Well-posedness of a fluid-particle interaction model

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Contents

Abstract

This thesis considers a model of a scalar partial differential equation in the presence of a singular source term, modeling the interaction between an inviscid fluid represented by the Burgers equation and an arbitrary, finite amount of particles $N(t)$ moving inside the fluid, each one acting as a point-wise drag force with a particle related friction constant λ.

$$\partial_t u + \partial_x(u^2/2) = \sum_{i \in N(t)} \lambda_i \Big(h_i'(t) - u(t, h_i(t))\Big)\delta(x - h_i(t))$$

The model was introduced for the case of a single particle by Lagoutière, Seguin and Takahashi in [60], is a first step towards a better understanding of interaction between fluids and solids on the level of partial differential equations and has the unique property of considering entropy admissible solutions and the interaction with shockwaves.
The model is extended to an arbitrary, finite number of particles and interactions like merging, splitting and crossing of particle paths are considered.
The theory of entropy admissibility is revisited for the cases of interfaces and discontinuous flux conservation laws, existing results are summarized and compared, and adapted for regions of particle interactions. To this goal, the theory of germs introduced by Andreianov, Karlsen and Risebro [8] is extended to this case of non-conservative interface coupling.
Exact solutions for the Riemann Problem of particles drifting apart are computed and analysis on the behavior of entropy solutions across the particle related interfaces is used to determine physically relevant and consistent behavior for merging and splitting of particles. Well-posedness of entropy solutions to the Cauchy problem is proven, using an explicit construction method, L^∞ bounds, an approximation of the particle paths and compactness arguments to obtain existence of entropy solutions. Uniqueness is shown in the class of weak entropy solutions using almost classical Kruzkov-type analysis and the notion of L^1-dissipative germs.
Necessary fundamentals of hyperbolic conservation laws, including weak solutions, shocks and rarefaction waves and the Rankine-Hugoniot condition are briefly recapitulated.

Zusammenfassung

Diese Arbeit befasst sich mit dem Modell einer skalaren partiellen Differentialgleichung mit singulärem Quellterm, das die Interaktion zwischen einem reibungsfreiem Fluid, dargestellt durch die Burgers Gleichung, und einer gegebenen, endlichen Menge von sich in dem Fluid bewegenden Partikeln $N(t)$ beschreibt, die eine punktweise Zugkraft auf das Fluid auswirken und durch eine entsprechende Reibungskonstante λ charakterisiert sind.

$$\partial_t u + \partial_x(u^2/2) = \sum_{i \in N(t)} \lambda_i \Big(h_i'(t) - u(t, h_i(t))\Big) \delta(x - h_i(t))$$

Das Modell wurde für den Fall der Interaktion mit einem einzelnen Partikel durch Lagoutière, Seguin and Takahashi in [60] eingeführt, stellt einen ersten Schritt zu einem besseren Verständnis der Interaktion zwischen einem Fluid und Festkörpern auf dem Level der partiellen Differentialgleichungen dar und hat die einzigartige Eigenschaft, dass Entropielösungen und die Interaktion mit Schockwellen berücksichtigt werden.
Das Modell wird zu einer beliebigen, endlichen Anzahl von Partikeln erweitert und Interaktionen wie das Verschmelzen und Spaltung von Partikeln werden behandelt.
Existierende Theorie der Entropie-Zulässigkeit im Hinblick auf Interfaces und Erhaltungsgleichungen mit unstetiger Flussfunktion wird zusammengefasst, die Resultate werden verglichen und für die Regionen mit Partikelinteraktionen angepasst. Zu diesem Zweck wird die Theorie der Germs, eingeführt von Andreianov, Karlsen und Risebro [8], auf den vorliegenden Fall eines nicht-erhaltenden Interfaces erweitert.
Für das Riemann Problem von auseinanderdriftenden Partikeln werden die exakten Lösungen berechnet und eine Analyse des Verhaltens von Entropielösungen über die von den Partikeln erzeugten Interface wird genutzt, um ein physikalisch sinnvolles und mit der Theorie eines einzelnen Partikels konsistentes Verhalten beim Verschmelzen und Spalten von Partikeln herzuleiten. Mit Hilfe einer expliziten Konstruktionsmethode, hergeleiteten L^∞ Beschränkungen, einer Approximation der Partikelpfade und Kompaktheitsargumenten wird gezeigt, dass das entsprechende Cauchy Problem wohlgestellt ist. Eindeutigkeit im Raum der schwachen Entropielösungen wird mit beinahe klassischen Argumenten der Theorie von Kruzkov sowie der Theorie von L^1-dissipativen Germs gezeigt.
Notwendige Grundlagen zu hyperbolischen Erhaltungsgleichungen, unter anderem die Theorie schwacher Lösungen, Schock- und Verdünnungswellen sowie die Rankine-Hugoniot Bedingung, werden in Grundzügen am Anfang der Arbeit wiederholt.

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Introduction

Hyperbolic conservation laws have been of ever increasing interest over the last decades, offering numerous obstacles connected to the selection of physically meaningful solutions, proving existence and uniqueness in more and more general classes of solutions while extensively being used to describe physical applications. The model of a scalar conservation law with singular source term considered in this thesis lines up with various applications of the underlying equation, and is used to describe phenomena like traffic flow, including constrained flux problems [36, 37, 64, 65], sedimentation in clarifier-thickener units [24, 25, 26, 38] and the motion of particles and rigid bodies inside a fluid or gas [3, 4, 9, 10, 27, 41, 60, 78].

One particular theoretical obstacle for hyperbolic conservation laws in general is that the structure of solutions demands for adjustments, relaxing the notion of derivatives and, more importantly, for an additional admissibility condition to select physically correct solutions. Although this additional constraint, mostly referred to as entropy condition, is well understood for the case of scalar, continuous flux conservation laws due to Kruzkov [58], and has been extended to the case of spatially discontinuous flux type scalar equations under certain assumptions on the flux [1, 2, 12], the latter case still lacks a theory capturing all kinds of general flux functions.

After multiple approaches to define well-posed solutions to discontinuous flux equations have been developed over the last decade, ranging from extending the Kruzkov formulation [58] and corresponding entropy inequality from the continuous case [12, 66], using a kinetic approach [14, 68] to defining a dissipative connection across the interface using the theory of (A, B)-connections [1, 2], Andreianov, Karlsen and Risebro came up with a very general viewpoint on admissibility using so called germs, investigating interface behavior by comparing to elementary solutions of the given equation [8].

Before introducing the model of fluid-particle interaction, which is the main part of this thesis, the different approaches to entropy admissibility for discontinuous flux scalar conservation laws are revisited and the different notions are compared regarding their suitability for dealing with interfaces.

The model considered in this work is an extension of inviscid fluid-particle interaction and is particularly interesting, as it is the first fluid-solid interaction model considering entropy admissible solutions in the presence of shockwaves. The fluid is modeled by the Burgers equation, while the particle acts as a point-wise drag force through the source term. One could think of a water pipe with a small ball moving inside, neglecting the friction and flux constraints towards the boundary of the pipe. Depending on their friction constant, the balls are partly permeable by the fluid.

It was originally introduced in [60] with a study of the Riemann problem for the case of a single particle and extended to the Cauchy problem in [9, 11, 10]. Some numerical results for the case of a single particle have been achieved in [4, 10] using mostly the wave-front tracking method by Holden and Risebro [49].

It is extended to the case of an arbitrary, finite amount of particles and particle interactions like merging, splitting and the crossing of particle paths are considered. The particles act as traveling interfaces, and new difficulties from the interaction and creation of interfaces are solved extending the theory of germs to fit the non-conservative, time-dependent interface coupling.
Well-posedness of the Cauchy problem for multiple particles is achieved and the study of interface behavior using a special Riemann problem, where the particles drift apart, is a new example for the usefulness and generality of the theory of L^1-dissipative germs.
The work is structured in the following way. Chapter 1 introduces the fundamentals connected to scalar conservation laws and entropy admissibility of weak solutions in the case of continuous flux functions. In order to be able to study the interfaces created by the particles and to extend the necessary entropy admissibility condition to the model, Chapter 2 revisits the theory of flux or source term induced interfaces for solutions of conservation laws. This includes especially the theory of germs, which though very useful, has not been investigated much after the original paper, see [5, 11].
Chapter 3 starts by briefly recalling the known theory of the model for a single particle, as the corresponding existence result, L^∞-bounds and bounds in total variation are used later to obtain well-posedness of the Cauchy problem. The second part of Chapter 3 contains a study of the Riemann problem for multiple particles, where the study of a domain of influence of the particles and the introduction of a new notion of generalized germs allows for an existence and uniqueness result for the Riemann problem as well as giving the basis to define entropy admissibility for interfaces influenced by multiple particles. Finally, the Cauchy problem is considered in the third part, admissible particles are defined in a physically meaningful way, such that the dissipative behavior of interactions like merging and splitting are consistent with analysis of germs for the simplified case of the Riemann problem. Existence and L^∞ bounds are proven using an explicit, time-stepping construction method and the results from the single-particle case. For the case of splitting, an approximation of the particle paths is used, and compactness is shown to obtain a weak entropy solution. The dissipativity of the particle related germs is used to obtain the Kato inequality, uniqueness of weak entropy solutions and the desired L^1-contraction property. Some conclusions and further possible extensions for future projects are made at the end.

1. Fundamentals

In this chapter, some fundamental notions of the hyperbolic partial differential equations are briefly revisited. As the problems considered in the later parts of this work are partially based on the fundamental problem of non-uniqueness of solutions of conservation laws, the notion of weak solution, the Rankine-Hugoniot condition and entropy admissibility will be presented in a very compact manner. Furthermore, it will be shown that the Burgers equation is the simplest example containing the critical behavior of conservation laws. For further details and a more dedicated introduction to hyperbolic conservation laws, the reader is kindly referred to well-recognized textbooks on the topic, for example [34, 47, 49].

1.1. Conservation laws

Hyperbolic conservation laws are time-dependent partial differential equations inspired by physical problems in fluid and gas dynamics. Their study dates back to Euler and has been of ongoing interest, not only due to the physical application, but also due to some stunning mathematical problems connected to the regularity of solutions.

Remark 1.1.1. *On of the most prominent examples is the regularity of solutions to the Euler equations. Although the issue of well-posedness was partially answered by recent studies of so called wild solutions [29, 35, 42, 51], proving Onsager's conjecture, the debate about whether or not it is possible to find suitable admissibility conditions to obtain well-posedness in a general framework is continuing, compare [22, 44].*

A hyperbolic conservation law is a balance law given by the following partial differential equation in divergence form:

$$\partial_t u(t, \mathbf{x}) + \operatorname{div} f(u) = 0 \tag{1.1}$$

where $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ is the spatial variable, $t \in \mathbb{R}^+$ and

$u(t, \mathbf{x})$	is the conserved variable and unknown
$f(u)$	is the flux function.

Integrating 1.1 with respect to space and using the notion of total derivative gives

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{R}^n} u(t, \mathbf{x}) \, \mathrm{dx} = 0,$$

which clarifies why u is called the conserved variable.

As the topic of this work is the study of well-posedness of scalar hyperbolic conservation laws, the fundamental problem is the following initial value problem:
Given $\Omega \subset \mathbb{R}$, $T > 0$, $u_0 : \Omega \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, search for a function $u : \Omega \to \mathbb{R}$ that satisfies

$$\begin{aligned} \partial_t u(t,x) + \partial_x f(u) &= 0 \qquad && \text{in } \Omega \times (0,T] \\ u(0,x) &= u_0(x) && x \in \Omega \end{aligned} \tag{1.2}$$

The intuitive approach is to search for smooth solutions, as the equation is demanding for existence of first derivatives in space and time. Following this idea, solutions $u \in C^1(\Omega \times [0,T])$, or of even higher regularity, are called classical or strong solutions to problem (1.2). If there exists a classical solution, it can be obtained using the method of characteristics, a well-known method to solve quasi-linear partial differential equations of first order. A rigorous introduction of its use in the context of conservation laws can for example be found in [52].

However, even for very smooth initial data $u_0 \in C^\infty$, solutions to problems of type (1.2) can easily develop discontinuities. This displays in the characteristic curves intersecting after a finite time, and can be seen in the following classical example, taken from [40]:

Example 1. Consider the initial value problem

$$\begin{aligned} \partial_t u(t,x) + \partial_x \left(\frac{u^2}{2}\right) &= 0 \qquad && \text{in } \mathbb{R} \times (0,\infty) \\ u(0,x) &= u_0(x) && \text{in } \mathbb{R} \times \{t=0\} \end{aligned} \tag{1.3}$$

with initial data

$$u_0(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 1-x, & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x \ge 1. \end{cases}$$

Computing the solution via characteristics gives

$$u(t,x) = \begin{cases} 1 & \text{if } x \le t \\ \frac{1-x}{1-t} & \text{if } t \le x \le 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

and clearly breaks down when $t \ge 1$. The solution for $t \ge 1$ embraces a discontinuity, called shock, and writes for $s(t) = \frac{1+t}{2}$

$$u(t,x) = \begin{cases} 1, & \text{if } x < s(t) \\ 0, & \text{if } x > s(t). \end{cases}$$

Therefore, it is clear that the search for classical solutions is not sufficient, and a generalized class of solutions has to be considered.

1.2. Weak solutions and the Rankine-Hugoniot condition

The necessity of including solutions of a certain non-regularity into a broader class of generalized solutions to (1.2) has led to the consideration of so called weak solutions. Integrating (1.2) with respect to space and time, multiplying with a testfunction $\phi \in C_0^\infty$ and using partial integration to have the derivatives on ϕ, one has redesigned the problem to incorporate solutions, where the non-regularity is restricted to jump-type discontinuities.

Definition 1.2.1. *Let $u_0 \in L^\infty(\Omega)$. A function $u \in L^\infty(\Omega \times (0,T))$ is called weak solution or solution in the distributional sense to the scalar conservation law (1.2), if it satisfies*

$$\int_\Omega \int_0^T (u\partial_t \phi + f(u)\partial_x \phi)\ dt\, dx + \int_\Omega u_0(x)\phi(0,x)\ dx = 0$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$.

Investigating which non-smooth solutions are allowed following this new definition, one recovers a condition on the pointwise discontinuities, called the Rankine-Hugoniot condition. This is done, also for higher spatial dimension, studying the flow across an interface between two constant states and can be found for example in [40]. The result is stated in the following proposition.

Proposition 1.2.2. *Given a two domain partition D_1, D_2 of $[0,T] \times \Omega$, where D_1, D_2 are separated by a smooth curve $\Sigma : t \mapsto (\sigma(t), t)$, and further $u \in L^\infty((0,T) \times \Omega)$ satisfying (1.2) locally in D_1, D_2 in a classical sense, with $u|_{D_1} = u_1$, $u|_{D_2} = u_2$. Then u is a weak solution in the sense of definition 1.2.1 if and only if it satisfies the Rankine-Hugoniot condition at the interface*

$$\Big(u_1(t,\sigma(t)) - u_2(t,\sigma(t))\Big)\sigma'(t) = f(u_1(t,\sigma(t)) - f(u_2(t,\sigma(t)). \tag{1.4}$$

At last, it is also interesting to remark that given certain smoothness criteria are satisfied, solutions to (1.2) are smooth, thus of strong or classical type, at least for a finite time. This property holds even for some of the more delicate systems concerning well-posedness, like the Euler or Navier-Stokes equations. Additionally, it has been proven even for many systems of conservation laws, that for as long as a strong solution exists, every weak solution coincides with the strong solution, answering the question of uniqueness at least for this finite time interval. This notion is called weak-strong uniqueness and is used vastly both in theoretical aswell as numerical approaches to systems like Euler or Navier-Stokes, compare for example [43, 45, 79]. The first proposition states the existence result for strong solutions in the case of scalar conservation laws. A proof can be found in [56].

Proposition 1.2.3. *Let $u_0 \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$, such that f'' and u_0 are bounded functions on $\mathbb{R}$. Then there exists a finite time interval $[0, t^*)$, such that problem (1.2) has a classical solution $u \in C^1([0, t^*) \times \mathbb{R})$.*

Concerning weak-strong uniqueness, the following proposition provides the result for the incompressible Euler equations, and acts as the basis for further results in the direction of the Navier-Stokes equations. A proof and further details on weak-strong uniqueness and relative energy methods can be found for example in a recent summary on the topic, [79].

Proposition 1.2.4. *Let $u \in L^\infty((0,T); L^2(\Omega))$ be a weak solution and $U \in C^1(\Omega \times [0,T])$ a strong solution of the incompressible Euler equations*

$$\begin{aligned} \partial_t u + div(u \otimes u) + \nabla p &= 0, \\ div\ u &= 0, \end{aligned}$$

and assume that u and U share the same initial datum u_0. Assume moreover that

$$\frac{1}{2}\int_\Omega |u(\tau,x)|^2 dx \leq \frac{1}{2}\int_\Omega |u_0(x)|^2 dx$$

for almost every $\tau \in (0,T)$. Then $u(\tau,x) = U(\tau,x)$ for almost every $(\tau,x) \in (0,T) \times \Omega$.

1.3. Entropy admissibility

It is widely known in the field of conservation laws, that weak solutions to (1.1), although naturally satisfying the Rankine-Hugoniot condition, are not unique in general. Whenever there is no smooth solution to given initial data, an additional condition has to be applied in order to prevent non-physical shocks from appearing. The multitude of weak solutions can be seen in the following example.

Example 2. Consider the Burgers equation with given initial data

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \qquad u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases} .$$

Given the solution only needs to satisfy the Rankine-Hugoniot condition, one can give a whole family of solutions by considering 2-shock solutions of the form

$$u_\alpha(t,x) = \begin{cases} 0, & \text{if} \quad x < \alpha t/2, \\ \alpha, & \text{if} \quad \alpha t/2 \leq x \leq (1+\alpha)t/2, \\ 1, & \text{if} \quad x > (1+\alpha)t/2, \end{cases}$$

It is easily checked that the shocks satisfy the Rankine-Hugoniot condition independent of $\alpha \in (0,1)$ and the solution outside of the shocks is stationary and indeed a solution to the given initial data. The immediate consequence is the need of another condition to select the physically meaningful solution. The following smooth solution is the correct one, which is stated for completeness:

$$u(t,x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ \frac{x}{t}, & \text{if} \quad 0 \leq x \leq t, \\ 1, & \text{if} \quad x > t, \end{cases}$$

Solutions of this type, resolving a discontinuity into a smooth fan of characteristics is called and will be referred to as a rarefaction wave.

