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VON MISES CONDITIONS REVISITED

BY MICHAEL FALK AND FRANK MAROHN

Katholische Universität Eichstätt

It is shown that the rate of convergence in the von Mises conditions of extreme value theory determines the distance of the underlying distribution function F from a generalized Pareto distribution. The distance is measured in terms of the pertaining densities with the limit being ultimately attained if and only if F is ultimately a generalized Pareto distribution.

Consequently, the rate of convergence of the extremes in an iid sample, whether in terms of the distribution of the largest order statistics or of corresponding empirical truncated point processes, is determined by the rate of convergence in the von Mises condition. We prove that the converse is also true.

0. Introduction. Let $X_1, \dots, X_n$ be iid random variables with common distribution function ($\equiv$ df) F and denote by $X_{1:n} \le \cdots \le X_{n:n}$ the corresponding order statistics.

It is well known that $(X_{n:n} - b_n)/a_n$ converges in distribution to some nondegenerate limiting distribution G for some choice of constants $a_n > 0$, $b_n \in \mathbb{R}$ if and only if for any integer k the random vector $((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k$ converges in distribution to $G^{(k)}$ having Lebesgue density $g^{(k)}(x_1, \dots, x_k) = G(x_k)\prod_{i=1}^k G'(x_i)/G(x_i)$ if $x_1 > \cdots > x_k$ and zero elsewhere [Dwass (1966) and Weissman (1975); a corresponding result for the variational distance is proved in Theorem 5.3.4 in Reiss (1989)]. In this case we say that *F is in the domain of attraction of G*, that is, $F \in \mathscr{D}(G)$.

Since the classical article by Gnedenko (1943) it is known that G must be of one of the following types, where $\alpha > 0$: $G_{1,\alpha}(x) := \exp(-x^{-\alpha})$, $x > 0$, (Fréchet), $G_{2,\alpha}(x) := \exp(-(-x)^{\alpha})$, $x \le 0$, (reversed Weibull) and $G_3(x) := \exp(-e^{-x})$, $x \in \mathbb{R}$, (Gumbel distribution).

Gnedenko (1943) gave in addition necessary and sufficient conditions for F to belong to the domain of attraction of each of the above limits. The specification of the auxiliary function in Gnedenko's characterization is due to de Haan (1970). These necessary and sufficient conditions are a bit complex, whereas the following sufficient conditions which are due to von Mises (1936) proved to be widely and easily applicable.

Assume that F has a positive derivative f on $[x_0, \omega(F))$, where $x_0 < \omega(F) := \sup\{x \in \mathbb{R}\colon F(x) < 1\}$. Then $F \in \mathscr{D}(G_{1,\alpha})$, $\mathscr{D}(G_{2,\alpha})$, $\mathscr{D}(G_3)$ if (VM1),

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(VM2) and (VM3) is satisfied, respectively, where

(VM1) $$\omega(F) = \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{xf(x)}{1 - F(x)} = \alpha,$$

(VM2) $$\omega(F) < \infty \quad \text{and} \quad \lim_{x \uparrow \omega(F)} \frac{(\omega(F) - x) f(x)}{1 - F(x)} = \alpha,$$

(VM3) $$\int_{-\infty}^{\omega(F)} (1 - F(u))\, du < \infty,$$
$$\lim_{x \uparrow \omega(F)} f(x) \int_x^{\omega(F)} (1 - F(u))\, du / (1 - F(x))^2 = 1.$$

It will later turn out that the following strengthening of (VM3) will be crucial for the case $F \in \mathscr{D}(G_3)$. Suppose that F has a positive derivative f on $[x_0, \omega(F))$ such that for some $c \in (0, \infty)$

(VM3′) $$\lim_{x \to \omega(F)} \frac{f(x)}{1 - F(x)} = c.$$

Note that $(1 - F(x))' = -f(x)$ and $(\int_x^{\omega(F)}(1 - F(u))\, du)' = -(1 - F(x))$ and thus, (VM3′) implies (VM3) by l'Hôpital's rule.

By using the well-known fact that $\lim_{x \to \infty} x(1 - \Phi(x))/\varphi(x) = 1$ for the standard normal df Φ and its density φ, it is, for example, straightforward to show that Φ satisfies (VM3) but not (VM3′). On the other hand, distributions with differentiable upper tail of Gamma type, that is, $\lim_{x \to \infty} F'(x)/((b^p/\Gamma(p))e^{-bx}x^{p-1}) = 1$ with $b, p > 0$ satisfy (VM3′), whereas F with F' of Weibull type, that is, $\lim_{x \to \infty} F'(x)/(bpx^{p-1}\exp(-bx^p)) = 1$ with $b, p > 0$ and $p \neq 1$ satisfies (VM3) but not (VM3′). With $p = 1$ we obtain the particular (gamma) case of an exponential type distribution which satisfies (VM3′).

(VM1) is, for example, satisfied for F with differentiable upper tail of Cauchy type, whereas triangular-type distributions satisfy (VM2).

Since the von Mises conditions require differentiability of F, one can analyze the implications of these conditions among those df's whose densities exist. Consequently, it was shown by Falk (1985) that (VM1)–(VM3) imply pointwise convergence of the density $f_n^{(k)}(x)$ of the distribution of $((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k$ to $g^{(k)}(x)$ as n tends to infinity for any $k \in \mathbb{N}$, which, by Scheffé's lemma, entails its uniform convergence, that is, uniformly over all Borel sets, to $G^{(k)}$. By using the concept of regularly varying functions, Sweeting (1985) showed that (VM1)–(VM3) are actually equivalent to the locally uniform convergence of $f_n^{(k)}$ to $g^{(k)}$. A closely related characterization requiring higher derivatives of F was established by Pickands (1986). For a discussion of earlier related results we refer to Section 2.11 of the monograph by Galambos (1987).

Denote now by $\{W_{1,\alpha}, W_{2,\alpha}, W_3\colon\ \alpha > 0\}$ the class of generalized Pareto distribution functions ($\equiv$ gPdfs), that is, $W(x) := 1 + \log(G(x))$ if $1/e \le G(x) \le 1$, yielding

$$\begin{aligned} W_{1,\alpha}(x) &= 1 - x^{-\alpha}, && x > 1,\\ W_{2,\alpha}(x) &= 1 - (-x)^{\alpha}, && x \in [-1,0],\\ W_3(x) &= 1 - \exp(-x), && x \ge 0. \end{aligned}$$

Notice that $W_{1,\alpha}$ is the standard Pareto distribution, W_3 the standard exponential and $W_{2,1}$ the uniform distribution on $[-1,0]$. The importance of gPds in extreme value theory was first observed by Pickands (1975), who showed that, roughly speaking, F belongs to the domain of attraction of an extreme value distribution G if, and only if, the conditional distribution $(F(x) - F(x_0))/(1 - F(x_0))$ is approximately given by an appropriately shifted gPd if x_0 is large [see also Reiss (1989), Theorem 5.1.1].

