x^*)^T d & = 0 \quad (j \in J), \\ \nabla H_i(x^*)^T d & = 0 \quad (i : \eta_i^H \neq 0), \\ \nabla G_i(x^*)^T d & = 0 \quad (i : \eta_i^G \neq 0)\}. \end{array}
$$

Note that, obviously, the critical cone $C(x^*)$ also depends substantially on the multipliers $(\lambda, \mu, \eta^G, \eta^H)$ but for our purposes we will always assume MPVC-LICQ, which implies that the multipliers are unique and thus, it will always be clear which multipliers the cone refers to. Mind also that the critical cone that we use here to define MPVC-SSOSC is, under SULSCC, a larger set than the critical cone that was used in Chapter 7 to establish second-order optimality conditions. Thus, under SULSCC, MPVC-SSOSC is a stronger assumption than the sufficient condition from Chapter 7 . In turn, under SULSCC, the MPVC-SSOSC is exactly the second-order condition to be used used in [31, Th. 5.4], which is the comparable spot to where we employ MPVC-SSOSC, see also the discussion following the next theorem.

In the upcoming result, the notion of a *piecewise smooth function* is used. For a definition and extensive treatment we refer the reader to [57].

Theorem 10.2.10 *Let* $(x^*, \lambda, \mu, \eta^G, \eta^H)$ *be a strongly stationary point of* (1.1) *such that MPVC-SSOSC, MPVC-LICQ and SULSCC are satisfied. Then there exists an open neighbourhood U of* x^*, *a scalar* $\bar{t} > 0$ *and a piecewise smooth function* $x : (-\bar{t}, \bar{t}) \to U$ *such that, for all* $t \in (0, \bar{t})$, *the vector* $x(t)$ *is the unique KKT point of NLP(t) in U, also satisfying strong second-order sufficient conditions (SSOSC).*

Proof. For $t > 0$ consider the parametric nonlinear program $P(t)$

$$
\begin{array}{llll}
\min & f(x) \\
\text{s.t.} & g(x) & \leq & 0, \\
& h(x) & = & 0, \\
& H_i(x) & \geq & 0 \quad (i : \eta_i^H > 0), \\
& G_j(x) H_j(x) & \leq & t \quad (j : \eta_j^H < 0), \\
& G_k(x) H_k(x) & \leq & t \quad (k : \eta_k^G > 0).
\end{array}
$$

The Lagrangian of $P(t)$ is given by

$$
\begin{aligned}
L_{P(t)}(x, \lambda, \mu, \alpha, \beta, \gamma) \quad := \quad & f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j \in J} \mu_j h_j(x) - \sum_{i:\eta_i^H > 0} \alpha_i H_i(x) \\
& + \sum_{j:\eta_j^H < 0} \beta_j(\theta_j(x) - t) + \sum_{k:\eta_k^G > 0} \gamma_k(\theta_k(x) - t).
\end{aligned}
$$

Put

$$
\begin{aligned}
\lambda_i^* &:= \lambda_i && (i = 1, \ldots, m), \\
\mu_j^* &:= \mu_j && (j \in J), \\
\alpha_i^* &:= \eta_i^H && (i : \eta_i^H > 0), \\
\beta_j^* &:= -\frac{\eta_j^H}{G_j(x^*)} && (j : \eta_j^H < 0), \\
\gamma_k^* &:= \frac{\eta_k^G}{H_k(x^*)} && (k : \eta_k^G > 0).
\end{aligned}
\tag{10.9}
$$

We are not dividing by zero here, since we have $j \in I_{0+}$ if $\eta_j^H < 0$ and $k \in I_{+0}$ if $\eta_k^G > 0$ at a strongly stationary point. With the MPVC-Lagrangian from (7.1), we can easily calculate that