Remark 1.3.1. *The initial value problem given in Example 2, where the initial data is composed of two constant states, is called a Riemann problem. Riemann problems are of great use when studying the behaviour of solutions of partial differential equations, as they single out how the solution evolves around an initial discontinuity. Many numerical approaches, including the well-known Roe-Solver, are built on solving sequences of Riemann problems locally [71, 70].*

There are multiple conditions granting the necessary dissipativity of shocks, compare for examples [21, 34], however, as this work later deals with effects of interfaces and discontinuous flux functions, the Kruzkov entropy condition and related L^1-dissipativity will be introduced here and was originally published in [58].
The general form of the additional constraint was discovered by Crandall and Lions [31], by adding a small diffusion term, representing viscosity, to the conservation law and considering only weak solutions that are limits of series of solutions to the approximated system

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx} u^\epsilon. \tag{1.5}$$

Given a convex function $\eta \in C^2(\mathbb{R})$, called entropy, and $\Phi \in C^2(\mathbb{R})$, called entropy flux, such that $\Phi' = \eta' f'$, multiplication of (1.5) with $\eta'(u^\epsilon)$ yields

$$\begin{aligned} \partial_t \eta(u^\epsilon) + \partial_x \Phi(u^\epsilon) &= \epsilon\, \eta'(u^\epsilon)\, \partial_{xx} u^\epsilon \\ &= \epsilon \left[\partial_{xx} \eta(u^\epsilon) - \eta''(u^\epsilon) \cdot (u^\epsilon)^2 \right] \end{aligned}$$

Notice that the last term on the right hand side is positive due to the convexity of η. Multiplying the above with a nonnegative smooth function $\phi \in C_0^\infty(\mathbb{R})$ with compact support and integrating by parts yields

$$\int_0^T \int_{\mathbb{R}} \{\eta(u^\epsilon)\partial_t \phi + \Phi(u^\epsilon)\partial_x \phi\} \mathrm{dxdt} \geq -\epsilon \int_0^T \int_{\mathbb{R}} \eta(u^\epsilon) \partial_{xx} \phi \, \mathrm{dxdt}.$$

Letting $\epsilon \to 0$ one obtains the entropy inequality

$$\int_0^T \int_{\mathbb{R}} \{\eta(u)\partial_t \phi + \Phi(u)\partial_x \phi\} \mathrm{dxdt} \geq 0. \tag{1.6}$$

Remark 1.3.2. *Notice that the conservation porperty for the entropy variables can easily be recovered using the original equation and*

$$\eta'(u) \cdot f'(u) = \Phi'(u).$$

It follows

$$\partial_t \eta(u) + \partial_x \Phi(u) = \eta'(u)\partial_t u + \Phi'(u)\partial_x u = \eta'(u)(-f'(u)\partial_x u) + \Phi'(u)\partial_x u = 0.$$

Remark 1.3.3. *It is worthwhile to note that the idea of Crandall and Lions to consider limits of solutions to parabolic problems, like (1.5), is not restricted to approximating the viscous case by using a diffusion term. In fact, any higher order differential operator term could be used to obtain an admissibility condition. The choice of viscosity is due to physical context, compare the discussion on this in the second chapter of [21].*

In order to obtain the physically correct solution, it is not enough if (1.6) is satisfied for a single convex entropy. In his ground breaking work [58], Kruzkov introduces a set of entropies that give a sufficient restriction to the solutions of the approximated equation (1.5), using the entropy inequality (1.6) to reach the desired L^1-dissipativity property. Given $\kappa \in \mathbb{R}$, he introduced the entropy - entropy flux pair

$$\begin{aligned} \eta(u) &= |u - \kappa|, \\ \Phi(u, \kappa) &= \operatorname{sgn}(u - \kappa)(f(u) - f(\kappa)). \end{aligned}$$

which is referred to as Kruzkov entropy and Kruzkov entropy flux. Using (1.6), the definition of weak (Kruzkov) entropy solution follows.

Definition 1.3.4.
Let $u_0 \in L^\infty(\Omega)$. A function $u \in L^\infty(\Omega \times (0,T))$ is called generalized weak entropy solution to the scalar conservation law (1.2) if it satisfies

$$\int_{\mathbb{R}^+ \times \mathbb{R}} |u - \kappa| \partial_t \phi(x,t) dxdt + \int_{\mathbb{R}} |u_0 - \kappa| \partial_t \phi(x,0) dxdt + \int_{\mathbb{R}^+ \times \mathbb{R}} \Phi(u,\kappa) \partial_x \phi dxdt \geq 0 \tag{1.7}$$

for all $\kappa \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R})$.

The notion of entropy admissibility is, as mentioned before, inspired by physical application, especially thermodynamics, but although it is well-known for the scalar case with continuous flux functions, it is far more difficult and problem-dependent for non-smooth fluxes, which will be discussed in the next chapter, and in many cases an open problem in higher dimensional spaces for systems of conservation laws, examples and discussion on this can be found in [20, 43, 44].

At last, weak entropy solutions of scalar conservation laws (1.2) are L^1-dissipative in the following sense.

Proposition 1.3.5. *Let $u_0, v_0 \in L^\infty(\mathbb{R})$, $f \in Lip(\mathbb{R})$. Weak entropy solutions $u, v \in L^\infty((0,T) \times \mathbb{R})$ satisfying (1.7), also satisfy the L^1 contraction property*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}. \tag{1.8}$$

This directly implies uniqueness of weak entropy solutions given initial data $u_0 \in L^\infty((0,T) \times \mathbb{R})$.

The proof is considered classical, following the idea of Kruzkov, and uses so called doubling of variables to merge the entropy inequalities for both entropy solutions u, v. For details in this method, the reader is pointed to the original paper [58] or [47]. One recovers the Kato inequality

$$\partial_t |u - v| + \partial_x \Big(\operatorname{sgn}(u - v)(f(u) - f(v)) \Big) \leq 0. \tag{1.9}$$

Integrating this inequality along the Cone $C_{R,T} := \{(x,t) \in \mathbb{R} \times (0,\infty), |x| \leq R + M(T-t), t \in [0,T]\}$ gives a local version and by extension (1.8). This clearly implies uniqueness of entropy solutions (assuming two solutions with the same initial data, with their L^1-distance only decreasing).

2. Interfaces, pointwise disturbances and L^1-dissipativity

Conservation laws with discontinuous flux functions or source terms corresponding to point-wise disturbances have been a field of particular interest over the last years. The question how to define entropy admissible solutions was tackled in many different ways, ranging from extending the Kruzkov formulation and corresponding entropy inequality from the continuous case [12, 66], using a kinetic approach [14, 68] to defining a dissipative connection across the interface using the theory of (A, B)-connections [1, 2].
Summarizing and including many of the known results into a new framework, Andreianov, Karlsen and Risebro developed the theory of germs in their well-known paper from 2011 [8]. The theory is based on the comparison of constant states, forming so called elementary solutions to the given initial value problem. Besides giving interesting insights into the different admissibility criteria, the theory allows for comparison and, to a certain extent, proof of equivalency of different entropy conditions.
This chapter summarizes the existing theory on flux or source-term induced discontinuities for conservation laws and corresponding well-posedness of weak solutions. The first section gives an introduction to discontinuous flux conservation laws and related problems, giving different solutions to approach dissipativity and uniqueness in the space of weak solutions. The second section introduces and summarizes the theory of germs, the most general framework to deal with interfaces in the context of conservation laws and will be extended to a case of non-conservative coupling for the application of particles induced interfaces in chapter 4. Some examples for better understanding will be given and equivalency of the different entropy admissibility formulations is proven under some typical convexity assumptions on the flux function.

2.1. Discontinuous flux conservation laws

This section deals with the Cauchy Problem for discontinuous flux conservation laws, where the flux function admits a single discontinuity with respect to the spacial variable. The model equation is

$$\partial_t u + \partial_x f(u, x) = 0, \tag{2.1}$$

where the flux function is assumed to be piecewise Lipschitz continuous with a single discontinuity. Note that this could without effort be extended to multiple discontinuities, as the effects remain local. The model (2.1) has numerous applications, ranging from industrial sedimentation in clarifier-thickener units [24, 25, 26, 38], traffic flow [36, 64, 65] and modelling gravity [53] to fluid-solid interactions [10, 60]. In order to allow for a physically meaningful definition of solution with respect to usual spaces of initial data, like $u_0 \in L^\infty$, some assumptions on the flux function have to be made. As in many of the mentioned works on the topic, the flux function is assumed to be a continuous, convex or concave function multiplied with a discontinuous, piecewise constant coefficient function, resulting in the model

$$\begin{aligned} \partial_t u + \partial_x F(u,x) &= 0 \\ u(0,x) &= u_0(x) \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} F(u,x) &= k(x)f(u), && f \in \mathrm{Lip}(\mathbb{R}) \\ k(x) &= \begin{cases} k_L & \text{for } x \leq 0 \\ k_R & \text{for } x > 0 \end{cases}, && k_L, k_R \in \mathbb{R}. \end{aligned}$$

It should be mentioned that among the other investigated models, a lot of theory has been done for $F(u,x) = H(x)f(u) + (1 - H(x))g(u)$, with H the Heaviside function and f, g of some regularity (for example $f, g \in C^2$, strictly convex or concave and of super-linear growth). For more details on models of this type, the reader is referred to [2, 46]. Some studies on generalizations of the flux function towards not strictly convex fluxes have been done in [1, 6, 63]. Panov has extended some of the well-posedness theory to the case of fluxes with discontinuities with respect to the conserved variable, compare [48, 66].

Note that even under the strong assumptions of (2.2), given certain flux functions, the problem is not necessarily solvable for arbitrary initial data, as the following example shows. In multiple space dimension, though not the topic of this work, the matter becomes even more difficult. In fact, even for continuous flux problems of conservation laws in multiple space dimensions, uniqueness of weak entropy solutions is only guaranteed under additional conditions, see [18, 57] for counterexamples in the general case.

Example 3. Consider model (2.2) with

$$\begin{aligned} f(u) &= (u+1)^2 \\ k(x) &= \begin{cases} 1 & \text{for } x \leq 0 \\ -1 & \text{for } x > 0, \end{cases} \\ u_0(x) &= \begin{cases} u_0^1 & \text{for } x \leq 0,\ u_0^1 \in \mathbb{R}^- \\ u_0^2 & \text{for } x > 0,\ u_0^2 \in \mathbb{R}^+ \end{cases} \end{aligned}$$

Clearly, away from the interface at $x = 0$, constant states should still be stationary solutions to the problem, such that u_0^1, u_0^2 solve the equation (which can also be seen by using the method of characteristics, realizing all characteristics are straight lines towards the origin) and therefore, the only possible solution should be (u_0^1, u_0^2) with a stationary shock at the origin. However, it is not possible for this shock to satisfy the Rankine-Hugoniot condition (1.4) as this would mean

$$f(u_0^1) = -f(u_0^2)$$

which is impossible due to the choice of $f(\cdot) \in \mathbb{R}^+$.

There are multiple ways to solve equations of type (2.2) for given initial data, computing the vanishing viscosity limit [8], numerically using schemes like wave-front tracking, see for example [23, 24, 25], or by solving the Hamilton-Jacobi equation, compare [2, 46]. This work focuses on the first approach.

2.1.1. Kruzkov entropy condition for discontinuous flux problems

As in the continuous case, an entropy condition is mandatory to exclude non-physical (non-dissipative) shockwaves. However, an adaptation has to be made to credit the additionaly discontinuity induced by the flux function. Using the Kruzkov entropy-entropy flux pair derived in the continuous case, following the same procedure to derive the entropy inequality involves spatial derivatives on the flux, which make no sense for the discontinuous coefficient $k(x)$. To resolve this issue, one defines functions $k_\epsilon(x) \in C^1(\mathbb{R})$ sufficiently smooth with $k_\epsilon(x) \to k(x)$ as $\epsilon \to 0$. Therefore, starting from the entropy inequality (1.6) with flux $f_\epsilon(u,x) = k_\epsilon(x) f(u)$ and multiplying with $\operatorname{sgn}(u-\kappa)$ gives

$$
\begin{aligned}
& \partial_t |u-\kappa| + \partial_x \big(k_\epsilon(x) q(u,\kappa)\big) + k'_\epsilon(x) f(\kappa) \operatorname{sgn}(u-\kappa) && \leq 0. && (2.3)\\
\Leftrightarrow\quad & \partial_t |u-\kappa| + \partial_x \big(k_\epsilon(x) q(u,\kappa)\big) && \leq -k'_\epsilon(x) f(\kappa) \operatorname{sgn}(u-\kappa) \\
\Rightarrow\quad & \partial_t |u-\kappa| + \partial_x \big(k_\epsilon(x) q(u,\kappa)\big) && \leq |k'_\epsilon(x)| f(\kappa),
\end{aligned}
$$

the last step being a rough estimate on the right side. Rewriting this in the distributional form, writing the derivatives on the testfunction by partial differentiation and letting $\epsilon \to 0$ leads to a first definition of admissible solutions.

Definition 2.1.1.
Let $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty((0,T)\times\mathbb{R})$ with traces $\gamma^-(u), \gamma^+(u)$ is called weak (Kruzkov) entropy solution to the initial value problem (2.2) if it is a weak solution of (2.2) satisfying

$$
\begin{aligned}
\int_{\mathbb{R}+}\int_{\mathbb{R}} |u(t,x)-\kappa| \partial_t \phi(t,x)\,dx\;dt + \int_{\mathbb{R}} |u_0(x)-\kappa| \phi(0,x)\,dx \\
+ \int_{\mathbb{R}+}\int_{\mathbb{R}} k(x) \Phi(u,v) \partial_x \phi\,dx\;dt + |k_L - k_R| \int_{\mathbb{R}+} f(\kappa)\phi(t,0)\,dt \;\geq 0
\end{aligned}
\tag{2.4}
$$

for all testfunctions $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $\kappa \in \mathbb{R}$.

Multiplying with a cut-off function[1] $\zeta_\epsilon = \frac{1}{\epsilon}\zeta(\frac{x}{\epsilon})$ centered at the origin reveals the condi-

[1] For more details on the cut-off function, see the Appendix

tion governing the interface discontinuity

$$k_R \Phi(\gamma^-(u), \kappa) - k_L \Phi(\gamma^+(u), \kappa) - |k_L - k_R| f(\kappa) \ \leq \ 0.$$

This corresponds exactly to the dissipativity in the continuous case, with a relaxation due to the discontinuity of the flux. The following theorem states the L^1-contractivity of corresponding entropy solutions.

Theorem 2.1.2. *Given $u_0, v_0 \in L^\infty(\mathbb{R})$, weak entropy solutions $u, v \in L^\infty((0,T) \times \mathbb{R})$ to (2.2), that is, solutions that satisfy (2.4) and (2.2) in the weak sense, satisfy the L^1-contraction property (1.8).*

The proof is done in several steps, where the goal is to recover the Kato inequality (1.9), which gives the L^1-contraction integrating along a cone $C_{R,T}$, compare section 1.3.

Proof.
Proving uniqueness follows the same steps as in the continuous flux case, with the difference that an additional left and right boundary term appears at the interface. Starting from the entropy inequality (2.4), the proof is done in three steps.
Step 1. Choose a testfunction $\phi \in C_0^\infty$ to have compact support away from the origin $\{x = 0\}$. Plugging this testfunction into (2.4) excludes the problematic interface, because $\phi(t, 0) = 0$. Given Riemann initial data

$$u_0(x) = \begin{cases} u_L & \text{for } x \leq 0 \\ u_R & \text{for } x > 0 \end{cases} .$$

the problem separates into two continuous flux problems left and right of the origin

$$\begin{aligned} \partial_t w^1 + \partial_x (k_L f(w^1)) &= 0, \\ w^1(0, x) &= u_L \end{aligned} \qquad \text{for} \quad x < 0 \tag{2.5}$$

$$\begin{aligned} \partial_t w^2 + \partial_x (k_R f(w^2)) &= 0, \\ w^2(0, x) &= u_R \end{aligned} \qquad \text{for} \quad x > 0 \tag{2.6}$$

Both problems satisfy $f \in \mathrm{Lip}(\mathbb{R})$ and therefore, following the same calculations as in the proof of Proposition 1.3.5, namely doubling of variables, gives the Kato inequality locally left and right of the origin. Given u, v are each weak entropy solutions to both local problems (2.5), (2.6), one recovers

$$\begin{aligned} \int_{\mathbb{R}+} \int_{\mathbb{R}} |u(t,x) - v(x)| \partial_t \psi(t,x) \text{ dx dt} + \int_{\mathbb{R}} |u_0(x) - v_0(x)| \psi(0,x) \text{ dx} \\ + \int_{\mathbb{R}+} \int_{\mathbb{R}} k(x) \Phi(u,v) \partial_x \psi \text{ dx dt } \geq 0. \end{aligned} \tag{2.7}$$

Step 2. In the second step, the restriction on the testfunction ψ needs to be removed to recover the behavior at the interface $\{x = 0\}$. Therefore one defines a new testfunction $\psi_\epsilon \in C_0^\infty(\mathbb{R})$ in the following way

$$\psi_\epsilon(t,x) = \phi(t,x)\big(1 - \omega_\epsilon(x)\big) \tag{2.8}$$

with a testfunction $\phi(t,x) \in C_0^\infty(\mathbb{R})$, such that $0 \in \operatorname{supp} \phi$ and

$$\omega_\epsilon(x) = \begin{cases} 0, & \text{if} \quad |x| > 2\epsilon \\ \frac{2\epsilon - |x|}{\epsilon}, & \text{if} \quad \epsilon \le |x| \le 2\epsilon \\ 1, & \text{if} \quad |x| < \epsilon \end{cases} \tag{2.9}$$

