Mathematical Programs with Complementarity Constraints: Theory, Methods, and Applications

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Preface

Graduation is an exciting journey and in my case took some unexpected turns. I would like to thank my advisor and tour guide Christian Kanzow for making sure I did not get lost in the wide field of bilevel optimization. He admirably managed to find a balance between leaving me enough freedom to explore interesting directions on my own and at the same time constantly pointing out new areas worth investigating. Whenever I had a problem, his door stood open and he always managed to come up with an idea, a reference or at least a new angle how to look at the problem. I am greatly indebted to him for his support and guidance not only throughout the years of my graduation but already during my undergraduate studies.

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Apart from Christian Kanzow, many other people from out institute influenced this thesis. I would like to thank my colleague Tim Hoheisel for introducing me to MPVCs and many cooperations. Some of his results found their way into the numerical part of this thesis. My thanks also go to Jürgen Grahl for many interesting discussions and for an advice that enabled me to prove the central result of Chapter 2. And of course there are many other colleagues I wish to thank for the great atmosphere at our institute. I will not list all of your names here, but I think you know who I am talking about.

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Contents

Abbreviations

ACQ	Abadie constraint qualification
AMPL	a mathematical programming language
CPLD	constant positive linear dependence constraint qualification
CQ	constraint qualification
CRCQ	constant rank constraint qualification
GCQ	Guignard constraint qualification
KKT	Karush-Kuhn-Tucker (conditions)
LICQ	linear independence constraint qualification
MATLAB	a programming language
MFCQ	Mangasarian Fromovitz constraint qualification
MPCC	mathematical program with complementarity constraints
MPEC	mathematical program with equilibrium constraints
NCP	nonlinear complementarity problem
NLP	nonlinear program

Notation

Spaces

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the nonnegative real numbers
$\mathbb{R}_{++}$	the positive real numbers
$\mathbb{R}_-$	the nonpositive real numbers
$\mathbb{R}_{--}$	the negative real numbers
$\mathbb{R}^n$	the n-dimensional real vector space
$\mathbb{R}^n_+$	the nonnegative orthant in $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	the positive orthant in $\mathbb{R}^n$
$\mathbb{R}^n_-$	the nonpositive orthant in $\mathbb{R}^n$
$\mathbb{R}^n_{--}$	the negative orthant in $\mathbb{R}^n$

Sets

$\{x\}$	the set consisting of the vector x
$\mathrm{cl}(S)$	the closure of the set S
$\mathrm{bd}(S)$	the boundary of the set S
$\lvert S\rvert$	the cardinality of S
$S_1 \subseteq S_2$	S_1 is a subset of S_2
$S_1 \subsetneq S_2$	S_1 is a proper subset of S_2
$S_1 \backslash S_2$	the set of elements contained in S_1 but not in S_2
$S_1 \cap S_2$	the intersection of S_1 and S_2
$S_1 \cup S_2$	the union of S_1 and S_2
$S_1 \times S_2$	the cartesian product of S_1 and S_2
$\mathbb{B}(x;\varepsilon)$	the open ball of radius ε around x
(x_1, x_2)	an open interval in $\mathbb{R}$
$[x_1, x_2]$	a closed interval in $\mathbb{R}$
$\mathcal{X}$	the feasible set of the NLP
X	the feasible set of the MPCC

Vectors and Matrices

$x \in \mathbb{R}^n$	a column vector in $\mathbb{R}^n$
(x, y)	the column vector $(x^T, y^T)^T$
x_i	the i-th component of x
x_I	the vector in $\mathbb{R}^{\lvert I\rvert}$ consisting of the components x_i, $i \in I$
$\mathrm{supp}(x)$	the support of x, $\mathrm{supp}(x) = \{i \mid x_i \neq 0\}$
$x \geq y$	componentwise comparison $x_i \geq y_i$, $i = 1, \ldots, n$

$x > y$	componentwise comparison $x_i > y_i$, $i = 1, \ldots, n$
$\min\{x, y\}$	the vector whose i-th component is $\min\{x_i, y_i\}$
$\max\{x, y\}$	the vector whose i-th component is $\max\{x_i, y_i\}$
$\|\|x\|\|_2$	the euclidean norm of x, $\|\|x\|\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$
$\|\|x\|\|_1$	the Manhattan norm of x, $\|\|x\|\|_1 = \sum_{i=1}^n \|x_i\|$
$\|\|x\|\|_\infty$	the infinity norm of x, $\|\|x\|\|_\infty = \max \|x_i\| \mid i = 1, \ldots, n$
$e_i \in \mathbb{R}^n$	the i-th element of the canonical basis of $\mathbb{R}^n$
0_n	the zero element of $\mathbb{R}^n$
$I_{n\times n} \in \mathbb{R}^{n\times n}$	the identity matrix of size $n \times n$

Cones

$T_X(x)$	the Bouligand tangent cone to X in x
$L_X(x)$	the linearized tangent cone to X in x
$L_{MPCC}(x)$	the MPCC linearized tangent cone to X in x
$N_X^F(x)$	the Fréchet normal cone to X in x
$N_X(x)$	the limiting normal cone to X in x
X°	the polar cone to X

Functions

$f : \mathbb{R}^n \to \mathbb{R}^m$	a function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$
$f_i : \mathbb{R}^n \to \mathbb{R}$	the i-th component of f
$\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$	a multifunction that maps $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$
$\mathrm{gph}(\Phi)$	the graph of the multifunction Φ
$\nabla f(x)$	the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x, column vector
$\nabla_x f(x, y)$	the gradient of f with respect to x only
$f'(x)$	the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ in x
$f''(x)$	the second derivative of f in x
$\partial^F f(x)$	the Fréchet subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x
$\partial f(x)$	the limiting subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x
$\mathrm{dist}_X(x)$	the distance between x and a closed set X
$\mathrm{Proj}_X(x)$	the (not necessarily unique) projection of x to a closed set X

Sequences

$\{x^k\} \subseteq \mathbb{R}^n$	a sequence in R^n
$\{x^k\}_K$	a subsequence containing only those x^k with $k \in K$
$x^k \to x^*$	a convergent sequence with limit x^*
$x^k \to_X x^*$	a convergent sequence with limit x^* and $\{x^k\} \subseteq X$
$\lim_{k\to\infty} x^k$	limit of the convergent sequence x^k
$\{t_k\} \subseteq \mathbb{R}$	a real-valued sequence

$t_k \downarrow t^*$ a convergent sequence with limit t^* and $t_k > t^*$ for all $k \in \mathbb{N}$

1. Introduction

Solving optimization problems such as finding the shortest connection between two spots or deciding which grain to sow in order to maximize the harvest has always been a part of human life. Thanks to mathematics, more abstract problems could be formulated and at the same time they became more and more complicated, sometimes too complicated to solve them by hand. This gave rise to a new field in mathematics in which numerical methods for the solution of such problems were developed. While at first only linear problems could be solved, since the middle of the last century an in-depth theory and very efficient numerical methods for nonlinear optimization problems have been developed. However, it turned out that not all optimization problems can be treated the same way. This lead to the definition of several classes of nonlinear optimization problems distinguished for example by their constraints.

In this thesis, we focus on the class of *mathematical programs with equilibrium constraints*, MPEC for short, which are optimization problems of the form

$$\min_x f(x) \quad \text{subject to} \quad x \in X, x_2 \in S(x_1).$$

Here, an objective function $f : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ depending on a variable $x = (x_1, x_2)$ shall be minimized subject to standard constraints represented by $x \in X$ with $X \subseteq \mathbb{R}^{n_1+n_2}$ and so called equilibrium constraints $x_2 \in S(x_1)$. The set valued map $S : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ usually represents the solution set of a lower level problem which may for example be another optimization problem or a Nash equilibrium problem. For theoretical and numerical purposes, the condition $x_2 \in S(x_1)$ is usually formulated as a variational inequality, generalized equation or complementarity conditions.

This type of problems appears frequently in economics for example if one generalizes the Nash equilibrium problem to the so called leader-follower or Stackelberg problem, see [71, 22, 61, 86, 97] for economic MPECs. As we will see in this thesis, another economic problem that can be formulated as such a Stackelberg game is the effort maximization problem. MPECs appear also in engineering and physics, see [68, 3, 9, 10, 116, 112, 100, 120] for examples. For more applications we would like to refer to the monographs [82, 95, 30, 27].

In the theoretical and the numerical part of this thesis, we focus on the case, where the lower level problem can be represented by a nonlinear complementarity problem. These MPECs are of the form

$$\begin{aligned} \min_x f(x) \quad \text{subject to} \quad & g_i(x) \le 0 && \forall i = 1, \dots, m, \\ & h_i(x) = 0 && \forall i = 1, \dots, p, \\ & G_i(x) \ge 0 && \forall i = 1, \dots, q, \\ & H_i(x) \ge 0 && \forall i = 1, \dots, q, \\ & G_i(x)H_i(x) = 0 && \forall i = 1, \dots, q, \end{aligned} \tag{1.1}$$

where the functions $f, g_i, h_i, G_i, H_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable. This special type of MPECs is called *mathematical program with complementarity constraints*

(MPCC). Although this problem looks very close to a standard nonlinear optimization problem, the combinatorial nature of the complementarity conditions causes both theoretical and numerical trouble if one wishes to apply the standard theory. The feasible set of MPCCs is usually nonconvex and has an empty interior. Most standard constraint qualifications are violated and therefore the Karush-Kuhn-Tucker conditions are not necessary optimality conditions. Since these are the basis of most numerical methods to solve nonlinear programs, standard algorithms applied to MPCCs are likely to fail. For this reason a special theory for MPCCs has been developed over the last 20 years including several stationarity concepts and MPCC tailored constraint qualifications. This theory was then used as a basis for numerous numerical methods to solve MPCCs. For more information, we refer the interested reader to the monographs [95, 82, 30] and the introductions to the respective parts of this thesis.

The aim of this thesis is threefold: First, we consider the economic problem of effort maximization in asymmetric n-person contest games in Part I which turns out to be an MPEC. For the case of constant returns to scale, we extend the existing knowledge in Chapter 2 by proving the existence of a solution of this problem and deriving an explicit formula for this solution. This was so far only done for two players or n homogeneous players. We also provide some examples illustrating our results and a brief comparison with the all-pay auction model. Then we turn to more general production technologies in Chapter 3 and provide a reformulation as an MPCC for a class of production technologies, which can be used to solve the effort maximization problem numerically.

Before we can proceed to the numerical solution of MPCCs however, we need to talk about theory. This is done in Part II. First, we recall some basic facts about standard nonlinear programs in Chapter 4 and then we define constraint qualifications and stationarity concepts for MPCCs in Chapter 5. At that point, we introduce an MPCC analogue of the constant positive linear dependence constraint qualification which is a very weak constraint qualification that will turn out to be very useful in the numerical part of this thesis. In contrast to standard nonlinear programs there are several prominent stationarity concepts for MPCCs which are necessary optimality conditions under different assumptions. To circumvent this, one might want to turn to the Fritz-John conditions which are known to be necessary optimality conditions without any further assumptions. In the context of MPCCs however, the standard Fritz-John conditions are rather useless. For this reason, we derive special MPCC Fritz-John conditions in Chapter 6. These enhanced Fritz-John conditions give rise to two new constraint qualifications for MPCCs which each have an interesting application. One of them can be used to obtain a very simple proof for the fact that M-stationarity is a necessary optimality condition under most of the common MPCC constraint qualifications. The other one forms the basis of an exact penalty result for MPCCs under much weaker assumptions than commonly used. Additionally, we discuss how the three new MPCC constraint qualifications fit into the system of existing ones.

In Part III we finally come to the numerical solution of MPCCs. By now, there are several different approaches to this problem but we focus only on the relaxation approach. We consider this idea very interesting since it allows the usage of the very efficient nonlinear program solvers, that have been developed over the last decades, for MPCCs with very little effort. Since several relaxation methods have been introduced in the last ten years, we chose four of them [105, 79, 67, 109], which are very similar, and take a closer look at their theoretical properties in Chapter 7.

We improve the respective convergence results and analyze the existence of Lagrange multipliers to local minima of the relaxed problems. In Chapter 8, we present a new relaxation method which can be seen as an enhancement of [67] and analyze its theoretical properties. Chapter 9 is devoted to the comparison of these five relaxation methods. First, we gather the theoretical results from the previous two chapters and then we provide a numerical comparison of these methods based on the MacMPEC collection of test problems [73] which is composed of both academic examples and real life applications. To close the circle, we finally use the new relaxation method from Chapter 8 so solve the effort maximization problem from the first part of this thesis. At first, we test our algorithm on the case of constant returns to scale and regain the theoretically derived results. Then, we try to solve the effort maximization problem with a more difficult production technology and obtain results that give hope for a further theoretical investigation of this topic.

Parts of the new results in this thesis have already been published in [69, 60, 58] or are contained in the preprints [48, 59, 70]. All of them are are joint work with my advisor Christian Kanzow. The papers [60, 58, 59] were created in cooperation with my colleague Tim Hoheisel and [48] is a cooperation with Jörg Franke and Wolfgang Leininger from the Department of Economics, University of Dortmund (TU). Wherever other sources have been used in this thesis, this is documented by an appropriate reference.

Part I.

Effort Maximization in Asymmetric n-Person Contest Games – an Economic Example

Lobbying, public procurement, affirmative action, promotion tournaments: What do all these have in common? Although the topics themselves are not closely related, the underlying mechanism is the same in all cases. On the one side, there is a group of people or companies exerting effort and competing against each other for a prize which has a certain value to them. On the other side, we have a contest designer or administrator who has the power to set the rules of the competition and thus influences the effort exerted by the participants. This results in several coupled optimization problems. The participants choose their effort in order to maximize their individual gain from the contest which, however, does not only depend on their own effort but also on the effort exerted by the other contestants and, of course, on the contest itself. The contest designer, in turn, wishes to maximize his earnings from the contest which depend on the effort of the participants. Thus, from the contest designer's point of view, these processes can be modeled as bilevel optimization problems. On the upper level, the contest designer sets the rules of the contest in order to maximize his gain, and on the lower level, the participants of the contest solve a Nash equilibrium problem to determine their optimal efforts. Before we delve into the examples mentioned above, we would like to state our mathematical model.

In order to incorporate the characteristics of the afore mentioned situations, which will be discussed more in-depth afterwards, we decided to model these processes as an asymmetric lottery contest game under complete information in the style of Tullock [118], see also [25]. We assume that there are n participants and denote the set of all players by $N := \{1, \dots, n\}$. Every contestant ν chooses a nonnegative effort $x_\nu \geq 0$ in order to maximize his utility function θ_ν. However, his utility function does not only depend on his decision but also on the chosen efforts of the other players subsumed under $x_{-\nu} \in \mathbb{R}^{n-1}$, i.e., $\theta_\nu : \mathbb{R}^n_+ \to \mathbb{R}$ is given by

$$\theta_\nu(x_\nu, x_{-\nu}) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\alpha_\nu c(x_\nu)}{\sum_{\mu \in N} \alpha_\mu c(x_\mu)} V_\nu - \beta_\nu x_\nu & \text{else.} \end{cases}$$

These utility functions can be interpreted as follows. All players take part in a lottery where a prize with the value $V_\nu > 0$ for player ν can be won. Every player can increase his probability of winning, which is given by his relative effort $\frac{\alpha_\nu c(x_\nu)}{\sum_{\mu \in N} \alpha_\mu c(x_\mu)}$, by increasing his effort x_ν which, however, also leads to higher costs $\beta_\nu x_\nu$. Here, $\beta_\nu > 0$ are the individual costs of the ν-th contestant for exerting effort. The function $c : \mathbb{R}_+ \to \mathbb{R}$ indicates what kind of rent-seeking technology is considered. In the simplest case, one assumes $c(x) \equiv x$ and speaks of constant returns to scale. This is the case we will deal with most of the time, since only there closed formulas for both the underlying Nash game and the designer's optimal solution are known. However, more general functions are also possible. Usually, one assumes at least $c(0) = 0$ and $c'(x) > 0$ for all $x > 0$ which reflects that zero effort leads to a winning probability of zero and increasing the effort also increases the probability of winning. We will give a small outlook on this case. The parameters $\alpha_\mu > 0$ are the designer's variables which he can use to bias the contest in a way that is favorable for him. His aim is to choose his variable $\alpha \in \mathbb{R}^n_{++}$ in order to maximize his gain

$$\theta_0(\alpha) = \sum_{\mu=1}^{n} x^*_\mu(\alpha),$$

where $x^*(\alpha)$ is the Nash equilibrium of the contest with the parameters α_μ. For simplicity, we will only consider contest games that posses a unique solution. One might wonder why we do not allow the designer to explicitly exclude someone from the contest by choosing the corresponding $\alpha_\nu = 0$. However, as we will see in the case of constant returns to scale, this possibility does not lead to a more favorable contest from the designer's point of view but instead causes technical problems. In the optimal contest, unwanted players can also be excluded by a sufficiently small but positive α_ν.

This problem is obviously an MPEC as we defined it in the introduction since we have to solve

$$\max_{\alpha,x} \sum_{\mu=1}^{n} x_\mu \quad \text{subject to} \quad \alpha > 0, x = x(\alpha).$$

The standard constraints here are the positivity condition on the designer's variable α and the, usually set valued, equilibrium constraint reduces to the equation $x = x(\alpha)$ due to the fact that we consider only contest games with a unique solution. It turns out that, in the case of constant returns to scale, the effort maximization problem is one of the rare nontrivial MPECs where one can prove the existence of a solution and provide a closed form of this solution. For general functions c however, no such solution is known. Thus, in this case one has to use numerical methods in order to obtain solutions of this MPEC. For this reason, we will provide a reformulation of the effort maximization problem as MPCC and some numerical results in Part III.

Before we apply this model to the processes mentioned above, let us briefly summarize the results we are going to derive for the case of constant returns to scale. We will see that in the optimally biased contest all players, that would have been active if the contest was unbiased, will exert a positive effort as well, i.e., there is no exclusion of strong players. Instead, the playing field is leveled to some extend such that some weaker contestants might also decide to become active. However, whenever there are more than two players and the contestants are not homogeneous a priori, the playing field is never leveled to the full extend. Finally, we found out that at least three players will be active in the optimal contest independent of their heterogeneity. This is somewhat surprising since two active players suffice to obtain a Nash equilibrium of the contest game.

According to the Center for Responsive Politics, a total of 3.49 Billion US Dollars were spent by 13.664 registered lobbyists in the year 2009 in the United States of America, not including campaign contributions, cf. [46]. This figure highlights how important it is to understand the mechanisms behind lobbying, especially since lobbying to influence political decision making is of course not a phenomenon solely appearing in the USA. Lots of effort has already been made in this direction, see for example [54, 15] for two articles on the motivation for firms to engage in corporate political activities both in the United States of America and the European Union. In contrast to this, we aim to provide a mathematical model that can be used to describe the relations between several lobby groups on the one side and the person whose decision these groups try to influence on the other side. The resulting lobbying process has interesting properties: On the one hand, it is public since for example in the United States, the "Lobbying Disclosure Act" (1995) and the "Honest Leadership and Open Government Act" (2007) require lobbyists to register and disclose their spendings. On the other hand, it is informal, since of course there is no guarantee that the effort exerted by a certain lobby group leads to the desired decision of the politician. Also, the competition might be biased since the politician might favor a certain decision or lobbying party. By contrast, an effort - how big it might be - by an opposing political fraction is not very likely to influence the politicians decision. These properties are reflected in our model. We assume that everyone participating in the lobbying process has full information about his rivals and can thus try to anticipate their behavior. The uncertainty of the effectiveness of exerted effort is represented by the lottery character of our contest success function and the possibility of a biased contest by the weights α_μ. Our results allow conclusions on the behavior of a politician who aims to maximize the aggregated effort of all lobby groups and thus could be used as a basis for countermeasures to constrain excessive lobbying activities. Such countermeasures and the difficulties thereof are also discussed in [20, 33], however based on a different model. Another popular model for the lobbying process is the all-pay auction model, cf. [11, 35, 32]. This, however, leads to completely different results than our model as we will see in a special section on the difference between those two approaches.

Another topic which is sometimes closely related to lobbying is public procurement. Here, several companies compete for one government contract and the government has to decide on a rule how to choose best contractor. However, the best contractor is not necessarily the one with the cheapest offer but there might also be other requirements. Sometimes, smaller companies are preferred or domestic firms are favored over foreign ones, see [88, 87]. Our results prove that favoring weaker contestants is not necessarily a contradiction to the primary goal of choosing a cheap contractor but may even improve the result of the competition.

What holds for companies is even more true for single persons. Even nowadays, people often face competitive situations in which they have advantages or disadvantages due to their race, origin or sex. In Germany, for example, women still earn 16 to 20 percent less than men with the same qualifications. A study by the German Institute for Economic Research [76] explains this as follows: Women perceive lower wages as fair and therefore accept lower wages than men. It is argumented that women compare themselves to other women in order to determine what would be a fair wage. And since many women still work in low-paid branches, this leads to the effect described above. Hence, according to this study, the gender wage gap is not due to present but past discrimination. One approach to resolve such imbalances is called affirmative action. The idea here is that in competitive situations such as employment weaker participants should be favored, i.e., the playing field should be leveled such that all contestant have the same chance, see for example [49, 47] for a work on affirmative action in college admissions and a more general work on affirmative action compared to the neutral equal treatment approach. These two papers prove that affirmative action may have an additional positive effect, namely it can increase the effort of all participants. By applying the model described above, we will see that, if we look at such a contest from the contest organizer's point of view and wish to maximize the effort of all participants, we automatically end up with a playing field that is leveled to some extend. Consequently, effort maximization and affirmative action are not as contrary as it is often assumed.

To provide effort incentives is also a central topic when it comes to designing labor contracts. One possibility to motivate workers are promotion tournaments, see [72, 117] for two works on rank-order tournaments and on how to choose a CEO (chief executive officer). These two papers analyze the effects of handicapping in such tournaments, however based on a different model. Nonetheless, our lottery model as described above can also be used to reflect situations in which for example several employees compete for a better payed position or a bonus payment and, as it was already mentioned before, we will see that handicapping can have positive effects on the overall effort in our model as well.

Since there are so many possible applications for this model, it is not surprising that this and similar contest success functions have been analyzed before, see [108, 23] for an axiomatization and [65] for a stochastic foundation. So far, the focus of most papers was not the effort maximization problem but only the underlying contest game or they considered either the two-player case or the homogeneous n-player case. In [91], the asymmetric two-player case with arbitrary returns to scale is considered and results on the existence and form of solutions are given. In [92], the same author considers the asymmetric two-player case with a more general contest success function and addresses the question of effort maximization. The existence of equilibria in the homogeneous n-player case is analyzed in [98] and an effort maximization problem for the homogeneous n-player case with constant returns to scale is solved in [28]. However, in the latter case, in contrast to our model, the contest designer is allowed to choose a positive probability in which he keeps the prize. Existence, uniqueness and properties of solutions of the asymmetric n-player contest game are the focus of [26, 113], sometimes with again more general contest success functions and [111] provides an interesting characterization of the active contestants in the case of constant returns to scale. However, a closed form of the solution to the asymmetric n-player contest game, which is needed to solve the effort maximization problem, is so far only available for constant returns to scale.

Another model which is very closely related to the lottery contest considered here is the so called all-pay auction. The most prominent difference is that in an all-pay auction the player with the highest bid or effort wins the price with certainty. Therefore, this type of contests is usually not influenced by scaling the efforts but by explicitly deciding who is allowed to participate. Although these models are very similar, they exhibit a completely different behavior. In [12] it is shown that whenever there are more than two players, the auction has not one unique equilibrium but a whole continuum. The same authors show in [11] that a contest designer, whose incentive is to maximize the overall effort, will exclude the strongest players, i.e. those with the highest valuation and the lowest costs, up to a certain threshold from the auction. As we mentioned before, this is completely opposed to the results we obtained for the optimal lottery contest. We will go more into detail about these differences later on. Finally, we would like to refer the interested reader to [4, 32, 35] for a more extensive discussion on relations between the all-pay auction and the lottery model.

This part of the thesis is structured as follows: Chapter 2 is all about the case of constant returns to scale. We provide an analysis of the underlying Nash equilibrium, the existence of an optimally biased contest is proven and we derive a closed form of the optimal weights, the resulting equilibrium efforts and other interesting figures. Afterwards, we illustrate our theory on some well known examples and compare our results to those known for the all-pay auction model. In Chapter 3, we consider more general rent-seeking technologies, provide some theoretical results, and reformulate the effort maximization problem as an MPCC.

2. Constant Returns to Scale

In this chapter, we are concerned with the case of constant returns to scale, i.e., we assume that $c(x) \equiv x$. The underlying contest game has already been thoroughly analyzed, see for example [26, 113, 111], but for completeness sake, we will derive existence and uniqueness of the Nash equilibrium together with a formula for the equilibrium efforts step by step in Section 2.1. Although there are already results on the optimal contest in the 2-player case and the homogeneous case where $\beta_1 = \ldots = \beta_n$, cf. [92, 28], the inhomogeneous n-player case has not yet been analyzed. Thus, we will prove the existence of an optimal contest in Section 2.2 and give a closed form of the optimal solution in Section 2.3. After providing some well-known examples in Section 2.4, we conclude by comparing our results to those known for the all-pay auction in Section 2.5.

2.1. The Underlying Contest Game

Once the designer has decided on his variables, every player $\nu \in N$ tries to maximize his utility function

$$\theta_\nu(x_\nu, x_{-\nu}) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\alpha_\nu x_\nu}{\sum_{\mu \in N} \alpha_\mu x_\mu} V_\nu - \beta_\nu x_\nu & \text{else} \end{cases}$$

under the restriction that his effort is nonnegative, i.e. $x_\nu \geq 0$. Since his outcome also depends on the efforts chosen by the other players, he has to predict their choice. Of course, all other participants try to do the same, hence we end up with n coupled optimization problems. This type of problem is known as Nash equilibrium problem, cf. [107] for a brief introduction to game theory, and one defines its solution as follows:

Definition 2.1 *A vector $x^* \in \mathbb{R}^n$ is called a* solution of the Nash equilibrium problem *or* Nash equilibrium *for short, if for all players $\nu \in N$*

$$\theta_\nu(x_\nu^*, x_{-\nu}^*) \geq \theta_\nu(x_\nu, x_{-\nu}^*) \quad \forall x_\nu \geq 0.$$

Obviously, this solution depends on the parameters α_μ chosen by the contest designer. In this section, we will assume that these parameters are fixed, i.e., the designer has made his decision and thus we will not indicate this dependence in the notation. It is easy to verify that the solutions of the contest game described above do not change if we replace the utility functions θ_ν by

$$\tilde{\theta}_\nu(x_\nu, x_{-\nu}) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\alpha_\nu x_\nu}{\sum_{\mu \in N} \alpha_\mu x_\mu} - \frac{\beta_\nu}{V_\nu} x_\nu & \text{else.} \end{cases}$$

Hence, we may, redefining β_ν if necessary, assume without loss of generality $V_\nu = 1$ for all players $\nu \in N$. Although the solution of this simplified Nash game still requires some work, the following lemma states some obvious properties which facilitate the calculation of the solution.

Lemma 2.2 *The vector* $(0, \dots, 0)$ *is never a Nash equilibrium and in every Nash equilibrium* x^* *at least two components are positive.*

Proof. If $(0, \dots, 0)$ was a Nash equilibrium, then $x_\nu = 0$ had to be a solution of

$$\max_{x_\nu} \theta_\nu(x_\nu, 0) \text{ subject to } x_\nu \geq 0$$

for every player $\nu \in N$. However, in this case, the utility function attains the form

$$\theta_\nu(x_\nu, 0) = \begin{cases} 0 & \text{if } x_\nu = 0, \\ 1 - \beta_\nu x_\nu & \text{else} \end{cases} \tag{2.1}$$

and it is easy to see that $\theta_\nu(x_\nu, 0) > 0$ if $x_\nu > 0$ is chosen sufficiently small. Hence, the vector $(0, \dots, 0)$ cannot be a Nash equilibrium.

If there was a Nash equilibrium x^* with only one positive component $x_\nu^* > 0$, player ν would face the utility function (2.1) and could hence increase his gain by choosing any $x_\nu \in (0, x_\nu^*)$ which contradicts the definition of a Nash equilibrium. □

This lemma implies that a contest with $n = 1$ players, which is not actually a contest anymore, does not have a solution. Hence, we will assume $n \geq 2$ from now on.

In order to calculate the Nash equilibria of this contest game, we first calculate the best answer function for every player $\nu \in N$ and use it to derive some properties of the Nash equilibria in Section 2.1.1. These properties allow us to identify every Nash equilibrium x^* with the corresponding set of active players $K := \{\nu \in N \mid x_\nu^* > 0\}$. A thorough analysis of these sets in Section 2.1.2 finally yields that the contest game always has a unique Nash equilibrium which is the basis for our analysis of the optimal contest in the Sections 2.2 and 2.3.

2.1.1. The Best Answer Function

The *best answer function* $S_\nu : \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ assigns every possible choice $x_{-\nu}$ of the other players the best answer of player ν, i.e. the solutions of the optimization problem

$$\max_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}) \text{ subject to } x_\nu \geq 0.$$

Note that it is theoretically possible that this optimization problem has more than one solution or no solution at all. However, we will see that the problem considered here has exactly one solution in all interesting cases. To this end, we introduce the following abbreviation

$$c_\nu(x_{-\nu}) := \sum_{\mu \in N \setminus \{\nu\}} \alpha_\mu x_\mu$$

for every player $\nu \in N$ where $x_{-\nu} \in \mathbb{R}_+^{n-1}$ is a feasible combination of strategies. The following lemma provides the best answer function for all feasible strategies $x_{-\nu} \neq 0$. As we have seen in the proof of Lemma 2.2, there is no best answer to $x_{-\nu} = (0, \dots, 0)$. This, however, is not a problem since Lemma 2.2 also implies $x^*_{-\nu} \neq (0, \dots, 0)$ in every Nash equilibrium x^*.

Lemma 2.3 *For every player $\nu \in N$ and every combination of strategies $x_{-\nu} \in \mathbb{R}_+^{n-1}$ with $x_{-\nu} \neq (0, \dots, 0)$, the best answer function is given by*

$$S_\nu(x_{-\nu}) = \max \left\{ 0, -\frac{c_\nu(x_{-\nu})}{\alpha_\nu} + \sqrt{\frac{c_\nu(x_{-\nu})}{\alpha_\nu \beta_\nu}} \right\}$$

Proof. Choose an arbitrary but fixed player $\nu \in N$ and an arbitrary combination of strategies $x_{-\nu} \in \mathbb{R}_+^{n-1}$ with $x_{-\nu} \neq 0$. Using the abbreviation introduced above, the player's utility function can then be written as

$$\theta_\nu(x_\nu, x_{-\nu}) = \frac{\alpha_\nu x_\nu}{c_\nu(x_{-\nu}) + \alpha_\nu x_\nu} - \beta_\nu x_\nu$$

and has the properties

$$\lim_{x_\nu \to \infty} \theta_\nu(x_\nu, x_{-\nu}) = -\infty \quad \text{and} \quad \lim_{x_\nu \downarrow -\frac{c_\nu(x_{-\nu})}{\alpha_\nu}} \theta_\nu(x_\nu, x_{-\nu}) = -\infty.$$

The derivative of θ_ν with respect to x_ν is given by

$$\nabla_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}) = \frac{\alpha_\nu c_\nu(x_{-\nu})}{(\alpha_\nu x_\nu + c_\nu(x_{-\nu}))^2} - \beta_\nu$$

and has the two roots

$$x_{\nu,1/2} = -\frac{c_\nu(x_{-\nu})}{\alpha_\nu} \pm \sqrt{\frac{c_\nu(x_{-\nu})}{\alpha_\nu \beta_\nu}}.$$

Here, we have

$$\lim_{x_\nu \to \infty} \nabla_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}) = -\beta_\nu < 0 \quad \text{and} \quad \lim_{x_\nu \downarrow -\frac{c_\nu(x_{-\nu})}{\alpha_\nu}} \nabla_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}) = +\infty.$$

Hence, $\theta_\nu(\cdot, x_{-\nu})$ is strictly increasing on the interval

$$\left(-\frac{c_\nu(x_{-\nu})}{\alpha_\nu}, -\frac{c_\nu(x_{-\nu})}{\alpha_\nu} + \sqrt{\frac{c_\nu(x_{-\nu})}{\alpha_\nu \beta_\nu}} \right)$$

and strictly decreasing on the interval

$$\left(-\frac{c_\nu(x_{-\nu})}{\alpha_\nu} + \sqrt{\frac{c_\nu(x_{-\nu})}{\alpha_\nu \beta_\nu}}, +\infty \right)$$

and consequently attains its maximum over the union of these intervals in

$$\bar{x}_\nu = -\frac{c_\nu(x_{-\nu})}{\alpha_\nu} + \sqrt{\frac{c_\nu(x_{-\nu})}{\alpha_\nu\beta_\nu}}.$$

Therefore, the unique best answer of player ν is $\bar{x}_\nu$ if this value is nonnegative and 0 else. □

Now, we are going to use the best answer function to determine the Nash equilibria of this contest. To this end note that the definition of Nash equilibria implies that x^* is a solution of the contest if and only if $x^*_\nu = S_\nu(x^*_{-\nu})$ for all players $\nu \in N$. We are going to exploit this relation in the proof of the following result which has already been derived by different means in [111].

Theorem 2.4 *Let $x^* \in \mathbb{R}^n_+$ be a Nash equilibrium and $K := \{\nu \in N \mid x^*_\nu > 0\}$ the set of active players with $k := |K|$. Then for all $\nu \in N$*

$$x^*_\nu = \max\left\{0, \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha_\mu}}\right\}$$

and the set of active players is characterized by the equivalence

$$\nu \in K \quad\Longleftrightarrow\quad (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu\in K}\frac{\beta_\mu}{\alpha_\mu}. \tag{2.2}$$

Proof. Since x^* is a Nash equilibrium, Lemma 2.2 implies $k \geq 2$. As already mentioned above, x^* is a solution of the contest if and only if $x^*_\nu = S_\nu(x^*_{-\nu})$ for all players $\nu \in N$. Together with Lemma 2.3, this implies for all $\nu \in K$

$$\begin{aligned} \sum_{\mu\in N}\alpha_\mu x^*_\mu &= \sqrt{\frac{\alpha_\nu}{\beta_\nu}}\sqrt{\sum_{\mu\in N\setminus\{\nu\}}\alpha_\mu x^*_\mu} \\ \Longleftrightarrow\quad \left(\sum_{\mu\in K}\alpha_\mu x^*_\mu\right)^2 &= \frac{\alpha_\nu}{\beta_\nu}\left(\sum_{\mu\in K}\alpha_\mu x^*_\mu - \alpha_\nu x^*_\nu\right) \\ \Longleftrightarrow\quad \alpha_\nu x^*_\nu &= \sum_{\mu\in K}\alpha_\mu x^*_\mu - \frac{\beta_\nu}{\alpha_\nu}\left(\sum_{\mu\in K}\alpha_\mu x^*_\mu\right)^2. \end{aligned} \tag{2.3}$$

By adding up this equation for all $\nu \in K$, we obtain

$$\begin{aligned} \sum_{\nu\in K}\alpha_\nu x^*_\nu &= k\left(\sum_{\mu\in K}\alpha_\mu x^*_\mu\right) - \left(\sum_{\nu\in K}\frac{\beta_\nu}{\alpha_\nu}\right)\left(\sum_{\mu\in K}\alpha_\mu x^*_\mu\right)^2 \\ \Longleftrightarrow\quad \left(\sum_{\nu\in K}\frac{\beta_\nu}{\alpha_\nu}\right)\left(\sum_{\mu\in K}\alpha_\mu x^*_\mu\right) &= k-1 \end{aligned}$$

$$\iff \sum_{\mu\in K} \alpha_\mu x_\mu^* = \frac{k-1}{\left(\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}\right)}.$$

Inserting this expression into (2.3) yields for all $\nu \in K$

$$x_\nu^* = \frac{1}{\alpha_\nu}\left(\sum_{\mu\in K} \alpha_\mu x_\mu^* - \frac{\beta_\nu}{\alpha_\nu}\left(\sum_{\mu\in K} \alpha_\mu x_\mu^*\right)^2\right) = \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}.$$

For all $\nu \in N \setminus K$ we have $x_\nu^* = 0$ by definition of K and according to the best answer function

$$\begin{aligned} 0 &> -\frac{\sum_{\mu\in K} \alpha_\mu x_\mu^*}{\alpha_\nu} + \sqrt{\frac{\sum_{\mu\in K} \alpha_\mu x_\mu^*}{\alpha_\nu \beta_\nu}} \\ \iff \quad 0 &> \frac{1}{\alpha_\nu}\left(\frac{1}{\beta_\nu} - \frac{1}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}} \\ \iff \quad 0 &> \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}. \end{aligned}$$

Hence, we obtain the postulated formula

$$x_\nu^* = \max\left\{0, \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right\}$$

for all $\nu \in N$. Obviously, $\nu \in K$ if and only if

$$1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}} > 0 \quad \iff \quad (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu},$$

which gives us the characterization of the set of active players K. □

This theorem should not be overinterpreted. It does not include any statement concerning existence or uniqueness of a solution and the *implicit* characterization of the set of active players is also not very helpful. In the following, we will see that the contest game indeed has a unique solution and we will also give a more practicable characterization of the set K. But first, we give some inversion of Theorem 2.4.

Theorem 2.5 *Let $K \subseteq N$ be a subset with $k := |K| \geq 2$ such that*

$$\nu \in K \quad \iff \quad (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}.$$

Then the vector $x^* \in \mathbb{R}^n$ with the components

$$x_\nu^* = \max\left\{0, \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha_\mu}}\right\}$$

is a solution of the contest game.

Proof. The assumptions on K guarantee $x_\nu^* = 0$ for all $\nu \in N \setminus K$ and $x_\nu^* > 0$ for all $\nu \in K$. To prove that x^* is a solution, we will employ Lemma 2.3 and show $x_\nu^* = S_\nu(x_{-\nu}^*)$ for all $\nu \in N$. To this end, define

$$c(x) := \sum_{\mu \in N} \alpha_\mu x_\mu$$

and note

$$c(x^*) = \sum_{\nu \in K} \left(1 - \frac{\beta_\nu}{\alpha_\nu} \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}} \right) \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}} = \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}} (k - (k-1)) = \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}}.$$

Now consider an inactive player $\nu \in N \setminus K$. The properties of K then yield $\frac{\alpha_\nu}{\beta_\nu} \leq c(x^*)$ and hence $S_\nu(x_{-\nu}^*) > 0$ would imply

$$0 < -\frac{c(x^*)}{\alpha_\nu} + \sqrt{\frac{c(x^*)}{\alpha_\nu \beta_\nu}} \quad \Longleftrightarrow \quad c(x^*) < \sqrt{\frac{\alpha_\nu c(x^*)}{\beta_\nu}} \quad \Longleftrightarrow c(x^*) < \frac{\alpha_\nu}{\beta_\nu},$$

a contradiction. If we consider an active player $\nu \in K$, we know $\frac{\alpha_\nu}{\beta_\nu} > c(x^*)$ and can prove analogously to the considerations above that $S_\nu(x_{-\nu}^*) > 0$ since $S_\nu(x_{-\nu}^*) = 0$ would imply

$$\begin{aligned} 0 \geq -\frac{c(x^*) - \alpha_\nu x_\nu^*}{\alpha_\nu} + \sqrt{\frac{c(x^*) - \alpha_\nu x_\nu^*}{\alpha_\nu \beta_\nu}} \quad &\Longleftrightarrow \quad c(x^*) - \alpha_\nu x_\nu^* \geq \sqrt{\frac{\alpha_\nu}{\beta_\nu}} \sqrt{c(x^*) - \alpha_\nu x_\nu^*} \\ &\Longleftrightarrow \quad c(x^*) - \alpha_\nu x_\nu^* \geq \frac{\alpha_\nu}{\beta_\nu} \\ &\Longleftrightarrow \quad c(x^*) - \left(1 - \frac{\beta_\nu}{\alpha_\nu} c(x^*)\right) c(x^*) \geq \frac{\alpha_\nu}{\beta_\nu} \\ &\Longleftrightarrow \quad c(x^*)^2 \geq \left(\frac{\alpha_\nu}{\beta_\nu}\right)^2, \end{aligned}$$

again a contradiction. Here, we used the fact that $k \geq 2$, ergo $c(x^*) - \alpha_\nu x_\nu^* > 0$, in the second equivalence and the definition of x^* in the last but one equivalence. It remains to prove that x_ν^* is in fact equal to $S_\nu(x_{-\nu}^*)$. However, due to Lemma 2.3 and the definition of x_ν^*, we have

$$\begin{aligned} S_\nu(x_{-\nu}^*) &= -\frac{c_\nu(x_{-\nu}^*)}{\alpha_\nu} + \sqrt{\frac{c_\nu(x_{-\nu}^*)}{\alpha_\nu \beta_\nu}} \\ &= x_\nu^* - \frac{c(x^*)}{\alpha_\nu} + \sqrt{\frac{c(x^*) - \left(1 - \frac{\beta_\nu}{\alpha_\nu} c(x^*)\right) c(x^*)}{\alpha_\nu \beta_\nu}} \\ &= x_\nu^*. \end{aligned}$$

This proves that the vector x^* is indeed a solution of the contest game. □

Thanks to the Theorems 2.4 and 2.5, we can now characterize the solutions of this contest game in terms of the subset $K \subseteq N$ satisfying the conditions in Theorem 2.5. This is formalized in the following corollary.

Corollary 2.6 *The set of all Nash equilibria x^* can be identified with the set of all subsets $K \subseteq N$ satisfying the conditions in Theorem 2.5 as follows: If x^* is a Nash equilibrium, then the set of active players $K := \{\nu \in N \mid x^*_\nu > 0\}$ satisfies the conditions in Theorem 2.5. If on the other hand a subset $K \subseteq N$ satisfies the conditions in Theorem 2.5, then the vector x^* defined in this theorem is a Nash equilibrium.*

This, however, does still not provide any information about existence and uniqueness of solutions. But is does propose a way to obtain these results, namely by analyzing the sets K satisfying the conditions in Theorem 2.5, which is exactly what we are going to do in the next section.

2.1.2. Existence and Uniqueness of a Solution

To obtain more information about the sets $K \subseteq N$ satisfying the conditions in Theorem 2.5, we assume from now on without loss of generality that the players are ordered in such a way that

$$\frac{\beta_1}{\alpha_1} \leq \frac{\beta_2}{\alpha_2} \leq \ldots \leq \frac{\beta_n}{\alpha_n} \tag{2.4}$$

holds. This ordering enables us to simplify the proof of the following result which guarantees that, if there is a solution, this solution is unique.

Lemma 2.7 *There is at most one set $K \subseteq N$ with $k := |K| \geq 2$ and*

$$\nu \in K \quad \Longleftrightarrow \quad (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}.$$

Proof. Due to the properties of K, we know

$$\max_{\nu \in K} \frac{\beta_\nu}{\alpha_\nu} < \min_{\mu \in N \setminus K} \frac{\beta_\mu}{\alpha_\mu}.$$

Together with the assumption in (2.4), this implies $K = \{1, 2, \ldots, k\}$. Now assume that there was a second set $L \neq K$ with the same properties as K. This would imply $L = \{1, 2, \ldots, l\}$ with $l < k$ or $l > k$. Without loss of generality, we consider only the case $l > k$. Due to $l \notin K$, we have

$$(k-1)\frac{\beta_l}{\alpha_l} \geq \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}.$$

This together with the ordering of the players (2.4) implies

$$(l-1)\frac{\beta_l}{\alpha_l} = (l-k)\frac{\beta_l}{\alpha_l} + (k-1)\frac{\beta_l}{\alpha_l} \geq \sum_{\mu=k+1}^{l} \frac{\beta_\mu}{\alpha_\mu} + \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu} = \sum_{\mu \in L} \frac{\beta_\mu}{\alpha_\mu},$$

which is a contradiction to $l \in L$. Ergo, there is at most one set K with these properties. □

Obviously, this lemma implies that the contest game has at most one solution. The next result provides a more useful characterization of K, which was already obtained in [111], and at the same time guarantees the existence of a Nash equilibrium.

Lemma 2.8 *If the players are ordered according to* (2.4)*, then the set*

$$K := \left\{ \nu \in N \,\middle|\, (\nu - 1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu=1}^{\nu} \frac{\beta_\mu}{\alpha_\mu} \right\}$$

satisfies the conditions in Theorem 2.5. In particular, there is always at least one set K satisfying the conditions in Theorem 2.5.

Proof. Due to the definition of this set K, obviously $\nu = 1 \in K$ and $\nu = 2 \in K$. Hence, we know $k := |K| \geq 2$. Also, it is easy to see that $\nu \in K$ directly implies $\nu - 1 \in K$, hence $K = \{1, 2, \ldots, k\}$. It remains to verify that

$$(k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu=1}^{k} \frac{\beta_\mu}{\alpha_\mu}$$

if and only if $\nu \in K$. However, due to the ordering of the players, this condition is satisfied for all $\nu \in K$, since by definition of K it is satisfied for the biggest fraction $\frac{\beta_k}{\alpha_k}$. Analogously, this condition is not satisfied for all $\nu \in N \setminus K$ since it is not satisfied for the smallest fraction $\frac{\beta_{k+1}}{\alpha_{k+1}}$ due to

$$k\frac{\beta_{k+1}}{\alpha_{k+1}} \geq \sum_{\mu=1}^{k+1} \frac{\beta_\mu}{\alpha_\mu} \quad \Longleftrightarrow (k-1)\frac{\beta_{k+1}}{\alpha_{k+1}} \geq \sum_{\mu=1}^{k} \frac{\beta_\mu}{\alpha_\mu}$$

This proves that K satisfies all conditions from Theorem 2.5. □

Lemma 2.7 together with Lemma 2.8 proves that there is exactly one set K satisfying the conditions in Theorem 2.5 and hence, the contest game has exactly one solution which has already been derived by different means in [26, 113]. Lemma 2.8 provides a way to calculate the set of active players in the Nash equilibrium and Theorem 2.5 gives a formula for the equilibrium efforts. These results are collected in the following corollary.

Corollary 2.9 *There is exactly one set $K \subseteq N$ with $k := |K| \geq 2$ and*

$$\nu \in K \quad \Longleftrightarrow \quad (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}.$$

In the unique Nash equilibrium x^* of the contest game, we have

$$x_\nu^* = \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu \in K}\frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu \in K}\frac{\beta_\mu}{\alpha_\mu}}$$

for all active players $\nu \in K$ and $x_\nu^* = 0$ else.

We would like to conclude this analysis with another direct corollary of Lemma 2.8. The upper approximation of the set K derived in this corollary can be useful to calculate this set and will be used in the subsequent analysis.

Corollary 2.10 *If the players are ordered according to* (2.4) *and K is the unique set satisfying the conditions in Theorem 2.5, then*

$$K \subseteq \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right\}.$$

Proof. Due to the ordering of the players, $K = \{1, 2, \dots, k\}$. The inequality $\frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}$ obviously holds for $\nu = 1, 2$. For $\nu = 3, \dots, k$ this inequality follows inductively from

$$(\nu - 2)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu=1}^{\nu-1} \frac{\beta_\mu}{\alpha_\mu} < (\nu - 2)\left(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right),$$

where we used the characterization of K given in Lemma 2.8. □

The following example illustrates that the inclusion in this corollary can be an inequality but, in general, will be a strict inclusion.

Example 2.11 Consider a game with four players and $\alpha = (1, 1, 1, 1)$.

(a) If $\beta = (2, 3, 3.5, 4)$, we have

$$K = \{1, 2, 3, 4\} = \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\}.$$

(b) If, however, $\beta = (2, 3, 3.5, 4.5)$, we have

$$K = \{1, 2, 3\} \subsetneq \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\},$$

i.e., in this case K is a strict subset. ◇

2.2. Existence of an Optimal Contest

Having proven the existence of a unique solution of the contest game in Section 2.1, we can now focus on the contest designers problem. To this end, we now assume $\alpha \in \mathbb{R}^n_{++}$ to be a variable again and denote the – due to Corollary 2.9 unique – solution of the contest game with the variable α by $x^*(\alpha)$, the corresponding set of active players by $K(\alpha)$ and its number of elements by $k(\alpha) := |K(\alpha)|$. The values β_ν, however, are still fixed since they represent the inherent cost parameters of the participating players and thus cannot be altered neither by the contest designer nor by the players themselves. Hence, the contest designers problem is

$$\max_{\alpha} \theta_0(\alpha) \quad \text{subject to} \quad \alpha > 0, \tag{2.5}$$

where

$$\theta_0(\alpha) = \sum_{\nu=1}^{n} x_\nu^*(\alpha) = \sum_{\nu \in K(\alpha)} x_\nu^*(\alpha) = \frac{k(\alpha)-1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \left(\sum_{\mu \in K(\alpha)} \frac{1}{\alpha_\mu} - \frac{k(\alpha)-1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu^2} \right) \tag{2.6}$$

and the set of active players is defined by

$$\nu \in K(\alpha) \quad \Longleftrightarrow \quad (k(\alpha)-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}. \tag{2.7}$$

The aim of this section is to prove that the designers optimization problem (2.5) has a solution. This is not clear a priori since the feasible set is both unbounded and open and the objective function might be noncontinuous. In Section 2.2.1 we are going to prove that the value of the designer's utility function remains unchanged under certain variations of the variable α. On the one hand, this enables us to restrict the feasible area for (2.5) to a bounded set, but on the other hand, it implies that an optimal solution, if it exists, can never be unique. The next step is then to analyze the continuity properties of θ_0 on this bounded set and to prove that the function can be continuously extended onto the closure of this set. This is done in Section 2.2.2.

2.2.1. Nonuniqueness of the Optimal Contest

Our next step is to analyze some manipulations of the argument α that leave the value of θ_0 unchanged. These manipulations are interesting since they shed some light on the influence of the weights α_ν on the behavior of the contestants and also imply that, if there is an optimal contest, it cannot be uniquely determined. On the other hand, these manipulations are also useful for the analysis of the continuity of θ_0. The first result in this section states that the value of the designer's utility function does not change if one scales α with a nonnegative factor.

Lemma 2.12 *For all $\alpha \in \mathbb{R}^n_{++}$ and all $c > 0$, we have*

$$K(c\alpha) = K(\alpha) \quad \textit{and} \quad \theta_0(c\alpha) = \theta_0(\alpha).$$

This is quite obvious since such a variation of α does not change the participants's utility functions θ_ν ($\nu \in N$) and consequently the Nash equilibrium does not change either. However, this result enables us to restrict the feasible set of the designer's optimization problem to a bounded set. To this end, consider an arbitrary $\alpha \in \mathbb{R}^n_{++}$. Lemma 2.12 then implies

$$\theta_0(\alpha) = \theta_0\left(\frac{1}{\sum_{\nu \in N} \alpha_\nu}\alpha\right).$$

Thus, defining

$$A := \left\{ \alpha \in \mathbb{R}^n_{++} \,\middle|\, \sum_{\nu \in N} \alpha_\nu = 1 \right\}, \tag{2.8}$$

we obtain $\theta_0(\mathbb{R}^n_{++}) = \theta_0(A)$ and the designer's utility function attains a global maximum on the unbounded set $\mathbb{R}^n_{++}$ if and only if it attains a global maximum on the bounded set A. The set A, however, is still not closed and the function θ_0 is a priori not defined on the closure of A. We will deal with this problem in the next section. But first, we would like to provide another manipulation of α that leaves the value of θ_0 unchanged. Roughly, this result says that we may choose the weights α_ν of inactive players arbitrarily, as long as they remain below a certain threshold depending on the weights of the active players. In particular, we do not need to put $\alpha_\nu = 0$ to exclude a certain player ν, it suffices to choose $\alpha_\nu > 0$ sufficiently small.

Lemma 2.13 *Let $\alpha^* \in \mathbb{R}^n_{++}$ be arbitrarily given. Then $K(\alpha^*) = K(\alpha)$ and $\theta_0(\alpha^*) = \theta_0(\alpha)$ hold for all $\alpha \in \mathbb{R}^n_{++}$ satisfying the following properties:*

(a) For all $\nu \in K(\alpha^*)$, we have

$$\alpha_\nu = \alpha^*_\nu.$$

(b) For all $\nu \notin K(\alpha^*)$, we have

$$\alpha_\nu \in \left(0, \frac{(k(\alpha^*) - 1)\beta_\nu}{\sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu}}\right].$$

Proof. Choose $\alpha \in \mathbb{R}^n_{++}$ in such a way that the two properties (a) and (b) hold. Due to Corollary 2.9, the set of active players $K(\alpha)$ corresponding to the choice of α as weights is uniquely defined. Using property (a), we obtain for all $\nu \in K(\alpha^*)$

$$(k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu} = (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha^*_\nu} < \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

On the other hand, property (b) implies for all $\nu \notin K(\alpha^*)$

$$(k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu} \geq \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

The uniqueness of $K(\alpha)$ therefore gives $K(\alpha) = K(\alpha^*)$. Together with property (a) we then obtain $\theta_0(\alpha) = \theta_0(\alpha^*)$. □

This lemma will be used in the next section to prove the existence of a global maximum. It also provides some insight in the relation between the choice of the weights α_ν and the behavior of the players.

2.2.2. Continuous Extension of the Designer's Utility Function

In order to prove the existence of a global maximum, we first verify that the designer's utility function θ_0 is continuous on $\mathbb{R}^n_{++}$. This is not obvious, since the set $K(\alpha)$, which plays a critical role in the definition of θ_0, may change with α.

Theorem 2.14 *The objective function θ_0 is continuous on $\mathbb{R}^n_{++}$. Moreover, this function is continuously differentiable in an open neighborhood of any vector $\alpha^* \in \mathbb{R}^n_{++}$ having the following property:*

$$\nu \notin K(\alpha^*) \Longrightarrow (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha^*_\nu} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu}. \tag{2.9}$$

Proof. The statement regarding the continuous differentiability is clear since condition (2.9) guarantees that, locally, the index set $K(\alpha^*)$ is constant, hence $K(\alpha) = K(\alpha^*)$ for all α from a sufficiently small neighborhood of α^*. In particular, θ_0 is continuous in these points.

In order to verify the continuity of θ_0 on the whole set $\mathbb{R}^n_{++}$, it therefore remains to consider a point $\alpha^* \in \mathbb{R}^n_{++}$ such that the index set

$$L := \left\{ \nu \in N \;\middle|\; (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha^*_\nu} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu} \right\}$$

is nonempty. Now, it is not difficult to see that there is a neighborhood $U \subseteq \mathbb{R}^n_{++}$ of α^* such that

$$K \subseteq K(\alpha) \subseteq K \cup L \quad \forall \alpha \in U,$$

where, for simplicity of notation, we use the abbreviation $K := K(\alpha^*)$. Let us further write $k := |K|$ und $l := |L|$. Then, for each $\alpha \in U$, we have $K(\alpha) = M$ for one of the 2^l sets M satisfying $K \subseteq M \subseteq K \cup L$. Setting $m := |M|$ and using

$$\frac{\beta_\nu}{\alpha^*_\nu} = \frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu}}{k - 1}$$

for all $\nu \in M \setminus K$, we obtain for all these index sets M

$$\begin{aligned}
\theta_0^M(\alpha^*) \;&:=\; \frac{m-1}{\sum_{\mu \in M} \frac{\beta_\mu}{\alpha^*_\mu}} \left(\sum_{\mu \in M} \frac{1}{\alpha^*_\mu} - \frac{m-1}{\sum_{\mu \in M} \frac{\beta_\mu}{\alpha^*_\mu}} \sum_{\mu \in M} \frac{\beta_\mu}{(\alpha^*_\mu)^2} \right) \\
&=\; \frac{m-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu} \cdot \left(1 + \frac{m-k}{k-1}\right)} \left[\sum_{\mu \in K} \frac{1}{\alpha^*_\mu} + \sum_{\mu \in M \setminus K} \frac{\sum_{\lambda \in K} \frac{\beta_\lambda}{\alpha^*_\lambda}}{(k-1)\beta_\mu} \right. \\
&\qquad \left. - \frac{m-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu} \cdot \left(1 + \frac{m-k}{k-1}\right)} \left(\sum_{\mu \in K} \frac{\beta_\mu}{(\alpha^*_\mu)^2} + \sum_{\mu \in M \setminus K} \frac{\left(\sum_{\lambda \in K} \frac{\beta_\lambda}{\alpha^*_\lambda}\right)^2}{(k-1)^2 \beta_\mu} \right) \right] \\
&=\; \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu}} \left[\sum_{\mu \in K} \frac{1}{\alpha^*_\mu} + \frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu}}{(k-1)} \sum_{\mu \in M \setminus K} \frac{1}{\beta_\mu} \right. \\
&\qquad \left. - \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu}} \left(\sum_{\mu \in K} \frac{\beta_\mu}{(\alpha^*_\mu)^2} + \left(\frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha^*_\mu}}{k-1} \right)^2 \sum_{\mu \in M \setminus K} \frac{1}{\beta_\mu} \right) \right]
\end{aligned}$$

$$= \quad \theta_0(\alpha^*).$$

Since the 2^l functions θ_0^M are continuous in α^*, we obtain for an arbitrary $\varepsilon > 0$ and all M a suitable $\delta_M > 0$ such that $|\theta_0^M(\alpha) - \theta_0^M(\alpha^*)| < \varepsilon$ for all $\alpha \in \mathbb{B}(\alpha^*; \delta_M)$. Define

$$\delta := \min\{\delta_M \mid K \subseteq M \subseteq K \cup L\}.$$

Then we obtain for all $\alpha \in \mathbb{B}(\alpha^*; \delta)$ that $K(\alpha) = M$ for one of the above index sets M and, therefore, $\left|\theta_0(\alpha) - \theta_0(\alpha^*)\right| = \left|\theta_0^M(\alpha) - \theta_0^M(\alpha^*)\right| < \varepsilon$. This proves continuity of θ_0 in α^*. □

Now, we know that θ_0 is continuous on the feasible set of the contest designer's optimization problem (2.5). However, this set is unbounded and open, hence continuity of the objective function alone is not enough to guarantee the existence of a global maximum. To solve the problem with the unboundedness of the feasible set, we are now coming back to the set A as defined in (2.8). Recall that $\theta_0(\mathbb{R}^n_{++}) = \theta_0(A)$ and hence the designer's utility function has a global maximum on $\mathbb{R}^n_{++}$ if and only if it has a global maximum on A. Note that the set A is not closed and thus not compact. Since we want to exploit that continuous functions attain at least one global maximum on compact sets, our next step is to extend the function θ_0 continuously onto the closure $\mathrm{cl}(A)$. Note that, a priori, the designer's utility function is not defined on $\mathrm{cl}(A) \setminus A$. In order to simplify our notation in the following results, let us define the index set

$$J(\alpha) := \{\nu \in N \mid \alpha_\nu = 0\}$$

for a given $\alpha \in \mathbb{R}^n_+$. Note that by definition of A, one has $|J(\alpha)| \in \{0, 1, \dots, n-1\}$ for all $\alpha \in \mathrm{cl}(A)$. The next result extends θ_0 onto those $\alpha \in \mathrm{cl}(A)$ with $|J(\alpha)| \leq n - 2$.

Lemma 2.15 *The function θ_0, viewed as a mapping from A to $\mathbb{R}$, can be extended continuously onto the set $\{\alpha \in cl(A) \mid |J(\alpha)| \leq n-2\}$.*

Proof. Recall from the proof of Theorem 2.14 that θ_0 is continuous on the set

$$A = \{\alpha \in \mathrm{cl}(A) \mid |J(\alpha)| = 0\}$$

(in fact, it is continuous on $\mathbb{R}^n_{++}$). Now, let $\alpha^* \in \mathrm{cl}(A)$ with $|J(\alpha^*)| \in \{1, \dots, n-2\}$ be arbitrarily given. Then let us define the set of players $N^* := N \setminus J(\alpha^*)$. Since we have $|N^*| \geq 2$, it follows that the Nash game with the set of players N^* replacing the set of players N has all the properties that were already shown. Consequently, if we let

$$\theta_0^*(\alpha) := \sum_{\nu \in N^*} x_\nu^*(\alpha)$$

be the objective function of this new game, we, in particular, obtain from Theorem 2.14 that θ_0^* is continuous in a sufficiently small neighborhood of α^* simply since we eliminated the critical players ν with $\alpha_\nu^* = 0$ from the set N. We will show in the next paragraph that, for all α from a sufficiently small neighborhood U of α^*, we have $K(\alpha) \subseteq N^*$. This then implies $\theta_0(\alpha) = \theta_0^*(\alpha)$ for all $\alpha \in U$ and, in this way, we obtain the desired continuous extension of θ_0 in α^*.

To verify the above claim, we have to find a sufficiently small neighborhood U of α^* such that $K(\alpha) \subseteq N^*$ for all $\alpha \in U$, i.e., for all $\alpha \in U$ and all indices ν with $\nu \in K(\alpha)$ we necessarily have $\alpha_\nu > 0$. By contraposition, this is equivalent to showing that, for all $\alpha \in U$ and all indices ν with $\alpha_\nu = 0$, we have $\nu \notin K(\alpha)$. To see this, we first choose a sufficiently small neighborhood of α^* such that $|J(\alpha)| \in \{0, 1, \dots, n-2\}$ for all $\alpha \in U$. We then define a function $c(\alpha)$ on U as the sum of the two smallest quotients $\frac{\beta_\mu}{\alpha_\mu}$ ($\mu \in N$). Then $c(\alpha)$ is continuous and finite. Moreover, Corollary 2.10 shows that we always have $K(\alpha) \subseteq \{\nu \in N \mid \frac{\beta_\nu}{\alpha_\nu} < c(\alpha)\}$. By taking a possibly smaller neighborhood U, we may assume by continuity that $c(\alpha) < 2c(\alpha^*)$ for all $\alpha \in U$ and, in addition, that $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*)$ for all $\nu \in J(\alpha^*)$. This implies the desired claim since, now, we obtain $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*) > c(\alpha)$ for all $\alpha \in U$ and all $\nu \in J(\alpha^*)$, hence $\nu \notin K(\alpha)$. □

Now, it remains to consider those $\alpha \in \text{cl}(A)$ with $|J(\alpha)| = n-1$. It turns out that these points are a little more complicated but it is still possible to extend θ_0 continuously into those remaining points.

Lemma 2.16 *The function θ_0, viewed as a mapping from $\{\alpha \in cl(A) \mid |J(\alpha)| \leq n-2\}$ to $\mathbb{R}$, can be extended continuously onto the set $cl(A)$ by setting $\theta_0(\alpha^*) = 0$ for all $\alpha^* \in cl(A)$ with $|J(\alpha^*)| = n-1$.*

Proof. We begin with some preliminary comments. In order to verify our claim, we have to show that, given an arbitrary vector $\alpha^* \in \text{cl}(A)$ with $|J(\alpha^*)| = n-1$ as well as a sequence $\{\alpha\} \to \alpha^*$ with $\alpha \in \text{cl}(A)$ satisfying $|J(\alpha)| \leq n-2$ for all α, we have $\theta_0(\alpha) \to \theta_0(\alpha^*)$. Now, for all $\alpha \in A$ (so all components of α are positive), we have the representation

$$\theta_0(\alpha) = \sum_{\nu \in K(\alpha)} x_\nu^*(\alpha)$$

of our objective function, where $K(\alpha)$ is the set of active players, cf. (2.6). On the other hand, if one or more (at most $n-2$) components of α are equal to zero, we obtained θ_0 by a continuous extension in the proof of Lemma 2.15, hence the representation (2.6) does not necessarily hold in this case. However, we showed in the proof of Lemma 2.15 that $K(\alpha) \cap J(\alpha) = \emptyset$ so that players ν with $\alpha_\nu = 0$ are certainly not active. This means that for all $\alpha \in \text{cl}(A)$ with $|J(\alpha)| \leq n-2$, the representation (2.6) is still valid, and we will work with it throughout this proof.

Now, take an arbitrary $\alpha^* \in \text{cl}(A)$ with $|J(\alpha^*)| = n-1$, i.e. $\alpha^* = e_j$ for some $j \in \{1, \dots, n\}$. Then we obtain for all $\alpha \in \text{cl}(A) \setminus \{\alpha^*\}$ sufficiently close to α^* that, on the one hand, $|J(\alpha)| \in \{0, \dots, n-2\}$ and, on the other hand,

$$\frac{\beta_j}{\alpha_j} = \min_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu},$$

hence $j \in K(\alpha)$. Consider an arbitrary sequence $\{\alpha\} \subset \text{cl}(A) \setminus \{\alpha^*\}$ with $\alpha \to \alpha^*$. We can divide the sequence into finitely many subsequences such that, within each subsequence, the set $K(\alpha)$ is constant. We verify the statement for each of these subsequences which then, obviously, implies that the statement holds for the entire sequence. We now consider one of these subsequences and call it, once again, $\{\alpha\}$. In view of the previous remark, we have $K(\alpha) \equiv K$ and $k(\alpha) \equiv k$ for all α.

We now verify the limit $\theta_0(\alpha) = \sum_{\nu\in K} x^*_\nu(\alpha) \to 0$ by showing that $x^*_\nu(\alpha) \to 0$ holds for all $\nu \in K$. For $\nu = j$, this follows immediately from

$$x^*_j(\alpha) = \left(1 - \frac{\beta_j(k-1)}{\sum_{\mu\in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}}\right) \frac{(k-1)}{\sum_{\mu\in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}} \;\to\; (1-0)0 = 0.$$

Moreover, for $k = 2$, the statement also follows easily for $\nu \in K \setminus \{j\}$:

$$x^*_\nu(\alpha) = \left(1 - \frac{\beta_\nu}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}}\right) \frac{1}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}} \;\to\; (1-1)\frac{1}{\beta_\nu} = 0.$$

It therefore remains to verify $x^*_\nu(\alpha) \to 0$ for all $\nu \in K \setminus \{j\}$ in the case $k \geq 3$. To this end, we show that, for all $k = 3, 4, \ldots$ and all $\nu, \mu \in K \setminus \{j\}$ with $\nu \neq \mu$, we have

$$\lim_{\alpha\to e_j} \frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}. \tag{2.10}$$

Using (2.10), we then obtain for all $\nu \in K \setminus \{j\}$ and all $k \geq 3$

$$x^*_\nu(\alpha) = \left(1 - \frac{(k-1)}{\sum_{\mu\in K} \frac{\beta_\mu}{\beta_\nu}\frac{\alpha_\nu}{\alpha_\mu}}\right) \frac{(k-1)}{\beta_\nu \sum_{\mu\in K} \frac{\beta_\mu}{\beta_\nu}\frac{\alpha_\nu}{\alpha_\mu}} \;\to\; (1-1)\frac{1}{\beta_\nu} = 0$$

and therefore the desired statement. To verify (2.10), it suffices to show that, for all $k = 3, 4, \ldots$ and all $\nu, \mu \in K \setminus \{j\}$ with $\nu \neq \mu$, we have

$$\limsup_{\alpha\to e_j} \frac{\alpha_\nu}{\alpha_\mu} \leq \frac{\beta_\nu}{\beta_\mu}. \tag{2.11}$$

Exchanging the roles of ν and μ then yields (2.10).