$$
\begin{aligned}
\nabla_x L_{P(0)}(x^*, \lambda^*, \mu^*, \alpha^*, \beta^*, \gamma^*) \quad = \quad & \nabla f(x^*) + \sum_{i=1}^{l} \lambda^* \nabla g_i(x^*) + \sum_{j \in J} \mu_j^* \nabla h_j(x^*) \\
& - \sum_{i:\eta_i^H > 0} \alpha_i^* \nabla H_i(x^*) + \sum_{j:\eta_j^H < 0} \beta_j^* \nabla \theta_j(x^*) + \sum_{k:\eta_k^G > 0} \gamma_k^* \nabla \theta_k(x^*) \\
= \quad & \nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{j \in J} \mu_j \nabla h_j(x^*) \\
& - \sum_{i:\eta_i^H > 0} \eta_i^H \nabla H_i(x^*) - \sum_{j:\eta_j^H < 0} \eta_j^H \nabla H_j(x^*) + \sum_{k:\eta_k^G > 0} \eta_k^G \nabla G_k(x^*) \\
= \quad & \nabla_x L(x^*, \lambda, \mu, \eta^G, \eta^H) \\
= \quad & 0,
\end{aligned}
$$

due to the fact that $(x^*, \lambda, \mu, \eta^G, \eta^H)$ is a strongly stationary point of (1.1). Taking into account the properties of the multipliers defined in (10.9), we see that $(x^*, \lambda^*, \mu^*, \alpha^*, \beta^*, \gamma^*)$ is a KKT point of $P(0)$. In addition to that it is immediately clear from the MPVC-LICQ assumption that standard LICQ holds at x^* for $P(0)$. We will now verify that it also satisfies the strong second order sufficient

condition (SSOSC) in the sense of [51] and Definition 2.3.2. To this, we note that

$$
\begin{aligned}
\nabla^2_{xx} & L_{P(0)}(x^*, \lambda^*, \mu^*, \alpha^*, \beta^*, \gamma^*) \\
= \ & \nabla^2 f(x^*) + \sum_{i:\lambda_i>0} \lambda_i \nabla^2 g_i(x^*) + \sum_{j\in J} \mu_j \nabla^2 h_j(x^*) \\
& - \sum_{i:\eta_i^H>0} \eta_i^H \nabla^2 H_i(x^*) - \sum_{i:\eta_i^H<0} \eta_i^H \nabla^2 H_i(x^*) + \sum_{k:\eta_k^G>0} \eta_k^G \nabla^2 G_i(x^*) \\
& + \sum_{j:\eta_j^H<0} \beta_j^* (\nabla G_j(x^*)\nabla H_j(x^*)^T + \nabla H_j(x^*)\nabla G_j(x^*)^T) \\
& + \sum_{k:\eta_k^G>0} \gamma_k^* (\nabla G_j(x^*)\nabla H_j(x^*)^T + \nabla H_j(x^*)\nabla G_j(x^*)^T) \\
= \ & \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) \\
& + \sum_{j:\eta_j^H<0} \beta_j^* (\nabla G_j(x^*)\nabla H_j(x^*)^T + \nabla H_j(x^*)\nabla G_j(x^*)^T) \\
& + \sum_{k:\eta_k^G>0} \gamma_k^* (\nabla G_k(x^*)\nabla H_k(x^*)^T + \nabla H_k(x^*)\nabla G_k(x^*)^T),
\end{aligned}
\tag{10.10}
$$

while the critical cone for $P(0)$ at $(x^*, \lambda^*, \mu^*, \alpha^*, \beta^*, \gamma^*)$ is given by

$$
\begin{aligned}
C_{P(0)}(x^*) \ = \ & \{d \in \mathbb{R}^n \ \mid \
\begin{aligned}
\nabla g_i(x^*)^T d \ &= 0 \quad (i : \lambda_i^* > 0), \\
\nabla h_j(x^*)^T d \ &= 0 \quad (j \in J), \\
\nabla H_i(x^*)^T d \ &= 0 \quad (i : \alpha_i^* > 0), \\
\nabla \theta_j(x^*)^T d \ &= 0 \quad (j : \beta_j^* > 0), \\
\nabla \theta_k(x^*)^T d \ &= 0 \quad (k : \gamma_k^* > 0)\}
\end{aligned} \\[2mm]
= \ & \{d \in \mathbb{R}^n \ \mid \
\begin{aligned}
\nabla g_i(x^*)^T d \ &= 0 \quad (i : \lambda_i > 0), \\
\nabla h_j(x^*)^T d \ &= 0 \quad (j \in J), \\
\nabla H_i(x^*)^T d \ &= 0 \quad (i : \eta_i^H > 0), \\
\nabla H_j(x^*)^T d \ &= 0 \quad (j : \eta_j^H < 0), \\
\nabla G_k(x^*)^T d \ &= 0 \quad (k : \eta_k^G > 0)\}
\end{aligned} \\[2mm]
= \ & \{d \in \mathbb{R}^n \ \mid \
\begin{aligned}
\nabla g_i(x^*)^T d \ &= 0 \quad (i : \lambda_i > 0), \\
\nabla h_j(x^*)^T d \ &= 0 \quad (j \in J), \\
\nabla H_i(x^*)^T d \ &= 0 \quad (i : \eta_i^H \neq 0), \\
\nabla G_i(x^*)^T d \ &= 0 \quad (i : \eta_i^G \neq 0)\}
\end{aligned} \\[2mm]
= \ & C(x^*).
\end{aligned}
$$