Note that for $\epsilon > 0$, $0 \notin \operatorname{supp} \psi_\epsilon$ and choosing $\psi = \psi_\epsilon$ in (2.7) followed by passing to the limit in $\epsilon \to 0$ holds

$$\begin{aligned} \int_{\mathbb{R}+} \int_{\mathbb{R}} |u - v| \partial_t \psi(t,x) \text{ dx dt} + \int_{\mathbb{R}} |u_0 - v_0| \psi(0,x) \text{ dx} + \int_{\mathbb{R}+} \int_{\mathbb{R}} k(x)\Phi(u,v)\partial_x \psi \text{ dx dt} \\ \ge \int_{\mathbb{R}+} k_L \Phi\big(\gamma^-(u), \gamma^-(v)\big) - k_R \Phi\big(\gamma^+(u), \gamma^+(v)\big) \text{ dt} \end{aligned} \tag{2.10}$$

Note that the term on the right side, involving traces on u and v, which are denoted by $\gamma^\pm(u), \gamma^\pm(v)$ respectively, appears due to the spatial dependance of $\Phi(u,v,x) = k(x)\Phi(u,v)$.
Step 3. To obtain the Kato inequality, all that remains to prove, is

$$0 \;\le\; k_L \Phi\big(\gamma^-(u), \gamma^-(v)\big) - k_R \Phi\big(\gamma^+(u), \gamma^+(v)\big) \tag{2.11}$$

which corresponds to a dissipative behavior at the interface. This is done carefully studying (2.11) case-by-case with respect to $\kappa, \gamma^\pm(u)$ and $\gamma^\pm(v)$, which can be found in the Appendix A.2. The starting point however, lies in extracting the good interface behaviour out of the original entropy inequality (2.3) after applying the smoothened coefficient function k_ϵ

$$\partial_t |u - \kappa| + \partial_x\big(k_\epsilon(x)\Phi(u,\kappa)\big) + k'_\epsilon(x) f(\kappa) \operatorname{sgn}(u - \kappa) \le \; 0.$$

Taking this equation in distributional sense, multiplying with a testfunction $\psi_\epsilon(t,x)$ and passing to the limit in order to get the interface term, one obtains

$$k_L \Phi\big(\gamma^-(u), \gamma^-(v)\big) - k_R \Phi\big(\gamma^+(u), \gamma^+(v)\big) - |k_L - k_R| f(\kappa) \;\le\; 0. \tag{2.12}$$

By symmetry, one can assume $k_L > k_R$ without loss of generality and a study of the remaining cases to obtain (2.11) from (2.12) can be found in the Appendix. This implies the Kato inequality and therefore uniqueness of weak solutions of (2.2), entropy admissible in the sense of (2.4), and, by classical arguments (compare section 1.3), the L^1-contraction property, finishing the proof. □

Remark 2.1.3. *It is possible to recover the Rankine-Hugoniot condition from (2.12), by using the boundaries of the entropy constant,* $\kappa = 0$ *and* $\kappa = 1$*. Indeed, taking* $\kappa = 0$ *in (2.12) leads to*

$$k_L f\big(\gamma^-(u)(t)\big) - k_R f\big(\gamma^+(u)(t)\big) \qquad \geq 0$$

and plugging $\kappa = 1$ *into it gives*

$$\begin{aligned} -k_L f(\gamma^-(u)(t)) - f(1) - (-k_R f(\gamma^+(u)(t)) - f(1)) & \qquad \geq 0 \\ \Rightarrow \qquad k_L f(\gamma^-(u)(t)) - k_R f(\gamma^+(u)(t)) & \qquad \leq 0 \end{aligned}$$

Using the fact that the flux vanishes at the boundaries $f(0) = f(1) = 0$*, gives in the Rankine-Hugoniot condition at* $\{x = 0\}$

$$k_L\big(f(\gamma^-(u)(t)\big) = k_R\big(f(\gamma^+(u)(t)\big).$$

Remark 2.1.4. *The proof is strongly based on the assumption of existence of traces* $\gamma^{\pm}(u), \gamma^{\pm}(v)$ *towards the interface. Even though the existence of strong traces of the flux* $f(\gamma^{\pm}(u))$ *can be proven without assumptions on the flux function, see [8], the result can not be extended to measure-valued solutions. It was shown however in [7], that as long as a weak entropy solution exists, any measure-valued entropy solution coincides with the weak entropy solution.*

2.1.2. Adapted Kruzkov entropies

Following the idea, that given a new set of stationary solutions, a new notion of entropy, entropy flux pair has to be considered, Audusse and Perthame introduced a new admissibility criteria in [12], which is a well-recognized approach to entropy admissibility for discontinuous flux problems nowadays, compare for example [2, 8, 11, 15, 21, 28, 48, 66].

The theory is based on the comparison to piece-wise constant functions $a(x)$, which admit jump-type discontinuities at the points where the flux function is discontinuous. It is easy to see that all trivial stationary solutions are solutions of this type, given the correct, dissipative shock at the interface created by the flux. Thus one defines new entropies

$$\eta(u, a) = |u - a(x)|, \qquad \Phi(u, a, x) = \operatorname{sgn}(u - a)\big(F(u, x) - F(a, x)\big)$$
$$a(x) = \begin{cases} a_L & \text{for } x < 0 \\ a_R & \text{for } x > 0 \end{cases}, \qquad a_R, a_R \in \mathbb{R}.$$

$\eta(u, a)$ is called adapted Kruzkov entropy and $\Phi(u, a, x)$ adapted Kruzkov entropy flux. Recreating the same steps as in the previous section, one obtains the entropy inequality

$$\int_{\mathbb{R}+} \int_{\mathbb{R}} |u(t, x) - a(x)| \partial_t \phi(t, x) \text{ dx dt} + \int_{\mathbb{R}} |u_0(x) - a(x)| \phi(0, x) \text{ dx} + \int_{\mathbb{R}+} \int_{\mathbb{R}} \Phi(u, a, x) \partial_x \phi \text{ dx dt} \geq 0.$$

Searching for the condition describing the behaviour of solutions across the interface, multiplication with a cut-off function ζ_ϵ reveals the necessary dissipativity condition at the interface

$$k_R\Phi(\gamma^+(u), a_R) - k_L\Phi(\gamma^-(u), a_L) \ \leq \ 0 \tag{2.13}$$

As all solutions of (2.2) need to satisfy (2.13) in order to be L^1-dissipative, which will be shown at the end of this section, the inequality characterizes the stationary, piece-wise constant, dissipative solutions giving the necessary information on the functions to compare with (a_L, a_R), compare [12], where this second notion of entropy admissible weak solutions to (2.2) was originally derived.

The choice of a pair $(a_L, a_R) \in \mathbb{R}^2$ to define a suitable (stationary) solution to use for comparison, needs to satisfy the Rankine Hugoniot condition, thus

$$k_L f(a_L) = k_R f(a_R).$$

The admissibility condition for the interface, see [8], compare also section 2.2, defining admissible piece-wise constant states to compare with was found using a truncation in the Kato inequality to the interface $\{x = 0\}$:
A pair $(a_L, a_R) \in \mathbb{R}^2$ is suitable to define a stationary solution to (2.2), if for any other suitable pair (b_L, b_R), inequality (2.13) holds, that is

$$k_R\Phi(a_R, b_R) - k_L\Phi(a_L, b_L) \leq 0. \tag{2.14}$$

Definition 2.1.5.
Let $u_0 \in L^\infty(\mathbb{R})$ and $a(x)$ a piecewise constant stationary solution to (2.2) . A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ with traces $\gamma^+(u), \gamma^-(u)$ is called weak (adapted Kruzkov) entropy solution to the initial value problem (2.2) if it is a weak solution of (2.2) satisfying

$$\int_{\mathbb{R}+}\int_{\mathbb{R}} |u(t,x) - a(x)|\partial_t\phi(t,x) \ dx \ dt + \int_{\mathbb{R}} |u_0(x) - a(x)|\phi(0,x) \ dx + \int_{\mathbb{R}+}\int_{\mathbb{R}} \Phi(u,a,x)\partial_x\phi \ dx \ dt \ \geq \ 0 \tag{2.15}$$

for all testfunctions $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, all pairs (a_L, a_R) satisfying (2.14) and additionally

$$k_R\Phi(\gamma^+(u), a_R) - k_L\Phi(\gamma^-(u), a_L) \ \leq \ 0. \tag{2.16}$$

The following theorem holds, which was shown in [12].

Theorem 2.1.6. *Weak (adapted Kruzkov) entropy solutions to (2.2), defined by the previous definition, statisfy the L^1-contraction property (1.8), and therefore are unique in the class of weak solutions.*

Proof.
The proof follows the same steps as the proof of theorem 2.1.2. After truncation with a testfunction and thereafter recovering the interface by passing to the limit, one recovers directly the Kato inequality (1.9), as the interface term vanishes due to (2.13). Following the classical doubling of variables of Kruzkov gives the L^1-contraction. □

2.1.3. Kinetic formulation of conservation laws

This section gives an additional, interesting point of view towards entropy admissibility, considering a kinetic formulation for (2.2). It has the advantage of not treating the interface directly, and does not require any assumption on the existence of traces towards the interface. However, even though a necessary condition is shown to give the L^1-contraction, it was not yet possible to link it to the kinetic entropy equation. The theory of kinetic formulation for conservation laws was introduced by Lions, Perthame and Tadmor [62], later summarized in [68] for the case of different continuous flux type models and has rarely seen application in the context of discontinuous flux conservation laws. In the continuous case, the elegant approach of embedding the nonlinearity of the equation inside the argument of the operating function, resulting in a linear equation allows for regularization and a very nice, simplified well-posedness theory. Many proofs, including L^1-dissipativity, are possible in a compact manner, offering an alternative to the Kruzkov entropy approach. For a rigorous introduction to kinetic formulation for conservation laws, the reader is referred to [68].
In contrary to the continuous case, it is not as easy to derive the kinetic entropy formulation as a spatial derivative on the flux function again. As in (2.3), the discontinuous coefficient function $k(x)$ is smoothened. Starting from the entropy inequality in the distributional sense (1.6), choosing as entropy $\eta(u) = |u - \xi| - |\xi|$, existence of a non-negative measure $m(t, x, \xi)$ can be assumed without loss of generality, such that one obtains an equality with a (arbitrary, finite) negative coefficient on the right side

$$\begin{aligned} \partial_t\big((|u-\xi|-|\xi|\big) + \partial_x\Big[k_\epsilon(x)\big((f(u)-f(\xi)\big)\mathrm{sgn}(u-\xi) - \mathrm{sgn}(\xi)f(\xi)\Big] \\ + k'_\epsilon(x)f(\xi)\mathrm{sgn}(|u-\xi|-|\xi|) \;=\; -2m(t,x,\xi). \end{aligned} \tag{2.17}$$

Note that equation (2.17) as well as all following equations need to be treated in the weak (distributional sense). However, for better readability, the integral signs and testfunctions are omitted here. The measure $m(t, x, \xi)$ is called kinetic entropy defect measure, and holds some useful properties, where the important ones are continuity with respect to ξ and $\forall t > 0$

$$m(t, x, u(t, x)) = 0. \tag{2.18}$$

Additionally, the kinetic indicator function χ needs to be introduced. It is defined by

$$\chi(\xi; u) = \begin{cases} 1, & \text{for } 0 < \xi < u, \\ -1, & \text{for } u < \xi < 0, \\ 0, & \text{otherwise} \end{cases}$$

and holds the following properties, which are proven in [68]. A short proof as well as more details on the kinetic entropy defect measure are included in the Appendix A.3.

Theorem 2.1.7 (**Main properties of** χ).
Let $S' \in L^\infty_{loc}$, then the following properties related to χ hold true.

$$\partial_\xi \chi(\xi;u) = \delta(\xi) - \delta(\xi - u) \tag{2.19}$$

$$\int_{\mathbb{R}} S'(\xi)\chi(\xi;u)\ d\xi = S(u) - S(0) \tag{2.20}$$

$$\int_{\mathbb{R}} |\chi(\xi;u) - \chi(\xi;v)|\ d\xi = |u - v| \tag{2.21}$$

To rewrite (2.17) with respect to the kinetic function $\chi(\xi;u)$, one first recognizes

$$-\frac{1}{2}\frac{\partial}{\partial \xi}(|u-\xi| - |\xi|) = -\frac{1}{2}\frac{\xi - u}{|\xi - u|} - \frac{\xi}{|\xi|} = \left\{\begin{matrix} 1, & \text{if } 0 < \xi < u,\ u > 0 \\ -1, & \text{if } u < \xi < 0,\ u < 0 \\ 0, & \text{else} \end{matrix}\right\} = \chi(\xi;u). \tag{2.22}$$

Now multiplication of (2.17) by $\left(-\frac{1}{2}\right)$ and taking the derivative with respect to ξ gives

$$\partial_t\left(-\frac{1}{2}\frac{\partial}{\partial\xi}(|u-\xi| - |\xi|)\right) + \partial_x\left(-\frac{1}{2}\frac{\partial}{\partial\xi}k_\epsilon(x)([f(u) - f(\xi)]\text{sgn}(u-\xi) - \text{sgn}(\xi)f(\xi)\right)$$
$$-k'_\epsilon(x)f'(\xi) = \frac{\partial}{\partial\xi}m(t,x,\xi).$$

Using property (2.22) of $\chi(\xi;u)$, one obtains

$$\partial_t\chi(\xi;u) + k(x)f'(\xi)\partial_x\chi(\xi;u) - k'_\epsilon(x)f'(\xi) = \partial_\xi m(t,x,\xi).$$

Dropping the regularization on $k(x)$ again by letting $\epsilon \to 0$ and integrating by parts on both sides, one reaches the kinetic entropy formulation of (2.2).

Definition 2.1.8. *Let $u \in C(\mathbb{R}^+; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$. Then u is called kinetic solution to (2.2) if there exists a nonnegative bounded measure $m(t,x,\xi)$ such that for all $\xi \in \mathbb{R}$ and all testfunctions $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}_x \times \mathbb{R}_\xi)$*

$$\int\limits_{\mathbb{R}_t^+ \times \mathbb{R}_x \times \mathbb{R}_\xi} \chi(\xi,u)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi\ dtdxd\xi + \int\limits_{\mathbb{R}_x \times \mathbb{R}_\xi} \chi(\xi;u_0)\phi_{\{t=0\}}dxd\xi \tag{2.23}$$
$$-(k_L - k_R)\int\limits_{\Sigma \times \mathbb{R}_\xi} f'(\xi)\phi_{\{x=0\}}dtd\xi \quad = \int\limits_{\mathbb{R}_t^+ \times \mathbb{R}_x \times \mathbb{R}_\xi} \partial_\xi\phi\ dm(x,t,\xi).$$

with $\chi(\xi;u(0,x)) = \chi(\xi;u_0)$ and $\Sigma = \{0\} \times \mathbb{R}_+$ the line of the interface.

Remark 2.1.9. *It is also possible to derive a kinetic formulation for so called sub-, supersolutions using the notion of (Kruzkov) semi-entropies $(u-\xi)^+, (u-\xi)^-$, where $u^\pm$ denotes the positive/negative part of u respectively. This approach was used in [13, 14] in an attempt to prove L^1-dissipativity.*

Remark 2.1.10. *So far, it has not yet been to accomplished to connect the notions of adapted Kruzkov entropies and kinetic formulation directly, as the derivation of the kinetic formulation requires a derivative with respect to the newly interpreted, former constant ξ, which is discontinuous for the adapted Kruzkov entropies. The only link is due to proving equivalency to standard Kruzkov entropy solutions, compare section 2.3, a formulation, which is only possible if the solution admits strong traces towards the interface.*

The result in this section is the following condition on the interface, which, if statisfied, implies L^1-dissipativity.

Theorem 2.1.11. *Given $u_0 \in L^\infty(\mathbb{R})$, then any kinetic entropy solution u in the sense of Definition 2.1.8, that satisfy (2.23) and additionally for every kinetic solution v*

$$(k_L - k_R) \int\limits_{\Sigma\times\mathbb{R}_\xi} \Big(\big(\chi(\xi;u) + \chi(\xi;v)\big) f'(\xi) - |f'(\xi)| \Big) \phi_{\{x=0\}} \; dtd\xi \quad \geq \; 0. \tag{2.24}$$

is L^1-dissipative in the sense of (1.8).