To verify (2.11), we first consider the case $k = 3$. Therefore, let $\nu, \mu \in K \setminus \{j\}$ be given with $\nu \neq \mu$. We then obtain for an arbitrary α, exploiting the characteristic property (2.2) of $\mu \in K$, that

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_\mu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{1}{2}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu}\right).$$

Rewriting this expression gives

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\left(\frac{\alpha_\nu\beta_j}{\beta_\nu\alpha_j} + 1\right).$$

Taking into account $\alpha \to e_j$, we obtain (2.11).

Next, consider the case $k = 4$. To this end, choose arbitrary $\nu, \mu \in K \setminus \{j\}$ with $\nu \neq \mu$, and let $K = \{j, \nu, \mu, \lambda\}$. Using $\lambda \in K$, we have

$$\frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{3}\sum_{\rho\in K} \frac{\beta_\rho}{\alpha_\rho} \iff \frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{2}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu}\right). \tag{2.12}$$

Exploiting once again (2.2), we obtain from $\mu \in K$ the inequality

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_\mu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{1}{3}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu} + \frac{\beta_\lambda}{\alpha_\lambda}\right).$$

Estimating the right-hand side by using (2.12) and rearranging the resulting terms, we obtain the same inequality

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\left(\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_j}{\alpha_j} + 1\right)$$

as above, so that $\alpha \to e_j$ also yields (2.11) for the case $k = 4$. For $k = 5, 6, \ldots$, the statement can be verified in an analogous way. □

Lemma 2.15 and 2.16 together yield a first result concerning the existence of a global maximum of the designer's utility function θ_0 on the set $\mathrm{cl}(A)$. However, this is not yet the desired result, since we want a global maximum on the set A, which is a strict subset of $\mathrm{cl}(A)$.

Theorem 2.17 *The function $\theta_0 : A \to \mathbb{R}$ has a continuous extension onto the closure $cl(A)$ of A and, therefore, attains a global maximum on $cl(A)$. Moreover, no vector $\alpha \in cl(A)$ with $|J(\alpha)| = n - 1$ is a global maximum.*

Proof. The fact that θ_0 can be continuously extended from A onto $\mathrm{cl}(A)$ follows from Lemmas 2.15 and 2.16, where, in particular, it is shown that this extension satisfies $\theta_0(\alpha) = 0$ for all $\alpha \in \mathrm{cl}(A)$ with $|J(\alpha)| = n - 1$, hence none of these vectors is a global maximum of θ_0 since θ_0 attains positive values on A. The existence of a global maximum then follows immediately from the fact that $\mathrm{cl}(A)$ is a compact set. □

Note, however, that the global maximizer whose existence is proven in the theorem above, might belong to the set $\mathrm{cl}(A) \setminus A$. Since the feasible set of the contest designer's optimization problem (2.5) is A or without scaling $\mathbb{R}^n_{++}$, it remains to verify that Theorem 2.17 implies the existence of a global maximum in $\mathbb{R}^n_{++}$. This is done in the final result of this section using the variations of α discussed in Section 2.2.1. Note that the following result shows that we can choose the maximizer in such a way that it also has some additional differentiability properties that will be exploited in Section 2.3.

Corollary 2.18 *The function θ_0 attains a global maximum in $\mathbb{R}^n_{++}$. Moreover, this global maximum can be chosen in such a way that condition* (2.9) *from Theorem 2.14 holds.*

Proof. By Theorem 2.17, the function θ_0 attains a global maximum in $\mathrm{cl}(A)$, and this maximum necessarily belongs to the set

$$\{\alpha \in \mathrm{cl}(A) \mid |J(\alpha)| \in \{0, \ldots, n-2\}\}.$$

However, as a consequence of Lemma 2.12, we have $\theta_0(c\alpha) = \theta_0(\alpha)$ for all α from this set and for all scalars $c > 0$. Consequently, the function θ_0 attains a global maximum α^* on the set

$$\{\alpha \in \mathbb{R}^n_+ \mid |J(\alpha)| \in \{0, \ldots, n-2\}\}.$$

If, for this maximum, we have $|J(\alpha^*)| \in \{1, \dots, n-2\}$, i.e. $\alpha^* \notin \mathbb{R}^n_{++}$, Lemma 2.13 shows that there is a point $\alpha^{**} \in \mathbb{R}^n_{++}$ with the same function value so that α^{**} is also a global maximizer. Consequently, θ_0 has a global maximum in the set $\mathbb{R}^n_{++}$, too. If this maximum does not satisfy condition (2.9) from Theorem 2.14, we can apply Lemma 2.13 once more and get another point in $\alpha^{***} \in \mathbb{R}^n_{++}$ with the same function value (which, therefore, is also a maximum) such that (2.9) holds. □

The results in this section, apart from guaranteeing the existence of an optimal contest, also back up our decision to choose $\mathbb{R}^n_{++}$ as feasible set for the designer's optimization problem instead of $\mathbb{R}^n_{+}$. The latter choice would have enabled the contest designer to explicitly exclude participants from the contest by choosing their $\alpha_\nu = 0$. This, however, would have complicated the analysis in Section 2.1 significantly since most formulae there include terms of the form $\frac{1}{\alpha_\nu}$ and consequently are not well-defined for $\alpha_\nu = 0$. On the other hand, the results in Section 2.2.2 imply that it is possible to extend our model meaningful to the case where the designer can choose his variables from $\mathbb{R}^n_{+}$, but also that this extension does not lead to a better optimal contest. Hence, to save ourselves the technical difficulties, we restricted the feasible set in our model to $\mathbb{R}^n_{++}$, knowing that this does not impair the possibility to design an optimal contest.

2.3. Closed Form of the Optimal Contest

Having proven the existence of a solution to the contest designer's optimization problem (2.5) in the last section, we are now going to derive an explicit formula for the global maximum. Since Corollary 2.18 implies the existence of a solution $\alpha^* \in \mathbb{R}^n_{++}$ such that θ_0 is continuously differentiable in α^*, we know $\nabla\theta_0(\alpha^*) = 0$ from standard analysis. This is the basis of the subsequent analysis. As a first step, we calculate all stationary points of θ_0 in Section 2.3.1. i.e. all solutions of the nonlinear system $\nabla\theta_0(\alpha) = 0$ in those points, where the derivative exists. It is possible to identify these stationary points with certain sets of active players which are analyzed in Section 2.3.2. Due to the results of Section 2.2, we can determine the set of active players corresponding to the optimal contest by simply comparing the associated values of θ_0. Finally, we use this knowledge in Section 2.3.3 to determine the optimal weights, the equilibrium effort and other interesting characteristics of the optimal contest.

Unfortunately, computing the zeros of $\nabla\theta_0(\alpha) = 0$ is not an easy task, especially since the derivative with respect to α leads to complicated formulas. In order to simplify our calculations we therefore use the transformation $\gamma : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ defined by

$$\gamma_\nu(\alpha) := \frac{\beta_\nu}{\alpha_\nu} \tag{2.13}$$

for all $\nu \in N$. Since $\beta \in \mathbb{R}^n_{++}$, the mapping γ is a diffeomorphism from $\mathbb{R}^n_{++}$ onto $\mathbb{R}^n_{++}$. We further write $\gamma = \frac{\beta}{\alpha}$ for the vector whose components are given by $\frac{\beta_\nu}{\alpha_\nu}$. For some $\gamma \in \mathbb{R}^n_{++}$, we also write

$$K(\gamma) := \left\{ \nu \in N \;\middle|\; (k(\gamma) - 1)\gamma_\nu < \sum_{\mu \in K(\gamma)} \gamma_\mu \right\}$$

with $k(\gamma) := |K(\gamma)|$. Using Corollary 2.9, it follows that for each $\gamma \in \mathbb{R}^n_{++}$, there is precisely one such set $K(\gamma)$. Based on the set $K(\gamma)$, we now define the function $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}$ by

$$\varphi(\gamma) := \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \left(\sum_{\mu \in K(\gamma)} \frac{\gamma_\mu}{\beta_\mu} - \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \sum_{\mu \in K(\gamma)} \frac{\gamma_\mu^2}{\beta_\mu} \right).$$

Since

$$K(\gamma(\alpha)) = K(\alpha)$$

for all $\alpha \in \mathbb{R}^n_{++}$, we have $\varphi = \theta_0 \circ \gamma^{-1}$. Hence, for all global maxima α^* of the function θ_0 satisfying condition (2.9) of Theorem 2.14, the function φ has a global maximum in $\gamma^* := \frac{\beta}{\alpha^*}$ and is continuously differentiable in a neighborhood of γ^*, since

$$(k(\gamma^*) - 1)\gamma_\nu^* = (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu^*} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\gamma^*)} \gamma_\mu^* \quad \forall \nu \notin K(\gamma^*).$$

Conversely, if γ^* denotes a global maximum of φ with the property

$$\nu \notin K(\gamma^*) \quad \Longrightarrow \quad (k(\gamma^*) - 1)\gamma_\nu^* > \sum_{\mu \in K(\gamma^*)} \gamma_\mu^*, \tag{2.14}$$

then $\alpha^* = \frac{\beta}{\gamma^*}$ is a global maximum of θ_0 such that condition (2.9) of Theorem 2.14 holds. Hence we have the following result.

Lemma 2.19 *$\alpha^* \in \mathbb{R}^n_{++}$ is a global maximum of θ_0 satisfying property* (2.9) *of Theorem 2.14 if and only if $\gamma^* = \frac{\beta}{\alpha^*}$ is a global maximum of φ satisfying condition* (2.14).

Consequently, instead of looking for the global maxima of θ_0 satisfying condition (2.9), we can also search for the global maxima of φ satisfying (2.14) and a simple retransformation then yields the optimal weights for the contest. Analogously to the original objective function θ_0, the function φ is continuously differentiable around every global maximum γ^* satisfying (2.14) and thus satisfies $\nabla\varphi(\gamma^*) = 0$. Hence, we are going to calculate all solutions of the nonlinear system $\nabla\varphi(\gamma) = 0$ in those points, where φ is differentiable. But before we do so, we would like to state two properties of φ whose analogues have already been proven for θ_0 in Section 2.2.1.

Lemma 2.20 *(a) For all $\gamma \in \mathbb{R}^n_{++}$ and all $c > 0$, we have $K(\gamma) = K(c\gamma)$ and $\varphi(\gamma) = \varphi(c\gamma)$.*

(b) Let $\gamma^* \in \mathbb{R}^n_{++}$ be arbitrary. Then $K(\gamma^*) = K(\gamma)$ and $\varphi(\gamma^*) = \varphi(\gamma)$ hold for all $\gamma \in \mathbb{R}^n_{++}$ satisfying

$$\gamma_\nu = \gamma_\nu^* \quad \forall \nu \in K(\gamma^*) \quad \textit{and} \quad \gamma_\nu \geq \frac{1}{k(\gamma^*) - 1} \sum_{\mu \in K(\gamma^*)} \gamma_\mu^* \quad \forall \nu \notin K(\gamma^*).$$

Lemma 2.20 (a) implies that it suffices to calculate those maxima γ^* of φ that satisfy

$$\sum_{\mu \in K(\gamma^*)} \gamma_\mu^* = 1. \tag{2.15}$$

Note that in contrast to the set A defined in Section 2.2, which was used to prove the existence of a maximum, this time only the variables corresponding to active contestants are normalized. The reason for this is part (b) of the lemma above, where it is stated that the variables of inactive players can be varied in certain bounds without changing the set of active players or the collective effort. Hence we can restrict our interest at first to those maxima satisfying the scaling (2.15) and the differentiability condition (2.14). Later on, we obtain all maxima by applying the variations from the previous lemma. Since the optimization problem considered here is essentially unconstrained, all maxima satisfying the differentiability condition are stationary points, i.e. solutions of the equation $\nabla\varphi(\gamma) = 0$. Therefore, our next step is to calculate all stationary points satisfying the two conditions mentioned above.

2.3.1. The Stationary Points

As explained above, in this section we are only interested in points γ satisfying (2.15) and (2.14). In these points, the objective function φ is differentiable and we can calculate the stationary points γ^*, i.e. the solutions of the system $\nabla\varphi(\gamma) = 0$. The following result collects some properties of these points.

Theorem 2.21 *Let $\gamma^* \in (0,\infty)^n$ be a stationary point of the function φ satisfying* (2.15) *and* (2.14). *Then the following statements hold:*

(a) For all active players $\nu \in K(\gamma^*)$, we have

$$\gamma^*_\nu = \frac{1}{2(k(\gamma^*)-1)}\left(1 + (k(\gamma^*)-2)\frac{\beta_\nu}{\sum_{\mu\in K(\gamma^*)}\beta_\mu}\right).$$

(b) For all inactive players $\nu \notin K(\gamma^*)$, we have

$$\gamma^*_\nu > \frac{1}{k(\gamma^*)-1}.$$

(c) For all active players $\nu \in K(\gamma^*)$, we have

$$(k(\gamma^*)-2)\beta_\nu < \sum_{\mu\in K(\gamma^*)}\beta_\mu.$$

(d) The total equilibrium effort is given by:

$$\varphi(\gamma^*) = \frac{1}{4}\left(\sum_{\mu\in K(\gamma^*)}\frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu\in K(\gamma^*)}\beta_\mu}\right).$$

Proof. Since γ^* satisfies condition (2.14), there is a neighborhood U of γ^* with

$$K(\gamma) = K(\gamma^*) =: K \quad \text{and} \quad k(\gamma) = k(\gamma^*) =: k$$

and since it is stationary, we have $\nabla\varphi(\gamma^*) = 0$.

The only statement we obtain for the components γ_ν with $\nu \notin K$ follows from (2.14) together with the scaling (2.15):

$$\gamma_\nu^* > \frac{1}{k-1}\sum_{\mu\in K}\gamma_\mu^* = \frac{1}{k-1}.$$

This shows that statement (b) holds.

Moreover, for all $\nu \in K$, we have

$$\begin{aligned}
0 &= \frac{\partial}{\partial\gamma_\nu}\varphi(\gamma^*) \\
&= -\frac{k-1}{\left(\sum_{\mu\in K}\gamma_\mu^*\right)^2}\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} + \frac{k-1}{\sum_{\mu\in K}\gamma_\mu^*}\frac{1}{\beta_\nu} + \frac{2(k-1)^2}{\left(\sum_{\mu\in K}\gamma_\mu^*\right)^3}\sum_{\mu\in K}\frac{(\gamma_\mu^*)^2}{\beta_\mu} - \frac{2(k-1)^2}{\left(\sum_{\mu\in K}\gamma_\mu^*\right)^2}\frac{\gamma_\nu^*}{\beta_\nu} \\
&= -(k-1)\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} + (k-1)\frac{1}{\beta_\nu} + 2(k-1)^2\sum_{\mu\in K}\frac{(\gamma_\mu^*)^2}{\beta_\mu} - 2(k-1)^2\frac{\gamma_\nu^*}{\beta_\nu}. \qquad (2.16)
\end{aligned}$$

Summing up equation (2.16) over all $\nu \in K$, we get

$$-(k-1)\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} + 2(k-1)^2\sum_{\mu\in K}\frac{(\gamma_\mu^*)^2}{\beta_\mu} = \frac{1}{k}\left(2(k-1)^2\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} - (k-1)\sum_{\mu\in K}\frac{1}{\beta_\mu}\right).$$

Inserting this again into (2.16) and cancelling the factor $k-1$, we obtain for all $\nu \in K$:

$$\begin{aligned}
\frac{2(k-1)}{k}\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} - \frac{1}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu} + \frac{1}{\beta_\nu} - 2(k-1)\frac{\gamma_\nu^*}{\beta_\nu} &= 0 \\
\iff \quad \gamma_\nu^* - \frac{1}{k}\sum_{\mu\in K}\frac{\beta_\nu}{\beta_\mu}\gamma_\mu^* &= \frac{1}{2(k-1)}\left(1 - \frac{\beta_\nu}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu}\right).
\end{aligned}$$

Consequently, the vector $\gamma_K^* := (\gamma_\nu^*)_{\nu\in K}$ is a solution of the linear system of equations

$$\left(I_{k\times k} - \frac{1}{k}\left[\frac{\beta_\nu}{\beta_\mu}\right]_{\nu,\mu\in K}\right)\left[\gamma_\mu\right]_{\mu\in K} = \frac{1}{2(k-1)}\left[1 - \frac{\beta_\nu}{k}\sum_{\lambda\in K}\frac{1}{\beta_\lambda}\right]_{\nu\in K}.$$

Using the abbreviations $\beta_K := (\beta_\nu)_{\nu\in K}$ and $\beta_K^{-1} := (\frac{1}{\beta_\nu})_{\nu\in K}$, the matrix of this linear system can be written as

$$I_{k\times k} - \frac{1}{k}\left[\frac{\beta_\nu}{\beta_\mu}\right]_{\nu,\mu\in K} = I_{k\times k} - \frac{1}{k}\beta_K(\beta_K^{-1})^T =: M.$$

This matrix M is singular, more precisely, it has rank $k-1$ and its null space (kernel) is given by $\ker(M) = \operatorname{span}\{\beta_K\}$ (this singularity reflects the fact that the function value $\varphi(\gamma)$ is independent of the scaling of γ, cf. Lemma 2.20, hence M cannot be expected to be nonsingular at an arbitrary stationary point). Now it is easy to see that one particular solution of the above linear system of equations is the vector from the right-hand side:

$$\tilde{\gamma}_\nu = \frac{1}{2(k-1)}\left(1 - \frac{\beta_\nu}{k}\sum_{\lambda\in K}\frac{1}{\beta_\lambda}\right) \quad \forall \nu \in K.$$

Therefore, the vector γ_K^* is of the form $\gamma_K^* = \tilde{\gamma}_K + c\beta_K$, where $c \in \mathbb{R}$ has to be chosen in such a way that $\sum_{\mu\in K}\gamma_\mu^* = 1$. It follows that

$$c = \frac{1}{2(k-1)}\left(\frac{k-2}{\sum_{\mu\in K}\beta_\mu} + \frac{1}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu}\right)$$

and, therefore,

$$\gamma_\nu^* = \frac{1}{2(k-1)}\left(1 + (k-2)\frac{\beta_\nu}{\sum_{\mu\in K}\beta_\mu}\right) \ (>0)$$

for all $\nu \in K$. Hence statement (a) holds.

By the definition of $K = K(\gamma^*)$, we have for all $\nu \in K$:

$$(k-1)\gamma_\nu^* \ < \ \sum_{\mu\in K}\gamma_\mu^* \ = \ 1 \quad \Longleftrightarrow \quad (k-2)\beta_\nu \ < \ \sum_{\mu\in K}\beta_\mu.$$

This verifies statement (c). Inserting the representation of γ_K^* gives the desired representation of $\varphi(\gamma^*)$ from assertion (d). □

Note that Theorem 2.21 does not imply that the condition

$$(k(\gamma^*)-2)\beta_\nu < \sum_{\mu\in K(\gamma^*)}\beta_\mu$$

is violated for all $\nu \notin K(\gamma^*)$ in a stationary point γ^*. However, we will see later that the global maximum can be characterized by the fact that this condition is satisfied only for $\nu \in K(\gamma^*)$. But first, we are going to prove that, in some sense, the converse of Theorem 2.21 also holds, which allows us to identify stationary points with certain sets of active players.

Lemma 2.22 *Let $K \subseteq N$ be arbitrarily given, let $k := |K| \geq 2$ and suppose that*

$$(k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu \quad \forall \nu \in K. \tag{2.17}$$

Define the vector $\gamma^* \in (0,\infty)^n$ in such a way that $\gamma_\nu^* > \frac{1}{k-1}$ is arbitrary for all $\nu \notin K$ and

$$\gamma_\nu^* = \frac{1}{2(k-1)}\left(1 + (k-2)\frac{\beta_\nu}{\sum_{\mu\in K}\beta_\mu}\right) \quad \forall \nu \in K.$$

Then the following statements hold:

(a) $\sum_{\mu\in K}\gamma_\mu^* = 1$.

(b) $K(\gamma^*) = K$ *and* γ^* *satisfies condition* (2.14).

(c) The function φ is continuously differentiable in a neighborhood of γ^* with $\nabla\varphi(\gamma^*) = 0$.

(d) $\varphi(\gamma^*) = \frac{1}{4}\left(\sum_{\mu\in K}\frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu}\right)$.

Proof. Statement (a) can be verified easily using the definition of γ^*_μ for $\mu \in K$. Assertions (c) and (d), on the other hand, follow in essentially the same way as in the proof of Theorem 2.21 since our *definition* of γ^*_ν is exactly the same as the *representation* of γ^*_ν obtained in Theorem 2.21 for γ^*_ν ($\nu \in K$). To see that statement (b) holds, we verify that

$$(k-1)\gamma^*_\nu < \sum_{\mu\in K}\gamma^*_\mu = 1 \iff \nu \in K. \tag{2.18}$$

The definition of the index set $K(\gamma^*)$ together with the uniqueness of this index set then shows $K = K(\gamma^*)$. Now, for $\nu \in K$, we obtain from the definition of γ^*_ν together with (2.17) that

$$(k-1)\gamma^*_\nu = \frac{1}{2}\left(1 + (k-2)\frac{\beta_\nu}{\sum_{\mu\in K}\beta_\mu}\right) < \frac{1}{2}(1+1) = 1,$$

hence the implication "$\Longleftarrow$" holds in (2.18). On the other hand, for $\nu \notin K$, we have $(k-1)\gamma^*_\nu > 1$ which, by contraposition, shows that also the implication "$\Longrightarrow$" holds in (2.18). □

Lemma 2.22 and Theorem 2.21 are the foundation of the following idea how to find all global maxima which satisfy the scaling (2.15) and the differentiability condition (2.14). As mentioned above, all other global maxima can be derived from those using the variations from Lemma 2.20.

Theorem 2.21 allows the following interpretation: If γ^* is a stationary point (and hence a candidate for a global maximum) of φ satisfying (2.15) and (2.14), then we necessarily have

$$|K(\gamma^*)| \geq 2 \quad \text{and} \quad (k(\gamma^*) - 2)\beta_\nu < \sum_{\mu\in K(\gamma^*)}\beta_\mu \quad \forall \nu \in K(\gamma^*),$$

whereas statement (d) calculates the corresponding function value $\varphi(\gamma^*)$. Now, Lemma 2.22 takes an arbitrary index set $K \subseteq N$ with

$$k := |K| \geq 2 \quad \text{and} \quad (k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu \quad \forall \nu \in K, \tag{2.19}$$

defines corresponding values for γ^*_ν ($\nu \in N$) and then states that, in particular, we have $K = K(\gamma^*)$ and that γ^*_ν satisfies (2.15) as well as (2.14) and $\nabla\varphi(\gamma^*) = 0$. Consequently, the vector γ^* corresponding to K is a stationary point. Hence, we can compute all stationary points satisfying the additional conditions from Theorem 2.21 by searching for those index sets $K \subseteq N$ with (2.19) and then obtain the global maxima by comparing the function value $\varphi(\gamma^*)$. Remember that there are only finitely many index sets $K \subseteq N$. This is done in the next section.

2.3.2. The Optimal Set of Active Players

To make the idea introduced in the last section more precise, we have to introduce some notation first. Using the formula for $\varphi(\gamma^*)$ at a stationary point γ^* given in Theorem 2.21 (d), we define the function

$$\psi(K) := \frac{1}{4}\left(\sum_{\mu\in K}\frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu}\right)$$

for all $K \subseteq N$ with $k := |K| \geq 2$ and $(k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu$ for all $\nu \in K$. Furthermore, we want to introduce the following terminology that will simplify our subsequent discussion to some extent.

Definition 2.23 *A set $K \subseteq N$ with $k := |K|$ is called*

(a) feasible*, if $k \geq 2$ and $(k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu$ for all $\nu \in K$.*

(b) maximal*, if K is feasible and there is no feasible superset $\tilde{K} \subseteq N$ of K.*

(c) optimal*, if K is feasible and $\psi(K) \geq \psi(\tilde{K})$ for all feasible sets $\tilde{K}$.*

We stress that a feasible set K still allows the existence of players $\nu \notin K$ such that the inequality

$$(k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu$$

holds. A feasible set is maximal if it is not strictly contained in another feasible set. Furthermore, an optimal set K is a feasible set such that the expression $\psi(K)$ is maximal among all feasible sets. The existence of such a set is clear since the number of feasible sets is finite (though typically exponentially large).

With this terminology, we can state our idea from the end of Section 2.3.1 more formally: According to Lemma 2.22 and Theorem 2.21, γ^* is a global maximum of φ satisfying the conditions of Theorem 2.21 if and only if $K(\gamma^*)$ is optimal, i.e., $K(\gamma^*)$ is a solution of

$$\max\ \psi(K) \quad \text{subject to} \quad K \text{ is feasible.} \tag{2.20}$$

So far, we only explained why we introduced feasible and optimal sets. The reason, why we are interested in maximal sets, is the following proposition.

Proposition 2.24 *Let $K, M \subseteq N$ be feasible sets such that $M \subsetneq K$. Then we have $\psi(M) < \psi(K)$.*

Proof. Using the well-known inequality between the arithmetic and harmonic mean together with some elementary calculations, we obtain

$$\begin{aligned}
\psi(K) - \psi(M) &= \frac{1}{4}\left(\sum_{\mu\in K\setminus M}\frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu} + \frac{(m-2)^2}{\sum_{\mu\in M}\beta_\mu}\right) \\
&\geq \frac{1}{4}\left(\frac{(k-m)^2}{\sum_{\mu\in K\setminus M}\beta_\mu} - \frac{(k-m)^2 + 2(k-m)(m-2) + (m-2)^2}{\sum_{\mu\in K\setminus M}\beta_\mu + \sum_{\mu\in M}\beta_\mu} + \frac{(m-2)^2}{\sum_{\mu\in M}\beta_\mu}\right) \\
&= \frac{1}{4}\left(\frac{\left((k-m)\sum_{\mu\in M}\beta_\mu - (m-2)\sum_{\mu\in K\setminus M}\beta_\mu\right)^2}{\sum_{\mu\in K\setminus M}\beta_\mu \sum_{\mu\in K}\beta_\mu \sum_{\mu\in M}\beta_\mu}\right)
\end{aligned}$$

$$\geq \quad 0,$$

and equality $\psi(K) = \psi(M)$ holds if and only if

$$\sum_{\mu \in K \setminus M} \frac{1}{\beta_\mu} = \frac{(k-m)^2}{\sum_{\mu \in K \setminus M} \beta_\mu} \quad \text{and} \quad \sum_{\mu \in M} \beta_\mu = (m-2) \frac{\sum_{\mu \in K \setminus M} \beta_\mu}{k-m}.$$

Since all β_μ ($\mu \in K \setminus M$) are positive, the harmonic mean and the arithmetic mean coincide if and only if all β_μ ($\mu \in K \setminus M$) coincide, i.e. if $\beta_\mu = \beta$ for all $\mu \in K \setminus M$ and a suitable $\beta > 0$. Hence, the second equation implies that, for all $\mu \in K \setminus M$, we have

$$(k-2)\beta_\mu \;=\; (k-m)\beta + (m-2)\beta \;=\; \sum_{\mu \in K \setminus M} \beta_\mu + \sum_{\mu \in M} \beta_\mu \;=\; \sum_{\mu \in K} \beta_\mu,$$

which, however, is a contradiction to the feasibility of K. □

Proposition 2.24 implies that a feasible set K, which is not maximal, can never be optimal. Hence, from now on, we can restrict our analysis to maximal sets K. This observation directly leads to the following conclusion: Consider the case of $n \geq 3$ players and note that every subset $M \subseteq N$ consisting of two players is feasible. Then, take an arbitrary element $\nu \in N \setminus M$ and define $K := M \cup \{\nu\}$. This set K consists of three players containing M as a strict subset and is feasible, too. In view of Proposition 2.24, it follows that M cannot be an optimal set. Hence we obtain the following result.

Theorem 2.25 *Consider the effort maximization problem* (2.5) *with $n \geq 3$. Then there are at least three active players in every global maximum.*

This result is somewhat surprising. From Corollary 2.9, we know that in the Nash equilibrium of the contest game always at least two contestants exert a positive effort. If a player ν is active in the equilibrium depends on whether his effective cost parameter $\gamma_\nu = \frac{\beta_\nu}{\alpha_\nu}$ is small enough compared to the other players. Theorem 2.25 says that, independent from the given cost parameters β_ν of the contestants, the contest designer will always choose the weights α_μ (or γ_μ respectively) such that at least three players are active. We will come back to this point later when comparing our model with the all-pay auction model.

As we will prove later, one of the maximal subsets mentioned above is

$$K^* := \left\{ \nu \in N \;\middle|\; (|K^*| - 2)\beta_\nu < \sum_{\mu \in K^*} \beta_\mu \right\}. \tag{2.21}$$

Our aim is to prove that K^* is the unique optimal set. To this end, first note that the definition of K^* is given in an implicit form since K^* also occurs in the expression within the parenthesis. Therefore, it is neither clear whether this object is well-defined and unique nor whether it is a useful expression for the explicit calculation of the set K^*. The following result gives an alternative (and explicit) expression for K^* (provided that, without loss of generality, the coefficients β_μ are ordered in such a way that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$). This expression also implies that K^* exists. In fact, it turns out that there is precisely one set K^* satisfying the definition in (2.21).

Lemma 2.26 *(a) If the coefficients β_μ are ordered in such a way that $\beta_1 \le \beta_2 \le \ldots \le \beta_n$, then the definition of K^* is equivalent to the following definition:*

$$K^* := \left\{ \nu \in N \;\middle|\; (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\}.$$

(b) The set K^* exists and is unique.

Proof. We verify statements (a) and (b) simultaneously, first under the assumption that $\beta_1 \le \beta_2 \le \ldots \le \beta_n$. To this end, let us denote the set mentioned above by

$$M := \left\{ \nu \in N \;\middle|\; (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\},$$

and define $m := |M|$. Obviously, M exists and is unique. Furthermore, one can use the ordering of β_μ to verify the implications

$$\nu \in M \implies \nu - 1 \in M \quad \text{or, equivalently,} \quad \nu \notin M \implies \nu + 1 \notin M.$$

Hence, the set M is of the form $M = \{1, \ldots, m\}$.

Our first step is to show that the set M has the same properties as K^*. To this end, choose an arbitrary $\nu \in M$. Then the ordering of β_μ implies

$$(m-2)\beta_\nu = (\nu - 2)\beta_\nu + (m - \nu)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu + \sum_{\mu=\nu+1}^{m} \beta_\mu = \sum_{\mu \in M} \beta_\mu.$$

On the other hand, we can use the definition of M to obtain

$$((m+1) - 2)\beta_{m+1} \ge \sum_{\mu=1}^{m+1} \beta_\mu \quad \iff \quad (m-2)\beta_{m+1} \ge \sum_{\mu \in M} \beta_\mu.$$

This, however, implies that the following is true for all $\nu \notin M$:

$$(m-2)\beta_\nu \ge (m-2)\beta_{m+1} \ge \sum_{\mu \in M} \beta_\mu.$$

Consequently, M satisfies the conditions imposed on K^*.

So far, we have shown that there is at least one set which suffices the definition of K^*, namely the set M. Furthermore, the ordering of β_μ implies that every set K^* has to be of the form $K^* = \{1, \ldots, k^*\}$. Now, it remains to prove that every set K^* has the same properties as M. To this end, we choose an arbitrary $\nu \notin K^*$. Then the ordering of β_μ implies

$$(\nu - 2)\beta_\nu = (k^* - 2)\beta_\nu + (\nu - k^*)\beta_\nu \ge \sum_{\mu \in K^*} \beta_\mu + \sum_{\mu=k^*+1}^{\nu} \beta_\mu = \sum_{\mu=1}^{\nu} \beta_\mu.$$

On the other hand, we know

$$(k^* - 2)\beta_{k^*} < \sum_{\mu \in K^*} \beta_\mu \quad \Longleftrightarrow \quad ((k^* - 1) - 2)\beta_{k^*} < \sum_{\mu=1}^{k^*-1} \beta_\mu.$$

Using this reformulation and the ordering of the β_μ, we obtain

$$(\nu - 2)\beta_\nu \leq (\nu - 2)\beta_{\nu+1} < \sum_{\mu=1}^{\nu} \beta_\mu$$

inductively for all $\nu = k^* - 1, \ldots, 2$ (the case $\nu = 1$ is trivial). Altogether, we have shown that every set satisfying the definition of K^* also satisfies the definition of M.

Therefore, we have shown that the definitions of K^* and M coincide whenever the β_μ are ordered in an increasing way. Note that this implies existence and uniqueness of K^* in the ordered case.

Now, let us consider the case where β_ν are not necessarily sorted in increasing order. The definition of K^* is obviously independent of the numbering of the coefficients β_μ. Hence, we can use a permutation $\pi : N \to N$ to obtain an increasing ordering of the form $\beta_{\pi(1)} \leq \beta_{\pi(2)} \leq \ldots \leq \beta_{\pi(n)}$. To shorten the notation, we define $\tilde{\beta}_\mu := \beta_{\pi(\mu)}$ and

$$\tilde{K} := \left\{ \nu \in N \;\middle|\; (\nu - 2)\tilde{\beta}_\nu < \sum_{\mu \in \tilde{K}} \tilde{\beta}_\mu \right\}.$$

We are now in a position to apply the first part of our proof and obtain existence and uniqueness of $\tilde{K}$. On the other hand, we have $\nu \in K^*$ if and only if $\pi^{-1}(\nu) \in \tilde{K}$, i.e. $K^* = \pi(\tilde{K})$, where the permutation is meant to be applied elementwise on the set $\tilde{K}$. By combining these two facts, we can derive existence and uniqueness of the set K^* as well. □

Note that the assumption $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ can be stated without loss of generality. Then Lemma 2.26 shows that K^* consists precisely of the $k^* := |K^*|$ smallest elements of the coefficients β_μ, i.e. $K^* = \{1, 2, \ldots, k^*\}$. This is an intuitive efficiency property of any solution: only the most able contestants, i.e. those with the lowest cost to provide effort, are chosen to be active by the contest organizer. This expression of K^* is very useful for the actual computation of this set. On the other hand, in our subsequent analysis, we typically exploit the implicit definition of K^* from (2.21).

Since we want to show that K^* is an optimal (in fact, the optimal) set, we know from Proposition 2.24 that it has to be at least a maximal set. The following result therefore verifies that K^* is indeed a maximal set, so it remains a candidate for being an optimal set.

Lemma 2.27 *The set K^* is maximal in the sense of Definition 2.23.*

Proof. By definition, the set K^* is obviously feasible. In order to prove maximality, we will assume without loss of generality that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. Then K^* has to be of the form $\{1, 2, \ldots, k^*\}$. Assume that K^* is not maximal. Then there is a set $M \subseteq N \setminus K^*$ such that $m :=$

$|M| \neq 0$ and $K^* \cup M$ is feasible. Let $\tilde{\mu}$ be the largest index in $K^* \cup M$. Then $\tilde{\mu} \notin K^*$ and thus the definition of K^* and the increasing order of the β_μ imply

$$((k+m)-2)\beta_{\tilde{\mu}} = (k-2)\beta_{\tilde{\mu}} + m\beta_{\tilde{\mu}} \geq \sum_{\mu \in K^*} \beta_\mu + \sum_{\mu \in M} \beta_\mu = \sum_{\mu \in K^* \cup M} \beta_\mu,$$

a contradiction to the feasibility of $K^* \cup M$. □

The next result is a technical lemma that will be exploited in our subsequent analysis. It is, however, also of some interest on its own by giving a necessary condition on the size of the β_μ that belong to any feasible set K: These β_μ have to be strictly smaller than the sum of the three smallest elements β_ν with $\nu \in K$. A comparison with the respective condition from Section 2.1.2, presented in Corollary 2.10, indicates that the active set of players under the optimal weights might be larger than, for instance, under the neutral weighting scheme, see Section 2.4 for a detailed example of this comparison.

Lemma 2.28 *Let K be feasible with $k := |K| \geq 3$. Then all β_ν ($\nu \in K$) are strictly smaller than the sum of the three smallest β_μ ($\mu \in K$).*

Proof. Assume once again, without loss of generality, that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ and let $K = \{\mu_1, \ldots, \mu_k\}$ in increasing order, i.e. $\beta_{\mu_1} \leq \beta_{\mu_2} \leq \ldots \leq \beta_{\mu_k}$. Then the three smallest β_μ ($\mu \in K$) are $\beta_{\mu_1}, \beta_{\mu_2}, \beta_{\mu_3}$. The definition of K implies

$$(k-2)\beta_{\mu_k} < \sum_{j=1}^{k} \beta_{\mu_j} \quad \Longleftrightarrow \quad ((k-1)-2)\beta_{\mu_k} < \sum_{j=1}^{k-1} \beta_{\mu_j}.$$

Using this reformulation and the ordering of the β_{μ_j}, we obtain

$$(i-2)\beta_{\mu_i} \leq (i-2)\beta_{\mu_{i+1}} < \sum_{j=1}^{i} \beta_{\mu_j}$$

inductively for all $i = k-1, \ldots, 2$ (there is nothing to prove for $i = 1$). Now, the assertion is obviously true for $\beta_{\mu_1}, \beta_{\mu_2}, \beta_{\mu_3}$. For $\beta_{\mu_4}, \ldots, \beta_{\mu_k}$, the statement can be derived inductively using the formula above. □

In order to state our main result, we need another technical lemma whose proof is quite lengthy and technical.

Lemma 2.29 *Suppose $n \geq 4$. For every feasible set K with $k := |K| \geq 3$ and $K \setminus K^* \neq \emptyset$ the following estimation holds:*

$$-\sum_{\mu \in K \setminus K^*} \frac{1}{\beta_\mu} + \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \geq -\frac{(k-d)(k^*-2)}{\sum_{\mu \in K^*} \beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2+k^*-d)\sum_{\mu \in K^*} \beta_\mu - (k^*-2)\sum_{\mu \in K^* \setminus K} \beta_\mu},$$

where $d := |K^* \cap K|$.

Proof. Our aim is to find a suitable lower bound for the expression

$$-\sum_{\mu\in K\setminus K^*}\frac{1}{\beta_\mu}+\frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu}=-\sum_{\mu\in K\setminus K^*}\frac{1}{\beta_\mu}+\frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu+\sum_{\mu\in K\setminus K^*}\beta_\mu},$$

because this lower bound is going to play a very crucial role in the proof of our main result, Theorem 2.30. Standard estimates for this expression do not work in the proof of that result, so we need a very sharp lower bound. To this end, we compute analytically the solution of a related optimization problem. More precisely, we will prove that the lower bound given in Lemma 2.29 is the global minimum of the problem

$$\begin{aligned}\min_{b_\nu,\nu\in K\setminus K^*}\quad & -\sum_{\mu\in K\setminus K^*}\frac{1}{b_\mu}+\frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu+\sum_{\mu\in K\setminus K^*}b_\mu}\\ \text{s.t.}\quad & -(k^*-2)b_\nu+\sum_{\mu\in K^*}\beta_\mu\le 0\quad\forall\nu\in K\setminus K^*,\\ & (k-2)b_\nu-\sum_{\mu\in K\setminus K^*}b_\mu-\sum_{\mu\in K\cap K^*}\beta_\mu\le 0\quad\forall\nu\in K\setminus K^*,\end{aligned}\tag{2.22}$$

where β_ν ($\nu\in K\cap K^*$) are viewed as fixed and b_ν ($\nu\in K\setminus K^*$) are the variables. Obviously, β_ν ($\nu\in K\setminus K^*$) is feasible for this optimization problem due to the feasibility of K and the definition of K^*. To do so, we will proceed in two steps: First, we will prove the existence of a global minimum and then we will calculate it explicitly.

Existence of a global minimum:
Note that the feasible set is nonempty since the definitions of the index sets K and K^* immediately imply that the vector $\beta_{K\setminus K^*}$ is feasible. Since $n\ge 4$, it follows from Theorem 2.25 that $k^*\ge 3$. The maximality of K^* together with $K\setminus K^*\neq\emptyset$ implies $K^*\setminus K\neq\emptyset$, i.e. $k^*-d>0$, where $d:=|K^*\cap K|$. At first, we will deal with the case $d\ge 3$.

We claim that, under this additional assumption, the feasible set of (2.22) is bounded, hence compact. To this end, let $b_{K\setminus K^*}$ be any feasible vector for this program. This implies that for all $\gamma\in K^*\cap K$ and all $\nu\in K\setminus K^*$

$$0<\beta_\gamma<\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\le b_\nu.$$

Because of $d\ge 3$, this implies that the three smallest elements of $\{b_\nu\mid\nu\in K\setminus K^*\}\cup\{\beta_\nu\mid\nu\in K\cap K^*\}$ belong to indices $\nu\in K^*\cap K$. Define c as the sum over the three smallest β_ν ($\nu\in K^*\cap K$). This constant is independent from $b_{K\setminus K^*}$, and one can prove analogously to Lemma 2.28 that every feasible $b_{K\setminus K^*}$ satisfies $b_\nu\le c$ for all $\nu\in K\setminus K^*$. So the feasible set of problem (2.22) is not only closed but also bounded, i.e., it is compact. Hence, the continuous objective function attains a global minimum in the feasible set.

Now, we have to deal with the remaining case $d<3$. Unfortunately, the feasible set is unbounded for $d\in\{0,1,2\}$, so we have to use a slightly different argumentation here. We can find a sequence of feasible vectors $\{b^m_{K\setminus K^*}\}_m$ such that the corresponding values of the objective function converge to the infimum of the function on the feasible set (which could be $-\infty$). If any subsequence of $\{b^m_{K\setminus K^*}\}_m$ converges to a finite limit point, the closedness of the feasible set guarantees that this limit point is feasible and thus a global minimum.

Now let us assume that for every subsequence, at least one component b^m_ν ($\nu \in K \setminus K^*$) is unbounded. Then we can find a subsequence of $\{b^m_{K\setminus K^*}\}_m$ such that every component b^m_ν ($\nu \in K \setminus K^*$) either converges to a finite b_ν or diverges to $+\infty$. Denote by I_f the index set of the converging components and by I_∞ the index set of the diverging components. Then $I_\infty \neq \emptyset$. This, however, implies $d + i_f \leq 2$, where $i_f := |I_f|$. Otherwise, an argument similar to Lemma 2.28 would yield that all components of the limit point were bounded by the sum of the three smallest elements of $\{b_\nu \mid \nu \in K \setminus K^*\} \cup \{\beta_\nu \mid \nu \in K \cap K^*\}$, hence finite. Using this information, we can compare the infimum of the objective function, which is then given by

$$-\sum_{\mu\in I_f} \frac{1}{b_\mu}$$

with

$$b_\nu \geq \frac{\sum_{\mu\in K^*} \beta_\mu}{k^* - 2} \qquad \forall \nu \in I_f, \tag{2.23}$$

(by continuity) with the value of the objective function in the point $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\setminus K^*}$, which is feasible as we will show below. The value of the objective function corresponding to this vector is given by

$$\begin{aligned}
&-\sum_{\mu\in K\setminus K^*} \frac{k^*-2}{\sum_{j\in K^*}\beta_j} + \frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*}\frac{\sum_{j\in K^*}\beta_j}{k^*-2}} \\
&= -(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k^*-2)\sum_{\mu\in K\cap K^*}\beta_\mu + (k-d)\sum_{\mu\in K^*}\beta_\mu} \\
&= -(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta},
\end{aligned}$$

where we used the abbreviation

$$\delta := (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu.$$

Using the definition of K^* together with $K^* \setminus K \neq \emptyset$, it is not difficult to see that $\delta > 0$. This yields

$$\begin{aligned}
&\left(-(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}\right) - \left(-\sum_{\mu\in I_f}\frac{1}{b_\mu}\right) \\
&\leq -\frac{(k-d-i_f)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta} \\
&\leq -\frac{(k-2)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}
\end{aligned}$$

$$\begin{aligned} &= \frac{(k-2)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{-\delta}{(k-2)\sum_{\mu\in K^*}\beta_\mu+\delta} \\ &< 0, \end{aligned}$$

where the first expression was motivated above, the first inequality follows by estimating the second term based on (2.23), the second inequality is a consequence of the fact that $d + i_f \leq 2$, the subsequent equation follows by direct calculation using some cancellations, and the final inequality uses the fact that $\delta > 0$.

It remains to prove the feasibility of $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\setminus K^*}$. Obviously, for all $\nu \in K \setminus K^*$

$$b_\nu = \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} \geq \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}.$$

On the other hand, we have

$$\frac{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu}{k-2} = \frac{\sum_{\mu\in K^*}\beta_\mu - \sum_{\mu\in K^*\setminus K}\beta_\mu + (k-d)\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}}{k-2} > \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} = b_\nu$$

for all $\nu \in K \setminus K^*$, where the strict inequality can be obtained using the fact $\delta > 0$. This proves the feasibility.

This, however, is a contradiction, because the objective function attains a smaller value in $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\setminus K^*}$ than in its infimum. Hence, our assumption was wrong and the objective function always attains a global minimum. In fact, we will prove in the next part that $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\setminus K^*}$ is this global minimum.

Calculation of the global minimum:
As all constraints in (2.22) are linear, the global minimum has to be a stationary point, see Chapter 4 for a definition. Therefore, our next step is to calculate all stationary points of this problem. Assume that $b_{K\setminus K^*}$ is such a stationary point. Then we know that there are multipliers λ^u, λ^l such that the following equations hold for all $\nu \in K \setminus K^*$:

$$\begin{aligned} \frac{1}{b_\nu^2} - \frac{(k-2)^2}{(\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu)^2} - \lambda_\nu^l(k^*-2) + \lambda_\nu^u(k-2) - \sum_{\mu\in K\setminus K^*}\lambda_\mu^u &= 0, \\ (k^*-2)b_\nu \geq \sum_{\mu\in K^*}\beta_\mu,\ \ \lambda_\nu^l \geq 0,\ \ \Big((k^*-2)b_\nu - \sum_{\mu\in K^*}\beta_\mu\Big)\lambda_\nu^l &= 0, \\ (k-2)b_\nu \leq \sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu,\ \ \lambda_\nu^u \geq 0,\ \ \Big((k-2)b_\nu - \sum_{\mu\in K\cap K^*}\beta_\mu - \sum_{\mu\in K\setminus K^*} b_\mu\Big)\lambda_\nu^u &= 0. \end{aligned}$$

The feasibility of $b_{K\setminus K^*}$ implies that

$$\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} \leq b_\nu \leq \frac{1}{k-2}\left(\sum_{\mu\in K\setminus K^*} b_\mu + \sum_{\mu\in K\cap K^*}\beta_\mu\right) \quad \forall \nu \in K \setminus K^*$$

(recall that $k^* - 2 > 0$ and $k - 2 > 0$, so the denominators above are well-defined). However, the lower estimate for b_ν ($\nu \in K \setminus K^*$) given in the previous formula is strictly smaller than the upper estimate. This can be seen in the following way: Using the definition of K^* together with $K^* \setminus K \neq \emptyset$ as well as the feasibility of b_ν ($\nu \in K \setminus K^*$), we obtain

$$b_\nu \geq \frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2} \quad \forall \nu \in K \setminus K^* \implies \sum_{\nu \in K \setminus K^*} b_\nu \geq \frac{k - d}{k^* - 2} \sum_{\mu \in K^*} \beta_\mu,$$
$$\beta_\nu < \frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2} \quad \forall \nu \in K^* \implies \sum_{\nu \in K^* \setminus K} \beta_\nu < \frac{k^* - d}{k^* - 2} \sum_{\mu \in K^*} \beta_\mu,$$

where the implications follow by taking the summations over all $\nu \in K \setminus K^*$ and all $\nu \in K^* \setminus K$, respectively. Using these estimates, we indeed obtain

$$\begin{aligned} \frac{\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}{k - 2} &= \frac{\sum_{\mu \in K^*} \beta_\mu - \sum_{\mu \in K^* \setminus K} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}{k - 2} \\ &> \frac{\sum_{\mu \in K^*} \beta_\mu ((k^* - 2) - (k^* - d) + (k - d))}{(k^* - 2)(k - 2)} \\ &= \frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2}. \end{aligned} \tag{2.24}$$

Hence, every $\nu \in K \setminus K^*$ belongs to exactly one of the following three cases:
Case 1: If $b_\nu = \frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2}$, then the KKT conditions together with (2.24) imply $\lambda_\nu^u = 0$. We define I_l as the set of all indices $\nu \in K \setminus K^*$ that belong to this case.
Case 2: If $b_\nu = \frac{\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}{k - 2}$, then the KKT conditions together with (2.24) imply $\lambda_\nu^l = 0$ and thus

$$\lambda_\nu^u (k - 2) - \sum_{\mu \in K \setminus K^*} \lambda_\mu^u = 0. \tag{2.25}$$

We define I_u as the set of all indices $\nu \in K \setminus K^*$ that belong to this case.
Case 3: If $b_\nu \in \left(\frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2}, \frac{\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}{k - 2} \right)$, then the KKT conditions together with (2.24) imply $\lambda_\nu^l = \lambda_\nu^u = 0$ and

$$\frac{1}{b_\nu^2} - \frac{(k - 2)^2}{(\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu)^2} - \sum_{\mu \in K \setminus K^*} \lambda_\mu^u = 0. \tag{2.26}$$

Our next step is to show that Case 3 cannot occur. To this end, let $i_u := |I_u|$ and add (2.25) for all $\nu \in I_u$. This yields (taking into account that $\lambda_\nu^u = 0$ for all $\nu \in I_u$)

$$(k - 2 - i_u) \sum_{\nu \in I_u} \lambda_\nu^u = 0.$$

We will show that $k - 2 - i_u > 0$. Then the nonnegativity of all λ_ν^u ($\nu \in I_u$) implies $\lambda_\nu^u = 0$ for all $\nu \in I_u$ and therefore $\lambda_\nu^u = 0$ for all $\nu \in K \setminus K^*$. But then (2.26) gives a formula for b_ν which is in contradiction to the value of b_ν in Case 3. Hence Case 3 cannot occur. To prove the assertion, assume that $k - 2 - i_u \leq 0$ or, equivalently, $i_u \geq k - 2$. Summation over all b_ν ($\nu \in I_u$) with the expression for b_ν as in Case 2 and using the fact that $K \cap K^*$ is nonempty would then imply

$$\sum_{\nu \in I_u} b_\nu = \frac{i_u}{k-2}\left(\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu\right) \geq \sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu > \sum_{\mu \in K \setminus K^*} b_\mu \geq \sum_{\mu \in I_u} b_\mu$$

which gives the desired contradiction.

Hence, every stationary point is of the form

$$b_\nu = \begin{cases} \frac{\sum_{\mu \in K^*} \beta_\mu}{k^* - 2}, & \nu \in I_l, \\ \frac{\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}{k-2}, & \nu \in I_u, \end{cases}$$

with a characteristic partition $K \setminus K^* = I_l \cup I_u$. Using this and the abbreviation $i_l := |I_l|$, we can resolve the implicit definition of b_ν ($\nu \in I_u$) in the following way:

$$\begin{aligned} (k-2)b_\nu &= \sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu \\ &= \sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in I_u} b_\mu + \sum_{\mu \in I_l} b_\mu \\ &= \sum_{\mu \in K^*} \beta_\mu - \sum_{\mu \in K^* \setminus K} \beta_\mu + \sum_{\mu \in I_u} b_\mu + \frac{i_l}{k^* - 2} \sum_{\mu \in K^*} \beta_\mu \\ &= \sum_{\mu \in K^*} \beta_\mu - \sum_{\mu \in K^* \setminus K} \beta_\mu + i_u b_\nu + \frac{k - d - i_u}{k^* - 2} \sum_{\mu \in K^*} \beta_\mu, \end{aligned}$$

where the first equation follows from the previous implicit expression of b_ν, the second equation takes into account the partition of the set $K \setminus K^*$ into the union $I_u \cup I_l$, the third equation uses a trivial identity together with the previous explicit representation of b_ν for all $\nu \in I_l$, and the fourth equation takes into account that all b_μ ($\mu \in I_u$) have a constant value (the same as b_ν) as well as the fact that $i_l + i_u = |I_l \cup I_u| = |K \setminus K^*| = k - d$. The representation we got in this way can now be solved for b_ν in order to get the explicit expression

$$\begin{aligned} b_\nu &= \frac{(k - d - i_u)\sum_{\mu \in K^*} \beta_\mu + (k^* - 2)\sum_{\mu \in K^*} \beta_\mu - (k^* - 2)\sum_{\mu \in K^* \setminus K} \beta_\mu}{(k^* - 2)(k - 2 - i_u)} \\ &= \frac{(k - 2 - i_u)\sum_{\mu \in K^*} \beta_\mu + (k^* - d)\sum_{\mu \in K^*} \beta_\mu - (k^* - 2)\sum_{\mu \in K^* \setminus K} \beta_\mu}{(k^* - 2)(k - 2 - i_u)} \quad (\nu \in I_u). \end{aligned}$$

Using this and the abbreviations $b_l := b_\nu$ for $\nu \in I_l$ and $b_u := b_\nu$ for $\nu \in I_u$ (recall that both numbers are constant within their corresponding index sets), we can express the value of the objective function in a stationary point as follows:

$$-\sum_{\mu \in K \setminus K^*} \frac{1}{b_\mu} + \frac{(k-2)^2}{\sum_{\mu \in K \cap K^*} \beta_\mu + \sum_{\mu \in K \setminus K^*} b_\mu}$$

$$
\begin{aligned}
&= -\sum_{\mu\in I_u}\frac{1}{b_\mu} - \sum_{\mu\in I_l}\frac{1}{b_\mu} + (k-2)\underbrace{\frac{(k-2)}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu}}_{=\frac{1}{b_u}} \\
&= \frac{k-2-i_u}{b_u} - (k-d-i_u)\frac{1}{b_l} \\
&= \frac{(k-2-i_u)^2(k^*-2)}{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu} - \frac{(k-d-i_u)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} \\
&= \frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{(k-2-i_u)(d-2)\sum_{\mu\in K^*}\beta_\mu - (k-d-i_u)\left[(k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu\right]}{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu},
\end{aligned}
$$

here the first equation uses the partition of $K \setminus K^*$ into $I_u \cup I_l$, the second equation takes into account that b_μ is constant on the two index sets I_u and I_l, respectively, as well as the implicit representation of b_u, the third equation substitutes the explicit values for b_l and b_u, respectively, and the final equation can be verified by direct calculation.

We already know that, for every stationary point, we have $i_u \in \{0, 1, \ldots, k-3\}$. Hence, we are interested in the minimum of the term above for $i_u \in [0, k-3]$ (viewed as a continuous variable, for the moment). To this end, remember the abbreviation

$$
\delta := (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu.
$$

Obviously, δ does not depend on i_u and we have seen before that $\delta > 0$. Differentiation of the term above with respect to i_u then yields (after some algebraic manipulations)

$$
\begin{aligned}
&\frac{\partial}{\partial i_u}\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{(k-2-i_u)(d-2)\sum_{\mu\in K^*}\beta_\mu - (k-d-i_u)\delta}{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + \delta} \\
&= \frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu}\cdot\left(\frac{\left(-(d-2)\sum_{\mu\in K^*}\beta_\mu + \delta\right)\left((k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + \delta\right)}{\left((k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + \delta\right)^2}\right. \\
&\qquad \left. - \frac{\left((k-2-i_u)(d-2)\sum_{\mu\in K^*}\beta_\mu - (k-d-i_u)\delta\right)\left(-\sum_{\mu\in K^*}\beta_\mu\right)}{\left((k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + \delta\right)^2}\right) \\
&= \frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{\delta^2}{\left((k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + \delta\right)^2},
\end{aligned}
$$

which is strictly positive for all $i_u \in [0, k-3]$ because of $\delta > 0$. Hence the objective function is strictly increasing with respect to i_u. Therefore, the stationary point corresponding to the global minimum is the one with the smallest i_u possible, i.e. the one with $i_u = 0$ (which, fortunately, turned out to be an integer, though within our intermediate calculations i_u was assumed to be a real number). While proving the existence of a global minimum, we have already shown that the vector $b_{K\setminus K^*}$ with $I_u = \emptyset$ and $I_l = K \setminus K^*$ is indeed feasible for (2.22) and thus the global minimum.

As we mentioned at the beginning of the proof, the vector $\beta_{K\setminus K^*}$ also is feasible for (2.22). Thus, we obtain the assertion by using the value of the objective function in its global minimum as a lower bound. □

Based on the previous results, we are now in a position to state the main theorem of this section which gives the set of active contestants in the optimal contest.

Theorem 2.30 *The set K^* is the unique optimal set (for the given parameters $\beta_\mu, \mu \in N$).*

Proof. For $n = 2$, $K^* = \{1, 2\}$ is the only feasible and thus optimal set. For $n = 3$, the assertion follows from Theorem 2.25. Now suppose $n \geq 4$. We show that for every feasible set $K \neq K^*$, there exists another feasible set $\tilde{K}$ with $\psi(K) < \psi(\tilde{K})$. To this end, let $K \neq K^*$ be an arbitrary feasible set. If $k := |K| = 2$ or $K \subsetneq K^*$ we can find such a set $\tilde{K}$ according to Proposition 2.24 (note that, in particular, every subset of N consisting of three players is feasible, so every feasible set K consisting of just two players can be enlarged to a set with three players, and then Proposition 2.24 can be applied also to this case). Hence, the only remaining case is $k \geq 3$ and $K \setminus K^* \neq \emptyset$. Define $k^* := |K^*|$ and $d := |K^* \cap K|$. In this case, we are in the situation where Lemma 2.29 can be applied, and we obtain

$$\begin{aligned}
4(\psi(K^*) - \psi(K)) &= \sum_{\mu \in K^* \setminus K} \frac{1}{\beta_\mu} - \frac{(k^* - 2)^2}{\sum_{\mu \in K^*} \beta_\mu} - \sum_{\mu \in K \setminus K^*} \frac{1}{\beta_\mu} + \frac{(k - 2)^2}{\sum_{\mu \in K} \beta_\mu} \\
&\geq \frac{(k^* - d)^2}{\sum_{\mu \in K^* \setminus K} \beta_\mu} - \frac{(k^* - 2)^2}{\sum_{\mu \in K^*} \beta_\mu} - \frac{(k - d)(k^* - 2)}{\sum_{\mu \in K^*} \beta_\mu} \\
&\quad + \frac{(k - 2)^2 (k^* - 2)}{(k - d + k^* - 2) \sum_{\mu \in K^*} \beta_\mu - (k^* - 2) \sum_{\mu \in K^* \setminus K} \beta_\mu} \\
&= \frac{(k^* - 2 + k - d)\big((k^* - d) \sum_{\mu \in K^*} \beta_\mu - (k^* - 2) \sum_{\mu \in K^* \setminus K} \beta_\mu\big)^2}{\big(\sum_{\mu \in K^* \setminus K} \beta_\mu\big)\big(\sum_{\mu \in K^*} \beta_\mu\big)\big((k - d + k^* - 2) \sum_{\mu \in K^*} \beta_\mu - (k^* - 2) \sum_{\mu \in K^* \setminus K} \beta_\mu\big)} \\
&> 0,
\end{aligned}$$

where the first equation follows from the definition of the function $\psi(K)$, the first inequality uses both the inequality between the arithmetic and the harmonic mean applied to the first term and Lemma 2.29 applied to the last two terms, the second equation follows by direct computation using a common denominator for all terms, and the final strict inequality exploits the feasibility of K^*, $k^* \geq 3$ and $k > d$ (the latter holds since $K \setminus K^* \neq \emptyset$). This shows $\psi(K) < \psi(K^*)$ for the remaining case. Consequently, K^* is the unique optimal set. □

Thus, we have proven what we promised at the beginning of this section, namely that the unique set of active players in the optimum is given by

$$K^* = \left\{ \nu \in N \,\middle|\, (k^* - 2)\beta_\nu < \sum_{\mu \in K^*} \beta_\mu \right\}.$$

We are going to use this result in the next section to calculate optimal weights and other interesting properties of the optimal contest.

2.3.3. The Optimal Bias and Other Figures

Using Theorem 2.30 together with Lemma 2.22 and recalling the definition of $\gamma = \frac{\beta}{\alpha}$, we are now able to calculate one solution α^* of the contest designer's problem (2.5), namely

$$\alpha_\nu^* = \frac{2(k^*-1)\beta_\nu}{1 + \frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}} \quad \text{for } \nu \in K^*, \qquad \alpha_\nu^* < (k^*-1)\beta_\nu \quad \text{for } \nu \notin K^*. \tag{2.27}$$

Due to the Lemmas 2.12 and 2.13, we already knew that the contest designer's optimization problem does not have a unique solution, but all other solution can be derived from (2.27) using the variations from these two Lemmas. Nonetheless, the optimal contest is unique in the sense that the set of active players and their individual equilibrium efforts are uniquely determined independent from the particular choice of optimal weights. We can also infer from formula (2.27) that all optimal weights share an equal treatment property, i.e., participants with equal costs β_ν get equal weights α_ν^*.

The expression for α_ν^* is clearly increasing in β_ν for active players. This implies that under the optimal weighting scheme players with high costs are favored relatively more than players with low costs. Hence, with optimally specified weights the heterogeneity between active players is reduced to some extent. A closer look at the formula from Theorem 2.21 (a) reveals, however, that the effective cost parameter,

$$\gamma_\nu^* = \frac{\beta_\nu}{\alpha_\nu^*} = \frac{1}{2(k^*-1)}\left(1 + (k^*-2)\frac{\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}\right),$$

is still increasing in β_ν, whenever there are more than two players active under the optimal weighting scheme. Hence, although the heterogeneity is reduced to some extent, the disadvantage of players with a higher cost factor still remains.

Based on the explicit characterization of the active set of players and the corresponding optimal weighting scheme we are now in a position to derive explicit formulae for equilibrium values. The optimal effort of player ν is then given by

$$x_\nu(\alpha^*) = \begin{cases} \frac{1}{4\beta_\nu}\left[1 - \left(\frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}\right)^2\right] & \text{for } \nu \in K^*, \\ 0 & \text{for } \nu \notin K^*, \end{cases} \tag{2.28}$$

and, after some elementary calculations, one obtains

$$\theta_\nu(x_\nu(\alpha^*), x_{-\nu}(\alpha^*)) = \begin{cases} \frac{1}{4}\left(1 - \frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}\right)^2 & \text{for } \nu \in K^*, \\ 0 & \text{for } \nu \notin K^* \end{cases} \tag{2.29}$$

as the corresponding payoff for player ν. Note that the expression for equilibrium utility in (2.29) is never identical for players with different cost parameters, in fact it is decreasing in β_ν. Hence, the playing field is not leveled to the full extent (with the exception of the two-player case as presented in Example 2.31). Finally, the contest designer's payoff is given by

$$\theta_0(\alpha^*) = \frac{1}{4}\left(\sum_{\mu\in K^*}\frac{1}{\beta_\mu} - \frac{(k^*-2)^2}{\sum_{\mu\in K^*}\beta_\mu}\right). \tag{2.30}$$

To sum it up, by now we have all information the contest designer needs to design the optimal, effort maximizing, contest and formulae for the resulting reactions of the contestants such as their equilibrium effort and utility. We are going to illustrate these results on a few examples in the next section.

2.4. Examples

To illustrate the results given in the last section, we first consider two examples which are well known from the literature, cf. [92, 28], and calculate all interesting figures such as the optimal set of active players (2.21), one set of optimal weights (2.27), the corresponding equilibrium effort of the individual players (2.28) and their resulting utility (2.29), and finally the designer's payoff (2.30). The first example deals with the smallest possible contest with only two participants and illustrates the gain of optimally leveling the playerfield.

Example 2.31 In the 2-player case, the set of active players in the global maximum is $K^* = \{1, 2\}$ and one possible choice for optimal weighting parameters is

$$\alpha_\nu^* = 2\beta_\nu, \quad \text{hence} \quad \gamma_\nu^* = \frac{1}{2} \quad \forall \nu = 1, 2.$$

Since both contestants have the same effective cost parameter γ_ν^*, heterogeneity between the players is completely removed in the optimum. The optimal set of weighting parameters yields the following equilibrium results:

$$\begin{aligned} x_\nu^* &= \frac{1}{4\beta_\nu} \quad \forall \nu = 1, 2, \\ \theta_\nu(x^*) &= \frac{1}{4} \quad \forall \nu = 1, 2, \\ \theta_0(\alpha^*) &= \frac{\beta_1 + \beta_2}{4\beta_1\beta_2}. \end{aligned}$$

The complete removal of heterogeneity is also reflected by the fact that expected payoff in equilibrium is identical for both players.

For comparison, an equal treatment approach with neutral weights, i.e. $\alpha_\nu = 1$ for both players $\nu = 1, 2$, would lead to an aggregated effort of

$$\theta_0^{\text{equal}} = \frac{1}{\beta_1 + \beta_2}.$$

Since $\theta_0(\alpha^*)$ is one half of the inverse of the harmonic mean of β_1 and β_2 and θ_0^{equal} is one half of the inverse of the respective arithmetic mean, the optimally designed contest always yields a greater aggregated effort than the equal treatment approach except, of course, for the homogeneous case where $\beta_1 = \beta_2$ and consequently $\alpha_1^* = \alpha_2^*$. ◇

The following example generalizes the homogeneous two-player case to an arbitrary number of homogeneous players. We will see that, in this case, the optimally designed contest is exactly the one with neutral weights where all participants are active and exert the same effort.