One of the significant properties of gPd's is the fact that only in this case the uniform distance between the distribution $F_n^{(k)}$ of $((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k$ and its limiting distribution $G^{(k)}$ is of order $O(k/n)$ for any $k \in \{1,\dots,n\}$, $n \in \mathbb{N}$ [Reiss (1981), Theorem 2.6 and 3.2 and Falk (1989), Theorem 10]. Actually, gPd's are the only distributions under which the Kolmogorov–Smirnov distance between $F_n^{(k)}$ and $G^{(k)}$ converges to zero for *any* sequence $k = k(n)$ of integers satisfying $k/n \to_{n\to\infty} 0$ [Falk (1990), Theorem 2].

It is therefore obvious that gPd's do not only play a crucial role in extreme value theory if the joint distribution of the k largest observations is considered, which is usually done in statistical applications [see, e.g. Smith (1987)]; in addition, gPd's provide a useful unifying approach to the rate of convergence of extremes, since the rate of convergence of $((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k$ to $G^{(k)}$ is determined by the (one-dimensional) distance of the underlying df F from the corresponding gPd $W = 1 + \log G$ [Falk (1985), (1986) and Reiss (1989), Section 5.2].

Next observe that the limiting values α and 1 are *attained* for x near $\omega(F)$ if a suitable gPd is plugged into the von Mises condition (VM1)–(VM3′). In view of the preceding considerations one might therefore conjecture that the rate at which the limit is attained in (VM1)–(VM3′) determines the distance of F from a corresponding gPd. The verification of this conjecture, which we will carry out in Section 2, is a major goal of the present manuscript. As an immediate consequence, the rate of convergence in (VM1)–(VM3′) determines the rate of convergence of extremes, measured by the uniform distance

$$d\big(F_n^{(k)}, G^{(k)}\big) := \sup_{B\in\mathbb{B}^k} \Big| P\Big\{((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k \in B\Big\} - G^{(k)}(B)\Big|,$$

where $\mathbb{B}^k$ denotes the Borel-σ-algebra on $\mathbb{R}^k$ and $F_n^{(k)}$ the distribution of the vector $((X_{n-i+1:n} - b_n)/a_n)_{i=1}^k$ (see Section 3).

In Section 1 we will establish in Lemma 1.1 an equivalent formulation of the von Mises conditions (VM1)–(VM3) in terms of expansions of densities, which will be a basic auxiliary result for all subsequent ones. As a consequence of Lemma 1.1 we can show that the limits in (VM1)–(VM3) are attained for x near $\omega(F)$ if, and only if, F is ultimately a gPd. In Theorem 2.2 of the present paper we will then utilize Lemma 1.1 to prove that the distance $d(F_n^{(k)}, G^{(k)})$ is determined by the rate of convergence in (VM1)–(VM3′). In Section 3 we will prove that the converse is also true.

1. An alternative formulation of von Mises conditions via densities of gPds. Denote by $w_{1,\alpha}, w_{2,\alpha}, w_3$ the densities of $W_{1,\alpha}, W_{2,\alpha}, W_3$, that is,

$$\begin{aligned} w_{1,\alpha}(x) &= \alpha x^{-(1+\alpha)}, && x \geq 1, \\ w_{2,\alpha}(x) &= \alpha(-x)^{\alpha-1}, && x \in [-1,0], \\ w_3(x) &= \exp(-x), && x \geq 0. \end{aligned}$$

The following equivalent formulations of (VM1)–(VM3′) will be a basic tool for the derivation of our subsequent results.

1.1 Lemma. *Assume that F has a positive density f on $[x_0, \omega(F))$ for some $x_0 < \omega(F)$ and let the remainder functions η: $[x_0, \omega(F)) \to \mathbb{R}$ defined below satisfy $\eta(x) \to 0$ as $x \to \omega(F)$.*

(i) *Suppose that $\omega(F) = \infty$. Then*

$$\frac{xf(x)}{1-F(x)} = \alpha(1+\eta_1(x)), \qquad x \geq x_0,$$

if and only if the conditional density $f(x)/(1-F(x_0))$ admits the expansion

$$\frac{f(x)}{1-F(x_0)} = \frac{w_{1,\alpha}(x)}{1-W_{1,\alpha}(x_0)}(1+\eta_1(x))l_{1,\alpha}(x), \qquad x \geq x_0,$$

where

$$l_{1,\alpha}(x) = \exp\left\{-\alpha\int_{x_0}^{x}(\eta_1(t)/t)\,dt\right\}.$$

(ii) *Suppose that $\omega(F) < \infty$. Then*

$$\frac{(\omega(F)-x)f(x)}{1-F(x)} = \alpha(1+\eta_2(x)), \qquad x \in [x_0, \omega(F)),$$

if and only if

$$\frac{f(x)}{1-F(x_0)} = \frac{w_{2,\alpha}(x-\omega(F))}{1-W_{2,\alpha}(x_0-\omega(F))}(1+\eta_2(x))l_{2,\alpha}(x), \qquad x \in [x_0, \omega(F)),$$

where

$$l_{2,\alpha}(x) = \exp\left\{-\alpha\int_{x_0}^{x}\eta_2(t)/(\omega(F)-t)\,dt\right\}.$$

(iii) *Suppose that* $\int_{x_0}^{\omega(F)}(1 - F(u))\,du < \infty$. *Define*

$$U(x) := \int_x^{\omega(F)}(1 - F(t))\,dt/(1 - F(x))$$

and

$$H(x) := (1 + \eta_3(x))/U(x).$$

Then

$$f(x)\int_x^{\omega(F)}(1 - F(u))\,du/(1 - F(x))^2 = 1 + \eta_3(x), \qquad x \in [x_0, \omega(F)),$$

if and only if

$$\frac{f(x)}{1 - F(x_0)} = \frac{w_3(x)}{1 - W_3(x_0)} H(x)\exp\left\{-\int_{x_0}^{x}(H(t) - 1)\,dt\right\}, \qquad x \in [x_0, \omega(F)).$$