Proof. The goal will be to reach the L^1-contraction, implying uniqueness, compare section 1.3. Given initial data $u_0, v_0 \in L^\infty(\mathbb{R})$ and two kinetic entropy solutions u, v, equation (2.23) holds for two respective entropy functions $\chi(\xi;u)$ and $\chi(\xi;v)$ with kinetic defect measures $m(t,x,\xi)$ and $q(t,x,\xi)$. Thus the starting point is (2.23) and

$$\begin{aligned} \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \chi(\xi;v)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi dtdxd\xi + \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} \chi(\xi;v_0)\phi_{\{t=0\}}dxd\xi \\ - (k_L - k_R) \int\limits_{\Sigma\times\mathbb{R}_\xi} f'(\xi)\phi_{\{x=0\}}dtd\xi = \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi\phi \; dq(x,t,\xi). \end{aligned} \tag{2.25}$$

Making use of the fact that the kinetic formulation is a linear equation, one applies a regularization to the kinetic functions $\chi_\epsilon(\xi;u) = \chi(\xi;u)\cdot\tau_\epsilon$ and $\chi_\epsilon(\xi;v) = \chi(\xi;v)\cdot\tau_\epsilon$ with a regularizing kernel τ_ϵ

$$\tau_\epsilon = \frac{1}{\epsilon_1}\phi_1\left(\frac{t}{\epsilon_1}\right)\frac{1}{\epsilon_2}\phi_2\left(\frac{x}{\epsilon_2}\right)\frac{1}{\epsilon_3}\phi_3\left(\frac{\xi}{\epsilon_3}\right),$$

where $\phi_{1,2,3} \in C_0^\infty(\mathbb{R})$ are testfunctions. Now multiplying equation (2.23) with $\chi_\epsilon(\xi;v)$

and (2.25)) with $\chi_\epsilon(\xi;u)$ and adding the two resulting equations together, one obtains

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \chi_\epsilon(\xi;u)\chi_\epsilon(\xi;v)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi dtdxd\xi + \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} \chi_\epsilon(\xi;u_0)\chi_\epsilon(\xi;v_0)\phi_{\{t=0\}}dxd\xi$$
$$-(k_L-k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi}\big(\chi_\epsilon(\xi;u)+\chi(\xi;v)\big)f'(\xi)\phi_{\{x=0\}}dtd\xi$$
$$= \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \chi_\epsilon(\xi;v)\partial_\xi\phi \; dm_\epsilon(x,t,\xi) + \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \chi_\epsilon(\xi;u)\partial_\xi\phi \; dq_\epsilon(x,t,\xi).$$

Integration by parts on the right side yields

$$= -\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi\chi_\epsilon(\xi;v)\phi \; dm_\epsilon(x,t,\xi) - \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi\chi_\epsilon(\xi;u)\phi \; dq_\epsilon(x,t,\xi).$$
$$\geq -\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \delta(\xi)\phi dm_\epsilon(x,t,\xi) - \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \delta(\xi)\phi \; dq_\epsilon(x,t,\xi). \tag{2.26}$$

We discover another two equations by multiplying equations (2.23) and (2.24) by sgn(ξ)

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} |\chi_\epsilon(\xi;u)|\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi dtdxd\xi + \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} |\chi_\epsilon(\xi;u_0)|\phi_{\{t=0\}}dxd\xi$$
$$-(k_L-k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi} |f'(\xi)|\phi_{\{x=0\}}dtd\xi \;=\; -2\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \phi \; dm_\epsilon(x,t,\xi). \tag{2.27}$$

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} |\chi_\epsilon(\xi;v)|\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi dtdxd\xi + \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} |\chi_\epsilon(\xi,v_0)|\phi_{\{t=0\}}dxd\xi$$
$$-(k_L-k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi} |f'(\xi)|\phi_{\{x=0\}}dtd\xi \;=\; -2\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \phi \; dq_\epsilon(x,t,\xi). \tag{2.28}$$

Multiplication of (2.26) by (-2) and adding (2.27) and (2.28), one obtains

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \big(|\chi_\epsilon(\xi;u)| + |\chi_\epsilon(\xi;v)| - 2\chi_\epsilon(\xi;u)\chi_\epsilon(\xi;v)\big)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi\ dtdxd\xi$$
$$+ \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} (|\chi_\epsilon(\xi;u_0)| + |\chi_\epsilon(\xi;v_0)| - 2\chi_\epsilon(\xi;u_0)\chi_\epsilon(\xi;v_0))\phi_{\{t=0\}}\ dxd\xi$$
$$- 2(k_L - k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi} |f'(\xi)|\phi_{\{x=0\}}\ dtd\xi$$
$$+ 2(k_L - k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi} \big(\chi_\epsilon(\xi;u) + \chi_\epsilon(\xi;v)\big)f'(\xi)\phi_{\{x=0\}}\ dtd\xi \quad \le\ 0.$$

Using (2.24), this becomes

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \big(|\chi_\epsilon(\xi;u)| + |\chi_\epsilon(\xi;v)| - 2\chi_\epsilon(\xi;u)\chi_\epsilon(\xi;v)\big)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi\ dtdxd\xi$$
$$+ \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} (|\chi_\epsilon(\xi;u_0)| + |\chi_\epsilon(\xi;v_0)| - 2\chi_\epsilon(\xi;u_0)\chi_\epsilon(\xi;v_0))\phi_{\{t=0\}}\ dxd\xi \quad \le\ 0.$$

where one can already observe that the second term vanishes as soon as $v_0 = u_0$. Droppping regularization and using the special form of the χ indicator function, one recovers

$$\frac{d}{dt}\int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi} |\chi(\xi;u) - \chi(\xi;v)|\, dxd\xi \le 0$$

which, due to (2.20), (2.21), implies

$$\frac{d}{dt}\int_{\mathbb{R}} |u(t,x) - v(t,x)|\ dx\ \le\ 0.$$

□

Remark 2.1.12. *In the continuous case, the kinetic entropy equation follows directly from the Kruzkov entropy inequality and reduces the space of weak solutions to the ones satisfying the* L^1*-contraction. Thus one would not expect the necessity for another condition, such that it should be possible to show, that (2.24) follows from (2.23) or (2.4).*

2.2. The notion of germs

As the previous sections have shown, there have been many approaches to a physically relevant notion of (entropy) solution to discontinuous flux type problems like (2.2) over the last decades. The underlying idea to focus on entropy admissibility originally brought up by Kruzkov and well understood in the context of continuous flux problems, has ultimately led to the consideration of the divided problem (2.5), (2.6), such that every region could be treated as a continuous flux problem in the sense of Kruzkov (1.3.4) and the study of the behavior of solutions across the interfaces dividing the original problem. Generalizing existing studies, compare [1, 2, 12, 15, 17, 26, 54, 75], B. Andreianov, K.H. Karlsen and N.H. Risebro introduced a theory revolving around characterizing admissible, dissipative behavior at the interface [8] by comparison to so called elementary solutions, which are piece-wise constant stationary solutions to the original discontinuous flux problem.
This section summarizes the existing theory and results, giving some explanatory examples and leads towards the extension of the theory to non-conservative coupling, which is the case needed for the particle-fluid interaction in the second part of this work.

2.2.1. Germ based entropy solutions and L^1-dissipative germs

Most of the work to define dissipative behavior at an interface has already been done by applying a centered testfunction on the Kruzkov entropy formulations stated in the Definitions 2.1.1, and 2.1.5. Knowing the behavior leading to dissipativity, the definition of a set capturing all piece-wise constant stationary solutions following this behavior at the interface follows directly.

Definition 2.2.1. *Any set $\mathcal{G} \in \mathbb{R}^2$ is called a dissipative (or L^1D) germ, if for all $(u_L, u_R), (v_L, v_R) \in \mathcal{G}$*

$$\Phi_R(u_R, v_R) - \Phi_L(u_L, v_L) \ \leq \ 0 \tag{2.29}$$

holds for Kruzkov entropy fluxes $\Phi_{L,R}(a, b) = sgn(a - b)\big(k_{L,R} f(a) - k_{L,R} f(b)\big)$, compare section 1.3.

Remark 2.2.2. *This definition can of course also be done for other discontinuous fluxes, such that more generally $\Phi_{L,R}(a, b) = sgn(a - b)\big(f_{L,R}(a) - f_{L,R}(b)\big)$. However, the definition above stays with the introduced model (2.2).*

Using this set, one is able to define entropy admissible solutions, by applying Kruzkov entropy admissibility away from the interface and the dissipative behavior at the interface. Using the notion of admissiblity inspired by the continuous flux problems, one obtains the following new definition.

Definition 2.2.3. *A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ admitting strong traces $\gamma^\pm(u)$ in $\{x = 0\}$ is a weak entropy solution to problem (2.2) if and only if it satisfies the following two properties:*

1. *For all $\kappa \in \mathbb{R}$ and all non-negative test functions $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ with $\phi(t,0) = 0$ for all $t \geq 0$, it satisfies*

$$\int_{\mathbb{R}+} \int_{\mathbb{R}} |u(t,x) - \kappa| \partial_t \phi(t,x) dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa| \phi(0,x) dx + \int_{\mathbb{R}+} \int_{\mathbb{R}} \Phi(u(t,x), \kappa, x) \partial_x \phi(t,x) dx dt \geq 0 \tag{2.30}$$

2. *The traces of u satisfy*

$$\text{for a.e. } t > 0, \quad \big(\gamma^-(u)(t), \gamma^+(u)(t)\big) \in \mathcal{G}.$$

Remark 2.2.4. *Comparing this definition to the original definition inspired by Kruzkov, definition 2.1.1, it is at first glimpse not easy to see that both are very close. The entropy inequality (2.30) is much easier to treat, giving the L^1 contraction property following the lines of section 1.3. However, it is necessary to explicitly state another condition, which was directly implied by the entropy inequality (2.4) in definition 2.1.1. Another advantage of definition 2.2.3 is the possibility of changing the search criteria for dissipative behavior, by simply changing the set of admissible jumps. The model considered in the second part of this work is an example of a very explicit construction of an admissibility (L^1D) germ.*

A second definition was also introduced in [8], which has the distinct advantage of not directly using the traces $\gamma^\pm(u)$ towards the interface. It has proven very useful for results on well-posedness of measure valued solutions, compare [39, 14, 7], where existence of these traces can not easily be assumed and is based on the adapted Kruzkov entropies and a penalization term in the entropy inequality.

Definition 2.2.5. *A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is a weak entropy solution to problem (2.2) if, given an L^1D germ $\mathcal{G}$, there exists a constant $M > 0$, such that for all $(a_L, a_R) \in \mathbb{R}^2$ and all non-negative test functions $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, it satisfies*

$$\int_{\mathbb{R}+} \int_{\mathbb{R}} |u(t,x) - a(x)| \partial_t \phi(t,x) dx dt + \int_{\mathbb{R}} |u_0(x) - a(x)| \phi(0,x) dx + \int_{\mathbb{R}+} \int_{\mathbb{R}} \Phi(u(t,x), a(x), x) \partial_x \phi(t,x) dx dt + M \int_{\mathbb{R}+} dist\big((a_L, a_R), \mathcal{G}\big) \phi(t,0) dt \geq 0$$

where $a(x) = \begin{cases} a_L, & x < 0 \\ a_R, & x > 0 \end{cases}$ *and $dist(\cdot,\cdot)$ is the standard Euclidean distance in $\mathbb{R}^2$.*

As the wording already suggests, both definitions are equivalent if the same L^1D germ $\mathcal{G}$ is chosen, which was proven in [8] for the case of conservative problems, but can be extended to germs representing non-conservative coupling. The proof will be included in section 2.3, see also [5, 7]. Being the whole point of interest, both definitions imply uniqueness of the considered admissible solutions, which is stated in the following theorem. Although this theorem was proven in [8], a short proof is included in this work.

Theorem 2.2.6. *Given an initial value problem (2.2) and $u_0 \in L^\infty(\mathbb{R})$, solutions admissible in the sense of definition 2.2.3 or 2.2.5 are unique and satisfy the L^1 contraction property 1.8.*

Proof. Given two solutions u, v, admissible in the sense of definition 2.2.3, one starts from the entropy inequality (2.30). The method used to obtain the Kato inequality in the continuous case, i.e. doubling of variables, can be applied in the exact same manner provided that the testfunction ϕ vanishes in a neighborhood of $\{x = 0\}$. After reintroducing generalized testfunctions, as in the proof of theorem 2.1.2, and using $\Phi(u, \kappa, x) = k(x)\Phi(u, \kappa)$ on obtains (2.10). In order to further follow the steps to obtain the Kato inequality, all that needs to be satisfied is

$$k_R\Phi(\gamma^+(u), \gamma^+(v)) - k_L\Phi(\gamma^-(u), \gamma^-(v)) \leq 0.$$

which is exactly the dissipativity of $\mathcal{G}$. Thus definition 2.2.3 implies uniqueness and the L^1-contraction property.
Now considering definition 2.2.5. Applying the same method as before, one obtains as interface condition the following properties of $\mathrm{dist}\big((a_L, a_R), \mathcal{G}\big)$[2]

$$\begin{aligned} (a_L, a_R) \in \mathcal{G} \;&\Rightarrow\; \mathrm{dist}\big((a_L, a_R), \mathcal{G}\big) = 0 \\ \forall (u_L, u_R) \in \mathbb{R}^2, (a_L, a_R) \in \mathcal{G} \;&\Rightarrow\; k_R\Phi(u_R, a_R) - k_L\Phi(u_L, a_L) \leq \mathrm{dist}\big((a_L, a_R), \mathcal{G}\big) \end{aligned}$$

which follow as $\mathcal{G}$ is an L^1D germ. □

Example 4. It might be interesting to note, that the theory of germs could also be applied to the continuous flux case, where it would just recover the good behavior across shockwaves, i.e. the Lax entropy condition, see for example [40]. As stationary solutions in this case are simply constants $a \in \mathbb{R}$, the corresponding dissipative germ is of the form

$$\mathcal{G}^c = \big\{(a, a) \in \mathbb{R}^2\big\} \cup \big\{(a_L, a_R), f(a_L) = f(a_R) \text{ and } a_R > a_L\big\}.$$

The first part is dictated by the stationary solutions, the second part ensuring the conservation and dissipativity across stationary shockwaves. Entropy solutions to problems involving continuous flux functions can also be defined by demanding that all traces $\gamma^\pm(u)$ of towards every point in space are in the set $\mathcal{G}^c$ above.

Remark 2.2.7. *The notion of germ can be extended to an interface coupling changing in time, resulting in time-dependent germs. An example of such an extension is done in the second part of this work, where due to merging or splitting of particles, the friction constant λ can change, which changes the corresponding admissibility germ $\mathcal{G}_\lambda$.*

[2] A more in-depth study of this term and the proof of these properties can be found in [8].

2.2.2. Definite germs

Analysis of L^1D germs, see [8, 11], as well as consideration of the continuous flux case, where the germ is composed of solutions that satisfy the Rankine-Hugoniot condition and are entropy admissible, revealed that germs are often composed of subsets, which might be interesting to study. This led to the introduction of so called definite germs, sets which capture one or multiple key properties of dissipative germs, but lead to unique solutions if and only if they fully coincide with their corresponding L^1D germ.

Definition 2.2.8. *Let $\mathcal{G}$ be a dissipative germ in the sense of definition 2.2.1. A set $\mathcal{G}_0 \subset \mathcal{G}$ is called a definite germ with respect to $\mathcal{G}$, if the set of pairs (u_L, u_R), such that (2.29) holds for all $(v_L, v_R) \in \mathcal{G}_0$, is $\mathcal{G}$.*

The following lemma about non-uniqueness of definite germs holds true.

Lemma 2.2.9. *Let $\mathcal{G}$ be a dissipative germ in the sense of definition 2.2.1 and G_0 a definite germ in the sense of definition 2.2.8. Then, any set $\mathcal{G}_1$, such that $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}$ is a definite germ.*

Proof. To prove the lemma, every such set $\mathcal{G}_1$ needs to require the whole germ $\mathcal{G}$, such that (2.29) holds for every pair $(u_L, u_R) \in \mathcal{G}_1$. This is obviously true, as the subset $\mathcal{G}_0 \subset \mathcal{G}_1$ already needs the whole germ $\mathcal{G}$, which is unique. Therefore every such set $\mathcal{G}_1$ is a definite germ. □

Example 4 (extended). To give an example for a definite germ, it is sufficient to consider the continuous flux problem again, returning to Example 4. Taking the set $\mathcal{G}_0^c = \{(a, a) \in \mathbb{R}^2\}$, and checking relation (2.29) gives

$$\Phi(u_R, a) - \Phi(u_L, a) \ \leq \ 0.$$

This condition is exactly what can be derived by the Kruzkov formulation, and is therefore true if and only if the pair of traces (u_L, u_R) is entropy admissible, namely $(u_L, u_R) \in \mathcal{G}^c$. As a result, $\mathcal{G}_0^c$ is a definite germ with respect to $\mathcal{G}^c$.

2.2.3. Maximal germs

It is well known in the context of conservation laws, that problems of type (2.2), without specification on the flux function, i.e. generally considering fluxes $F(u, x)$, sometimes generate multiple L^1-contractive semigroups, see [2, 1, 8]. Therefore one would not necessarily expect for L^1D germs to be unique, and, as proven in [8], they aren't. This is not necessarily a bad thing, as multiple L^1D germs can correspond to different physical interpretations, but in order to clarify, the notion of maximal (L^1D) germ was introduced.

Definition 2.2.10. *A dissipative (L^1D) germ $\mathcal{G}^*$ is called a maximal (L^1D) germ, if there exists no other dissipative germ $\mathcal{G}$, such that $\mathcal{G}^* \subset \mathcal{G}$. If there exists a dissipative germ $\mathcal{G}^d$, such that $\mathcal{G}^d \subset \mathcal{G}^*$, then $\mathcal{G}^*$ is called maximal (L^1D) extension of $\mathcal{G}^d$.*

Example 4 (extended). Clearly, as other interface couplings are ruled out by the Kruzkov entropy approach (and Rankine-Hugoniot condition), the germ $\mathcal{G}^c$ is a maximal germ.

More examples of germs corresponding to known admissibility criteria can be found in [8], for example the Gelfand germ, which corresponds to the vanishing viscosity limit of a continuous flux problem, germs for the cases of increasing surjective and monotone fluxes and the germ corresponding to the Karlsen-Risebro-Towers entropy condition, which introduced a crossing condition for general discontinuous fluxes, see [54, 55, 75, 76].

2.2.4. Non-conservative coupling

As mentioned before, it is possible to use the notion of germs not only for discontinuous flux type problems, but also for interfaces where conservation might not hold for different reasons. The model of fluid-particle interaction considered in the second part of this work is one of the first examples for this result, where a source term corresponding to the particles creates an interface not so different to the ones created by a jump in the flux function, compare [10, 60]. This example will be discussed in chapter 4, which is why we focus on another example here, which corresponds to road traffic with a point constraint and was introduced in [30], compare also [5].
The model problem is a typical constrained traffic model

$$\partial_t u + \partial_x f(u) = 0, \qquad \text{with constraint} \quad f(u)(t,0) \leq F(t)$$

with flux $f(0) = 0 = f(1)$, f nonnegative, $f \in \mathrm{Lip}(\mathbb{R})$, such that $u_{\max} := \arg\max f$. Choosing $0 \leq u_0 \leq 1$ and $F \in [0, F(u_{\max})]$, it is possible to define entropy admissibility using the notion of germs. The admissible points $B_F \leq u_{\max} \leq A_F$ to jump at the F-level set of f characterize the germ, thus

$$\mathcal{G}(t) = \big\{(A_{F(t)}, B_{F(t)})\big\}$$

and one can compute the maximal extension of $\mathcal{G}(t)$ by allowing for other dissipative jumps to obtain

$$\mathcal{G}(t)^* = \mathcal{G}(t) \cup \big\{(a_L, a_R) \in \mathbb{R}^2 : f(a_{L,R}) \leq F \text{ and } a_L \leq a_R\big\}$$

Defining entropy solutions to be Kruzkov entropy solutions u to the continuous flux problem with constraint on the traces

$$\text{for } a.e.\ t > 0, \quad \big(\gamma^-(u)(t), \gamma^+(u)(t)\big) \in \mathcal{G}(t)^*,$$

uniqueness and L^1-contraction can be shown almost classically following the ideas of the proofs of the previous section. In fact, it is easy to see, that all stationary solutions jumping below the F-level set at the interface are Kruzkov solutions of the continuous flux problem without constraint.