Example 2.32 In the homogeneous n-player case, where $\beta_\nu = \beta_\mu$ (=: β) for all $\nu, \mu \in N$, all subsets $K \subseteq N$ with $k := |K| \geq 2$ are feasible. Hence, as the optimal set has to be a maximal set, the set of active players in the global maximum is $K^* = N$, and the corresponding optimal parameters are

$$\alpha^*_\nu = n\beta, \quad \text{hence} \quad \gamma^*_\nu = \frac{\beta}{\alpha^*_\nu} = \frac{1}{n} \quad \forall \nu \in N.$$

In particular, all players are active in the optimum and have the same effective cost parameter. The equilibrium results are as follows:

$$\begin{aligned} x^*_\nu &= \frac{n-1}{n^2\beta} \quad \forall \nu \in N, \\ \theta_\nu(x^*) &= \frac{1}{n^2} \quad \forall \nu \in N, \\ \theta_0(\alpha^*) &= \frac{n-1}{n\beta}. \end{aligned}$$

◇

As we have proven in the last section, the set of active players in the optimal contest is given by

$$K^* = \left\{ \nu \in N \;\middle|\; (|K^*| - 2)\beta_\nu < \sum_{\mu \in K^*} \beta_\mu \right\}.$$

On the other hand, the set of active players in the non-biased contest with neutral weights $\hat{\alpha} := (1, \ldots, 1)$ is

$$K(\hat{\alpha}) = \left\{ \nu \in N \;\middle|\; (|K(\hat{\alpha})| - 1)\beta_\nu < \sum_{\mu \in K(\hat{\alpha})} \beta_\mu \right\},$$

Hence, we always have $K(\hat{\alpha}) \subseteq K^*$, but it is possible that the set of active contestants in the optimal contest is actually bigger than in the contest with neutral weights. Thus, the optimal contest exhibits some kind of inclusion principle since it is possible that motivating weak players to participate increases the aggregated effort. As we will see in the next section, this is completely different when interpreting the contest as an all-pay auction. But first, we want to illustrate this behavior on a few examples. To simplify these examples, we consider only cases where $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. In these cases, we know

$$K^* = \left\{ \nu \in N \;\middle|\; (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\} \quad \text{and} \quad K(\hat{\alpha}) = \left\{ \nu \in N \;\middle|\; (\nu - 1)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\}.$$

Example 2.33 We consider a contest with four participants with the following distribution of cost parameters: $\beta = (1, 2, 2, 4)^T$. Using the above mentioned characterization of the active sets, it is obvious that the player with $\beta_4 = 4$ is not active with neutral weights but that he is induced to participate under the optimal weighting scheme. Hence, in this case there is inclusion of players due to the optimal weighting scheme. Equilibrium results are presented in Table 2.1 where the weighting factors are normalized to facilitate the comparison between the two weighting schemes.

	Neutral Weights $\hat{\alpha}$	Optimal Weights α^*
α	$(0.25, 0.25, 0.25, 0.25)$	$(0.143, 0.243, 0.243, 0.371)$
$K(\alpha)$	$\{1, 2, 3\}$	$\{1, 2, 3, 4\}$
x^*	$(0.24, 0.08, 0.08, 0)$	$(0.238, 0.100, 0.100, 0.013)$
$\frac{\alpha_\nu x_\nu^*}{\sum_{\mu=1}^n \alpha_\mu x_\mu^*}$	$(0.6, 0.2, 0.2, 0)$	$(0.389, 0.278, 0.278, 0.056)$
$\theta_\nu(x^*)$	$(0.36, 0.04, 0.04, 0)$	$(0.151, 0.077, 0.077, 0.003)$
$\theta_0(\alpha)$	0.4	0.451

Table 2.1.: Results for Example 2.33

Under the optimal weighting scheme the last player is induced to participate due to the relatively large weight that he obtains in comparison to the other players. Note also that the dispersion in winning probabilities between the other players is reduced in comparison to the neutral weighting scheme which illustrates that the playing field is more leveled. Both effects, i.e. additional inclusion in combination with the balanced competition, result in higher total equilibrium effort under the optimal weighting scheme. ◇

Of course, if the difference between the contestants is too big, the optimal weighting scheme might not lead to the inclusion of additional active players. (The one exception to this is the three-player case, which is considered in Example 2.35.) Nonetheless, the balancing effect of the optimal weights can still increase the aggregated effort as illustrated in the following example.

Example 2.34 The distribution of cost parameters is slightly altered for the last player: $\beta = (1, 2, 2, 6)^T$. In this case the last player even remains inactive under the optimal weights, i.e., the set of active players coincides for both weighting schemes, see Table 2.2.

	Neutral Weights $\hat{\alpha}$	Optimal Weights α^*
α	$(0.25, 0.25, 0.25, 0.25)$	$(0.226, 0.387, 0.387, 0)$
$K(\alpha)$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
x^*	$(0.24, 0.08, 0.08, 0)$	$(0.24, 0.105, 0.105, 0)$
$\frac{\alpha_\nu x_\nu^*}{\sum_{\mu=1}^n \alpha_\mu x_\mu^*}$	$(0.6, 0.2, 0.2, 0)$	$(0.4, 0.3, 0.3, 0)$
$\theta_\nu(x^*)$	$(0.36, 0.04, 0.04, 0)$	$(0.16, 0.09, 0.09, 0)$
$\theta_0(\alpha)$	0.4	0.45

Table 2.2.: Results for Example 2.34

As in the previous example the dispersion in winning probabilities of active players declines, i.e., the heterogeneity among the players is reduced under the optimal weighting scheme. Again, setting optimal weights results in higher total equilibrium. A comparison with the previous example also implies that there is a positive effect from the inclusion of players because the total effort under the optimal weighting scheme is slightly higher in Example 2.33 where all players are successfully encouraged to participate. ◇

As mentioned before, Theorem 2.25 implies that there will always be at least three active players in the optimal contest. If we consider a contest with $n = 3$ participants, then all of them will be active in the optimal contest independent from their cost distribution. This effect is illustrated in the following example.

Example 2.35 Consider a contest with $n = 3$ participants and cost parameters $\beta = (1, 1, c)$, where $c \geq 2$. Then it is easy to verify that only the first two players are active in the unbiased contest whereas all three players have to be active in the optimal contest. One readily verifies the following results for the unbiased contest with $\hat{\alpha} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$:

$$\begin{aligned} x^* &= \left(\frac{1}{4}, \frac{1}{4}, 0\right), \\ (\theta_\nu(x^*))_{\nu=1,2,3} &= \left(\frac{1}{4}, \frac{1}{4}, 0\right), \\ \theta_0(\alpha^*) &= \frac{1}{2}. \end{aligned}$$

On the other hand, we obtain the following formulae for these figures in the optimal contest:

$$\begin{aligned} \alpha_\nu^* &= \begin{cases} 4\left(1 - \frac{1}{c+3}\right) & \text{if } \nu = 1, 2, \\ 4c\left(1 - \frac{c}{2c+2}\right) & \text{if } \nu = 3, \end{cases} \\ x_\nu(\alpha^*) &= \begin{cases} \frac{1}{4}\left(1 - \frac{1}{(c+2)^2}\right) & \text{if } \nu = 1, 2, \\ \frac{1}{4c}\left(1 - \frac{c^2}{(c+2)^2}\right) & \text{if } \nu = 3, \end{cases} \\ \theta_\nu(x^*) &= \begin{cases} \frac{1}{4}\left(1 - \frac{1}{c+2}\right)^2 & \text{if } \nu = 1, 2, \\ \frac{1}{4}\left(1 - \frac{c}{c+2}\right)^2 & \text{if } \nu = 3, \end{cases} \\ \theta_0(\alpha^*) &= \frac{1}{2}\left(1 + \frac{1}{c(c+2)}\right). \end{aligned}$$

Hence, the bigger c becomes, the bigger α_3^* has to be in order to motivate the third player to take part in the contest. But it can also be seen that, although including the third player increases the overall effort, this positive effect is very small when the third player has too high costs compared to the other two, i.e., when c is large. In fact, the figures above (except for α^*) converge to those from the non-biased contest for $c \to \infty$. This is illustrated on a few values of c in Table 2.3. Note that we normalized the weights α^* in this table in order to simplify comparison with the unbiased case. ◇

c	2	10	100
α^*	$(0.273, 0.273, 0.455)$	$(0.126, 0.126, 0.47)$	$(0.019, 0.019, 0.962)$
$x^*(\alpha^*)$	$(0.234, 0.234, 0.094)$	$(0.248, 0.248, 0.008)$	$(0.250, 0.250, 0.970 * 10^{-4})$
$\theta_\nu(x^*)$	$(0.141, 0, 141, 0, 063)$	$(0.210, 0, 210, 0, 007)$	$(0, 245, 0.245, 0.961 * 10^{-4})$
$\theta_0(\alpha^*)$	0.563	0.504	0.500

Table 2.3.: Results for Example 2.35

2.5. Comparison: The All-Pay Auction

A different approach, which is popular to model the situations mentioned in the introduction, is the so called all-pay auction (with complete information), see for example [11, 12, 35]. In an all-pay auction, all participants simultaneously place a bid for a certain prize. The player with the highest bid wins the prize, but all participants have to pay what they bid. Usually, if there are several highest bids, it is supposed that the prize is split evenly between all participants who placed one of these bids. The difference to our model is that in the all-pay auction, the participant with the highest bid or effort is awarded the prize with certainty whereas in our model, every participant has a chance to obtain the prize proportional to his relative effort. Consequently, the contest designer cannot influence the participants' effort by manipulating the winning probabilities. Instead, he will maximize the overall effort by deciding who is allowed to take part in the contest and who is not. As we will see, this leads to substantially different results than those we obtained for the lottery approach.

To substantiate these elaborations, consider again a contest with n participants and denote the set on contestants by N. Let us define a utility function for the participants of the all-pay auction, that is very close to the lottery model we analyzed in the last sections. This function $\theta^A : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\theta_\nu^A(x_\nu, x^{-\nu}) = \begin{cases} -\beta_\nu x_\nu & \text{if there is a } \mu \in N \text{ such that } x_\nu < x_\mu, \\ \frac{V_\nu}{m} - \beta_\nu x_\nu & \text{if } \nu \text{ ties the highest bid with } m-1 \text{ others}, \\ V_\nu - \beta_\nu x_\nu & \text{if } x_\nu > x_\mu \text{ for all } \mu \neq \nu. \end{cases} \tag{2.31}$$

Again, we denote by $\beta_\nu > 0$ the individual cost parameter and by $V_\nu > 0$ the personal valuation of the prize of a participant $\nu \in N$. Analogous to the lottery model, we can assume without loss of generality that all players have the same valuation $V_\nu = 1$ of the prize. Otherwise, we could scale their utility functions by $\frac{1}{V_\nu}$. A closer look at the utility function reveals that θ_ν^A is discontinuous. This hints at the fact that analysis of this contest game is not as straight-forward as in the lottery case. And in fact, it was proven in [12] that the all-pay auction does not have a Nash equilibrium in pure strategies but instead usually a whole continuum of Nash equilibria in mixed strategies. Fortunately, things become easier when we are only interested in the effort maximizing strategy of the contest designer and the thus generated overall effort. If we assume that all participants are ordered according to their cost parameter, i.e.

$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n,$$

then the following result from [11] holds.

Theorem 2.36 *(a) The contest designer maximizes the overall effort by excluding all players v with costs $\beta_v < \beta_{k^A}$, where $k^A \in N$ is chosen such that*

$$\left(1 + \frac{\beta_{k^A}}{\beta_{k^A+1}}\right)\frac{1}{2\beta_{k^A+1}} \geq \left(1 + \frac{\beta_\mu}{\beta_{\mu+1}}\right)\frac{1}{2\beta_{\mu+1}} \tag{2.32}$$

for all $\mu \in N$.

(b) Using the strategy above, an overall effort of

$$\theta_0^A = \left(1 + \frac{\beta_{k^A}}{\beta_{k^A+1}}\right)\frac{1}{2\beta_{k^A+1}} \tag{2.33}$$

is generated.

Hence, someone designing an optimal all-pay auction has an incentive to exclude strong players, i.e. players with low costs, from the contest. In contrast to this, we have proven that in an optimal lottery the strongest players are active, weaker players up to a certain threshold are motivated to participate by choosing the weights in their favor and only the weakest players are inactive. Consequently, designing an optimal all-pay auction requires a totally different strategy than the one we derived before for an optimal lottery contest.

Although this is not the scope of this analysis, we would like to point out an interesting and quite obvious question: Which one of the two models – lottery contest versus all-pay auction – allows the designer to elicit the highest aggregated effort from the contestants? The answer is probably not straightforward since the highest possible effort in both models depends heavily on the heterogeneity of the contestants. In the homogeneous n-player case where $\beta_v = \beta$ for all contestants, we know that the optimal lottery yields a maximal effort of $\frac{n-1}{n}\frac{1}{\beta}$ whereas the auction model yields the higher optimal effort $\frac{1}{\beta}$. In the heterogeneous 2-player case, however, the situation is already more complicated. If we assume $\beta_1 \leq \beta_2$, the maximal effort obtained from the lottery model is $\frac{\beta_1+\beta_2}{4\beta_1\beta_2}$ and the maximal effort that can be elicited from the auction model is $\frac{\beta_1+\beta_2}{2\beta_2^2}$. Therefore, if $\beta_2 \in [\beta_1, 2\beta_1)$, the auction model is better for the contest designer, if $\beta_2 = 2\beta_1$, there is no difference, and in the remaining case $\beta_2 > 2\beta_1$, the lottery model yields the higher total effort. Some answers to this question can be found in [4, 32, 35].

3. Outlook

In this chapter, we come back to the general case where the utility functions of the contestants are given by

$$\theta_\nu(x_\nu, x_{-\nu}) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\alpha_\nu c(x_\nu)}{\sum_{\mu\in N}\alpha_\mu c(x_\mu)} V_\nu - \beta_\nu x_\nu & \text{else,} \end{cases}$$

and $c(x)$ is not necessarily as simple as $c(x) = x$. Another popular function for rent-seeking technologies is $c(x) = x^r$ with either $0 < r < 1$ (decreasing returns to scale) or $r > 1$ (increasing returns to scale). In contrast to the case of constant returns to scale which we considered in the last section, in this general case usually no closed form of the Nash equilibrium is known or it cannot even be proven that the contest game has a Nash equilibrium. We will collect the available theoretical results in Section 3.1. Section 3.2 instead is devoted to the reformulation of the effort maximization problem as a mathematical program with complementarity constraints such that, even if no closed form of the Nash equilibrium problem is known, we can attempt to solve the effort maximization problem numerically using one of the algorithms from Part III.

3.1. General Rent-Seeking Technologies

The minimum assumptions usually imposed on the function $c : \mathbb{R}_+ \to \mathbb{R}$ representing the rent-seeking or production technology are $c(0) = 0$ and c should be continuously differentiable with $c'(x) > 0$ for all $x > 0$. These conditions reflect that zero effort results in zero production and that a higher effort always yields a higher production. Under these assumptions the function c possesses a continuous inverse c^{-1} which is continuously differentiable on $c(\mathbb{R}_{++})$. Thus we can rewrite the ν-th player's optimization problem

$$\max_{x_\nu} \theta_\nu(x_\nu, x_{-\mu}) \quad \text{subject to} \quad x_\nu \geq 0 \tag{3.1}$$

using the substitution $y_\mu := c(x_\mu)$ for all $\mu \in N$ as

$$\max_{y_\nu} \phi_\nu(y_\nu, y_{-\mu}) \quad \text{subject to} \quad y_\nu \geq 0,$$

where the new utility functions of the contestants are given by

$$\phi_\nu(y_\nu, y_{-\nu}) = \begin{cases} 0 & \text{if } y = 0, \\ \frac{\alpha_\nu y_\nu}{\sum_{\mu\in N}\alpha_\mu y_\mu} V_\nu - \beta_\nu c^{-1}(y_\nu) & \text{else.} \end{cases}$$

We could now again try to calculate the best answer function of every player ν and then search for Nash equilibria but without more information on c this attempt will most likely be futile. In fact, in some cases players may not even have a best answer or the contest game have no solution. To illustrate this, we would like to refer to [26], where the production function $c(x) = x^r$ with $r > 1$ is analyzed.

The situation improves if we impose the additional condition that the production function c is twice continuously differentiable with $c''(x) < 0$ for all $x > 0$. Then we know that the inverse function c^{-1} satisfies

$$(c^{-1})'(y) = \frac{1}{c'(c^{-1}(y))} > 0 \quad \text{and} \quad (c^{-1})''(y) = -\frac{c''(c^{-1}(y))}{c'(c^{-1}(y))^3} > 0$$

for all $y \geq 0$, where c^{-1} is defined. To obtain an existence result for the lower level contest game, we employ the following simplifying assumptions: We assume again $V_\nu = 1$ for all contestants $\nu = 1, \ldots, n$ without loss of generality. Otherwise, we could divide the utility functions ϕ_ν by V_ν and redefine β_ν appropriately. We will also assume without loss of generality $\alpha_\nu = 1$ for all $\nu = 1, \ldots, n$. Otherwise, we could substitute $z_\nu := \alpha_\nu y_\nu$ and redefine c^{-1} or c respectively. Note that the necessary redefinition of these functions does not change the sign of their derivatives due to the condition $\alpha_\nu > 0$. Finally, we substitute $d_\nu(y) := \beta_\nu c^{-1}(y)$ for all $\nu \in N$ with $d'_\nu(y) = \beta_\nu (c^{-1})'(y) > 0$ and $d''_\nu(y) = \beta_\nu (c^{-1})''(y) > 0$ for all $y > 0$. This leads to the simplified utility functions

$$\phi_\nu(y_\nu, y_{-\nu}) = \begin{cases} 0 & \text{if } y = 0, \\ \frac{y_\nu}{\sum_{\mu \in N} y_\mu} - d_\nu(y_\nu) & \text{else.} \end{cases}$$

Now, one can again try to calculate the best answer function of each player and although one does not obtain explicit formulae for the best answer function or the Nash equilibrium, it is still possible to prove the existence and uniqueness of a solution to the contest game. This was done in [113] and their result is restated for our setting in the following theorem.

Theorem 3.1 *For all parameters $\alpha, \beta \in \mathbb{R}^n_{++}$ and all twice continuously differentiable production functions $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(0) = 0$, $c'(x) > 0$ and $c''(x) < 0$ for all $x > 0$ the corresponding contest game* (3.1) *has a unique Nash equilibrium x^*, where $x^*_\nu > 0$ for all contestants $\nu \in N$.*

In contrast to the situation we considered in the last chapter, the additional condition $c''(x) > 0$ guarantees that all contestants will actively participate in the solution of the contest game. This simplifies the situation to some extent since calculating the set of active players was one of the crucial parts in our analysis of the case of constant returns to scale. On the other hand, we do not have a closed form of the Nash equilibrium anymore and therefore, one would need a totally different technique in order to prove the existence of solutions to the effort maximization problem and we cannot expect to obtain any closed formula for the solution. If we are interested in concrete instances of the effort maximization problem however, we can also try to solve it numerically. For this reason, we will reformulate the effort maximization problem in the next section to facilitate the application of numerical solution methods.

3.2. Reformulation as MPCC

Most numerical methods for the solution of MPECs expect the equilibrium constraints to be represented by complementarity conditions. Therefore, if we want to solve the effort maximization problem numerically, we have to reformulate the equilibrium condition as a complementarity condition. From game theory it is known that the Nash equilibrium problem (3.1) is equivalent to the following variational inequality if all utility functions θ_ν are concave as functions of x_ν, see [107] for more information: Find $x^* \in \mathbb{R}^n_+$ such that

$$-F(x^*)^T(x - x^*) \geq 0$$

for all $x \in \mathbb{R}^n_+$, where

$$F(x) := \begin{pmatrix} \nabla_{x_1}\theta_1(x) \\ \nabla_{x_2}\theta_2(x) \\ \vdots \\ \nabla_{x_N}\theta_N(x) \end{pmatrix}.$$

Now, it is easy to verify that x^* is a solution of this variational inequality if and only if x^* solves the complementarity system

$$x \geq 0, -F(x) \geq 0, -F(x)^T x = 0.$$

This is exactly the type of constraints we were looking for. Therefore, it remains to verify the concavity of the utility functions. This can be done using the fact that a function is concave on a set if and only if its second derivative is negative semidefinite on this set. We will show that this condition is satisfied in the case of constant returns to scale and for all functions c with $c(0) = 0$, $c'(x) > 0$, $c''(x) < 0$ for all $x > 0$.

Let us first consider the case of constant returns to scale $c(x) = x$. In this case, the second order derivative of θ_ν with respect to x_ν is given by

$$\nabla^2_{x_\nu x_\nu}\theta_\nu(x) = \frac{-2\alpha_\nu^2 \sum_{\mu \neq \nu} \alpha_\mu x_\mu}{\left(\sum_{\mu \in N} \alpha_\mu x_\mu\right)^3} \leq 0 \quad \forall x \in \mathbb{R}^n_+,$$

consequently θ_ν is concave with respect to x_ν on $\mathbb{R}^n_+$.

If we consider the case of an arbitrary function c with $c(0) = 0$, $c'(x) > 0$, $c''(x) < 0$ for all $x > 0$, the second order derivative is given by

$$\nabla^2_{x_\nu x_\nu}\theta_\nu(x) = \alpha_\nu \sum_{\mu \neq \nu} \alpha_\mu c(x_\mu) \frac{c''(x_\nu) \sum_{\mu \in N} \alpha_\mu c(x_\mu) - 2\alpha_\nu c'(x_\nu)^2}{\left(\sum_{\mu \in N} \alpha_\mu c(x_\mu)\right)^3} \leq 0 \quad \forall x \in \mathbb{R}^n_{++}.$$

Hence, we have concavity also in this case. One can also see that the condition $c''(x) \leq 0$ would have been enough to be able to reformulate the Nash equilibrium problem as complementarity condition, whereas for Theorem 3.1 we need the stronger assumption $c''(x) < 0$.

With the above considerations we have verified that in both cases considered here, we can reformulate the effort maximization problem equivalently as

$$\max_{\alpha,x} \sum_{\mu \in N} x_\mu \quad \text{subject to} \quad \begin{aligned} &\alpha > 0, \\ &x \geq 0, \\ &-F(\alpha, x) \geq 0, \\ &-F(\alpha, x)^T x = 0, \end{aligned}$$

where $F(\alpha, x)$ is defined as above, but α is now mentioned explicitly as a variable since it is not fixed anymore. We will use this reformulation in Part III to obtain numerical solutions of some effort maximization problems.

3.3. Concluding Remarks

In this part, we considered the problem of effort maximization in asymmetric n-person contest games, an economic example for an MPEC. We first analyzed the case of constant returns to scale and were able to solve the effort maximization analytically. This was so far only done for $n = 2$ or for n homogeneous players and it turns out that these two cases are degenerate in a certain sense. The 2-player case is the only one where only two contestants participate actively in the solution. Whenever there are more than two players, at least three of them are active in the solution of the effort maximization problem. Also, the 2-player and the homogeneous n-player case are the only ones where all differences between the players are equalized. For $n \geq 3$ heterogeneous players, the optimal weights α^* are chosen such that the playing field is leveled to a certain extend but never completely. We also included a short comparison of our model with the similar all-pay auction model since we obtain quite different results than those known for the auction. For the designer of an all-pay auction, it is favorable to exclude strong players from the contest. In contrast to this, we do not have exclusion of any players in the considered lottery model but rather inclusion of weak players. These effects are also illustrated on a few examples.

Secondly, we gave a brief outlook on contest games with general rent-seeking or production technologies. We introduced a class of rent-seeking technologies for which one can prove existence and uniqueness of solutions to the contest game and provided a reformulation of the corresponding effort maximization problems and those corresponding to linear returns to scale as an MPCC, which will be used for numerical experiments in Part III.

Part II.

Theoretical Results

Although an MPCC (1.1) is at first glance nothing more than a standard optimization problem, the special structure of its constraints causes almost all standard optimization theory to fail. As we will see later, most of the constraint qualifications known from optimization have very little or no chance of being satisfied in a feasible point of (1.1). The well known linear independence constraint qualification and Mangasarian-Fromovitz constraint qualification for example are violated in every feasible point. Even the Abadie constraint qualification, which is one of the weakest known for nonlinear optimization problems, cannot be expected to hold. The only standard constraint qualification applicable in the context of MPCCs is the even weaker Guignard constraint qualification, see [40]. Without constraint qualifications, however, the KKT conditions cannot be guaranteed to be necessary optimality conditions. And in fact, there are even examples where all constraint functions g, h, G, H are linear but the global minimum of the corresponding MPCC is not a stationary point.

For this reason, a number of stationarity concepts for MPCCs have emerged over the last years. The oldest ones are probably strong stationarity and C-stationarity introduced in [104]. Strong stationarity however can be proven to be equivalent to the KKT conditions of the MPCC (1.1) and therefore is a necessary optimality condition only under strong assumptions. Clarke stationarity on the other hand is weaker than strong stationarity and a necessary optimality condition under very mild conditions. With Mordukhovich's limiting calculus, see [89, 90, 103], another stationarity concept called M-stationarity was born, cf. [93, 94, 122, 124]. M-stationarity can be shown to be a necessary optimality condition under the same assumptions as C-stationarity but is a stronger condition. Apart from these three, there is at least one other stationarity concept for MPCCs called A-stationarity, which was introduced in [37] but will not play a role in this thesis.

In order to guarantee that a local minimum of the MPCC (1.1) is stationary in one of the above senses, conditions on the representation of the feasible set, so called constraint qualifications, are needed. Since standard constraint qualifications do not work for MPCCs, a whole zoo of specialized MPCC constraint qualifications has been developed, most of them analogues of standard constraint qualifications. We are going to introduce three new MPCC constraint qualifications. One of them is an MPCC version of the constant positive linear dependence condition which was introduced in [99] and proven to be a constraint qualification in [5]. This constraint qualification will play a role in the convergence results of some relaxation methods in Part III.

A different approach to cope with the fact that standard constraint qualifications are usually not satisfied and the KKT conditions therefore are not necessary optimality conditions is to turn to the Fritz-John conditions since these are always necessary optimality conditions, i.e., no additional conditions are required. However, we will show that a direct application of the standard Fritz-John conditions to the MPCC (1.1) is not really helpful. Fortunately, it is possible to derive a suitable analogue of the Fritz-John conditions for MPCCs. This was for example done in [104, 123].

We improve the results from [123] using a completely different approach based on an idea from [17]. There, the authors consider a standard nonlinear optimization problem with an additional abstract constraint. Exploiting the special structure of the complementarity constraints, we obtain a result which is stronger than what we would obtain by direct application of [17] to the MPCC (1.1). These enhanced Fritz-John conditions motivate the introduction of some new constraint qualifications for MPCCs. One of these is then used to obtain M-stationarity as a necessary optimality condition under various known MPCC constraint qualifications. While this is, in principle, a known result, we have for the first time a completely elementary proof of this fact. All other proofs known to us are based on the limiting subdifferential and the limiting coderivative by Mordukhovich which, albeit being powerful instruments for variational analysis, are not known to the whole community. Nevertheless, we will also make use of the limiting subdifferential afterwards in order to obtain an exact penalty result for MPCCs under weaker assumptions than for example those in [106, 80]. Finally, we close by discussing the relations between the new MPCC constraint qualifications we introduced and those commonly used for MPCCs.

The theoretical part of this thesis is structured as follows. We begin by recalling some results and definitions from standard optimization in Chapter 4. In Chapter 5, we define the MPCC-analogues of constraint qualifications and some stationarity concepts for MPCCs. The new Fritz-John result is then derived in Chapter 6, where we also state some new constraint qualifications related to the improved Fritz-John conditions and prove an exact penalty result for MPCCs.

4. Standard Nonlinear Optimization Programs

Although the focus of this thesis are not standard nonlinear optimization programs (NLP for short) but mathematical programs with equilibrium constraints, we wish to recall some results from the theory of NLPs first. These results will be used to illustrate the difference between MPCCs and NLPs, as a basis for the theory on MPCCs and, last but not least, will be needed in the numerical part where we solve MPCCs by replacing them by a sequence of NLPs. For more information on standard nonlinear optimization we refer to [52]. To us, an NLP is an optimization problem of the form

$$\min_x f(x) \quad \text{subject to} \quad \begin{array}{l} g_i(x) \leq 0 \quad \forall i = 1, \dots, m, \\ h_i(x) = 0 \quad \forall i = 1, \dots, p, \end{array} \tag{4.1}$$

where the functions $f, g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable. We denote the set of all feasible points by $\mathcal{X}$ and for a point $x \in \mathcal{X}$

$$I_g(x) := \{i \mid g_i(x) = 0\}$$

is the set of active inequalities. One of the basic results from standard optimization theory is the following necessary optimality condition, see for example [52, Theorem 2.53].

Theorem 4.1 *Let x^* be a local minimum of* (4.1)*. Then there are multipliers α, λ, μ such that $(x^*, \alpha, \lambda, \mu)$ is a* Fritz-John point*, i.e., the multipliers satisfy $\alpha \geq 0$, $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ with $(\alpha, \lambda, \mu) \neq 0$, $\lambda_i g_i(x^*) = 0$ for all $i = 1, \dots, m$ and*

$$\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0.$$

Another well-known necessary optimality criterion is the following: When x^* is a local minimum, then the directional derivative in all feasible directions has to be nonnegative or more precisely

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in T_{\mathcal{X}}(x^*),$$

where $T_{\mathcal{X}}(x^*)$ is the *Bouligand tangent cone to $\mathcal{X}$ in x^**

$$T_{\mathcal{X}}(x^*) := \left\{ d \in \mathbb{R}^n \,\middle|\, \exists \{x^k\} \subseteq \mathcal{X}, \exists \{\tau_k\} \downarrow 0 \text{ such that } x^k \to x^* \text{ and } \frac{x^k - x^*}{\tau_k} \to d \right\},$$

see [52, Lemma 2.30] for a proof. Since the tangent cone is difficult to calculate, for practical purposes one usually uses the *linearized tangent cone*

$$L_{\mathcal{X}}(x^*) := \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \ (i \in I_g(x^*)), \nabla h_i(x^*)^T d = 0 \ (i = 1 \ldots, p)\}.$$

Between the tangent cone and the linearized tangent cone one always has the relation $T_{\mathcal{X}}(x^*) \subseteq L_{\mathcal{X}}(x^*)$, see for example [52, Lemma 2.32]. The reverse inclusion, however, is in general not true. By the Farkas Lemma [52, Lemma 2.27],

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in L_{\mathcal{X}}(x^*)$$

is then equivalent to the following condition:

Definition 4.2 *A point x^* is called a* stationary point *if there are multipliers λ, μ such that (x^*, λ, μ) is a* Karush-Kuhn-Tucker point *(KKT point), i.e., the multipliers satisfy $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ with $\lambda_i g_i(x^*) = 0$ for all $i = 1, \ldots, m$ and*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0.$$

Due to the fact that $T_{\mathcal{X}}(x^*)$ can be a strict subset of $L_{\mathcal{X}}(x^*)$, being a stationary point is not a necessary optimality condition. Hence, in order to guarantee that local minima are stationary points so called *constraint qualifications* have been introduced. A constraint qualification, or CQ for short, is a condition on the constraints representing the feasible set which ensures a certain regularity of the feasible set such that local minima are also stationary points. This can be done for example by ensuring the equality of both cones $T_{\mathcal{X}}(x^*) = L_{\mathcal{X}}(x^*)$ or at least that the corresponding polar cones coincide $T_{\mathcal{X}}(x^*)^\circ = L_{\mathcal{X}}(x^*)^\circ$.

Definition 4.3 *Let $C \subseteq \mathbb{R}^n$ be a nonempty set. The* polar cone *of C is defined as*

$$C^\circ := \{s \in \mathbb{R}^n \mid s^T c \leq 0 \ \forall c \in C\}.$$

The most well-known constraint qualifications for NLPs are probably the following.

Definition 4.4 *A point x^* feasible for* (4.1) *is said to satisfy the*

(a) linear independence constraint qualification (LICQ) *if the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_g(x^*)\} \cup \{\nabla h_i(x^*) \mid i = 1, \ldots, p\}$$

are linearly independent;

(b) Mangasarian-Fromovitz constraint qualification (MFCQ) *if the gradients $\{\nabla h_i(x^*) \mid i = 1, \ldots, p\}$ are linearly independent and there is a $d \in \mathbb{R}^n$ such that*

$$\nabla g_i(x^*)^T d < 0 \ (i \in I_g(x^*)) \text{ and } \nabla h_i(x^*)^T d = 0 \ (i = 1, \ldots, p);$$

(c) constant rank constraint qualification *(CRCQ) if, for any subsets* $I_1 \subseteq I_g(x^*)$ *and* $I_2 \subseteq \{1, \ldots, p\}$ *such that the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_1\} \cup \{\nabla h_i(x^*) \mid i \in I_2\}$$

are linearly dependent, there exists a neighborhood $U(x^*)$ *of* x^* *such that these gradients remain linearly dependent;*

(d) Abadie constraint qualification (ACQ) *if* $T_{\mathcal{X}}(x^*) = L_{\mathcal{X}}(x^*)$*;*

(e) Guignard constraint qualification (GCQ) *if* $T_{\mathcal{X}}(x^*)^\circ = L_{\mathcal{X}}(x^*)^\circ$.

Whereas LICQ, MFCQ, and ACQ are very famous, CRCQ, which was introduced in [64], and GCQ, which goes back to [53], might be less familiar. We would like to mention that our definition of CRCQ is not the classical one but is equivalent to it as one easily verifies. However, in the context of MPCCs, GCQ turned out to be the only useful standard constraint qualification, see for example [40]. In order to define another recently introduced constraint qualification, we need the notion of positive-linearly dependent vectors.

Definition 4.5 *A finite set of vectors* $\{a^i \mid i \in I_1\} \cup \{b^i \mid i \in I_2\}$ *is said to be* positive-linearly dependent *if there exist scalars* α_i $(i \in I_1)$ *and* β_i $(i \in I_2)$*, not all of them being zero, with* $\alpha_i \geq 0$ *for all* $i \in I_1$ *and*

$$\sum_{i \in I_1} \alpha_i a^i + \sum_{i \in I_2} \beta_i b^i = 0.$$

Otherwise, we say that these vectors are positive-linearly independent.

Using this concept, we can now define the constant positive-linear dependence constraint qualification, which was introduced in [99] and proven to be a constraint qualification in [5].

Definition 4.6 *A point* x^* *feasible for* (4.1) *is said to satisfy the* constant positive-linear dependence constraint qualification *(CPLD) if, for any subsets* $I_1 \subseteq I_g(x^*)$ *and* $I_2 \subseteq \{1, \ldots, p\}$ *such that the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_1\} \cup \{\nabla h_i(x^*) \mid i \in I_2\}$$

are positive-linearly dependent, there exists a neighborhood $N(x^*)$ *of* x^* *such that the gradients*

$$\{\nabla g_i(x) \mid i \in I_1\} \cup \{\nabla h_i(x) \mid i \in I_2\}$$

are linearly dependent for all $x \in N(x^*)$.

Positive-linear dependent vectors can also be used to give a different characterization of MFCQ based on Motzkin's Theorem of alternatives [84].

Lemma 4.7 *A point* $x^* \in \mathcal{X}$ *satisfies MFCQ if and only if the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_g(x^*)\} \cup \{\nabla h_i(x^*) \mid i = 1, \ldots, p\}$$

are positive-linearly independent.

Apart from the constraint qualifications introduced above, there are two more, which usually appear in the context of Fritz-John points and can be found for example in [56, 16].

Definition 4.8 *A point x^* feasible for* (4.1) *is said to satisfy*

(a) pseudonormality *if there are no multipliers λ, μ such that*

  (i) $\sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0$,

  (ii) $\lambda \in \mathbb{R}^m_+$, $\lambda_i g_i(x^*) = 0$ *for all* $i = 1, \ldots, m$,

  (iii) there is a sequence $x^k \to x^*$ such that the following is true for all $k \in \mathbb{N}$:

$$\sum_{i=1}^{m} \lambda_i g_i(x^k) + \sum_{i=1}^{p} \mu_i h_i(x^k) > 0;$$

(b) quasinormality *if there are no multipliers λ, μ such that*

  (i) $\sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0$,

  (ii) $\lambda \in \mathbb{R}^m_+$, $\lambda_i g_i(x^*) = 0$ *for all* $i = 1, \ldots, m$,

  (iii) there is a sequence $x^k \to x^*$ such that the following is true for all $k \in \mathbb{N}$: For all $\lambda_i > 0$ we have $\lambda_i g_i(x^k) > 0$ and for all $\mu_i \neq 0$ we have $\mu_i h_i(x^k) > 0$.

The relations that hold between the constraint qualifications introduced above are depicted in Figure 4.1.

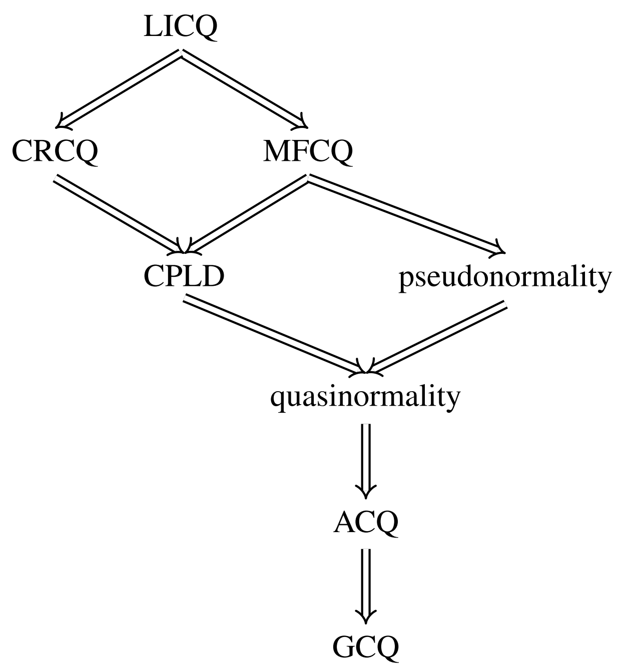


Figure 4.1.: Relations between standard CQs

The implications between LICQ, MFCQ, CRCQ and CPLD follow directly from the respective definitions as does the relation between MFCQ, pseudonormality and quasinormality and between ACQ and GCQ. The fact that CPLD implies quasinormality can be found in [5]. They also give examples illustrating that CPLD is weaker than both MFCQ and CRCQ, stronger than quasinormality and that there is no relation between CPLD and pseudonormality. The relation between quasinormality and ACQ was proven in [16].

As it was already mentioned before, the reason why constraint qualifications are important is the following well-known result.

Theorem 4.9 *A local minimum* x^* *of* (4.1) *that satisfies GCQ or any stronger constraint qualification is a KKT point.*

5. Mathematical Programs with Complementarity Constraints

Although the MPCC (1.1) is by definition nothing more than an NLP, the special structure of the equilibrium constraints, in our case complementarity constraints, causes trouble if one wants to apply standard NLP theory. But before this is illustrated on an few examples, we need some more notation. The feasible set of the MPCC (1.1) defined in the introduction is denoted by X and similarly to the NLP we define the following sets of active constraints in an arbitrary $x^* \in X$:

$$\begin{aligned} I_g(x^*) &:= \{i \mid g_i(x^*) = 0\}, \\ I_{00}(x^*) &:= \{i \mid G_i(x^*) = 0, H_i(x^*) = 0\}, \\ I_{0+}(x^*) &:= \{i \mid G_i(x^*) = 0, H_i(x^*) > 0\}, \\ I_{+0}(x^*) &:= \{i \mid G_i(x^*) > 0, H_i(x^*) = 0\}. \end{aligned}$$

Note that the first (second) subscript indicates whether $G_i(x^*)$ ($H_i(x^*)$) is zero or positive at the given point x^*.

As it was introduced in the last section, being a Fritz-John point is a necessary optimality criterion for NLPs and thus also for MPCCs. However, in the context of MPCCs, this criterion is rendered completely useless by the following result.

Lemma 5.1 *Let x^* be feasible for* (1.1)*. Then there are multipliers such that x^* together with these is a Fritz-John point.*

Proof. Since $I_{00}(x^*) \cup I_{0+}(x^*) \cup I_{+0}(x^*) = \{1, \ldots, q\}$, at least one of these index sets is nonempty.

In case $I_{00}(x^*) \neq \emptyset$ pick an arbitrary $i^* \in I_{00}(x^*)$ and define multipliers $\alpha = 0$, $\lambda = 0_m$, $\mu = 0_p$, $\gamma = 0_q$, $\nu = 0_q$ and $\delta \in \mathbb{R}^q$ with $\delta_i = 0$ for all $i \neq i^*$ and $\delta_{i^*} = 1$. Then it is easy to verify that x^* together with these multipliers is a Fritz-John point, i.e., that the equation

$$\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*) + \sum_{i=1}^{q} \delta_i \nabla (G_i \cdot H_i)(x^*) = 0$$

is satisfied together with all other conditions on the multipliers.

In case $I_{0+}(x^*) \neq \emptyset$ pick an arbitrary $i^* \in I_{0+}(x^*)$ and define the multipliers as above with the only difference that in this case we want $\gamma_{i^*} = H_i(x^*)$ and not $\gamma_{i^*} = 0$. The case $I_{+0}(x^*) \neq \emptyset$ can be handled analogously. □

While the Fritz-John conditions are useless because they are satisfied in too many points, standard constraint qualifications in turn are usually not satisfied in feasible points.

Lemma 5.2 *Let x^* be feasible for* (1.1). *Then MFCQ is not satisfied in x^*.*

This result can easily be verified using the characterization of MFCQ given in Lemma 4.7, see also [21]. Even very weak constraint qualifications like ACQ are usually violated. The only standard constraint qualification which has a reasonable chance of being satisfied is GCQ, cf. [40]. Without constraint qualifications however, the KKT conditions are not necessary optimality conditions and indeed there are simple MPCCs where the solutions are not stationary points. To illustrate this consider the following MPCC with linear constraints which is originally due to [104].

Example 5.3 Consider the following three-dimensional MPCC

$$\min_{x_1,x_2,x_3} f(x) = x_1 + x_2 - x_3 \quad \text{subject to} \quad \begin{aligned} g_1(x) &= -4x_1 + x_3 \leq 0, \\ g_2(x) &= -4x_2 + x_3 \leq 0, \\ G(x) &= x_1 \geq 0, \\ H(x) &= x_2 \geq 0, \\ G(x)H(x) &= x_1 x_2 = 0. \end{aligned}$$

One can easily verify that the global minimum is $x^* = (0, 0, 0)^T$, where all inequalities are active. In order to prove that x^* is a stationary point, we would therefore have to find multipliers $\lambda \in \mathbb{R}^2_+$, $\gamma \geq 0$, $\nu \geq 0$ and $\delta \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} - \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \nu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

This would imply

$$\begin{aligned} \gamma &= 1 - 4\lambda_1 \geq 0 \implies \lambda_1 \in [0, 0.25], \\ \nu &= 1 - 4\lambda_2 \geq 0 \implies \lambda_2 \in [0, 0.25], \\ 1 &= \lambda_1 + \lambda_2, \end{aligned}$$

which is obviously not possible. ◇

Since the usual stationarity concepts and constraint qualifications do not work for MPCCs, specialized versions have been introduced for MPCCs. We will present those relevant in the context of this thesis in the following two sections.

5.1. Constraint Qualifications

By now, there is a whole zoo of constraint qualifications specially designed to deal with the structure of the complementarity constraints. We will therefore introduce only those needed later on and refer for example to [123] for the definition of even more MPCC constraint qualifications.

Most of the constraint qualifications for MPCCs that will appear in this thesis are derived from standard constraint qualifications via the following NLP. For an arbitrary feasible point $x^* \in X$ define the auxiliary problem

$$\begin{array}{lll} \min & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & \forall i = 1, \ldots, m, \\ & h_i(x) = 0 & \forall i = 1, \ldots, p, \\ & G_i(x) = 0, H_i(x) \geq 0 & \forall i \in I_{0+}(x^*), \\ & G_i(x) \geq 0, H_i(x) = 0 & \forall i \in I_{+0}(x^*), \\ & G_i(x) = 0, H_i(x) = 0 & \forall i \in I_{00}(x^*), \end{array} \tag{5.1}$$

which is called *tightened nonlinear program* TNLP(x^*). Note that TNLP(x^*) substantially depends on the chosen point x^* which, by the way, is feasible for TNLP(x^*). Its feasible set, however, is always a subset of X which is the reason for its name. This NLP can now be utilized to define constraint qualifications for MPCCs in the following way:

Definition 5.4 *A point x^* feasible for* (1.1) *is said to satisfy* MPCC-LICQ *(*MPCC-MFCQ, MPCC-CRCQ, MPCC-CPLD*) if standard LICQ (MFCQ, CRCQ, CPLD) for the corresponding TNLP(x^*) is satisfied in x^*.*

Since we are going to need these MPCC constraint qualifications quite a lot, let us state them explicitly which can be done by applying the definitions of the standard constraint qualifications to TNLP(x^*).

Corollary 5.5 *A point x^* feasible for* (1.1) *satisfies*

(a) MPCC-LICQ if and only if the gradients

$$\begin{array}{ll} \nabla g_i(x^*) & (i \in I_g(x^*)), \\ \nabla h_i(x^*) & (i = 1, \ldots, p), \\ \nabla G_i(x^*) & (i \in I_{00}(x^*) \cup I_{0+}(x^*)), \\ \nabla H_i(x^*) & (i \in I_{00}(x^*) \cup I_{+0}(x^*)), \end{array}$$

are linearly independent;

(b) MPCC-MFCQ if and only if the gradients

$$\begin{array}{ll} \nabla h_i(x^*) & (i = 1, \ldots, p), \\ \nabla G_i(x^*) & (i \in I_{0+}(x^*) \cup I_{00}(x^*)), \\ \nabla H_i(x^*) & (i \in I_{+0}(x^*) \cup I_{00}(x^*)) \end{array}$$

are linearly independent, and there exists a vector $d \in \mathbb{R}^n$ such that

$$\begin{array}{ll} \nabla g_i(x^*)^T d < 0 & \forall i \in I_g(x^*), \\ \nabla h_i(x^*)^T d = 0 & \forall i = 1, \ldots, p, \\ \nabla G_i(x^*)^T d = 0 & \forall i \in I_{0+}(x^*) \cup I_{00}(x^*), \\ \nabla H_i(x^*)^T d = 0 & \forall i \in I_{+0}(x^*) \cup I_{00}(x^*); \end{array}$$

(c) *MPCC-CRCQ if and only if, for any subsets* $I_1 \subseteq I_g(x^*)$*,* $I_2 \subseteq \{1, \dots, p\}$*,* $I_3 \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$ *and* $I_4 \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$ *such that the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_1\} \cup \{\nabla h_i(x^*) \mid i \in I_2\} \cup \{\nabla G_i(x^*) \mid i \in I_3\} \cup \{\nabla H_i(x^*) \mid i \in I_4\}$$

are linearly dependent, there exists a neighborhood $U(x^*)$ *of* x^* *such that the gradients*

$$\{\nabla g_i(x) \mid i \in I_1\} \cup \{\nabla h_i(x) \mid i \in I_2\} \cup \{\nabla G_i(x) \mid i \in I_3\} \cup \{\nabla H_i(x) \mid i \in I_4\}$$

remain linearly dependent for all $x \in U(x^*)$*;*

(d) *MPCC-CPLD if and only if, for any subsets* $I_1 \subseteq I_g(x^*)$*,* $I_2 \subseteq \{1, \dots, p\}$*,* $I_3 \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$ *and* $I_4 \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$ *such that the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_1\} \cup \{\{\nabla h_i(x^*) \mid i \in I_2\} \cup \{\nabla G_i(x^*) \mid i \in I_3\} \cup \{\nabla H_i(x^*) \mid i \in I_4\}\}$$

are positive-linearly dependent, there exists a neighborhood $U(x^*)$ *of* x^* *such that the gradients*

$$\{\nabla g_i(x) \mid i \in I_1\} \cup \{\nabla h_i(x) \mid i \in I_2\} \cup \{\nabla G_i(x) \mid i \in I_3\} \cup \{\nabla H_i(x) \mid i \in I_4\}$$

remain linearly dependent for all $x \in U(x^*)$*.*

Note that we used an extra pair of curly brackets in the characterization of MPCC-CPLD in order to indicate those vectors for which no sign restriction applies in the definition of positive-linear dependence. While MPCC-LICQ and MPCC-MFCQ have been around for quite some time now, see [104], MPCC-CRCQ was introduced in [109] and MPCC-CPLD just recently in [58]. MPCC-CPLD can be seen as a relaxation of both MPCC-MFCQ and MPCC-CPLD and the following example illustrates that it is in fact strictly weaker than both of them.

Example 5.6 Consider the following two-dimensional MPCC

$$\begin{aligned}
\min_{x_1, x_2} f(x) = 2x_2 \quad \text{subject to} \quad & g_1(x) = x_1 + x_2^2 \leq 0, \\
& g_2(x) = x_1 \leq 0, \\
& G(x) = x_2 \geq 0, \\
& H(x) = x_1 + x_2 \geq 0, \\
& G(x)H(x) = x_2(x_1 + x_2) = 0.
\end{aligned}$$

It it easy to see that the feasible region is $X = \{x \in \mathbb{R}^2 \mid x_1 \in [-1, 0], x_2 = -x_1\}$ and that the global minimum is $x^* = (0, 0)^T$. All constraints are active in x^* and the gradients are

$$\nabla g_1(x^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nabla g_2(x^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nabla G(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nabla H(x^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, MPCC-MFCQ is violated in x^* and MPCC-CRCQ does not hold either since the gradients of g_1 and g_2 are linearly dependent in x^* but linearly independent everywhere else.

On the other hand, the weaker constraint qualification MPCC-CPLD is satisfied in x^*. To see this, note first that every subset of gradients that does not include $\nabla g_1(x)$, is independent of x. Thus, we only have to consider those subsets of gradients that include $\nabla g_1(x)$. Now, it is easy to see that there are only two subsets such that the included gradients are positive-linearly dependent in x^*, namely $\{\nabla g_1(x)\} \cup \{\nabla G(x), \nabla H(x)\}$ and $\{\nabla g_1(x), \nabla g_2(x)\} \cup \{\nabla G(x), \nabla H(x)\}$, and that those remain linearly dependent in a whole neighborhood. ◇

Analogously to standard MFCQ, we can also give an alternative characterization of MPCC-MFCQ using positive-linear independence.

Lemma 5.7 *A point $x^* \in X$ satisfies MPCC-MFCQ if and only if the gradients*

$$\{\nabla g_i(x^*) \mid i \in I_g(x^*)\} \cup$$
$$\{\{\nabla h_i(x^*) \mid i = 1, \dots, p\} \cup \{\nabla G_i(x^*) \mid i \in I_{00}(x^*) \cup I_{0+}(x^*)\} \cup \{\nabla H_i(x^*) \mid i \in I_{00}(x^*) \cup I_{+0}(x^*)\}\}$$

are positive-linearly independent.

Since these MPCC constraint qualifications are defined via TNLP(x^*), they inherit some properties from the standard versions, especially the relations between them are the same. Before we illustrate these relations in Figure 5.1, let us also define analogues of ACQ and GCQ.

If we wanted to apply ACQ directly to MPCCs, we would have to demand $T_X(x^*) = L_X(x^*)$. However, $L_X(x^*)$ is a linear cone and thus always convex. On the other hand, $T_X(x^*)$ is usually not convex when X is the feasible set of an MPCC. This is illustrated in the following example.

Example 5.8 Consider the most basic MPCC

$$\min_{x_1, x_2} f(x) \quad \text{subject to} \quad \begin{aligned} G(x) &= x_1 \geq 0, \\ H(x) &= x_2 \geq 0, \\ G(x)H(x) &= x_1 x_2 = 0. \end{aligned}$$

If we consider $x^* = (0, 0)^T$, then we can see that

$$T_X(x^*) = \{d \in \mathbb{R}^2_+ \mid d_1 d_2 = 0\},$$

which is exactly the feasible set X and thus nonconvex, whereas $L_X(x^*) = \mathbb{R}^2_+$. Consequently, $T_X(x^*)$ and $L_X(x^*)$ do not coincide and thus standard ACQ is not satisfied. However, the corresponding polar cone is in both cases $\mathbb{R}^2_-$, so standard GCQ is satisfied in x^*. ◇

The behavior of Example 5.8 is typical for MPCCs. Note that MPCC-LICQ is satisfied in x^*. But even under this strong MPCC constraint qualification, standard ACQ is not necessarily satisfied. In contrast to this, it was proven in [40] that standard GCQ is always satisfied under MPCC-LICQ. Note, however, that this result is not true anymore, if we replace MPCC-LICQ by any weaker MPCC constraint qualification such as for example MPCC-MFCQ. This effect can be seen in Example 5.3. Therefore, one uses a different linearized tangent cone for MPCCs, the so called *MPCC-linearized tangent cone*

$$\begin{aligned} L_{MPCC}(x^*) := \{d \in \mathbb{R}^n \mid \; & \nabla g_i(x^*)^T d \leq 0 \; \forall i \in I_g(x^*), \\ & \nabla h_i(x^*)^T d = 0 \; \forall i = 1, \dots, p, \\ & \nabla G_i(x^*)^T d = 0 \; \forall i \in I_{0+}(x^*), \\ & \nabla H_i(x^*)^T d = 0 \; \forall i \in I_{+0}(x^*), \\ & \nabla G_i(x^*)^T d \geq 0, \nabla H_i(x^*)^T d \geq 0 \; \forall i \in I_{00}(x^*), \\ & (\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) = 0 \; \forall i \in I_{00}(x^*)\}. \end{aligned}$$

Note that this cone, which was introduced in [104, 96, 38], is not a linear cone anymore and consequently not necessarily convex. However due to [38], one always has the following inclusions

$$T_X(x^*) \subseteq L_{MPCC}(x^*) \subseteq L_X(x^*)$$

and thus $L_{MPCC}(x^*)$ is an appropriate replacement for $L_X(x^*)$ in the context of MPCCs.

Example 5.9 (Example 5.8 continued) If we consider once again $x^* = (0,0)^T$, we can easily verify that the MPCC-linearized tangent cone is

$$L_{MPCC}(x^*) = \{d \in \mathbb{R}^2_+ \mid d_1 d_2 = 0\}$$

and thus coincides with the tangent cone $T_X(x^*)$. ◇

For this reason, ACQ and GCQ for MPCCs are defined as follows.

Definition 5.10 *A point x^* feasible for* (1.1) *is said to satisfy*

(a) MPCC-ACQ *if $T_X(x^*) = L_{MPCC}(x^*)$;*

(b) MPCC-GCQ *if $T_X(x^*)^\circ = L_{MPCC}(x^*)^\circ$.*

Obviously, MPCC-ACQ implies MPCC-GCQ. The connection to those MPCC constraint qualifications defined before is made in the following lemma.

Lemma 5.11 *Let x^* be a feasible point for* (1.1) *where MPCC-CPLD is satisfied. Then MPCC-ACQ is also satisfied there.*

Proof. Consider the auxiliary nonlinear program NLP(J_1, J_2)

$$\begin{aligned} \min_x f(x) \quad \text{subject to} \quad & g_i(x) \leq 0 && \forall i = 1, \dots, m, \\ & h_i(x) = 0 && \forall i = 1, \dots, p, \\ & G_i(x) \geq 0, H_i(x) = 0 && \forall i \in I_{+0}(x^*) \cup J_1, \\ & G_i(x) = 0, H_i(x) \geq 0 && \forall i \in I_{0+}(x^*) \cup J_2, \end{aligned}$$

where J_1 and J_2 form a partition of $I_{00}(x^*)$. Obviously, x^* is feasible for NLP(J_1, J_2) and the active gradients sorted by inequality constraints and equality constraints are

$$\begin{aligned} &\{\{\nabla g_i(x^*) \mid i \in I_g(x^*)\} \cup \{-\nabla G_i(x^*) \mid i \in J_1\} \cup \{-\nabla H_i(x^*) \mid i \in J_2\}\} \\ \cup \quad &\{\{\nabla h_i(x^*) \mid i = 1, \ldots, p\} \cup \{\nabla G_i(x^*) \mid i \in I_{0+}(x^*)\} \cup \{\nabla H_i(x^*) \mid i \in I_{+0}(x^*)\}\}. \end{aligned}$$

Since MPCC-CPLD is satisfied in x^*, standard CPLD for NLP(J_1, J_2) and thus also standard ACQ holds in x^* for any partition J_1, J_2 of $I_{00}(x^*)$. This was proven to be a sufficient condition for MPCC-ACQ in [38]. □

Using this result, we now know that exactly the same relations hold between the MPCC constraint qualifications as they do between standard constraint qualifications, see Figure 5.1. Later on, we will also introduce MPCC analogues to pseudonormality and quasinormality and show how they fit into the set of MPCC constraint qualifications introduced so far.

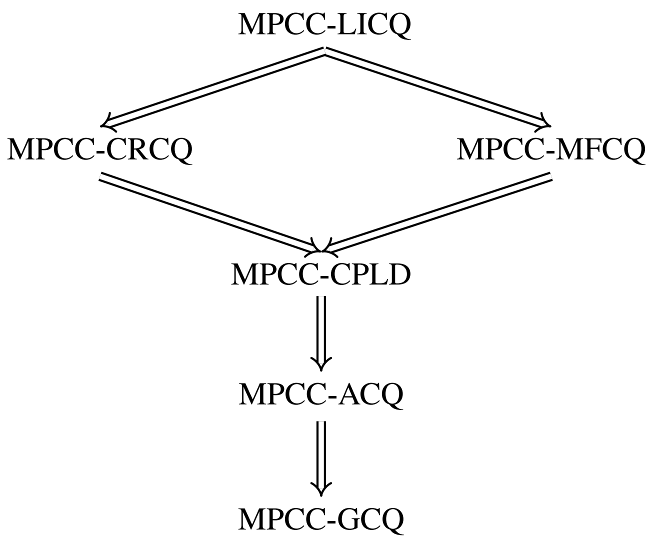


Figure 5.1.: Relations between MPCC-CQs

5.2. Stationarity Concepts

Let us begin with some MPCC Fritz-John conditions which were introduced by Scheel and Scholtes in [104] and Ye in [123].

Theorem 5.12 *Let x^* be a local minimum of* (1.1).

(a) [104] Then there are multipliers $\alpha \geq 0$, $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p$, $\gamma \in \mathbb{R}^q$, and $\nu \in \mathbb{R}^q$ such that

$(\alpha, \lambda, \mu, \gamma, \nu) \neq 0$,

$$\begin{aligned} \lambda_i &= 0 && \forall i \notin I_g(x^*), \\ \gamma_i &= 0 && \forall i \in I_{+0}(x^*), \\ \nu_i &= 0 && \forall i \in I_{0+}(x^*), \\ \gamma_i \nu_i &\geq 0 && \forall i \in I_{00}(x^*), \end{aligned}$$

and

$$\alpha \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0.$$

(b) [123] Then there are multipliers $\alpha \geq 0$, $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p$, $\gamma \in \mathbb{R}^q$, and $\nu \in \mathbb{R}^q$ such that $(\alpha, \lambda, \mu, \gamma, \nu) \neq 0$,

$$\begin{aligned} \lambda_i &= 0 && \forall i \notin I_g(x^*), \\ \gamma_i &= 0 && \forall i \in I_{+0}(x^*), \\ \nu_i &= 0 && \forall i \in I_{0+}(x^*), \\ \gamma_i, \nu_i &> 0 \text{ or } \gamma_i \nu_i = 0 && \forall i \in I_{00}(x^*), \end{aligned}$$

and

$$\alpha \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0.$$

The difference between these two Fritz-John results lies only in the conditions on the multipliers corresponding to the biactive set $I_{00}(x^*)$. This is due to the fact that the result from [104] is based on Clarke's subdifferential calculus whereas [123] employs Mordukhovich's limiting calculus and thus leads to stronger conditions. In the next chapter, we will extend the above result from Jane Ye by adding more conditions for the case $\alpha = 0$. These conditions will give rise to two new MPCC constraint qualifications which are useful for example in the context of exact penalty functions.

Whereas KKT points are the one prominent stationarity concept for NLPs, there are lots of stationarity concepts for MPCCs. We introduce only those needed subsequently but would like to mention that there is at least one more, called A-stationarity, which was introduced in [37].

Definition 5.13 *Let x^* be feasible for* (1.1). *Then x^* is said to be*

(a) weakly stationary (W-stationary)*, if there are multipliers $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p, \gamma, \nu \in \mathbb{R}^q$ such that*

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0$$

and

$$\begin{aligned} &\lambda_i \geq 0,\ \lambda_i g_i(x^*) = 0 \ (i = 1, \ldots, m) \\ &\gamma_i = 0 \ (i \in I_{+0}(x^*)),\ \nu_i = 0 \ (i \in I_{0+}(x^*)); \end{aligned}$$

(b) Clarke stationary (C-stationary), *if it is W-stationary and* $\gamma_i \nu_i \geq 0$ *for all* $i \in I_{00}(x^*)$*;*

(c) Mordukhovich stationary (M-stationary), *if it is W-stationary and either* $\gamma_i > 0, \nu_i > 0$ *or* $\gamma_i \nu_i = 0$ *for all* $i \in I_{00}(x^*)$*;*

(d) strongly stationary (S-stationary) *if it is W-stationary and* $\gamma_i, \nu_i \geq 0$ *for all* $i \in I_{00}(x^*)$.

(e) Bouligand stationary (B-stationary) *if*

$$\nabla f(x^*)^T d \geq 0 \ \forall d \in T_X(x^*).$$

W-, C- and S-stationarity were introduced in [104], M-stationarity independently in [124, 93, 94, 122], and S-stationarity may also be found in [82]. A few words on the idea behind these stationarity concepts: W-stationarity is equivalent to the KKT conditions applied to the nonlinear program TNLP(x^*) we used to define MPCC constraint qualifications, whereas S-stationarity can be shown to be equivalent to the KKT conditions applied directly to the MPCC (1.1), cf. [40]. C- and M-stationarity can be motivated by the respective Fritz-John conditions stated in Theorem 5.12. B-stationarity on the other hand is just the necessary optimality condition already known to us from standard nonlinear programs.

Obviously, the first four stationarity concepts differ only in the conditions on the multipliers corresponding to the biactive set $I_{00}(x^*)$ and thus coincide when this set is empty. These conditions are visualized in Figure 5.2.

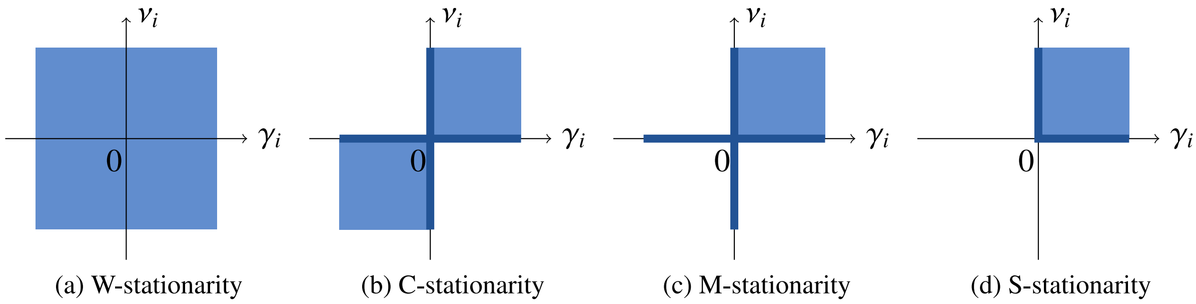


Figure 5.2.: Feasible sets for multipliers corresponding to $i \in I_{00}(x^*)$

Since S-stationarity is equivalent to the KKT conditions applied to (1.1) it implies B-stationarity. Thus, the relations depicted in Figure 5.3 hold between these stationarity concepts.
Conversely, if MPCC-LICQ holds in x^*, then B-stationarity also implies S-stationarity. This is due to the fact that MPCC-LICQ implies GCQ [40] and thus B-stationary points are also S-stationary. Thus S-stationarity is a necessary optimality condition under MPCC-LICQ. This, however, is not true anymore if we replace MPCC-LICQ by any weaker MPCC constraint qualification as the following example illustrates.

Example 5.14 (Example 5.3 continued) Consider again the global optimum $x^* = (0, 0, 0)^T$. We have already proven in Example 5.3 that x^* is not a KKT point and thus is not S-stationary

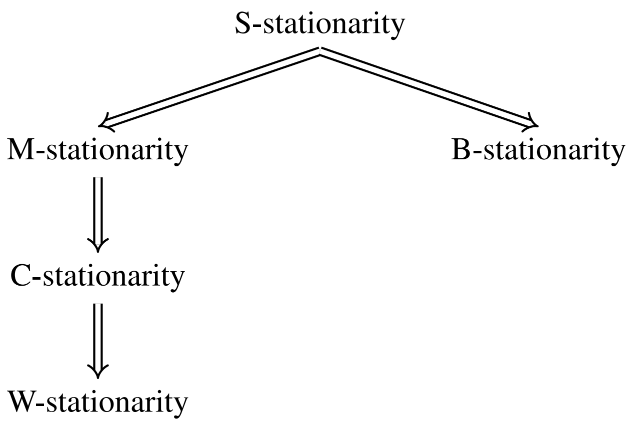


Figure 5.3.: Relations between stationarity concepts for MPCCs

although it is the global minimum. On the other hand, it can easily be seen that MPCC-MFCQ is satisfied and therefore x^* is M-stationary for example with multipliers $\lambda = (0.25, 0.75)$, $\gamma = 0$ and $\nu = -2$. ◇

The M-stationarity of the solution in the example above is not a coincidence but due to the following result from [41].

Theorem 5.15 *A local minimum of* (1.1) *that satisfies MPCC-GCQ or any stronger MPCC constraint qualification is an M-stationary point.*

6. Enhanced Fritz John Conditions

6.1. Normal Cones and Subgradients

Before we can derive the Fritz-John conditions, we first need to define some normal cones and subdifferentials that will appear in the proofs. Let us begin with the normal cones.

Definition 6.1 *Let $C \subseteq \mathbb{R}^n$ be a closed nonempty set and $c^* \in C$.*

(a) The Fréchet normal cone to C in c^* *is*

$$N_C^F(c^*) := T_C(c^*)^\circ.$$

(b) The limiting (Mordukhovich) normal cone to C in c^* *is*

$$N_C(c^*) := \left\{ s \in \mathbb{R}^n \,\middle|\, \exists \{c^k\} \to_C c^*, s^k \in N_C^F(c^k) : \; s^k \to s \right\}.$$

We would like to illustrate these definitions in the following example.