In this case we have

$$H(x) = \frac{f(x)}{1 - F(x)}.$$

(iv) *Let* $\omega(F)$ *be arbitrary. We have for some* $c > 0$,

$$\frac{f(x)}{1 - F(x)} = c(1 + \eta_{3'}(x)), \qquad x \in [x_0, \omega(F))$$

if and only if

$$\frac{f(x)}{1 - F(x_0)} = \frac{cw_3(cx)}{1 - W_3(cx_0)}(1 + \eta_{3'}(x))l_{3'}(x),$$

where

$$l_{3'}(x) = \exp\left\{-c\int_{x_0}^{x}\eta_{3'}(t)\,dt\right\}.$$

Lemma 1.1 entails a formulation of the von Mises conditions in terms of gPd's. Note that the functions l are slowly varying. We remark that an alternative formulation relating F to a gPd W in terms of differentiably slow varying remainders might be possible; for the definition of the class of regularly varying functions of higher differentiable order and their implications for asymptotic expansions in extreme value theory we refer to Sweeting (1989). The present formulation of the auxiliary result 1.1 with specified remainder function $l(x)$ is, however, especially tailored for later purposes; in particular the present formulation implies that the density $f(x)$ of F is asymptotically equivalent with the density of a suitably shifted gPd as x tends to $\omega(F)$, if and only if the remainder function $l(x)$ has a limit in $\mathbb{R}$ as $x \to \omega(F)$ (cf. Proposition 2.1).

Letting η of Lemma 1.1 be identically zero, we obtain that the limits in (VM1)–(VM3′) are attained for x near $\omega(F)$ if and only if F is ultimately a gPd. Notice that the function U defined in (iii) satisfies $(d/dx)U(x) = \eta_3(x)$ for $x \geq x_0$.

1.2 COROLLARY.

(i) *The condition* $xf(x)/(1 - F(x)) = \alpha$ *for all* $x \geq x_0$ *is equivalent to*

$$f(x) = c_0^{-1/\alpha} w_{1,\alpha}(c_0^{-1/\alpha} x)$$

for $x \in [x_0, \infty)$, *where* $c_0 = (1 - F(x_0))/(1 - W_{1,\alpha}(x_0))$.

(ii) *The condition* $(\omega(F) - x)f(x)/(1 - F(x)) = \alpha$ *for all* $x \in [x_0, \omega(F))$ *is equivalent to*

$$f(x) = c_0^{1/\alpha} w_{2,\alpha}(c_0^{1/\alpha}(x - \omega(F)))$$

for $x \in [x_0, \omega(F))$, *where* $c_0 = (1 - F(x_0))/(1 - W_{2,\alpha}(x - \omega(F)))$.

(iii) *The condition* $f(x)\int_x^{\omega(F)}(1 - F(u))\,du/(1 - F(x))^2 = 1$ *for all* $x \in [x_0, \omega(F))$ *is equivalent to*

$$f(x) = c_0 w_3(c_0(x - x_0) - \log(1 - F(x_0)))$$

for $x \in [x_0, \omega(F))$ *with* $c_0 = f(x_0)/(1 - F(x_0))$.

(iv) *The condition* $f(x)/(1 - F(x)) = c \in (0, \infty)$ *for all* $x \in [x_0, \omega(F))$ *is equivalent to*

$$f(x) = cw_3(c(x - x_0) - \log(1 - F(x_0))), \qquad x \in [x_0, \omega(F)).$$

For the proof of Lemma 1.1 we need the following lemma.

1.3 LEMMA. *Let g and h be positive measurable functions defined on some interval $[x_0, x_1)$ such that $g(u)/u$ and $h(u)/u$ are integrable on $[x_0, x_1)$. If*

$$g(x)\exp\left\{\int_{x_0}^x g(u)/u\,du\right\} = h(x)\exp\left\{\int_{x_0}^x h(u)/u\,du\right\}, \qquad x_0 \leq x < x_1,$$

then $g = h$.

PROOF. Define for $x \in [x_0, x_1)$

$$G(x) = \exp\left\{\int_{x_0}^x g(u)/u\,du\right\}$$

and

$$H(x) = \exp\left\{\int_{x_0}^x h(u)/u\,du\right\}.$$

Then G and H are absolutely continuous functions with derivative

$$G'(x) = \frac{g(x)}{x}\exp\left\{\int_{x_0}^x \frac{g(u)}{u}\,du\right\},$$

$$H'(x) = \frac{h(x)}{x}\exp\left\{\int_{x_0}^x \frac{h(u)}{u}\,du\right\},$$

for Lebesgue almost any $x \in [x_0, x_1)$. Hence $(G - H)' = 0$ almost everywhere and $G(x_0) = H(x_0) = 0$ implies $G \equiv H$. But then we have

$$\int_{x_0}^{x} g(u)/u\,du = \int_{x_0}^{x} h(u)/u\,du$$

and thus

$$g(x)/x = h(x)/x \quad \text{almost everywhere.}$$

Since g/h is a continuous function we conclude that $g(x) = h(x)$ for all $x \in [x_0, x_1)$. □

PROOF OF LEMMA 1.1. (i): We have

$$\begin{aligned}\frac{f(x)}{1 - F(x_0)} &= \frac{\alpha x^{-(1+\alpha)}}{x_0^{-\alpha}} \frac{xf(x)}{\alpha(1 - F(x))} \frac{1 - F(x)}{1 - F(x_0)} \frac{x_0^{-\alpha}}{x^{-\alpha}} \\ &= \frac{w_{1,\alpha}(x)}{1 - W_{1,\alpha}(x_0)} (1 + \eta_1(x)) \frac{1 - F(x)}{1 - F(x_0)} \frac{1 - W_{1,\alpha}(x_0)}{1 - W_{1,\alpha}(x)}.\end{aligned}$$

By elementary calculations, we get

$$-\alpha \int_{x_0}^{x} \eta_1(t)/t\,dt = \log \frac{1 - F(x)}{1 - F(x_0)} + \log \frac{1 - W_{1,\alpha}(x_0)}{1 - W_{1,\alpha}(x)}.$$

Taking into account the definition of $l_{1,\alpha}$, the "if" part of the assertion is shown.