2.3. Equivalency and comparison of different entropy formulations

After deriving several viewpoints on entropy admissibility for solutions of discontinuous flux conservation laws in the previous sections, this section summarizes and compares the results. Equivalency of the different notions is not direct and they bear different advantages connected to analysis, complexity of proofs and assumptions needed for example in the presence of measure valued solutions. The following definition summarizes the different entropy admissibility conditions.

Definition 2.3.1. *Given $u_0 \in L^\infty(\mathbb{R})$. A function u is called weak entropy solution to the initial value problem (2.2), if it is a weak solution to (2.2) satisfying one of the following conditions*

1. **Classical Kruzkov approach**

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(t,x)-\kappa|\partial_t\phi(t,x)dxdt + \int_{\mathbb{R}} |u_0(x)-\kappa|\phi(0,x)dx + \int_{\mathbb{R}^+}\int_{\mathbb{R}} k(x)\Phi(u,\kappa)\partial_x\phi dxdt + |k_L-k_R|\int_{\mathbb{R}^+} f(\kappa)\phi(t,0)dt \geq 0$$

for all testfunctions $\phi \in C_0^\infty(\mathbb{R}^+\times\mathbb{R})$, all $\kappa\in\mathbb{R}$ with u admitting traces $\gamma^\pm(u)$ towards the interface.

2. **Adapted Kruzkov approach**

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(t,x)-a(x)|\partial_t\phi(t,x)dxdt + \int_{\mathbb{R}} |u_0(x)-a(x)|\phi(0,x)dx + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Phi(u,a,x)\partial_x\phi dxdt \;\geq\; 0$$

for all testfunctions $\phi \in C_0^\infty(\mathbb{R}^+\times\mathbb{R})$ and all $a(x) = a_L\mathbb{1}_{x\leq 0} + a_R\mathbb{1}_{x>0}$ defined by (2.14).

3. **Germ based admissibility using traces**

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(t,x)-\kappa|\partial_t\phi(t,x)dxdt + \int_{\mathbb{R}} |u_0(x)-\kappa|\phi(0,x)dx + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Phi(u,\kappa,x)\partial_x\phi(t,x)dxdt \;\geq\; 0$$

for all $\kappa\in\mathbb{R}$, $t\geq 0$, all non-negative test functions $\phi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R})$ with $\phi(t,0)=0$ and the traces of u satisfy for a.e. $t>0$: $\big(\gamma^-(u)(t),\gamma^+(u)(t)\big)\in\mathcal{G}$.

4. **Germ based admissibility with penalization term**
if there exists a constant $M > 0$, such that for all $(a_L, a_R) \in \mathbb{R}^2$ and all nonnegative test functions $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ it satisfies

$$\int_{\mathbb{R}+}\int_{\mathbb{R}}|u(t,x)-a(x)|\partial_t\phi(t,x)dxdt + \int_{\mathbb{R}}|u_0(x)-a(x)|\phi(0,x)dx$$
$$+\int_{\mathbb{R}+}\int_{\mathbb{R}}\Phi(u,a(x),x)\partial_x\phi(t,x)dxdt + M\int_{\mathbb{R}+} dist\big((a_L,a_R),\mathcal{G}\big)\phi(t,0)dt \geq 0$$

where $a(x) = a_L\mathbb{1}_{x\leq 0} + a_R\mathbb{1}_{x>0}$.

5. **Kinetic entropy admissibility**
if there exists a nonnegative bounded measure $m(t,x,\xi)$ such that for all $\xi \in \mathbb{R}$ and all testfunctions $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}_x \times \mathbb{R}_\xi)$

$$\int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi} \chi(\xi,u)\big(\partial_t + \big(k(x)f'(\xi)\big)\partial_x\big)\phi \; dtdxd\xi + \int\limits_{\mathbb{R}_x\times\mathbb{R}_\xi}\chi(\xi;u_0)\phi_{\{t=0\}}dxd\xi$$
$$-(k_L-k_R)\int\limits_{\Sigma\times\mathbb{R}_\xi} f'(\xi)\phi_{\{x=0\}}dtd\xi \quad = \int\limits_{\mathbb{R}_t^+\times\mathbb{R}_x\times\mathbb{R}_\xi}\partial_\xi\phi \; dm(x,t,\xi).$$

with $\chi(\xi;u(0,x)) = \chi(\xi;u_0)$ and $\Sigma = \{0\}\times\mathbb{R}^+$ the line of the interface.

Reviewing the differences, there are a couple of things to notice. The first formulation, using classical Kruzkov entropies, was the first approach, introduced in [75], to discontinuous flux conservation laws after the Kruzkov formulation turned out to be massively useful in the continuous flux case. As shown in section 2.1.1, it allows for a uniqueness proof, but there are some disadvantages. It drops the original idea of Kruzkov, which was comparison to (trivial) stationary solutions, and the appearance of the resulting interface term complicates computations. However, it allows for κ to be treated as a new variable, opening the possibility to derive the kinetic entropy formulation 5.
The kinetic approach to linearize the equation, even though offering simple alternatives for the proofs of many results in the continuous case, enabling tools like regularization, compare [32, 68], has proven to be difficult to treat in the discontinuous flux case, compare [13, 14]. It seems to be advantageous when dealing with measure valued solutions, as the existence of traces towards the interface on the kinetic level holds true even for functions with very low regularity.
The adapted Kruzkov formulation 2 extends the idea of Kruzkov, but as it makes it necessary to study left and right traces at the interface anyway and embedding it in the much more general framework of germs even holds some simplifications, it is often easier to directly define the corresponding admissibility germ moving to formulations 3 and 4.

The germ based formulations carry the distinct advantage of the easiest way to consider the problem and prove uniqueness. Also, as was demonstrated in section 2.2 and can also be seen following the specific problem in chapter 3, the theory extends directly to constrained flux, source-term and other non-conservative problems, compare

[5, 10, 30, 54, 55, 60, 75, 76]. The formulation with the penalization also proves very useful for analysis on convergence of approximated solutions, as the germ is directly embedded in the formulation, compare section 3.3.7 or [11].

Regarding equivalency of the different formulations, the following theorem can be proven, following mostly the lines of [11].

Theorem 2.3.2. *The notions of entropy admissibility regarding solutions of (2.2) by formulations* $2-4$ *are equivalent.*

Proof. The structure of the proof will be to show 2. ⇒ 3. ⇒ 4. ⇒ 2.,
starting with formulation 2 ⇒ formulation 3.
Thus the starting point is the entropy inequality given in formulation 2 with the additional condition for the traces. It follows directly by (2.14) that $(a_L, a_R) \in \mathcal{G}$. Clearly, compare section 2.1.2, the Kruzkov inequality is implied left and right of the interface. Choosing a testfunction with compact support in a neighborhood of $\{x = 0\}$, after dropping regularization, such that the testfunction becomes a delta distribution, the interface term is obtained:

$$\forall (a_L, a_R) \in \mathcal{G} : \qquad \int_{\mathbb{R}+} \Big(\Phi(\gamma^-(u), a_L) - \Phi((\gamma^+(u), a_R)\Big)\phi\, dt \geq 0. \tag{2.31}$$

Comparing this again to (2.14), it follows that also $(\gamma^-(u), \gamma^+(u)) \in \mathcal{G}$.

The next step is to show formulation 3 ⇒ formulation 4.
Given $(c_L, c_R) \in \mathbb{R}^2$ instead of $(c_L, c_R) \in \mathcal{G}$ in (2.31), taking the closest pair $(b_L, b_R) \in \mathcal{G}$ to (c_L, c_R), then by computation

$$\begin{aligned}\Phi(\gamma^-(u), c_L) - \Phi(\gamma^+(u), c_R) \leq{}& \Phi(\gamma^-(u), b_L) - \Phi(\gamma^+(u), b_R) \\ &- \Big|\Phi(\gamma^-(u), c_L) - \Phi(\gamma^+(u), c_R) - \Phi(\gamma^-(u), b_L) + \Phi(\gamma^+(u), b_R)\Big|.\end{aligned}$$

The first term of the right side is negative by (2.14) and choosing a constant
$M(|\gamma^{\pm}(u)|, |b_{L,R}|, |c_{L,R}|)$ large enough, the second term is controlled by
$M \text{ dist}\big((c_L, c_R), (b_L, b_R)\big)$, which corresponds, due to the choice of (b_L, b_R) exactly to
$M \text{ dist}\big((c_L, c_R), \mathcal{G}\big)$.

It remains to show formulation 4 ⇒ formulation 2.
First of all notice that the solutions of [4] satisfy the entropy inequality of formulation 2, because obviously $M \geq 0$ and $\text{dist}((a_L, a_R), \mathcal{G}) \geq 0$. To prove that the additional condition of formulation 2 holds, one uses again a testfunction with compact support around $\{x = 0\}$, to obtain

$$\int_{\mathbb{R}+} \big(\Phi(\gamma^-(u)(t), a_L) - \Phi(\gamma^+(u)(t), a_R)\big)\phi(t, 0) dt \;\geq\; -M \int_{\mathbb{R}+} \text{dist}((a_L, a_R), \mathcal{G}))\phi(t, 0) dt$$

Now one takes $(a_L, a_R) \in G$ obtaining the desired inequality
$\Phi(\gamma^-(u)(t), a_L) - \Phi(\gamma^+(u)(t), a_R) \geq 0$, i.e. (2.14). □

Remark 2.3.3. *Notice that it was not possible so far to clarify if formulations 1 and 2 are equivalent, compare [12, 14]. Clearly both select a unique dissipative solution, meaning the interface coupling defined by the formulations are L^1D germs, however, it is not clear if they coincide or if one of them is maximal. Equivalency of the kinetic formulation in the case of discontinuous flux problems would be very interesting however, as the way the kinetic formulation derives from the Kruzkov formulation and the fact that it does not make use of the traces of the original solution make it very attractive for dealing with generalized or measure valued solutions, where the other formulations can not be used a priori.*

3. A model for fluid-particle interaction

The third chapter and following part of this thesis contains the study of a model of inviscid fluid-particle interaction. The model is particularly interesting as it is the first fluid-solid interaction model considering entropy admissible solutions in the presence of shockwaves and, as mentioned in the previous section, an example where well-posedness can be achieved for a non-conservative, singular balance law using the notion of germs. The model was originally introduced in [60] with a study of the Riemann problem for the case of a single particle and extended to the Cauchy problem in [10, 11]. Some numerical results for the case of a single particle have been achieved in [4, 10].
The influence of the particle towards the fluid is achieved by a singular source term and corresponds to friction between the fluid and the particle. While the particle could be considered moving according to an ordinary differential equation, compare [60], it is sufficient to demand certain dissipative behavior at the position of a particle in order to study well-posedness of solutions to the fluid equation. This work builds on the initial model in [60], where the fluid is modeled by the Burgers equation and the particles act as a point-wise drag force.
In the first section of this chapter, the preceding results of [9, 10, 11, 60] are summarized, some of which will be used as building blocks for the results of the later sections. This includes well-posedness of the Cauchy problem, L^∞ and TV-bounds and a study of the behavior of the fluid at the position of a particle. In section 3.2, the existence and uniqueness results are extended to the Riemann problem for an arbitrary, finite amount of particles. A simplification for the study of the interface terms is shown, using a new notion of generalized admissibility germ. In the special case of all particles drifting apart with given, constant velocity, exact entropy solutions are computed for all possible initial data. The interaction between the particles and corresponding germs gives the critical information used in section 3.3.2 to determine the change in the germs when particles split or merge, which should clearly impact the friction coefficient.
Most of the analysis is done for the case of two particles, which contains most of the difficulties to extend the model, with details on how to extend to an arbitrary number N of particles in sections 3.2.4 and 3.3.8. Section 3.3 extends the study to the Cauchy problem for N particles, where admissibility and the behavior of the special germs is discussed in sections 3.3.2 and 3.3.3. Then the main result of this work is stated, which is well-posedness and an L^∞ bound for entropy solutions of the Cauchy problem in the case of N particles, which is then proven in the remaining sections using an explicit construction algorithm for existence, an approximation for the case of splitting or particles and tools from chapter 2 as well as functional analysis and measure theory.

3.1. The model with a single particle

The model for interaction with a single particle was introduced in [60] with a study of well-posedness of the Riemann problem and further studied in [11, 10], where well-posedness for generel Cauchy initial data, aswell as L^∞ and TV bounds were proven. This section summarizes the results of these papers. The model in the presence of a single particle with friction constant $\lambda \in \mathbb{R}$ reads

$$\begin{aligned} \partial_t u + \partial_x (u^2/2) &= \lambda (h(t) - u(t,h(t)) \delta(x - h(t)), \\ u(0,x) &= u_0(x). \end{aligned} \tag{3.1}$$

There are multiple things to note. The source term is singular, and contains a non-conservative product $u(t, h_i(t))\delta(x - h_i(t))$. This problem was tackled in [10] by a regularization of the particle, using a non-negative compactly supported density function instead. However, an analysis of the behaviour of the fluid at the position of the particle allows for a well-posedness proof considering the influence of the particle as a condition on the behaviour of the fluid at a moving interface located at the particle position.
The influence of the source term is dictated by the friction constant λ, leaving the Burgers equation in the case $\lambda = 0$, corresponding to no friction between the particle and the fluid and a strictly decoupled problem with a wall-like boundary condition in the case $\lambda \to \infty$.

Remark 3.1.1. *Even though it is clearly desirable to also consider different equations to govern the fluid in this model, for example Eulers equations or the Navier-Stokes equations, so far, the Burgers equation seems the only model equipped with sufficient results regarding well-posedness in order to study the notion of entropy admissibility and interaction with shocks. The only results for the coupling with the Euler equations, for a specific case of the Riemann problem, i.e. a fixed particle, is [3]. Recent developments, compare [22, 35, 44, 51], have led to a discussion within the community, if it is even possible to sufficiently define admissible (dissipative) solutions to the Euler equations.*

The first obstacle is to define entropy admissible solutions. The approach chosen in [60] follows the ideas of section 2.2, and therefore needs to define the behavior at the particle induced interface in a physically meaningful (and dissipative) way. Following ideas of [50, 73], they study a traveling wave at the position of the particle $h(t)$, determining all possible solutions across the interface. Away from the interface, Kruzkov entropy admissibility is sufficient, where for the study of the interface, Lax entropy condition was used.
The following proposition describes the possible behavior of the fluid across the particle, therefore giving the set of admissible left and right states, thus the germ, which is stated below.

Proposition 3.1.2. *Let* $\mathcal{U}_1(\bar{U}, \lambda, v) \subset \mathbb{R}$ *be the set defined by*

$$\mathcal{U}_1(\bar{U}, \lambda, v) = \begin{cases} \{\bar{U} - \lambda\} & \text{if } \bar{U} < v, \\ [2v - \bar{U} - \lambda, v] & \text{if } v \leq \bar{U} \leq v + \lambda, \\ \{\bar{U} - \lambda\} \cup [2v - \bar{U} - \lambda, 2v - \bar{U} + \lambda] & \text{if } \bar{U} > v + \lambda. \end{cases} \tag{3.2}$$

Then the set defines dissipative behavior across the interface by a single particle.

Proof. The proof is done in [60] and we will only recreate the most important steps here. One searches for solutions following the particle $u^\epsilon(t,x) = U^\epsilon(x - h(t))$. Restricting to the neighborhood of the particle $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ and regularizing, the equation of interest becomes

$$-h'(t)(U^\epsilon)'(\xi) + ((U^\epsilon)^2/2)'(\xi) - \lambda(h'(t) - U^\epsilon(\xi))(H^\epsilon)'(\xi) = 0$$

where the left trace towards the observed region is $U^\epsilon(-\frac{\epsilon}{2}) = u_L$ and H^ϵ is the regularized delta distribution, i.e. the particle. Demanding entropy admissible jumps gives the conditions at the interface

$$\begin{aligned} (U^\epsilon - h')(U^\epsilon + \lambda H^\epsilon)' = 0, \qquad & \text{if } U^\epsilon \text{ is smooth and} \\ U^\epsilon(\xi_0^+) + U^\epsilon(\xi_0^-) = 2h'(t) \ \text{ and } \ U^\epsilon(\xi_0^-) > U^\epsilon(\xi_0^+), \qquad & \text{if } U^\epsilon \text{ is discontinuous in } \xi_0. \end{aligned}$$

With these conditions, one obtains case-by-case with respect to the appearance of a discontinuity that $U^\epsilon(\frac{\epsilon}{2}) \in \mathcal{U}_1(U_L, \lambda, h'(t))$ and respectively for any $U_R \in \mathcal{U}_1(U_L, \lambda, h'(t))$ existence of a unique solution U^ϵ with $U^\epsilon(\frac{\epsilon}{2}) = U_R$. □

Therefore one can give the set of admissible traces, i.e. the corresponding germ.

Definition 3.1.3. *The admissibility Germ* $\mathcal{G}_\lambda \subset \mathbb{R}^2$ *associated with the interface resulting from one particle is defined by*

$$\mathcal{G}_\lambda = \left\{(c_-, c_+) \in \mathbb{R}^2;\ c_- - c_+ = \lambda\right\} \cup \left\{(c_-, c_+) \in \mathbb{R}_+ \times \mathbb{R};\ -\lambda \leq c_- + c_+ \leq \lambda\right\}.$$

As the connection across the particle determined by this condition is unique, the germ needs to be dissipative, which is rigorously written down in [11] and following the theory of [8], compare section 2.2, this means the following proposition holds true.