Now, let $d \in C_{P(0)}(x^*)$ be chosen arbitrarily. Then, in view of (10.10) and since MPVC-SSOSC holds, we have that

$$
\begin{aligned}
d^T \nabla^2_{xx} L_{P(0)}(x^*, \lambda^*, \mu^*, \alpha^*, \beta^*, \gamma^*) d &= d^T \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) d \\
&\quad + 2 \sum_{i:\eta_i^H < 0} \beta_i^* (\nabla G_i(x^*)^T d) \underbrace{(\nabla H_i(x^*)^T d)}_{=0} \\
&\quad + 2 \sum_{i:\eta_i^G > 0} \gamma_i^* \underbrace{(\nabla G_i(x^*)^T d)}_{=0} (\nabla H_i(x^*)^T d) \\
&= d^T \nabla^2_{xx} L(x^*, \lambda, \mu, \eta^G, \eta^H) d \\
&> 0,
\end{aligned}
\tag{10.11}
$$

and thus, SSOSC holds for $P(0)$ at x^*. Now, since x^* also satisfies LICQ for $P(0)$, as was argued above, we may now invoke [57, Th. and Prop. 5.2.1] to obtain a locally (around x^*) unique and piecewise smooth KKT point function $x(t)$ and a piecewise smooth multiplier function $(\lambda, \mu, \alpha, \beta, \gamma)(t)$ for $P(t)$. Furthermore, $x(t)$ is a local minimizer of $P(t)$ satisfying SSOSC, see [57, Prop. 5.2.1] and its proof. Now, we show that $x(t)$ is also feasible for $NLP(t)$ for $t > 0$ sufficiently small:

First, let $i \in I_{00} \cup I_{0-}$. Then, by SULSCC, we know that $\eta_i^H > 0$, thus by continuity we have $\alpha_i(t) > 0$ for t sufficiently small. This yields $H_i(x(t)) = 0$ and thus $\theta_i(x(t)) = 0 < t$.

Now, choose $i \in I_{0+}$. Then, by SULSCC, we get $\eta_i^H \neq 0$. If $\eta_i^H > 0$, we may argue as before to obtain $H_i(x(t)) = \theta_i(x(t)) = 0 < t$. On the other hand, if $\eta_i^H < 0$, we get $\beta_i(t) > 0$ for t sufficiently small and thus $\theta_i(x(t)) = t$. Since $G_i(x(t)) > 0$ for all sufficiently small $t > 0$, this also implies $H_i(x(t)) > 0$ for all these t.

For $i \in I_{+-}$, it follows immediately from the continuity of $x(\cdot)$ that $H_i(x(t)) > 0$ and $G_i(x(t)) < 0$ and thus $\theta_i(x(t)) < 0 < t$ for t sufficiently small.

Eventually, we pick $i \in I_{+0}$. Then, by continuity arguments it follows that $H_i(x(t)) > 0$ for $t > 0$ sufficiently small. Furthermore, since we have $\eta_i^G > 0$ due to SULSCC, we get $\gamma_i(t) > 0$ for t sufficiently small and thus $\theta_i(x(t)) = t$.