The converse is proved if

$$\begin{aligned}&\frac{xf(x)}{1 - F(x)} \exp\left\{ -\int_{x_0}^{x} \frac{f(u)}{1 - F(u)}\,du \right\} \\ &\qquad = \alpha(1 + \eta_1(x)) \exp\left\{ -\int_{x_0}^{x} \frac{\alpha(1 + \eta_1(u))}{u}\,du \right\}\end{aligned}$$

implies

$$\frac{xf(x)}{1 - F(x)} = \alpha(1 + \eta_1(x)).$$

But this follows from Lemma 1.3.

(ii) and (iv): These parts are proven in a similar way to (i).

(iii): If $1 + \eta_3(x) = f(x) \int_x^{\omega(F)} (1 - F(t))\,dt / (1 - F(x))^2$, then we get $H(x) = f(x)/(1 - F(x))$. The asserted expansion is now obtained from elementary calculations.

The converse follows from Lemma 1.3, since

$$\frac{f(x)}{1 - F(x)} \exp\left\{ -\int_{x_0}^{x} \frac{f(t)}{1 - F(t)}\,dt \right\} = H(x) \exp\left\{ -\int_{x_0}^{x} H(t)\,dt \right\}$$

implies $f(x)/(1 - F(x)) = H(x)$, $x \geq x_0$. □

It was shown in Falk (1985, 1986) and Reiss [(1989), Section 5], that the distance $d(F_n^{(k)}, G^{(k)})$ is determined by the rate at which the remainder function $h(x)$ tends to zero as x tends to $\omega(F)$, if the underlying density f satisfies the expansion

$$f(x) = \omega(x)(1 + h(x)) \tag{1}$$

for $x \in (x_0, \omega(F))$. The preceding Lemma 1.1 entails that f formally admits such an expansion if one of the von Mises conditions (VM1)–(VM3′) is satisfied. The remaining is left whether the remainder term h actually converges to zero, and if so, at what rate. This will be investigated in the next section.

2. Expansions of densities involving the remainder term of von Mises conditions. In this section we utilize the results of the preceding section to investigate the rate at which the remainder term h in the representation (1) converges to zero, depending on the remainder term occurring in the von Mises condition. The following result shows that the underlying df F is asymptotically tail equivalent with a gPd if and only if the remainder function η in the von Mises conditions (VM1)–(VM3′) converges to zero fast enough.

2.1 PROPOSITION. (i) *Suppose that F satisfies* (VM1). *Then*

$$\lim_{x\to\infty} \frac{f(x)}{w_{1,\alpha}(x)} = \lim_{x\to\infty} \frac{1-F(x)}{1-W_{1,\alpha}(x)} = \begin{cases} \lambda, & \textit{for some } \lambda > 0, \\ 0, \\ \infty, \end{cases}$$

$$\Leftrightarrow \quad \int_{x_0}^{\infty} \frac{\eta_1(t)}{t}\,dt = \begin{cases} -\dfrac{1}{\alpha}\log\left(\dfrac{\lambda(1-W_{1,\alpha}(x_0-\omega(F)))}{1-F(x_0)}\right), & \textit{for some } \lambda > 0, \\ \infty, \\ -\infty. \end{cases}$$

(ii) *Suppose that F satisfies* (VM2). *Then*

$$\lim_{x\uparrow\omega(F)} \frac{f(x)}{w_{2,\alpha}(x-\omega(F))}$$

$$= \lim_{x\uparrow\omega(F)} \frac{1-F(x)}{1-W_{2,\alpha}(x-\omega(F))} = \begin{cases} \lambda, & \textit{for some } \lambda > 0, \\ 0, \\ \infty, \end{cases}$$

$$\Leftrightarrow \quad \int_{x_0}^{\omega(F)} \frac{\eta_2(t)}{\omega(F)-t}\,dt$$

$$= \begin{cases} -\dfrac{1}{\alpha}\log\left(\dfrac{\lambda(1-W_{2,\alpha}(x_0-\omega(F)))}{1-F(x_0)}\right), & \textit{for some } \lambda > 0, \\ \infty, \\ -\infty. \end{cases}$$

(iii) *Suppose that F satisfies* (VM3) *with $\omega(F) = \infty$ and put again for $x \geq x_0$*

$$U(x) = \int_x^\infty (1 - F(t))\, dt/(1 - F(x))$$

and

$$H(x) = f(x)/(1 - F(x)) = (1 + \eta_3(x))/U(x).$$

Then we have for some $\lambda, c \in (0, \infty)$,

$$\lim_{x\to\infty} \frac{f(x)}{cw_3(cx)} = \lambda$$

$$\Leftrightarrow \quad \lim_{x\to\infty} \frac{1 - F(x)}{1 - W_3(cx)} = \lambda \quad \textit{and} \quad \lim_{x\to\infty} U(x) = \frac{1}{c}$$

$$\Leftrightarrow \quad \lim_{x\to\infty} \frac{1 - F(x)}{1 - W_3(cx)} = \lambda \quad \textit{and} \quad \int_{x_0}^\infty \eta_3(t)\, dt \in \mathbb{R}$$

$$\Leftrightarrow \quad \lim_{x\to\infty} H(x) = c \quad \textit{such that} \quad \int_{x_0}^\infty (H(t) - c)\, dt \in \mathbb{R}.$$

(iv) *Suppose that F satisfies* (VM3′). *Then we have for some $c > 0$,*

$$\lim_{x\to\omega(F)} \frac{f(x)}{cw_3(cx)} = \lim_{x\to\omega(F)} \frac{1 - F(x)}{1 - W_3(cx)} = \begin{cases} \lambda, & \textit{for some } \lambda > 0, \\ 0, & \\ \infty, & \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \omega(F) = \infty \quad \textit{and} \quad \int_{x_0}^\infty \eta_{3'}(t)\, dt = -\frac{1}{c} \log\left(\frac{\lambda(1 - W_3(cx_0))}{1 - F(x_0)}\right), \\ \omega(F) < \infty \quad \textit{or} \quad \omega(F) = \infty \quad \textit{and} \quad \int_{x_0}^\infty \eta_{3'}(t)\, dt = \infty, \\ \omega(F) = \infty \quad \textit{and} \quad \int_{x_0}^\infty \eta_{3'}(t)\, dt = -\infty. \end{cases}$$

REMARKS.