Example 6.2 Let us consider those sets to which one typically needs a normal cone in optimization, namely the feasible sets for equality constraints $\{0\} \subseteq \mathbb{R}$ and for inequality constraints $\mathbb{R}_-$. In these two cases, the Fréchet and the limiting normal cone coincide and one easily verifies

$$\begin{array}{rclcl} N_{\{0\}}^F(0) & = & N_{\{0\}}(0) & = & \mathbb{R}, \\ N_{\mathbb{R}_-}^F(x) & = & N_{\mathbb{R}_-}(x) & = & \{0\} \quad \forall x < 0, \\ N_{\mathbb{R}_-}^F(0) & = & N_{\mathbb{R}_-}(0) & = & \mathbb{R}_+. \end{array}$$

For MPCCs one also needs the normal cone to the feasible set for complementarity conditions

$$C := \{(a, b) \in \mathbb{R}^2 \mid a \geq 0, b \geq 0, ab = 0\}.$$

Here, the two cones do not always coincide. If only one of both values a, b is zero, one has

$$\begin{array}{rclcl} N_C^F((a,0)) & = & N_C((a,0)) & = & \{0\} \times \mathbb{R} \quad \forall a > 0, \\ N_C^F((0,b)) & = & N_C((0,b)) & = & \mathbb{R} \times \{0\} \quad \forall b > 0, \end{array}$$

whereas in the biactive case $(a, b) = (0, 0)$ the normal cones are

$$\begin{array}{rcl} N_C^F((0,0)) & = & \mathbb{R}^2_-, \\ N_C((0,0)) & = & \{(a,b) \in \mathbb{R}^2 \mid a, b < 0 \text{ or } ab = 0\}. \end{array} \qquad \diamond$$

In order to prove the Fritz-John result we only need the Fréchet normal cone, but to verify exactness of the penalty function we employ the limiting subdifferential, which is closely related to the limiting normal cone, see [89, 103] for more information.

Definition 6.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous.*

(a) The Fréchet subdifferential *of f in x^* is defined as*

$$\partial^F f(x^*) := \left\{ s \in \mathbb{R}^n \,\middle|\, \liminf_{x \to x^*} \frac{f(x) - f(x^*) - s^T(x - x^*)}{\|x - x^*\|} \geq 0 \right\}.$$

(b) The limiting subdifferential *of f in x^* is defined as*

$$\partial f(x^*) := \left\{ s \in \mathbb{R}^n \,\middle|\, \exists \{x^k\} \to x^*, s^k \in \partial^F f(x^k) : \; s^k \to s \right\}.$$

We would like to mention that there are more subdifferentials used in the theory of MPCCs, the most prominent among them being probably Clarke's subdifferential from [24]. However, since they will not play a role in the subsequent analysis, we do not introduce them in this thesis.

6.2. Enhanced Fritz-John Result

As we have seen in Chapter 4, there exist different ways to obtain first-order optimality conditions for standard nonlinear programs. One is a geometric approach which requires that the tangent cone is equal to a suitable linearized cone (or, at least, that the polar cones of these two sets are identical) and leads to the KKT conditions. Another way is via the Fritz-John conditions which do not require any constraint qualifications, but have the disadvantage that they involve a multiplier also in front of the gradient of the objective function. However, under suitable standard constraint qualifications (like MFCQ), one can show that this multiplier is nonzero and again ends up at the KKT conditions. A third way is via an exact penalty function P, since the unconstrained first-order optimality condition $0 \in \partial P(x^*)$ for this penalty function can be used to obtain corresponding optimality conditions and again one arrives at the KKT conditions. The only difference between these three approaches is that other CQs are required to ensure the necessity of the KKT conditions.

The situation is different for MPCCs. Different approaches lead to different optimality conditions (besides the fact that also different MPCC-tailored CQs are needed). Here, similar to [37, 123, 39], we take the Fritz-John approach. However, as we have seen in Lemma 5.1, a direct application of the Fritz-John conditions to an MPCC is not very effective. Therefore, we are interested in an MPCC-tailored Fritz-John condition. The following is the main result of this kind and motivated by similar ideas (for standard nonlinear programs with an additional abstract constraint set) from [17].

Theorem 6.4 *Let x^* be a local minimum of the MPCC* (1.1)*. Then, there are multipliers α, λ, μ, γ, ν such that*

(i)

$$\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*) = 0,$$

(ii) $\alpha \geq 0$, $\lambda_i \geq 0$ *for all* $i \in I_g(x^*)$, $\lambda_i = 0$ *for all* $i \notin I_g(x^*)$, $\gamma_i = 0$ *for all* $i \in I_{+0}(x^*)$, $\nu_i = 0$ *for all* $i \in I_{0+}(x^*)$ *and either* $\gamma_i > 0, \nu_i > 0$ *or* $\gamma_i \nu_i = 0$ *for all* $i \in I_{00}(x^*)$*,*

(iii) $\alpha, \lambda, \mu, \gamma, \nu$ *are not all equal to zero,*

(iv) if $\lambda, \mu, \gamma, \nu$ *are not all equal to zero, then there is a sequence* $\{x^k\} \to x^*$ *such that for all* $k \in \mathbb{N}$*:*
$f(x^k) < f(x^*)$,
if $\lambda_i > 0$ *(*$i \in \{1, \dots, m\}$*), then* $\lambda_i g_i(x^k) > 0$*,*
if $\mu_i \neq 0$ *(*$i \in \{1, \dots, p\}$*), then* $\mu_i h_i(x^k) > 0$*,*
if $\gamma_i \neq 0$ *(*$i \in \{1, \dots, q\}$*), then* $\gamma_i G_i(x^k) < 0$*,*
if $\nu_i \neq 0$ *(*$i \in \{1, \dots, q\}$*), then* $\nu_i H_i(x^k) < 0$*.*

Proof. We first formulate our MPCC (1.1) equivalently as

$$\begin{aligned} \min_{x,y,z} f(x) \quad \text{subject to} \quad & g(x) \leq 0, \\ & h(x) = 0, \\ & y - G(x) = 0, \\ & z - H(x) = 0, \\ & (x, y, z) \in C, \end{aligned} \tag{6.1}$$

where the set

$$C := \{(x, y, z) \in \mathbb{R}^{n+q+q} \mid y_i \geq 0, z_i \geq 0, y_i z_i = 0 \text{ for all } i = 1, \dots, q\} \tag{6.2}$$

is nonempty and closed and we have a local minimum in (x^*, y^*, z^*) with $y^* = G(x^*), z^* = H(x^*)$. Now, we can apply the idea behind [17, Proposition 2.1]: Choose $\varepsilon > 0$ such that $f(x) \geq f(x^*)$ for all $(x, y, z) \in S$ that are feasible for the reformulated MPCC (6.1), where

$$S := \{(x, y, z) \mid \|(x, y, z) - (x^*, y^*, z^*)\|_2 \leq \varepsilon\}.$$

Then consider the penalized problem

$$\min_{x,y,z} F_k(x, y, z) \quad \text{subject to} \quad (x, y, z) \in S \cap C,$$

with

$$\begin{aligned} F_k(x, y, z) \quad := \quad & f(x) + \frac{k}{2} \sum_{i=1}^{m} \max\{0, g_i(x)\}^2 + \frac{k}{2} \sum_{i=1}^{p} h_i(x)^2 \\ & + \frac{k}{2} \sum_{i=1}^{q} (y_i - G_i(x))^2 + \frac{k}{2} \sum_{i=1}^{q} (z_i - H_i(x))^2 + \frac{1}{2} \|(x, y, z) - (x^*, y^*, z^*)\|_2^2 \end{aligned}$$

for every $k \in \mathbb{N}$. Because $S \cap C$ is compact and F_k is continuous, this problem has at least one solution (x^k, y^k, z^k) for all $k \in \mathbb{N}$. Our next step is to show that the sequence $\{(x^k, y^k, z^k)\}$ converges to (x^*, y^*, z^*). To this end, note that

$$\begin{aligned} & f(x^k) + \frac{k}{2}\sum_{i=1}^{m} \max\{0, g_i(x^k)\}^2 + \frac{k}{2}\sum_{i=1}^{p} h_i(x^k)^2 \\ & +\frac{k}{2}\sum_{i=1}^{q} (y_i^k - G_i(x^k))^2 + \frac{k}{2}\sum_{i=1}^{q} (z_i^k - H_i(x^k))^2 + \frac{1}{2}\|(x^k, y^k, z^k) - (x^*, y^*, z^*)\|_2^2 \\ = \;\; & F_k(x^k, y^k, z^k) \;\le\; F_k(x^*, y^*, z^*) \;=\; f(x^*) \end{aligned}$$

for all $k \in \mathbb{N}$. Because $S \cap C$ is compact, the sequence $\{f(x^k)\}$ is bounded. This yields

$$\begin{aligned} & \lim_{k\to\infty} \max\{0, g_i(x^k)\} = 0 \quad \forall i = 1, \ldots, m, \\ & \lim_{k\to\infty} h_i(x^k) = 0 \quad \forall i = 1, \ldots, p, \\ & \lim_{k\to\infty} y_i^k - G_i(x^k) = 0 \quad \forall i = 1, \ldots, q, \\ & \lim_{k\to\infty} z_i^k - H_i(x^k) = 0 \quad \forall i = 1, \ldots, q \end{aligned}$$

because otherwise the left-hand side of the inequality above would become unbounded. Thus, every accumulation point of $\{(x^k, y^k, z^k)\}$ is feasible for the reformulated MPCC (6.1). The compactness of $S \cap C$ ensures that there is at least one accumulation point. Let $(\bar{x}, \bar{y}, \bar{z})$ be an arbitrary accumulation point of the sequence. Then we know by continuity that

$$f(\bar{x}) + \frac{1}{2}\|(\bar{x}, \bar{y}, \bar{z}) - (x^*, y^*, z^*)\|_2^2 \le f(x^*)$$

and, on the other hand, by the feasibility of $(\bar{x}, \bar{y}, \bar{z})$

$$f(x^*) \le f(\bar{x}).$$

Together, this yields $\|(\bar{x}, \bar{y}, \bar{z}) - (x^*, y^*, z^*)\|_2 = 0$. Thus, the entire sequence $\{(x^k, y^k, z^k)\}$ converges to (x^*, y^*, z^*).

Consequently, we may assume without loss of generality that (x^k, y^k, z^k) is an interior point of S for all $k \in \mathbb{N}$. Then, the standard necessary optimality condition says that

$$-\nabla F_k(x^k, y^k, z^k) \in N_C^F(x^k, y^k, z^k)$$

for all $k \in \mathbb{N}$, where the gradient of F_k is given by

$$\begin{aligned} & -\nabla F_k(x^k, y^k, z^k) \\ = \;\; & -\left[\begin{pmatrix} \nabla f(x^k) \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{m} k \max\{0, g_i(x^k)\} \begin{pmatrix} \nabla g_i(x^k) \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{p} k h_i(x^k) \begin{pmatrix} \nabla h_i(x^k) \\ 0 \\ 0 \end{pmatrix} \right. \end{aligned}$$

$$
- \sum_{i=1}^{q} k(y_i^k - G_i(x^k)) \begin{pmatrix} \nabla G_i(x^k) \\ -e_i \\ 0 \end{pmatrix} - \sum_{i=1}^{q} k(z_i^k - H_i(x^k)) \begin{pmatrix} \nabla H_i(x^k) \\ 0 \\ -e_i \end{pmatrix}
$$
$$
+ \left(\begin{pmatrix} x^k \\ y^k \\ z^k \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \right) \Bigg]
$$

and the Fréchet normal cone of C in (x^k, y^k, z^k) is easily seen to be given by

$$
N_C^F(x^k, y^k, z^k) = \left\{ \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} : \begin{array}{ll} \xi_i = 0, \zeta_i \in \mathbb{R} & \text{if } y_i^k > 0 \\ \zeta_i = 0, \xi_i \in \mathbb{R} & \text{if } z_i^k > 0 \\ \xi_i \leq 0, \zeta_i \leq 0 & \text{if } y_i^k = z_i^k = 0 \end{array} \right\},
$$

see Example 6.2. This yields

$$
\begin{aligned}
0 \;\; = \;\; & \nabla f(x^k) + \sum_{i=1}^{m} k \max\{0, g_i(x^k)\} \nabla g_i(x^k) + \sum_{i=1}^{p} k h_i(x^k) \nabla h_i(x^k) \\
& - \sum_{i=1}^{q} k(y_i^k - G_i(x^k)) \nabla G_i(x^k) - \sum_{i=1}^{q} k(z_i^k - H_i(x^k)) \nabla H_i(x^k) + (x^k - x^*)
\end{aligned}
$$

for all $k \in \mathbb{N}$ and also

$$
\begin{array}{lcll}
k(y_i^k - G_i(x^k)) & = & -(y_i^k - y_i^*) & \text{if } \; y_i^k > 0, z_i^k = 0, \\
k(z_i^k - H_i(x^k)) & = & -(z_i^k - z_i^*) & \text{if } \; y_i^k = 0, z_i^k > 0, \\
\left.\begin{array}{l} k(y_i^k - G_i(x^k)) \\ k(z_i^k - H_i(x^k)) \end{array}\right. & \begin{array}{c} \geq \\ \geq \end{array} & \left.\begin{array}{l} -(y_i^k - y_i^*) \\ -(z_i^k - z_i^*) \end{array}\right\} & \text{if } \; y_i^k = z_i^k = 0.
\end{array}
$$

Now define the multipliers

$$
\delta_k := \sqrt{1 + \sum_{i=1}^{m} (k \max\{0, g_i(x^k)\})^2 + \sum_{i=1}^{p} (k h_i(x^k))^2 + \sum_{i=1}^{q} (k(y_i^k - G_i(x^k)))^2 + \sum_{i=1}^{q} (k(z_i^k - H_i(x^k)))^2}
$$

and

$$
\begin{aligned}
\alpha_k &:= \frac{1}{\delta_k}, \\
\lambda_i^k &:= \frac{k \max\{0, g_i(x^k)\}}{\delta_k} \quad \forall i = 1, \ldots, m, \\
\mu_i^k &:= \frac{k h_i(x^k)}{\delta_k} \quad \forall i = 1, \ldots, p, \\
\gamma_i^k &:= \frac{k(y_i^k - G_i(x^k))}{\delta_k} \quad \forall i = 1, \ldots, q, \\
\nu_i^k &:= \frac{k(z_i^k - H_i(x^k))}{\delta_k} \quad \forall i = 1, \ldots, q.
\end{aligned}
$$

Because of $\|(\alpha_k, \lambda^k, \mu^k, \gamma^k, \nu^k)\|_2 = 1$ for all $k \in \mathbb{N}$, we may assume without loss of generality that the sequence of multipliers converges to some limit $(\alpha, \lambda, \mu, \gamma, \nu) \neq 0$. Now, we are interested in some properties of this limit. Because of the convergence of $\alpha_k \to \alpha$, we know that the sequence $\{\delta_k\}$ either diverges to $+\infty$ or converges to some positive value (greater or equal to one). We will use this fact later to obtain more information about the signs of γ and ν. By continuity and because of $x^k \to x^*$, we obtain

$$\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*) = 0.$$

Furthermore, it is easy to see that $\alpha \geq 0$ and $\lambda \geq 0$. Additionally, we have $\lambda_i = 0$ for all $i \notin I_g(x^*)$ because this implies $g_i(x^k) < 0$ for all $k \in \mathbb{N}$ sufficiently large. Now remember $(y^*, z^*) = (G(x^*), H(x^*))$ and $(x^k, y^k, z^k) \in C$ for all $k \in \mathbb{N}$. If $i \in I_{+0}(x^*)$, this yields $y_i^k > 0, z_i^k = 0$ for all k sufficiently large. Thus, we know

$$\gamma_i = \lim_{k\to\infty} \frac{k(y_i^k - G_i(x^k))}{\delta_k} = \lim_{k\to\infty} \frac{-(y_i^k - y_i^*)}{\delta_k} = 0$$

for all $i \in I_{+0}(x^*)$. Analogously, one can prove $\nu_i = 0$ for all $i \in I_{0+}(x^*)$. For $i \in I_{00}(x^*)$ at least one of the following three cases has to occur: If $y_i^k > 0, z_i^k = 0$ for infinitely many k, the same argumentation as above yields $\gamma_i = 0$. Analogously, if $y_i^k = 0, z_i^k > 0$ for infinitely many k, we obtain $\nu_i = 0$. If, however, $y_i^k = z_i^k = 0$ for infinitely many k, we obtain

$$\begin{aligned} \gamma_i &= \lim_{k\to\infty} \frac{k(y_i^k - G_i(x^k))}{\delta_k} \geq \lim_{k\to\infty} \frac{-(y_i^k - y_i^*)}{\delta_k} = 0, \\ \nu_i &= \lim_{k\to\infty} \frac{k(z_i^k - H_i(x^k))}{\delta_k} \geq \lim_{k\to\infty} \frac{-(z_i^k - z_i^*)}{\delta_k} = 0. \end{aligned}$$

Thus, for all $i \in I_{00}(x^*)$ we have either $\gamma_i > 0, \nu_i > 0$ or $\gamma_i \nu_i = 0$.

Finally, let us assume $(\lambda, \mu, \gamma, \nu) \neq 0$. Then $(\lambda^k, \mu^k, \gamma^k, \nu^k) \neq 0$ for all $k \in \mathbb{N}$ sufficiently large. Using the definition of these multipliers, it therefore follows that $(x^k, y^k, z^k) \neq (x^*, y^*, z^*)$ for all k sufficiently large. Consequently, we have

$$f(x^k) < f(x^k) + \frac{1}{2}\|(x^k, y^k, z^k) - (x^*, y^*, z^*)\|_2^2 \leq f(x^*)$$

for all $k \in \mathbb{N}$ sufficiently large. Furthermore, we have the following implication for all i and all k sufficiently large:

$$\begin{aligned} &\lambda_i > 0 \implies \lambda_i^k > 0 \implies g_i(x^k) > 0 \implies \lambda_i g_i(x^k) > 0, \\ &\mu_i \neq 0 \implies \mu_i \mu_i^k > 0 \implies \mu_i h_i(x^k) > 0. \end{aligned}$$

Now let $i \in \{1, \ldots, q\}$ be an index with $\gamma_i \neq 0$. This implies $\gamma_i \gamma_i^k > 0$ or equivalently

$$\gamma_i(y_i^k - G_i(x^k)) > 0 \tag{6.3}$$

for all k sufficiently large. We have seen above that if $y_i^k > 0$ for infinitely many k the multiplier γ_i has to be zero. Therefore, in our case $y_i^k = 0$ for all k sufficiently large and consequently

$$\gamma_i G_i(x^k) < 0$$

for all those k. One can prove the implication $\nu_i \neq 0 \Longrightarrow \nu_i H_i(x^k) < 0$ for all k sufficiently large analogously. □

Statements (i)–(iii) of Theorem 6.4 were shown previously in [123] (see also [39]) using a completely different technique of proof based on the limiting co-derivative. Here we improve the result from [123, 39] by showing that statement (d) also holds. The idea for this proof (and the corresponding statements) is inspired by a corresponding result from [17]. We stress, however, that we did not simply apply the result from [17], but that we exploit the particular structure of the complementarity constraints within our MPCC in order to obtain suitable sign constraints on the multipliers. Direct application of [17, Proposition 2.1] would have lead to condition (6.3) for γ and an analogous condition for ν. However, these conditions are less favorable than ours as they do not only depend on x but also on the artificial slack variables y and z.

6.3. New Constraint Qualifications

Motivated by Theorem 6.4 and the related discussion in [17], we now define MPCC-analogues of some constraint qualifications we already introduced in Chapter 4 for standard nonlinear programs.

Definition 6.5 *A vector $x^* \in X$ is said to satisfy*

(a) MPCC generalized MFCQ, *if there is no multiplier* $(\lambda, \mu, \gamma, \nu) \neq (0, 0, 0, 0)$ *such that*

   (i) $\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0,$

   (ii) $\lambda_i \geq 0$ *for all* $i \in I_g(x^*)$, $\lambda_i = 0$ *for all* $i \notin I_g(x^*)$, $\gamma_i = 0$ *for all* $i \in I_{+0}(x^*)$, $\nu_i = 0$ *for all* $i \in I_{0+}(x^*)$ *and either* $\gamma_i > 0, \nu_i > 0$ *or* $\gamma_i \nu_i = 0$ *for all* $i \in I_{00}(x^*)$.

(b) MPCC generalized pseudonormality, *if there is no multiplier* $(\lambda, \mu, \gamma, \nu)$ *such that*

   (i) $\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0,$

   (ii) $\lambda_i \geq 0$ *for all* $i \in I_g(x^*)$, $\lambda_i = 0$ *for all* $i \notin I_g(x^*)$, $\gamma_i = 0$ *for all* $i \in I_{+0}(x^*)$, $\nu_i = 0$ *for all* $i \in I_{0+}(x^*)$ *and either* $\gamma_i > 0, \nu_i > 0$ *or* $\gamma_i \nu_i = 0$ *for all* $i \in I_{00}(x^*)$,

   (iii) there is a sequence $\{x^k\} \to x^*$ *such that the following is true for all* $k \in \mathbb{N}$*:*

$$\sum_{i=1}^m \lambda_i g_i(x^k) + \sum_{i=1}^p \mu_i h_i(x^k) - \sum_{i=1}^q \gamma_i G_i(x^k) - \sum_{i=1}^q \nu_i H_i(x^k) > 0.$$

(c) MPCC generalized quasinormality, *if there is no multiplier* $(\lambda, \mu, \gamma, \nu)$ *such that*

   (i) $\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \gamma_i \nabla G_i(x^*) - \sum_{i=1}^q \nu_i \nabla H_i(x^*) = 0,$

(ii) $\lambda_i \geq 0$ *for all* $i \in I_g(x^*)$, $\lambda_i = 0$ *for all* $i \notin I_g(x^*)$, $\gamma_i = 0$ *for all* $i \in I_{+0}(x^*)$, $\nu_i = 0$ *for all* $i \in I_{0+}(x^*)$ *and either* $\gamma_i > 0, \nu_i > 0$ *or* $\gamma_i \nu_i = 0$ *for all* $i \in I_{00}(x^*)$,

(iii) $(\lambda, \mu, \gamma, \nu) \neq (0, 0, 0, 0)$,

(iv) there is a sequence $\{x^k\} \to x^*$ *such that the following is true for all* $k \in \mathbb{N}$: *For all* $\lambda_i > 0$ *we have* $\lambda_i g_i(x^k) > 0$, *for all* $\mu_i \neq 0$ *we have* $\mu_i h_i(x^k) > 0$, *for all* $\gamma_i \neq 0$ *we have* $-\gamma_i G_i(x^k) > 0$, *and for all* $\nu_i \neq 0$ *we have* $-\nu_i H_i(x^k) > 0$.

MPCC generalized MFCQ was already introduced in [123] under a different name, namely NNAMCQ (No Nonzero Abnormal Multiplier Constraint Qualification). The term MPCC GMFCQ, where G stands for generalized, also appears there and it is shown that, although MPCC GMFCQ is defined differently from our MPCC generalized MFCQ, both are equivalent. Furthermore, it is not difficult to see that it is weaker than the standard MPCC-MFCQ condition, see Lemma 5.7. Note that MPCC generalized MFCQ is motivated by statements (i)–(iii) of Theorem 6.4 since this CQ guarantees that a local minimum is an M-stationary point for our MPCC, see also Theorem 6.11 below. On the other hand, MPCC generalized quasinormality is motivated by statements (i)–(iv). We added the term "generalized" to the names of MPCC generalized pseudonormality and MPCC generalized quasinormality to distinguish them from those MPCC constraint qualifications defined via TNLP(x^*). In theory, it is also possible to define MPCC analogues of pseudonormality and quasinormality via TNPL(x^*). This, however, leads to different (stronger) conditions and, at least in our opinion, does not properly reflect the idea behind these constraint qualifications. Nonetheless, this idea can be used to prove the following result.

Lemma 6.6 *Let* x^* *be a feasible point of* (1.1) *where MPCC-CPLD is satisfied. Then MPCC generalized quasinormality is also satisfied there.*

Proof. Recall that MPCC-CPLD is satisfied in x^* if and only if standard CPLD for TNLP(x^*) is satisfied there as well. By [5], we know that then standard quasinormality for TNLP(x^*) also has to be satisfied. This amounts to the following condition: There is no multiplier $(\lambda, \mu, \gamma, \nu)$ such that

(i) $\sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*) = 0$,

(ii) $\lambda_i \geq 0$ for all $i \in I_g(x^*)$, $\lambda_i = 0$ for all $i \notin I_g(x^*)$, $\gamma_i = 0$ for all $i \in I_{+0}(x^*)$, $\nu_i = 0$ for all $i \in I_{0+}(x^*)$,

(iii) $(\lambda, \mu, \gamma, \nu) \neq (0, 0, 0, 0)$,

(iv) there is a sequence $\{x^k\} \to x^*$ such that the following is true for all $k \in \mathbb{N}$: For all $\lambda_i > 0$ we have $\lambda_i g_i(x^k) > 0$, for all $\mu_i \neq 0$ we have $\mu_i h_i(x^k) > 0$, for all $\gamma_i \neq 0$ we have $-\gamma_i G_i(x^k) > 0$, and for all $\nu_i \neq 0$ we have $-\nu_i H_i(x^k) > 0$.

The only difference between these conditions and MPCC generalized quasinormality lies in (ii), where the restrictions on the multipliers corresponding to the biactive set $I_{00}(x^*)$ are missing. Therefore, the conditions above imply MPCC generalized quasinormality. □

We already mentioned that MPCC-MFCQ implies MPCC generalized MFCQ. From the definitions, MPCC generalized MFCQ obviously implies MPCC generalized pseudonormality which in turn is a sufficient condition for MPCC generalized quasinormality. Thus, we can extend Figure 5.1 by some implications. All implications known so far are depicted in Figure 6.1.

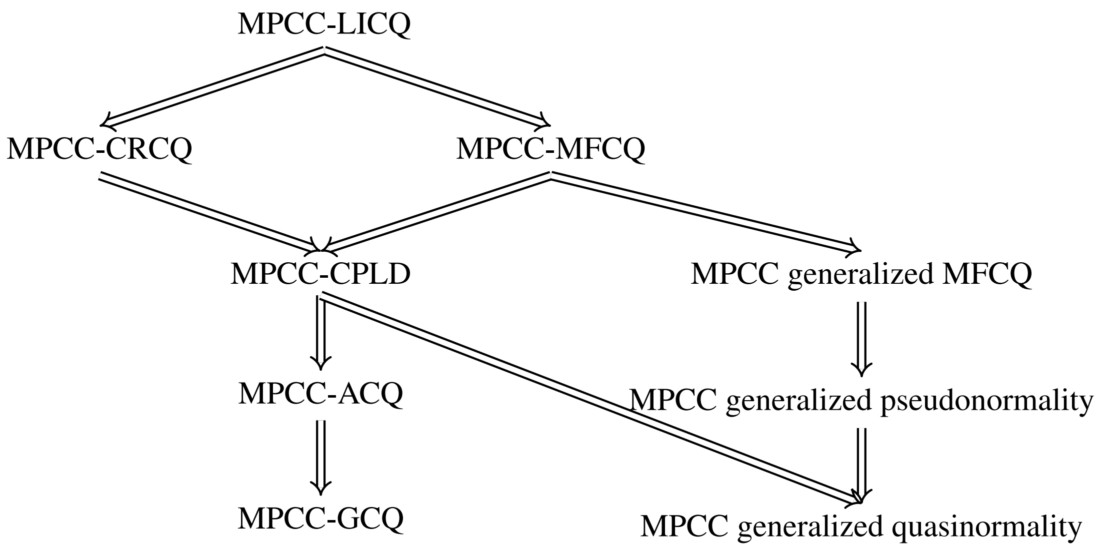


Figure 6.1.: Relations between MPCC-CQs

Now, we want to give some examples illustrating that the reverse implications are not true. Let us begin with MPCC-MFCQ and MPCC generalized MFCQ.

Example 6.7 Consider the following 2-dimensional example with linear constraints

$$\min_{x_1,x_2} f(x) \quad \text{subject to} \quad \begin{aligned} g(x) &:= x_2 \leq 0, \\ G(x) &:= x_1 \geq 0, \\ H(x) &:= x_1 + x_2 \geq 0, \\ G(x)H(x) &= x_1(x_1 + x_2) = 0, \end{aligned}$$

where f will be specified in a subsequent example. The feasible set is obviously given by

$$X = \{(t, -t) \in \mathbb{R}^2 \mid t \geq 0\},$$

hence $x^* := (0, 0)^T$ is feasible. Obviously, MPCC-MFCQ is violated in x^* as there are multipliers $(\lambda, \gamma, \nu) \neq (0, 0, 0)$ with $\lambda \geq 0$ such that

$$\lambda \nabla g(x^*) - \gamma \nabla G(x^*) - \nu \nabla H(x^*) = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \nu \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

On the other hand, it is easy to see that MPCC generalized MFCQ is satisfied. ◇

The next example illustrates that MPCC generalized pseudonormality is strictly weaker than MPCC generalized MFCQ.

Example 6.8 Consider the 2-dimensional minimization problem

$$\begin{aligned} \min_{x_1,x_2} f(x) \quad \text{subject to} \quad & g(x) := x_1 + x_2 \le 0, \\ & G(x) := x_1 \ge 0, \\ & H(x) := x_2 \ge 0, \\ & G(x)H(x) = x_1 x_2 = 0. \end{aligned}$$

The origin $x^* = (0,0)^T$ is feasible, and all constraints are active at x^*. To prove that MPCC generalized MFCQ is violated, we have to find $(\lambda,\gamma,\nu) \neq 0$ such that $\lambda \geq 0$, either $\gamma\nu = 0$ or $\gamma,\nu > 0$ and

$$\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Obviously, all vectors with these properties are of the form $(\lambda,\gamma,\nu) = c(1,1,1)$ with $c > 0$. Hence, MPCC generalized MFCQ is violated. MPCC generalized pseudonormality, on the other hand, is satisfied, because we have

$$\lambda g(x^k) - \gamma G(x^k) - \nu H(x^k) = c(x_1^k + x_2^k) - c x_1^k - c x_2^k = 0$$

for all sequences $x^k \to x^*$. ◇

We have already seen that MPCC generalized MFCQ is strictly weaker than MPCC-MFCQ. The following example, which is based on [5, Counterexample 1], illustrates that it has to be significantly weaker than MPCC-MFCQ since it is not strong enough to imply MPCC-CPLD anymore.

Example 6.9 Consider the two-dimensional MPCC

$$\begin{aligned} \min_{x_1,x_2} f(x) \quad \text{subject to} \quad & G(x) = e^{x_1} x_2 \ge 0, \\ & H(x) = x_2 \ge 0, \\ & G(x)H(x) = e^{x_1} x_2^2 = 0. \end{aligned}$$

Obviously, the feasible set is $X = \mathbb{R} \times \{0\}$. The active gradients in $x^* = (0,0)^T$ are

$$\nabla G(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nabla H(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

hence they are linearly dependent in x^* but linearly independent as soon as $x_2 \neq 0$. Consequently, MPCC-CPLD is violated in x^*. MPCC generalized MFCQ on the other hand holds since there are no multipliers γ,ν such that $\gamma,\nu > 0$ or $\gamma\nu = 0$ with $\gamma\nabla G(x^*) + \nu\nabla H(x^*) = 0$. ◇

Since MPCC generalized MFCQ is not strong enough to imply MPCC-CPLD, of course MPCC generalized pseudonormality and MPCC generalized quasinormality are not strong enough either. We have proven above that MPCC-CPLD implies MPCC generalized quasinormality. The following example based on [17, Example 3.1] shows that it does not imply MPCC generalized pseudonormality, in fact it shows that even MPCC-CRCQ does not imply MPCC generalized pseudonormality.

Example 6.10 Consider the two-dimensional MPCC

$$\begin{aligned} \min_{x_1,x_2} f(x) \quad \text{subject to} \quad & g(x) = x_1^2 + (x_2-1)^2 - 1 \le 0, \\ & G(x) = 1 - (x_1 - \cos(\pi/6))^2 - (x_2 + \sin(\pi/6))^2 \ge 0, \\ & H(x) = 1 - (x_1 + \cos(\pi/6))^2 - (x_2 + \sin(\pi/6))^2 \ge 0, \\ & G(x)H(x) = 0. \end{aligned}$$

The only feasible point here is $x^* = (0,0)^T$ and the active gradients are

$$\nabla g(x^*) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \nabla G(x^*) = \begin{pmatrix} 2\cos(\pi/6) \\ -2\sin(\pi/6) \end{pmatrix}, \nabla H(x^*) = \begin{pmatrix} -2\cos(\pi/6) \\ -2\sin(\pi/6) \end{pmatrix}.$$

The only subset of linearly dependent gradients contains all three gradients and of course, they remain linearly dependent, consequently MPCC-CRCQ holds in x^*. On the other hand, we have

$$\nabla g(x^*) - \nabla G(x^*) - \nabla H(x^*) = 0$$

due to $\sin(\pi/6) = 0.5$ and

$$g(x) - G(x) - H(x) = 3(x_1^2 + x_2^2) > 0$$

for all $x \neq 0$. Thus, MPCC generalized pseudonormality is violated in x^*.

Since MPCC-CRCQ holds in x^*, we already know that MPCC generalized quasinormality is satisfied as well. This can be seen directly since for all $x \neq 0$ sufficiently close to x^* at least one of the constraints $g(x) \le 0$, $-G(x) \le 0$ or $-H(x) \le 0$ is satisfied. ◇

Thus we have shown two things in the example above: Firstly, that MPCC-CRCQ does not imply MPCC generalized pseudonormality, and secondly, as an immediate consequence, that MPCC generalized pseudonormality is strictly stronger than MPCC generalized quasinormality.

Some of the implications for which we gave counter examples in this section and in Section 5.1 are gathered in Figure 6.2. We chose not depict all of them for clarity reasons.
We will come back to the relations between our new constraint qualifications and the existing ones in Section 6.5. Let us close this section with a result which is a direct consequence of Theorem 6.4.

Theorem 6.11 *Let x^* be a local minimum of* (1.1) *satisfying MPCC generalized quasinormality. Then x^* is an M-stationary point of* (1.1).

Proof. Suppose that x^* is a local minimum of our MPCC. Then Theorem 6.4 implies the existence of multipliers $\alpha, \lambda, \mu, \gamma, \nu$ such that statements (i)–(iv) of that result hold. Assume that $\alpha = 0$. Then the MPCC generalized quasinormality condition implies that $\lambda = \mu = \gamma = \nu = 0$,

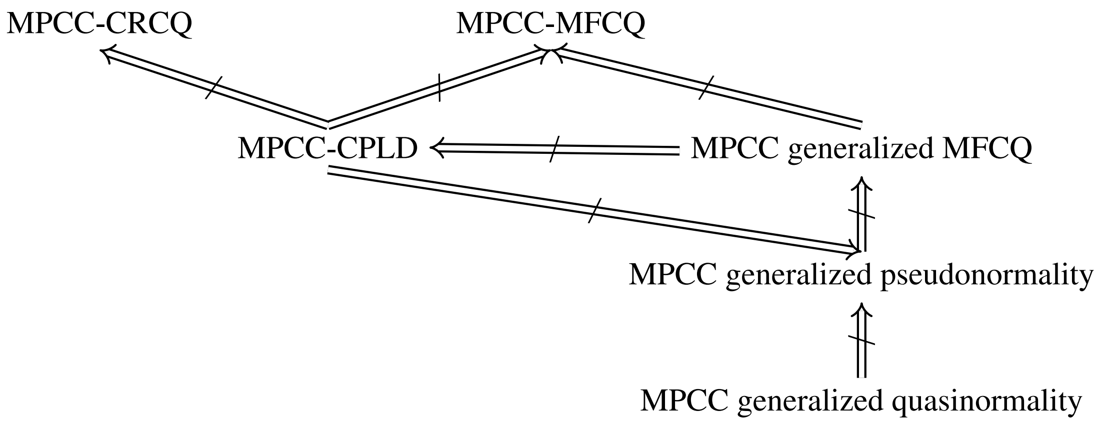


Figure 6.2.: Counter examples

contradicting the fact that not all multipliers are zero. Hence $\alpha > 0$, and we may assume without loss of generality that $\alpha = 1$, showing that x^* is indeed an M-stationary point. □

Since we have shown that MPCC generalized quasinormality is implied by MPCC-CPLD, this also implies that local minima are M-stationary under MPCC-LICQ, MPCC-MFCQ, MPCC-CRCQ and of course MPCC-CPLD. However, this was already known to us due to Theorem 5.15. But, in contrast to all previous results known to us where it is shown that a local minimum is an M-stationary point under suitable MPCC constraint qualifications, the proof of Theorem 6.11 is completely elementary and does not assume any knowledge of the limiting subdifferential or the limiting co-derivative by Mordukhovich.

At the moment, it is not clear why we also introduced the MPCC generalized pseudonormality condition since, so far, it is not really used anywhere. In the following section, however, this constraint qualification will play a fundamental role in the proof an exact penalty result.

6.4. An Exact Penalty Result

Exact penalty results for MPCCs are known in the literature, see [82, 83, 125, 106, 80] for example. In particular, it is known that MPCC-MFCQ implies exactness of a certain penalty function that will also appear in our context as a side-product. Some authors also use a partial penalization only, in fact, they sometimes penalize the standard constraints only, whereas (simplified) complementarity-type constraints are left as constraints (see [80] and also Remark 6.19 below). However, we believe that one should at least penalize the (complicated) complementarity constraints. For the sake of simplicity, we penalize all constraints in this section and obtain an exactness result for our penalty function under MPCC generalized pseudonormality which, we recall, is weaker than the usual MPCC-MFCQ condition.

In order to derive the exactness result, let us first rewrite our MPCC equivalently as

$$\min f(x) \quad \text{subject to} \quad F(x) \in \Lambda, \tag{6.4}$$

where

$$F(x) := \begin{pmatrix} g_i(x)_{i=1,\dots,m} \\ h_i(x)_{i=1,\dots,p} \\ \begin{pmatrix} G_i(x) \\ H_i(x) \end{pmatrix}_{i=1,\dots,q} \end{pmatrix}$$

and

$$\Lambda := \begin{pmatrix} (-\infty, 0]^m \\ \{0\}^p \\ C^q \end{pmatrix}$$

with

$$C := \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, b \geq 0, ab = 0\}.$$

The penalty function associated to (6.4) is

$$P_\alpha(x) := f(x) + \alpha \mathrm{dist}_\Lambda(F(x)). \tag{6.5}$$

Here, the distance function is defined by

$$\mathrm{dist}_\Lambda(F(x)) = \inf\{\|y - F(x)\| \mid y \in \Lambda\}, \tag{6.6}$$

where, in principle, the norm can be chosen arbitrarily. Our goal is to prove that the penalty function (6.5) is exact in the sense of the following definition in every local minimum x^* satisfying a suitable constraint qualification.

Definition 6.12 *The penalty function P_α is said to be* exact *in a local minimum x^* of* (1.1) *if there is a finite $\bar{\alpha} \geq 0$ such that x^* is an unconstrained local minimum of $P_\alpha(x)$ for all $\alpha \geq \bar{\alpha}$.*

It is well-known that exactness of this function using a specific norm implies exactness for all other norms as well. Therefore, we will restrict ourselves to the l_1-norm. In this case, $P_\alpha(x)$ is of the form

$$P_\alpha(x) = f(x) + \alpha \left(\sum_{i=1}^{m} \mathrm{dist}_{(-\infty,0]}(g_i(x)) + \sum_{i=1}^{p} \mathrm{dist}_{\{0\}}(h_i(x)) + \sum_{i=1}^{q} \mathrm{dist}_C((G_i(x), H_i(x))) \right) \tag{6.7}$$

and elementary calculations lead to the following explicit formulas for the corresponding distance functions.

Lemma 6.13 *Under the l_1-norm, the distance functions are given by the following expressions for $a, b \in \mathbb{R}$:*

$$\begin{aligned} dist_{(-\infty,0]}(a) &= \max\{a, 0\}, \\ dist_{\{0\}}(a) &= |a|, \\ dist_C((a,b)) &= \max\{-a, -b, -(a+b), \min\{a,b\}\} = \begin{cases} a \text{ or } b & \text{if } a = b \geq 0, \\ b & \text{if } a > b > 0, \\ -b & \text{if } a > 0, b \leq 0, \\ -(a+b) & \text{if } a \leq 0, b \leq 0, \\ -a & \text{if } a \leq 0, b > 0, \\ a & \text{if } b > a > 0. \end{cases} \end{aligned}$$

It follows that the penalty function we consider in this section is explicitly given by

$$\begin{aligned} P_\alpha(x) = f(x) + \alpha \Big(& \textstyle\sum_{i=1}^m \max\{0, g_i(x)\} + \sum_{i=1}^p |h_i(x)| + \\ & \textstyle\sum_{i=1}^q \max\{-G_i(x), -H_i(x), -(G_i(x) + H_i(x)), \min\{G_i(x), H_i(x)\}\}\Big), \end{aligned} \tag{6.8}$$

see also the discussion at the end of this section for the relation between this penalty function and another one which is used more frequently in the context of MPCCs.

In order to prove the exactness result, we need to calculate the limiting subdifferentials of the distance functions stated in Lemma 6.13.

Lemma 6.14 *The limiting subdifferentials of the distance functions from Lemma 6.13 (recall that we use the l_1-norm here) are given by*

$$\partial dist_{(-\infty,0]}(a) = \begin{cases} \{0\} & \text{if } a < 0, \\ [0,1] & \text{if } a = 0, \\ \{1\} & \text{if } a > 0, \end{cases}$$

$$\partial dist_{\{0\}}(a) = \begin{cases} \{-1\} & \text{if } a < 0, \\ [-1,1] & \text{if } a = 0, \\ \{1\} & \text{if } a > 0, \end{cases}$$

$$\partial dist_C((a,b)) = \begin{cases} \{(\xi,0)^T, (0,\zeta)^T \mid \xi,\zeta \in [0,1]\} \cup \{(\xi,\zeta)^T \mid \xi,\zeta \in [-1,0]\} & \text{if } a = b = 0, \\ \{(1,0)^T, (0,1)^T\} & \text{if } a = b > 0, \\ \{(0,1)^T\} & \text{if } a > b > 0, \\ \{(0,\zeta)^T \mid \zeta \in [-1,1]\} & \text{if } a > 0, b = 0, \\ \{(0,-1)^T\} & \text{if } a > 0, b < 0, \\ \{(\xi,-1)^T \mid \xi \in [-1,0]\} & \text{if } a = 0, b < 0, \\ \{(-1,-1)^T\} & \text{if } a < 0, b < 0, \\ \{(-1,\zeta)^T \mid \zeta \in [-1,0]\} & \text{if } a < 0, b = 0, \\ \{(-1,0)^T\} & \text{if } a < 0, b > 0, \\ \{(\xi,0)^T \mid \xi \in [-1,1]\} & \text{if } a = 0, b > 0, \\ \{(1,0)^T\} & \text{if } b > a > 0. \end{cases}$$

Proof. For convex functions, both the Fréchet and the limiting subdifferential coincide with the standard subdifferential from convex analysis, cf. [103]. This gives the expressions for the first two distance functions.

In order to get the expression for the limiting subdifferential of the third distance function $(a,b) \mapsto \text{dist}_C((a,b))$, we also recall that the Fréchet and the limiting subdifferentials of a locally continuously differentiable function are equal to a single set, consisting of the gradient of that function, cf. [103, Example 8.8]. Together with the previous comment regarding (locally) convex functions, we obtain all statements except for the first two cases $a = b = 0$ and $a = b > 0$ which we will now treat separately.

First consider the case $a = b > 0$. We claim that the Fréchet subdifferential is empty. To see this, assume there exists an element $s = (s_1, s_2) \in \partial^F \mathrm{dist}_C((a, b))$. Then consider the particular sequence $\{(a^k, b^k)\}$ with $(a^k, b^k) := (a + \frac{1}{k}, b)$. An elementary calculation then shows that

$$\frac{\mathrm{dist}_C((a^k, b^k)) - \mathrm{dist}_C((a, b)) - s^T \begin{pmatrix} a^k - a \\ b^k - b \end{pmatrix}}{\left\| \begin{pmatrix} a^k - a \\ b^k - b \end{pmatrix} \right\|} = \frac{b - b - \frac{1}{k} s_1}{\frac{1}{k}} = -s_1,$$

hence the limit inferior of this expression is nonnegative if and only if $s_1 \leq 0$. On the other hand, consider the particular sequence $\{(a^k, b^k)\}$ with $(a^k, b^k) := (a - \frac{1}{k}, b)$. Again, a simple calculation gives

$$\frac{\mathrm{dist}_C((a^k, b^k)) - \mathrm{dist}_C((a, b)) - s^T \begin{pmatrix} a^k - a \\ b^k - b \end{pmatrix}}{\left\| \begin{pmatrix} a^k - a \\ b^k - b \end{pmatrix} \right\|} = \frac{a - \frac{1}{k} - a + \frac{1}{k} s_1}{\frac{1}{k}} = -1 + s_1,$$

and the limit inferior of this term is nonnegative if and only if $s_1 \geq 1$. This contradiction shows that s cannot belong to the Fréchet subdifferential. Hence, to obtain the elements s of the limiting subdifferential $\partial \mathrm{dist}_C((a, b))$, we only need to consider sequences $\{s^k\}$ converging to s with s^k being an element of the Fréchet subdifferential $\partial^F \mathrm{dist}_C((a^k, b^k))$ at points (a^k, b^k) satisfying $a^k, b^k > 0$ and $a^k \neq b^k$. The corresponding expressions were already calculated and show that the limiting subdifferential consists of the two vectors $(0, 1)^T$ and $(1, 0)^T$.

Finally, consider the case $a = b = 0$. First, let us calculate the Fréchet subdifferential at this point. We claim that this Fréchet subdifferential is given by the rectangle $[-1, 0] \times [-1, 0]$. To see this, note that it is not difficult to see that the numerator

$$\mathrm{dist}_C((a^k, b^k)) - \mathrm{dist}_C((0, 0)) - (s_1, s_2) \begin{pmatrix} a^k - 0 \\ b^k - 0 \end{pmatrix}$$

occurring in the definition of the Fréchet subdifferential is always nonnegative for all $(s_1, s_2) \in [-1, 0] \times [-1, 0]$, so these elements certainly belong to $\partial^F \mathrm{dist}_C((0, 0))$. On the other hand, there cannot exist any other elements since, by taking the particular sequence $\{(a^k, b^k)\}$ with $(a^k, b^k) := (\frac{1}{k}, 0)$, we obtain

$$\frac{\mathrm{dist}_C((a^k, b^k)) - \mathrm{dist}_C((0, 0)) - (s_1, s_2) \begin{pmatrix} a^k - 0 \\ b^k - 0 \end{pmatrix}}{\left\| \begin{pmatrix} a^k - 0 \\ b^k - 0 \end{pmatrix} \right\|} = \frac{0 - 0 - \frac{1}{k} s_1}{\frac{1}{k}} = -s_1,$$

whereas for the particular sequence $\{(a^k, b^k)\}$ with $(a^k, b^k) := (-\frac{1}{k}, 0)$, we get

$$\frac{\mathrm{dist}_C((a^k, b^k)) - \mathrm{dist}_C((0,0)) - (s_1, s_2)\begin{pmatrix} a^k - 0 \\ b^k - 0 \end{pmatrix}}{\left\| \begin{pmatrix} a^k - 0 \\ b^k - 0 \end{pmatrix} \right\|} = \frac{\frac{1}{k} - 0 + \frac{1}{k}s_1}{\frac{1}{k}} = 1 + s_1,$$

so that the definition of the Fréchet subdifferential shows that we necessarily have $s_1 \in [-1, 0]$. A symmetric argument shows that also $s_2 \in [-1, 0]$ is necessary for the vector $s = (s_1, s_2)$ belonging to $\partial^F \mathrm{dist}_C((0,0))$. Altogether, we therefore have $\partial^F \mathrm{dist}_C((0,0)) = [-1, 0] \times [-1, 0]$. Since the Fréchet subdifferential is a subset of the limiting subdifferential, it follows that $[-1,0]\times[-1,0] \in \partial \mathrm{dist}_C((0,0))$. The other elements $s \in \partial \mathrm{dist}_C((0,0))$ can be easily obtained by taking sequences $s^k \to s$ with $s^k \in \partial^F \mathrm{dist}_C((a^k, b^k))$ with $(a^k, b^k) \to (0,0)$ and $(a^k, b^k) \neq (0,0)$ together with the already known expressions for the Fréchet subdifferentials in these points. □

To prove the central result of this section, we will proceed in three steps. First, we need an auxiliary result, then we will prove that MPCC generalized pseudonormality implies the existence of local error bounds and, finally, we will use this fact to obtain exactness of our penalty function. Remember that we use the l_1-norm to measure distances.

Lemma 6.15 *Let x^* be feasible for* (1.1) *such that MPCC generalized pseudonormality holds in x^*. Then there are $\delta, c > 0$ such that for all $x \in \mathbb{B}(x^*; \delta)$ with $x \notin X$ and all $\xi \in \partial dist_\Lambda(F(x))$ the following estimate holds:*

$$\|\xi\|_1 \geq \frac{1}{c}.$$

Proof. Assume that the statement is wrong. Then, one can find a sequence $\{x^k\} \to x^*$ with $x^k \notin X$ and $\xi^k \in \partial \mathrm{dist}_\Lambda(F(x^k))$ for all $k \in \mathbb{N}$ such that $\|\xi^k\|_1 \to 0$. To calculate $\partial \mathrm{dist}_\Lambda(F(x^k))$ we may apply the sum rule from [19, Theorem 6.4.4] because the distance functions are Lipschitz continuous. Furthermore, we can use the chain rule stated in [19, p. 151], again because of the Lipschitz continuity of distance functions. This yields the existence of multipliers

$$\begin{aligned} \lambda_i^k &\in \partial \mathrm{dist}_{(-\infty,0]}(g_i(x^k)) \quad \forall\, i = 1, \dots, m, \\ \mu_i^k &\in \partial \mathrm{dist}_{\{0\}}(h_i(x^k)) \quad \forall\, i = 1, \dots, p, \\ (\gamma_i^k, \nu_i^k) &\in -\partial \mathrm{dist}_C(G_i(x^k), H_i(x^k)) \quad \forall\, i = 1, \dots, q \end{aligned}$$

such that

$$\xi^k = \sum_{i=1}^m \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^p \mu_i^k \nabla h_i(x^k) - \sum_{i=1}^q (\gamma_i^k \nabla G_i(x^k) + \nu_i^k \nabla H_i(x^k)) \tag{6.9}$$

for all $k \in \mathbb{N}$. Using Lemma 6.14, it is easy to see that the sequence $\{(\lambda^k, \mu^k, \gamma^k, \nu^k)\}$ is bounded. Hence, we may assume without loss of generality that it converges to some limit $(\lambda, \mu, \gamma, \nu)$. Taking the limit $k \to \infty$ in (6.9) and using the smoothness of g, h, G, H then yields

$$0 = \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q (\gamma_i \nabla G_i(x^*) + \nu_i \nabla H_i(x^*)).$$

Furthermore, Lemma 6.14 yields

$$\begin{aligned} \lambda_i &\geq 0 && \forall i = 1, \ldots, m, \\ \lambda_i &= 0 && \forall i \notin I_g(x^*), \\ \gamma_i &= 0 && \forall i \in I_{+0}(x^*), \\ \nu_i &= 0 && \forall i \in I_{0+}(x^*), \\ \gamma_i \nu_i = 0 \text{ or } \gamma_i &> 0, \nu_i > 0 && \forall i \in I_{00}(x^*). \end{aligned}$$

Additionally, it is easy to see that, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \lambda_i g_i(x^k) &\geq 0 && \forall i = 1, \ldots, m, \\ \mu_i h_i(x^k) &\geq 0 && \forall i = 1, \ldots, p, \\ -\gamma_i G_i(x^k) &\geq 0 && \forall i = 1, \ldots, q, \\ -\nu_i H_i(x^k) &\geq 0 && \forall i = 1, \ldots, q. \end{aligned}$$

Because of $x^k \notin X$ for all $k \in \mathbb{N}$, at least one constraint has to be violated infinitely many times. Using Lemma 6.14, it is easy to see that the corresponding product is strictly positive for all $k \in \mathbb{N}$ such that the constraint is violated, i.e., if the constraint $g_i(x^k) \leq 0$ is violated for infinitely many k we have $\lambda_i g_i(x^k) > 0$ for all those k, if the constraint $h_i(x^k) = 0$ is violated for infinitely many k we have $\mu_i h_i(x^k) > 0$ for all those k and finally, if the constraint $(G_i(x^k), H_i(x^k)) \in C$ is violated for infinitely many k we have $-(\gamma_i G_i(x^k) + \nu_i H_i(x^k)) > 0$ for all those k. This yields

$$\sum_{i=1}^{m} \lambda_i g_i(x^k) + \sum_{i=1}^{p} \mu_i h_i(x^k) - \sum_{i=1}^{q} (\gamma_i G_i(x^k) + \nu_i H_i(x^k)) > 0$$

at least on a subsequence $K \subseteq \mathbb{N}$. This, however, implies that MPCC generalized pseudonormality is violated in x^*, a contradiction. □

The following result about local error bounds is based on [121, Theorem 2.2], where a more general setting was considered. Hence, the proof for our special case is easier and therefore stated here. Again, distances are measured in the l_1-norm.

Lemma 6.16 *Let x^* be feasible for* (1.1) *and $\delta, c > 0$ such that $\|\xi\|_1 \geq \frac{1}{c}$ for all $x \in \mathbb{B}(x^*; \delta) \setminus X$ and all $\xi \in \partial dist_\Lambda(F(x))$. Then the following estimate holds for all $x \in \mathbb{B}(x^*; \frac{\delta}{2})$:*

$$dist_X(x) \leq nc \; dist_\Lambda(F(x)).$$

Proof. Assume that the statement is wrong. Then, there is an $\tilde{x} \in \mathbb{B}(x^*; \frac{\delta}{2})$ with

$$\text{dist}_X(\tilde{x}) > nc \; \text{dist}_\Lambda(F(\tilde{x})).$$

Obviously, this implies $\tilde{x} \notin X$. Furthermore, we can choose a $t > 1$ such that

$$d := tnc\, \mathrm{dist}_\Lambda(F(\tilde{x})) < \mathrm{dist}_X(\tilde{x}). \tag{6.10}$$

Because of $\tilde{x} \notin X$, we have $d > 0$. Furthermore, $\mathrm{dist}_\Lambda(F(\tilde{x})) = \frac{d}{tnc}$ and thus

$$\mathrm{dist}_\Lambda(F(\tilde{x})) \leq \inf_{x \in \mathbb{R}^n} \mathrm{dist}_\Lambda(F(x)) + \frac{d}{tnc}.$$

Application of Ekeland's variational principle [24, Theorem 7.5.1] to the continuous nonnegative function $x \mapsto \mathrm{dist}_\Lambda(F(x))$ with $\varepsilon = \frac{d}{tnc}$ and $\lambda = d$ yields the existence of an $\bar{x}$ with the following properties:

$$\|\bar{x} - \tilde{x}\|_1 \leq d, \tag{6.11}$$

$$\mathrm{dist}_\Lambda(F(\bar{x})) \leq \mathrm{dist}_\Lambda(F(\tilde{x})), \tag{6.12}$$

$$\mathrm{dist}_\Lambda(F(x)) + \frac{1}{tnc}\|x - \bar{x}\|_1 > \mathrm{dist}_\Lambda(F(\bar{x})) \quad \forall x \in \mathbb{R}^n, x \neq \bar{x}. \tag{6.13}$$

Equations (6.11) and (6.10) imply

$$\|\bar{x} - \tilde{x}\|_1 \leq d < \mathrm{dist}_X(\tilde{x}),$$

thus $\bar{x} \notin X$. According to (6.13), the function $x \mapsto \mathrm{dist}_\Lambda(F(x)) + \frac{1}{tnc}\|x - \bar{x}\|_1$ attains a global minimum in $\bar{x}$. Thus, by Fermat's rule, we have

$$0 \in \partial\left[\mathrm{dist}_\Lambda(F(x)) + \frac{1}{tnc}\|x - \bar{x}\|_1\right]_{x=\bar{x}}.$$

To calculate this subdifferential, we may invoke the sum rule from [19, Theorem 6.4.4] because distance functions are Lipschitz continuous. Furthermore, it is easy to see that the limiting subdifferential of the convex function $x \mapsto \frac{1}{tnc}\|x - \bar{x}\|_1$ in $\bar{x}$ is given by

$$\partial\left(\frac{1}{tnc}\|\cdot - \bar{x}\|_1\right)(\bar{x}) = \frac{1}{tnc}\{\zeta \in \mathbb{R}^n \mid \|\zeta\|_\infty \leq 1\}.$$

Hence, we can find $\xi \in \partial \mathrm{dist}_\Lambda(F(\bar{x}))$ and ζ with $\|\zeta\|_\infty \leq 1$ such that

$$0 = \|\xi + \frac{1}{tnc}\zeta\|_1 \geq \|\xi\|_1 - \frac{1}{tnc}\|\zeta\|_1 \geq \|\xi\|_1 - \frac{1}{tnc}n,$$

consequently

$$\|\xi\|_1 \leq \frac{1}{tc} < \frac{1}{c}.$$

On the other hand, we have $\bar{x} \notin X$ and, using equations (6.11), (6.10) together with $x^* \in X$,

$$\|\bar{x} - x^*\|_1 \leq \|\bar{x} - \tilde{x}\|_1 + \|\tilde{x} - x^*\|_1 \leq d + \|\tilde{x} - x^*\|_1 < \mathrm{dist}_X(\tilde{x}) + \|\tilde{x} - x^*\|_1 \leq 2\|\tilde{x} - x^*\|_1 \leq 2\frac{\delta}{2} = \delta.$$

This, however, is a contradiction to our assumptions. □

Taking into account the previous two results, we can now follow [57] and get an exact penalty result for MPCCs.

Theorem 6.17 *Let x^* be a local minimizer of* (1.1) *with f locally Lipschitz-continuous around x^* with modulus $L > 0$. If MPCC generalized pseudonormality holds in x^*, then the penalty function P_α defined in* (6.8) *is exact in x^*.*

Proof. According to Lemma 6.15 and Lemma 6.16, we can find constants $\delta, c > 0$ such that

$$\mathrm{dist}_X(x) \leq c\ \mathrm{dist}_\Lambda(F(x))$$

holds for all $x \in \mathbb{B}(x^*; \delta)$ (note that we redefined the constants δ, c from Lemma 6.16 to shorten the notation). Now choose $\varepsilon > 0$ such that $2\varepsilon < \delta$ and that f attains a global minimum in x^* on $\mathbb{B}(x^*; 2\varepsilon) \cap X$. Furthermore, we assume without loss of generality that f is Lipschitz continuous in $\mathbb{B}(x^*; 2\varepsilon)$ with the Lipschitz constant L . Then the following holds for every $x \in \mathbb{B}(x^*; \varepsilon)$: Choose $x^p \in \mathrm{Proj}_X(x)$ arbitrary. This implies

$$\|x^p - x\|_1 \leq \|x^* - x\|_1 \leq \varepsilon \quad \Longrightarrow \quad \|x^p - x^*\|_1 \leq \|x^p - x\|_1 + \|x - x^*\|_1 \leq 2\varepsilon$$

and consequently we have

$$f(x^*) \leq f(x^p) \leq f(x) + L\|x^p - x\|_1 = f(x) + L\ \mathrm{dist}_X(x) \leq f(x) + cL\ \mathrm{dist}_\Lambda(F(x)).$$

Thus, the penalty function P_α is exact with $\bar{\alpha} = cL$. □

In the proof above we only used that f is locally Lipschitz-continuous around x^*, so it is not necessary to demand f to be smooth. On the other hand, mere continuity of f is not enough to guarantee exactness as the following example illustrates:

Example 6.18 (Example 6.7 continued) Since we have already seen that MPCC generalized MFCQ is satisfied in $x^* = (0, 0)^T$, we know that the penalty function P_α is exact in x^* for every locally Lipschitz continuous function f. Now let us consider the objective function

$$f(x) := -\sqrt{|x_1 + x_2|}$$

which is well defined and continuous in $\mathbb{R}^2$ but not locally Lipschitz continuous around x^*. Note that x^* is a local minimizer of f over X. However, x^* is not a local minimizer of P_α for any $\alpha > 0$. Evaluation of P_α at $x^k := (\frac{1}{k^2}, 0)$, $k \in \mathbb{N}$, for example yields

$$P_\alpha(x^k) = \frac{1}{k}\left(-1 + \alpha\frac{1}{k}\right),$$

which eventually becomes negative for all $\alpha > 0$. Hence, x^* with $P_\alpha(x^*) = 0$ is not a local minimizer of P_α for any $\alpha > 0$ or, equivalently, P_α is not exact in x^*. ◇

Theorem 6.17 is particularly interesting, because it also works for nonstrict local minima x^*. A similar exact penalty result based on pseudonormality can be found in [17, Proposition 4.2], however, that result requires x^* to be a strict local minimum, and it is stated in [17, Example 7.7] that this assumption might be crucial. In our case, we do not need a strict local minimum to guarantee exactness. We stress, however, that our technique of proof is also completely different from the one in [17], and it is not clear whether this technique can also be used to improve the result from [17].

Remark 6.19 Under the stronger MPCC generalized MFCQ condition, exactness of P_α can also be proven using a result from [80]: If x^* is a local minimum of (1.1), we know that $(x^*, G(x^*), H(x^*))$ is a local minimizer of

$$\min_{(x,y,z)} f(x) \quad \text{subject to} \quad g(x) \leq 0,\ h(x) = 0, G(x) - y = 0,\ H(x) - z = 0,$$
$$(y, z) \in C^q,$$

where C denotes the set from (6.2) (recall that we have used this reformulation before in the proof of Theorem 6.4). Now suppose that MPCC generalized MFCQ holds in x^*. According to [80, Theorem 3.5], one can then find a $\mu > 0$ such that $(x^*, G(x^*), H(x^*))$ is a local minimizer of the partially penalized problem

$$\begin{aligned} \min_{(x,y,z)} \quad & f(x) + \mu\left(\sum_{i=1}^m \max\{0, g_i(x)\} + \sum_{i=1}^p |h_i(x)| + \sum_{i=1}^q |G_i(x) - y_i| + \sum_{i=1}^q |H_i(z) - z_i|\right) \\ \text{subject to} \quad & (y, z) \in C^q, \end{aligned}$$

too. By [24, Proposition 2.4.3], there exists an $L > 0$ such that $(x^*, G(x^*), H(x^*))$ is a local minimizer of the now completely penalized and thus unconstrained problem

$$\min_{(x,y,z)} f(x) + \mu\left(\sum_{i=1}^m \max\{0, g_i(x)\} + \sum_{i=1}^p |h_i(x)| + \sum_{i=1}^q |G_i(x) - y_i| + \sum_{i=1}^q |H_i(z) - z_i|\right) + L\sum_{i=1}^q \operatorname{dist}_C((y_i, z_i)).$$

If we restrict the feasible area to those (x, y, z), where $G(x) - y = 0$ and $H(x) - z = 0$, we obtain that x^* is a local minimizer of

$$\min_x \ f(x) + \mu\left(\sum_{i=1}^m \max\{0, g_i(x)\} + \sum_{i=1}^p |h_i(x)| +\right) + L\sum_{i=1}^q \operatorname{dist}_C(G_i(x), H_i(x)).$$

Thus, we have proven that P_α is exact with $\alpha = \max\{\mu, L\}$.

We recall that (6.8) gives the explicit representation of the penalty function used within this section. Another popular penalty function that is typically taken by the authors in the MPCC-setting, see [106, 36], takes into account the equivalence

$$G_i(x) \geq 0,\ H_i(x) \geq 0,\ G_i(x)H_i(x) = 0 \Longleftrightarrow \min\{G_i(x), H_i(x)\} = 0,$$

so that it is a natural idea to add the absolute value of the min-function to our penalty term, resulting into the mapping

$$\tilde{P}_\alpha(x) := f(x) + \alpha\left(\sum_{i=1}^{m} \max\{0, g_i(x)\} + \sum_{i=1}^{p} |h_i(x)| + \sum_{i=1}^{q} \left|\min\{G_i(x), H_i(x)\}\right|\right). \tag{6.14}$$

As a consequence of our previous results, we may also obtain an exact penalty result for $\tilde{P}_\alpha$. To this end, let us reconsider the original penalty function from (6.6). Using the l_1-norm, we then obtain the expression (6.7) for this distance-based penalty function. Using once again (for the sake of consistency) the l_1-norm to calculate the distances for all the terms that occur in (6.7) (cf. Lemma 6.13), we end up with the representation from (6.8). However, we could alternatively calculate the distances for each term using the l_∞-norm. Then it is not difficult to see that the last expression in Lemma 6.13 becomes

$$\mathrm{dist}_C((a, b)) = \left|\min\{a, b\}\right|,$$

i.e., also this mapping may be viewed as a distance function. Taking into account that all norms are equivalent in finite dimensions, we immediately see that $\tilde{P}_\alpha$ is also an exact penalty function under the assumption of Theorem 6.17. This proves the following result.

Corollary 6.20 *Let x^* be a local minimum of* (1.1) *such that MPCC generalized pseudonormality holds in x^*. Then the penalty function $\tilde{P}_\alpha(x)$ from* (6.14) *is exact in x^*.*

We would like to close this section with a few words on the constraint qualification we used to ensure exactness of the penalty function. The constraint qualification most commonly used in the context of exact penalty functions is MPCC-MFCQ or MPCC generalized MFCQ, see for example [106, 80]. The reason for this is as follows: According to our proof of Theorem 6.17, the existence of local error bounds, i.e. the existence of constants $\delta, c > 0$ such that

$$\mathrm{dist}_{F^{-1}(\Lambda)}(x) \leq c\,\mathrm{dist}_\Lambda(F(x)) \quad \forall x \in \mathbb{B}(x^*; \delta),$$

is a sufficient condition for exactness. It was shown in [55, Corollary 1] that the existence of local error bounds is equivalent to the calmness of the perturbation map

$$M(r) := \{x \in \mathbb{R}^n \mid F(x) + r \in \Lambda\}$$

at $(0, x^*)$, where calmness of a multifunction is defined as follows.

Definition 6.21 *Let $\Phi : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be a multifunction with closed graph and $(u, v) \in gph(\Phi)$. Then we say that Φ is calm at (u, v) if there are neighborhoods U of u,V of v, and a modulus $L \geq 0$ such that*

$$\Phi(u') \cap V \subseteq \Phi(u) + L\|u' - u\|_2\mathbb{B}(0; 1) \quad \forall u' \in U.$$

Furthermore, it is well-known that the following condition

$$\left.\begin{array}{l}\nabla F(x^*)^T\omega = 0,\\ \omega \in N_\Lambda(F(x^*))\end{array}\right\} \Longrightarrow \omega = 0 \tag{6.15}$$

guarantees calmness of M in $(0, x^*)$, cf. for example [42, Proposition 3.8], and thus exactness of P_α. In the next result, we want to relate this condition to one of our constraint qualifications.