This yields the feasibility of $x(t)$ for $NLP(t)$ for $t > 0$ sufficiently small. And since the feasible set of $NLP(t)$ is contained in the feasible set of $P(t)$ and $x(t)$ is a local minimizer of $P(t)$ for $t > 0$ sufficiently small, $x(t)$ is also a local minimizer of $NLP(t)$ for t sufficiently small. Due to Lemma 9.4.5, $x(t)$ also satisfies LICQ for $NLP(t)$ and thus, $x(t)$ is a KKT point of $NLP(t)$ for t sufficiently small with unique multipliers. At this, $x(t)$ satisfies SSOSC for $NLP(t)$, too, since it fulfills these conditions for the program $P(t)$, which is obtained from $NLP(t)$ by deleting some constraints. It only remains to show that $x(t)$ is the unique KKT point of $NLP(t)$ near x^*. For these purposes, suppose that $(x^t, \lambda^t, \mu^t, \rho^t, \nu^t)$ is a KKT point of $NLP(t)$ with $x^t \to x^*$ and $x^t \neq x(t)$. Then the KKT

conditions for *NLP(t)* and Corollary 10.2.5 yield that for *t* sufficiently small we have

$$
\begin{aligned}
0 &= \nabla f(x^t) + \sum_{i:g_i(x^t)=0} \lambda^t \nabla g_i(x^t) + \sum_{j \in J} \mu_j \nabla h_j(x^t) \\
&\quad - \sum_{i:H_i(x^t)=0} \rho_i^t \nabla H_i(x^t) + \sum_{i:\theta_i(x^t)=t} \nu_i^t \nabla \theta_i(x^t) \\
&= \nabla f(x^t) + \sum_{i:g_i(x^t)=0} \lambda^t \nabla g_i(x^t) + \sum_{j \in J} \mu_j \nabla h_j(x^t) \\
&\quad - \sum_{i:\eta_i^H>0} \rho_i^t \nabla H_i(x^t) + \sum_{i:\eta_i^G>0} \nu_i^t \nabla \theta_i(x^t) + \sum_{i:\eta_i^H<0} \nu_i^t \nabla \theta_i(x^t) \\
&= \nabla_x L_{P(t)}(x^t, \lambda^t, \mu^t, \alpha^t, \beta^t, \gamma^t),
\end{aligned}
\tag{10.12}
$$

where we define

$$
\alpha_i^t := \rho_i^t \ (i : \eta_i^H > 0), \quad \beta_i^t := \nu_i^t \ (i : \eta_i^H < 0), \quad \gamma_i^t := \nu_i^t \ (i : \eta_i^G > 0).
$$

This shows that x^t is a KKT point of $P(t)$ for t sufficiently small, in contradiction to the fact that $x(t)$ is the unique KKT point for $P(t)$ near x^*. This eventually concludes the proof. $\qquad \square$

The above result and its proof are borrowing from ideas that were established on the MPEC field in [58]. In [31, Th. 5.4] one can find another result for MPVCs, very similar to ours. Thus now, we want to discuss the similarities and differences between these two results: At first glance, we formulate our results in terms of stationary points, whereas in [31, Th. 5.4], the accent lies on local solutions. But it is quickly argued, that under the second-order conditions which both theorems assume, these two concepts are equivalent and hence, so are large parts of the assumptions and assertions of the two theorems. Moreover, the just mentioned second-order conditions are, under the assumed SULSCC or ULSCC plus $I_{0+} = \emptyset$, the same, as was already mentioned earlier. One (minor) difference, where our result somehow exceeds the assertions of [31, Th. 5.4], is the fact that our local KKT-point-mapping (or solution mapping) $x(t)$ is shown to be piecewise smooth, which is stronger than local Lipschitz continuity. In turn, we are quite sure that one could also extend the proof of [31, Th. 5.4] in a way such that it yields piecewise smoothness of the solution function, too. A more important advantage of our theorem is that we assume SULSCC, which is strictly weaker than ULSCC plus $I_{0+} = \emptyset$. The authors of [31] also discuss the case of assuming SULSCC, but for some reasons they lose local uniqueness of there solutions then, which is, in a sense, a more serious drawback.

We would like to finish this section with a convergence result which combines Theorem 10.2.1, Corollary 10.2.2 and Theorem 10.2.10. At this, we formulate our statement, more or less, from the viewpoint of an algorithm, by using iterates x^k for $k \in \mathbb{N}$ instead of x^t for $t > 0$.