(i) The preceding result reveals that the underlying density f is, for $x \to \omega(F)$, proportional to the density w of a suitably scaled gPd

$$f(x) = w(x)(1 + h(x)), \qquad x \geq x_0, \quad \text{with } h(x) \to_{x\to\omega(F)} 0,$$

if and only if the remainder term $\eta(x)$ in (VM1)–(VM3′) converges to zero fast enough, that is, if and only if $\int_{x_0}^{\omega(F)} \eta_1(t)/t\, dt \in \mathbb{R}$, $\int_{x_0}^{\omega(F)} \eta_2(t)/(\omega(F) - t)\, dt \in \mathbb{R}$ or $\int_{x_0}^{\omega(F)} \eta_3(t)\, dt \in \mathbb{R}$. By Lemma 1.1 these conditions can be summarized as the condition: $\lim_{x\to\infty} l(x)$ exists in $\mathbb{R}$ with necessarily $\omega(F) = \infty$ in case (VM3′). In particular we have in this case

$$(2)\qquad \begin{aligned} w_1(x) &= \lambda w_{1,\alpha}(x) = \lambda^{-1/\alpha} w_{1,\alpha}(\lambda^{-1/\alpha}x),\\ w_2(x) &= \lambda w_{2,\alpha}(x-\omega(F)) = \lambda^{1/\alpha} w_{2,\alpha}(\lambda^{1/\alpha}(x-\omega(F))),\\ w_3(x) &= \lambda c w_3(cx) = c w_3(cx - \log\lambda). \end{aligned}$$

(ii) Suppose that F is differentiable near $\omega(F) = \infty$ with positive derivative f. By Proposition 2.1(iii) and l'Hôpital's rule we have for some $\lambda, c > 0$ and x_0 large:

$$F \text{ satisfies (VM3)} \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{cw_3(cx)} = \lambda$$

$$\Leftrightarrow \quad \lim_{x\to\infty} H(x) = c \quad \text{and} \quad \int_{x_0}^{\infty} (H(x) - c)\,dx \in \mathbb{R}$$

$$\Leftrightarrow \quad F \text{ satisfies (VM3}') \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{cw_3(cx)} = \lambda.$$

This result shows that (VM3′) is the appropriate von Mises condition on F to ensure the expansion of its density via a gPd density.

PROOF OF PROPOSITION 2.1. (i): From (VM1) and Lemma 1.1(i) we obtain the two representations

$$\begin{aligned} \frac{f(x)}{1-F(x_0)} &= \frac{w_{1,\alpha}(x)}{1-W_{1,\alpha}(x)} \frac{1-F(x)}{1-F(x_0)}(1+\eta_1(x))\\ &= \frac{w_{1,\alpha}(x)}{1-W_{1,\alpha}(x_0)}(1+\eta_1(x))l_{1,\alpha}(x). \end{aligned}$$

These two equations imply

$$\begin{aligned} \frac{f(x)}{w_{1,\alpha}(x)} &= \frac{1-F(x)}{1-W_{1,\alpha}(x)}(1+\eta_1(x))\\ &= \frac{1-F(x_0)}{1-W_{1,\alpha}(x_0)}(1+\eta_1(x))l_{1,\alpha}(x), \end{aligned}$$

from which the assertion follows since $\eta_1(x) \to_{x\to\infty} 0$.

(ii) and (iv) follow in complete analogy.

(iii): From the representation $f(x) = (1-F(x))(1+\eta_3(x))/U(x)$, $x \geq x_0$, and the equality $w_3(x) = 1 - W_3(x)$, $x > 0$, we obtain that

$$\frac{f(x)}{cw_3(cx)} = \frac{1-F(x)}{1-W_3(cx)} \frac{1+\eta_3(x)}{cU(x)}, \quad x \geq x_0.$$

Moreover, from Lemma 1.1(iii) we obtain the representation

$$\frac{f(x)}{cw_3(cx)} = \frac{1-F(x_0)}{1-W_3(cx)}\frac{H(x)}{c}\exp\left\{-\int_{x_0}^{x}(H(y)-c)\,dy\right\}.$$

The assertion is now immediate from the fact that $\eta_3(x) \to_{x\to\infty} 0$, $U(x) = \int_{x_0}^{x}\eta_3(t)\,dt + C_0$, $x > x_0$, with $C_0 = U(x_0)$ and l'Hôpital's rule. □

The following main result of this section is now a consequence of the representation of f via a gPd. It will in particular entail the fact that the distance $d(F_n^{(k)}, G^{(k)})$ is determined by the rate at which $\eta(x)$ converges to zero as x tends to $\omega(F)$ (see Theorem 3.1 below).

2.2 THEOREM.

(i) *Suppose that F satisfies one of the von Mises conditions* (VM1), (VM2) *and* (VM3′) *with $\omega(F) = \infty, 0, \infty$ in case $i = 1, 2, 3'$ and remainder term η satisfying*

$$\eta_i(x) = \begin{cases} O(x^{-a}), & \text{if } i=1,\\ O((-x)^{a}), & \text{if } i=2,\\ O(e^{-ax}), & \text{if } i=3', \end{cases}$$

for some $a > 0$ as x tends to $\omega(F)$. Then there exist $c > 0$, $d \in \mathbb{R}$ with $d = 0$ in case $i = 1, 2$ such that

$$f(x) = cw(cx-d)(1+h(x))$$

as $x \to \omega(F)$ with

$$h(x) = \begin{cases} O(x^{-a}), & \text{if } i=1,\\ O((-x)^{a}), & \text{if } i=2,\\ O(e^{-ax}), & \text{if } i=3'. \end{cases}$$

(ii) *Suppose that F satisfies* (VM3) *with $\omega(F) = \infty$ such that*

$$\eta_3(x) = O(e^{-ax})$$

for some $a > 0$ as x tends to infinity. Then

$$\lim_{x\to\infty} U(x) = \lim_{x\to\infty}\left(\int_x^{\infty}(1-F(t))\,dt/(1-F(x))\right) =: 1/c$$

exists in $[0,\infty)$ and there exist $x_0, d \in \mathbb{R}$ such that for $x \ge x_0$,

$$f(x) = cw_3(cx-d)(1+h_3(x)) \quad \text{with } h_3(x) \to_{x\to\infty} 0,$$

if and only if $1/c > 0$. In this case we have

$$h_3(x) = O(e^{-ax})$$

as x tends to infinity.