Lemma 6.22 *Condition* (6.15) *is equivalent to MPCC generalized MFCQ.*

Proof. Due to [103, Proposition 6.41], we may rewrite the limiting normal cone as

$$N_\Lambda(F(x^*)) = \bigtimes_{i=1}^m N_{(-\infty,0]}(g_i(x^*)) \times \bigtimes_{i=1}^p N_{\{0\}}(h_i(x^*)) \times \bigtimes_{i=1}^q N_C(G_i(x^*), H_i(x^*)).$$

Hence, condition (6.15) is equivalent to

$$\left.\begin{array}{l}\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q (\gamma_i \nabla G_i(x^*) + \nu_i \nabla H_i(x^*)) = 0,\\ \lambda_i \in N_{(-\infty,0]}(g_i(x^*)) \quad \forall i = 1,\dots,m,\\ \mu_i \in N_{\{0\}}(h_i(x^*)) \quad \forall i = 1,\dots,p,\\ (\gamma_i, \nu_i) \in -N_C(G_i(x^*), H_i(x^*)) \quad \forall i = 1,\dots,q,\end{array}\right\} \Longrightarrow (\lambda, \mu, \gamma, \nu) = 0,$$

which is exactly MPCC generalized MFCQ. □

Because MPCC-MFCQ implies MPCC generalized MFCQ, MPCC-MFCQ also is a sufficient condition for exactness of P_α. However, recall from Examples 6.7 and 6.8 that MPCC-MFCQ is strictly stronger than MPCC generalized pseudonormality. Thus, MPCC-MFCQ is a sufficient condition for exactness of P_α, but it is by far too restrictive.

6.5. New Constraint Qualifications revisited

In this section, we come back to the question how our new constraint qualifications fit into the system of existing ones. It is well-known that ACQ is not strong enough to guarantee exactness, cf. [17, Example 7.3]. Thus, it is not surprising that MPCC generalized pseudonormality is strictly stronger than MPCC-ACQ. To see this, we first need a technical result concerning the tangent cone. The proof of this result is rather straightforward, nevertheless, we stress that it is not a priori clear that this result holds since the set C is not regular in the sense of [103], see, in particular, Proposition 6.41 and the subsequent discussion in that reference.

Lemma 6.23 *Let x^* be feasible for* (1.1). *Then the tangent cone is given by*

$$T_\Lambda(F(x^*)) = \bigtimes_{i=1}^m T_{(-\infty,0]}(g_i(x^*)) \times \bigtimes_{i=1}^p T_{\{0\}}(h_i(x^*)) \times \bigtimes_{i=1}^q T_C(G_i(x^*), H_i(x^*)).$$

Proof. The inclusion "⊆" follows directly from [103, Proposition 6.41]. To prove the inclusion "⊇" consider arbitrary elements $d_{g_i} \in T_{(-\infty,0]}(g_i(x^*))$, $d_{h_i} \in T_{\{0\}}(h_i(x^*))$ and $(d_{G_i}, d_{H_i}) \in T_C(G_i(x^*), H_i(x^*))$, and define

$$d := (d_{g_i,i=1,\dots,m}, d_{h_i,i=1,\dots,p}, (d_{G_i}, d_{H_i})_{i=1,\dots,q}).$$

According to the definition of the tangent cone, there are sequences

$$\begin{aligned} d_{g_i}^k \to d_{g_i}, t_{g_i}^k \downarrow 0 \quad &\text{with} \quad g_i(x^*) + t_{g_i}^k d_{g_i}^k \leq 0, \\ d_{h_i}^k \to d_{h_i}, t_{h_i}^k \downarrow 0 \quad &\text{with} \quad h_i(x^*) + t_{h_i}^k d_{h_i}^k = 0, \\ (d_{G_i}^k, d_{H_i}^k) \to (d_{G_i}, d_{H_i}), t_{GH_i}^k \downarrow 0 \quad &\text{with} \quad 0 \leq G_i(x^*) + t_{GH_i}^k d_{G_i}^k \perp H_i(x^*) + t_{GH_i}^k d_{H_i}^k \geq 0 \end{aligned}$$

for all $k \in \mathbb{N}$. Consequently, we have

$$d^k := (d_{g_i,i=1,\dots,m}^k, d_{h_i,i=1,\dots,p}^k, (d_{G_i}^k, d_{H_i}^k)_{i=1,\dots,q}) \to d.$$

To prove $d \in T_\Lambda(F(x^*))$ it suffices to find a sequence $t^k \downarrow 0$ such that $F(x^*) + t^k d^k \in \Lambda$ for all $k \in \mathbb{N}$. Define

$$t^k := \min\{t_{g_i,i=1,\dots,m}^k, t_{GH_i,i=1,\dots,q}^k\}$$

for all $k \in \mathbb{N}$. Then we know $t^k \downarrow 0$ and it remains to show $F(x^*) + t^k d^k \in \Lambda$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary but fixed and recall that x^* is feasible for (1.1). For all $i = 1, \dots, m$ two cases can occur: If $d_{g_i}^k < 0$, we have

$$g_i(x^*) + t^k d_{g_i}^k < g_i(x^*) \leq 0,$$

and if $d_{g_i}^k \geq 0$, we have

$$g_i(x^*) + t^k d_{g_i}^k \leq g_i(x^*) + t_{g_i}^k d_{g_i}^k \leq 0.$$

For all $i = 1, \dots, p$ we have $d_{h_i}^k = 0$ because of $h_i(x^*) = 0$ and $t_{h_i}^k > 0$. Consequently, we obtain

$$h_i(x^*) + t^k d_{h_i}^k = 0.$$

Now consider an $i \in I_{+0}(x^*)$. Because of $d_{G_i}^k \to d_{G_i}$ and $t_{GH_i}^k \downarrow 0$ this implies

$$G_i(x^*) + t_{GH_i}^k d_{G_i}^k > 0$$

for all $k \in \mathbb{N}$ sufficiently large and, consequently, $d_{H_i}^k = 0$ for these k. This implies

$$0 \leq G_i(x^*) + t^k d_{G_i}^k \perp H_i(x^*) + t^k d_{H_i}^k \geq 0$$

for all $k \in \mathbb{N}$ sufficiently large. By symmetry, we also obtain

$$0 \leq G_i(x^*) + t^k d_{G_i}^k \perp H_i(x^*) + t^k d_{H_i}^k \geq 0$$

for all $i \in I_{0+}(x^*)$. In remains to consider $i \in I_{00}(x^*)$. In this case, we know

$$0 \le d^k_{G_i} \perp d^k_{H_i} \ge 0,$$

which directly implies

$$0 \le G_i(x^*) + t^k d^k_{G_i} \perp H_i(x^*) + t^k d^k_{H_i} \ge 0$$

for all $k \in \mathbb{N}$. Together, this proves $F(x^*) + t^k d^k \in \Lambda$ for all $k \in \mathbb{N}$ sufficiently large. □

With this lemma, we can prove that MPCC generalized pseudonormality implies MPCC-ACQ.

Lemma 6.24 *Let x^* be feasible for* (1.1) *such that MPCC generalized pseudonormality holds in x^*. Then MPCC-ACQ also holds in x^*.*

Proof. As we have proven in Lemma 6.15 and Lemma 6.16, MPCC generalized pseudonormality implies the existence of local error bounds. According to [55, Corollary 1], the existence of local error bounds is equivalent to calmness of the perturbation map

$$M(r) := \{x \in \mathbb{R}^n \mid F(x) + r \in \Lambda\}$$

in $(0, x^*)$. Thus, we can apply Proposition 1 from the same paper and obtain $T_X(x^*) = L(x^*)$, where $L(x^*)$ is defined as

$$L(x^*) := \{d \in \mathbb{R}^n \mid \nabla F(x^*)^T d \in T_\Lambda(F(x^*))\}.$$

Because of Lemma 6.23, we may write $T_\Lambda(F(x^*))$ as

$$T_\Lambda(F(x^*)) = \bigtimes_{i=1}^{m} T_{(-\infty,0]}(g_i(x^*)) \times \bigtimes_{i=1}^{p} T_{\{0\}}(h_i(x^*)) \times \bigtimes_{i=1}^{q} T_C(G_i(x^*), H_i(x^*)).$$

Now, we can apply this knowledge to $L(x^*)$ and obtain

$$\begin{aligned}
L(x^*) = \{d \in \mathbb{R}^n \mid\ & \nabla g_i(x^*)^T d \in T_{(-\infty,0]}(g_i(x^*))\ \forall i = 1,\dots,m,\\
& \nabla h_i(x^*)^T d \in T_{\{0\}}(h_i(x^*))\ \forall i = 1,\dots,p,\\
& (\nabla G_i(x^*)^T d, \nabla H_i(x^*)^T d) \in T_C(G_i(x^*), H_i(x^*))\ \forall i = 1,\dots,q\}\\
= \{d \in \mathbb{R}^n \mid\ & \nabla g_i(x^*)^T d \le 0\ \forall i \in I_g(x^*),\\
& \nabla h_i(x^*)^T d = 0\ \forall i = 1,\dots,p,\\
& \nabla G_i(x^*)^T d = 0\ \forall i \in I_{0+}(x^*),\\
& \nabla H_i(x^*)^T d = 0\ \forall i \in I_{+0}(x^*),\\
& \nabla G_i(x^*)^T d \ge 0\ \forall i \in I_{00}(x^*),\\
& \nabla H_i(x^*)^T d \ge 0\ \forall i \in I_{00}(x^*),\\
& (\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) = 0\ \forall i \in I_{00}(x^*)\}\\
= L_{MPCC}(x^*),\ &
\end{aligned}$$

where $L_{MPCC}(x^*)$ denotes the MPCC-linearized cone from the definition of MPCC-ACQ. Consequently, we have $T_X(x^*) = L(x^*) = L_{MPCC}(x^*)$, which is exactly MPCC-ACQ. □

Conversely, the following example based on [17, Example 7.1] shows that MPCC-ACQ does not even imply MPCC generalized quasinormality.

Example 6.25 Consider the two-dimensional MPCC

$$\min_{x_1,x_2} f(x) \quad \text{subject to} \quad \begin{aligned} h(x) &= x_2 = 0, \\ G(x) &= 1-(x_1-1)^2-x_2^2 \geq 0, \\ H(x) &= 1-(x_1+1)^2-x_2^2 \geq 0, \\ G(x)H(x) &= 0. \end{aligned}$$

The only feasible point here is $x^* = (0,0)^T$ and the active gradients are

$$\nabla h(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nabla G(x^*) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \nabla H(x^*) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Consequently $T_X(x^*) = \{0\} = L_{MPCC}(x^*)$, i.e., MPCC-ACQ is satisfied in x^*. On the other hand, if we choose $\mu = 0$ and $\gamma = \nu = 1$, it is easy to see that MPCC generalized quasinormality is violated since there are sequences $x^k \to x^*$ with both $G(x^k) < 0$ and $H(x^k) < 0$. ◇

We have seen that calmness of the multifunction M in $(0, x^*)$ implies both MPCC-ACQ and exactness of P_α if x^* is a local minimum of (1.1). The following example, which can be found in [17, Example 7.3] illustrates, that MPCC-ACQ does not imply exactness. Note that this example is a standard nonlinear program, however our MPCC constraint qualifications reduce to the standard ones in absence of complementarity conditions.

Example 6.26 Consider the two-dimensional NLP

$$\min_{x_1,x_2} f(x) \quad \text{subject to} \quad \begin{aligned} g_1(x) &= x_2 \leq 0, \\ g_2(x) &= x_1^6 + x_2^3 \leq 0. \end{aligned}$$

One readily verifies that in $x^* = (0,0)^T$ the equality $T_X(x^*) = \mathbb{R} \times \mathbb{R}_- = L_X(x^*)$, hence ACQ holds in x^*. If, however, we consider the function $f(x) = -x_1^4 - x_2$, which has a strict local minimum in x^*, the corresponding penalty function

$$P_\alpha(x) = f(x) + \alpha(\max\{0, g_1(x)\} + \max\{0, g_2(x)\})$$

does not have a unconstrained local minimum in x^* for any $\alpha > 0$. To see this, one can for example approach x^* with a sequence $(x_1, 0)$, where P_α takes the form

$$P_\alpha(x_1, 0) = -x_1^4 + \alpha x_1^6$$

and thus has a local maximum in x^*. Therefore, ACQ does not imply exactness of P_α. ◇

Note that this example also allows the conclusion that MPCC-ACQ cannot imply calmness of M. We would have liked to give another example illustrating that exactness does not imply MPCC-ACQ. However, the respective example in [17, Example 7.2] seems to be wrong and we were unfortunately not able to come up with a correct one.

The following example, which can be found in [55, Example 2] illustrates that calmness of M in $(0, x^*)$ does not imply MPCC generalized pseudonormality, in fact it does not even imply MPCC generalized quasinormality.

Example 6.27 Consider the one-dimensional MPCC

$$\min_x f(x) \quad \text{subject to} \quad G(x) = x \geq 0,$$
$$H(x) = \begin{cases} -x^2 & \text{if } x < 0, \\ 0 & \text{if } x \in [0,1], \\ (x-1)^2 & \text{if } x > 1 \end{cases} \geq 0,$$
$$G(x)H(x) = 0.$$

Note that, although H is piecewise defined, it is continuously differentiable. The feasible set here is $X = [0, 1]$ and we consider the point $x^* = 0$, where both constraints are active. The active gradients here are

$$\nabla G(x^*) = 1, \nabla H(x^*) = 0.$$

Consequently, MPCC generalized quasinormality does not hold in x^* since we can for example choose the multipliers $(\gamma, \nu) = (0, 1) \neq 0$ and the sequence $x^k := -\frac{1}{k} \uparrow x^*$ with $-\nu G(x^k) = \frac{1}{k} > 0$ for all $k \in \mathbb{N}$. On the other hand, it was proven in [55] that the corresponding map M is calm in $(0, x^*)$. ◇

All in all, we have seen in this chapter that the relations in Figure 6.3 hold and have provided examples illustrating that the relations in Figure 6.4 do not hold.
Two open questions remain however: Is MPCC generalized quasinormality strong enough to imply MPCC ACQ? This is true for standard nonlinear optimization problems and problems with an additional abstract constraint set which is regular, i.e., where the limiting and the Fréchet normal cone coincide, and was proven in [18]. Unfortunately, we were neither able to prove this implication nor to provide a counterexample for MPCCs, where the constraint set defined by the complementarity conditions is not regular. The second question is also connected to MPCC generalized quasinormality. We know that MPCC generalized pseudonormality implies calmness of the map M, but we do not know if this also holds for the weaker MPCC generalized quasinormality.

6.6. Concluding Remarks

In this chapter, we derived an enhanced version of the Fritz-John conditions for MPCCs with additional conditions for the case where the multiplier corresponding to the gradient of the objective function is zero. These conditions led to the introduction of two new constraint qualifications. One of these was used to obtain a very simple proof for local minima of MPCCs to be M-stationary under most of the common MPCC constraint qualifications such as MPCC-LICQ, MPCC-MFCQ, and MPCC-CRCQ whereas all previous proofs for this result rely on the limiting calculus by Mordukhovich. Another one of these new MPCC constraint qualifications could be

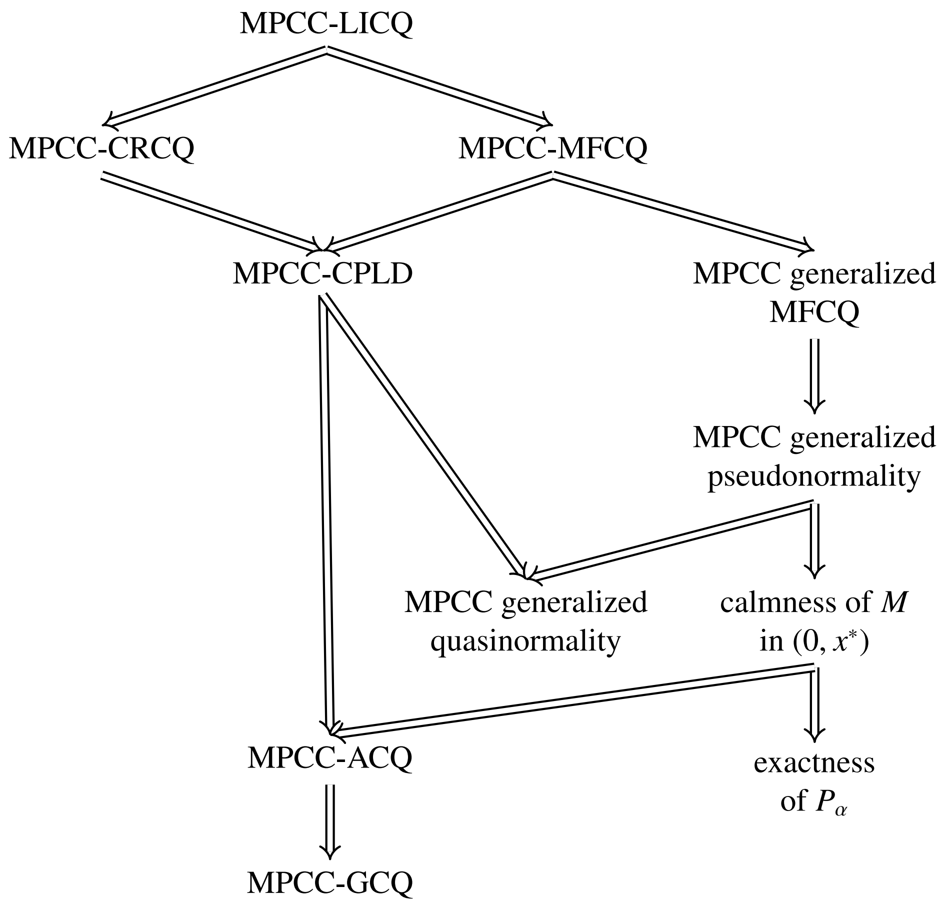


Figure 6.3.: Relations between MPCC-CQs

utilized to prove an exact penalty result for a penalty function based on the l_1-norm, which is interesting since this constraint qualification is significantly weaker than the assumptions usually required in this context. Additionally, this result also implies exactness of a more common MPCC penalty function under the same weak condition.

We also introduced an MPCC analogue of CPLD which will prove to be a very useful constraint qualification in the context of relaxation methods for the numerical solution of MPCCs. We will see in the numerical part that MPCC-CPLD can be used to guarantee M-stationarity of limit points of a new relaxation method whereas up to now the constraint qualification needed for such results was MPCC-LICQ. Additionally, we provided numerous relations and counter examples illustrating how the new constraint qualifications MPCC CPLD, MPCC generalized pseudonormality, and MPCC generalized quasinormality fit into the existing system of MPCC constraint qualifications.

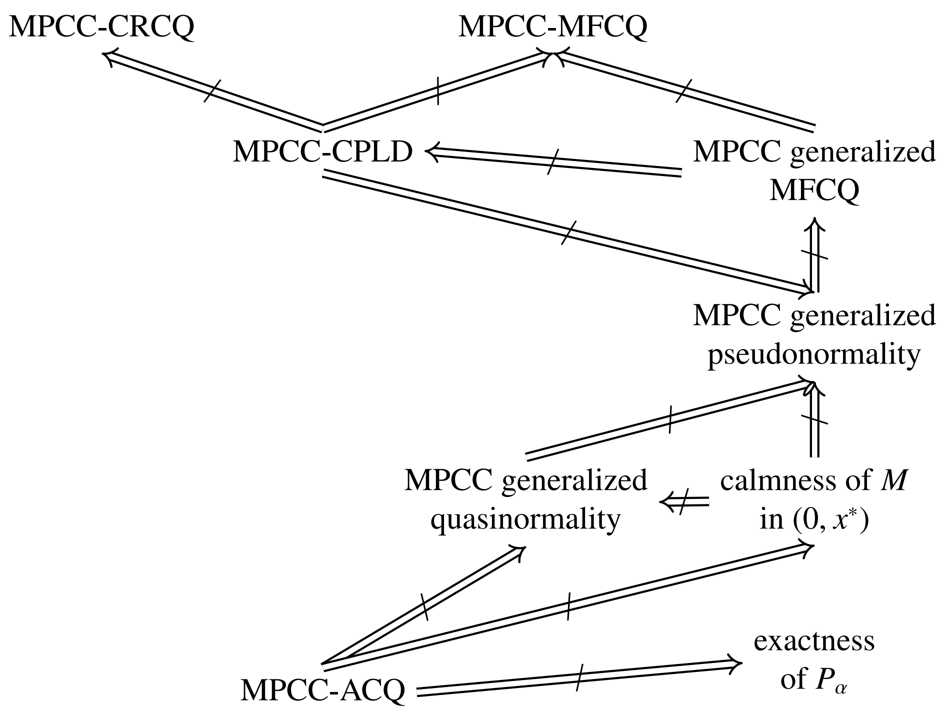


Figure 6.4.: Counter examples

Part III.

Numerical Methods

As we have seen in the last part, standard optimization theory is usually not working for MPCCs. Consequently, we have to expect trouble when applying numerical methods for standard NLPs to MPCCs since certain key assumptions from the convergence theory, like for example LICQ in the solution, are not satisfied. Another problem is that standard solvers usually converge to KKT points whereas we have seen that even simple MPCCs may have minima which are M-stationary but not S-stationary. Hence, even if the MPCC has a solution, a standard solver may not find it. For this reason, several approaches to solve MPCCs have been developed over the last years. We will give a brief overview of these methods and then focus for the rest of this part on relaxation methods. Similar and partly more extensive overviews of existing approaches to solve MPCCs numerically can be found in [36, 119, 115].

The idea behind a relaxation or regularization method is to find a suitable way to enlarge the feasible set of the MPCC such that the problem of linearly dependent gradients and the kink in the origin are removed and then iteratively solve the relaxed nonlinear programs while driving the relaxation down to zero. To us, this approach is very promising since by now there is a number of very effective and well tested solvers for NLPs that have been improved over the years to a level we could not obtain if we wrote an MPCC solver from scratch. A collection of some of these can be found on the NEOS server [2]. Different relaxations with different theoretical properties of the feasible set were suggested by [105, 79, 29, 67, 109, 77]. In this part of the thesis, we will improve the theoretical results known for four of these methods, introduce a new one with strong convergence properties and provide a numerical comparison of the four existing methods and the new one. We discarded the approach described in [29] from our analysis since the central idea in this paper is different. Whereas in all other approaches, there is one relaxation parameter for all complementarity constraints, the authors of [29] suggest a two-sided relaxation of the feasible set combined with some kind of active set technique to individually update the relaxation parameter for each constraint. Similarly, the approach in [77] is not discussed any further since it is based on different ideas than the remaining four methods.

Another approach which is closely related to the relaxation idea are the so-called smoothing methods, where the complementarity conditions are reformulated using an NPC-function such as the minimum function or the Fischer-Burmeister function which is then replaced by its smoothed counterpart and again a sequence of nonlinear problems with decreasing smoothing parameter is solved. This is done for example in [34, 66, 127]. These methods differ in the used NCP function, the underlying NLP solver and the technique for handling the smoothing parameter.

An intuitive idea to deal with the complicated complementarity conditions is to move them from the constraints to the objective function via a suitable penalty term as it was done in [83, 82, 78, 106, 62]. The differences here lie in the choice of the penalty term and the suggested way how to solve the penalized problem. In [106] for example, it is suggested to use a modification of the Sl_1QP method by Fletcher [43] where in the quadratic subproblems all linearized constraints are added to the objective function using an appropriate l_1 penalty term. Yet another idea comes from the authors of [80] who suggest to introduce slack variables for the complementarity constraints, replace all constraints but the now simplified complementarity conditions by a penalty term and then smooth the resulting penalty function. Compared to the other methods mentioned so far, this approach has the disadvantage that it is necessary to solve a sequence of MPCCs (with very simple complementarity constraints however).

Wherever there are penalty methods, there are also barrier methods and these combined with the relaxation idea give rise to interior point methods as the ones proposed in [81, 101]. Here, the complementarity constraints are relaxed and then all inequalities replaced by a slack variable for which a barrier term is added to the objective function. Another interior point method which is rather a combination of the penalty and the barrier idea can be found in [75], where the complementarity constraints are not relaxed but added to the objective as a penalty term.

Although we mentioned that the direct application of standard solvers to MPCCs is not very promising, we have to admit that there are some state of the art NLP solvers that work quite well when applied to MPCCs. In [45] for example it is suggested to apply a standard SQP method to the MPCC (1.1) interpreted as standard nonlinear program and in [44] corresponding numerical results are presented. Similarly, it is suggested to replace the complementarity constraints by a nonsmooth NPC function and then again apply a standard SQP method in [74]. We wanted to focus on those numerical approaches relevant in the context of this thesis, but of course there are much more than those mentioned above. For example there is the implicit approach from [95] and special SQP methods for MPCCs can be found in [50, 126]. In [7, 6, 8] an elastic mode formulation is proposed and [110, 63] suggest a lifting approach.

The numerical part of this thesis is structured as follows: In Chapter 7, we improve the convergence results of four existing relaxation methods and analyze what kind of constraint qualification the relaxed problems inherit from the MPCC. In Chapter 8, we introduce a new relaxation for MPCCs and verify that this relaxation has strong convergence properties under relatively weak assumptions. Having introduced all these methods, we collect the theoretical results and also give a numerical comparison based on the MacMPEC test suite in Chapter 9. Finally, we return to the effort maximization problem from Part I in Chapter 10 and solve it numerically using the new relaxation method from Chapter 8.

7. Improved Results for Existing Relaxation Methods

The basic idea of all relaxation schemes is to get rid of the complicated complementarity constraints

$$G_i(x) \geq 0, \; H_i(x) \geq 0, \; G_i(x)H_i(x) = 0 \quad \forall i = 1, \ldots, q$$

by replacing these conditions in a suitable way such that the corresponding relaxed problem has nicer properties. The relaxed problem depends on a parameter $t > 0$ which has to be driven to zero in order to reobtain the underlying MPCC.

For all relaxation schemes discussed here (we will discuss them in chronological order of their date of publication), suitable convergence results are already known. Typically, the most basic convergence results are as follows: Given a sequence $t_k \downarrow 0$ and a corresponding sequence of stationary points x^k of the relaxed problems $R(t_k)$ such that x^k converges to x^* and such that a suitable MPCC constraint qualification holds at x^*, then x^* is a C-stationary point (for three of the methods to be discussed below) or an M-stationary point (for the remaining two methods). Furthermore, under additional conditions, one can verify that the limit point x^* has further properties, like being M- or even S-stationary.

In our subsequent analysis, we try to improve only the most basic convergence results by relaxing the corresponding MPCC constraint qualifications. The additional results which guarantee stronger properties of the limit point x^* are not discussed here since a corresponding generalization of the existing results are usually straightforward. However, we also show under which conditions the relaxed problems actually have a stationary point. In some cases, our results generalize existing ones, in other cases, we prove completely new results.

The following technical lemma will be used in some of the subsequent convergence results. It can be found in [109, Lemma A.1] but we will present a different, more direct proof here.

Lemma 7.1 *Let $\{a_i \mid i = 1, \ldots, m\}$, $\{b_i \mid i = 1, \ldots, p\}$ and c be vectors in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}^m_+$, $\beta \in \mathbb{R}^p$ multipliers such that*

$$\sum_{i=1}^{m} \alpha_i a^i + \sum_{i=1}^{p} \beta_i b^i = c.$$

Then there exist multipliers $\alpha^* \in \mathbb{R}^m_+$ and β^* with $supp(\alpha^*) \subseteq supp(\alpha)$, $supp(\beta^*) \subseteq supp(\beta)$ and

$$\sum_{i=1}^{m} \alpha^*_i a^i + \sum_{i=1}^{p} \beta^*_i b^i = c$$

such that the vectors

$$\{a^i \mid i \in supp(\alpha^*)\} \cup \{b^i \mid i \in supp(\beta^*)\}$$

are linearly independent.

Proof. If the vectors

$$\{a^i \mid i \in \operatorname{supp}(\alpha)\} \cup \{b^i \mid i \in \operatorname{supp}(\beta)\}$$

are already linearly independent, we can choose $\alpha^* = \alpha$, $\beta^* = \beta$ and are done. Otherwise, there are scalars $\delta_i, i \in \operatorname{supp}(\alpha)$ and $\tau_i, i \in \operatorname{supp}(\beta)$ not all equal to zero such that

$$\sum_{i \in \operatorname{supp}(\alpha)} \delta_i a^i + \sum_{i \in \operatorname{supp}(\beta)} \tau_i b^i = 0.$$

If all δ_i are equal to zero, we can choose an arbitrary $i^* \in \operatorname{supp}(\tau) \subseteq \operatorname{supp}(\beta)$ and define

$$\tilde{\alpha} := \alpha, \quad \text{and} \quad \tilde{\beta} := \begin{cases} \beta_i - \frac{\beta_{i^*}}{\tau_{i^*}}\tau_i & \text{if } i \in \operatorname{supp}(\beta), \\ 0 & \text{else.} \end{cases}$$

Otherwise, we can assume without loss of generality that there is at least one $\alpha_i > 0$ and choose i^* as an index with

$$\frac{\alpha_{i^*}}{\delta_{i^*}} = \min\left\{ \frac{\alpha_i}{\delta_i} \,\middle|\, i \in \operatorname{supp}(\alpha), \delta_i > 0 \right\}.$$

In this case, we define the new multipliers as

$$\tilde{\alpha} := \begin{cases} \alpha_i - \frac{\alpha_{i^*}}{\delta_{i^*}}\delta_i & \text{if } i \in \operatorname{supp}(\alpha), \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \tilde{\beta} := \begin{cases} \beta_i - \frac{\alpha_{i^*}}{\delta_{i^*}}\tau_i & \text{if } i \in \operatorname{supp}(\beta), \\ 0 & \text{else.} \end{cases}$$

In both cases we have $\tilde{\alpha} \geq 0$ and $\operatorname{supp}(\tilde{\alpha}, \tilde{\beta}) \subsetneq \operatorname{supp}(\alpha, \beta)$. Additionally, these multipliers still have the property that

$$\sum_{i=1}^{m} \tilde{\alpha}_i a^i + \sum_{i=1}^{p} \tilde{\beta}_i b^i = c.$$

If the vectors

$$\{a^i \mid i \in \operatorname{supp}(\tilde{\alpha})\} \cup \{b^i \mid i \in \operatorname{supp}(\tilde{\beta})\}$$

are linearly independent, we can finish here. Otherwise, we have to repeat the procedure above. Since the support of $(\tilde{\alpha}, \tilde{\beta})$ decreases each time, after a finite number of iteration either the vectors corresponding to nonvanishing multipliers are linearly independent or $\operatorname{supp}(\tilde{\alpha}, \tilde{\beta}) = \emptyset$, in which case the assertion is trivially satisfied. □

Additionally, we need the following lemma that facilitates the calculation of polar cones to linearized cones. It can be found, for example, in [13, Theorem 3.2.2].

Lemma 7.2 *Consider the cones*

$$\begin{aligned} C_1 &:= \{d \in \mathbb{R}^n \mid a_i^T d \leq 0,\ \forall i = 1, \dots, m,\ b_i^T d = 0\ \forall i = 1, \dots, p\}, \\ C_2 &:= \{s \in \mathbb{R}^n \mid s = \sum_{i=1}^{m} \alpha_i a_i + \sum_{i=1}^{p} \beta_i b_i,\ \alpha_i \geq 0\ \forall i = 1, \dots, m\}. \end{aligned}$$

Then $C_2 = C_1^\circ$ and $C_1 = C_2^\circ$.

7.1. The Global Relaxation by Scholtes

Probably the first attempt to use a relaxation idea for solving MPCCs goes back to Scholtes [105]. It is closely related to the smoothing-type method by Facchinei et al. [34]. Some local properties of Scholtes' approach around an S-stationary point can also be found in Ralph and Wright [102].

The basic idea of the relaxation scheme by Scholtes is to replace the MPCC by a sequence of parametrized NLPs of the form

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m, \\ & h_j(x) = 0 \quad \forall j = 1, \dots, p, \\ & G_i(x) \geq 0 \quad \forall i = 1, \dots, q, \\ & H_i(x) \geq 0 \quad \forall i = 1, \dots, q, \\ & G_i(x)H_i(x) \leq t \quad \forall i = 1, \dots, q. \end{array}$$

see Figure 7.1 for a geometric illustration.

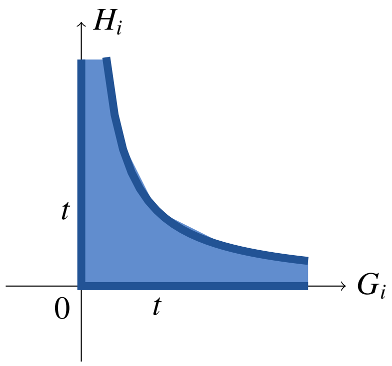


Figure 7.1.: Geometric interpretation of the relaxation method by Scholtes

We denote the relaxed problem by $R^S(t)$ and its feasible set by $X^S(t)$. Since, geometrically, this is a global relaxation of the complementarity conditions, we call this approach the *global relaxation method*.

7.1.1. Convergence to C-Stationary Points

For the convergence analysis, some index sets are needed:

$$\begin{aligned} I_g(x) &:= \{i \mid g_i(x) = 0\}, \\ I_G(x) &:= \{i \mid G_i(x) = 0\}, \\ I_H(x) &:= \{i \mid H_i(x) = 0\}, \\ I_{GH}(x;t) &:= \{i \mid H_i(x)G_i(x) = t\}. \end{aligned}$$

The following is the most basic convergence result for the global relaxation method.

Theorem 7.3 *Let $\{t_k\} \downarrow 0$ and let x^k be a stationary point of $R^S(t_k)$ with $x^k \to x^*$ such that MPCC-MFCQ holds at x^*. Then x^* is a C-stationary point of* (1.1).

Proof. Since x^k is a stationary point of $R^S(t_k)$ there exist multipliers $(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^k)$ such that

$$\begin{aligned} 0 \;=\;& \nabla f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^p \mu_i^k \nabla h_i(x^k) - \sum_{i=1}^l \gamma_i^k \nabla G_i(x^k) \\ & - \sum_{i=1}^q \nu_i^k \nabla H_i(x^k) + \sum_{i=1}^q \delta_i^k [H_i(x^k)\nabla G_i(x^k) + G_i(x^k)\nabla H_i(x^k)] \end{aligned} \tag{7.1}$$

with

$$\begin{aligned} \lambda^k \geq 0 \quad &\text{and} \quad \text{supp}(\lambda^k) \subseteq I_g(x^k), \\ \gamma^k \geq 0 \quad &\text{and} \quad \text{supp}(\gamma^k) \subseteq I_G(x^k), \\ \nu^k \geq 0 \quad &\text{and} \quad \text{supp}(\nu^k) \subseteq I_H(x^k), \\ \delta^k \geq 0 \quad &\text{and} \quad \text{supp}(\delta^k) \subseteq I_{GH}(x^k; t_k) \end{aligned}$$

for all $k \in \mathbb{N}$. This implies

$$\text{supp}(\gamma^k) \cap \text{supp}(\delta^k) = \emptyset, \quad \text{supp}(\nu^k) \cap \text{supp}(\delta^k) = \emptyset \tag{7.2}$$

for all $k \in \mathbb{N}$. Moreover, for all $k \in \mathbb{N}$ sufficiently large, we have $I_g(x^k) \subseteq I_g(x^*)$, $I_G(x^k) \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$, and $I_H(x^k) \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$.

Our next step is to define suitable new multipliers

$$\tilde{\gamma}_i^k = \begin{cases} \gamma_i^k, & \text{if } i \in \text{supp}(\gamma^k), \\ -\delta_i^k H_i(x^k), & \text{if } i \in \text{supp}(\delta^k) \setminus I_{+0}(x^*), \\ 0, & \text{else,} \end{cases}$$

and

$$\tilde{\nu}_i^k = \begin{cases} \nu_i^k, & \text{if } i \in \text{supp}(\nu^k), \\ -\delta_i^k G_i(x^k), & \text{if } i \in \text{supp}(\delta^k) \setminus I_{0+}(x^*), \\ 0, & \text{else.} \end{cases}$$

With these multipliers, we can rewrite (7.1) as

$$\begin{aligned} 0 \;=\;& \nabla f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^p \mu_i^k \nabla h_i(x^k) - \sum_{i=1}^q \tilde{\gamma}_i^k \nabla G_i(x^k) - \sum_{i=1}^q \tilde{\nu}_i^k \nabla H_i(x^k) \\ & + \sum_{i \in I_{+0}(x^*)} \delta_i^k H_i(x^k) \nabla G_i(x^k) + \sum_{i \in I_{0+}(x^*)} \delta_i^k G_i(x^k) \nabla H_i(x^k). \end{aligned}$$

If we assume that the sequence $\{(\lambda^k, \mu^k, \tilde{\gamma}^k, \tilde{\nu}^k, \delta^k_{I_{+0}(x^*) \cup I_{0+}(x^*)})\}$ was unbounded, then we can find a subsequence K such that the normed sequence converges:

$$\frac{(\lambda^k, \mu^k, \tilde{\gamma}^k, \tilde{\nu}^k, \delta^k_{I_{+0} \cup I_{0+}})}{\|(\lambda^k, \mu^k, \tilde{\gamma}^k, \tilde{\nu}^k, \delta^k_{I_{+0} \cup I_{0+}})\|} \to_K (\lambda, \mu, \tilde{\gamma}, \tilde{\nu}, \delta_{I_{+0} \cup I_{0+}}) \neq 0.$$

The equation above then yields

$$0 = \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \tilde{\gamma}_i \nabla G_i(x^*) - \sum_{i=1}^{q} \tilde{\nu}_i \nabla H_i(x^*)$$

where $\lambda \geq 0$ and for all $k \in K$ sufficiently large

$$\begin{aligned}
\operatorname{supp}(\lambda) &\subseteq I_g(x^k) \subseteq I_g(x^*),\\
\operatorname{supp}(\tilde{\gamma}) &\subseteq I_G(x^k) \cup I_{GH}(x^k; t_k) \setminus I_{+0}(x^*) \subseteq I_{00}(x^*) \cup I_{0+}(x^*),\\
\operatorname{supp}(\tilde{\nu}) &\subseteq I_H(x^k) \cup I_{GH}(x^k; t_k) \setminus I_{0+}(x^*) \subseteq I_{00}(x^*) \cup I_{+0}(x^*).
\end{aligned}$$

Additionally, $(\lambda, \mu, \tilde{\gamma}, \tilde{\nu}) \neq 0$ has to hold. Otherwise, $\delta_i > 0$ would have to hold for at least one $i \in I_{+0}(x^*) \cup I_{0+}(x^*)$. Assume without loss of generality $\delta_i > 0$ for an $i \in I_{+0}(x^*)$. This implies $\delta_i^k > 0$ for all k sufficiently large and consequently $\tilde{\nu}_i^k = -\delta_i^k G_i(x^k)$ for those k. Because of $i \in I_{+0}(x^*)$, this yields $\tilde{\nu}_i = \lim_{k \in K} -\delta_i^k G_i(x^k) < 0$, a contradiction to our assumption $\tilde{\nu} = 0$.

However, due to Lemma 5.7 $(\lambda, \mu, \tilde{\gamma}, \tilde{\nu}) \neq 0$ is a contradiction to the prerequisite that MPCC-MFCQ holds in x^*. Thus, we may assume without loss of generality that the sequence is convergent to some vector $(\lambda^*, \mu^*, \tilde{\gamma}^*, \tilde{\nu}^*, \delta^*_{I_{+0}(x^*) \cup I_{0+}(x^*)})$. It is easy to see that $\lambda^* \geq 0$ and $\operatorname{supp}(\lambda^*) \subseteq I_g(x^*)$. According to the definition of $\tilde{\gamma}^k$ and $\tilde{\nu}^k$, we have

$$\operatorname{supp}(\tilde{\gamma}^*) \subseteq I_{00}(x^*) \cup I_{0+}(x^*), \quad \operatorname{supp}(\tilde{\nu}^*) \subseteq I_{00}(x^*) \cup I_{+0}(x^*).$$

The continuous differentiability of f, g, h, G, H then implies

$$0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i^* \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i^* \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i^* \nabla H_i(x^*).$$

To prove the C-stationarity of x^*, it remains to show that $\gamma_i^* \nu_i^* \geq 0$ for all $i \in I_{00}(x^*)$. Assume that there is an $i \in I_{00}(x^*)$ with $\gamma_i^* < 0$ and $\nu_i^* > 0$ or with $\nu_i^* > 0$ and $\gamma_i^* < 0$. We consider only the first case, the second one can be treated similarly. Because of $\gamma_i^k \geq 0$, the condition $\gamma_i^* < 0$ implies $i \in \operatorname{supp}(\delta^k)$ for all $k \in \mathbb{N}$ sufficiently large. This implies $i \notin \operatorname{supp}(\nu^k)$ in view of (7.2) and, therefore, $\nu_i^* \leq 0$ in contradiction to our assumption. □

Note that the corresponding result in [105] assumes MPCC-LICQ and shows that the sequence of multipliers corresponding to the stationary points x^k converges, whereas here we assume the weaker MPCC-MFCQ which, obviously, does not guarantee convergence of the corresponding sequence of multipliers, but the proof shows that one can extract a sequence of multipliers which stays bounded and is, therefore, convergent at least on a subsequence.

7.1.2. Existence of Multipliers

The assumption of x^k being a stationary point of the relaxed problem $R^S(t_k)$ is based on the existence of multipliers. A priori, it is not clear that these multipliers really exist. The following result essentially guarantees the existence of these multipliers by showing that MPCC-MFCQ at a feasible point x^* of the original MPCC implies that standard MFCQ holds for the relaxed problems $R^S(t)$, at least locally around x^*.

Theorem 7.4 *Let x^* be feasible for* (1.1) *such that MPCC-MFCQ is satisfied at x^*. Then there exists a neighborhood $U(x^*)$ of x^* and $\bar{t} > 0$ such that the following holds for all $t \in (0, \bar{t}]$: If $x \in U(x^*)$ is feasible for $R^S(t)$, then standard MFCQ for $R^S(t)$ holds in x.*

Proof. First note that, by continuity, for all $x \in X^S(t)$ sufficiently close to x^*, we have

$$\begin{aligned} I_g(x) &\subseteq I_g(x^*), \\ I_G(x) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\ I_H(x) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*), \\ I_{GH}(x) \cap I_G(x) &= \emptyset, \\ I_{GH}(x) \cap I_H(x) &= \emptyset. \end{aligned} \tag{7.3}$$

Since MPCC-MFCQ holds, the gradients

$$\begin{aligned} &\{\nabla g_i(x^*) \mid i \in I_g(x^*)\} \cup \\ &\{\{\nabla h_i(x^*) \mid i = 1, \dots, p\} \cup \{\nabla G_i(x^*) \mid i \in I_{00}(x^*) \cup I_{0+}(x^*)\} \cup \{\nabla H_i(x^*) \mid i \in I_{00}(x^*) \cup I_{+0}(x^*)\}\} \end{aligned}$$

are positive-linearly independent by Lemma 5.7. In view of [99, Prop. 2.2], this implies that the set of gradients

$$\begin{aligned} &\{\nabla g_i(x) \mid i \in I_g(x^*)\} \cup \\ &\{\{\nabla h_i(x) \mid i = 1, \dots, p\} \cup \{\nabla G_i(x) \mid i \in I_{00}(x^*) \cup I_{0+}(x^*)\} \cup \{\nabla H_i(x) \mid i \in I_{00}(x^*) \cup I_{+0}(x^*)\}\} \end{aligned}$$

is also positive-linearly independent for all $x \in X^S(t)$ sufficiently close to x^*. Taking into account that

$$I_G(x) \cup \big(I_{GH}(x) \cap I_{0+}(x^*)\big) \cup \big(I_{GH}(x) \cap I_{00}(x^*)\big) \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$$

and

$$I_H(x) \cup \big(I_{GH}(x) \cap I_{+0}(x^*)\big) \cup \big(I_{GH}(x) \cap I_{00}(x^*)\big) \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$$

for all $x \in X^S(t)$ sufficiently close to x^* and using the fact that $G_i(x) > 0, H_i(x) \approx 0$ for all $i \in I_{+0}(x^*)$ as well as $G_i(x) \approx 0, H_i(x) > 0$ for all $i \in I_{0+}(x^*)$ whenever x is close to x^*, it follows that there is a neighborhood $U(x^*)$ such that the set of vectors

$$\begin{aligned} \nabla g_i(x) \quad & (i \in I_g(x)), \\ \nabla h_i(x) \quad & (i = 1, \dots, p), \\ \nabla G_i(x) \quad & (i \in I_G(x)), \\ \nabla H_i(x) \quad & (i \in I_H(x)), \\ G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) \quad & (i \in I_{GH}(x) \cap I_{0+}(x^*)), \\ G_i(x)\nabla H_i(x) + G_i(x)\nabla H_i(x) \quad & (i \in I_{GH}(x) \cap I_{+0}(x^*)), \\ \nabla G_i(x) \quad & (i \in I_{GH}(x) \cap I_{00}(x^*)), \\ \nabla H_i(x) \quad & (i \in I_{GH}(x) \cap I_{00}(x^*)) \end{aligned} \tag{7.4}$$

is positive-linearly independent for all $x \in X^S(t) \cap U(x^*)$.

We now claim that standard MFCQ holds for the relaxed program $R^S(t)$ whenever $x \in X^S(t) \cap U(x^*)$. To this end, take an arbitrary $x \in X^S(t) \cap U(x^*)$. In view of Lemma 4.7, we have to show that

$$\begin{aligned} 0 \; = \; & \sum_{i \in I_g(x)} \lambda_i \nabla g_i(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) - \sum_{i \in I_G(x)} \alpha_i \nabla G_i(x) \\ & - \sum_{i \in I_H(x)} \beta_i \nabla H_i(x) + \sum_{i \in I_{GH}(x)} \gamma_i (G_i(x) \nabla H_i(x) + H_i(x) \nabla G_i(x)) \end{aligned} \tag{7.5}$$

with $\mu \in \mathbb{R}^p$ and $\lambda, \alpha, \beta, \gamma \geq 0$ holds only for the null vector. To see this, we rewrite (7.5) as

$$\begin{aligned} 0 \; = \; & \sum_{i \in I_g(x)} \lambda_i \nabla g_i(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) - \sum_{i \in I_G(x)} \alpha_i \nabla G_i(x) - \sum_{i \in I_H(x)} \beta_i \nabla H_i(x) \\ & + \sum_{i \in I_{GH}(x) \cap (I_{0+}(x^*) \cup I_{+0})} \gamma_i (G_i(x) \nabla H_i(x) + H_i(x) \nabla G_i(x)) \\ & + \sum_{i \in I_{00}(x^*) \cap I_{GH}(x)} (\gamma_i G_i(x)) \nabla H_i(x) + \sum_{i \in I_{00}(x^*) \cap I_{GH}(x)} (\gamma_i H_i(x)) \nabla G_i(x). \end{aligned} \tag{7.6}$$

Applying the positive-linear independence of the vectors from (7.4) to (7.6) and using (7.3), we immediately obtain that all coefficients from (7.5) are zero, and this completes the proof. □

7.2. The Smooth Relaxation by Lin and Fukushima

The relaxation scheme proposed by Lin and Fukushima in [79] employs the following relaxation, see Figure 7.2 for a geometric illustration of the relaxed feasible set:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad \forall i = 1, \ldots, m, \\ & h_i(x) = 0 \quad \forall i = 1, \ldots, p, \\ & G_i(x) H_i(x) - t^2 \leq 0 \quad \forall i = 1, \ldots, q, \\ & (G_i(x) + t)(H_i(x) + t) - t^2 \geq 0 \quad \forall i = 1, \ldots, q. \end{aligned}$$

The relaxed problem is denoted by $R^{LF}(t)$ and its feasible set by $X^{LF}(t)$. Obviously, the idea behind this relaxation is very close to the previous one by Scholtes, however with the advantage of needing less constraints. We call it a smooth relaxation since the kink of the original feasible set in the origin is completely removed in the relaxed feasible sets.

In order to investigate the modified relaxation scheme in-depth, we need to introduce some further index sets. To this end, let $x \in X^{LF}(t)$ for $t > 0$. Then we put:

$$\begin{aligned} I_g(x) & := \{i \mid g_i(x) = 0\}, \\ I_{GH}^+(x;t) & := \{i \mid G_i(x) H_i(x) - t^2 = 0\}, \\ I_{GH}^-(x;t) & := \{i \mid (G_i(x) + t)(H_i(x) + t) - t^2 = 0\}. \end{aligned}$$

To facilitate the following proofs, we want to take a closer look at these index sets. Let x be feasible for $R^{LF}(t)$ and $i \in I_{GH}^+(x;t)$. This implies $G_i(x) H_i(x) = t^2 > 0$, i.e., $G_i(x), H_i(x) \neq 0$ and

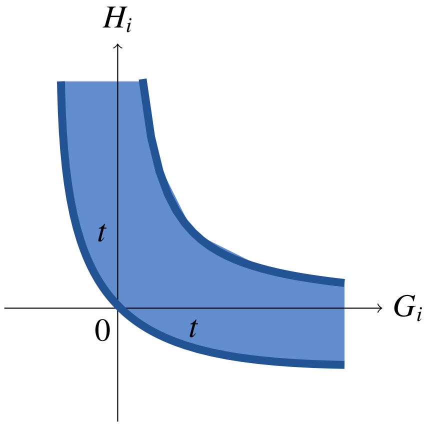


Figure 7.2.: Geometric interpretation of the relaxation method by Lin and Fukushima

both have the same sign. Now, assume that both values were negative. This would imply

$$G_i(x)H_i(x) + t(G_i(x) + H_i(x)) < t^2$$

in contradiction to the feasibility of x. Thus, we have the following implication:

$$i \in I^+_{GH}(x;t) \quad \Longrightarrow \quad G_i(x) > 0, H_i(x) > 0. \tag{7.7}$$

Now consider the case where $i \in I^-_{GH}(x;t)$. This implies $(G_i(x) + t)(H_i(x) + t) = t^2 > 0$, so either both values $G_i(x) + t, H_i(x) + t$ are strictly greater or smaller than zero. Assume that both values are negative. This implies $G_i(x), H_i(x) < 0$ and thus

$$G_i(x)H_i(x) - t^2 = -t(G_i(x) + H_i(x)) > 0$$

in contradiction to the feasibility of x. Hence, we obtain the following implication:

$$i \in I^-_{GH}(x;t) \quad \Longrightarrow \quad G_i(x) + t > 0, H_i(x) + t > 0. \tag{7.8}$$

7.2.1. Convergence to C-Stationary points

In this section we state the main convergence result which can be viewed as a refinement of [79, Theorem 3.3]. It shows that MPCC-MFCQ is sufficient to guarantee that a limit point of a sequence of stationary points of the relaxed programs $R^{LF}(t)$ is C-stationary. The corresponding result in [79] requires the stronger MPCC-LICQ condition in order to obtain this statement.

Theorem 7.5 *Let $\{t_k\} \downarrow 0$ and let x^k be a stationary point of $R^{LF}(t)$ with $x^k \to x^*$ such that MPCC-MFCQ holds in x^*. Then x^* is C-stationary.*

Proof. Since x^k is a stationary point of $R^{LF}(t_k)$, we have multipliers $\lambda^k, \mu^k, \delta^{+,k}, \delta^{-,k}$ such that

$$0 \;=\; \nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(x^k)$$

$$+\sum_{i=1}^{q}\delta_i^{+,k}[H_i(x^k)\nabla G_i(x^k)+G_i(x^k)\nabla H_i(x^k)]$$
$$-\sum_{i=1}^{q}\delta^{-,k}[(H_i(x^k)+t_k)\nabla G_i(x^k)+(G_i(x^k)+t_k)\nabla H_i(x^k)]$$

with

$$\begin{aligned}\lambda^k\geq 0 \quad &\text{and} \quad \operatorname{supp}(\lambda^k)\subseteq I_g(x^k),\\ \delta^{+,k}\geq 0 \quad &\text{and} \quad \operatorname{supp}(\delta^{+,k})\subseteq I_{GH}^{+}(x^k;t_k),\\ \delta^{-,k}\geq 0 \quad &\text{and} \quad \operatorname{supp}(\delta^{-,k})\subseteq I_{GH}^{-}(x^k;t_k)\end{aligned}$$

for all $k\in\mathbb{N}$. This implies

$$\operatorname{supp}(\delta^{+,k})\cap\operatorname{supp}(\delta^{-,k})=\emptyset \tag{7.9}$$

for all $k\in\mathbb{N}$. Hence the following new multipliers are at least well-defined:

$$\gamma_i^k=\begin{cases}-\delta_i^{+,k}H_i(x^k), & \text{if } i\in\operatorname{supp}(\delta^{+,k})\setminus I_{+0}(x^*),\\ \delta_i^{-,k}(H_i(x^k)+t_k) & \text{if } i\in\operatorname{supp}(\delta^{-,k})\setminus I_{+0}(x^*),\\ 0, & \text{else}\end{cases}$$

and

$$\nu_i^k=\begin{cases}-\delta_i^{+,k}G_i(x^k), & \text{if } i\in\operatorname{supp}(\delta^{+,k})\setminus I_{0+}(x^*),\\ \delta_i^{-,k}(G_i(x^k)+t_k) & \text{if } i\in\operatorname{supp}(\delta^{-,k})\setminus I_{0+}(x^*)\\ 0, & \text{else.}\end{cases}$$

With these multipliers, we can rewrite the equation from the beginning as

$$\begin{aligned}0 \;=\; &\nabla f(x^k)+\sum_{i=1}^{m}\lambda_i^k\nabla g_i(x^k)+\sum_{i=1}^{p}\mu_i^k\nabla h_i(x^k)-\sum_{i=1}^{q}\gamma_i^k\nabla G_i(x^k)\\ &-\sum_{i=1}^{q}\nu_i^k\nabla H_i(x^k)+\sum_{i\in I_{+0}(x^*)}\delta_i^{+,k}H_i(x^k)\nabla G_i(x^k)+\sum_{i\in I_{0+}(x^*)}\delta_i^{+,k}G_i(x^k)\nabla H_i(x^k)\\ &-\sum_{i\in I_{+0}(x^*)}\delta_i^{-,k}(H_i(x^k)+t_k)\nabla G_i(x^k)-\sum_{i\in I_{0+}(x^*)}\delta_i^{-,k}(G_i(x^k)+t_k)\nabla H_i(x^k).\end{aligned}$$

If we assume that the sequence $\{(\lambda^k,\mu^k,\gamma^k,\nu^k,\delta^{+,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)},\delta^{-,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)})\}$ is unbounded, then we can find a subsequence K such that the normed sequence converges:

$$\frac{(\lambda^k,\mu^k,\gamma^k,\nu^k,\delta^{+,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)},\delta^{-,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)})}{\|(\lambda^k,\mu^k,\gamma^k,\nu^k,\delta^{+,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)},\delta^{-,k}_{I_{+0}(x^*)\cup I_{0+}(x^*)})\|}\to_K(\lambda,\mu,\gamma,\nu,\delta^{+}_{I_{+0}(x^*)\cup I_{0+}(x^*)},\delta^{-}_{I_{+0}(x^*)\cup I_{0+}(x^*)})\neq 0.$$

The equation above then yields

$$0 \;=\; \sum_{i=1}^{m}\lambda_i\nabla g_i(x^*)+\sum_{i=1}^{p}\mu_i\nabla h_i(x^*)-\sum_{i=1}^{q}\gamma_i\nabla G_i(x^*)-\sum_{i=1}^{q}\nu_i\nabla H_i(x^*)$$

where $\lambda \geq 0$ and for all $k \in K$ sufficiently large

$$\begin{aligned} \text{supp}(\lambda) &\subseteq I_g(x^k) \subseteq I_g(x^*), \\ \text{supp}(\gamma) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\ \text{supp}(\nu) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*). \end{aligned}$$

Additionally, $(\lambda, \mu, \gamma, \nu) \neq 0$ has to hold. Otherwise, $\delta_i^+ > 0$ or $\delta_i^- > 0$ would have to hold for at least one index $i \in I_{+0}(x^*) \cup I_{0+}(x^*)$. Assume first, without loss of generality, that $\delta_i^+ > 0$ for an $i \in I_{+0}(x^*)$. This implies $\delta_i^{+,k} > 0$ for all k sufficiently large and consequently $\nu_i^k = -\delta_i^{+,k} G_i(x^k)$ for those k. Because of $i \in I_{+0}(x^*)$, this yields $\nu_i = \lim_{k\in K} -\delta_i^{+,k} G_i(x^k) < 0$, a contradiction to our assumption $\nu = 0$. Now assume $\delta_i^- > 0$ for an $i \in I_{+0}(x^*)$. This implies $\delta_i^{-,k} > 0$ for all k sufficiently large and thus $\nu_i^k = \delta_i^{-,k}(G_i(x^k) + t_k)$ for those k. Because of $i \in I_{+0}(x^*)$, this yields $\nu_i = \lim_{k\in K} \delta_i^{-,k}(G_i(x^k) + t_k) > 0$, again a contradiction to our assumption $\nu = 0$.

However, because of Lemma 5.7 $(\lambda, \mu, \gamma, \nu) \neq 0$ is a contradiction to the prerequisite that MPCC-MFCQ holds in x^*. Thus, we may assume without loss of generality that the sequence is convergent to some vector $\{(\lambda^*, \mu^*, \gamma^*, \nu^*, \delta^{+,*}_{I_{+0}(x^*)\cup I_{0+}(x^*)}, \delta^{-,*}_{I_{+0}(x^*)\cup I_{0+}(x^*)})\}$. It is easy to see that $\lambda^* \geq 0$ and $\text{supp}(\lambda^*) \subseteq I_g(x^*)$. According to the definition of γ^k and ν^k, we have

$$\text{supp}(\gamma^*) \subseteq I_{00}(x^*) \cup I_{0+}(x^*), \quad \text{supp}(\nu^*) \subseteq I_{00}(x^*) \cup I_{+0}(x^*).$$

The continuous differentiability of f, g, h, G, H then implies

$$0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i^* \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i^* \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i^* \nabla H_i(x^*).$$

To prove the C-stationarity of x^*, it remains to show $\gamma_i^* \nu_i^* \geq 0$ for all $i \in I_{00}(x^*)$. Assume that there is an $i \in I_{00}$ with $\gamma_i^* < 0$ and $\nu_i^* > 0$ or with $\gamma_i^* > 0$ and $\nu_i^* < 0$. We consider only the first case, the second one can be treated similarly. Since $\gamma_i^* < 0$, we have $\gamma_i^k < 0$ for all $k \in \mathbb{N}$ sufficiently large. But this implies $i \in \text{supp}(\delta^{+,k})$ since, otherwise, the definition of γ_i^k would imply $i \in \text{supp}(\delta^{-,k})$, hence $i \in I_\Phi^-(x^k; t_k)$, hence $\delta_i^{-,k} > 0$ and $H_i(x^k) + t_k > 0$ by (7.8) and, therefore, $\gamma_i^k > 0$ due to the definition of γ_i^k. Knowing that $i \in \text{supp}(\delta^{+,k})$, we have $i \notin \text{supp}(\delta^{-,k})$ in view of (7.9). This implies that either $\nu_i^k = 0$ or $\nu_i^k = -\delta_i^{+,k} G_i(x^k)$ for all $k \in \mathbb{N}$ sufficiently large. However, $i \in \text{supp}(\delta^{+,k})$ gives $i \in I_{GH}^+(x^k; t_k)$ and, therefore, $G_i(x^k) > 0$ by (7.7). This shows that, in any case, we have $\nu_i^k \leq 0$ which, in turn, gives the contradiction $\nu_i^* \leq 0$. □

Note that the previous proof actually shows that a suitable sequence of multipliers remains bounded and, therefore, converges at least on a subsequence under the MPCC-MFCQ condition. The related result in [79], on the other hand, shows convergence of a corresponding sequence of multipliers under the stronger MPCC-LICQ assumption.

7.2.2. Existence of Multipliers

The subsequent result shows that the relaxed problem $R^{LF}(t)$ is in fact less ill-posed with respect to constraint qualifications than the original MPCC (1.1).

Theorem 7.6 *Let x^* be feasible for* (1.1) *such that MPCC-MFCQ (-LICQ) is satisfied at x^*. Then there exists $\bar{t} > 0$ and a neighborhood $U(x^*)$ of x^* such that standard MFCQ (LICQ) for $R^{LF}(t)$ is satisfied at all $x \in U(x^*) \cap X^{LF}(t)$ and for all $t \in (0, \bar{t}]$.*

Proof. The LICQ part is due to [79, Th. 2.3].

Due to MPCC-MFCQ at x^*, in view of [99, Prop. 2.2], one can see that the following set of vectors is positive-linearly independent for all $x \in X^{LF}(t)$ sufficiently close to x^* and t sufficiently close to 0:

$$\begin{array}{rl} \nabla g_i(x) & (i \in I_g(x)), \\ \nabla h_i(x) & (i = 1, \dots, p), \\ G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) & (i \in I_{GH}^+(x;t) \cap (I_{+0}(x^*) \cup I_{0+}(x^*))), \\ \nabla H_i(x) & (i \in I_{GH}^+(x;t) \cap I_{00}(x^*)), \\ \nabla G_i(x) & (i \in I_{GH}^+(x;t) \cap I_{00}(x^*)), \\ (G_i(x) + t)\nabla H_i(x) + (H_i(x) + t)\nabla G_i(x) & (i \in I_{GH}^-(x;t) \cap (I_{+0}(x^*) \cup I_{0+}(x^*))), \\ \nabla H_i(x) & (i \in I_{GH}^-(x;t) \cap I_{00}(x^*)), \\ \nabla G_i(x) & (i \in I_{GH}^-(x;t) \cap I_{00}(x^*)). \end{array} \tag{7.10}$$

We now claim that standard MFCQ holds for the relaxed program $R^{LF}(t)$ whenever $x \in X^{LF}(t) \cap U(x^*)$ for some sufficiently small neighborhood $U(x^*)$ of x^*. To this end, let x be such an element. In view of Lemma 4.7, we have to show that

$$\begin{aligned} 0 \;=\; & \sum_{i \in I_g(x)} \lambda_i \nabla g_i(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) + \sum_{i \in I_{GH}^+(x,t)} \alpha_i (G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x)) \\ & - \sum_{i \in I_{GH}^-(x,t)} \beta_i [(G_i(x) + t)\nabla H_i(x) + (H_i(x) + t)\nabla G_i(x)] \end{aligned}$$

with suitable multipliers $\mu \in \mathbb{R}^p$ and $\lambda, \alpha, \beta \geq 0$ holds only for the zero vector. In order to see this, let us rewrite the above equation as

$$\begin{aligned} 0 \;=\; & \sum_{i \in I_g(x)} \lambda_i \nabla g_i(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) \\ & + \sum_{i \in I_{GH}^+(x;t) \cap (I_{+0}(x^*) \cup I_{0+}(x^*))} \alpha_i (G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x)) \\ & + \sum_{i \in I_{GH}^+(x;t) \cap I_{00}(x^*)} \alpha_i G_i(x)\nabla H_i(x) + \sum_{i \in I_{GH}^+(x;t) \cap I_{00}(x^*)} \alpha_i H_i(x)\nabla G_i(x) \\ & - \sum_{i \in I_{GH}^-(x;t) \cap (I_{+0}(x^*) \cup I_{0+}(x^*))} \beta_i [(G_i(x) + t)\nabla H_i(x) + (H_i(x) + t)\nabla G_i(x)] \\ & - \sum_{i \in I_{GH}^-(x;t) \cap I_{00}(x^*)} \beta_i (G_i(x) + t)\nabla H_i(x) - \sum_{i \in I_{GH}^-(x;t) \cap I_{00}(x^*)} \beta_i (H_i(x) + t)\nabla G_i(x). \end{aligned} \tag{7.11}$$

Now, using the positive-linear independence of the vectors from (7.10) and applying this observation to (7.11), taking into account (7.7) and (7.8), we immediately see that $(\lambda, \mu, \alpha, \beta) = 0$, and this completes the proof. □

Altogether, it follows that the relaxation scheme by Lin and Fukushima has the same theoretical properties as the previous relaxation method by Scholtes.

7.3. The Nonsmooth Relaxation by Kadrani et al.

The approach in [67] proposes the following relaxation, see also Figure 7.3:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad \forall i = 1, \dots, m, \\ & h_j(x) = 0 \quad \forall j = 1, \dots, p, \\ & G_i(x) \geq -t \quad \forall i = 1, \dots, q, \\ & H_i(x) \geq -t \quad \forall i = 1, \dots, q, \\ & (G_i(x) - t)(H_i(x) - t) \leq 0 \quad \forall i = 1, \dots, q. \end{aligned}$$

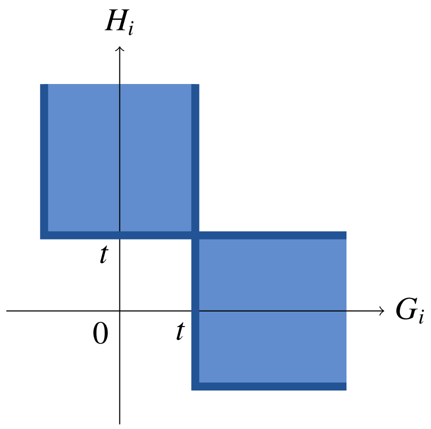


Figure 7.3.: Geometric interpretation of the relaxation method by Kadrani et al.

The relaxed problems are denoted by $R^{KDB}(t)$ and the corresponding feasible set by $X^{KDB}(t)$. We call this a nonsmooth relaxation since in contrast to the previous two approaches the feasible set of the MPCC (1.1) is relaxed but the kink is not smoothed out. As we will see, this has a positive effect on the convergence properties of this relaxation method.