Corollary 10.2.11 *Let the assumptions of Theorem 10.2.10 hold. Let $\{t_k\}_{k \in \mathbb{N}}$ be a sequence with $t_k \downarrow 0$ and $\{(x^k, \lambda^k, \mu^k, \rho^k, \nu^k)\}_{k \in \mathbb{N}}$ a corresponding sequence of KKT points of $NLP(t_k)$. Then there exists an open neighbourhood U of x^* such that, if for any k we have $x^k \in U$, it holds that $(x^k, \lambda^k, \mu^k, \eta^{G,k}, \eta^{H,k}) \to (x^*, \lambda, \mu, \eta^G, \eta^H)$, where $\eta_i^{G,k} := H_i(x^k)\nu_i^k$ and $\eta_i^{H,k} := \rho_i^k - G_i(x^k)\nu_i^k$.*

Final remarks

This thesis contains an exhaustive treatment of the very new class of mathematical programs with vanishing constraints, also being the first comprehensive text on this topic.

Starting off, it is shown that MPVCs are a proper framework to model (and solve) problems from truss topology optimization, displaying its relevance from the viewpoint of applications. Moreover, MPVCs are compared to mathematical programs with equilibrium constraints, coining the fact that MPECs are even more ill-posed and hence, the possible reformulation of an MPVC as an MPEC is not recommended and the analysis of the MPVC itself is additionally justified.

One emphasis in the analysis of MPVCs lies on constraint qualifications and stationarity concepts. At this, it is argued that all standard CQs but the Guignard CQ are too restrictive for MPVCs and hence, the KKT conditions do not offhand provide necessary optimality conditions. In turn, the situation is not quite as bad as for MPECs. Nevertheless, new and more problem-tailored constraint qualifications are established, their relations, also to standard CQs, are analyzed and it is investigated which stationarity conditions they yield. In this context the concept of M-stationarity, being weaker than KKT conditions, comes into play and it is shown that all MPVC-tailored constraint qualifications yield, at least, M-stationarity as a first-order optimality condition.

In addition to first-order necessary criteria, first-order sufficient optimality results for convex-type (but still nonconvex) MPVCs are proven.

Complementing the first-order analysis, second-order optimality conditions are presented showing that one can use the same critical cone for both necessary and sufficient conditions.

Furthermore, an MPVC-tailored penalty function is constructed, which is shown to be exact under MPVC-MFCQ. This penalty function is then used to recover M-stationarity as a necessary optimality condition for MPVCs.

In order to tackle the MPVC in terms of numerical computations two algorithms are presented and investigated. The first one is based on smoothing and regularization techniques, where the basic idea is borrowed from a comparable algorithm for the numerical solution of MPECs. The MPVC-tailored algorithm, however, is shown to have substantially better convergence properties than its MPEC analogon, another hint for the fact that MPECs are the more difficult class of problems.

The second algorithm is a pure relaxation scheme which was, in a similar fashion, also investigated for MPECs. At this, the convergence theory, like for MPECs, also allows for a satisfactory stability result.

Abbreviations

ACQ	Abadie constraint qualification
CQ	constraint qualification
GCQ	Guignard constraint qualification
KKT	Karush-Kuhn-Tucker
KTCQ	Kuhn-Tucker constraint qualification
lsc	lower semicontinuous
LICQ	linear independence constraint qualification
MFCQ	Mangasarian-Fromovitz constraint qualification
MPEC	mathematical program with equilibrium constraints
MPVC	mathematical program with vanishing constraints
NLP	nonlinear program
SCQ	Slater constraint qualification
SSOSC	strong second-order sufficient condition
TNLP	tightened nonlinear program
WSCQ	weak slater constraint qualification

Notation

Number sets

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the nonnegative real numbers
$\mathbb{R}_-$	the nonpositive real numbers