PROOF. Part $i = 1$: The assertion follows from the expansion

$$f(x) = w_{1,\alpha,\lambda}(x)\left[\frac{1-F(x_0)}{1-W_{1,\alpha,\lambda}(x_0)} l_{1,\alpha}(x)(1+\eta_1(x))\right]$$

$$= w_{1,\alpha,\lambda}(x)\left[\exp\left\{\alpha\int_x^\infty \frac{\eta_1(t)}{t}\,dt\right\}(1+\eta_1(x))\right]$$

with $w_{1,\alpha,\lambda}(x) = \lambda w_{1,\alpha}(x) = \lambda^{-1/\alpha} w_{1,\alpha}(\lambda^{-1/\alpha}x)$ if we show that

$$\exp\left\{\alpha\int_x^\infty \frac{\eta_1(t)}{t}\,dt\right\}(1+\eta_1(x)) - 1 = O(x^{-\alpha}).$$

But this is immediate from a Taylor expansion of $\exp(z)$ at zero, the condition $\eta_1(x) = O(x^{-\alpha})$ and elementary computations. The other parts are proved in complete analogy. □

It is worth mentioning that under suitable conditions also the reverse implications in Theorem 2.2 hold. For the proof of this result, which is Theorem 2.4 below, we need the following lemma.

2.3 LEMMA. *Suppose that F, G are df's having positive derivatives f, g near $\omega(F) = \omega(G)$. If $\psi \ge 0$ is a decreasing function defined on a neighborhood of $\omega(F)$ with $\lim_{x\to\omega(F)}\psi(x) = 0$ such that*

$$|f(x)/g(x) - 1| = O(\psi(x)),$$

then

$$|(1-G(x))/(1-F(x)) - 1| = O(\psi(x)).$$

PROOF. The assertion is immediate from the inequalities

$$\left|\frac{1-G(x)}{1-F(x)} - 1\right| \le \int_x^{\omega(F)} \left|\frac{f(t)}{g(t)} - 1\right| \frac{dG(t)}{1-F(x)}$$

$$\le C\int_x^{\omega(F)} \frac{\psi(t)\,dG(t)}{1-F(x)}$$

$$\le C\psi(x)\frac{1-G(x)}{1-F(x)},$$

where C is some positive constant. Recall that ψ is decreasing. □

2.4 THEOREM. *Suppose that F satisfies one of the von Mises conditions (VM1)–(VM3), (VM3′) with $\omega(F) = \infty, 0, \infty$ in case $i = 1, 2, 3$ together with the expansion*

$$f(x) = w(x)(1 + h(x))$$

for x near $\omega(F)$ for some suitably scaled gPd w with $\omega(F) = \omega(W)$. If the *remainder term h satisfies for some* $a > 0$,

$$h(x) = \begin{cases} O(x^{-a}), & \text{if } i = 1, \\ O((-x)^{a}), & \text{if } i = 2, \\ O(e^{-ax}), & \text{if } i = 3, 3', \end{cases}$$

as $x \to \omega(F)$, *then the remainder term* η *in* (VM1)–(VM3), (VM3′) *is of the same order, that is,*

$$\eta(x) = \begin{cases} O(x^{-a}), & \text{if } i = 1, \\ O((-x)^{a}), & \text{if } i = 2, \\ O(e^{-ax}), & \text{if } i = 3, 3'. \end{cases}$$

PROOF. Part $i = 1$: Write $h_{1,\alpha}(x) = f(x)/w(x) - 1$ and

$$\begin{aligned} \eta_1(x) &= \frac{1 - W(x)}{1 - F(x)} \frac{f(x)}{w(x)} - 1 \\ &= \left(\frac{1 - W(x)}{1 - F(x)} - 1 \right) \frac{f(x)}{w(x)} + h_{1,\alpha}(x). \end{aligned}$$

The condition $h_{1,\alpha}(x) = O(x^{-a})$ then obviously implies $\eta_1(x) = O(x^{-a})$ by utilizing Lemma 2.3 with $\psi(x) = x^{-a}$. Parts $i = 2$ and $i = 3'$ are shown in complete analogy. Part $i = 3$: In this case we have the representations $h_3(x) = f(x)/w(x) - 1$ and

$$\eta_3(x) = \left(\frac{1 - W(x)}{1 - F(x)} \frac{U(x)}{1/c} - 1 \right) \frac{f(x)}{w(x)} + h_3(x),$$

where $1/c = \lim_{x \to \infty} U(x)$ exists in $(0, \infty)$ by Proposition 2.1(iii). The assertion now follows from Lemma 2.3 with $\psi(x) = e^{-ax}$ and elementary computations showing that $|U(x) - 1/c| = O(\psi(x))$ as well. □

3. Rates of convergence of extremes. In this section we investigate the relationship between the rate of convergence of extremes and the remainder terms in the von Mises conditions. The following result is an immediate consequence of Theorem 2.2 and Corollary 5.5.5 in Reiss (1989).

3.1 THEOREM. *Suppose that F satisfies one of the von Mises conditions* (VM1), (VM2) *or* (VM3′) *with the remainder term* η *satisfying*

$$\eta_i(x) = \begin{cases} O(x^{-\delta\alpha}), & \text{if } i = 1, \\ O((-x)^{\delta\alpha}), & \text{if } i = 2, \\ O(e^{-\delta x}), & \text{if } i = 3', \end{cases}$$

for some $\delta > 0$ *as x tends to* $\omega(F) = \infty, 0, \infty$, $i = 1, 2, 3'$. *Then there exists a* *positive constant D such that for any* $k \in \mathbb{N}$,

$$\sup_{B \in \mathbb{B}^k} \left| P\left\{ \left((X_{n-i+1:n} - d_n)/c_n \right)_{i=1}^k \in B \right\} - G^{(k)}(B) \right|$$
$$\le D\left((k/n)^{\delta} k^{1/2} + k/n \right),$$

where $d_n = 0$, $c_n = (\lambda n)^{1/\alpha}$ *if* $i = 1$, $d_n = 0$, $c_n = (\lambda n)^{-1/\alpha}$ *if* $i = 2$, $d_n = \log(\lambda n)/c$, *and* $c_n = 1/c$ *if* $i = 3'$.

REMARKS.

(i) The conclusion of the preceding result remains valid if we replace the condition that F satisfies (VM3′) by (VM3) together with the assumption that $\lim_{x \to \infty} U(x) > 0$. Recall that by Proposition 2.1(iii), $\eta_3(x) = O(e^{-\delta x})$ implies that $\lim_{x \to \infty} U(x)$ exists in $[0, \infty)$ anyway.