Proposition 3.1.4. *The Germ* $\mathcal{G}_\lambda$ *is maximal and dissipative, in the sense that*

$$(c_L.c_R) \in \mathcal{G}_\lambda(v) \ \Leftrightarrow\ \left[\forall (b_L, b_R) \in \mathcal{G}_\lambda(v),\ \bar{\Phi}(v; c_L, b_L) \geq \bar{\Phi}(u; c_R, b_R)\right];$$

with

$$\bar{\Phi}(u; a, b) := sgn(a - b)(a^2/2 - b^2/2) - v|a - b| \ (a, b \in \mathbb{R}).$$

Note that the entropy flux $\bar{\Phi}$ contains an interface term explicitly given by the form of the source term and is obtained by the methods of section 2.1.1 using the form of the germ $\mathcal{G}_\lambda$.

3.1.1. Definition of entropy solutions

With the behavior of the solution at the position of the particle known and the admissibility germ $\mathcal{G}_\lambda$ defined, one can define entropy admissible solutions to problem 3.1, following the lines in [10].

Definition 3.1.5. *Given $u_0 \in L^\infty, N > 0, h(t) \in W^{1,\infty}([0,T])$. We call $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ weak entropy solution to the Cauchy Problem (3.1), if it satisfies for all piecewise constant functions $c(t,x)$ and almost every time t*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} (|u-c|\partial_t\phi + \Phi(u,c)\partial_x\phi)\,dtdx + \int_{\mathbb{R}^+} |u_0 - c|\phi(0,x)dx \;\geq\; 0 \tag{3.3}$$

with $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+), \phi(t,h(t)) = 0$, and additionally

$$(\gamma^-(u(t,h(t)), \gamma^+(u(t,h(t))) \in \mathcal{G}_\lambda(t), \quad \textit{for a.e. } t \in (0,T)$$

Note that in [10], the second definition using a penalization term was also used, i.e. definition 2.1.5 with $\mathcal{G} = \mathcal{G}_\lambda$, which will be extended to the case of multiple particles in section 3.3.3.

3.1.2. Well-posedness and L^∞ bounds

This section summarizes the results of [9, 10, 11, 60] regarding well-posedness of entropy solutions. The proofs will again only briefly be sketched, fur further details, the reader is referred to [10]. The first theorem states the existence and uniqueness of entropy solutions, as well as an L^∞-bound.

Theorem 3.1.6. *Given $u_0 \in L^\infty(\mathbb{R})$ and $h \in W^{1,\infty}([0,T])$, then there exists a unique solution u of (3.1), entropy admissible in the sense of Definition 3.1.5. Moreover, u satisfies*

$$\forall t \in (0,T) \qquad \|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty + \lambda.$$

Proof. The proof of uniqueness follows the ideas of the previous chapter and can also be found in [11] for the case of a fixed particle, i.e. $h \equiv 0$. It is done by comparison of two entropy admissible solutions u, v due to definition 3.1.5. Truncation around the particle path using suitable testfunctions, as in the proof of theorem 2.1.2, gives at the interface

$$\int_0^T \int_{\mathbb{R}} |u-v|\partial_t\phi\,dx\,dt - \int_0^T \int_{\mathbb{R}} \Phi(u,v)\partial_x\phi\,dx\,dt$$
$$\leq \int_0^T \Big(\bar{\Phi}(h'(t), \gamma^+(u(t,h(t)), \gamma^+(v(t,h(t))) - \bar{\Phi}(h'(t), \gamma^-(u(t,h(t)), \gamma^-(v(t,h(t))\Big)\phi_{h(t)}dt$$

where the entropy flux $\bar{\Phi}(h'(t), u, v) = \Phi(u,v) - h'(t)|u-v|$ was used, and the term on the right side of the inequality turns out to be strictly non-positive, as it corresponds to

the dissipativity of the germ, compare Proposition 3.1.4. Thus the inequality is exactly the Kato inequality, giving uniqueness following the classical arguments introduced in section 1.3.
The existence result is reached by approximation. In [11], existence of entropy solutions to problem 3.1 was shown under the constraint of a piece-wise affine particle path $h(t)$ using a well-balanced finite volume scheme. Therefore, after approximating $h(t)$ by a series of piece-wise affine paths, existence of entropy solutions to the approximated problem can be deduced. The L^∞ bound is proven in [10] for the approximated problem as well, using the change of variables $v(t,y) = u(t,y+V_l t) - V_l$, where V_l is given by the approximation of the particle path $h_l'(t) = \sum_{n=1}^{N_l} V_l^n \, \mathbb{1}_{(t^{n-1},t^n]}(t)$. Using strong compactness results derived in [62, 67], one can pass to the limit and obtain the (unique) entropy solution. □

The second result of [10] is the BV control of entropy solutions, which is stated in the following theorem.

Theorem 3.1.7. *Given $u_0 \in BV(\mathbb{R})$, $h \in W^{1,\infty}([0,T])$, $h' \in BV([0,T])$. The unique entropy solution u of (3.1) belongs to $L^\infty([0,T]; BV(\mathbb{R}))$ and satisfies*

$$TV\, u(t,\cdot) \;\le\; TV\, u_0 + 2\,dist\Big(\big(u_0(0^-), u_0(0^+)\big), \mathcal{G}_\lambda\big(h'(0)\big)\Big) + 4\, TV_{[0,T]} h'. \tag{3.4}$$

Proof. The proof is based on numerical analysis using the method of wave front-tracking introduced by Holden and Risebro in [49]. For details on the proof, the reader is referred to the Appendix of [10]. The resulting estimate is

$$\sup_{t\in(0,T]} \mathrm{TV}\, u(t,\cdot) \;\le\; \mathrm{TV}\, u_0 + 2\mathrm{dist}\Big(\big(u_0(0^-), u_0(0^+)\big), \mathcal{G}_\lambda\Big).$$

This can be directly applied for the approximated problem with piece-wise affine path. Applying this estimate on each region in time $[t_l^{n-1}, t_l^n]$ where the approximated particle path is straight, a penalization term of the form $\mathrm{dist}\Big(\big(u(h^-), u(h^+)\big), \mathcal{G}_\lambda\Big)$ is added whenever a new Riemann problem is solved due to the change in particle velocity. Adding up over the whole time span gives (3.4). □

3.2. The Riemann Problem for multiple particles

In this section the Riemann problem in the presence of multiple particles will be discussed. Riemann problems are a very important simplification of problems in fluid mechanics and theory of partial differential equations in general, as they often provide critical information about the behaviour of waves. There are many examples, like [49, 75, 76], where the solution of the Riemann problem acts as a building block towards proving existence of the Cauchy problem. In our case, although we will not use the solutions of the Riemann problem to specifically construct solutions for the Cauchy problem later, the study of the Riemann problem resolves the issue of how to define admissible particles and will allow us to give a good admissibility criteria for entropy solutions when particles merge or split. In addition, we are able to see some nice properties of the solution at the position of the particles and introduce a new mathematical object, called generalized Germ, which allows to give a very compact formulation of the problem and is easily extended to an arbitrary number of particles.
Most of the analysis will be done for the case of two particles and later extended to arbitrarily many particles.

3.2.1. The model problem

We consider the following Riemann Problem for multiple particles. The model writes

$$\begin{aligned} \partial_t u + \partial_x(u^2/2) &= \sum_{i=1}^{N} \lambda_i(v_i - u(t,h_i(t))\delta_{x-h_i(t)}, \qquad N \in \mathbb{N}_0^+ \\ u(0,x) = u_0(x) &:= \begin{cases} u_L, & \text{for } x \leq 0, \\ u_R, & \text{for } x > 0, \end{cases} \\ h_i(0) &= 0. \\ h_i'(t) &= v_i = \text{constant} \quad \forall i \in [1,N] \end{aligned} \tag{3.5}$$

and we assume the particle speeds to be ordered, such that without loss of generality $v_i < v_{i+1}$ and

$u(x,t)$	the velocity of the fluid
$h_i(t)$	the path of the i-th particle
v_i	the velocity of the i-th particle
λ_i	a friction constant corresponding to the i-th particles.

3.2.2. Generalized Germs

We search for stationary solutions to the Riemann Problem with two particles initially located at the origin and moving with given, constant velocities v_1 and v_2. Note that left of the first and right of the second particle, the fluid is governed by the Burgers equation and allows for constant states as stationary solutions in these regions. We derive the possible connections over the two particles and the region in between them from the analysis of the behaviour of the solution across a single particle. Given a left state u_L, we recall that the following set of states describes the possible right states that can be connected across a single particle, compare section 3.1.

$$\mathcal{U}_1(u_L, \lambda, v) = \begin{cases} \{u_L - \lambda\} & \text{if } u_L < v, \\ [2v - u_L - \lambda, v] & \text{if } v \le u_L \le v + \lambda, \\ \{u_L - \lambda\} \cup [2v - u_L - \lambda, 2v - u_L + \lambda] & \text{if } u_L > v + \lambda. \end{cases} \tag{4.2}$$

This set defines the Germ covering the connection across a single particle, and the connection between the sets is stated in the following definition. From here on, the left and right traces of the solution at the position of the particle will be called u^- and u^+ respectivly.

Definition 3.2.1. *A couple of traces at the position of a single particle is called admissible in the sense of Germs, i.e.* $(u^-, u^+) \in \mathcal{G}_\lambda$, *if* $u^+ \in \mathcal{U}_1(u^-, \lambda, v)$.

As the selection of solutions using this Germ is unique, which is proven in [60], the Germ $\mathcal{G}_\lambda$ is, by definition, dissipative.

In order to give an equivalent condition for the interface created by two particles, one has not only to consider the jumps at the position of the particles, but also the possibility of a simple wave in the region between the two particles. The additional condition (besides the Rankine-Hugoniot condition) for waves in between the particles is, that its speed must also lie in between the particles (as they are located at the same position initially).

Given a state u_L, the following states can be connected to the right by waves with speed less than v_1.

Proposition 3.2.2. *Let* $\mathcal{U}_-(u_L, v_1) \subset \mathbb{R}$ *be the set defined by*

$$\mathcal{U}_-(u_L, v_1) = \begin{cases} (-\infty, v_1] & \text{if } u_L < v_1, \\ \{u_L\} \cup (-\infty, 2v_1 - u_L] & \text{if } u_L \ge v_1. \end{cases} \tag{3.6}$$

Then $\mathcal{U}_-(u_L, v_1)$ *describes the states that can be connected to* u_L *by a simple wave with speed* $v_{SW} < v_1$.

Proof. The proof is simple and based upon the Rankine-Hugoniot condition (1.4), which restricts the possible states that can be connected through a shockwave, as $v_{SW} < v_1$.

The case of $u_L < v_1$ allows for shockwaves to reach all states $u < u_L$. It is easy to check that for those waves the wavespeeds $v_{\text{SW}} \leq v_1$ and the entropy admissibility condition (1.6) is satisfied. By rarefaction waves, all states $u \in [u_L, v_1]$ can be reached. v_1 is the limit, as otherwise, the wavespeed of the rarefaction would exceed the velocity of the particle, therefore not being located before the particle.
In the case of $u_L \geq v_1$, shockwaves are somewhat more restricted. To match the desired wavespeed $v_{\text{SW}} \leq v_1$, we can only connect to states $u \leq 2v_1 - u_L$. Rarefaction waves are not possible at all, as the speed of their fan would begin at $u_L > v_1$. Finally, we recover $\{u_L\}$ as a single possible state in the event of no wave.
Summing up, recover exactly $\mathcal{U}_-(u_L, v_1)$. □

Given a state u_R, the following states can be connected to the left by waves with speed greater than v_2

Proposition 3.2.3. *Let $\mathcal{U}_+(u_R, v_2) \subset \mathbb{R}$ be the set defined by*

$$\mathcal{U}_+(u_R, v_2) = \begin{cases} [v_2, \infty) & \text{if } u_R \geq v_2, \\ \{u_R\} \cup (2v_2 - u_R, \infty) & \text{if } u_R < v_2. \end{cases} \tag{3.7}$$

Then $\mathcal{U}_-(u_R, v_2)$ describes the states that can be connected to u_R to the left by a simple wave with speed $v_{SW} > v_2$.

Proof. The proof works the same way as for the previous proposition.
We start with $u_R \geq v_2$. As the wavespeed of possible shockwaves must now exceed v_2, every state $u \geq u_R$ can be connected to u_R by a shockwave. The rarefaction waves are restricted by $v_{\text{RW}} \geq v_2$, and can therefore only connect to states $u \in [v_2, u_R]$.
In the second case $u_R < v_2$, rarefaction waves are again not possible, as the speed of the edge of their fan would be $u_R < v_2$. Shockwaves must be fast enough, making it possible to connect to any state $u > 2v_2 - u_R$. Lastly, there could be no wave at all, leaving the state $\{u_R\}$.
We recover exactly $\mathcal{U}_+(u_R, v_2)$. □

With these results, we can extend the idea of Germs to a connection across a whole domain in the space of solutions. For two particles, we start by computing the set $\mathcal{U}_2(u_L, \lambda_1, \lambda_2, v_1, v_2)$, which describes the admissible set of states , that can be connected to a given state u_L across the fan created by two particles initially located at the origin and moving with constant velocity.

Theorem 3.2.4. *Let* $\mathcal{U}_2(u_L, \lambda_1, \lambda_2, v_1, v_2) \subset \mathbb{R}$ *be the set defined by*

$$\mathcal{U}_2(u_L, \lambda, v) = \begin{cases} \{u_L - \lambda_1 - \lambda_2\} & \text{if } u_L < v_1, \\ [2v_1 - u_L - \lambda_1 - \lambda_2, v_2] & \text{if } v_1 \leq u_L \leq v_2 + \lambda_1 + \lambda_2, \\ \{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, 2v_2 - u_L + \lambda_1 + \lambda_2] & \text{if } u_L > v_2 + \lambda_1 + \lambda_2. \end{cases} \tag{3.8}$$

The set of states gives the good connections across the two particles and given Riemann initial data (u_L, u_R) *for the problem (3.5) with* $N = 2$*, the connection through the Germ*

$$(u_L, u_R) \in \mathcal{G}_2 \;\Leftrightarrow\; u_R \in \mathcal{U}_2(u_L, \lambda_1, \lambda_2, v_1, v_2)$$

is unique.

Proof. The proof is done by a case-by-case study with respect to $u_L, v_1, v_2, \lambda_1, \lambda_2$ and the possible positions of shocks and rarefaction waves.

- $u_L < v_1$.
  The set of possible states after the interface corresponding to the first particle is given by (3.2) and in this case is just $\{u_L - \lambda_1\}$. At the position of the particle, there could be a stationary shockwave with speed v_1. The only possible wave with speed v_1 is not an entropy admissible shockwave, due to $u_L - \lambda_1 < u_L < v_1$ and therefore we stay at $\{u_L - \lambda_1\}$. Now there is possibly a simple wave between the two Germ-related interfaces. The only restriction besides the usual ones coming from the Rankine-Hugoniot condition and entropy admissibility, is that the speed of the wave v_{SW} must be between v_1 and v_2. It is easy to check that a shockwave is not possible due to the entropy condition and a rarefaction is also not possible, because $\{u_L - \lambda_1\} \notin [v_1, v_2]$ and the fan would go across the interfaces. Therefore the only remaining connection is across the second particle, which gives, using (3.2) again, $\{u_L - \lambda_1 - \lambda_2\}$.

- $u_L \in [v_1, v_1 + \lambda_1]$
  The set of states after the first interface, given by (3.2), is $[2v_1 - u_L - \lambda_1, v_1 + \lambda_1]$. Now this set can be connected by a shockwave with speed $v_1 < \sigma < v_2$ to the set of states $[2v_1 - u_L - \lambda_1, v_1 + \lambda_1]$ (no wave or shock) and by a rarefaction wave to the set of states $[u_L, v_2]$ giving the overall set of possible states after the simple wave of $[2v_1 - u_L - \lambda_1, v_2]$. Using again (3.2) for the second particle, we obtain as the set of possible right states $[2v_1 - u_L - \lambda_1 - \lambda_2, v_2]$. Note that states up to v_2 can only be recovered because of the special case that v_2 was included in the set before the interface (therefore ending up in the second case from (3.2).

- $u_L \in (v_1 + \lambda_1, v_2 + \lambda_1 + \lambda_2]$
  For the set of states after the first particle, we obtain from the third case of (3.2), $\{u_L - \lambda_1\} \cup [2v_1 - u_L - \lambda_1, 2v_1 - u_L + \lambda_1]$. By a simple wave, we reach the set of states $[2v_1 - u_L - \lambda_1, v_2 - \lambda_1]$ (shock or no wave) and $[v_1, v_2]$ (rarefaction wave). Additionally, it could be that the state we have before the simple wave is

greater than v_2. In this case, besides a shockwave, there could be no wave, giving $\{u_L - \lambda_1\}$. Now we apply the last interface and obtain for the set of right states $[2v_1 - u_L - \lambda_1 - \lambda_2, v_2]$. The single state $\{u_L - \lambda_1\}$ also went inside this set, due to $u_L < v_2 + \lambda_1 + \lambda_2$.

- $u_L > v_2 + \lambda_1 + \lambda_2$
  For this last set of left states, the Germ for the first particle gives for the set of intermediate states $\{u_L - \lambda_1\} \cup [2v_1 - u_L - \lambda_1, 2v_1 - u_L + \lambda_1]$. The second set can be dealt with similar to before and gives the set of right states after the second particle $[2v_1 - u_L - \lambda_1 - \lambda_2, 2v_2 - u_L + \lambda_1 + \lambda_2]$. For $\{u_L - \lambda_1\}$ after the first particle, there can be a shockwave, giving the set of states after the simple wave of $[2v_1 - u_L + \lambda_1, 2v_2 - u_L + \lambda_1]$ which like before goes into the set of right states $[2v_1 - u_L - \lambda_1 - \lambda_2, 2v_2 - u_L + \lambda_1 + \lambda_2]$ or no wave, which gives by (3.2) the single right state $\{u_L - \lambda_1 - \lambda_2\}$.