7.3.1. Convergence to M-Stationary Points

For a refined convergence result, we need to define certain index sets. To this end, let x be feasible for $R^{KDB}(t)$. Then we set

$$\begin{array}{rcl} I_g(x) & := & \{i \mid g_i(x) = 0\}, \\ I_G(x;t) & := & \{i \mid G_i(x) + t = 0\}, \\ I_H(x;t) & := & \{i \mid H_i(x) + t = 0\}, \\ I_{GH}(x;t) & := & \{i \mid (G_i(x) - t)(H_i(x) - t) = 0\}, \\ I_{GH}^{0*}(x;t) & := & \{i \in I_{GH}(x;t) \mid G_i(x) - t = 0\}, \\ I_{GH}^{0+}(x;t) & := & \{i \in I_{GH}^{0*}(x;t) \mid H_i(x) - t > 0\}, \\ I_{GH}^{0-}(x;t) & := & \{i \in I_{GH}^{0*}(x;t) \mid H_i(x) - t < 0\}, \\ I_{GH}^{00}(x;t) & := & \{i \in I_{GH}(x;t) \mid G_i(x) - t = H_i(x) - t = 0\}, \\ I_{GH}^{*0}(x;t) & := & \{i \in I_{GH}(x;t) \mid H_i(x) - t = 0\}, \\ I_{GH}^{+0}(x;t) & := & \{i \in I_{GH}^{*0}(x;t) \mid G_i(x) - t > 0\}, \\ I_{GH}^{-0}(x;t) & := & \{i \in I_{GH}^{*0}(x;t) \mid G_i(x) - t < 0\}. \end{array} \tag{7.12}$$

The following is the main convergence result for the relaxation method by Kadrani et al. It generalizes a corresponding result from [67] by replacing the MPCC-LICQ assumption by the much weaker MPCC-CPLD condition.

Theorem 7.7 *Let $\{t_k\} \downarrow 0$ and assume that x^k is a stationary point of $R^{KDB}(t_k)$ for all $k \in \mathbb{N}$. Moreover, suppose that $x^k \to x^*$ such that MPCC-CPLD holds at x^*. Then x^* is an M-stationary point of* (1.1).

Proof. Note that in this proof we skip the standard constraints to keep the notation as compact as possible. The proof can be extended to the case with standard constraints without any serious changes.

Since x^k is a stationary point of $R^{KDB}(t_k)$ for all k, there exist multipliers $\alpha^k, \beta^k, \gamma^k \geq 0$ such that

$$0 = \nabla f(x^k) - \sum_{i=1}^{l} \alpha_i^k \nabla G_i(x^k) - \sum_{i=1}^{q} \beta_i^k \nabla H_i(x^k) + \sum_{i=1}^{q} \delta_i^k [(H_i(x^k) - t_k)\nabla G_i(x^k) + (G_i(x^k) - t_k)\nabla H_i(x^k)]$$

and

$$\alpha_i^k (G_i(x^k) + t_k) = 0, \quad \beta_i^k (H_i(x^k) + t_k) = 0, \quad \delta_i^k (G_i(x^k) - t_k)(H_i(x^k) - t_k) = 0.$$

Now, put

$$\eta_i^{G,k} := -\delta_i^k (H_i(x^k) - t_k), \quad \eta_i^{H,k} := -\delta_i^k (G_i(x^k) - t_k).$$

Then we infer from the equations above that

$$0 = \nabla f(x^k) - \sum_{i=1}^{q} \alpha_i^k \nabla G_i(x^k) - \sum_{i=1}^{q} \beta_i^k \nabla H_i(x^k) - \sum_{i=1}^{q} \eta_i^{G,k} \nabla G_i(x^k) - \sum_{i=1}^{q} \eta_i^{H,k} \nabla H_i(x^k) \tag{7.13}$$

and, for all k sufficiently large,

$$\begin{array}{rcccl} \mathrm{supp}(\alpha^k) & \subseteq & I_G(x^k;t_k) & \subseteq & I_{00}(x^*) \cup I_{0+}(x^*), \\ \mathrm{supp}(\beta^k) & \subseteq & I_H(x^k;t_k) & \subseteq & I_{00}(x^*) \cup I_{+0}(x^*), \\ \mathrm{supp}(\eta^{G,k}) & \subseteq & I_{GH}^{0*}(x^k;t_k) & \subseteq & I_{00}(x^*) \cup I_{0+}(x^*), \\ \mathrm{supp}(\eta^{H,k}) & \subseteq & I_{GH}^{*0}(x^k;t_k) & \subseteq & I_{00}(x^*) \cup I_{+0}(x^*). \end{array} \tag{7.14}$$

Moreover, one sees that

$$\begin{array}{rcl} \mathrm{supp}(\alpha^k) \cap \mathrm{supp}(\eta^{G,k}) & = & \emptyset, \\ \mathrm{supp}(\beta^k) \cap \mathrm{supp}(\eta^{H,k}) & = & \emptyset, \\ \mathrm{supp}(\eta^{G,k}) \cap \mathrm{supp}(\eta^{H,k}) & = & \emptyset. \end{array} \tag{7.15}$$

In addition, we have

$$\begin{array}{rcl} i \in \mathrm{supp}(\eta^{G,k}) \cap \mathrm{supp}(\beta^k) & \Longrightarrow & \eta_i^{G,k} > 0, \\ i \in \mathrm{supp}(\eta^{H,k}) \cap \mathrm{supp}(\alpha^k) & \Longrightarrow & \eta_i^{H,k} > 0. \end{array} \tag{7.16}$$

Without loss of generality, cf. Lemma 7.1, we may assume that the following gradients

$$\left\{\nabla G_i(x^k) \mid \mathrm{supp}(\alpha^k) \cup \mathrm{supp}(\eta^{G,k})\right\} \cup \left\{\nabla H_i(x^k) \mid \mathrm{supp}(\beta^k) \cup \mathrm{supp}(\eta^{H,k})\right\} \tag{7.17}$$

are linearly independent. Now, we want to prove that the sequence $\{(\alpha^k, \beta^k, \eta^{G,k}, \eta^{H,k})\}$ is bounded. For this purpose, we assume the contrary. Nevertheless, we may suppose, without loss of generality, that there is a vector $(\tilde{\alpha}, \tilde{\beta}, \tilde{\eta}^G, \tilde{\eta}^H)$ such that

$$\frac{(\alpha^k, \beta^k, \eta^{G,k}, \eta^{H,k})}{\|(\alpha^k, \beta^k, \eta^{G,k}, \eta^{H,k})\|} \to (\tilde{\alpha}, \tilde{\beta}, \tilde{\eta}^G, \tilde{\eta}^H) \neq 0,$$

and, clearly, for all k (sufficiently large) one has

$$\begin{array}{ll} \mathrm{supp}(\tilde{\alpha}) \subseteq \mathrm{supp}(\alpha^k), & \mathrm{supp}(\tilde{\beta}) \subseteq \mathrm{supp}(\beta^k), \\ \mathrm{supp}(\tilde{\eta}^G) \subseteq \mathrm{supp}(\eta^{G,k}), & \mathrm{supp}(\tilde{\eta}^H) \subseteq \mathrm{supp}(\eta^{H,k}). \end{array} \tag{7.18}$$

By passing to the limit, (7.13) therefore yields

$$0 = \sum_{i=1}^{q} \tilde{\alpha}_i \nabla G_i(x^*) + \sum_{i=1}^{q} \tilde{\beta}_i \nabla H_i(x^*) + \sum_{i=1}^{q} \tilde{\eta}_i^G \nabla G_i(x^*) + \sum_{i=1}^{q} \tilde{\eta}_i^H \nabla H_i(x^*),$$

i.e., the gradients

$$\left\{\nabla G_i(x^*) \mid \mathrm{supp}(\tilde{\alpha}) \cup \mathrm{supp}(\tilde{\eta}^G)\right\} \cup \left\{\nabla H_i(x^*) \mid \mathrm{supp}(\tilde{\beta}) \cup \mathrm{supp}(\tilde{\eta}^H)\right\}$$

are (positive-) linearly dependent. This, in view of MPCC-CPLD, remains true for x^k instead of x^*. But in view of (7.18), this contradicts the linear independence in (7.17). Thus, we can infer that $\{(\alpha^k, \beta^k, \eta^{G,k}, \eta^{H,k})\}$ is bounded, that is, at least on a subsequence, we have

$$(\alpha^k, \beta^k, \eta^{G,k}, \eta^{H,k}) \to (\alpha, \beta, \eta^G, \eta^H)$$

for some vectors $\alpha, \beta, \eta^G, \eta^H$ satisfying

$$\begin{aligned} &\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\alpha^k), && \mathrm{supp}(\beta) \subseteq \mathrm{supp}(\beta^k), \\ &\mathrm{supp}(\eta^G) \subseteq \mathrm{supp}(\eta^{G,k}), && \mathrm{supp}(\eta^H) \subseteq \mathrm{supp}(\eta^{H,k}). \end{aligned} \tag{7.19}$$

Now, for $i = 1, \ldots, l$, put

$$\gamma_i := \begin{cases} \alpha_i, & \text{if } i \in \mathrm{supp}(\alpha), \\ \eta_i^G, & \text{if } i \in \mathrm{supp}(\eta^G), \\ 0, & \text{else}, \end{cases} \quad \text{and} \quad \nu_i := \begin{cases} \beta_i, & \text{if } i \in \mathrm{supp}(\beta), \\ \eta_i^H, & \text{if } i \in \mathrm{supp}(\eta^H), \\ 0, & \text{else}. \end{cases}$$

In view of (7.19) and (7.15), γ and ν are at least well-defined. We now show that x^*, together with the multipliers γ, ν, is an M-stationary point. To this end, first note that (7.13) implies

$$0 = \nabla f(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*).$$

Furthermore, note that, for $i \in I_{+0}(x^*)$, we have $i \notin \mathrm{supp}(\alpha^k) \cup \mathrm{supp}(\eta^{G,k})$ in view of (7.14). Using (7.19), this implies $i \notin \mathrm{supp}(\alpha) \cup \mathrm{supp}(\eta^G)$, hence the definition of γ gives $\gamma_i = 0$. A symmetric argument shows that $\nu_i = 0$ for all $i \in I_{0+}(x^*)$. This means that x^* is at least weakly stationary. Furthermore, if either $\gamma_i = 0$ or $\nu_i = 0$, the M-stationarity conditions are satisfied for such an index i. Consequently, taking into account the definitions of γ_i, ν_i, it remains to consider four cases.

Case 1: $i \in \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$. Then $\gamma_i = \alpha_i \geq 0$ and $\nu_i = \beta_i \geq 0$, so that the M-stationarity conditions hold for such an index.

Case 2: $i \in \mathrm{supp}(\alpha) \cap \mathrm{supp}(\eta^H)$. Then $i \in \mathrm{supp}(\alpha^k) \cap \mathrm{supp}(\eta^{H,k})$ for all $k \in \mathbb{N}$ sufficiently large, cf. (7.19). Hence (7.16) implies $\eta_i^{H,k} > 0$ for all $k \in \mathbb{N}$ sufficiently large which, in turn, gives $\eta_i^H \geq 0$, hence $\nu_i \geq 0$. Furthermore, since $i \in \mathrm{supp}(\eta^{H,k})$, we have $i \notin \mathrm{supp}(\eta^{G,k})$ by (7.15), hence $i \notin \mathrm{supp}(\eta^G)$ by (7.16). This implies $\gamma_i \geq 0$ and shows that the M-stationarity conditions also hold for an index i from Case 2.

Case 3: $i \in \mathrm{supp}(\eta^G) \cap \mathrm{supp}(\beta)$. Here a symmetric reasoning to Case 2 shows that the M-stationarity conditions also hold in this case.

Case 4: $i \in \mathrm{supp}(\eta^G) \cap \mathrm{supp}(\eta^H)$. Then (7.16) implies that $i \in \mathrm{supp}(\eta^{G,k}) \cap \mathrm{supp}(\eta^{H,k})$ for all $k \in \mathbb{N}$ sufficiently large. In view of (7.15), we see that this case cannot occur.

Altogether, this shows that x^* is an M-stationary point. □

7.3.2. Existence of Multipliers

The question regarding the existence of KKT multipliers for the relaxed problems, as needed in the above convergence result, cannot be answered as satisfactory and quickly as for the foregoing approaches. To illustrate this, let us consider the following example.

Example 7.8 (Example 5.8 continued) Consider once again the two-dimensional MPCC

$$\min_{x_1, x_2} f(x) \quad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 = 0. \tag{7.20}$$

This MPCC satisfies MPCC-LICQ, and hence in particular MPCC-CPLD, at any feasible point. Now, choose $t_k := \frac{1}{k}$ and $x^k := (t_k, t_k)$ for $k \in \mathbb{N}$. Then $x^k \to x^* := (0,0)$ and geometric arguments and a quick calculation show that $T_{X^{KDB}(t_k)}(x^k) = \{d \in \mathbb{R}^2 \mid d_1 d_2 \leq 0\} \neq \mathbb{R}^2 = L_{X^{KDB}(t_k)}(x^k)$. Hence, ACQ is violated at x^k, in particular, all stronger concepts like CPLD, MFCQ or LICQ are also violated at x^k. On the other hand, it is easy to see that GCQ is satisfied. ◇

Despite this counterexample, Kadrani et al. were able to verify existence of KKT multipliers for the relaxed problem under the MPCC-LICQ assumption. The following result is a refinement of their observation and partly motivated by Example 7.8.

Theorem 7.9 *Let x^* be feasible for* (1.1) *such that MPCC-LICQ holds at x^*. Then there exists $\bar{t} > 0$ and a neighborhood $U(x^*)$ of x^* such that (standard) GCQ for $R^{KDB}(t)$ is fulfilled at all $\hat{x} \in U(x^*) \cap X^{KDB}(t)$ for all $t \in (0, \bar{t})$.*

Proof. Again, we skip the standard constraints from the proof without loss of generality.

Let $t > 0$ and $\hat{x} \in X^{KDB}(t)$. Furthermore, let $I \subseteq I_{GH}^{00}(\hat{x}; t)$ and put $\bar{I} := I_{GH}^{00}(\hat{x}; t) \setminus I$. Herewith, define the program $NLP(I)$

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & G_i(x) + t \geq 0 \quad (i = 1, \dots, q), \\ & H_i(x) + t \geq 0 \quad (i = 1, \dots, q), \\ & G_i(x) - t \leq 0 \quad (i \in I_{GH}^{0+}(\hat{x}; t) \cup I), \\ & G_i(x) - t \geq 0 \quad (i \in I_{GH}^{0-}(\hat{x}; t) \cup \bar{I}), \\ & H_i(x) - t \leq 0 \quad (i \in I_{GH}^{+0}(\hat{x}; t) \cup \bar{I}), \\ & H_i(x) - t \geq 0 \quad (i \in I_{GH}^{-0}(\hat{x}; t) \cup I), \end{aligned}$$

and denote its feasible set by $\hat{X}(I)$. Then we have $\hat{x} \in \hat{X}(I)$ and, locally around $\hat{x}$, we have $\hat{X}(I) \subseteq X^{KDB}(t)$. We now claim that

$$T_{X^{KDB}(t)}(\hat{x}) = \bigcup_{I \subseteq I_{GH}^{00}(\hat{x}, t)} T_{\hat{X}(I)}(\hat{x}). \tag{7.21}$$

The $\supseteq$-inclusion is obvious. For the converse direction let $d \in T_{X^{KDB}(t)}(\hat{x})$, i.e., there exists $\{x^k\} \subseteq X^{KDB}(t)$ with $x^k \to \hat{x}$ and $\{t_k\} \downarrow 0$ such that $\frac{x^k - \hat{x}}{t_k} \to d$. By continuity, for k sufficiently large, we have

$$\begin{aligned} G_i(x^k) - t \leq 0 \quad & (i \in I_{GH}^{0+}(\hat{x}; t)), \\ G_i(x^k) - t \geq 0 \quad & (i \in I_{GH}^{0-}(\hat{x}; t)), \\ H_i(x^k) - t \leq 0 \quad & (i \in I_{GH}^{+0}(\hat{x}; t)), \\ H_i(x^k) - t \geq 0 \quad & (i \in I_{GH}^{-0}(\hat{x}; t)), \end{aligned}$$

since $x^k \in X^{KDB}(t)$. Moreover, we also have $H_i(x^k) + t \geq 0, G_i(x^k) + t \geq 0$ $(i = 1, \dots, l)$ anyway. Due to the fact that $I_{GH}^{00}(\hat{x}; t)$ is finite, there exists an infinite subset $K \subseteq \mathbb{N}$ and $I \subseteq I_{GH}^{00}(\hat{x}, t)$ such that

$$G_i(x^k) - t \leq 0 \quad (i \in I_{GH}^{0+}(\hat{x}; t) \cup I),$$

$$\begin{aligned} G_i(x^k) - t &\geq 0 \quad (i \in I_{GH}^{0-}(\hat{x};t) \cup \bar{I}), \\ H_i(x^k) - t &\leq 0 \quad (i \in I_{GH}^{+0}(\hat{x};t) \cup \bar{I}), \\ H_i(x^k) - t &\geq 0 \quad (i \in I_{GH}^{-0}(\hat{x};t) \cup I), \end{aligned}$$

for all $k \in K$. Therefore, $\{x^k\}_K \subseteq \hat{X}(I)$ and hence, $d \in T_{\hat{X}(I)}(\hat{x})$, which gives the desired inclusion.

Now, for an arbitrary $I \subseteq I_{GH}^{00}(\hat{x};t)$, the active gradients of $NLP(I)$ at $\hat{x}$ are

$$\begin{aligned} &\nabla G_i(\hat{x}) \quad (i \in I_G(\hat{x};t) \subseteq I_{00}(x^*) \cup I_{0+}(x^*)), \\ &\nabla H_i(\hat{x}) \quad (i \in I_H(\hat{x};t) \subseteq I_{00}(x^*) \cup I_{+0}(x^*)), \\ &\nabla G_i(\hat{x}) \quad (i \in I_{GH}^{0+}(\hat{x};t) \cup I \subseteq I_{00}(x^*) \cup I_{0+}(x^*)), \\ &\nabla G_i(\hat{x}) \quad (i \in I_{GH}^{0-}(\hat{x};t) \cup \bar{I} \subseteq I_{00}(x^*) \cup I_{0+}(x^*)), \\ &\nabla H_i(\hat{x}) \quad (i \in I_{GH}^{+0}(\hat{x};t) \cup \bar{I} \subseteq I_{00}(x^*) \cup I_{+0}(x^*)), \\ &\nabla H_i(\hat{x}) \quad (i \in I_{\Phi}^{-0}(\hat{x};t) \cup I \subseteq I_{00}(x^*) \cup I_{+0}(x^*)), \end{aligned}$$

where for the index set inclusions, in particular, we exploit the fact that $I_{GH}^{00}(\hat{x};t) \subseteq I_{00}(x^*)$.

The above gradients are, in view of MPCC-LICQ at x^*, linearly independent if $\hat{x}$ is sufficiently close to x^*, i.e., LICQ and hence ACQ holds for $NLP(I)$ at $\hat{x}$. This means that $T_{\hat{X}(I)}(\hat{x}) = L_{\hat{X}(I)}(\hat{x})$ and in view of (7.21) and invoking [13, Th. 3.1.9], this yields

$$T_{X^{KDB}(t)}(\hat{x})^\circ = \bigcap_{I \subseteq I_{GH}^{00}(\hat{x};t)} L_{\hat{X}(I)}(\hat{x})^\circ, \tag{7.22}$$

where, by means of Lemma 7.2, we have

$$\begin{aligned} L_{\hat{X}(I)}(\hat{x})^\circ = \{v \in \mathbb{R}^n \mid \exists_{\alpha,\beta,\gamma,\delta,\epsilon,\rho \geq 0} : v \;=\; & - \sum_{i \in I_G(\hat{x};t)} \alpha_i \nabla G_i(\hat{x}) - \sum_{i \in I_H(\hat{x};t)} \beta_i \nabla H_i(\hat{x}) \\ & + \sum_{i \in I_{GH}^{0+}(\hat{x};t) \cup I} \gamma_i \nabla G_i(\hat{x}) - \sum_{i \in I_{GH}^{0-}(\hat{x};t) \cup \bar{I}} \delta_i \nabla G_i(\hat{x}) \\ & + \sum_{i \in I_{GH}^{+0}(\hat{x};t) \cup \bar{I}} \epsilon_i \nabla H_i(\hat{x}) - \sum_{i \in I_{GH}^{-0}(\hat{x};t) \cup I} \rho_i \nabla H_i(\hat{x})\}. \end{aligned}$$

In order to verify GCQ for $R^{KDB}(t)$ at $\hat{x}$, i.e. $T_{X^{KDB}(t)}(\hat{x})^\circ \subseteq L_{X^{KDB}(t)}(\hat{x})^\circ$, let $v \in T_{X^{KDB}(t)}(\hat{x})^\circ$. Using (7.22), it then follows that, for some $I \subseteq I_{GH}^{00}(\hat{x};t)$, we have both $d \in L_{\hat{X}(I)}(\hat{x})^\circ$ and $d \in L_{\hat{X}(\bar{I})}(\hat{x})^\circ$. Then exploiting the representation above and the linear independence of the occurring gradients, it is quickly argued that the multipliers with indices in I and $\bar{I}$ must vanish and hence, v can be expressed as

$$\begin{aligned} v \;=\; & - \sum_{i \in I_G(\hat{x};t)} \alpha_i \nabla G_i(\hat{x}) - \sum_{i \in I_H(\hat{x};t)} \beta_i \nabla H_i(\hat{x}) + \sum_{i \in I_{GH}^{0+}(\hat{x};t)} \gamma_i \nabla G_i(\hat{x}) \\ & - \sum_{i \in I_{GH}^{0-}(\hat{x};t)} \delta_i \nabla G_i(\hat{x}) + \sum_{i \in I_{GH}^{+0}(\hat{x};t)} \epsilon_i \nabla H_i(\hat{x}) - \sum_{i \in I_{GH}^{-0}(\hat{x};t)} \rho_i \nabla H_i(\hat{x}) \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta, \epsilon, \rho \geq 0$, and this means that $v \in L_{X^{KDB}(t)}(\hat{x})^\circ$, again by [13, Th. 3.2.2]. This concludes the proof. □

Although this result and the preceding example are discouraging, we can obtain a better result in most feasible points.

Theorem 7.10 *Let x^* be feasible for the MPCC* (1.1) *such that MPCC-CPLD (MPCC-LICQ) holds at x^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(x^*)$ of x^* such that the following holds for all $t \in (0, \bar{t}]$: If $x \in U(x^*) \cap X^{KDB}(t)$ with $I_{GH}^{00}(x; t) = \emptyset$ then standard CPLD (LICQ) for $R^{KDB}(t)$ holds at x.*

Proof. Note that we only need to prove the CPLD part, since the assertion on LICQ follows from [67, Th. 2.4].

Now, suppose our assertion is false. Then there exist sequences $\{t_k\} \downarrow 0$ and $\{x^k\} \subseteq X^{KDB}(t_k)$ with $x^k \to x^*$ and $I_{GH}^{00}(x^k; t_k) = \emptyset$ such that CPLD for $R^{KDB}(t_k)$ is violated at x^k. This yields subsets $I_1^k \subseteq I_g(x^k), I_2^k \subseteq \{1, \ldots, p\}, I_3^k \subseteq I_G(x^k; t_k), I_4^k \subseteq I_H(x^k; t_k), I_5 \subseteq I_{GH}^{0*}(x^k; t_k), I_6 \subseteq I_{GH}^{*0}(x^k; t_k)$ such that the gradients

$$\begin{aligned}\{\nabla h_i(x) \mid i \in I_2^k\} \quad \cup \quad &\Big\{\{\nabla g_i(x) \mid i \in I_1^k\} \cup \{-\nabla G_i(x) \mid i \in I_3^k\} \cup \{-\nabla H_i(x) \mid i \in I_4^k\} \\ &\cup \{(H_i(x) - t_k)\nabla G_i(x) \mid i \in I_5^k\} \cup \{(G_i(x) - t_k)\nabla H_i(x) \mid i \in I_6^k\}\Big\}\end{aligned}$$

are positive-linearly dependent at $x = x^k$, but linearly independent in x arbitrary close to x^k. Moreover, by a finiteness argument, we can assume without loss of generality that $I_i^k = I_i$ for $i = 1, \ldots, 6$ and all $k \in \mathbb{N}$. Then it is easy to see that $I_1 \subseteq I_g(x^*), I_3 \cup I_5 \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$, and $I_4 \cup I_6 \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$. The positive-linear dependence of the above gradients at x^k immediately implies the positive-linear dependence of the gradients

$$\{\nabla g_i(x) \mid i \in I_1\} \cup \Big\{\{\nabla h_i(x) \mid i \in I_2\} \cup \{\nabla G_i(x) \mid i \in I_3 \cup I_5\} \cup \{\nabla H_i(x) \mid i \in I_4 \cup I_6\}\Big\}.$$

at $x = x^k$. Due to the violation of CPLD at x^k this yields a sequence $\{y^k\} \to x^*$ such that these gradients are linearly independent at $x = y^k$. If they were positive-linearly independent at x^*, by [99, Theorem 2.2], they would remain positive-linearly independent nearby, which contradicts the existence of $\{x^k\}$. On the other hand, if they were positive-linearly dependent, by MPCC-CPLD, they would remain linearly dependent in a whole neighborhood, which contradicts the existence of $\{y^k\}$. This concludes the proof. □

7.4. The Local Relaxation by Steffensen and Ulbrich

The idea behind the subsequent method is as follows: Geometrically, the complementarity conditions are given by the two nonnegative half-axes in the two-dimensional space. If we rotate this set by 45° counterclockwise, we obtain the graph of the absolute value function which is nondifferentiable in the origin. The idea is to approximate this absolute value function locally (say, within the interval $[-1, 1]$ though this will later be scaled to smaller neighborhoods) by a suitable smooth function in such a way that it coincides with the absolute value function outside this local neighborhood. This is made more precise in the following definition going back to [109].

Definition 7.11 *$\theta : [-1,1] \to \mathbb{R}$ is called a* regularization function *if it satisfies the following conditions:*

(a) θ is twice continuously differentiable on $[-1,1]$;

(b) $\theta(-1) = \theta(1) = 1$;

(c) $\theta'(-1) = -1$ and $\theta'(1) = 1$;

(d) $\theta''(-1) = \theta''(1) = 0$;

(e) $\theta''(x) > 0$ for all $x \in (-1,1)$.

Note that condition (e) implies that θ is strictly convex on $[-1,1]$. The following result taken from [109, Lemma 3.1] reveals an immediate but crucial property of all regularization functions.

Lemma 7.12 *Let $\theta : [-1,1] \to \mathbb{R}$ be a regularization function. Then it holds that $\theta(x) > |x|$ for all $x \in (-1,1)$.*

Two simple examples of suitable regularization functions are

$$\theta(x) := \frac{2}{\pi} \sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right) + 1 \quad \text{and} \quad \theta(x) := \frac{1}{8}\left(-x^4 + 6x^2 + 3\right),$$

cf. [109, 119]. The second function is the Hermite interpolation polynomial satisfying the requirements from Definition 7.11.

This idea leads to the following relaxation from [109] (cf. Figure 7.4 for an illustration)

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad \forall i = 1,\dots,m, \\ & h_j(x) = 0 \quad \forall j = 1,\dots,p, \\ & G_i(x) \ge 0 \quad \forall i = 1,\dots,q, \\ & H_i(x) \ge 0 \quad \forall i = 1,\dots,q, \\ & \Phi^{SU}(x;t) \le 0 \quad \forall i = 1,\dots,q \end{array}$$

with

$$\Phi^{SU} : \mathbb{R}^n \to \mathbb{R}^q, \quad \Phi_i^{SU}(x;t) := G_i(x) + H_i(x) - \varphi(G_i(x) - H_i(x);t) \; \forall i = 1,\dots,q$$

where

$$\varphi(\cdot;t) : \mathbb{R} \to \mathbb{R}, \quad \varphi(a;t) := \begin{cases} |a|, & \text{if } |a| \ge t, \\ t\theta(\frac{a}{t}), & \text{if } |a| < t, \end{cases} \tag{7.23}$$

and θ is a regularization function.

Again, we denote the relaxed problem by $R^{SU}(t)$ and the feasible set by $X^{SU}(t)$. Before we state the convergence result for this regularization, let us take a closer look at the relaxed problems.

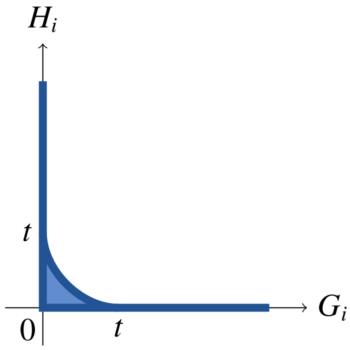


Figure 7.4.: Geometric interpretation of the relaxation method by Steffensen and Ulbrich

7.4.1. Properties of the Relaxed NLPs

First, we are interested in some properties of the function φ defined in (7.23).

Lemma 7.13 *Let φ be defined as in* (7.23). *Then we have*

(a) $\varphi(a;t) > |a|$ *for all* $a \in (-t, t)$ *and for all* $t > 0$*;*

(b) $\varphi(a;t) = |a|$ *for* $|a| \geq t$ *and for all* $t > 0$*;*

(c) $\lim_{t\to 0} \varphi(a;t) = |a|$ *for all* $a \in \mathbb{R}$*;*

(d) $\varphi(\cdot;t)$ *is twice continuously differentiable for all* $t > 0$.

Proof.

(a) Let $a \in (-t, t)$ for some $t > 0$. Then $\frac{|a|}{t} < 1$, and the definition of φ therefore implies

$$\varphi(a;t) = t\theta\left(\frac{a}{t}\right) > t\frac{|a|}{t} = |a|,$$

where the (strict) inequality comes from Lemma 7.12.

(b), (d) These statements follow directly from the definition of θ.

(c) Let $a \in \mathbb{R}$. If $a = 0$, then the boundedness of θ immediately gives

$$\varphi(0;t) = t\theta\left(\frac{a}{t}\right) \to 0 = a \quad \text{for } t \to 0.$$

On the other hand, if $a \neq 0$, we have $|a| \geq t$ for all $t > 0$ sufficiently small. Hence we obtain $\varphi(a;t) = |a|$ for all $t > 0$ sufficiently small, and this gives our assertion also for $a \neq 0$. □

With the aid of $\varphi(\cdot;t)$, we defined the function $\Phi^{SU}(\cdot;t) : \mathbb{R}^n \to \mathbb{R}^q$ by

$$\Phi_i^{SU}(x;t) := G_i(x) + H_i(x) - \varphi(G_i(x) - H_i(x);t) \quad \forall i = 1,\dots,q. \tag{7.24}$$

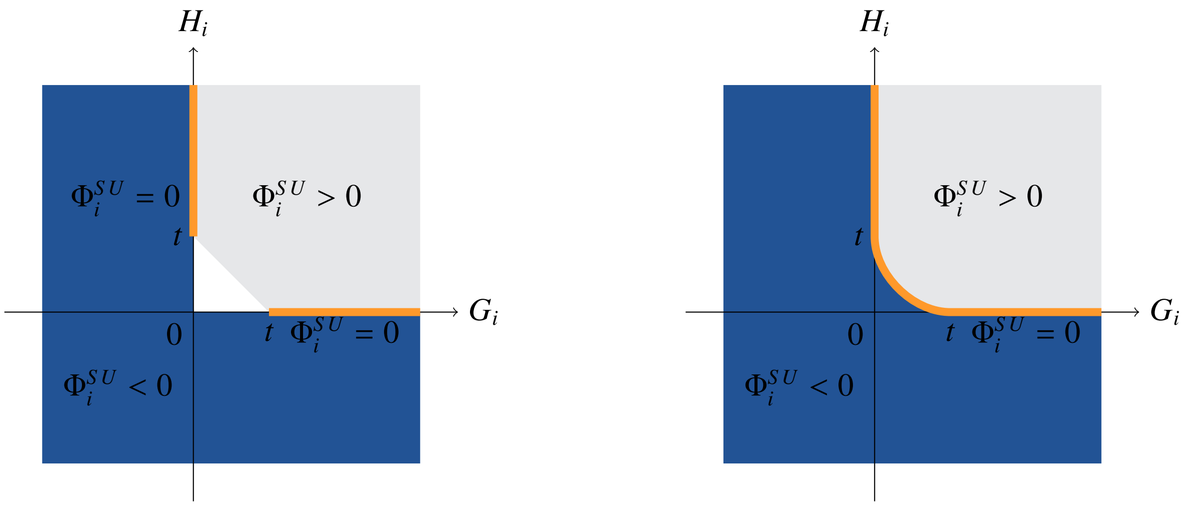
Some useful properties of the function $\Phi^{SU}(\cdot;t)$ are stated in the following results.

Lemma 7.14 *For $t > 0$ let $\Phi^{SU}(\cdot;t)$ be the function given in* (7.24)*. Then $\Phi_i^{SU}(\cdot;t)$ is twice continuously differentiable for all $i = 1,\dots,q$ with gradient*

$$\nabla\Phi_i^{SU}(x;t) = \begin{cases} 2\nabla G_i(x) & \text{if } G_i(x) - H_i(x) \le -t, \\ (1 - \theta'(\frac{G_i(x)-H_i(x)}{t}))\nabla G_i(x) + (1 + \theta'(\frac{x_1-x_2}{t}))\nabla H_i(x) & \text{if } |G_i(x) - H_i(x)| < t, \\ 2\nabla H_i(x) & \text{if } G_i(x) - H_i(x) \ge t. \end{cases}$$

Proof. The proof follows immediately from the definition of Φ^{SU} applying standard calculus rules. □

The following result indicates where the function value of $\Phi_i^{SU}(\cdot;t)$ is positive, negative, or zero depending on the sign of $G_i(x)$ and $H_i(x)$. The statement is also illustrated in Figure 7.5. More precisely, Figure 7.5a gives the sign structure for a function Φ_i^{SU} defined by an arbitrary regularization function θ (note that there is a white triangle in the middle where nothing can be said), whereas Figure 7.5b shows a typical sign structure including this triangle. Note that the precise location of the boundary in the formerly white triangle depends on the chosen regularization function.



(a) arbitrary regularization function θ (b) typical sign structure

Figure 7.5.: Illustration of Lemma 7.15

Lemma 7.15 *Let $\Phi^{SU}(\cdot;t)$ be given by* (7.24). *Then the following holds true for all $i = 1, \dots, q$:*

$$\Phi_i^{SU}(x;t) \begin{cases} < 0, & \text{if } G_i(x) < 0 \quad \text{or} \quad H_i(x) < 0, \\ < 0, & \text{if } G_i(x), H_i(x) \geq 0 \quad \text{and} \quad G_i(x) \cdot H_i(x) = 0 \quad \text{and} \quad |G_i(x) - H_i(x)| < t, \\ = 0, & \text{if } G_i(x), H_i(x) \geq 0 \quad \text{and} \quad G_i(x) \cdot H_i(x) = 0 \quad \text{and} \quad |G_i(x) - H_i(x)| \geq t, \\ > 0, & \text{if } G_i(x), H_i(x) > 0 \quad \text{and} \quad |G_i(x) - H_i(x)| \geq t, \\ \text{free}, & \text{if } G_i(x), H_i(x) > 0 \quad \text{and} \quad |G_i(x) - H_i(x)| < t. \end{cases}$$

Proof. The proof follows immediately from considering the following cases:

i) For $G_i(x), H_i(x) \leq 0$ we have

$$\Phi_i^{SU}(x;t) = \underbrace{G_i(x) + H_i(x)}_{\leq 0} - \underbrace{\varphi(G_i(x) - H_i(x);t)}_{>0 \text{ by Lemma 7.13}} < 0.$$

ii) For $G_i(x) > 0, H_i(x) < 0$ we obtain from Lemma 7.13

$$\Phi_i^{SU}(x;t) = G_i(x)+H_i(x)-\varphi(G_i(x)-H_i(x);t) \leq G_i(x)+H_i(x)-|G_i(x)-H_i(x)| = 2H_i(x) < 0.$$

iii) For $G_i(x) < 0, H_i(x) > 0$ it follows again from Lemma 7.13 that

$$\Phi_i^{SU}(x;t) = G_i(x)+H_i(x)-\varphi(G_i(x)-H_i(x);t) \leq G_i(x)+H_i(x)-|G_i(x)-H_i(x)| = 2G_i(x) < 0.$$

iv) For $G_i(x) > 0, H_i(x) = 0$ we obtain from Lemma 7.13

$$\Phi_i^{SU}(x;t) = G_i(x) - \varphi(G_i(x);t) \begin{cases} = 0 & \text{if } G_i(x) \geq t, \\ < 0 & \text{if } G_i(x) < t. \end{cases}$$

v) For $G_i(x) = 0, H_i(x) > 0$ it holds that

$$\Phi_i^{SU}(x;t) = H_i(x) - \varphi(-H_i(x);t) \begin{cases} = 0 & \text{if } H_i(x) \geq t, \\ < 0 & \text{if } H_i(x) < t. \end{cases}$$

vi) For $G_i(x), H_i(x) > 0$ with $|G_i(x) - H_i(x)| \geq t$ the definition of φ implies

$$\Phi_i^{SU}(x;t) = G_i(x) + H_i(x) - \varphi(G_i(x) - H_i(x);t) = G_i(x) + H_i(x) - |G_i(x) - H_i(x)| > 0.$$

The previous cases also show that, by continuity, the sign of Φ_i^{SU} can be both positive and negative in the remaining case where $G_i(x), H_i(x) > 0$ and $|G_i(x) - H_i(x)| < t$. □

Finally, it is now easy to see that the feasible sets $X^{SU}(t)$ of the relaxed problems have the properties we would expect.

Proposition 7.16 *For the feasible sets X of* (1.1) *and $X^{SU}(t)$ of $R^{SU}(t)$, we have the following relations:*

(a) For $t > 0$, we have $X \subset X^{SU}(t)$;

(b) $X^{SU}(0) = X$;

(c) For $t_1 < t_2$ we have $X^{SU}(t_1) \subset X^{SU}(t_2)$.

Proof. The proof of (a) follows directly from Lemma 7.15. Statement (b) is due to $\Phi_i^{SU}(x; 0) = G_i(x) + H_i(x) - |G_i(x) - H_i(x)|$ for all $x \in \mathbb{R}^n$ and all $i = 1, \ldots, q$, whereas statement (c) can be found in [109, Lemma 3.2]. □

7.4.2. Convergence to C-Stationary Points

The original convergence result from [109] states that, given a convergent sequence $x^k \to x^*$ of stationary points of the relaxed problems $R^{SU}(t_k)$ with $\{t_k\} \downarrow 0$, then the limit point x^* is C-stationary provided that MPCC-CRCQ holds at x^*. Actually, the following – stronger – result holds. However, since its proof is exactly the same as the one of the original result in [109], we do not restate it here. Note that the assertion holds, in particular, under MPCC-MFCQ.

Theorem 7.17 *Let $\{t_k\} \downarrow 0$ and let x^k be a stationary point of $R^{SU}(t_k)$ with $x^k \to x^*$ such that MPCC-CPLD holds in x^*. Then x^* is a C-stationary point of* (1.1).

Although at first glance, this result is not much of an improvement for MPCCs since MPCC-CRCQ and MPCC-CPLD differ only in the conditions on the multipliers corresponding to the inequality constraints, the following example shows that in fact there are simple MPCCs where MPCC-CPLD is the strongest MPCC constraint qualification which is satisfied.

Example 7.18 (Example 5.6 continued) Consider again the two-dimensional MPCC:

$$\min_{x_1,x_2} f(x) = 2x_2 \quad \text{subject to} \quad \begin{aligned} & g_1(x) = x_1 + x_2^2 \le 0, \\ & g_2(x) = x_1 \le 0, \\ & G(x) = x_2 \ge 0, \\ & H(x) = x_1 + x_2 \ge 0, \\ & G(x)H(x) = x_2(x_1 + x_2) = 0. \end{aligned} \tag{7.25}$$

We know already that both MPCC-MFCQ and MPCC-CRCQ are violated in the global minimum $x^* = (0, 0)^T$ whereas MPCC-CPLD holds.

Now consider a sequence of the corresponding relaxed problems $R^{SU}(t_k)$, $t_k \downarrow 0$, where the condition $G(x)H(x) = 0$ is replaced by $\Phi^{SU}(x; t_k) \le 0$. One can easily verify that x^* is also the global minimum of $R^{SU}(t_k)$ and that standard CPLD holds in x^* for all $k \in \mathbb{N}$. Thus, $\{x^k\}$ with $x^k := x^*$ for all $k \in \mathbb{N}$ is a sequence of stationary points of $R^{SU}(t_k)$ that trivially converges to x^*.

Hence, (7.25) is an example for an MPCC where the relaxation method converges although only MPCC-CPLD is satisfied. ◇

The local relaxation scheme discussed in this section also has the advantage that it might not be necessary to drive the relaxation parameter $\{t_k\}$ down to zero under suitable assumptions, in particular, when $G_i(x^*) + H_i(x^*) > 0$ holds for all $i = 1, \ldots, q$. This follows immediately from the observation that, in this case, the feasible sets of the MPCC itself and of the relaxed problem $R^{SU}(t)$ coincide locally.

7.4.3. Existence of Multipliers

On the other hand, the question under which assumptions one may expect to get multipliers for the relaxed problem has not been discussed in [109] and the answer is not trivial as the following simple example illustrates. At this point, I would like to thank my colleague Tim Hoheisel who came up with the results in this section.

Example 7.19 (Example 5.8 continued) Consider again the two-dimensional MPCC

$$\min_{x_1,x_2} f(x) \quad \text{subject to} \quad x_1 \geq 0, x_2 \geq 0, x_1 x_2 = 0.$$

Given a sequence $\{t_k\} \downarrow 0$, we define $\{x^k\}$ by $x^k := (t_k, 0)^T$. Then the active gradients at x^k are $-\binom{0}{1}, \nabla\Phi^{SU}(t_k, 0; t_k) = \binom{0}{2}$, which are obviously positive-linearly dependent. On the other hand, for $\varepsilon > 0$ sufficiently small, the above gradients evaluated at $x^k_\varepsilon := (t_k - \varepsilon, 0)^T$ become $-\binom{0}{1}, \binom{1-\theta'(\frac{t_k-\varepsilon}{t_k})}{1+\theta'(\frac{t_k-\varepsilon}{t_k})}$, which are obviously linearly independent. Hence, CPLD is violated at x^k for all k, although MPCC-LICQ holds at $x^* = (0,0)^T$. However, it is easy to see that ACQ is fulfilled. ◇

In order to prove our result on constraint qualifications, some index sets need to be defined. For these purposes, let $t > 0$ and $x \in X^{SU}(t)$. Then we put

$$\begin{aligned} I_g(x) &:= \{i \mid g_i(x) = 0\}, \\ I_G(x) &:= \{i \mid G_i(x) = 0\}, \\ I_H(x) &:= \{i \mid H_i(x) = 0\}, \\ I_\Phi(x;t) &:= \{i \mid \Phi_i(x;t) = 0\}. \end{aligned}$$

Theorem 7.20 *Let x^* be feasible for* (1.1) *such that MPCC-LICQ holds at x^*. Then there exists a neighborhood $U(x^*)$ of x^* and $\bar{t} > 0$ such that for all $t \in (0, \bar{t})$ and $\hat{x} \in X^{SU}(t) \cap U(x^*)$ (standard) ACQ for $R^{SU}(t)$ is satisfied at $\hat{x}$.*

Proof. Note that, again, we skip the standard constraints from the proof for clarity's sake.

Now, if $I_{0+}(x^*) \cup I_{+0}(x^*) \neq \emptyset$, first, put $\bar{t} := \frac{1}{2}\min\{G_i(x^*)\,(i \in I_{+0}(x^*)), H_i(x^*)\,(i \in I_{0+}(x^*))\}$. Then, in particular, one has $\bar{t} > 0$. Otherwise choose $\bar{t} > 0$ arbitrarily. Now, let $t \in (0, \bar{t})$ and

$\hat{x} \in X^{SU}(t)$. Then we define the program $NLP(\hat{x})$ by

$$
\begin{array}{lll}
\min & f(x) & \\
\text{s.t.} & G_i(x) \geq 0 & (i \notin I_\Phi(\hat{x};t)), \\
& H_i(x) \geq 0 & (i \notin I_\Phi(\hat{x};t)), \\
& \Phi_i(x;t) = 0 & (i \in I_\Phi(\hat{x};t) \cap (I_G(\hat{x}) \cup I_H(\hat{x}))), \\
& \Phi_i(x;t) \leq 0 & (i \in I_\Phi(\hat{x};t) \setminus (I_G(\hat{x}) \cup I_H(\hat{x}))),
\end{array}
$$

and denote its feasible region by $\hat{X}$. Then, clearly, $\hat{x} \in \hat{X}$. Moreover, using Lemma 7.14, the gradients for the active constraints of $R^{SU}(\hat{x})$ at $\hat{x}$ read to

$$
\begin{array}{ll}
\nabla G_i(\hat{x}) & (i \in I_G(\hat{x}) \setminus I_\Phi(\hat{x};t)), \\
\nabla H_i(\hat{x}) & (i \in I_H(\hat{x}) \setminus I_\Phi(\hat{x};t)), \\
2\nabla G_i(\hat{x}) & (i \in I_G(\hat{x}) \cap I_\Phi(\hat{x};t)), \\
2\nabla H_i(\hat{x}) & (i \in I_H(\hat{x}) \cap I_\Phi(\hat{x};t)), \\
\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x}) & (i \in I_\Phi(\hat{x};t) \setminus (I_G(\hat{x}) \cup I_H(\hat{x}))),
\end{array}
\tag{7.26}
$$

where $\alpha_i = 1 - \theta'(\frac{G_i(\hat{x})-H_i(\hat{x})}{t}), \beta_i = 1 + \theta'(\frac{G_i(\hat{x})-H_i(\hat{x})}{t})$. Now, for $\hat{x}$ sufficiently close to x^*, we have the inclusions

$$
I_G(\hat{x}) \subseteq I_{00}(x^*) \cup I_{0+}(x^*), \quad I_H(\hat{x}) \subseteq I_{00}(x^*) \cup I_{+0}(x^*),
$$

and by the choice of $\bar{t}$, we also get

$$
I_\Phi(\hat{x};t) \setminus \{I_G(\hat{x}) \cup I_H(\hat{x})\} \subseteq I_{00}(x^*).
$$

Hence, in view of MPCC-LICQ at x^*, the gradients in (7.26) are linearly independent for $\hat{x}$ sufficiently close to x^*, and thus, LICQ and in particular ACQ for $R^{SU}(\hat{x})$ holds at $\hat{x}$.

Moreover, we have

$$
\begin{array}{rcll}
L_{\hat{X}}(\hat{x}) & = & \Big\{ d \in \mathbb{R}^n \mid & \nabla G_i(\hat{x})^T d \geq 0 \quad (i \in I_G(\hat{x}) \setminus I_\Phi(\hat{x};t)), \\
& & & \nabla H_i(\hat{x})^T d \geq 0 \quad (i \in I_H(\hat{x}) \setminus I_\Phi(\hat{x};t)), \\
& & & \nabla G_i(\hat{x})^T d = 0 \quad (i \in I_G(\hat{x}) \cap I_\Phi(\hat{x};t)), \\
& & & \nabla H_i(\hat{x})^T d = 0 \quad (i \in I_H(\hat{x}) \cap I_\Phi(\hat{x};t)), \\
& & & \nabla(\Phi_i(\hat{x};t))^T d \leq 0 \quad (i \in I_\Phi(\hat{x},t) \setminus \{I_G(\hat{x}) \cup I_H(\hat{x})\}) \Big\} \\
& = & L_{X^{SU}(t)}(\hat{x}), &
\end{array}
\tag{7.27}
$$

where the last equality can easily be verified by direct calculation. We now want to show that, locally around $\hat{x}$, we have $\hat{X} \subseteq X^{SU}(t)$.

For these purposes, it remains to see that, for $x \in \hat{X}$ sufficiently close to $\hat{x}$, we have

$$
G_i(x) \geq 0 \ (i \in I_\Phi(\hat{x};t)) \quad \text{and} \quad H_i(x) \geq 0 \ (i \in I_\Phi(\hat{x};t)).
$$

To this end, consider first the case of $i \in I_\Phi(\hat{x};t) \cap \{I_G(\hat{x}) \cup I_H(\hat{x})\}$. Then, in particular, $\Phi_i(x;t) = 0$, which in view of Lemma 7.15 gives $G_i(x), H_i(x) \geq 0$. If otherwise $i \notin I_G(\hat{x}) \cup I_H(\hat{x})$, we get $G_i(x), H_i(x) > 0$ by continuity.

The local inclusion $\hat{X} \subseteq X^{SU}(t)$ yields $T_{\hat{X}}(\hat{x}) \subseteq T_{X^{SU}(t)}(\hat{x})$ and hence, by ACQ for $R^{SU}(\hat{x})$ at $\hat{x}$ and (7.27), we have

$$L_{\hat{X}}(\hat{x}) = T_{\hat{X}}(\hat{x}) \subseteq T_{X^{SU}(t)}(\hat{x}) \subseteq L_{X^{SU}(t)}(\hat{x}) = L_{\hat{X}}(\hat{x}),$$

which gives the assertion. □

In some feasible points, we get a stronger result.

Theorem 7.21 *Let x^* be feasible for* (1.1) *such that MPCC-LICQ holds at x^*. Then there exists a neighborhood $U(x^*)$ of x^* and a $\bar{t} > 0$ such that the following holds: If $t \in (0, \bar{t}]$ and $x \in U(x^*) \cap X^{SU}(t)$ with $I_\Phi(x; t) \cap \{I_G(x) \cup I_H(x)\} = \emptyset$ then standard LICQ holds for $R^{SU}(t)$ at x.*

Proof. Let $x \in X^{SU}(t)$. Then, if $I_\Phi(x; t) \cap \{I_G(x) \cup I_H(x)\} = \emptyset$, the active gradients for $R^{SU}(t)$ at x are,

$$\begin{array}{ll}
\nabla g_i(x) & (i \in I_g(x)), \\
\nabla h_i(x) & (i = 1, \dots, p), \\
\nabla G_i(x) & (i \in I_G(x)), \\
\nabla H_i(x) & (i \in I_H(x)), \\
\alpha_i \nabla G_i(x) + \beta_i \nabla H_i(x) & (i \in I_\Phi(x; t) \setminus \{I_G(x) \cup I_H(x)\}),
\end{array}$$

where $\alpha_i = 1 - \theta'(\frac{G_i(x)-H_i(x)}{t}), \beta_i = 1 + \theta'(\frac{G_i(x)-H_i(x)}{t})$. Moreover, cf. also the proof of Theorem 7.20, for x sufficiently close to x^* and t sufficiently close to 0, we have the inclusions

$$\begin{aligned}
I_g(x) &\subseteq I_g(x^*), \\
I_G(x) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\
I_H(x) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*), \\
I_\Phi(x; t) \setminus \{I_G(x) \cup I_H(x)\} &\subseteq I_{00}(x^*),
\end{aligned}$$

and in view of MPCC-LICQ the active gradients from above are linearly independent, i.e., LICQ holds at x. □

8. A New Relaxation Method

Among the relaxation methods analyzed above, the one with the by far best convergence properties is the relaxation by Kadrani et al. from Section 7.3. However, a closer look at the feasible set of the corresponding relaxed problems reveals two potential drawbacks. First, the feasible set of the relaxed problem is almost disconnected, so one has to expect severe problems when solving them by a standard optimization method. Kadrani et al. therefore propose a combination of their relaxation with a penalty approach and an active set algorithm. In our opinion however, one of the main advantages of relaxation methods is the possibility to employ the existing and highly efficient solvers for standard nonlinear problems rather effortless to solve MPCCs as well. Therefore, we prefer a relaxation method where it is not necessary to develop a special solver for the relaxed problems. The second disadvantage of the method by Kadrani et al. is the fact that the feasible set of the MPCC is not contained in the feasible set of the relaxed problems for any positive relaxation parameter. As we will see in Section 9.2, this is a problem when we have to decide upon a stopping criterion for the corresponding algorithm.

Inspired by the strengths and weaknesses of the relaxation method by Kadrani et al. we developed a new relaxation for MPCCs with similar convergence properties, but without the problems mentioned above. The disadvantage of the relaxation methods by Scholtes, Lin and Fukushima, and Steffensen and Ulbrich is that only C-stationarity of limit points can be guaranteed. The following well-known example however illustrates that there are MPCCs with C-stationary points that attract these relaxation methods but which are not local minima.

Example 8.1 Consider the following 2-dimensional MPCC

$$\min_{x_1,x_2} f(x) = \frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-1)^2 \quad \text{subject to} \quad \begin{aligned} G(x) &= x_1 \geq 0, \\ H(x) &= x_2 \geq 0, \\ G(x)H(x) &= x_1 x_2 = 0. \end{aligned}$$

This optimization problem has two global minima in $(0,1)^T$ and $(1,0)^T$, which are S-stationary, and a C-stationary point in $(0,0)^T$. Now it is easy to verify that the relaxed problems corresponding to the methods by Scholtes, Lin and Fukushima, and Steffensen and Ulbrich have a stationary point at the intersection of the line, where $x_1 = x_2$, with the north-eastern border of the relaxed feasible set, hence there exists a sequence of stationary points converging to the C-stationary point in the origin which is not a solution of the MPCC. The relaxed feasible sets of the scheme by Kadrani et al. however only have stationary points in $(t,1)^T$ and $(1,t)^T$ which are converging to the global minima of the MPCC. ◇

Another problem with some of these relaxations is that they seem to be quite sensitive to second-order conditions as the following example from [67] illustrates.

Example 8.2 Consider again the 2-dimensional MPCC from above, however this time with a different objective function:

$$\min_{x_1,x_2} f(x) = x_2 \quad \text{subject to} \quad \begin{aligned} G(x) &= x_1 \geq 0, \\ H(x) &= x_2 \geq 0, \\ G(x)H(x) &= x_1 x_2 = 0. \end{aligned}$$

This optimization problem has a continuum of local minima in $(x_1, 0)^T$, $x_1 > 0$, which are S-stationary, and an M-stationary point in $(0, 0)^T$. The relaxed problems corresponding to the methods by Scholtes and Lin and Fukushima, however, do not have a stationary point at all. The relaxation scheme by Steffensen and Ulbrich benefits from that fact that the feasible set X of the MPCC is only locally relaxed around the origin and thus the relaxed problems have stationary points in $(x_1, 0)^T$ for all $x_1 \geq t$. Similarly, the relaxed feasible sets of the scheme by Kadrani et al. have stationary points in $(x_1, t)^T$ for all $x_1 > t$. ◇

We will introduce a new relaxation scheme in this chapter which has the same positive properties as the one by Kadrani et al., that is convergence to M-stationary points under MPCC-CPLD and convergence to S-stationary points under an additional nondegeneracy condition without any second order condition. Additionally, we will prove that the relaxed problems inherit CPLD from the original MPCC almost everywhere and prove the existence of Lagrange multipliers also for the remaining points. Finally, we will provide a condition under which one can find local minima of the relaxed problems in a neighborhod of a solution of the MPCC. The advantage of our method compared to the one by Kadrani et al. is the shape of its feasible set. We do not have the problem that the feasible set is almost disconnected and additionally the feasible set of the original MPCC is contained in all relaxed feasible sets. This is an advantage when it comes to the numerical realization of the relaxation method since we may terminate the algorithm early when the solution of one of the relaxed problems is already feasible for the MPCC.

8.1. Properties of the Relaxed NLPs

Our relaxation is based on the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(a, b) = \begin{cases} ab, & \text{if } a + b \geq 0, \\ -\frac{1}{2}(a^2 + b^2), & \text{if } a + b < 0. \end{cases}$$

This function has the following elementary properties.

Lemma 8.3 *(a) φ is an NCP-function, i.e. $\varphi(a, b) = 0$ if and only if $a \geq 0, b \geq 0, ab = 0$.*

(b) φ is continuously differentiable with gradient

$$\nabla\varphi(a, b) = \begin{cases} \begin{pmatrix} b \\ a \end{pmatrix}, & \text{if } a + b \geq 0, \\ \begin{pmatrix} -a \\ -b \end{pmatrix}, & \text{if } a + b < 0. \end{cases}$$

(c) φ has the property that

$$\varphi(a,b)\begin{cases} > 0, & \text{if } a > 0 \text{ and } b > 0, \\ < 0, & \text{if } a < 0 \text{ or } b < 0. \end{cases}$$

Proof.

(a) First suppose that $a \geq 0, b \geq 0, ab = 0$. Then $a + b \geq 0$, and the definition of φ therefore gives $\varphi(a,b) = ab = 0$. Conversely, assume that $\varphi(a,b) = 0$. If $a + b \geq 0$, it then follows that $ab = 0$ and thus $a \geq 0, b = 0$ or $a = 0, b \geq 0$. On the other hand, if $a + b < 0$, we have $-\frac{1}{2}(a^2 + b^2) = 0$ which, in turn, implies $a = b = 0$, a contradiction to $a + b < 0$.

(b) This statement follows immediately from standard calculus rules.

(c) Using the continuity of φ together with the NCP-function property of part (a), it follows that φ has the same sign in all points of the positive orthant, as well as the same sign in all points in the other three orthants. Since $\varphi(1,1) = 1 > 0$ and $\varphi(-1,-1) = -1 < 0$, the statement follows. □

Based on this function, we define a continuously differentiable mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^q$ componentwise by

$$\begin{aligned} \Phi_i(x;t) &:= \varphi(G_i(x) - t, H_i(x) - t) \\ &= \begin{cases} (G_i(x) - t)(H_i(x) - t), & \text{if } G_i(x) + H_i(x) \geq 2t, \\ -\frac{1}{2}((G_i(x) - t)^2 + (H_i(x) - t)^2), & \text{if } G_i(x) + H_i(x) < 2t, \end{cases} \end{aligned}$$

where $t \geq 0$ is an arbitrary parameter. With this function, we can formulate the *relaxed or regularized problem $R^{KS}(t)$* for $t \geq 0$ as

$$\begin{aligned} \min f(x) \quad \text{subject to} \quad & g_i(x) \leq 0 \quad \forall i = 1,\dots,m, \\ & h_i(x) = 0 \quad \forall i = 1,\dots,p, \\ & G_i(x) \geq 0 \quad \forall i = 1,\dots,q, \\ & H_i(x) \geq 0 \quad \forall i = 1,\dots,q, \\ & \Phi_i(x;t) \leq 0 \quad \forall i = 1,\dots,q. \end{aligned} \tag{8.1}$$

Hence, in our approach, we replace the complementarity conditions

$$G_i(x) \geq 0,\ H_i(x) \geq 0,\ G_i(x)H_i(x) = 0 \quad \forall i = 1,\dots,q$$

by the conditions

$$G_i(x) \geq 0,\ H_i(x) \geq 0,\ \Phi_i(x;t) \leq 0 \quad \forall i = 1,\dots,q$$

which, from a geometric point of view, gives a set of the form shown in Figure 8.1.
Similar to the index sets used for MPCCs before, we define

$$I_g(x) \quad := \quad \{i \mid g_i(x) = 0\},$$

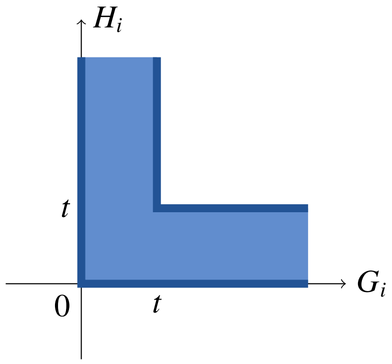


Figure 8.1.: Geometric interpretation of the new regularization

$$
\begin{aligned}
I_G(x) &:= \{i \mid G_i(x) = 0\},\\
I_H(x) &:= \{i \mid H_i(x) = 0\},\\
I_\Phi(x;t) &:= \{i \mid \Phi_i(x;t) = 0\}
\end{aligned}
$$

for $t \geq 0$ and x feasible for $R^{KS}(t)$. We also use a partition of the index set $I_\Phi(x;t)$ into the following three subsets:

$$
\begin{aligned}
I_\Phi^{00}(x;t) &:= \{i \in I_\Phi(x;t) \mid G_i(x) - t = 0, H_i(x) - t = 0\},\\
I_\Phi^{0+}(x;t) &:= \{i \in I_\Phi(x;t) \mid G_i(x) - t = 0, H_i(x) - t > 0\},\\
I_\Phi^{+0}(x;t) &:= \{i \in I_\Phi(x;t) \mid G_i(x) - t > 0, H_i(x) - t = 0\}.
\end{aligned}
$$

Note that these sets form a partition of $I_\Phi(x;t)$ since the definition of Φ implies that

$$\Phi_i(x;t) = 0 \quad \Longleftrightarrow \quad G_i(x) - t \geq 0,\; H_i(x) - t \geq 0,\; (G_i(x) - t)(H_i(x) - t) = 0.$$

In view of Lemma 8.3, the function Φ is continuously differentiable with its gradient given by

$$\nabla\Phi_i(x;t) = \begin{cases} (H_i(x) - t)\nabla G_i(x) + (G_i(x) - t)\nabla H_i(x), & \text{if } G_i(x) + H_i(x) \geq 2t,\\ -(G_i(x) - t)\nabla G_i(x) - (H_i(x) - t)\nabla H_i(x), & \text{if } G_i(x) + H_i(x) < 2t \end{cases} \tag{8.2}$$

for all $i = 1, \ldots, q$.

The following result summarizes some simple properties of the regularized program $R^{KS}(t)$.

Lemma 8.4 *For $t > 0$ let X and $X^{KS}(t)$ be the feasible sets of the MPCC* (1.1) *and $R^{KS}(t)$, respectively. Then the following three statements hold:*

(a) $X^{KS}(0) = X$.

(b) $X^{KS}(t_1) \subseteq X^{KS}(t_2)$ *for all* $0 \leq t_1 \leq t_2$.

(c) $\bigcap_{t\geq 0} X^{KS}(t) = X$.

Proof.

(a) Taking into account the properties of φ and the definition of Φ, the complementarity conditions

$$G_i(x) \geq 0,\ H_i(x) \geq 0,\ G_i(x)H_i(x) = 0 \quad \forall i = 1, \dots, q$$

are equivalent to

$$G_i(x) \geq 0,\ H_i(x) \geq 0,\ \Phi_i(x; 0) \leq 0$$

for all $i = 1, \dots, q$. This proves $X^{KS}(0) = X$.

(b) Let $0 \leq t_1 \leq t_2$ and x be an arbitrary element of $X^{KS}(t_1)$. To prove $x \in X^{KS}(t_2)$, we only have to show $\Phi_i(x; t_2) \leq 0$ for all $i = 1, \dots, q$. Let i be one of these indices. If $G_i(x)+H_i(x) < 2t_2$, we immediately obtain $\Phi_i(x; t_2) \leq 0$ since $\Phi_i(x; t)$ is always nonpositive in this case. Hence, the only case to consider is $G_i(x) + H_i(x) \geq 2t_2$. We want to prove $\Phi_i(x; t_2) = (G_i(x) - t_2)(H_i(x) - t_2) \leq 0$. Assume this is not true. Then either both values $G_i(x) - t_2$ and $H_i(x) - t_2$ would have to be positive or both negative. However, if both values were negative, we would have $G_i(x) + H_i(x) < 2t_2$, a contradiction. If both values were positive, $G_i(x) - t_1$ and $H_i(x) - t_1$ also were both positive and thus $\Phi_i(x; t_1) > 0$, a contradiction to $x \in X^{KS}(t_1)$.

(c) According to part (a) and (b), we know $X = X^{KS}(0) \subseteq X^{KS}(t)$ for all $t \geq 0$ and thus $X \subseteq \bigcap_{t \geq 0} X^{KS}(t)$. Now let $x \in \bigcap_{t \geq 0} X^{KS}(t)$ be an arbitrary element. To prove $x \in X$, we only have to show $\Phi_i(x; 0) \leq 0$ for all $i = 1, \dots, q$. Assume that there is an i such that $\Phi_i(x; 0) > 0$. This implies (in fact is equivalent to) $G_i(x) > 0$ and $H_i(x) > 0$. Now choose an arbitrary $\bar{t} > 0$ with $\bar{t} < \min\{G_i(x), H_i(x)\}$. This definition of $\bar{t}$ yields $G_i(x) + H_i(x) > 2\bar{t}$ and thus $\Phi_i(x; \bar{t}) = (G_i(x) - \bar{t})(H_i(x) - \bar{t}) > 0$. Consequently, $x \notin X^{KS}(\bar{t})$ which is a contradiction to $x \in \bigcap_{t \geq 0} X^{KS}(t)$. □

The previous result shows, in particular, that the feasible set X of the original MPCC is always contained in the feasible set $X^{KS}(t)$ of the regularized program $R^{KS}(t)$ (in contrast to the approach by Kadrani et al., and that our relaxation exhibits the desired behavior $\lim_{t \downarrow 0} X^{KS}(t) = X$. Note also that, from a geometric point of view, our regularized problem has a much nicer feasible set than the one by Kadrani et al. which, we recall, consists of almost disconnected pieces.

Remark 8.5 (a) The particular NCP-function φ used here can be replaced by other suitable NCP-functions. However, we stress that we cannot use an arbitrary NCP-function that, geometrically, gives the same feasible set for the regularized problem $R^{KS}(t)$ since the stationary point properties that will be shown in the subsequent section heavily depend on the particular representation of this feasible set. Nevertheless, one particular alternative is the mapping $\varphi(a, b) := \theta(a) + \theta(b) - \theta(|a - b|)$ with $\theta : \mathbb{R} \to \mathbb{R}$ given by

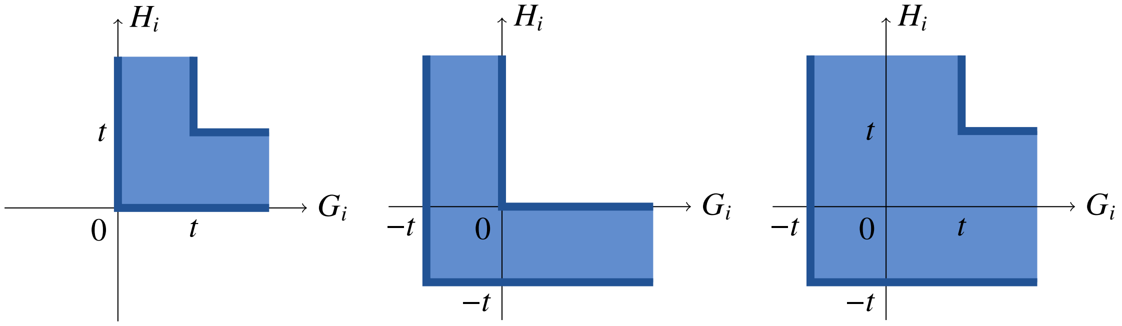
$$\theta(\tau) := \begin{cases} -\frac{1}{2}\tau^2, & \text{if } \tau < 0, \\ \frac{1}{2}\tau^2, & \text{if } \tau \geq 0 \end{cases} \qquad \text{or} \qquad \theta(\tau) = \frac{1}{3}\tau^3.$$

This function is a particular member of a class of NCP-functions introduced in [85]. It is not difficult to see that our analysis goes through also for this mapping.