MPVC-related sets

X	the feasible set of the MPVC
J	$\{1, \ldots, p\}$
J	$\{1, \ldots, p\}$
I_g	$\{i \mid g_i(x^*) = 0\}$
I_0	$\{i \mid H_i(x^*) = 0\}$
I_+	$\{i \mid H_i(x^*) > 0\}$
I_{0+}	$\{i \in I_0 \mid G_i(x^*) > 0\}$
I_{00}	$\{i \in I_0 \mid G_i(x^*) = 0\}$
I_{0-}	$\{i \in I_0 \mid G_i(x^*) < 0\}$
I_{+0}	$\{i \in I_+ \mid G_i(x^*) = 0\}$
I_{+-}	$\{i \in I_+ \mid G_i(x^*) < 0\}$
$\mathcal{P}(I_{00})$	the set of all partitions of I_{00}

Other set-related symbols

$S \cup T$	the union of the sets S and T
$S \setminus T$	the set consisting of the points which are in S and not in T
$S \times T$	the cartesian product of the sets S and T
S^n	the $n-$fold cartesian product of the set S
$\{x\}$	the set consisting of the point x
$\text{conv}(S)$	the convex hull of the set S
$B_\varepsilon(x)$	open ball with radius ε around x
$\mathbb{B} \subset \mathbb{R}^n$	closed unit ball (in $\mathbb{R}^n$) around the origin
$|S|$	cardinality of the set S

Vectors

$x \in \mathbb{R}^n$	column vector in $\mathbb{R}^n$
(x, y)	column vector $(x^T, y^T)^T$
$e^i \in \mathbb{R}^n$	the i-th unnit vector in $\mathbb{R}^n$
$e \in \mathbb{R}^n$	the vector (in $\mathbb{R}^n$) of all ones

Cones

$\mathcal{T}(x, S)$	the (Bouligand) tangent cone of S at x
$\mathcal{T}(x^*)$	tangent cone of the MPVC (1.1) at $x^* \in X$
$\mathcal{L}(x)$	the linearized cone (to a feasible set of an NLP) at x
$\mathcal{L}_{MPVC}(x^*)$	the MPVC-linearized cone at $x^* \in X$
$\mathcal{F}(x, S)$	the cone of feasible directions of S at x
$\mathcal{F}(x^*)$	the cone of feasible directions of the MPVC (1.1) at $x^* \in X$
$\mathcal{A}(x^*)$	the cone of attainable directions of the MPVC (1.1) at $x^* \in \mathbb{X}$
$\mathcal{A}(x, S)$	the cone of attainable directions of S at x
$\hat{N}(x, S)$	the Fréchet normal cone to S at x
$N(x, S)$	the limiting normal cone to S at x
S°	the polar cone of the set S
S^*	the dual cone of the set S

Functions

$f : \mathbb{R}^n \to \mathbb{R}^m$	a function that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$
$\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$	a multifunction that maps from $\mathbb{R}^n$ to the power set of $\mathbb{R}^m$
$\mathrm{gph}\Phi$	graph of the multifunction Φ
$\nabla f(x)$	gradient of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$\nabla^2 f(x)$	Hessian of a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$f'(x)$	Jacobian of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at x
$\hat{\partial} f(x)$	Fréchet subdifferential of an lsc function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$\partial f(x)$	limiting subdifferential of an lsc function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$\partial^B f(x)$	Bouligand subdifferential of locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$\partial^{Cl} f(x)$	Clarke's generalized gradient of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at x
$\|x\|$	(an arbitrary l_p-) norm of the vector x
$d_C(x)$	distance between the vector x and the closed set C (w.r.t. $\| \cdot \|$)
$\mathrm{Proj}_C(x)$	(possibly set-valued) projection of the vector x on the closed set C (w.r.t. $\| \cdot \|$)

Sequences

$\{a^k\}$	a sequence in $\mathbb{R}^n$
$\{a^k\} \to a$	a convergent sequence with limit a
$a^k \to a$	the sequence $\{a^k\}$ converges to a
$a^k \downarrow a$	a convergent sequence in $\mathbb{R}$ with limit a and $a^k > a$ for all $k \in \mathbb{N}$
$a^k \uparrow a$	a convergent sequence in $\mathbb{R}$ with limit a and $a^k < a$ for all $k \in \mathbb{N}$
$\lim\limits_{k \to \infty} a^k$	limit of a convergent sequence $\{a^k\}$

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