And the form of the Germ $\mathcal{G}_2$ follows as proposed. □

Remark 3.2.5. *In contrary to the dissipative Germ $\mathcal{G}_\lambda$, which is a Germ in the classical sense introduced by Andreianov, Karlsen and Risebro [8], where the left and right traces are just connected by an interface, the new set $\mathcal{G}_2$ stretches the dissipative connection of the traces across a domain (in our case a cone given by the particles). This means, additionally to the dissipativity of the generalized Germ $\mathcal{G}_2$, we had to analyse the solution inside this cone to ensure uniqueness of solutions.*

3.2.3. Existence and exact solutions in the case of two particles

We consider the Riemann Problem (3.5) with two particles, i.e. $N = 2$. Now we can give the exact solution for the Riemann Problem case-by-case with respect to $u_L, u_R, v_1, v_2, \lambda_1, \lambda_2$. Purely for computational reasons, we assume the particle velocities are not too close compared to their friction, i.e. $|v_2 - v_1| > \lambda_1 + \lambda_2$.

Hierarchy of the case-by-case study

- if $u_L < v_1$
  - then if $u_R > v_2$ (case 1)
  - else if $v_1 < u_R < v_2$ (case 2)
  - else if $2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < v_1$ (case 3)
  - else if $u_R < v_1 - u_L - \lambda_1 - \lambda_2$ (case 4)
- else if $v_1 < u_L < v_2 + \lambda_1 + \lambda_2$
  - then if $u_R > v_2$ (case 5)
  - else if $v_1 < u_R < v_2$ (case 6)
  - else if $2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < v_1$ (case 7)
  - else if $u_R < 2v_1 - u_L - \lambda_1 - \lambda_2$ (case 8)
- else if $u_L > v_2 + 2\lambda$
  - then if $u_R > u_L - \lambda_1 - \lambda_2$ (case 9)
  - else if $2v_2 - u_L - \lambda_1 - \lambda_2 < u_R < u_L - \lambda_1 - \lambda_2$ (case 10)
  - else if $2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < 2v_2 - u_L - \lambda_1 - \lambda_2$ (case 11)
  - else if $u_R < 2v_1 - u_L - \lambda_1 - \lambda_2$ (case 12)

Some examples for the corresponding cases, can be found in the adjacent figures, see Figure 3.1, 3.2, 3.3.

1. $u_L < v_1, u_R > v_2$.
   To obtain the state of the solution after the second particle, we construct the sets after each possible wave and interface starting from a given left state $u_L < v_1$

$$
\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2((-\infty, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap [v_2, \infty) \\
&= ((-\infty, v_1 - \lambda_1 - \lambda_2] \cup [v_1 - \lambda_1 - \lambda_2, v_2]) \cap [v_2, \infty) \\
&= (-\infty, v_2] \cap [v_2, \infty) \\
&= \{v_2\}
\end{aligned}
$$

Due to theorem 3.2.4, we know that the connection across the germ is unique given (u_L, u_R) and using (3.2), (3.6), (3.7), the state before the first particle follows

$$\mathcal{U}_-(u_L, v_1) = \{v_1\}$$

and the solution has one rarefaction wave where the speed of the fluid and the speed of the particles match at the particle positions.

2. $u_L < v_1, v_1 < u_R < v_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2((-\infty, v_1], \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= (-\infty, v_2] \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= \{u_R\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{v_1\}$$

and the solution has one shock at the position of the second particles, connecting $(\max(v_2, u_R + \lambda_2), u_R)$ and two rarefaction waves, one before the first particle, connecting (u_L, v_1) and one after the first particle, connecting $(v_1, \max(v_2, u_R + \lambda_2))$.

3. $u_L < v_1, 2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < v_1$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2((-\infty, v_1], \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= (-\infty, v_2] \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= \{u_R\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{v_1\}$$

and the solution has a rarefaction between the particles, connecting $(v_1, u_R + \lambda_2)$ and a shock at the position of the second particle connecting $(u_R + \lambda_2, u_R)$.

4. $u_L < v_1, u_R < 2v_1 - u_L - \lambda_1 - \lambda_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2((-\infty, v_1], \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= (-\infty, v_2] \cap (\{u_R\} \cup (2v_2 - u_R.\infty)) \\
&= \{u_R\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \min(\{u_R + \lambda_1 + \lambda_2\}, v_1)$$

and the solution has a simple wave before the first particle, connecting $(u_L, \max(u_R + \lambda_1 + \lambda_2, v_1))$ (shock if $u_R + \lambda_1 + \lambda_2 > u_L$ and a rarefaction if $u_R + \lambda_1 + \lambda_2 < u_L$), aswell as two shocks at the particle positions, connecting $(\min(u_R + \lambda_1 + \lambda_2, v_1), u_R + \lambda_2)$ and $(u_R + \lambda_2, u_R)$.

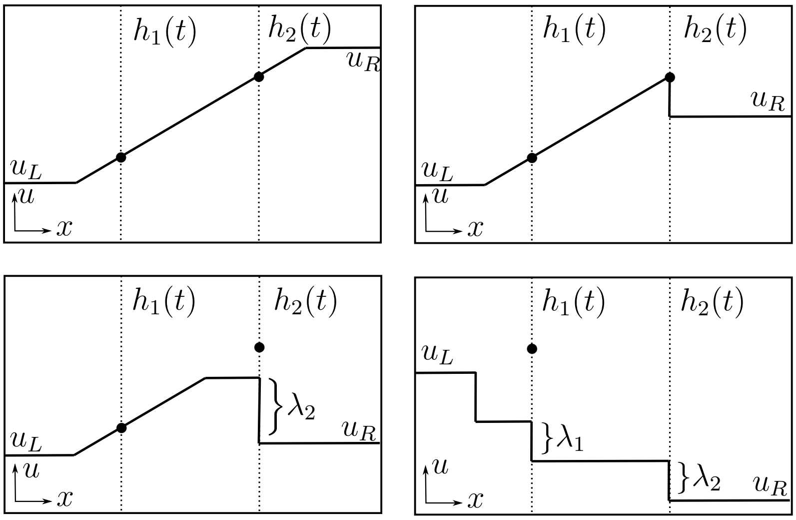

Figure 3.1.: Sample solutions for the cases 1 (upper left), 2 (upper right), 3 (lower left) and 4 (lower right).

5. $v_1 < u_L < v_2 + \lambda_1 + \lambda_2, u_R > v_2$.
   We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L), \lambda_1, \lambda_2, v_1, v_2) \cap [v_2, \infty) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, v_2]) \cap [v_2, \infty) \\
&= \{v_2\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

and the solution has one shockwave at the first particle, connecting $(u_L, \max(v_1, u_L - \lambda_1))$ and a rarefaction wave after the first particle, connecting $(\max(v_1, u_L - \lambda_1), v_2)$ where it matches the speed of the second particle at its position.

6. $v_1 < u_L < v_2 + \lambda_1 + \lambda_2, v_1 < u_R < v_2$.
   We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L), \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, v_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_R\}
\end{aligned}$$

   giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

   and the solution has one shockwave at the first particle, connecting $(u_L, \max(v_1, u_L - \lambda_1))$ and a simple wave after the first particle, connecting $(\max(v_1, u_L - \lambda_1), \max(v_2, u_R + \lambda_2))$ (either a rarefaction or shockwave, depending on $u_L \gtrless u_R$) and a shockwave at the position of the second particle, connecting $(\max(v_2, u_R + \lambda_2), u_R)$.

7. $v_1 < u_L < v_2 + \lambda_1 + \lambda_2, 2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < v_1$.
   We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L), \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, v_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_R\}
\end{aligned}$$

   giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

   and the solution has two shocks located at the particle interfaces connecting the states $(u_L, u_L - \lambda_1)$ and $(u_R + \lambda_2, u_R)$ and one simple wave in the region between the particles (either a shockwave or a rarefaction wave depending on if $2v_1 - u_L - \lambda_1 - \lambda_2 \gtrless u_R$), connecting $(u_L - \lambda_1, u_R + \lambda_2)$.

8. $v_1 < u_L < v_2 + \lambda_1 + \lambda_2, u_R < 2v_1 - u_L - \lambda_1 - \lambda_2$.
   We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L), \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, v_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_R\}
\end{aligned}$$

   giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_R + \lambda_1 + \lambda_2\}$$

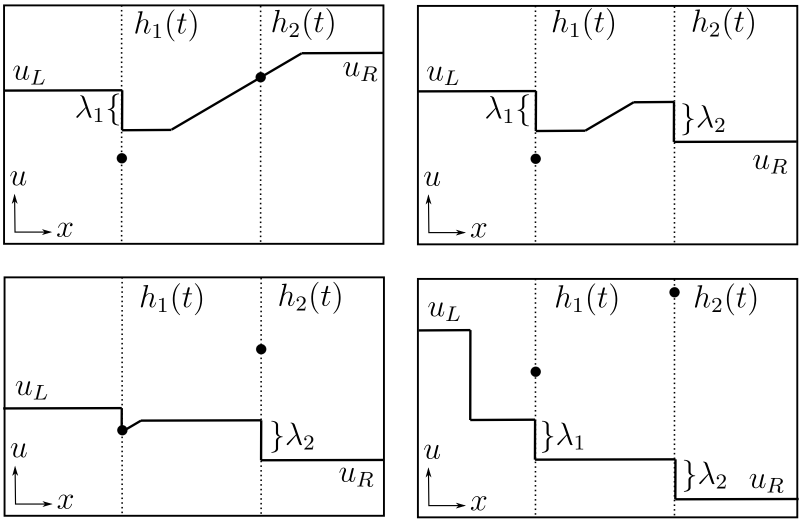

Figure 3.2.: Sample solutions for the cases 5 (upper left), 6 (upper right), 7 (lower left) and 8 (lower right).

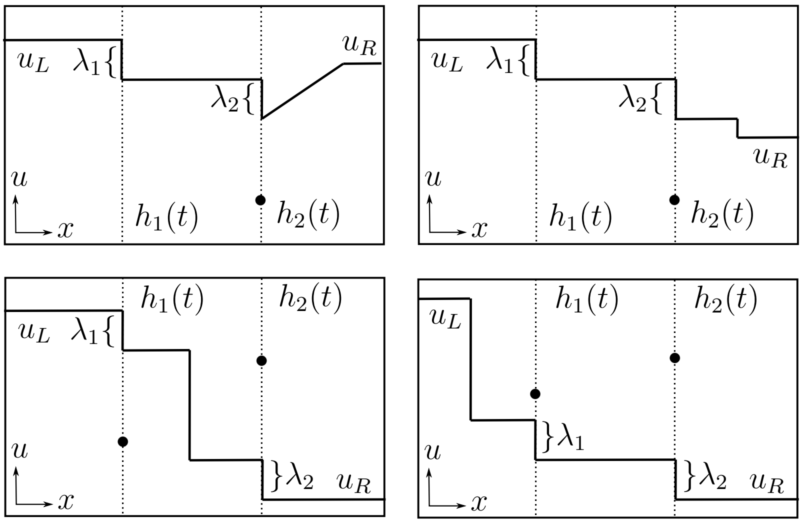

Figure 3.3.: Sample solutions for the cases 9 (upper left), 10 (upper right), 11 (lower left) and 12 (lower right).

and the solution has a shockwave before the first particle, connecting $(u_L, u_R + \lambda_1 + \lambda_2)$ and two shocks located at the particle interfaces connecting the states $(u_R + \lambda_1 + \lambda_2, u_R + \lambda_2)$ and $(u_R + \lambda_2, u_R)$.

9. $u_L > v_2 + \lambda_1 + \lambda_2, u_R > u_L - \lambda_1 - \lambda_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L], \lambda_1, \lambda_2, v_1, v_2) \cap [v_2, \infty) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup (-\infty, 2v_2 - u_L - \lambda_1 - \lambda_2]) \cap [v_2, \infty) \\
&= \{u_L - \lambda_1 - \lambda_2\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

and the solution admits two shockwaves of height λ_1, λ_2 at the positions of the particles and one rarefaction wave right of the second particle connecting $(u_L - \lambda_1 - \lambda_2, u_R)$.

10. $u_L > v_2 + \lambda_1 + \lambda_2, 2v_2 - u_L - \lambda_1 - \lambda_2 < u_R < u_L - \lambda_1 - \lambda_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L], \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup (-\infty, 2v_2 - u_L - \lambda_1 - \lambda_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_L - \lambda_1 - \lambda_2\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

and the solution admits two shockwaves of height λ_1, λ_2 at the positions of the particles followed by a shockwave after the second particle, connecting $(u_L - \lambda_1 - \lambda_2, u_R)$.

11. $u_L > v_2 + \lambda_1 + \lambda_2, 2v_1 - u_L - \lambda_1 - \lambda_2 < u_R < 2v_2 - u_L - \lambda_1 - \lambda_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L], \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, 2v_2 - u_L + \lambda_1 + \lambda_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (-\infty, v_2] \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_R\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_L\}$$

and the solution has two shocks at the particle positions, connecting $(u_L, u_L - \lambda_1)$ und $(u_R + \lambda_2, u_R)$ and a shock in between the particles, connecting $(u_L - \lambda_1, u_R + \lambda_2)$.

12. $u_L > v_2 + \lambda_1 + \lambda_2, u_R < 2v_1 - u_L - \lambda_1 - \lambda_2$.
We proceed as before and obtain as state after the second particle

$$\begin{aligned}
&\mathcal{U}_2(\mathcal{U}_-(u_L, v_1), \lambda_1, \lambda_2, v_1, v_2) \cap \mathcal{U}_+(u_R, v_2) \\
&= \mathcal{U}_2(\{u_L\} \cup (-\infty, 2v_1 - u_L), \lambda_1, \lambda_2, v_1, v_2) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= (\{u_L - \lambda_1 - \lambda_2\} \cup [2v_1 - u_L - \lambda_1 - \lambda_2, v_2]) \cap (\{u_R\} \cup (2v_2 - u_R, \infty)) \\
&= \{u_R\}
\end{aligned}$$

giving the (unique) state before the first particle

$$\mathcal{U}_-(u_L, v_1) = \{u_R + \lambda_1 + \lambda_2\}$$

and the solution has a shockwave before the first particles, connecting $(u_L, u_R + \lambda_1 + \lambda_2)$ and two shocks located at the particle interfaces connecting the states $(u_R + \lambda_1 + \lambda_2, u_R + \lambda_2)$ and $(u_R + \lambda_2, u_R)$.

It is easy to check that the given cases cover all possible left and right states (u_L, u_R).

3.2.4. Well-posedness in the case of N particles

We now investigate the problem for an arbitrary number of N particles. Following our previous analysis, we again rewrite the equations as

$$\begin{aligned}
\partial_t u + \partial_x (u^2/2) &= 0 && \text{for } x \neq h_i(t) \\
\big(u^-(t, h_i(t)), u^+(t, h_i(t))\big) &\in \mathcal{G}_{\lambda_i} && \forall i \in [1, N]
\end{aligned} \tag{3.9}$$

In order to be able to make statements about the behaviour of the solution, we first recall that by introducing the particles and thus pointwise disturbances, the set of stationary solutions changed drastically. In fact, constant states are no stationary solutions anymore, and the best we can hope for is a solution with stationary states moving in between the particles. The most intuitive solution of this type is

$$u(t, x) = \begin{cases} u_1 & \text{if } x < h_1(t) \\ u_2 & \text{if } h_1(t) < x < h_2(t) \\ \vdots & \\ u_N & \text{if } h_{N-1}(t) < x < h_N(t) \\ u_{N+1} & \text{if } h_N(t) < x \end{cases} \tag{3.10}$$

where $|u_{i+1} - u_i| = \lambda \; \forall i \in [1, N]$.
It can be easily checked that this actually is a stationary solution to problem (3.5).

This means however, that even without waves created by the movement of the fluid, we should expect jumps of height $|u^+ - u^-| \leq \lambda$ at the position of the particles. In the following analysis of the appearance of shockwaves, we discount those particle-induced shockwaves, which are stationary with respect to the particle position and are comparably small in size (depending on the friction constant). One could say we search for waves built on the new basic solution (3.10).
In the following we refer to the left trace of the solution at the position of a shock with u^-_{shock} and the corresponding right trace with u^+_{shock}, and we denote the left and right traces of the solution at the position of the i-th particle with $u_i^- := \gamma^-(u(t, h_i(t)))$, $u_i^+ := \gamma^+(u(t, h_i(t)))$ respectively. Furthermore, we will write $\sum \lambda_i$ as short for $\sum_{i=1}^N \lambda_i$.

Theorem 3.2.6. *Any solution $u(t, x)$ to problem (3.5) admits at most one shock not superpositioned with the particle paths $h_i(t)$. Furthermore, if there is a shockwave with height $|u_i^- - u_i^+| > \lambda$ at the position of a particle, then there is no shockwave elsewhere, meaning the jumps at the other particles are of heigth $|u^- - u^+| \leq \lambda$.*

Proof. We perform a case-by-case study with respect to the position of a possible shock and proof, that a second shock is not possible in every case.

- Let us assume the solution admits a shock before the first particle. By the nature of the self-similar solution, the shock must travel slower than the first particle, thus $s_{\text{shock}} < v_1$. By the Rankine-Hugoniot condition, the state right of the shock is $u^+_{\text{shock}} < 2v_1 - u_L$ and if $u_L < v_1$, to be entropy admissible, we also have $u^+_{\text{shock}} < u_L$.
  Note that by the analysis of the jump across a single particle, we have $u^+ \leq u^-$ and therefore $u_1^+ \leq 2v_1 - u_L < v_i \; \forall i \in [1, N]$. Note further that for given $u_i^+ < v_i$ there can neither be a rarefaction nor a shockwave, as both would violate the condition $v_i <$ speed $< v_{i+1}$ to be located between the particles. Therefore the only possible state after the particles is $u_N^+ = u_1^- - N\lambda$. This means, recalling the analysis of the jumps at the position of the particles (3.2), that the solution admits only jumps of heigth λ at the position of the particles. As a shockwave after the last particle should be faster than the last particle, meaning $v_N <$ speed and $u_N^+ < v_N$ by the previous analysis, there can not be an entropy admissible shock after the last particle. Therefore the initial shock was the only one not superpositioned with a particle.