(b) The regularization used in this thesis enlarges the feasible region coming from the complementarity constraints to the north-eastern direction. Alternatively, we may also use a regularization to the south-western direction by replacing the complementarity conditions by

$$G_i(x) \geq -t, \; H_i(x) \geq -t, \; \Phi_i(x;0) \leq 0 \quad \forall i = 1, \dots, q$$

We may also combine the two relaxations and regularize with respect to the north-eastern and the south-western direction simultaneously. Figure 8.2 illustrates the three possible regularizations.



(a) north-eastern direction (b) south-western direction (c) both directions combined

Figure 8.2.: The three possible relaxations

8.2. Convergence Properties

8.2.1. Convergence to M- and S-Stationary Points

In this section, we are concerned with stationarity properties of limit points of our relaxation method. If we solve $R^{KS}(t_k)$ for a sequence $\{t_k\} \downarrow 0$ and obtain KKT points $(x^k, \lambda^k, \mu^k, \gamma^k, \nu^k, \delta^k)$ of $R^{KS}(t_k)$, where $x^k \to x^*$, what kind of MPCC-stationarity can we expect in x^*? The next theorem gives an answer to this question.

Theorem 8.6 *Let $\{t_k\} \downarrow 0$ and $\{(x^k, \lambda^k, \mu^k, \gamma^k, \nu^k, \delta^k)\}$ be a sequence of KKT points of $R^{KS}(t_k)$ with $x^k \to x^*$. If MPCC-CPLD holds in x^*, then x^* is an M-stationary point of the MPCC* (1.1).

Proof. Obviously, x^* is feasible for the MPCC (1.1) and for all $k \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned} I_g(x^k) &\subseteq I_g(x^*), \\ I_G(x^k) \cup I_\Phi^{00}(x^k;t_k) \cup I_\Phi^{0+}(x^k;t_k) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\ I_H(x^k) \cup I_\Phi^{00}(x^k;t_k) \cup I_\Phi^{+0}(x^k;t_k) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*). \end{aligned} \tag{8.3}$$

Since all $(x^k, \lambda^k, \mu^k, \gamma^k, \nu^k, \delta^k)$ are KKT points of $R^{KS}(t_k)$, we have

$$\begin{aligned} 0 &= \nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(x^k) - \sum_{i=1}^{q} \gamma_i^k \nabla G_i(x^k) - \sum_{i=1}^{q} \nu_i^k \nabla H_i(x^k) \\ &\quad + \sum_{i=1}^{q} \delta_i^k \nabla \Phi_i(x^k; t_k) \end{aligned}$$

with

$$\begin{aligned} \lambda_i^k &= 0 \; \forall i \notin I_g(x^k) \quad \text{and} \quad \lambda_i^k \geq 0 \; \forall i \in I_g(x^k), \\ \gamma_i^k &= 0 \; \forall i \notin I_G(x^k) \quad \text{and} \quad \gamma_i^k \geq 0 \; \forall i \in I_G(x^k), \\ \nu_i^k &= 0 \; \forall i \notin I_H(x^k) \quad \text{and} \quad \nu_i^k \geq 0 \; \forall i \in I_H(x^k), \\ \delta_i^k &= 0 \; \forall i \notin I_\Phi(x^k; t) \quad \text{and} \quad \delta_i^k \geq 0 \; \forall i \in I_\Phi(x^k; t_k). \end{aligned}$$

Since the representation of $\nabla\Phi_i$ immediately gives $\nabla\Phi_i(x^k; t_k) = 0$ for all $i \in I_\Phi^{00}(x^k; t_k)$ and all $k \in \mathbb{N}$, we may also assume $\delta_i^k = 0$ for all $i \in I_\Phi^{00}(x^k; t_k)$ and all $k \in \mathbb{N}$. Thus, we can rewrite the equation above as

$$\begin{aligned} 0 &= \nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(x^k) - \sum_{i=1}^{q} \gamma_i^k \nabla G_i(x^k) - \sum_{i=1}^{q} \nu_i^k \nabla H_i(x^k) \\ &\quad + \sum_{i=1}^{q} \delta_i^{G,k} \nabla G_i(x^k) + \sum_{i=1}^{q} \delta_i^{H,k} \nabla H_i(x^k) \end{aligned}$$

where

$$\begin{aligned} \delta_i^{G,k} &= \begin{cases} \delta_i^k (H_i(x^k) - t_k), & \text{if } i \in I_\Phi^{0+}(x^k; t_k), \\ 0, & \text{else,} \end{cases} \\ \delta_i^{H,k} &= \begin{cases} \delta_i^k (G_i(x^k) - t_k), & \text{if } i \in I_\Phi^{+0}(x^k; t_k), \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Note that the multipliers $\delta^{G,k}$ and $\delta^{H,k}$ are nonnegative, too. According to Lemma 7.1, we may assume without loss of generality that the gradients corresponding to nonvanishing multipliers in this equation are linearly independent for all $k \in \mathbb{N}$ (note that this may change the multipliers, but a previously positive multiplier will stay at least nonnegative and a vanishing multiplier will remain zero).

Our next step is to prove that the sequence $\{(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})\}$ is bounded. Assuming the contrary, we can find a subsequence K such that

$$\frac{(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})}{\|(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})\|} \to_K (\lambda, \mu, \gamma, \nu, \delta^G, \delta^H) \neq 0.$$

Dividing by $\|(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})\|$ and passing to the limit in the equation above yields

$$\begin{aligned} 0 \quad = \quad & \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \gamma_i \nabla G_i(x^*) - \sum_{i=1}^{q} \nu_i \nabla H_i(x^*) \\ & + \sum_{i=1}^{q} \delta_i^G \nabla G_i(x^*) + \sum_{i=1}^{q} \delta_i^H \nabla H_i(x^*), \end{aligned}$$

i.e., the gradients

$$\begin{aligned} & \{\nabla g_i(x^*) \mid i \in \operatorname{supp}(\lambda)\} \cup \{\{\nabla h_i(x^*) \mid i \in \operatorname{supp}(\mu)\} \\ & \cup \{\nabla G_i(x^*) \mid i \in \operatorname{supp}(\gamma) \cup \operatorname{supp}(\delta^G)\} \cup \{\nabla H_i(x^*) \mid i \in \operatorname{supp}(\nu) \cup \operatorname{supp}(\delta^H)\}\} \end{aligned} \tag{8.4}$$

are positive-linearly dependent. MPCC-CPLD guarantees that they remain linearly dependent in a whole neighborhood. This, however, is a contradiction to the linear independence of these gradients in x^k. Here, we use

$$\operatorname{supp}(\lambda, \mu, \gamma, \nu, \delta^G, \delta^H) \subseteq \operatorname{supp}(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})$$

for all k sufficiently large and (8.3).

Consequently, our assumption was wrong and thus the sequence $\{(\lambda^k, \mu^k, \gamma^k, \nu^k, \delta^{G,k}, \delta^{H,k})\}$ is bounded. Therefore, we can assume without loss of generality that the whole sequence is convergent to some limit $(\lambda^*, \mu^*, \gamma^*, \nu^*, \delta^{G,*}, \delta^{H,*})$. Since $I_G(x^k) \cap I_\Phi^{0+}(x^k; t_k) = \emptyset$ and $I_H(x^k) \cap I_\Phi^{+0}(x^k; t_k) = \emptyset$ for all $k \in \mathbb{N}$, it is easy to see that the multipliers

$$\hat{\gamma}_i = \begin{cases} \gamma_i^* & \text{if } i \in \operatorname{supp}(\gamma^*), \\ -\delta_i^{G,*} & \text{if } i \in \operatorname{supp}(\delta^{G,*}), \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \hat{\nu}_i = \begin{cases} \nu_i^* & \text{if } i \in \operatorname{supp}(\nu^*), \\ -\delta_i^{H,*} & \text{if } i \in \operatorname{supp}(\delta^{H,*}), \\ 0 & \text{else} \end{cases}$$

are well-defined, and we obtain

$$0 \quad = \quad \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i^* \nabla h_i(x^*) - \sum_{i=1}^{q} \hat{\gamma}_i \nabla G_i(x^*) - \sum_{i=1}^{q} \hat{\nu}_i \nabla H_i(x^*).$$

Here, $\lambda^* \geq 0$ and

$$\begin{aligned} \operatorname{supp}(\lambda^*) \quad & \subseteq \quad I_g(x^k) \subseteq I_g(x^*), \\ \operatorname{supp}(\hat{\gamma}) \quad & = \quad \operatorname{supp}(\gamma^*) \cup \operatorname{supp}(\delta^{G,*}) \ \subseteq \ I_G(x^k) \cup I_\Phi^{0+}(x^k; t_k) \ \subseteq \ I_{00}(x^*) \cup I_{0+}(x^*), \\ \operatorname{supp}(\hat{\nu}) \quad & = \quad \operatorname{supp}(\nu^*) \cup \operatorname{supp}(\delta^{H,*}) \ \subseteq \ I_H(x^k) \cup I_\Phi^{+0}(x^k; t_k) \ \subseteq \ I_{00}(x^*) \cup I_{+0}(x^*) \end{aligned}$$

for all k sufficiently large. Consequently, we have $\hat{\gamma}_i = 0$ for all $i \in I_{+0}(x^*)$ and $\hat{\nu}_i = 0$ for all $i \in I_{0+}(x^*)$, i.e., $(x^*, \lambda^*, \mu^*, \hat{\gamma}, \hat{\nu})$ is at least a weakly stationary point of the MPCC (1.1). To prove M-stationarity, assume that there is an $i \in I_{00}(x^*)$ with $\hat{\gamma}_i < 0$ and $\hat{\nu}_i \neq 0$ (the case $\hat{\gamma}_i \neq 0$ and $\hat{\nu}_i < 0$ can be treated in a symmetric way). The condition $\hat{\gamma}_i < 0$ implies $i \in \operatorname{supp}(\delta^{G,*}) \subseteq I_\Phi^{0+}(x^k; t_k)$ for all k sufficiently large. Because of

$$I_\Phi^{0+}(x^k; t_k) \cap \left(I_H(x^k) \cup I_\Phi^{+0}(x^k; t_k)\right) = \emptyset$$

for all $k \in \mathbb{N}$, this yields $\hat{\nu}_i = 0$ in contradiction to our assumption. □

Under stronger assumptions like the one defined below, we can even obtain S-stationarity of the limit point.

Definition 8.7 *Let* $\{t_k\} \downarrow 0$ *and* $\{x^k\}$ *be a sequence of feasible points of* $R^{KS}(t_k)$ *with* $x^k \to x^*$*. If for all k sufficiently large*

$$\frac{G_i(x^k)}{H_i(x^k)} \leq 1 \quad \textit{for all} \quad i \in I_\Phi^{+0}(x^k; t_k)) \cap I_{00}(x^*), \qquad \textit{and}$$
$$\frac{H_i(x^k)}{G_i(x^k)} \leq 1 \quad \textit{for all} \quad i \in I_\Phi^{0+}(x^k; t_k) \cap I_{00}(x^*)$$

the sequence $\{x^k\}$ *is called* asymptotically weakly nondegenerate.

Related asymptotic weak nondegeneracy conditions were also used in [96, 81, 67]. A direct comparison of the different nondegeneracy conditions is not possible in general since they depend on the particular regularization. However, our impression is that our definition is a relatively weak assumption that will often be satisfied in practice.

The next result shows that MPCC-CPLD together with the asymptotic weak nondegeneracy condition already guarantees that the limit point is S-stationary.

Theorem 8.8 *Let* $\{t_k\} \downarrow 0$ *and* $\{(x^k, \lambda^k, \mu^k, \gamma^k, \nu^k, \delta^k)\}$ *be a sequence of KKT points of* $R^{KS}(t_k)$ *with* $x^k \to x^*$*. If MPCC-CPLD holds in* x^* *and the sequence* $\{x^k\}$ *is asymptotically weakly nondegenerate, then* x^* *is an S-stationary point of the MPCC* (1.1).

Proof. Using Theorem 8.6, we know that x^* is at least M-stationary. To verify S-stationarity, we use the proof of Theorem 8.6 again. The only change is that, in the very end, we now additionally apply the asymptotic weak nondegeneracy condition: Assume that $(x^*, \lambda^*, \mu^*, \hat{\gamma}, \hat{\nu})$ is not an S-stationary point of the MPCC (1.1). Then we can find an $i \in I_{00}(x^*)$, where $\hat{\gamma}_i < 0$ or $\hat{\nu}_i < 0$. Let us assume $\hat{\gamma}_i < 0$ without loss of generality. The second case can be treated in the same way. Then, by construction, $i \in \operatorname{supp}(\delta^{G,*}) \subseteq I_\Phi^{0+}(x^k; t_k)$ and consequently $G_i(x^k) = t_k$, $H_i(x^k) > t_k$ for all k sufficiently large. This however implies $\frac{H_i(x^k)}{G_i(x^k)} > 1$ for all those k in contradiction to the assumption of asymptotic weak nondegeneracy. □

We note that both Theorem 8.6 and Theorem 8.8 require significantly weaker assumptions than those regularization methods we discussed before, except for the one by Kadrani et al. These methods need MPCC-MFCQ (except for the one by Steffensen and Ulbrich) instead of MPCC-CPLD for convergence to C-stationary points, MPCC-LICQ plus an additional second-order condition, which is not needed here, for convergence to M-stationary points and even more conditions for convergence to S-stationary points. In this context, we also refer to the discussion in Section 9.1.

To close this section, we would like to briefly come back to Example 8.1 from the beginning of this chapter. Analogously to the relaxation scheme by Kadrani et al. the relaxed feasible sets corresponding to this new method only have stationary points in $(t, 1)^T$ and $(1, t)^T$, which are converging to the global minima. Convergence of a sequence of stationary points to the C-stationary origin is not possible due to Theorem 8.6.

8.2.2. Existence of Multipliers

There is an implicit assumption used in the previous two convergence results, namely that there exists a sequence of KKT points for the regularized problems $R^{KS}(t_k)$. In particular, we therefore require the existence of Lagrange multipliers. The aim of this section is to show that these Lagrange multipliers indeed exist under suitable assumptions. The most natural idea would be to show that the regularized problems $R^{KS}(t_k)$ (at least for $t_k > 0$ sufficiently small) inherit some constraint qualification from the original MPCC. However, this is not true in general. In fact, the following example shows that the MPCC itself might satisfy MPCC-LICQ at a feasible point x^*, whereas the corresponding regularized problem violates standard LICQ.

Example 8.9 (Example 5.8 continued) Consider again the two-dimensional MPCC

$$\min_{x_1,x_2} f(x) \quad \text{subject to} \quad 0 \leq x_1 \perp x_2 \geq 0.$$

Obviously, MPCC-LICQ holds at $x^* = (0,0)$. Now, consider the sequences $t_k = \frac{1}{k}$ and $x^k = (\frac{1}{k}, \frac{1}{k})$ for $k \in \mathbb{N}$. It is easy to see that $t_k \downarrow 0$ and $x^k \to x^*$. Furthermore, x^k is feasible for $R^{KS}(t_k)$ for all $k \in \mathbb{N}$. However, for all $k \in \mathbb{N}$ the only active gradient is

$$\nabla\Phi(x^k; t_k) = \begin{pmatrix} x_2^k - t_k \\ x_1^k - t_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence LICQ for the nonlinear program $R^{KS}(t_k)$ does not hold in x^k for all $k \in \mathbb{N}$. In fact, not even ACQ, one of the weakest constraint qualifications known for standard nonlinear programs, holds in x^k. However, the even weaker GCQ is satisfied. ◇

Inspired by this example, we are going to prove that, whenever MPCC-LICQ holds in a point x^* which is feasible for (1.1), there is a neighborhood of x^* such that for all $t > 0$ sufficiently small and all x in this neighborhood which are feasible for $R^{KS}(t)$, standard GCQ holds. In the proof of this result, we are going to work with some nonlinear programs that are closely related to $R^{KS}(t)$ but have better properties concerning constraint qualifications. Let $t > 0$ and $\hat{x}$ be feasible for $R^{KS}(t)$. Let I be an arbitrary subset of $I_\Phi^{00}(\hat{x}; t)$ and $\bar{I} := I_\Phi^{00}(\hat{x}; t) \setminus I$ its complement. We define the nonlinear program NLP(t, I) as

$$\begin{aligned} \min f(x) \quad \text{subject to} \quad & g_i(x) \leq 0 \ \forall i = 1, \ldots, m, \\ & h_i(x) = 0 \ \forall i = 1, \ldots, p, \\ & G_i(x) \geq 0, H_i(x) \geq 0, G_i(x) \leq t \ \forall i \in I_\Phi^{0+}(\hat{x}; t) \cup I, \\ & G_i(x) \geq 0, H_i(x) \geq 0, H_i(x) \leq t \ \forall i \in I_\Phi^{+0}(\hat{x}; t) \cup \bar{I}, \\ & G_i(x) \geq 0, H_i(x) \geq 0, \Phi_i(x; t) \leq 0 \ \forall i \notin I_\Phi(\hat{x}; t) \end{aligned} \tag{8.5}$$

and denote its feasible set by $X(t, I)$. Then it is easy to see that $X(t, I) \subseteq X^{KS}(t)$ and that $\hat{x}$ is feasible for NLP(t, I), too. The following lemma sheds some light on the relation between the tangent cone of $R^{KS}(t)$ and the tangent cones of NLP(t, I).

Lemma 8.10 *For all $t > 0$ and all $\hat{x}$ feasible for $R^{KS}(t)$,*

$$T_{X^{KS}(t)}(\hat{x}) = \bigcup_{I \subseteq I_\Phi^{00}(\hat{x};t)} T_{X(t,I)}(\hat{x}).$$

Proof. To prove the first inclusion, let d be an arbitrary element of $T_{X^{KS}(t)}(\hat{x})$. This implies that there exists a sequence $x^k \to_{X^{KS}(t)} \hat{x}$ and a sequence $\tau_k \downarrow 0$ such that $d = \lim_{k\to\infty} \frac{x^k - \hat{x}}{\tau_k}$. If we can find an $I \subseteq I_\Phi^{00}(\hat{x};t)$ such that $x^k \in X(t, I)$ for infinitely many $k \in \mathbb{N}$, we have proven $d \in \bigcup_{I \subseteq I_\Phi^{00}(\hat{x};t)} T_{X(t,I)}(\hat{x})$. However, for every $i \in I_\Phi^{00}(\hat{x};t)$ and all $k \in \mathbb{N}$, either $G_i(x^k) \le t$ or $H_i(x^k) \le t$. Hence, by choosing an appropriate subsequence $K \subseteq \mathbb{N}$ and defining I as the set of all indices i where $G_i(x^k) \le t$ for all $k \in K$, we can construct such a set I.

To prove the second inclusion, choose an arbitrary $I \subseteq I_\Phi^{00}(\hat{x}, t)$ and an arbitrary $d \in T_{X(t,I)}(\hat{x})$. This implies the existence of sequences $x^k \to_{X(t,I)} \hat{x}$ and $\tau_k \downarrow 0$ such that $d = \lim_{k\to\infty} \frac{x^k - \hat{x}}{\tau_k}$. Because of $X(t, I) \subseteq X^{KS}(t)$, this yields $d \in T_{X^{KS}(t)}(\hat{x})$. □

Now, we are in a position to state and prove the main result of this section.

Theorem 8.11 *Let x^* be feasible for the MPCC* (1.1) *such that MPCC-LICQ holds in x^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(x^*)$ such that the following holds for all $t \in (0, \bar{t}]$: If $x \in U(x^*)$ is feasible for $R^{KS}(t)$, then standard GCQ for $R^{KS}(t)$ holds in x.*

Proof. Since MPCC-LICQ holds in x^*, the gradients

$$\{\nabla g_i(x) \mid i \in I_g\} \cup \{\nabla h_i(x) \mid i = 1, \dots, p\} \cup \{\nabla G_i(x) \mid i \in I_{00} \cup I_{0+}\} \cup \{\nabla H_i(x) \mid i \in I_{00} \cup I_{+0}\} \quad (8.6)$$

are linearly independent in x^*. Because of the continuity of the derivatives, they remain linearly independent in a whole neighborhood. Hence, we can choose $\bar{t} > 0$ and $U(x^*)$ such that for all $t \in (0, \bar{t}]$ and all $x \in U(x^*)$ feasible for $R^{KS}(t)$ the gradients (8.6) are linearly independent in x, and the following inclusions hold, cf. (8.3):

$$\begin{aligned}
I_g(x) &\subseteq I_g(x^*), \\
I_G(x) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\
I_H(x) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*), \\
I_\Phi^{00}(x;t) \cup I_\Phi^{0+}(x;t) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\
I_\Phi^{00}(x;t) \cup I_\Phi^{+0}(x;t) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*).
\end{aligned}$$

Now, choose an arbitrary $t \in (0, \bar{t}]$ and $\hat{x} \in U(x^*)$ such that $\hat{x}$ is feasible for $R^{KS}(t)$. Then $\hat{x}$ is also feasible for NLP(t, I) for all $I \subseteq I_\Phi^{00}(\hat{x}, t)$ and the active gradients are

$$\begin{aligned}
&\{\nabla g_i(\hat{x}) \mid i \in I_g(\hat{x})\} \cup \{\nabla h_i(\hat{x}) \mid i = 1, \dots, p\} \cup \\
&\{\nabla G_i(\hat{x}) \mid i \in I_G(\hat{x}) \cup I_\Phi^{0+}(\hat{x};t) \cup I\} \cup \{\nabla H_i(\hat{x}) \mid i \in I_H(\hat{x}) \cup I_\Phi^{+0}(\hat{x};t) \cup \bar{I}\}.
\end{aligned}$$

Thus, by construction of $\bar{t}$ and $U(x^*)$, standard LICQ for NLP(t, I) holds in $\hat{x}$. Since LICQ implies ACQ, we have $T_{X(t,I)}(\hat{x}) = L_{X(t,I)}(\hat{x})$ for all $I \subseteq I_\Phi^{00}(\hat{x};t)$. Together with Lemma 8.10, this yields

$$T_{X^{KS}(t)}(\hat{x}) = \bigcup_{I \subseteq I_\Phi^{00}(\hat{x};t)} T_{X(t,I)}(\hat{x}) = \bigcup_{I \subseteq I_\Phi^{00}(\hat{x};t)} L_{X(t,I)}(\hat{x}).$$

Passing to the polar cone, we obtain

$$T_{X^{KS}(t)}(\hat{x})^\circ = \bigcap_{I \subseteq I_\Phi^{00}(\hat{x},t)} L_{X(t,I)}(\hat{x})^\circ, \tag{8.7}$$

see [13, Theorem 3.1.9]. To prove that GCQ for $R^{KS}(t)$ holds in $\hat{x}$, we only have to show the inclusion $T_{X^{KS}(t)}(\hat{x})^\circ \subseteq L_{X^{KS}(t)}(\hat{x})^\circ$, the opposite inclusion is always true. By definition, the linearized tangent cone of NLP(t, I) in $\hat{x}$ is given by

$$\begin{aligned} L_{X(t,I)}(\hat{x}) = \{d \in \mathbb{R}^n \quad | \quad & \nabla g_i(\hat{x})^T d \leq 0 \ \forall i \in I_g(\hat{x}), \\ & \nabla h_i(\hat{x})^T d = 0 \ \forall i = 1, \ldots, p, \\ & \nabla G_i(\hat{x})^T d \geq 0 \ \forall i \in I_G(\hat{x}), \\ & \nabla H_i(\hat{x})^T d \geq 0 \ \forall i \in I_H(\hat{x}), \\ & \nabla G_i(\hat{x})^T d \leq 0 \ \forall i \in I_\Phi^{0+}(\hat{x}; t) \cup I, \\ & \nabla H_i(\hat{x})^T d \leq 0 \ \forall i \in I_\Phi^{+0}(\hat{x}; t) \cup \bar{I}\}. \end{aligned}$$

Therefore, Lemma 7.2 yields

$$\begin{aligned} L_{X(t,I)}(\hat{x})^\circ = \{s \in \mathbb{R}^n \mid \quad s \quad = & \sum_{i \in I_g(\hat{x})} \lambda_i \nabla g_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\hat{x}) - \sum_{i \in I_G(\hat{x})} \gamma_i \nabla G_i(\hat{x}) \\ & - \sum_{i \in I_H(\hat{x})} \nu_i \nabla H_i(\hat{x}) + \sum_{i \in I_\Phi^{0+}(\hat{x};t) \cup I} \delta_i \nabla G_i(\hat{x}) + \sum_{i \in I_\Phi^{+0}(\hat{x};t) \cup \bar{I}} \sigma_i \nabla H_i(\hat{x}), \\ & \lambda, \gamma, \nu, \delta, \sigma \geq 0\}. \end{aligned}$$

Now, let s be an arbitrary element of $T_{X^{KS}(t)}(\hat{x})^\circ$. The representation of $T_{X^{KS}(t)}(\hat{x})^\circ$ in (8.7) then implies $s \in L_{X(t,I)}(\hat{x})^\circ$ for all $I \subseteq I_\Phi^{00}(\hat{x}, t)$. If we fix such an index set I, we obtain

$$\begin{aligned} s \quad = \quad & \sum_{i \in I_g(\hat{x})} \lambda_i \nabla g_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\hat{x}) - \sum_{i \in I_G(\hat{x})} \gamma_i \nabla G_i(\hat{x}) - \sum_{i \in I_H(\hat{x})} \nu_i \nabla H_i(\hat{x}) \\ & + \sum_{i \in I_\Phi^{0+}(\hat{x};t) \cup I} \delta_i \nabla G_i(\hat{x}) + \sum_{i \in I_\Phi^{+0}(\hat{x};t) \cup \bar{I}} \sigma_i \nabla H_i(\hat{x}) \end{aligned}$$

with some multipliers $\mu \in \mathbb{R}^p$ and $\lambda, \gamma, \nu, \delta, \sigma \geq 0$. On the other hand, $s \in L_{X(t,\bar{I})}(\hat{x})^\circ$ also holds, thus we also have

$$\begin{aligned} s \quad = \quad & \sum_{i \in I_g(\hat{x})} \bar{\lambda}_i \nabla g_i(\hat{x}) + \sum_{i=1}^{p} \bar{\mu}_i \nabla h_i(\hat{x}) - \sum_{i \in I_G(\hat{x})} \bar{\gamma}_i \nabla G_i(\hat{x}) - \sum_{i \in I_H(\hat{x})} \bar{\nu}_i \nabla H_i(\hat{x}) \\ & + \sum_{i \in I_\Phi^{0+}(\hat{x};t) \cup \bar{I}} \bar{\delta}_i \nabla G_i(\hat{x}) + \sum_{i \in I_\Phi^{+0}(\hat{x};t) \cup I} \bar{\sigma}_i \nabla H_i(\hat{x}) \end{aligned}$$

with some multipliers $\bar{\mu} \in \mathbb{R}^p$ and $\bar{\lambda}, \bar{\gamma}, \bar{\nu}, \bar{\delta}, \bar{\sigma} \geq 0$. However, by construction of $\bar{t}$ and $U(x^*)$, the gradients

$$\{\nabla g_i(\hat{x}) \mid i \in I_g(\hat{x})\} \cup \{\nabla h_i(\hat{x}) \mid i = 1, \ldots, p\} \cup$$

$$\{\nabla G_i(\hat{x}) \mid i \in I_G(\hat{x}) \cup I_\Phi^{0+}(\hat{x};t) \cup I_\Phi^{00}(\hat{x};t)\} \cup \{\nabla H_i(\hat{x}) \mid i \in I_H(\hat{x}) \cup I_\Phi^{+0}(\hat{x};t) \cup I_\Phi^{00}(\hat{x};t)\}$$

are linearly independent, hence the multipliers have to be the same. In particular, this implies $\delta_i = 0$ for all $i \in I$ and $\sigma_i = 0$ for all $i \in \bar{I}$. Since an elementary calculation shows that

$$\begin{aligned} L_{X^{KS}(t)}(\hat{x}) = \{d \in \mathbb{R}^n \mid\ & \nabla g_i(\hat{x})^T d \le 0\ \forall i \in I_g(\hat{x}), \\ & \nabla h_i(\hat{x})^T d = 0\ \forall i = 1, \ldots, p, \\ & \nabla G_i(\hat{x})^T d \ge 0\ \forall i \in I_G(\hat{x}), \\ & \nabla H_i(\hat{x})^T d \ge 0\ \forall i \in I_H(\hat{x}), \\ & \nabla G_i(\hat{x})^T d \le 0\ \forall i \in I_\Phi^{0+}(\hat{x};t), \\ & \nabla H_i(\hat{x})^T d \le 0\ \forall i \in I_\Phi^{+0}(\hat{x};t)\}, \end{aligned}$$

application of Lemma 7.2 yields $s \in L_{X^{KS}(t)}(\hat{x})^\circ$. Note that the representation of $L_{X^{KS}(t)}(\hat{x})$ above exploits the structure of $\nabla\Phi(x;t)$ as given in (8.2). Since $s \in T_{X^{KS}(t)}(\hat{x})^\circ$ was chosen arbitrarily, we have proven $T_{X^{KS}(t)}(\hat{x})^\circ \subseteq L_{X^{KS}(t)}(\hat{x})^\circ$, i.e., GCQ for $R^{KS}(t)$ holds in $\hat{x}$. □

The existence of Lagrange multipliers in local minima of $R^{KS}(t)$ is a direct consequence of Theorem 8.11.

Theorem 8.12 *Let x^* be feasible for the MPCC* (1.1) *such that MPCC-LICQ holds in x^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(x^*)$ such that the following holds for all $t \in (0, \bar{t}]$: If $x \in U(x^*)$ is a local minimizer of $R^{KS}(t)$, then there exist Lagrange multipliers such that x together with these multipliers is a KKT point of $R^{KS}(t)$.*

Note that Theorem 8.12 implies the existence of multipliers at a local minimum of the regularized problem $R^{KS}(t)$ since Theorem 8.11 shows that the standard GCQ holds for the regularized problem under the MPCC-LICQ assumption. Moreover, recall that Example 8.9 indicates that we cannot expect a stronger result, since even ACQ may not hold for $R^{KS}(t)$ under MPCC-LICQ. In a sense, this is similar to some results that are known for the MPCC itself, cf. [40]. However, the following result shows that there is a significant difference between MPCCs themselves and our regularized problem $R^{KS}(t)$. In fact, it is known that the MPCC does not satisfy standard LICQ (or even the weaker MFCQ) at an *arbitrary feasible point*. On the other hand, the next result shows that standard LICQ holds for $R^{KS}(t)$ if MPCC-LICQ is satisfied and, in addition, the index set $I_\Phi^{00}(x;t)$ is empty. The latter assumption excludes only points where $(G_i(x), H_i(x)) = (t, t)$ for at least one index i. In fact, this result also shows that MPCC-CPLD for the original MPCC implies standard CPLD for the regularized subproblems $R^{KS}(t)$.

Theorem 8.13 *Let x^* be feasible for the MPCC* (1.1) *such that MPCC-LICQ (MPCC-CPLD) holds in x^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(x^*)$ such that the following holds for all $t \in (0, \bar{t}]$: If $x \in U(x^*)$ is feasible for $R^{KS}(t)$ with $I_\Phi^{00}(x;t) = \emptyset$, then standard LICQ (CPLD) for $R^{KS}(t)$ holds in x.*

Proof. We first verify the assertion for MPCC-LICQ. Since MPCC-LICQ holds in x^*, the gradients

$$\begin{aligned} & \{\nabla g_i(x) \mid i \in I_g(x^*)\} \cup \{\nabla h_i(x) \mid i = 1, \ldots, p\} \\ \cup\ & \{\nabla G_i(x) \mid i \in I_{00}(x^*) \cup I_{0+}(x^*)\} \cup \{\nabla H_i(x) \mid i \in I_{00}(x^*) \cup I_{+0}(x^*)\} \end{aligned} \tag{8.8}$$

are linearly independent in $x = x^*$. Because of the continuity of the derivatives, they remain linearly independent in a whole neighborhood. Thus, we can choose $\bar{t} > 0$ and $U(x^*)$ such that for all $t \in (0, \bar{t}]$ and all $x \in U(x^*)$ feasible for $R^{KS}(t)$, the gradients (8.8) are linearly independent in x, and the following inclusions hold:

$$\begin{aligned} I_g(x) &\subseteq I_g(x^*), \\ I_G(x) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\ I_H(x) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*), \\ I_\Phi^{00}(x;t) \cup I_\Phi^{0+}(x;t) &\subseteq I_{00}(x^*) \cup I_{0+}(x^*), \\ I_\Phi^{00}(x;t) \cup I_\Phi^{+0}(x;t) &\subseteq I_{00}(x^*) \cup I_{+0}(x^*). \end{aligned} \tag{8.9}$$

Now, choose an arbitrary $t \in (0, \bar{t}]$. When $x \in U(x^*)$ is feasible for $R^{KS}(t)$ with $I_\Phi^{00}(x;t) = \emptyset$, the active gradients in x are

$$\begin{aligned} &\{\nabla g_i(x) \mid i \in I_g(x)\} \cup \{\nabla h_i(x) \mid i = 1, \dots, p\} \cup \\ &\cup \{-\nabla G_i(x) \mid i \in I_G(x)\} \cup \{(H_i(x) - t)\nabla G_i(x) \mid i \in I_\Phi^{0+}(x;t)\} \\ &\cup \{-\nabla H_i(x) \mid i \in I_H(x)\} \cup \{(G_i(x) - t)\nabla H_i(x) \mid i \in I_\Phi^{+0}(x;t)\}, \end{aligned}$$

where $G_i(x) - t > 0$ for $i \in I_\Phi^{+0}(x;t)$ and $H_i(x) - t > 0$ for $i \in I_\Phi^{0+}(x;t)$. Hence, the choice of $\bar{t}$ and $U(x^*)$ implies that these gradients are linearly independent, too. Therefore, standard LICQ holds in x.

It remains to prove the assertion under MPCC-CPLD. To this end, assume that there were sequences $t_k \downarrow 0$ and $x^k \to x^*$ with x^k feasible for $R^{KS}(t_k)$ and $I_\Phi^{00}(x^k; t_k) = \emptyset$ for all $k \in \mathbb{N}$ such that standard CPLD is not satisfied in x^k for all $k \in \mathbb{N}$. Violation of CPLD means that there are subsets $I_1^k \subseteq I_g(x^k)$, $I_2^k \subseteq \{1, \dots, p\}$, $I_3^k \subseteq I_G(x^k)$, $I_4^k \subseteq I_H(x^k)$, $I_5^k \subseteq I_\Phi^{0+}(x^k; t_k)$, $I_6^k \subseteq I_\Phi^{+0}(x^k; t_k)$ such that the gradients

$$\begin{aligned} &\Big\{\{\nabla g_i(x^k) \mid i \in I_1^k\} \cup \{-\nabla G_i(x^k) \mid i \in I_3^k\} \cup \{-\nabla H_i(x^k) \mid i \in I_4^k\} \\ &\cup \{(H_i(x^k) - t_k)\nabla G_i(x^k) \mid i \in I_5^k\} \cup \{(G_i(x^k) - t_k)\nabla H_i(x^k) \mid i \in I_6^k\}\Big\} \cup \{\nabla h_i(x^k) \mid i \in I_2^k\} \end{aligned}$$

are positive-linearly dependent in x^k, but linearly independent in points arbitrary close to x^k. We may assume without loss of generality $I_i^k = I_i$ for all $i = 1, \dots, 6$. For all k sufficiently large, we know $I_g(x^k) \subseteq I_g(x^*)$ and thus $I_1 \subseteq I_g(x^*)$. Analogously, we obtain $I_3 \cup I_5 \subseteq I_{00}(x^*) \cup I_{0+}(x^*)$ and $I_4 \cup I_6 \subseteq I_{00}(x^*) \cup I_{+0}(x^*)$. Positive-linear dependence in x^k as we stated it above also implies positive-linear dependence of the gradients

$$\{\nabla g_i(x^k) \mid i \in I_1\} \cup \Big\{\{\nabla h_i(x^k) \mid i \in I_2\} \cup \{\nabla G_i(x^k) \mid i \in I_3 \cup I_5\} \cup \{\nabla H_i(x^k) \mid i \in I_4 \cup I_6\}\Big\},$$

and because of the violation of CPLD, we can find a sequence $y^k \to x^*$ such that these gradients are linearly independent in y^k. If these gradients were positive-linearly independent in x^*, by continuity they would remain positive-linearly independent in a whole neighborhood. This, however, contradicts the existence of the sequence $x^k \to x^*$. On the other hand, if they were positive-linearly dependent in x^*, MPCC-CPLD would imply that they remain linearly dependent in a neighborhood, which contradicts the existence of $y^k \to x^*$. This concludes the proof. □

The previous result also holds for some other constraint qualifications. In fact, it is possible to show that MPCC-MFCQ for the original MPCC implies standard MFCQ for the regularized problem. Furthermore, MPCC-CRCQ for the MPCC itself also implies standard CRCQ for the regularized problem $R^{KS}(t)$. The corresponding proofs are very similar to the one of Theorem 8.13, so we skip the details.

We close this section with the following observation: Theorem 8.11 is based on the assumption that MPCC-LICQ holds in x^*. This assumption is quite strong in contrast to the one needed for Theorem 8.6, namely that MPCC-CPLD holds in x^*. However, the following example shows that it will not be easy to relax the assumptions in Theorem 8.11.

Example 8.14 (Example 5.3 continued) Consider again the following 3-dimensional MPCC

$$\min_{x_1,x_2,x_3} f(x) \quad \text{subject to} \quad \begin{aligned} &-4x_1 + x_3 \leq 0, \\ &-4x_2 + x_3 \leq 0, \\ &0 \leq x_1 \perp x_2 \geq 0 \end{aligned}$$

with $x^* = (0,0,0)^T$. We have already seen that MPCC-LICQ is violated in x^*, whereas MPCC-MFCQ is satisfied. For $t > 0$ we define $x^t := (t,t,4t)^T$. Then x^t is feasible for $R^{KS}(t)$ and the cones are given by

$$\begin{aligned} T_{X^{KS}(t)}(x^t) &= \{d \mid \min\{d_1,d_2\} \leq 0,\ d_3 \leq 4\min\{d_1,d_2\}\}, \\ L_{X^{KS}(t)}(x^t) &= \{d \mid (-4,0,1)d \leq 0,\ (0,-4,1)d \leq 0\}. \end{aligned}$$

Hence, the corresponding polar cones are

$$\begin{aligned} T_{X^{KS}(t)}(x^t)^\circ &= \{s \mid s_1 \leq 0,\ s_2 \leq 0,\ s_3 \geq -\frac{1}{4}(s_1+s_2)\}, \\ L_{X^{KS}(t)}(x^t)^\circ &= \{s \mid s_1 \leq 0,\ s_2 \leq 0,\ s_3 = -\frac{1}{4}(s_1+s_2)\}. \end{aligned}$$

Hence, GCQ is violated in all points x^t, $t > 0$. ◇

8.2.3. Existence of Stationary Points

In the last section, we have proven that, under the assumption that MPCC-LICQ holds in a point x^* feasible for the MPCC (1.1), every local minimum x of $R^{KS}(t)$ with $t > 0$ sufficiently small which is close to x^* is a stationary point of $R^{KS}(t)$. Naturally, the following question arises: When x^* is a local minimum of the MPCC, under what assumptions can we find local minima of $R^{KS}(t)$ close to x^*? Recall from Example 8.2 that the existence of local minima and thus stationary points of the relaxed problems cannot be taken for granted. However, we would like to point out that the new relaxation we consider here does, in contrast to some other methods, have stationary points in the example mentioned above, namely all points $(x_1, t)^T$ with $x_1 > t$ are stationary. Nonetheless, there are also MPCCs where we cannot find local minima or stationary points of the relaxed problems close to every solution of the MPCC without further assumptions. This is illustrated by the following example from [67].

Example 8.15 Consider the following 2-dimensional MPCC

$$\min_{x_1,x_2} f(x) = x_1(1 - x_2^2) \quad \text{subject to} \quad \begin{aligned} G(x) &= x_1 \geq 0, \\ H(x) &= x_2 \geq 0, \\ G(x)H(x) &= x_1 x_2 = 0. \end{aligned}$$

This optimization problem has only global minima, namely all points $(0, x_2)^T$ with $x_2 \geq 0$, which are S-stationary. For $t < 1$ however, the relaxed problems $R^{KS}(t)$ only have stationary points in $(0, x_2)^T$ with $x_2 \in [0, 1]$. Hence, we cannot find stationary points of $R^{KS}(t)$ close to the global minimum $(0, 2)$ for example, even though MPCC-LICQ is satisfied there. ◇

Analogously to [67], we can give the following answer to the question above. This theorem based on [67, Theorem 5.1] provides the existence of local minima of $R^{KS}(t)$ in a neighborhood of a solution of (1.1).

Theorem 8.16 *Let x^* be a strict local minimum of the MPCC* (1.1). *Then there is a $\bar{t} > 0$ and a neighborhood $U(x^*)$ such that $R^{KS}(t)$ has a local minimum in $U(x^*)$ for all $t \in (0, \bar{t}]$.*

Proof. Since x^* is a strict local minimum, we can find an $r > 0$ such that $f(x) > f(x^*)$ for all $x \in X \cap \text{cl}(\mathbb{B}(x^*; r))$. The existence of an $\varepsilon > 0$ such that

$$f(x) \geq f(x^*) + \varepsilon$$

for all $x \in X \cap \text{bd}(\mathbb{B}(x^*; r))$ then directly follows.

We are now going to prove that this implies the existence of an $\bar{t} > 0$ such that

$$f(x) \geq f(x^*) + \frac{\varepsilon}{2}$$

for all $x \in X^{KS}(t) \cap \text{bd}(\mathbb{B}(x^*; r))$, where $t \in (0, \bar{t}]$. To this end, assume by contradiction that there was a sequence $t_k \downarrow 0$ and a sequence $x^k \in X^{KS}(t_k) \cap \text{bd}(\mathbb{B}(x^*; r))$ with $f(x^k) < f(x^*) + \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$. Since all x^k are elements of the bounded set $\text{cl}(\mathbb{B}(x^*; r))$, we can assume without loss of generality that the sequence x^k converges to some limit $\bar{x}$. Because of $t_k \downarrow 0$, we can apply Lemma 8.4 and obtain $\bar{x} \in X$. By continuity, we know

$$f(\bar{x}) \leq f(x^*) + \frac{\varepsilon}{2} < f(x^*) + \varepsilon$$

and at the same time $\bar{x} \in X \cap \text{bd}(\mathbb{B}(x^*; r))$. Together, this is a contradiction to the choice of r and ε.

Now that we have proven the existence of a $\bar{t} > 0$ with the properties mentioned above, choose an arbitrary $t \in (0, \bar{t}]$. Since f is continuous, it attains a global minimum on the compact set $X^{KS}(t) \cap \mathrm{cl}(\mathbb{B}(x^*; r))$. Since $x^* \in X^{KS}(t) \cap \mathrm{cl}(\mathbb{B}(x^*; r))$ and by construction $f(x) > f(x^*)$ for all $x \in X^{KS}(t) \cap \mathrm{bd}(\mathbb{B}(x^*; r))$, this global minimum lies in the interior of $\mathbb{B}(x^*; r)$ and therefore is a local minimum of f on $X^{KS}(t)$. Thus, we have proven that the statement holds true with the neighborhood $U(x^*) = \mathbb{B}(x^*; r)$. □

Note that, in the result above, the neighborhood $U(x^*)$ can be chosen arbitrarily small. Hence, we can find local minima of the relaxed problems $R^{KS}(t)$ for sufficiently small parameters t in every neighborhood of x^*. When MPCC-LICQ holds in x^*, these local minima are also stationary points of the relaxed problems due to Theorem 8.11.

9. Theoretical and Numerical Comparison of the Relaxation Methods

Having analyzed the theoretical properties of five relaxation methods in the last two chapters, we are now going to compare these methods. After collecting the previous results and discussing the theoretical differences, we make a numerical experiment based on the MacMPEC collection of test problems to see if they also show a different numerical behavior.

9.1. Comparison of the Theoretical Properties

In Table 9.1, we try to summarize the results of the Chapters 7 and 8 in a very concise way. The columns contain the five relaxation schemes discussed here. The first two lines then state under which MPCC constraint qualification a limit point of a sequence generated by one of these methods is either C- or M-stationary. The second part of the table indicates under which MPCC constraint qualification the corresponding regularized problem satisfies one of the standard NLP constraint qualifications. Of course, this part only holds locally around a given feasible point x^* of the MPCC and for sufficiently small relaxation parameters t.

Relaxation	Scholtes	Lin–Fukush.	Kadrani et al.	Steff.–Ulbrich	new relax.
	stationary point results				
Assuming limit pts. are	MPCC-MFCQ C–stationary	MPCC-MFCQ C–stationary	MPCC-CPLD M–stationary	MPCC-CPLD C–stationary	MPCC-CPLD M–stationary
	existence of Lagrange multipliers				
Assuming R(t) satisf.	MPCC-MFCQ MFCQ	MPCC-MFCQ MFCQ	MPCC-LICQ GCQ	MPCC-LICQ ACQ	MPCC-LICQ GCQ

Table 9.1.: Summary of results regarding stationary points and constraint qualifications for the different relaxation methods

Note, however, that this second part does not cover the complete story. To this end, let us first note that MPCC-LICQ implies that the MPCC itself satisfies standard GCQ, cf. [40]. Therefore, when simply looking at the table, it seems that the feasible sets of some of the relaxation methods do not have better properties than the underlying MPCC, hence one might wonder why using such a regularization. In fact, GCQ or the slightly stronger ACQ are relatively weak conditions which, however, guarantee the existence of Lagrange multipliers at a local minimum. The main difference is that standard MFCQ (hence also standard LICQ) is violated at *any* feasible point of the MPCC itself, while the corresponding results for the three regularization methods by Kadrani et al. [67], Steffensen and Ulbrich [109], and the new relaxation method from Chapter 8 typically satisfy LICQ (hence also MFCQ) in many points under the MPCC-LICQ condition.

Let us discuss this point in more detail. The two relaxation methods by Scholtes [105] and Lin and Fukushima [79], besides being only convergent to C-stationary points, have no problems regarding constraint qualifications: MPCC-LICQ implies LICQ for the corresponding regularized problems, as shown in the original references, and MPCC-MFCQ also implies MFCQ, as shown in Section 7.1 for the Scholtes-relaxation and in Section 7.2 for the Lin-Fukushima-relaxation.

The situation is completely different with the other three relaxation schemes. These other three schemes have stronger convergence properties than the first two methods, more precisely, the relaxation scheme by Kadrani et al. and the one from Chapter 8 converge to M-stationary points, which is a much stronger property than C-stationarity, whereas the local regularization approach by Steffensen and Ulbrich only converges to C-stationary points, but has a nice finite termination property in the sense that it is not always necessary that the relaxation parameter t has to be driven down to zero. On the other hand, our analysis and the corresponding (counter-) examples show that the relaxed problems of any of these three methods usually do not inherit the corresponding standard constraint qualification from an MPCC constraint qualification.

In fact, it is easy to see that the relaxed problem by Steffensen and Ulbrich not only violates LICQ, but also CRCQ and CPLD, whereas ACQ (hence GCQ) is satisfied under MPCC-LICQ. Similarly, the relaxed problems by Kadrani et al. and those corresponding to the new relaxation do not even satisfy ACQ, whereas GCQ holds under MPCC-LICQ. Hence, from this point of view, it seems that the Steffensen-Ulbrich regularization is slightly better than the other two relaxations. However, also this is not true in general since, speaking in the $(G_i(x), H_i(x))$-space, both the Kadrani et al. and the new regularization satisfy standard LICQ in all points except for one (locally and assuming MPCC-LICQ, of course), whereas the Steffensen-Ulbrich relaxation violates standard LICQ in many points, namely in all points on the G_i- and H_i-axes where the feasible set of the MPCC is not changed by the local relaxation.

9.2. Numerical Comparison Based on the MacMPEC Collection

Since these five relaxation methods have different theoretical properties, it seems likely that their numerical realizations also exhibit a different behavior. We will analyze this conjecture using test problems from the MacMPEC collection by Sven Leyffer [73]. However, before we delve into the numerical details, let us clarify the aim of this section. So far, we have discussed the different theoretical properties of these five relaxation methods. Now, we want to find out what differences there are in the numerical behavior. Therefore, we tried to implement all five methods as similar as possible to ensure that different numerical results are caused by the different properties of the relaxations and not by algorithmic differences. As a consequence, we did not optimize our implementation individually for every relaxation, i.e., it is possible to obtain better results by tailoring the algorithms to the characteristics of the relaxations. For example, Steffensen and Ulbrich [109] and Kadrani et al. [67] proposed numerical approaches that deal with the specific characteristics of their relaxations. This, however, makes it nearly impossible to compare the numerical results, since a different performance may not be caused by the properties of the relaxation method but by the different numerical approaches. For this reason, we consider only a very simple realization of the relaxation methods and use the same NLP solver to solve the relaxed subproblems for each of them.

We implemented all five methods in MATLAB 7.11.0 and performed some tests on a 2.6 Ghz AMD Opteron and 64 GigaByte RAM running linux with a 2.6.27 kernel. The basic algorithm is Algorithm 9.1, where the maximum violation of all constraints

$$\mathrm{maxVio}(x_{opt}) = \max\{\max\{0, g(x_{opt})\}, |h(x_{opt})|, |\min\{G(x_{opt}), H(x_{opt})\}|\} \tag{9.1}$$

is used to measure the feasibility of the final iterate x_{opt}. For the method by Steffensen and Ulbrich, we used the regularization function

$$\theta(x) := \frac{2}{\pi} \sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right) + 1.$$

Algorithm 9.1 Basic relaxation algorithm $(x_0, t_0, \sigma, t_{\min})$

Require: a starting vector x^0, an initial relaxation parameter t_0, and parameters $\sigma \in (0, 1)$, $t_{\min} > 0$.

Set $k := 0$.
while $t_k \geq t_{\min}$ **do**
Find an approximate solution x^{k+1} of the relaxed problem $\mathrm{R}(t_k)$. To solve $\mathrm{R}(t_k)$, use x^k as starting vector.
Let $t_{k+1} \leftarrow \sigma \cdot t_k$ and $k \leftarrow k + 1$.
end while

Return: the final iterate $x_{opt} := x^k$, the corresponding function value $f(x_{opt})$, and the maximum constraint violation $\mathrm{maxVio}(x_{opt})$.

All relaxations except for the one by Kadrani et al. have the property that the following inclusion holds for all $0 \leq t_1 < t_2$

$$X(t_1) \subset X(t_2),$$

where $X(t)$ is the feasible area of the relaxed problem $\mathrm{R}(t)$ and $X = X(0)$ is the feasible area of the MPCC (1.1). This can be used in the numerical implementation in the following way: If a relaxed problem $\mathrm{R}(t_k)$ is infeasible, the MPCC is infeasible, too, and we can terminate the algorithm immediately. We can also terminate the algorithm early if the solution x^{k+1} of an iteration k is feasible for the MPCC because in this case, x^{k+1} is also a solution of the MPCC. Finally, if the solution x^{k+1} is also feasible for $\mathrm{R}(t_{k+1})$, it is a solution of $\mathrm{R}(t_{k+1})$ as well. Thus, we can skip the next iteration and reduce the relaxation parameter until x^{k+1} is not feasible for the next iteration anymore. These changes are incorporated into Algorithm 9.2, where feasibility of the iterate x^k for the original MPCC is measured by the violation of the complementarity constraints

$$\mathrm{compVio}(x^k) = \| \min\{G(x^k), H(x^k)\}\|_\infty.$$

Note, that the standard constraints $g(x) \leq 0$ and $h(x) = 0$ are part of the relaxed problems $\mathrm{R}(t)$ and therefore do not need to be checked here.

Algorithm 9.2 Improved relaxation algorithm $(x_0, t_0, \sigma, t_{\min}, \varepsilon)$

Require: a starting vector x^0, an initial relaxation parameter t_0, and parameters $\sigma \in (0, 1)$, $t_{\min} > 0$, and $\varepsilon > 0$.

Set $k := 0$.
while ($t_k \geq t_{\min}$ and $\mathrm{compVio}(x^k) > \varepsilon$) or $k = 0$ **do**
  Find an approximate solution x^{k+1} of the relaxed problem $\mathrm{R}(t_k)$. To solve $\mathrm{R}(t_k)$, use x^k as starting vector.
  If $\mathrm{R}(t_k)$ is infeasible, terminate the algorithm.
  Let $t_{k+1} \leftarrow \max_{l=1,2,3,\ldots}\{\sigma^l \cdot t_k \mid x^{k+1} \notin X(\sigma^l \cdot t_k) \text{ and } \sigma^l \cdot t \geq t_{\min}\}$ and $k \leftarrow k + 1$.
end while

Return: the final iterate $x_{opt} := x^k$, the corresponding function value $f(x_{opt})$, and the maximum constraint violation $\mathrm{maxVio}(x_{opt})$.

We used the improved Algorithm 9.2 for all relaxations except for the one by Kadrani et al., where we had to use the basic Algorithm 9.1 since the feasible area of the MPCC (1.1) is not included in the feasible area of the relaxed problems used in this method.

We used the parameters $t_{\min} = 10^{-15}$ and $\varepsilon = 10^{-6}$ for all relaxations and the TOMLAB 7.4.0 solver `snopt` to solve the relaxed problems $R(t_k)$. Determining the remaining two parameters t_0 and σ was a little more difficult. The relaxation parameter t enters the relaxed problems slightly different for some methods and therefore the relaxed feasible sets are of different size. To illustrate this, consider an arbitrary relaxation parameter $t > 0$ and calculate for every relaxation the intersection between the "$G = H$"-line and the north-eastern border of the relaxed feasible set. For the methods by Lin and Fukushima, Kadrani et al. and the new relaxation from Chapter 8, this intersection point is (t, t), for Scholtes' method it is $(\sqrt{t}, \sqrt{t})$ and for the method by Steffensen and Ulbrich it is $\frac{\pi-2}{2\pi}(t, t)$. Hence, if the same initial relaxation t_0 and the same reduction factor σ is chosen for all five methods, the initial feasible set of the Steffensen-Ulbrich relaxation is much smaller than the other ones and the feasible set of Scholtes' relaxation is shrinking more slowly. For this reason, our first approach was to pick two values $T > 0$ and $s \in (0, 1)$ and use the parameters $t_0 = \frac{2\pi}{\pi-2}T$, $\sigma = s$ for the Steffensen-Ulbrich method, $t_0 = T^2$ and $\sigma = s^2$ for Scholtes' relaxation and $t_0 = T, \sigma = s$ for the remaining three approaches. To test the relaxation algorithms, we chose 126 problems from the MacMPEC collection [73]. The other problems were discarded partly due to their size or form and partly because errors occurred during the evaluation of the objective function or the constraints by AMPL. Communication between AMPL and MATLAB was achieved using the mex function `amplfunc` [114], see also [51]. However, it turned out that there is no couple of values (T, s) such that all relaxation methods perform well simultaneously. If, for example, we chose $(T, s) = (1, 0.01)$, the relaxation method by Scholtes did not solve 11 problems, the relaxation method by Lin and Fukushima 62 problems, Kadrani et al. 19 problems. The relaxation method by Steffensen and Ulbrich failed to solve 18 problems and the new method from Chapter 8 did not solve 23 problems. Here, we consider a problem as solved if the maximum violation of all constraints is less than or equal to 10^{-6} (independent of the optimal function value found by the corresponding method). If, instead, we chose $(T, s) = (1, 0.1)$, the number of unsolved problems was, in the same order, 12, 73, 26, 24, and 21. For this reason, we decided to make a small parameter study for every method with $T \in \{10, 2, 10.5, 0.1, 0.01\}$ and $s \in \{0.5, 0.1, 0.01\}$ and counted the number of unsolved problems for every parameter combination. The results are summarized in Table 9.2.

Relaxation	Scholtes	Lin–Fukush.	Kadrani et al.	Steff.–Ulbrich	new relax.
min	10	61	14	3	12
max	23	86	32	36	23
median	15	66	19.5	14	17
std	3.95	8.37	5.08	9.11	3.36

Table 9.2.: Summary of parameter study

As we can see, the methods differ significantly in their success in solving the test problems. The method by Steffensen and Ulbrich for example is able to solve all but three problems with the right choice of parameters, whereas the method by Lin and Fukushima seems to have trouble independent of the parameters. However, not only the average number of solved problems differs from method to method but also the corresponding standard deviation. The method with the smallest standard deviation is our new relaxation whereas the method by Steffensen and Ulbrich has the highest one, i.e., the success of their method depends highly on the chosen parameters. Since it is usually not known a priori which are the best parameters for a problem, we chose T, s for every relaxation method such that the number of problems is equal to the respective median. Whenever there was more than one such parameter combination, we picked the one that produced the most feasible solutions in order to facilitate the subsequent comparison of optimal function values. The resulting parameters t_0, σ and the used Algorithm are given in Table 9.3.

To present the results, we use performance profiles as introduced by Dolan and Moré in [31]. All numerical results displayed in the performance profiles and the respective source code can be found in the appendix. A test problem, where the results are set to Inf, indicates that an error occurred during the evaluation of the problem data by AMPL. In Figure 9.1, the performance profile for the maximum violation of all constraints as defined in (9.1) is depicted.

It can be seen that the relaxation method by Steffensen and Ulbrich produces the smallest con-

Relaxation	Scholtes	Lin–Fukush.	Kadrani et al.	Steff.–Ulbrich	new relax.
Algorithm	Algorithm 9.2	Algorithm 9.2	Algorithm 9.1	Algorithm 9.2	Algorithm 9.2
t_0	0.5^2	0.5	1	$\frac{2\pi}{\pi-2}0.5$	2
σ	0.1^2	0.1	0.1	0.01	0.01

Table 9.3.: Parameters and algorithms used for the different relaxation methods

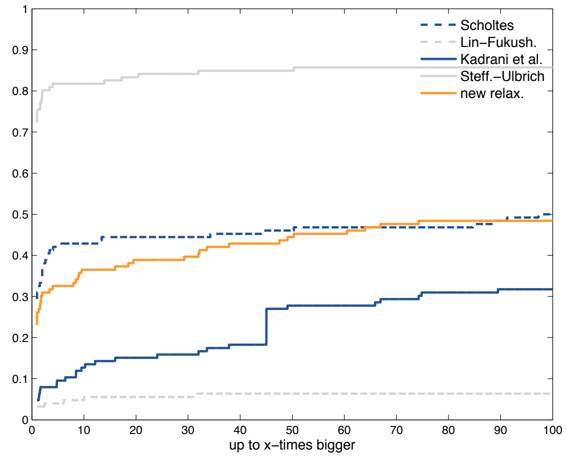


Figure 9.1.: Comparison of constraint violation

straint violation, followed by the relaxation of Scholtes and the new method. According to the criterion introduced above, the relaxation method by Scholtes did not solve 15 problems, the relaxation method by Lin and Fukushima 66 problems, Kadrani et al. 19 problems. The relaxation method by Steffensen and Ulbrich failed to solve 14 problems and the new method did not solve 17 problems. Trouble had to be expected for at least some test problems: `design-cent-1` is known to be infeasible, `ralphmod` has an unbounded set of feasible solutions and `ex9.2.2`, `qpec2`, `ralph1`, and `scholtes4` do not have S-stationary solutions, see for example [14, 101]. And in fact, many of them are among the unsolved problems for all approaches. We expected the relaxation method by Scholtes to be quite successful since it is the relaxation with the most regular subproblems. For the same reason, we are somewhat surprised by the results of the relaxation method by Steffensen and Ulbrich. Although the corresponding relaxed problems satisfy only very weak constraint qualifications, this method produces highly feasible solutions. This might be due to the fact that the relaxed feasible area is much smaller than for all other methods since it is only relaxed locally around the origin. However, in those cases where the maximum constraint violation is not less than or equal to 10^{-6}, it is mostly between 10^{-1} and 10, i.e., this small relaxed area sometimes leads to problems. For comparison, the maximum constraint violation for the majority of the unsolved problems is around 10^{-5} to 10^{-4} for the new relaxation method from Chapter 8. The two sided relaxation by Lin and Fukushima seems to cause serious numerical trouble. Here, the constraint violation in the unsolved problems covers the whole spectrum from 10^{-5} up to 10. This might be due to the fact that the functions $G_i(x)H_i(x) - t^2$ and $(G_i(x)+t)(H_i(x)+t)-t^2$ nearly coincide for small relaxation parameters $t > 0$. Perhaps a different approach, where we have an individual relaxation parameter for every one of the $2q$ constraints replacing the complementarity conditions, would work better for this method. Then we could take care that, for every couple of constraints $G_i(x)H_i(x) - (t_i^+)^2$ and $(G_i(x) + t)(H_i(x) + t) - (t_i^-)^2$, only one of the relaxation parameters t_i^+ or t_i^- is driven to zero while the other one is bounded away from zero.

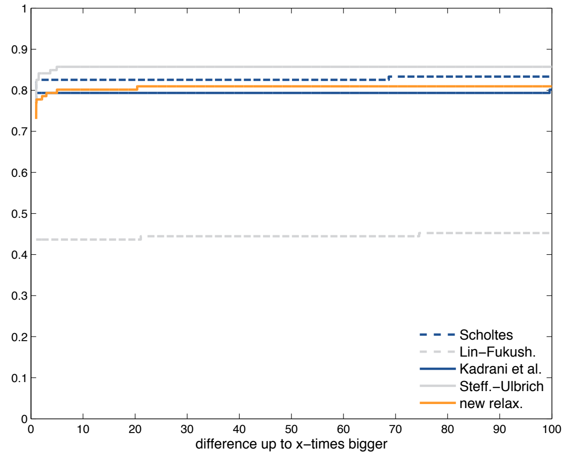
In the following performance profiles, we set the values corresponding to unsolved problems to $+\infty$. These four performance profiles compare the value of the objective function in the final iterate, the time needed for the calculation of this iterate and the number of necessary objective function and gradient evaluations. A few words on the performance profile comparing the optimal function value: As the optimal function value is negative for some test problems, we have to normalize the corresponding data slightly different than Dolan and Moré. Let f_R^k be the optimal function value for test problem k found by the relaxation method R, $R \in \{S, LF, KDB, SU, KS\}$. We then define the normalized data for the method by Scholtes as

$$\bar{f}_S^k := \frac{f_S^k - \min\{f_R \mid R \in \{S, LF, KDB, SU, KS\}\}}{|\min\{f_R \mid R \in \{S, LF, KDB, SU, KS\}\}|}$$

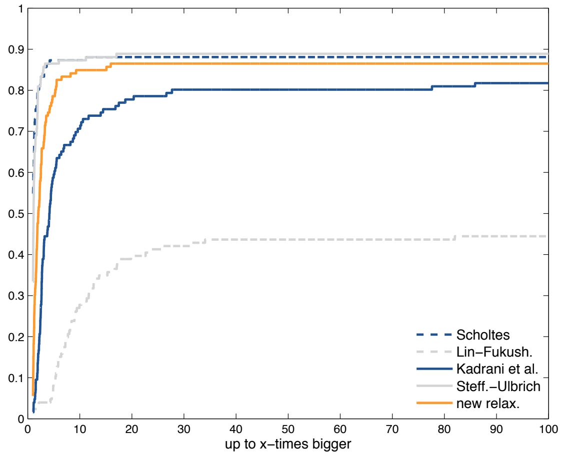
and analogously for all other methods, i.e., we consider the difference to the best value found by any of the five methods normalized by the absolute value of this best value.
Note that the highest possible value for a relaxation method in Figure 9.2 is the percentage of solved problems, e.g. 88.1% for the method by Scholtes. Figure 9.2b indicates that the relaxation methods by Scholtes and Steffensen and Ulbrich need the least time, closely followed by the new one. The relaxation method by Kadrani et al. is slower than these three but still significantly faster than the one by Lin and Fukushima. It had to be expected that the order here is about the same as the one in Figure 9.1 as we terminate the relaxation algorithm early if a solution feasible for the MPCC (1.1) is found. The only exception to this rule is the relaxation by Kadrani et al., see the discussion corresponding to Algorithm 9.2. Figure 9.2c and Figure 9.2d are very similar to Figure 9.2b and therefore will not be discussed separately. If we take a look at Figure 9.2a, we see that the relaxation by Steffensen and Ulbrich also finds the best function values, but is closely followed by Scholtes, the new relaxation, and Kadrani et al.

All in all, we have seen that the relaxation method by Steffensen and Ulbrich works extremely well without special tuning of the solver as it was proposed in the original work [109] although the feasible area of the relaxed problems does not have a strictly feasible interior and most constraint qualifications are violated. However, a certain instability can be observed since the final iterates either have an extremely small maximum constraint violation or they are barely feasible at all. The great success of this method may also be caused by the solver we used for the relaxed problems since `snopt` seems to be able to handle linearly dependent gradients very well. Hence, a relaxation of the kink in the origin might be all that was necessary to enable this solver to cope with MPCCs, which is exactly what the relaxation by Steffensen and Ulbrich does.

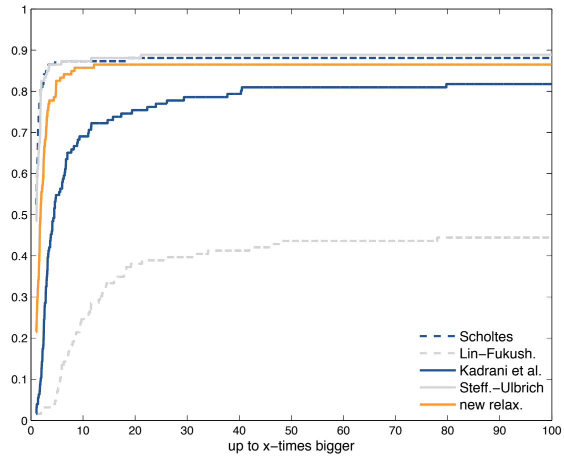
The oldest and simplest relaxation, namely the one by Scholtes, is still one of the most successful and stable numerical methods although most of the other methods have better theoretical properties. The relaxation method proposed by Lin and Fukushima is theoretically equivalent



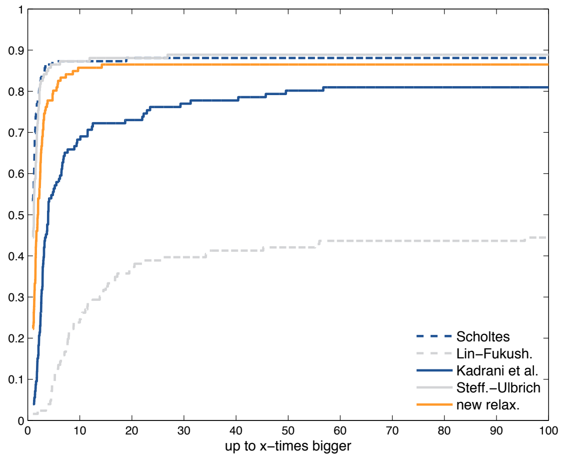
(a) Comparison of function value



(b) Comparison of elapsed time



(c) Comparison of objective function evaluations



(d) Comparison of gradient evaluations

Figure 9.2.: Comparison of function value and performance

to the one by Scholtes and has the advantage of needing less constraints. However, it has serious numerical problems if one uses only one relaxation parameter. Thus, we would propose to combine this method with an active set strategy like to the one used by Demiguel et al. in [29].