(ii) The two terms k/n and $(k/n)^{\delta} k^{1/2}$ of the bound in the preceding result appear for different reasons. While k/n is the best possible rate which is attained only by a gPd [Falk (1989, 1990)], the term $(k/n)^{\delta} k^{1/2}$ resembles the vicinity of the gPd W given by the order of the remainder function η; see Theorem 3.2 below. Notice that $(k/n)^{\delta} k^{1/2} \ge k/n$ if $\delta \le 1$.

(iii) Theorem 3.1 entails also that the Hellinger distance between empirical truncated point processes and Poisson point processes, as established in Falk and Reiss (1992), is determined by the rate at which the limit in the von Mises conditions (VM1)–(VM3′) is attained. A corresponding result has been established by Kaufmann and Reiss (1992). Asymptotic expansions of length 2 for the pointwise distance $|F^n(a_n x + b_n) - G(x)|$ under von Mises conditions were established by Radtke (1988) [see Problems 5.14 and 5.15 of Reiss (1989)]; for expansions of length one we refer to Section 2.4 of the book by Resnick (1987).

Next we will prove that under suitable regularity conditions also the reverse implication of Theorem 3.1 holds, that is, we will prove that if for some $a_n > 0$, $b_n \in \mathbb{R}$ and some $\delta \in (0, 1]$,

$$\sup_{x \in \mathbb{R}} \left| P\{ (X_{n:n} - b_n)/a_n \le x \} - G_i(x) \right| = O(n^{-\delta}), \tag{3}$$

then

$$\eta_i(x) = \begin{cases} O(x^{-\delta\alpha}), & x \ge x_0, \quad \text{if } i = 1, \\ O\left((-x)^{\delta\alpha} \right), & x_0 \le x < 0, \quad \text{if } i = 2, \\ O(e^{-\delta c x}), & x \ge x_0, \quad \text{if } i = 3'. \end{cases}$$

Consequently, we obtain from Theorem 2.2 that (3) already implies the expansion

$$f(x) = w(x)(1 + h(x)), \qquad x \in [x_0, \omega(F)),$$

for some suitably scaled gPd w with

$$h(x) = \begin{cases} O(x^{-\delta\alpha}), & x \geq x_0, & \text{if } i = 1, \\ O\big((-x)^{\delta\alpha}\big), & x_0 \leq x < 0, & \text{if } i = 2, \\ O(e^{-\delta c x}), & x \geq x_0, & \text{if } i = 3'. \end{cases}$$

These results show that the above growth conditions on the remainder function h occur naturally, if the extremes approach their limiting distribution G with a power of the sample size.

First we state the required conditions. Suppose that F satisfies one of the von Mises conditions (VM1) with $\alpha > 0$, (VM2) with $\alpha \geq 1$ and (VM3′) with $\omega(F) = \infty, 0, \infty$ such that

$$(4) \qquad \begin{aligned} &\int_{x_0}^{\infty} \eta_1(t)/t\, dt \in \mathbb{R}, \quad \text{in case (VM1)}, \\ &\int_{x_0}^{0} \eta_2(t)/t\, dt \in \mathbb{R}, \quad \text{in case (VM2)}, \\ &\int_{x_0}^{\infty} \eta_{3'}(t)\, dt \in \mathbb{R}, \quad \text{in case (VM3')}. \end{aligned}$$

We require further that η be proportional to some ultimately monotone function r, that is,

$$(5) \qquad \lim_{x \to \omega(F)} \eta(x)/r(x) = 1,$$

and $r(x)$ is monotone for x near $\omega(F)$.

We will need the preceding condition (5) to ensure that the rate at which $\int_x^{\omega(F)} \eta_1(t)/t\, dt$, $\int_x^0 \eta_2(t)/t\, dt$ or $\int_x^\infty \eta_{3'}(t)\, dt$ converge to zero as $x \to \omega(F)$ carries over to the rate at which $\eta_i(x)$ approaches 0, $i = 1, 2, 3'$.

Now we are ready to formulate the inverse of Theorem 3.1.

3.2 THEOREM. *Suppose that F satisfies conditions* (4) *and* (5). *If for some $a_n > 0$, $b_n \in \mathbb{R}$ and some $\delta \in (0, 1]$,*

$$\sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - G(x)| = O(n^{-\delta}),$$

and $|a_n/c_n - 1| + |b_n - d_n|/c_n = O(n^{-\delta})$, then

$$\eta_i(x) = \begin{cases} O(x^{-\delta\alpha}), & x \geq x_0, & \text{in case (VM1), that is, } i = 1, \\ O\big((-x)^{\delta\alpha}\big), & x_0 \leq x < 0, & \text{in case (VM2), that is, } i = 2, \\ O(e^{-\delta c x}), & x \geq x_0, & \text{in case (VM3'), that is, } i = 3', \end{cases}$$

where again $c = \lim_{x \to \infty} f(x)/(1 - F(x))$.

REMARK. Proposition 2.1 shows that if condition (4) is satisfied, the df F is tail equivalent to a suitably scaled gPd, that is, there exist $a > 0$, $b \in \mathbb{R}$ such that

$$\lim_{x \to \omega(F)} \frac{1 - F(x)}{1 - W((x-b)/a)} = 1. \tag{6}$$

From Lemma 2.2.3 in Galambos (1987) we therefore deduce that

$$\sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - G_i(x)| \to_{n \to \infty} 0 \tag{7}$$

necessarily implies

$$\frac{a_n}{c_n} \xrightarrow[n \to \infty]{} 1, \quad \frac{b_n - d_n}{c_n} \xrightarrow[n \to \infty]{} 0.$$

PROOF OF THEOREM 3.2. We prove only the case (VM3′); the other cases can be shown in complete analogy and are therefore omitted.

Define

$$R(x) := \int_x^\infty \eta_{3'}(t)\,dt, \qquad x \ge x_0.$$

From (VM3′) and (4) we deduce the representation

$$\begin{aligned}\log(1 - F(x)) &= -c\int_{x_0}^x 1 + \eta_{3'}(t)\,dt + C_0\\ &= -c(x - x_0) + C_0' + c\int_x^\infty \eta_{3'}(t)\,dt\\ &= -c(x - x_0') + c\int_x^\infty \eta_{3'}(t)\,dt\\ &= -c(x - x_0') + cR(x), \qquad x \ge x_0'\end{aligned}$$

for some fixed constants $C_0, C_0', x_0' \in \mathbb{R}$.

From the condition $\sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - G_3(x)| = O(n^{-\delta})$ we therefore obtain

$$\begin{aligned}&\sup_{x \ge x_0} \left| (1 - \exp\{-c(a_n x + b_n - x_0') + cR(a_n x + b_n)\}) - \exp(-e^{-x}) \right|\\ &\quad = \sup_{x \ge x_0} \Big| \big(1 - n^{-1} \exp\{-c(a_n x + b_n - x_0' - \log(n)/c)\\ &\qquad\qquad + cR(a_n x + b_n)\}\big)^n - \exp(-e^{-x}) \Big|\\ &\quad = O(n^{-\delta}),\end{aligned}$$

which implies $x_0' = \log(\lambda)/c$ (recall that $R(a_n x + b_n) \to_{n \to \infty} 0$ since $a_n x + b_n$

$\to \infty$). Consequently, we have

$$\sup_{x \geq x_0} \left| \left(1 - n^{-1} \exp\{-x + cR(a_n x + b_n)\}\right)^n - \exp(-e^{-x}) \right| = O(n^{-\delta}).$$

Since

$$\sup_{x \geq x_0} \left| \exp(-e^{-x}) - (1 - e^{-x}/n)^n \right| = O(n^{-1}),$$

we have

$$\sup_{x \geq x_0} \left| \left(1 - n^{-1} \exp\{-x + cR(a_n x + b_n)\}\right)^n - (1 - e^{-x}/n)^n \right| = O(n^{-\delta}).$$

Define now the auxiliary function $g\colon \mathbb{R} \to [0, \infty)$ by $g(t) := (1 - e^{-t}/n)^n$. Then $g'(t) = (1 - e^{-t}/n)^{n-1} e^{-t}$ and we can write, by using Taylor's formula,

$$\begin{aligned}
&\sup_{x \geq x_0} \left| \left(1 - n^{-1} \exp\{-x + cR(a_n x + b_n)\}\right)^n - (1 - e^{-x}/n)^n \right| \\
&\quad = \sup_{x \geq x_0} \left| g(x - cR(a_n x + b_n)) - g(x) \right| \\
&\quad = \sup_{x \geq x_0} \left| g'(\xi_x) cR(a_n x + b_n) \right| \\
&\quad \geq C \sup_{x \geq x_0} \left| \exp(-x) R(a_n x + b_n) \right|
\end{aligned}$$

for some appropriate fixed constant $C > 0$ if n is large, where ξ_x is between x and $x - cR(a_n x + b_n)$. Consequently, we have

$$\sup_{x \geq x_0} \left| \exp(-x) R(a_n x + b_n) \right| = O(n^{-\delta}). \tag{8}$$

We claim that (8) implies

$$R(x) = O(\exp(-\delta c x)) \tag{9}$$

as $x \to \infty$. Condition (5) then implies that

$$\eta_{3'}(x) = O(\exp(-\delta c x)), \qquad x \to \infty,$$

as well.

We will prove (9) by a contradiction. Suppose that (9) is not true; then there exists a sequence $x_n \to_{n \to \infty} \infty$ such that $|R(x_n) \exp(\delta c x_n)| \to_{n \to \infty} \infty$. We can find a sequence $m(n) \to_{n \to \infty} \infty$ of positive integers such that

$$x_n - \left(K x_0 + b_{m(n)}\right) =: \varepsilon_n \to_{n \to \infty} 0,$$

where $K/a > 1$. Such a sequence $m(n)$ exists since for *any* sequence $y_n \to_{n \to \infty} \infty$ we can find a sequence $m'(n) \to_{n \to \infty} \infty$ of integers such that $y_n - \log(m'(n)) \to_{n \to \infty} 0$ and since $b_n - x_0' + \log(n)/c \to 0$ as $n \to \infty$. From

(8) we obtain

$$\begin{aligned} O\big(m(n)^{-\delta}\big) &= \sup_n \big|\exp\big(-(Kx_0+\varepsilon_n)/a_{m(n)}\big)R(Kx_0+\varepsilon_n+b_{m(n)})\big| \\ &= \sup_n \big|\exp\big(-(Kx_0+\varepsilon_n)/a_{m(n)}\big)R(x_n)\big| \\ &= \sup_n \big|\exp\big(-(Kx_0+\varepsilon_n)/a_{m(n)}\big)R(x_n)\exp(\delta c x_n) \\ &\qquad \times\exp\big(-\delta c\big(Kx_0+\varepsilon_n+b_{m(n)}-x_0'-c^{-1}\log(m(u))\big)\big)m(n)^{-\delta}\big| \\ &\geq Cm(n)^{-\delta}\sup_n |R(x_n)\exp(\delta c x_n)| \end{aligned}$$

for some suitable fixed constant $C > 0$, since $a_n c \to 1$, $b_n - d_n \to 0$ as $n \to \infty$. Multiplying both sides of the above inequality by $m(n)^{\delta}$ we obtain that $\sup_n |R(x_n)\exp(\delta c x_n)|$ is finite; but this is the desired contradiction. Consequently, (9) is shown, which completes the proof. □

The following result is now immediate from Theorem 2.2.

3.3 COROLLARY. *Suppose that F satisfies conditions* (3) *and* (4). *If for some* $\delta \in (0,1]$ *and some* $a_n > 0$, $b_n \in \mathbb{R}$ *as in Theorem* 3.2,

$$\sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - G(x)| = O(n^{-\delta}),$$

then the density f of F admits the expansion

$$f(x) = w(x)(1+h(x)), \qquad x \in [x_0, \omega(F)),$$

where $w(x)$ *is the density of the scaled gPd* $W((x-b)/a)$ *with* $a > 0$, $b \in \mathbb{R}$ *as in* (2), *such that*

$$h(x) = \begin{cases} O(x^{-\delta\alpha}), & x \geq x_0 \quad \textit{in case (VM1)}, \\ O\big((-x)^{\delta\alpha}\big), & x_0 \leq x < 0, \quad \textit{in case (VM2)}, \\ O(e^{-\delta c x}), & x \geq x_0, \quad \textit{in case (VM3')}. \end{cases}$$

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MATHEMATISCH-GEOGRAPHISCHE FAKULTÄT
KATHOLISCHE UNIVERSITÄT EICHSTÄTT
D-8078 EICHSTÄTT
GERMANY