The relaxation method by Kadrani et al. and the new one have the same theoretical properties but behave slightly different when it comes to numerical results. The new relaxation method is faster, the final iterates have a smaller constraint violation, and it sometimes finds slightly smaller objective function values. However, the method by Kadrani et al. still works surprisingly well considering that the feasible set of the relaxed problems is almost disconnected and our algorithm does not incorporate any features to handle this difficulty. A longer running time compared to the other algorithms had to be expected due to the different termination criterion.

However, one should also take into account that the situation might be different when these solvers are applied to highly difficult MPCCs where C-stationary points attract those methods which, in general, converge to C-stationary points only. A corresponding (and sufficiently large) test suite of such problems is currently not available and, therefore, special tests on these kind of problems are not included.

10. Numerical Solution of the Effort Maximization Problem

Recall the effort maximization problem from the first part of this thesis. Here, we were able to give an analytic solution of the corresponding MPEC in the case of constant returns to scale. For another class of production technologies, we stated an existence result for the lower level contest game and we provided a reformulation as MPCC for both cases. Now, we want to apply the new relaxation method from Chapter 8 to these reformulated problems in order to solve them numerically. We begin with the case of constant returns to scale, where we already know the solution and thus are able to check whether the numerical method is successful or not. Afterwards, we try to solve the effort maximization problem with more complicated production technologies where no closed form of the solution is known.

Before we start with the numerical results, a few words on the implementation. As it was done throughout the first part of the thesis, we assume again that all contestants $\nu \in N$ have a valuation $V_\nu = 1$ of the prize. We use the MPCC reformulation derived in Section 3.2 and apply the new relaxation method from Chapter 8 using the algorithm and parameters described in Section 9.2. To avoid problems caused by the fact that the utility functions of the contestants are homogeneous in the designer's variable α, i.e., for every solution α^* of the effort maximization problem $c\alpha^*$ with $c > 0$ is a solution as well, we add the additional equality constraint

$$\sum_{\mu=1}^{n} \alpha_\nu = 1$$

to the reformulated problem. As initial point we use the vectors $\alpha_0 = \frac{1}{n}(1, \dots, 1)$ and $x_0 = (1, \dots, 1)$.

10.1. Verifying the Results for Constant Returns to Scale

In this section we will verify some results numerically that have already been derived analytically in Section 2.4. Let us begin with the 2-player case. Here, we know

$$\alpha^* = \frac{1}{\beta_1 + \beta_2}\big(\beta_1, \beta_2\big) \quad \text{and} \quad x^* = \frac{1}{4}\Big(\frac{1}{\beta_1}, \frac{1}{\beta_2}\Big).$$

We ran our algorithm for some combinations of cost parameters and obtained the results from Table 10.1.

As one can see, the numerical results coincide exactly with the theoretical ones.

β	α^*	x^*
(1, 1)	(0.500, 0.500)	(0.250, 0.250)
(1, 2)	(0.333, 0.667)	(0.250, 0.125)
(1, 3)	(0.250, 0.750)	(0.250, 0.083)
(1, 10)	(0.091, 0.909)	(0.250, 0.025)

Table 10.1.: Numerical results for the 2-player case

Next, we consider the homogeneous n-player case with different numbers of contestants n. For simplicity, we choose $\beta_\nu = 1$ for all contestants. This yields the results from Table 10.2. For comparison, the theoretically derived optimum values in this case are

$$\alpha_\nu^* = \frac{1}{n} \quad \text{and} \quad x_\nu^* = \frac{n-1}{n^2}$$

for all $\nu = 1, \dots, n$.

n	$\alpha_\nu^*, \nu \in N$	$x_\nu^*, \nu \in N$
3	0.333	0.222
4	0.250	0.1875
5	0.200	0.160
10	0.100	0.090

Table 10.2.: Numerical results for the homogeneous case

Here, again, the numerical results are exactly the ones we have already derived theoretically. These results encourage us to try and solve the effort maximization problem with more complicated production technologies numerically. This is done in the next section.

10.2. Solving the Problems from the Outlook

From the case of linear returns to scale where the production technology is given by $c(x) = x$ we now move on to more general production technologies. In Section 3.1, we imposed the assumptions $c(0) = 0$, $c'(x) > 0$ and $c''(x) < 0$ for all $x > 0$. One such function is $c(x) = \sqrt{x}$. We ran some numerical tests using this production technology and obtained the following results. Let us begin again with the 2-player case and the results given in Table 10.3.
Here, we see some resemblances to the case of constant returns to scale. A player with a higher cost parameter β_ν also gets a higher weight α_ν, but the difference is not completely removed as in the previous case. Analogously to the previous case, however, the optimal effort of the first player stays fixed with his costs whereas the optimal effort of the second player decreases with rising costs.

β	α^*	x^*
(1, 1)	(0.500, 0.500)	(0.125, 0.125)
(1, 2)	(0.414, 0.586)	(0.125, 0.0625)
(1, 3)	(0.367, 0.634)	(0.125, 0.042)
(1, 10)	(0.240, 0.760)	(0.125, 0.0125)

Table 10.3.: Numerical results for the 2-player case

n	$\alpha^*_\nu, \nu \in N$	$x^*_\nu, \nu \in N$
3	0.333	0.111
4	0.250	0.094
5	0.200	0.0800
10	0.100	0.045

Table 10.4.: Numerical results for the homogeneous case

Next, we considered again the homogeneous n-player case as above for different numbers of players and $\beta_\nu = 1$ for all of them. This led to the results from Table 10.4.
If we compare this table with the results for the homogeneous case with constant returns to scale, we see a striking resemblance. The optimal weights coincide, which could be expected since we consider homogeneous contestants and normalized the sum of the weights to 1, but the optimal effort in the case of constant returns to scale is exactly twice the optimal effort we obtain for the production technology $c(x) = \sqrt{x}$. This relation can also be observed in our results for the 2-player case. Since we know already that the homogeneous and the 2-player case are somewhat special, we also considered two inhomogeneous 4-player situations that were already discussed in Section 2.4. Using the production technology $c(x) = \sqrt{x}$, we obtained the results displayed in Table 10.5.

β	α^*	x^*
(1,2,2,4)	(0.214, 0.393, 0.393, 0.000)	(0.097, 0.058, 0.058, 0.000)
(1,2,2,6)	(0.221, 0.389, 0.389, 0.000)	(0.100, 0.058, 0.058, 0.000)

Table 10.5.: Numerical results for some inhomogeneous cases

If we compare these results with the ones obtained for the case on constant returns to scale, the optimal effort is not exactly half as big but in that scale. The difference may be due to numerical inaccuracy or due to the fact that the relation between the optimal efforts is not that simple. Nonetheless, the results of this section give hope. First of all, it seems to be possible to solve the effort maximization problem numerically if there is no analytical solution known. And secondly, the results exhibit a certain pattern that might be useful if one tries to obtain an analytical solution.

10.3. Concluding Remarks

In this part of the thesis, we gave a theoretical and numerical comparison of five different relaxation schemes for the solution of mathematical programs with complementarity constraints. First, we improved a number of existing convergence results, and also added some completely new results regarding the satisfaction of standard constraint qualifications for the relaxed problems for four existing methods. Then we introduced a new relaxation approach which can be seen as an enhancement of the method by Kadrani et al. and analyzed its theoretical properties. The new method was shown to converge at least to M-stationary points which is a much stronger property than what is known for the majority of other regularization methods. Moreover, convergence to these M-stationary points (and also to S-stationary points under an additional condition) is shown under significantly weaker assumptions than those used previously in related approaches. Additionally, we gave a condition under which the relaxed problems have local minima in the neighborhood of a solution of the MPCC.

The numerical comparison was surprisingly won by the method from Steffensen and Ulbrich although their relaxed subproblems have rather bad properties and we did not apply any of the techniques they suggested to handle this problem. However, we are not sure whether this success is partly due to the nonlinear program solver we used since their method is the one which changes the original problem the least. To answer this question fully, an extended comparison using different NLP solvers is necessary. On the other hand, when we focus on the found function values and the calculation time, this method is closely followed by the oldest and simplest relaxation we considered, namely the one from Scholtes, and by the new relaxation we introduced in this thesis, which are also both less dependent on the chosen parameters. Furthermore, we believe that the method by Kadrani et al. and, especially, the new method from this thesis will eventually outperform the other methods when applied to difficult MPCCs which have C-stationary points attracting the other methods which, in general, converge to C-stationary points only. Unfortunately, we are not aware of a sufficiently large collection of such problems and thus have to leave this topic for future research.

Finally, we used the new relaxation method introduced in this part to solve the effort maximization problem from Part I where we already provided a reformulation of this problem as MPCC. First, we successfully verified the theoretical results which we derived for the case of constant returns to scale in the first part of this thesis and then we attempted to solve more complicated effort maximization problems for which no solutions are known. The results are promising and can be used as a basis for future theoretical and numerical analysis of this problem class.

Final Remarks

To round this thesis off, we would like to collect the new results presented here and at the same time talk about open questions.

In the first part we considered an economic application of MPECs, the effort maximization problem in asymmetric n-person contest games. In the case of constant returns to scale, we were able to prove existence of a solution and give a closed form thereof. So far, this had only been done for two players or n homogeneous players. Afterwards, we defined a subclass of general production technologies, for which the existence of a solution of the underlying contest game is known, and provided a reformulation of these MPECs as MPCCs. This reformulation was then used to solve these effort maximization problems numerically using a new relaxation method introduced in this thesis. After recovering the known results for constant returns to scale, we also tested a more general production technology and the numerical results exhibited a certain pattern. This gives hope that a further theoretical analysis of these more general effort maximization problems might be successful. However, when we consider a more complicated production technology than in the constant returns to scale case, we cannot expect to obtain a closed form of the solution of the contest game anymore. For this reason, the implicit programming approach we used to solve the MPEC does not work anymore. Hence, one would have to use a different approach to prove the existence of a solution or even obtain a formula for it in this case.

In the second part of this thesis, we focused on MPCCs and derived enhanced Fritz-John conditions for this problem class. These conditions differ from the previously known ones by some additional conditions for the case in which the multiplier corresponding to the gradient of the objective function vanishes. These additional conditions gave rise to two new and comparatively weak constraint qualifications for MPCCs. One of them could be used for a very simple proof of the fact that most of the commonly used constraint qualifications for MPCCs imply M-stationarity of local minima. This result is not new, but all other proofs known to us are much more involved. The other constraint qualification could be used to prove exactness of a penalty function under weaker assumptions than the common ones. Additionally, we illustrated how the new constraint qualifications fit into the existing system of MPCC constraint qualifications. For future research, it might be interesting to analyze how the exactness of the penalty function can be used in numerical methods for the solution of MPCCs. From standard optimization we know that penalty functions can sometimes be used in combination with other algorithms as merit functions for example to determine steplengths.

In the final part, we turned to one of the numerous numerical approaches for the solution of MPCCs, the relaxation methods. We improved the convergence results of four existing methods and introduced a new relaxation method with very strong convergence properties where, at the same time, the relaxed problems maintain a nice feasible set. Additionally, we analyzed for all five methods what kind of constraint qualification the relaxed problems inherit from the MPCC which is important for the existence of Lagrange multipliers in solutions of the relaxed problems. For the new method, we also provided a conditions for the existence of local minima of the relaxed problems. Afterwards, we gave both a theoretical and numerical comparison of the five methods. The numerical comparison based on the MacMPEC testsuite was surprisingly won by the relaxation method with medium convergence properties where the feasible set is only locally relaxed around the kink in the origin and therefore still many constraint qualifications are violated in significant parts of the feasible set. The new relaxation method introduced in this thesis and the oldest one considered here, with weak convergence properties but very regular relaxed feasible sets, were also quite successful. To verify that the results of this comparison are actually due to the properties of the respective methods and not caused by the underlying NLP solver, it would be interesting to repeat the experiment with different solvers for the relaxed subproblems. Since the five methods considered here differ in the kind of stationarity that can be guaranteed in the limit points, it would also be interesting to compare their behavior when they are applied to complicated MPCCs which have many C-stationary points which are not local minima but could attract some of the methods. Unfortunately, we are not aware of any large enough collection of such problems.

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Results of the MacMPEC Collection

A.1. Numerical Results for the Relaxation by Scholtes

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
bar-truss-3	1.01666e+04	1.18216e-07	2.51238e-01	55	51
bard1	1.70000e+01	2.08232e-10	4.16520e-02	43	37
bard2	-6.59800e+03	1.42109e-14	1.26670e-02	10	8
bard3	-1.26787e+01	0.00000e+00	8.32100e-03	6	4
bard1m	1.70000e+01	1.68467e-07	1.89730e-02	17	11
bard2m	-6.59800e+03	1.42109e-14	1.14250e-02	10	8
bard3m	-1.26787e+01	7.13923e-10	2.08820e-02	19	13
bilevel1	5.00000e+00	7.05880e-07	3.81810e-02	40	36
bilevel2	-6.60000e+03	7.10543e-15	1.65390e-02	15	13
bilevel3	-1.26787e+01	7.13924e-10	2.37000e-02	22	16
bilevel2m	-6.60000e+03	7.10543e-15	1.67590e-02	15	13
bilin	-1.84000e+01	1.27548e-09	4.04860e-02	40	34
dempe	2.82501e+01	6.01502e-07	3.64673e-01	428	426
design-cent-1	-1.86065e+00	1.25774e-07	2.12840e-02	17	11
design-cent-2	-3.48382e+00	2.65475e-09	2.96950e-02	26	20
design-cent-21	-3.48382e+00	2.53828e-09	7.70970e-02	77	71
design-cent-3	-3.72337e+00	2.34118e-09	3.57070e-02	32	26
design-cent-31	-3.72337e+00	2.24635e-09	1.04793e-01	99	93
design-cent-4	-3.07920e+00	4.30808e-09	6.99680e-02	61	55
desilva	-1.00000e+00	6.59830e-08	1.06690e-02	7	5
df1	1.23260e-32	0.00000e+00	6.76000e-03	4	2
ex9.1.1	-1.30000e+01	7.10543e-15	1.25030e-02	11	9
ex9.1.2	-6.25000e+00	6.25000e-10	2.09950e-02	18	12
ex9.1.3	-2.92000e+01	1.56250e-09	6.09680e-02	56	50
ex9.1.4	-3.70000e+01	0.00000e+00	1.22710e-02	11	9
ex9.1.5	-1.00000e+00	6.25004e-10	6.08020e-02	22	16
ex9.1.6	-4.90000e+01	2.08414e-06	4.40747e-01	362	332
ex9.1.7	-2.30000e+01	1.24999e-09	3.84850e-02	38	32
ex9.1.8	-3.25000e+00	0.00000e+00	1.12080e-02	10	8

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
ex9.1.9	3.11111e+00	6.21742e-10	4.28360e-02	45	39
ex9.1.10	-3.25000e+00	0.00000e+00	1.12410e-02	10	8
ex9.2.1	1.70000e+01	2.08333e-10	4.51990e-02	48	42
ex9.2.2	9.99996e+01	4.08656e-05	1.83310e-01	204	174
ex9.2.3	5.00000e+00	0.00000e+00	1.22390e-02	11	9
ex9.2.4	5.00000e-01	6.24996e-10	2.79090e-02	28	22
ex9.2.5	9.00000e+00	6.24999e-10	3.15560e-02	32	26
ex9.2.6	-1.00000e+00	6.25000e-10	2.31750e-02	21	15
ex9.2.7	1.70000e+01	2.08333e-10	4.49990e-02	48	42
ex9.2.8	1.50000e+00	6.25000e-10	1.56380e-02	13	7
ex9.2.9	2.00000e+00	0.00000e+00	2.15700e-02	24	22
flp2	3.15879e-12	1.03807e-09	3.82120e-02	41	35
flp4-1	3.86094e-27	0.00000e+00	3.40950e-02	7	5
flp4-2	4.32422e-26	0.00000e+00	1.47258e-01	8	6
flp4-3	9.79091e-09	0.00000e+00	4.03728e-01	8	6
flp4-4	1.78075e-33	0.00000e+00	4.27507e-01	10	8
gauvin	2.00000e+01	1.56250e-10	2.36810e-02	20	14
gnash10	-2.30823e+02	9.12184e-08	2.55700e-02	24	20
gnash11	-1.29912e+02	8.08677e-08	2.24750e-02	20	16
gnash12	-3.69331e+01	6.88653e-08	2.18030e-02	19	15
gnash13	-7.06178e+00	6.21593e-08	2.79530e-02	26	22
gnash14	-1.79046e-01	5.78862e-08	3.48920e-02	34	30
gnash15	-3.54699e+02	1.42931e-09	2.83850e-02	26	20
gnash16	-2.41442e+02	4.07567e-10	2.62540e-02	23	17
gnash17	-9.07491e+01	5.80915e-10	3.04770e-02	28	22
gnash18	-2.56982e+01	8.57804e-09	5.29030e-02	53	47
gnash19	-6.11671e+00	1.25058e-09	3.63060e-02	34	28
hs044-i	1.56178e+01	1.30381e-04	8.75850e-02	74	44
incid-set1-8	2.42861e-17	4.68668e-08	5.02540e-02	9	7
incid-set1c-8	3.81639e-17	2.01993e-11	5.64760e-02	11	9
incid-set2-8	5.04269e-03	2.22963e-09	1.50505e-01	40	38
incid-set2c-8	5.63006e-03	5.12314e-07	1.35631e-01	35	33
jr1	5.00000e-01	0.00000e+00	9.90700e-03	7	5
jr2	5.00000e-01	1.25000e-09	3.75480e-02	41	35
kth1	0.00000e+00	0.00000e+00	6.14300e-03	4	2
kth2	0.00000e+00	0.00000e+00	9.31700e-03	8	6
kth3	5.00000e-01	6.25000e-10	2.69100e-02	27	21
liswet1-050	1.39943e-02	1.21014e-14	4.49090e-02	9	7

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
monteiro	-3.75300e+01	6.24497e-07	3.31502e-01	54	50
monteiroB	-8.27859e+02	4.69086e-10	6.67696e+00	2619	2613
nash1a	7.24562e-12	6.25003e-10	3.56360e-02	36	30
nash1b	6.17695e-12	6.25001e-10	3.20540e-02	33	27
nash1c	5.16568e-12	6.24998e-10	3.27480e-02	34	28
nash1d	6.56640e-12	6.24997e-10	3.51470e-02	37	31
nash1e	1.28871e-25	6.24996e-10	3.35820e-02	35	29
outrata31	3.20770e+00	4.20708e-10	2.46670e-02	24	18
outrata32	3.44940e+00	4.46441e-10	3.08820e-02	31	25
outrata33	4.60425e+00	5.28384e-10	3.12960e-02	32	26
outrata34	6.59268e+00	7.02093e-10	3.67460e-02	39	33
pack-comp1-8	6.00000e-01	9.93868e-09	1.38479e-01	28	18
pack-comp1c-8	6.00000e-01	1.26819e-12	1.50343e-01	28	18
pack-comp1p-8	6.00000e-01	5.37764e-17	1.12996e-01	25	17
pack-comp2-8	6.73117e-01	3.09118e-09	5.91920e-02	13	11
pack-comp2c-8	6.73458e-01	4.18032e-10	5.85980e-02	15	13
pack-comp2p-8	6.71475e-01	3.16368e-07	1.57937e-01	54	52
pack-rig1-8	7.87932e-01	3.29597e-17	1.09356e-01	36	32
pack-rig1c-8	7.88300e-01	1.11022e-16	5.69110e-02	19	15
pack-rig1p-8	7.87932e-01	1.47775e-05	6.63921e-01	102	72
pack-rig2-8	7.80404e-01	6.60809e-10	4.59340e-02	18	16
pack-rig2c-8	7.99306e-01	2.46743e-09	3.87920e-02	14	12
pack-rig2p-8	7.80404e-01	1.86846e-08	1.08243e-01	38	36
pack-rig3-8	7.35202e-01	1.04888e-13	1.86321e-01	32	28
pack-rig3c-8	7.53473e-01	2.48979e-06	7.23833e-01	198	170
portfl-i-1	1.50222e-05	3.83266e-07	9.22110e-02	55	49
portfl-i-2	1.45721e-05	3.79341e-07	8.06690e-02	51	45
portfl-i-3	6.26452e-06	1.59704e-07	7.24570e-02	40	34
portfl-i-4	2.16138e-06	9.96178e-05	2.56323e-01	143	113
portfl-i-6	2.34136e-06	1.20720e-04	2.48847e-01	148	118
qpec-100-1	9.90027e-02	1.15686e-08	6.60459e-01	66	60
qpec-100-2	-6.59073e+00	5.99660e-09	1.35432e+00	106	100
qpec-100-3	-5.47715e+00	3.37762e-05	1.52415e+00	90	60
qpec-100-4	-3.98212e+00	1.69352e-08	1.13249e+00	56	50
qpec-200-1	-1.93483e+00	1.93369e-07	5.96113e+00	67	61
qpec-200-2	-2.41037e+01	4.35327e-08	6.46649e+00	84	78
qpec-200-3	-1.93270e+00	2.63689e-07	1.77367e+01	113	107
qpec-200-4	-6.19870e+00	6.44300e-08	6.96887e+00	76	70

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
qpec1	8.00000e+01	0.00000e+00	4.32770e-02	8	6
qpec2	4.49624e+01	9.39909e-04	1.96946e-01	146	116
ralph1	-8.59881e-04	8.59881e-04	1.60232e-01	175	145
ralph2	-1.38716e-07	2.60777e-04	8.16220e-02	75	45
ralphmod	-5.12212e+02	6.79876e-06	1.31814e+01	1296	1288
scale1	1.00000e+00	6.25000e-10	1.12603e-01	37	31
scale2	1.00000e+00	6.25000e-10	8.09600e-02	26	20
scale3	1.00000e+00	6.25000e-10	2.87190e-02	28	22
scale4	1.00000e+00	1.74052e-10	4.33250e-02	47	41
scale5	1.00000e+02	6.25000e-10	5.84070e-02	66	60
scholtes1	2.00000e+00	0.00000e+00	1.58300e-02	15	13
scholtes2	1.50000e+01	0.00000e+00	1.23840e-02	11	9
scholtes3	5.00000e-01	6.25000e-10	1.08048e-01	130	124
scholtes4	-5.01113e-04	2.50556e-04	1.54063e-01	171	141
scholtes5	1.00000e+00	0.00000e+00	8.44700e-03	6	4
sl1	1.00000e-04	2.22045e-16	9.28200e-03	7	5
stackelberg1	-3.26667e+03	0.00000e+00	1.15780e-02	10	8
tap-09	1.09131e+02	5.08183e-06	1.75708e+00	879	849
tap-15	1.85141e+02	2.77441e-07	3.27859e+01	2700	2686
water-fl	3.33608e+03	6.19455e-07	5.13455e+00	724	718
water-net	9.27245e+02	9.04602e-04	5.85594e+01	55197	55167

A.2. Numerical Results for the Relaxation by Lin & Fukushima

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
bar-truss-3	1.01666e+04	5.96047e-07	4.03349e-01	195	145
bard1	2.50000e+01	5.96047e-07	2.09125e-01	215	165
bard2	-6.87800e+03	1.99691e+00	1.16404e+00	1391	1353
bard3	-1.26787e+01	5.96038e-07	1.42833e-01	128	78
bard1m	2.50000e+01	5.96047e-07	2.60148e-01	271	221
bard2m	-6.60000e+03	2.09491e+00	8.45236e-01	494	386
bard3m	-1.03600e+01	2.98025e-07	2.44483e-01	252	200
bilevel1	-1.06581e-13	5.96046e-07	1.91393e-01	187	137
bilevel2	-6.60000e+03	5.96047e-07	2.88137e-01	268	218
bilevel3	-1.03600e+01	2.98025e-07	1.93627e-01	185	133
bilevel2m	-6.60000e+03	5.96047e-07	2.86602e-01	268	218
bilin	-1.84000e+01	5.96057e-07	2.83771e-01	293	241
dempe	2.82517e+01	6.76109e-03	3.26023e+00	3962	3942
design-cent-1	-1.06765e+09	2.07351e-01	5.04317e+00	6101	6089
design-cent-2	-3.48382e+00	3.72335e-07	1.90869e-01	185	153
design-cent-21	-3.48382e+00	4.70463e-07	4.11091e-01	420	370
design-cent-3	-1.37402e+20	1.00682e+00	4.44433e-01	403	307
design-cent-31	-3.72337e+00	1.32214e-06	1.07950e+00	1119	1011
design-cent-4	-3.07920e+00	5.96047e-07	2.23787e-01	200	150
desilva	-1.50000e+00	1.00000e+00	1.00277e+00	1121	1013
df1	1.58718e-13	9.09498e-04	2.14897e-01	184	76
ex9.1.1	-1.30000e+01	2.98023e-07	2.82996e-01	283	231
ex9.1.2	-6.25000e+00	1.49013e-07	1.53422e-01	144	90
ex9.1.3	-2.92000e+01	1.40501e-05	2.72721e-01	227	123
ex9.1.4	-3.70000e+01	1.49012e-07	4.17892e-01	461	407
ex9.1.5	-1.00000e+00	5.96046e-07	2.18627e-01	213	163
ex9.1.6	-4.90000e+01	5.96047e-07	3.25755e-01	332	282
ex9.1.7	-2.60000e+01	2.98023e-07	4.89989e-01	515	463
ex9.1.8	-3.25000e+00	6.19243e-04	5.99201e-01	222	116
ex9.1.9	3.11111e+00	5.96046e-07	3.42201e-01	350	300
ex9.1.10	-3.25000e+00	6.19243e-04	2.52119e-01	222	116
ex9.2.1	2.50000e+01	5.96047e-07	2.21175e-01	211	161
ex9.2.2	9.99996e+01	3.71776e-05	5.19870e-01	523	415
ex9.2.3	-9.50001e+00	1.45000e+01	3.87427e-01	396	348
ex9.2.4	4.99999e-01	5.96046e-07	1.64536e-01	156	110

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
ex9.2.5	9.00000e+00	5.96046e-07	1.72437e-01	161	113
ex9.2.6	-1.50000e+00	1.00000e+00	3.69147e-01	331	223
ex9.2.7	2.50000e+01	5.96047e-07	2.22352e-01	211	161
ex9.2.8	1.50000e+00	2.32831e-08	6.69350e-02	56	28
ex9.2.9	2.00000e+00	1.49012e-07	1.68212e-01	156	102
flp2	1.98047e-12	2.98031e-07	2.96483e-01	301	249
flp4-1	-2.54020e+02	3.30265e+01	3.78494e+01	12073	11965
flp4-2	9.58084e+00	6.76348e-01	6.21233e+01	7191	7145
flp4-3	8.92240e+01	2.51648e+00	1.05447e+02	7224	7174
flp4-4	-3.98669e+02	7.12790e+00	2.30310e+02	4463	4411
gauvin	2.00000e+01	5.96046e-07	3.43414e-01	381	331
gnash10	-2.30823e+02	5.96046e-07	2.35009e-01	226	176
gnash11	-1.29912e+02	7.45057e-08	3.74871e-01	384	328
gnash12	-3.69331e+01	2.98023e-07	2.75715e-01	269	217
gnash13	-7.06179e+00	5.96046e-07	2.04610e-01	187	137
gnash14	-1.79047e-01	5.96046e-07	2.07360e-01	190	140
gnash15	-3.54699e+02	5.96046e-07	2.53182e-01	249	199
gnash16	-2.41442e+02	2.98023e-07	2.97546e-01	299	247
gnash17	-9.07491e+01	5.96046e-07	2.35129e-01	224	174
gnash18	-2.56982e+01	5.96046e-07	2.30584e-01	218	168
gnash19	-6.11671e+00	5.96046e-07	2.12378e-01	196	146
hs044-i	1.56178e+01	8.03642e-07	4.10909e-01	396	346
incid-set1-8	-3.85615e-04	9.58035e-06	9.68037e+00	2702	2594
incid-set1c-8	-3.94056e-04	9.98570e-06	5.06357e+00	1527	1419
incid-set2-8	Inf	Inf	Inf	Inf	Inf
incid-set2c-8	Inf	Inf	Inf	Inf	Inf
jr1	4.99999e-01	1.16992e-06	3.39003e-01	325	217
jr2	4.99997e-01	2.98024e-06	2.69912e-01	251	149
kth1	-2.19353e-08	2.05129e-08	1.09260e-02	9	7
kth2	-5.96046e-07	5.96046e-07	2.18116e-01	221	171
kth3	4.99999e-01	1.49012e-06	2.29263e-01	205	103
liswet1-050	1.39522e-02	5.96046e-07	3.68281e+00	702	668
monteiro	-3.75300e+01	1.27160e-05	9.26217e+00	3434	3326
monteiroB	-8.27860e+02	5.96047e-07	1.24456e+01	4828	4778
nash1a	1.95769e-12	2.98021e-07	2.96004e-01	275	225
nash1b	1.95769e-12	2.98021e-07	2.80430e-01	277	229
nash1c	1.95769e-12	2.98021e-07	2.61828e-01	265	219
nash1d	1.76853e-12	1.72675e-06	2.82036e-01	258	154

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
nash1e	2.39999e-12	2.98022e-07	2.56915e-01	257	209
outrata31	2.39121e+00	1.00601e+00	6.27278e-01	695	651
outrata32	3.44940e+00	5.96046e-07	2.23457e-01	212	162
outrata33	4.60425e+00	5.96047e-07	2.13537e-01	201	151
outrata34	6.59268e+00	5.96047e-07	2.15474e-01	198	148
pack-comp1-8	6.00000e-01	4.06231e-04	8.98300e-01	215	107
pack-comp1c-8	6.00000e-01	6.25060e-04	9.26873e-01	213	105
pack-comp1p-8	-8.32071e+02	2.83987e-02	4.32776e+01	18557	18449
pack-comp2-8	6.00000e-01	3.80123e-03	2.12724e+01	7358	7250
pack-comp2c-8	6.00000e-01	3.80469e-03	1.00404e+01	3399	3291
pack-comp2p-8	9.15823e-01	3.51348e-04	5.21385e+00	1791	1683
pack-rig1-8	7.87932e-01	5.78500e-13	1.80403e+01	10277	10185
pack-rig1c-8	7.88300e-01	5.68435e-13	7.89672e+00	4429	4337
pack-rig1p-8	6.00000e-01	3.28295e-02	2.40226e+01	11839	11731
pack-rig2-8	7.80405e-01	1.94519e-10	1.14807e+01	6620	6546
pack-rig2c-8	6.31943e-01	4.35760e-02	1.06680e+01	5456	5348
pack-rig2p-8	6.00000e-01	3.60353e-02	3.11635e+01	14877	14769
pack-rig3-8	7.34735e-01	2.17978e-05	8.11656e+00	4725	4617
pack-rig3c-8	7.53474e-01	7.52985e-05	1.82274e+00	838	730
portfl-i-1	1.50027e-05	5.96024e-07	4.25499e-01	227	187
portfl-i-2	1.45713e-05	3.52773e-05	1.12925e+00	460	362
portfl-i-3	6.25371e-06	5.96046e-07	4.48247e-01	234	194
portfl-i-4	2.16330e-06	1.08690e-04	9.03897e-01	473	377
portfl-i-6	2.35816e-06	3.41753e-05	8.26832e-01	446	350
qpec-100-1	2.52407e-01	1.61225e-02	5.03175e+00	446	338
qpec-100-2	-1.05204e+01	3.68689e+00	4.04193e+01	5165	5057
qpec-100-3	-5.48350e+00	1.00121e-02	6.60893e+00	452	344
qpec-100-4	-4.30216e+00	7.60920e-01	3.59191e+01	3666	3558
qpec-200-1	-1.88026e+00	7.25141e-03	1.15121e+02	2105	1997
qpec-200-2	-2.38497e+01	1.48627e-01	5.47664e+01	972	864
qpec-200-3	-1.95790e+00	6.88594e-03	1.49785e+02	959	851
qpec-200-4	-7.04238e+00	6.52085e-01	2.79958e+02	4658	4550
qpec1	7.99932e+01	3.39344e-04	2.11456e+00	1911	1803
qpec2	4.49756e+01	6.10368e-04	5.28532e-01	458	354
ralph1	-6.10392e-04	6.10392e-04	5.51721e-01	588	480
ralph2	-7.45510e-07	6.10513e-04	2.94721e-01	278	170
ralphmod	-6.83033e+02	7.45063e-08	5.21258e+01	6103	6047
scale1	9.99702e-01	1.49034e-06	2.63118e-01	248	152

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
scale2	9.99997e-01	1.49012e-06	2.30379e-01	207	105
scale3	9.99702e-01	1.49012e-06	2.21683e-01	201	105
scale4	9.92550e-01	3.73914e-05	2.17059e-01	199	109
scale5	9.99997e+01	1.49012e-06	3.00681e-01	290	188
scholtes1	2.00000e+00	1.89666e-06	4.33266e-01	440	332
scholtes2	1.17078e+01	1.33315e+00	4.57799e-01	466	358
scholtes3	4.99999e-01	1.49012e-06	2.62304e-01	245	143
scholtes4	-1.22111e-03	6.10557e-04	4.37020e-01	447	339
scholtes5	9.99998e-01	7.89520e-07	2.64167e-01	276	224
sl1	9.99047e-05	4.76769e-06	3.46571e-01	333	225
stackelberg1	-3.26667e+03	7.15228e-14	1.31566e+00	1552	1458
tap-09	1.09131e+02	6.03046e-07	1.76439e+00	884	834
tap-15	1.84295e+02	3.92242e-04	2.93098e+01	4593	4485
water-fl	3.35308e+03	6.44534e-07	5.32122e+00	1398	1376
water-net	9.30655e+02	1.93320e-03	2.86921e+02	263401	263313

A.3. Numerical Results for the Relaxation by Kadrani et al.

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
bar-truss-3	1.01666e+04	2.03499e-11	5.02446e-01	223	207
bard1	1.70000e+01	7.15428e-13	1.63081e-01	204	188
bard2	-6.59800e+03	1.00002e-08	5.16130e-02	46	30
bard3	-1.26787e+01	2.00000e-08	5.70170e-02	54	38
bard1m	1.70000e+01	9.97149e-11	8.26940e-02	92	76
bard2m	-6.59800e+03	1.00002e-08	5.07630e-02	46	30
bard3m	-1.26787e+01	3.00024e-08	4.49680e-02	40	24
bilevel1	5.00000e+00	4.00000e-08	4.77830e-02	43	27
bilevel2	-6.60000e+03	1.00000e-08	6.99150e-02	67	51
bilevel3	-1.26787e+01	9.48537e-10	5.87190e-02	55	39
bilevel2m	-6.60000e+03	1.00000e-08	7.04050e-02	67	51
bilin	-1.30000e+01	1.00000e-08	5.03960e-02	46	30
dempe	2.82501e+01	1.00000e-06	1.56570e-01	185	169
design-cent-1	-1.86065e+00	1.00414e-06	1.49536e-01	166	150
design-cent-2	-9.89239e+30	1.00000e+00	1.54463e-01	168	152
design-cent-21	-3.48382e+00	3.65610e-07	1.10932e-01	115	99
design-cent-3	-2.97484e+20	1.16020e+00	1.06672e-01	101	85
design-cent-31	-3.72337e+00	2.64948e-07	3.94197e-01	218	202
design-cent-4	-1.58523e-09	1.00091e-10	2.91712e-01	86	70
desilva	-1.00000e+00	1.00000e-06	1.49584e-01	44	28
df1	2.61961e-25	1.01364e-12	9.47270e-02	63	47
ex9.1.1	-1.30000e+01	5.50000e-08	5.59700e-02	52	36
ex9.1.2	-6.25000e+00	1.00000e-10	9.40500e-02	107	91
ex9.1.3	-2.92000e+01	4.85000e-07	7.35650e-02	68	52
ex9.1.4	-3.70000e+01	9.95892e-07	1.09194e-01	127	111
ex9.1.5	-1.00000e+00	2.00002e-08	5.84510e-02	54	38
ex9.1.6	-1.50000e+01	9.99743e-09	5.99940e-02	56	40
ex9.1.7	-2.60000e+01	2.00000e-08	5.99210e-02	54	38
ex9.1.8	-3.25000e+00	1.00000e-08	4.63910e-02	41	25
ex9.1.9	2.00000e+00	1.00000e+00	6.57770e-02	64	48
ex9.1.10	-3.25000e+00	1.00000e-08	4.58260e-02	41	25
ex9.2.1	1.70000e+01	9.99972e-09	5.09060e-02	47	31
ex9.2.2	9.99996e+01	3.63707e-05	1.59628e-01	182	166
ex9.2.3	5.00000e+00	3.00000e-08	4.72330e-02	42	26
ex9.2.4	4.99998e-01	1.00000e-06	5.71750e-02	56	40
ex9.2.5	9.00000e+00	4.99998e-08	5.78590e-02	55	39
ex9.2.6	-1.00000e+00	1.98387e-04	8.01560e-02	77	61

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
ex9.2.7	1.70000e+01	9.99972e-09	5.13720e-02	47	31
ex9.2.8	1.50000e+00	2.00000e-08	4.42010e-02	40	24
ex9.2.9	2.00000e+00	2.00000e-08	5.55890e-02	54	38
flp2	9.70693e-12	1.00027e-08	7.15770e-02	74	58
flp4-1	-3.00000e-13	1.00000e-14	1.57344e-01	42	26
flp4-2	-6.00000e-13	1.00000e-14	6.06144e-01	45	29
flp4-3	1.57836e-11	1.00000e-14	1.03926e+00	49	33
flp4-4	-1.00000e-12	1.00000e-14	2.59781e+00	48	32
gauvin	2.00000e+01	2.00004e-10	6.23850e-02	62	46
gnash10	-2.30823e+02	2.00001e-08	5.52450e-02	48	32
gnash11	-1.29912e+02	2.00001e-08	5.61210e-02	49	33
gnash12	-3.69331e+01	2.00001e-08	5.34890e-02	47	31
gnash13	-7.06178e+00	2.00002e-08	5.35120e-02	47	31
gnash14	-1.79046e-01	2.00002e-08	7.31970e-02	69	53
gnash15	-3.54699e+02	2.00000e-08	6.14690e-02	55	39
gnash16	-2.41442e+02	2.00000e-08	7.27290e-02	69	53
gnash17	-9.07491e+01	2.00000e-08	1.56364e-01	73	57
gnash18	-2.56982e+01	2.00001e-08	1.27311e-01	88	72
gnash19	-6.11671e+00	2.00001e-08	8.26170e-02	80	64
hs044-i	1.70901e+01	7.99958e-08	8.26090e-02	75	59
incid-set1-8	-7.73441e-08	2.62745e-08	1.02265e+00	235	219
incid-set1c-8	-7.73441e-08	1.01673e-08	9.51389e-01	194	178
incid-set2-8	4.51779e-03	1.07397e-08	1.51740e+00	373	357
incid-set2c-8	5.47117e-03	1.01837e-08	1.11261e+00	234	218
jr1	4.99999e-01	1.00000e-06	4.93920e-02	47	31
jr2	4.99999e-01	1.00000e-06	6.34700e-02	64	48
kth1	-1.00000e-06	1.00000e-06	3.34550e-02	28	12
kth2	-1.00000e-14	1.00000e-14	3.52540e-02	30	14
kth3	5.00000e-01	1.00000e-14	4.93730e-02	48	32
liswet1-050	1.39936e-02	1.00000e-08	3.85709e+00	363	347
monteiro	-3.75300e+01	3.80142e-08	8.82733e+00	2037	2021
monteiroB	-8.23398e+02	1.86340e-12	9.48233e-01	192	176
nash1a	2.14130e-12	7.43165e-07	6.23510e-02	61	45
nash1b	8.91444e-12	9.99999e-09	7.29530e-02	75	59
nash1c	3.92324e-12	9.99999e-09	7.04740e-02	70	54
nash1d	2.32366e-13	6.67998e-07	8.21450e-02	86	70
nash1e	8.91491e-12	9.99999e-09	6.89140e-02	69	53
outrata31	3.20770e+00	5.81895e-07	2.88354e-01	353	337

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
outrata32	3.44940e+00	1.00009e-08	9.99970e-02	99	83
outrata33	4.60425e+00	2.00016e-08	8.58000e-02	84	68
outrata34	6.59268e+00	9.50862e-07	8.89030e-02	84	68
pack-comp1-8	6.00000e-01	3.88919e-12	1.07482e+01	2233	2217
pack-comp1c-8	6.00000e-01	3.95367e-12	3.87859e+00	793	777
pack-comp1p-8	6.00000e-01	3.07969e-13	2.06913e+01	6658	6642
pack-comp2-8	6.73084e-01	8.37383e-07	8.61376e+00	2482	2466
pack-comp2c-8	6.73458e-01	1.00700e-08	7.04186e+00	2462	2446
pack-comp2p-8	6.70711e-01	2.44114e-08	2.95997e+00	1206	1190
pack-rig1-8	7.87932e-01	3.00000e-10	3.95374e-01	161	145
pack-rig1c-8	7.88300e-01	3.00000e-08	6.50915e+00	3899	3883
pack-rig1p-8	7.87932e-01	2.21509e-05	9.70800e-01	299	283
pack-rig2-8	7.80404e-01	3.00000e-10	4.41904e-01	200	184
pack-rig2c-8	7.99305e-01	3.00001e-08	3.56426e-01	154	138
pack-rig2p-8	7.80404e-01	1.70352e-05	2.79509e+00	1293	1277
pack-rig3-8	7.35202e-01	9.58816e-14	4.96512e-01	215	199
pack-rig3c-8	7.53472e-01	2.98037e-08	4.38923e-01	189	173
portfl-i-1	1.50242e-05	7.79413e-14	2.03787e-01	118	102
portfl-i-2	1.45728e-05	1.24123e-13	1.94817e-01	118	102
portfl-i-3	6.26499e-06	1.05249e-13	2.09141e-01	118	102
portfl-i-4	2.16138e-06	9.95999e-05	2.13220e-01	123	107
portfl-i-6	2.34136e-06	1.20713e-04	1.85264e-01	105	89
qpec-100-1	1.32964e-01	1.41975e-08	1.58140e+00	173	157
qpec-100-2	-6.59074e+00	1.58143e-08	2.10409e+00	242	226
qpec-100-3	-5.46914e+00	1.39908e-08	2.77310e+00	179	163
qpec-100-4	-3.91201e+00	1.13905e-10	1.92382e+00	145	129
qpec-200-1	-1.85113e+00	2.83332e-11	1.41649e+01	166	150
qpec-200-2	-2.40386e+01	1.32060e-08	1.70272e+01	225	209
qpec-200-3	-1.79444e+00	2.01666e-14	4.44811e+01	362	346
qpec-200-4	-5.72570e+00	1.27753e-12	3.09359e+01	487	471
qpec1	8.00000e+01	1.00000e-14	2.42456e-01	243	227
qpec2	4.49675e+01	8.11609e-04	3.26495e-01	329	313
ralph1	-1.00034e-03	1.00034e-03	2.57465e-01	319	303
ralph2	-3.00021e-08	1.00004e-04	1.83707e-01	221	205
ralphmod	-6.83033e+02	1.03231e-04	2.10464e+01	3272	3256
scale1	1.00000e+00	1.00000e-14	5.88710e-02	58	42
scale2	9.99998e-01	1.00000e-06	6.07600e-02	62	46
scale3	1.00000e+00	1.00000e-14	4.86170e-02	47	31

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
scale4	1.00000e+00	1.00000e-14	1.15322e-01	133	117
scale5	9.99998e+01	1.00000e-06	8.22970e-02	90	74
scholtes1	2.00000e+00	1.00000e-14	1.58200e-01	192	176
scholtes2	1.50000e+01	2.00000e-08	4.04630e-02	36	20
scholtes3	4.99999e-01	1.00000e-06	1.20994e-01	142	126
scholtes4	-7.04708e-04	3.52354e-04	2.47191e-01	302	286
scholtes5	9.99998e-01	1.00000e-06	4.46250e-02	42	26
sl1	9.99998e-05	9.99999e-08	9.53270e-02	104	88
stackelberg1	-3.26667e+03	9.98047e-07	1.00433e-01	116	100
tap-09	1.02582e+02	4.00000e+01	5.70136e+00	3382	3366
tap-15	1.74399e+02	2.40000e+01	1.70916e+01	2747	2731
water-fl	3.46145e+03	4.00000e-08	7.89264e+00	2522	2506
water-net	9.27264e+02	4.00000e-08	7.30097e-01	591	575

A.4. Numerical Results for the Relaxation by Steffensen & Ulbrich

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
bar-truss-3	1.01666e+04	2.27374e-13	1.90175e-01	34	32
bard1	1.70000e+01	0.00000e+00	3.93950e-02	23	21
bard2	-6.59407e+03	7.10543e-15	2.34730e-02	20	18
bard3	-1.26787e+01	8.88178e-16	1.16660e-02	7	5
bard1m	1.70000e+01	0.00000e+00	2.18090e-02	20	18
bard2m	-6.59407e+03	7.10543e-15	2.25730e-02	20	18
bard3m	-1.26787e+01	0.00000e+00	3.74150e-02	35	31
bilevel1	5.00000e+00	0.00000e+00	1.56970e-02	13	11
bilevel2	-6.60000e+03	3.55271e-15	3.14590e-02	27	23
bilevel3	-1.26787e+01	8.88178e-16	4.67050e-02	44	40
bilevel2m	-6.60000e+03	3.55271e-15	3.19000e-02	27	23
bilin	-1.60000e+01	2.22045e-16	1.85410e-02	16	14
dempe	2.82501e+01	8.83020e-11	8.61110e-02	92	90
design-cent-1	-1.86065e+00	9.82307e-08	1.71990e-02	13	9
design-cent-2	-3.48382e+00	2.98572e-11	5.30140e-02	49	45
design-cent-21	-1.19203e-03	1.67176e-04	6.06762e+00	6558	6552
design-cent-3	-2.95037e+14	1.00000e+00	1.15612e-01	99	83
design-cent-31	-1.89280e-04	3.54194e-09	1.18848e-01	113	111
design-cent-4	-3.07920e+00	1.11022e-16	2.74930e-02	21	17
desilva	-1.00000e+00	0.00000e+00	1.11700e-02	8	6
df1	1.23260e-32	0.00000e+00	7.13100e-03	4	2
ex9.1.1	-7.21788e+00	3.55271e-15	3.06240e-02	28	26
ex9.1.2	-6.25000e+00	2.77556e-17	1.70750e-02	13	9
ex9.1.3	-2.30000e+01	1.11022e-16	4.77380e-02	43	39
ex9.1.4	-3.70000e+01	2.84217e-14	2.32280e-02	21	19
ex9.1.5	-1.00000e+00	5.00000e-01	1.27840e-02	8	4
ex9.1.6	-2.67587e+01	7.10543e-15	1.48308e-01	62	60
ex9.1.7	-2.30000e+01	4.87329e-01	1.25829e-01	36	32
ex9.1.8	-3.25000e+00	4.44089e-16	6.70480e-02	18	14
ex9.1.9	2.00000e+00	1.00000e+00	7.00830e-02	57	53
ex9.1.10	-3.25000e+00	4.44089e-16	2.19970e-02	18	14
ex9.2.1	4.21323e+01	8.88178e-16	4.66430e-02	44	40
ex9.2.2	1.00000e+02	4.96114e-15	3.51050e-02	29	21
ex9.2.3	5.00000e+00	7.10543e-15	2.31570e-02	20	18
ex9.2.4	5.00000e-01	2.77556e-17	2.27840e-02	19	15

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
ex9.2.5	9.00000e+00	1.77636e-15	4.03070e-02	38	34
ex9.2.6	-1.00000e+00	2.77556e-16	1.78210e-02	13	9
ex9.2.7	4.21323e+01	8.88178e-16	4.70650e-02	44	40
ex9.2.8	1.50000e+00	1.73472e-18	1.43440e-02	10	6
ex9.2.9	2.00000e+00	2.22045e-16	2.05670e-02	18	16
flp2	2.62201e-14	0.00000e+00	2.52290e-02	22	18
flp4-1	1.21260e-20	0.00000e+00	4.04070e-02	8	6
flp4-2	3.20776e-33	0.00000e+00	8.66780e-02	9	7
flp4-3	8.92592e-07	0.00000e+00	2.16681e-01	14	12
flp4-4	6.44442e-07	0.00000e+00	6.29260e-01	18	16
gauvin	2.00000e+01	1.42109e-14	1.32350e-02	8	6
gnash10	-2.30823e+02	2.69296e-10	2.83710e-02	24	22
gnash11	-1.29912e+02	2.35949e-09	2.56390e-02	21	19
gnash12	-3.69331e+01	2.98758e-10	3.14240e-02	27	25
gnash13	-7.06178e+00	3.42837e-13	3.60880e-02	32	30
gnash14	-1.79046e-01	5.94969e-10	6.24130e-02	58	56
gnash15	-3.54699e+02	1.97176e-13	2.75200e-02	22	18
gnash16	-2.41442e+02	2.15934e-06	4.83680e-02	44	40
gnash17	-9.07491e+01	2.73559e-13	3.55430e-02	30	26
gnash18	-2.56982e+01	2.10056e-09	6.34670e-02	58	54
gnash19	-6.11671e+00	2.27818e-12	4.27160e-02	37	33
hs044-i	1.70901e+01	2.22045e-16	7.32070e-02	67	61
incid-set1-8	4.68375e-16	4.68668e-08	5.97500e-02	9	7
incid-set1c-8	3.81639e-17	1.50605e-12	6.25300e-02	10	8
incid-set2-8	5.04269e-03	2.22963e-09	1.89086e-01	40	38
incid-set2c-8	5.63006e-03	5.12314e-07	1.70813e-01	35	33
jr1	5.00000e-01	0.00000e+00	1.05770e-02	7	5
jr2	5.00000e-01	0.00000e+00	2.01070e-02	17	13
kth1	0.00000e+00	0.00000e+00	6.92400e-03	4	2
kth2	0.00000e+00	0.00000e+00	9.63500e-03	7	5
kth3	5.00000e-01	0.00000e+00	2.45760e-02	22	18
liswet1-050	1.47719e-02	4.21885e-15	7.64828e-01	190	188
monteiro	1.48022e+02	5.68434e-14	3.38792e-01	78	76
monteiroB	-8.23398e+02	2.19380e-13	6.27277e-01	142	138
nash1a	2.04357e-16	4.99275e-17	2.34290e-02	19	15
nash1b	8.86294e-14	7.10543e-15	2.72160e-02	24	20
nash1c	4.83475e-15	3.55271e-15	2.63480e-02	23	19
nash1d	0.00000e+00	0.00000e+00	3.00300e-02	27	23

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
nash1e	2.85523e-12	0.00000e+00	2.72310e-02	24	20
outrata31	3.20770e+00	7.24445e-09	8.18990e-02	84	80
outrata32	3.44940e+00	4.34208e-12	5.75640e-02	57	53
outrata33	4.60425e+00	1.08017e-08	4.76870e-02	46	42
outrata34	6.59268e+00	3.40386e-12	4.03750e-02	38	34
pack-comp1-8	6.00000e-01	3.25662e-07	1.74647e-01	41	33
pack-comp1c-8	6.00000e-01	3.25662e-07	1.83758e-01	41	33
pack-comp1p-8	6.00000e-01	3.25662e-07	1.91399e-01	47	39
pack-comp2-8	6.73117e-01	3.09118e-09	6.52590e-02	13	11
pack-comp2c-8	6.73458e-01	4.18032e-10	6.79510e-02	15	13
pack-comp2p-8	6.73902e-01	3.90104e-08	2.07451e-01	64	62
pack-rig1-8	7.87932e-01	1.56125e-17	9.49980e-02	36	32
pack-rig1c-8	7.88300e-01	1.11022e-16	6.01780e-02	19	15
pack-rig1p-8	7.87931e-01	3.36072e-07	8.60569e-01	298	294
pack-rig2-8	7.80404e-01	6.60809e-10	4.89980e-02	18	16
pack-rig2c-8	7.99306e-01	2.46743e-09	4.28010e-02	14	12
pack-rig2p-8	7.80404e-01	1.86846e-08	1.21753e-01	38	36
pack-rig3-8	7.35202e-01	4.16334e-17	8.28120e-02	32	28
pack-rig3c-8	7.53473e-01	2.39098e-12	7.89980e-02	30	26
portfl-i-1	1.50242e-05	2.22045e-16	2.31584e-01	167	161
portfl-i-2	1.45728e-05	2.37005e-13	6.28130e-02	32	26
portfl-i-3	6.26498e-06	1.11022e-16	6.69350e-02	32	26
portfl-i-4	2.17734e-06	2.22045e-16	7.44060e-02	40	34
portfl-i-6	2.36133e-06	2.22045e-16	6.58420e-02	33	27
qpec-100-1	4.66581e-01	2.21228e-09	7.49299e-01	74	70
qpec-100-2	-6.23680e+00	1.88738e-15	7.33854e-01	85	79
qpec-100-3	-5.33808e+00	1.55431e-15	9.03232e-01	63	57
qpec-100-4	-3.04666e+00	8.88178e-16	1.00641e+00	96	90
qpec-200-1	-1.02302e+00	1.19127e-13	8.74641e+00	194	188
qpec-200-2	-2.23792e+01	3.55271e-15	1.91722e+01	490	484
qpec-200-3	-1.90909e+00	4.21885e-15	2.22152e+01	180	174
qpec-200-4	-3.91717e+00	3.99680e-15	7.06768e+00	75	69
qpec1	8.00000e+01	0.00000e+00	4.17340e-02	8	6
qpec2	2.69396e+01	5.00000e-01	2.40070e-01	154	138
ralph1	-5.00000e-01	5.00000e-01	1.05954e-01	109	93
ralph2	-5.00000e-01	5.00000e-01	1.06693e-01	109	93
ralphmod	-4.87591e+02	1.03574e-06	2.61306e+01	4710	4694
scale1	1.00000e+00	2.08167e-17	3.10340e-02	28	24

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
scale2	1.00000e+00	5.55112e-17	2.33170e-02	20	16
scale3	1.00000e+00	1.73472e-17	2.45270e-02	21	17
scale4	1.00000e+00	0.00000e+00	8.45620e-02	83	69
scale5	5.00000e+01	5.00000e-01	1.08422e-01	109	93
scholtes1	2.00000e+00	0.00000e+00	1.09310e-02	8	6
scholtes2	1.50000e+01	0.00000e+00	8.91900e-03	6	4
scholtes3	5.00000e-01	0.00000e+00	6.15030e-02	59	49
scholtes4	-1.00000e+00	5.00000e-01	1.14372e-01	113	97
scholtes5	1.00000e+00	0.00000e+00	1.44260e-02	11	9
sl1	1.00000e-04	9.79993e-14	9.77900e-03	6	4
stackelberg1	2.13369e+02	0.00000e+00	3.26010e-02	31	29
tap-09	1.00000e+02	4.00000e+01	3.44370e-02	17	15
tap-15	1.74000e+02	2.39880e+01	1.10056e-01	18	16
water-fl	3.37091e+03	1.13864e-11	5.42586e+00	1111	1105
water-net	9.53230e+02	5.34504e-10	8.15950e+00	6840	6834

A.5. Numerical Results for the New Relaxation

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
bar-truss-3	1.01666e+04	2.00000e-08	3.03214e+00	107	97
bard1	1.70000e+01	2.00011e-08	5.02410e-02	37	27
bard2	-6.59800e+03	7.10543e-15	2.29020e-02	17	15
bard3	-1.26787e+01	8.88178e-16	1.48520e-02	6	4
bard1m	1.70000e+01	2.00000e-08	4.47150e-02	37	27
bard2m	-6.59800e+03	7.10543e-15	2.30970e-02	17	15
bard3m	-1.26787e+01	2.00029e-08	9.95510e-02	92	82
bilevel1	5.00000e+00	2.00000e-08	8.52180e-02	73	63
bilevel2	-6.60000e+03	2.00000e-08	8.75550e-02	72	62
bilevel3	-1.26787e+01	2.00029e-08	6.05000e-02	47	37
bilevel2m	-6.60000e+03	2.00000e-08	8.75550e-02	72	62
bilin	-4.40000e+00	2.00000e-08	5.85370e-02	47	37
dempe	2.82501e+01	2.00000e-08	1.07732e-01	105	95
design-cent-1	-1.05132e+08	2.63201e+01	2.65438e+01	30548	30538
design-cent-2	-1.65583e+28	1.00000e+00	2.53472e-01	253	237
design-cent-21	-3.48382e+00	8.22285e-07	1.27990e-01	121	111
design-cent-3	-8.51113e+27	1.00000e+00	2.97723e-01	301	285
design-cent-31	-3.72337e+00	2.00018e-08	2.39756e-01	237	227
design-cent-4	-3.07920e+00	2.00356e-08	7.33440e-02	61	51
desilva	-1.00000e+00	6.59830e-08	1.09020e-02	7	5
df1	1.23260e-32	0.00000e+00	8.76500e-03	4	2
ex9.1.1	-1.30000e+01	6.92779e-14	4.43570e-02	35	25
ex9.1.2	-6.25000e+00	0.00000e+00	4.48920e-02	24	14
ex9.1.3	-6.00000e+00	2.00000e-08	5.21070e-02	36	26
ex9.1.4	-3.70000e+01	2.84217e-14	2.72900e-02	22	20
ex9.1.5	4.00000e+00	2.00000e-08	4.48500e-02	34	24
ex9.1.6	-5.20000e+01	2.99880e+00	1.52981e-01	156	150
ex9.1.7	-6.00000e+00	2.00000e-08	9.73550e-02	31	21
ex9.1.8	-3.25000e+00	0.00000e+00	3.95900e-02	10	8
ex9.1.9	3.11111e+00	2.00000e-08	2.00782e-01	68	58
ex9.1.10	-3.25000e+00	0.00000e+00	1.27720e-02	10	8
ex9.2.1	1.70000e+01	1.99994e-08	5.12970e-02	46	36
ex9.2.2	9.99980e+01	2.00230e-04	1.47327e-01	154	138
ex9.2.3	5.00000e+00	1.42109e-14	3.01870e-02	28	26
ex9.2.4	5.00000e-01	2.00000e-08	6.14300e-02	57	47
ex9.2.5	9.00000e+00	2.00000e-08	1.42335e-01	154	144
ex9.2.6	-1.00000e+00	2.00000e-08	3.25100e-02	25	15

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
ex9.2.7	1.70000e+01	1.99994e-08	5.23780e-02	46	36
ex9.2.8	1.50000e+00	2.00000e-08	3.02310e-02	24	14
ex9.2.9	2.00000e+00	1.01252e-13	2.00130e-02	18	16
flp2	5.61158e-13	0.00000e+00	2.34544e-01	266	256
flp4-1	2.08447e-14	0.00000e+00	5.65160e-02	12	10
flp4-2	3.68576e-15	0.00000e+00	1.17332e-01	11	9
flp4-3	9.87033e-09	2.02741e-13	2.19124e-01	10	8
flp4-4	1.96277e-12	0.00000e+00	8.00033e-01	12	10
gauvin	2.00000e+01	2.00000e-08	2.00056e-01	62	52
gnash10	-2.30823e+02	2.00000e-08	2.10252e-01	68	58
gnash11	-1.29912e+02	2.00000e-08	9.38760e-02	88	78
gnash12	-3.69331e+01	2.00000e-08	1.20921e-01	120	110
gnash13	-7.06178e+00	2.00000e-08	9.16420e-02	86	76
gnash14	-1.79046e-01	2.00000e-08	1.07201e-01	104	94
gnash15	-3.54699e+02	2.00130e-08	1.78844e-01	184	174
gnash16	-2.41442e+02	2.00040e-08	1.11947e-01	109	99
gnash17	-9.07491e+01	2.00056e-08	1.00207e-01	94	84
gnash18	-2.56982e+01	2.00288e-08	8.90450e-02	81	71
gnash19	-6.11671e+00	2.00025e-08	1.09830e-01	103	93
hs044-i	1.70901e+01	2.00001e-08	1.35173e-01	127	117
incid-set1-8	4.68375e-16	4.68668e-08	5.95340e-02	9	7
incid-set1c-8	3.81639e-17	1.50577e-12	6.18460e-02	10	8
incid-set2-8	5.04269e-03	2.22963e-09	1.87413e-01	40	38
incid-set2c-8	5.63006e-03	5.12314e-07	1.69162e-01	35	33
jr1	5.00000e-01	0.00000e+00	1.05690e-02	7	5
jr2	4.99998e-01	2.00000e-06	7.47580e-02	66	50
kth1	0.00000e+00	0.00000e+00	7.41800e-03	4	2
kth2	0.00000e+00	0.00000e+00	9.69300e-03	7	5
kth3	5.00000e-01	0.00000e+00	4.34220e-02	39	29
liswet1-050	1.39943e-02	1.23457e-13	7.29290e-02	15	13
monteiro	-3.75300e+01	1.00000e-07	5.03145e-01	93	83
monteiroB	-7.68377e+02	6.04498e-08	9.78420e-01	210	200
nash1a	6.13352e-12	1.99999e-08	6.19960e-02	55	45
nash1b	1.22513e-11	2.00000e-08	6.24950e-02	58	48
nash1c	1.22512e-11	2.00000e-08	6.67300e-02	62	52
nash1d	3.42048e-12	2.00008e-08	6.93660e-02	65	55
nash1e	6.05709e-12	1.99999e-08	5.99730e-02	55	45
outrata31	3.20770e+00	2.00001e-08	4.90720e-02	43	33

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
outrata32	3.44940e+00	2.00000e-08	5.30270e-02	47	37
outrata33	4.60425e+00	2.00006e-08	6.40150e-02	60	50
outrata34	6.59268e+00	5.92702e-07	9.23500e-02	93	83
pack-comp1-8	6.00000e-01	1.83867e-09	1.47027e-01	30	18
pack-comp1c-8	6.00000e-01	1.11022e-16	1.40056e-01	27	17
pack-comp1p-8	6.00000e-01	4.11476e-15	1.79623e-01	41	25
pack-comp2-8	6.73117e-01	3.09118e-09	6.04580e-02	13	11
pack-comp2c-8	6.73458e-01	4.18032e-10	6.22740e-02	15	13
pack-comp2p-8	6.73902e-01	3.90057e-08	1.90144e-01	64	62
pack-rig1-8	7.87932e-01	3.98986e-17	9.24970e-02	36	32
pack-rig1c-8	7.88300e-01	2.94903e-17	5.71910e-02	19	15
pack-rig1p-8	7.87932e-01	2.03297e-05	3.49906e-01	70	56
pack-rig2-8	7.80404e-01	6.60809e-10	4.80240e-02	18	16
pack-rig2c-8	7.99306e-01	2.46743e-09	4.20550e-02	14	12
pack-rig2p-8	7.80404e-01	1.86846e-08	1.34997e-01	38	36
pack-rig3-8	7.35202e-01	1.04895e-13	7.71550e-02	32	28
pack-rig3c-8	7.53473e-01	8.24003e-06	2.28247e-01	86	74
portfl-i-1	1.50234e-05	2.00225e-08	1.03550e-01	49	41
portfl-i-2	1.45722e-05	2.01951e-08	1.68218e-01	43	35
portfl-i-3	6.26452e-06	2.97235e-08	2.18719e-01	42	34
portfl-i-4	2.16138e-06	9.96137e-05	1.46868e-01	68	54
portfl-i-6	2.34136e-06	1.20710e-04	1.69489e-01	94	80
qpec-100-1	3.09244e-01	2.00379e-08	2.56698e+00	232	222
qpec-100-2	-6.44521e+00	2.00542e-08	2.48895e+00	211	201
qpec-100-3	-5.47715e+00	2.12809e-08	2.06555e+00	144	134
qpec-100-4	-3.95017e+00	2.06910e-08	1.99337e+00	132	122
qpec-200-1	-1.85113e+00	2.26130e-08	8.39342e+00	131	121
qpec-200-2	-2.26580e+01	2.01831e-08	3.58690e+01	225	215
qpec-200-3	-1.95246e+00	2.13271e-10	5.64425e+01	289	277
qpec-200-4	-6.21133e+00	2.02058e-08	1.28960e+01	124	114
qpec1	8.00000e+01	0.00000e+00	1.38620e-02	6	4
qpec2	4.49675e+01	8.11593e-04	2.98753e-01	267	251
ralph1	-9.98361e-04	9.98361e-04	2.05589e-01	221	205
ralph2	-1.20000e-07	2.00000e-04	2.17528e-01	230	214
ralphmod	-6.78010e+02	3.98433e-05	2.56321e+01	2663	2647
scale1	1.00000e+00	0.00000e+00	5.61690e-02	45	37
scale2	1.00000e+00	2.00000e-08	5.44900e-02	50	40
scale3	1.00000e+00	0.00000e+00	4.02400e-02	35	27

Problem	f_{opt}	$constVio(x_{opt})$	time	func. eval.	grad. eval.
scale4	1.00000e+00	0.00000e+00	6.18280e-02	61	53
scale5	9.99997e+01	1.34008e-06	9.18960e-02	90	74
scholtes1	2.00000e+00	0.00000e+00	1.24350e-02	9	7
scholtes2	1.50000e+01	0.00000e+00	1.31530e-02	10	8
scholtes3	5.00000e-01	2.00000e-08	8.21510e-02	82	72
scholtes4	-8.96813e-04	4.48406e-04	1.96599e-01	214	198
scholtes5	1.00000e+00	0.00000e+00	1.08870e-02	7	5
sl1	1.00000e-04	1.77636e-15	1.46470e-02	11	9
stackelberg1	-3.26667e+03	0.00000e+00	1.42090e-02	11	9
tap-09	1.13325e+02	6.00000e-08	2.36251e+00	1033	1023
tap-15	1.89267e+02	1.00002e-07	9.56517e+00	799	789
water-fl	3.32286e+03	2.00002e-08	8.71570e+00	1440	1430
water-net	9.27259e+02	1.73890e-04	2.45446e+01	21039	21025

Source Code

The MATLAB source code of the algorithms implemented in this thesis as well as the test problems from the MacMPEC collection [73] and the auxiliary mex function `amplfunc` [114] are available at the Institutional Repository (OPUS) of the University of Würzburg [1].