- Let us assume that the solution admits a fluid induced shockwave at the position of the i-th particle, meaning $|u_i^- - u_i^+| > \lambda$. Using the previous case, we know that there is no shockwave before that. Knowing that the jump at the position of the i-th particle is bigger than λ, it is easy to check by (3.2), that the state after the i-th particle must be $u_i^+ < v_i$. By the same arguments as before, neither rarefaction nor shockwaves are possible between the following particles, and again by (3.2) it follows that $u_j^+ = u_j^- - \lambda < v_j \; \forall j \in [i+1, N]$. By the same arguments as in the previous cases, there can't be further shocks not superpositioned with a particle and the jumps at the position of the particles are of height λ.

- Let us assume now that there is a shockwave somewhere between the i-th and the $i+1$-th particle. By the previous cases, this means that there can not be a shockwave before, and the speed of the shock must be $v_i <$ speed $< v_{i+1}$. This means, that $u_{i+1}^- < v_{i+1}$ and by (3.2) it follows that $u_j^+ = u_j^- - \lambda < v_j \ \forall j \in [i+1, N]$. By the same arguments as in the previous cases, there can't be further shocks not superpositioned with a particle and the jumps at the position of the particles are of height λ.
- In the last case we have a shockwave after the last particle. The previous cases show that there can't be an entropy admissible shockwave before that.

□

Using this theorem, we can see that the structure of the solution remains the same, if we increase the number of particles. Now we extend the notion of generalized germs to N particles in the following proposition.

Theorem 3.2.7. *The generalized admissibility Germ $\mathcal{G}_N(v_1, \ldots, v_N, \sum \lambda_i) \subset \mathbb{R}^2$ associated with the cone $[h_1(t), h_N(t)]$ is given by*

$$(u_L, u_R) \in \mathcal{G}_N \ \Leftrightarrow \ u_R \in \mathcal{U}_N(u_L, v_1, \ldots, v_N, \sum \lambda_i) \ \textit{with} \tag{3.11}$$

$$\mathcal{U}_N = \begin{cases} \{u_L - \sum \lambda_i\} & \textit{if } u_L < v_1, \\ [2v_1 - u_L - \sum \lambda_i, v_N] & \textit{if } v_1 \le u_L \le v_N + \sum \lambda_i, \\ \{u_L - \sum \lambda_i\} \cup [2v_1 - u_L - \sum \lambda_i, 2v_N - u_L + \sum \lambda_i] & \textit{if } u_L > v_N + \sum \lambda_i. \end{cases}$$

Using this theorem, we can again rewrite the problem as

$$\begin{aligned} \partial_t u + \partial_x(u^2/2) = 0 \qquad & \text{for } x < h_1(t) \\ \partial_t u + \partial_x(u^2/2) = 0 \qquad & \text{for } x > h_N(t) \\ \big(u^-(t, h_1(t)), u^+(t, h_N(t))\big) \ \in \ & \mathcal{G}_N \end{aligned} \tag{3.12}$$

Theorem 3.2.8. *Weak entropy solutions u to problem (3.5), with admissibility Germ $\mathcal{G}_N$ (3.11) are unique.*

Proof. In the case of two particles, the proof of this theorem follows directly from our analysis of all possible cases of the Riemann problem, having uniqueness in each of them. As the problem for N particles can be rewritten in the form of a two particle problem (3.12), the solution is unique outside of the fan created by the particles. In the case of two particles, we already showed with theorem 3.2.4, that the connection through the fan is unique, given two unique left and right traces u_1^- and u_2^+. The proof of uniqueness inside the fan of N particles remains the same in structure, and we will not give a detailed proof here, which would be very lenghty and not significantly different from the proof for two particles. Instead, we refer to the uniqueness proof of the Cauchy-Problem for N particles in the next section, which includes this case. □

The final result is the following existence theorem, which captures also the structure of solutions including multiple particles.

Theorem 3.2.9. *Given the Riemann Problem with N particles (3.5) and the condition for the interface $\mathcal{G}_N$, there exists a self-similar solution $u(t,x)$ for all $t \in [0,T], x \in \mathbb{R}$. In fact, the problem outside of the particle-fan $[h_1(t), h_N(t)]$ can be considered a two particle problem with $\mathcal{G}_N(v_1, \ldots, v_N, \sum \lambda_i) = \mathcal{G}_2(v_1, v_N, \sum \lambda_i)$.*

Proof. The proof of this theorem is already done by our previous analysis. Theorem 3.2.7 gives the generalized admissibility Germ to rewrite the problem as (3.12). If we compare the generalized Germs $\mathcal{G}_2$ (3.8) and $\mathcal{G}_N$ (3.11), we clearly see the claimed correlation between the two and N particle problem. Using this connection, the existence result outside of the fan or particles extends from the case of two particles to the case of N particles. This gives us the required traces u_1^- and u_N^+ to determine the exact solution inside the fan using $\mathcal{G}_N$. □

3.3. Well-posedness of the Cauchy-problem for N particles

In this section we will analyse the Cauchy-problem. A good and natural notion of entropy admissibility is extended from the model with one particle to the model with finitely many particles. Existence and uniqueness of those entropy solutions, aswell as a L^∞-bound is shown. A good choice of admissible particles is developed, which extends the analysis to the case of merging and splitting of particles.

3.3.1. Extension of the model to multiple particles

We consider an inviscid fluid with velocity $u(t,x)$ and a finite number of particles moving inside. The fluid is modeled by the inviscid Burgers equation and the particles act as a point-wise drag force on the fluid, namely $\lambda_i(h_i'(t) - u(t,h_i(t)))$, where λ_i is the friction constant and $h_i(t)$ the given path of the i-th particle. The Cauchy problem writes

$$\begin{aligned} \partial_t u + \partial_x(u^2/2) &= \sum_{i\in N(t)} \lambda_i(h_i'(t) - u(t,h_i(t))\delta(x-h_i(t)), \\ u(0,x) &= u_0(x) \end{aligned} \tag{3.13}$$

with

(t,x)	$\in \mathbb{R}^+ \times \mathbb{R}$
$u(t,x)$	velocity of the one-dimensional fluid
$h_i(t)$	the given position of the i-th particle at time t
λ_i	the friction constant corresponding to the i-th particle
$\mathbf{N}(T)$	set of particles in $[0,T]\times\mathbb{R}$, with arbitrary, finite cardinality
$N(t) \subset \mathbf{N}$	the set of particles at a given time t

The non-conservative products $u(t,h_i(t))\delta(x-h_i(t))$ are again dealt with in the same manner as in [10], i.e. by a regularization of the particles. However, an analysis of the behaviour of the fluid at the position of the particle allows for a well-posedness proof considering the influence of the particle as a condition on the behaviour of the fluid at a moving interface located at the particle position.
The theory of this section extends the analysis of the fluid-solid interaction of [60, 9, 11, 10], where the original model also includes coupling to an ordinary differential equation, to the case of multiple particles. Models of this kind are of increasing interest theoretically, cf. [19], aswell as in applications like trajectory tracking in traffic flow, cf. [36], [37].
We proceed in the following way. In section 3.3.2 the notion of admissible particle is explained. In section 3.3.3, we give an admissibility condition for the selection of physical shockwaves and therefore a definition of entropy solutions to the problem. After

that, section 3.3.4 states the main theorem, which is the well-posedness result for problem (3.13) and a L^∞ bound. Section 3.3.5 to 3.3.9 give the proof to this theorem, first for the simplified case of only two particles at any given time, extending the result to N particles in section 3.3.8, where sections 3.3.5, 3.3.6 and 3.3.7 contain the building blocks for the existence proof as well as the L^∞ bound and section 3.3.9 is devoted to the uniqueness proof using almost classical Kruzkov-type arguments combined with the notion of germs.

3.3.2. Admissible particles

As the model allows for a variety of particle interactions to be considered, one clearly needs to specify what an admissible particle is. Certain conditions, like a finite speed of particles, are both physically reasonable as well as needed for computational reasons. This section contains the exact definition of the mathematical object that is considered an admissible particle in the framework of this model.
Including splitting and merging of particles leads to the number of particles being a time-dependant value, and therefore we want to recover some sort of conservation with respect to the particles. In fact, a property of this kind is contained in the system by the usage of interface admissibility, as discussed in section 1.2. Additionally, we do not want to allow for particles to be created out of nothing or existing particles to vanish inside the observed domain.
With those considerations in mind, we define the particle path of the i-th particle

$$h_i(t) : [t_i^1, t_i^2] \to \mathbb{R} \text{ with } [t_i^1, t_i^2] \subset [0, T] \text{ is Lipschitz continuous} \tag{P1}$$

where t_i^1 describes the beginning point in time of the particle path and t_i^2 the corresponding ending point. The Lipschitz continuity enforces the speed of a particle to remain finite.
Next, as we do not want to allow for sudden creation or deletion of particles, we restrict the starting point of a particle path to an arbitrary point at the initial time or the endpoint of another particle path. Respectivly we restrict the endpoints to the final time or starting points of new particles.

$$\text{Either } t_i^1 = 0 \text{ or } t_i^1 = t_j^2 \text{ s.t. there exists a } j \in [0, \mathbf{N}] \setminus \{i\} \text{ with } h_i(t_i^1) = h_j(t_j^2) \tag{P2}$$

$$\text{Either } t_i^2 = T \text{ or } t_i^2 = t_j^1 \text{ s.t. there exists a } j \in [0, \mathbf{N}] \setminus \{i\} \text{ with } h_i(t_i^2) = h_j(t_j^1) \tag{P3}$$

Let us continue by introducing some auxiliary sets.
The first set is the graph of the i-th particle, excluding starting and endpoint

$$\mathcal{C}_i := \{(t, h_i), t \in (t_i^1, t_i^2)\}.$$

The fourth admissibility condition for our particles is, that, excluding the starting and endpoint of the particle paths, we do not allow for particle intersections.

$$\mathcal{C}_i \cap \Big(\bigcup_{j \in [1, \mathbf{N}] \setminus \{i\}} \mathcal{C}_j \Big) = \emptyset \tag{P4}$$

where we recall that **N** is the number of particles in the whole domain. For a given problem, whenever two particle paths intersect, we define this point to be the endpoint of those particle paths and consider everything after this collision point as new particles. Finally, we have a conservation property regarding the influence of the particles towards the fluid. This becomes particularly interesting whenever particles split or merge, as the quesion about the relation between the friction of the new and the old particles naturally appears. We define for a given point (t, x) the sets of particles with starting or endpoint (t, x) as

$$\mathcal{I}^-(t,x) := \left\{k \mid \left(h_k(t_k^2) = x\right) \wedge \left(t_k^2 = t\right)\right\}$$
$$\mathcal{I}^+(t,x) := \left\{l \mid \left(h_l(t_l^1) = x\right) \wedge \left(t_l^1 = t\right)\right\}$$

and impose the following natural property, which corresponds to the stability preserving behavior found in the analysis of the particle related interface in section 3.2.2

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R} \qquad \sum_{k \in \mathcal{I}^-(t,x)} \lambda_k = \sum_{j \in \mathcal{I}^+(t,x)} \lambda_j \tag{P5}$$

Summarizing, we state the complete definition of admissible particles.

Definition 3.3.1 (admissible particles).
We say that $(h(t), \lambda)$, $\lambda \in \mathbb{R}_0^+$ is an admissible particle to our model, if it satisfies conditions (P1) - (P5).

Remark 3.3.2. *The definition of admissible particles corresponds to allowing particle paths that define a mesh on the domain $[0,T] \times \mathbb{R}$. Property 2 makes certain that whenever two particles cross, the particles after the crossing are considered new particles. That way particles never intersect except at the beginning or ending point of their paths. Property 3 is motivated by the goal to achieve stability for the case of two merging particles and can be seen as a conservation of influence, given by the friction constants. Whenever particles split or merge, the total friction disturbing the fluid remains the same. Finally, properties 4 and 5 exclude the cases of particles that just appear or vanish inside the domain, forcing the particle paths to either start at the end of another particle or at the initial time and end at the beginning point of another particle or the finite time.*

3.3.3. Definition of entropy solutions

Increasing the number of particles means that the behaviour of the fluid at each particle is governed by an interface admissibility condition $\mathcal{G}_1(v_i, \lambda_i)$ respectivly. Thus we are able to define entropy admissible solutions to the problem as long as the particle paths do not intersect using the notion of admissible particle-related jumps and the notion of adapted Kruzkov entropies. These adaptions of the classical Kruzkov entropies, which were already discussed in section 2.1.1, are not only able to be used in the presence of discontinuous flux problems, but all problems involving pointwise disturbances.
Note that complicated particle interactions like merging, splitting or crossing might

change the number of particles $N(t)$ as well as their order, such that the enumeration of particles used so far is time dependent.
In the case of multiple particles, an additional notation is needed to properly define the piecewise constant states for the adapted Kruzkov entropies, because the set $N(t)$, containing the particles at a given time, is not numbered yet. To circumvene difficulties, we define the mapping $\mathcal{N}(t)(\cdot)$

$$\mathcal{N} : N(t) \to [1, \ldots, |N(t)|]$$

such that $\mathcal{N}$ maps the particle numbers at a given time onto a numbered set, for example

$$\begin{array}{llll} \mathbf{N}(T): & & \{ & 1\ 2\ 3\ 4\ 5\ 6\ \ldots \\ N(t)\ : & & \{ & \quad\ \ 3\ 4\quad 6\ \ldots \\ & \mathcal{N}(t) & & \quad\ \ \downarrow\downarrow\quad \downarrow \\ & & & \quad\ \ 1\ 2\quad 3\ \ldots \end{array}$$

Remark 3.3.3. *The mapping is obviously bijective and allows to give the numbered set of particles at time t, which will be necessary to give a good definition of entropy solution. The use of this notation is mainly restricted to this section, where it is needed to be able to give the positions of the particles at any time t.*

It is now possible to define adapted Kruzkov entropies with respect to the interfaces located at the positions of the particles

$$\begin{array}{ll} \eta(u, c) = |u - c| & \text{adapted Kruzkov entropies} \\ \Phi(u, c) = \operatorname{sgn}(u - c)(f(u) - f(c)) & \text{corresponding Kruzkov entropy fluxes} \end{array}$$

with the piecewise constant function

$$\begin{aligned} c(t, x) := c_1 \mathbb{1}_{\{x<h_{\mathcal{N}^{-1}(t)(1)}\}} + \sum_{i=2}^{|N(t)|} c_i \mathbb{1}_{\{h_{\mathcal{N}^{-1}(t)(i-1)}<x<h_{\mathcal{N}^{-1}(t)(i)}\}} \\ + c_{|N(t)|+1} \mathbb{1}_{\{x>h_{\mathcal{N}^{-1}(t)(|N(t)|)}\}}. \end{aligned} \tag{3.14}$$

with $c_j \in \mathbb{R}$ for all $j \in [1, \ldots, |N(t)| + 1]$ and $\mathbb{1}_A$ the indicator function of A.

Definition 3.3.4. *Given $u_0 \in L^\infty, N(t) > 0, h_i(t) \in W^{1,\infty}([0,T])\ \forall t \in [0, T$. We call $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ weak entropy solution to the Cauchy Problem (3.13), if for $N(t) \in \mathbb{N}$, $h_i(t)$ the position and $h_i'(t)$ the velocity of particle i, with $i = 1, \ldots, N$, u satisfies for all piecewise constant functions $c(t,x)$ and almost every time t*

$$\int\int |u - c|\partial_t\phi + \Phi(u, c)\partial_x\phi\, dtdx + \int |u_0 - c|\phi(0, x)dx \ \geq\ 0 \tag{3.15}$$

with $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+), \phi(t, h_i(t)) = 0$, and additionally

$$(\gamma^-(u(t, h_i(t))), \gamma^+(u(t, h_i(t)))) \in \mathcal{G}_{\lambda_i}(t), \quad \textit{for a.e. } t \in (0, T)$$

where we denoted the left an right traces of $u(t,x)$ at the position of the particles by $\gamma^-(u(t,h_i(t))), \gamma^+(u(t,h_i(t)))$ respectivly. Due to the nature of the Burgers equation, these traces exist a priori, even for L^∞ initial data, cf [59, 77].

As we will make use of another, equivalent definition of entropy solution later, we state it here and refer to section 2.3 for the proof of equivalence.

Definition 3.3.5. *Given $u_0 \in L^\infty$ and a number $N \in \mathbb{R}$ of admissible particles in the sense of definition 3.3.1. We call $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ weak entropy solution to problem (3.13), if u is a weak solution to (3.13) and satisfies almost everywhere in $[0,T] \times \mathbb{R}$*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |u - c(x,t)|\partial_t \phi + \Phi(u, c(x,t))\partial_x \phi)\, dtdx + \int_{\mathbb{R}} |u_0 - c(x,0)|\phi(0,x)dx \\ + M \sum_{i \in N(t)} \int_0^T dist((c_{\mathcal{N}(t)(i)}, c_{\mathcal{N}(t)(i+1)}), \mathcal{G}_{\lambda_i})\phi|_{x=h_i(t)}\, ds \geq 0.\end{aligned} \tag{3.16}$$

for $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, $M > 0$, and all piecewise constant functions $c(t,x)$ of the form (3.14).

Remark 3.3.6. *Note that whenever two or more particles are located at the same position at a given time, the entropy admissibility condition (3.15) is not a meaningful condition, as two different interface conditions are enforced at the same point in space. In fact, it is more than likely, that the set of entropy solutions using this entropy condition is empty. However, this is not a problem for defining entropy solutions on the whole space-time domain, because the condition is only enforced almost anywhere in time.*
At this point one can also see the importance of ***N*** *to be finite, forbidding an infinite number of particle interactions. This ensures that the problem with defining entropy admissibility for particles located at the same position remains a local problem and allows for the given notion of entropy admissibility to be used.*
If one were to tackle that problem, recalling the analysis of the Riemann Problem in the previous section, it would be sensible to define a new interface condition at those interaction points x_I, using the germ $\mathcal{G}_{\lambda_\Sigma}$, with $\lambda_\Sigma = \sum_{\{i \in N(T) \,|\, h_i(t) = x_I\}} \lambda_i$.
This germ makes sure that the interface condition really applies the drag of both particles, and does not impose two (maybe contradictory) conditions at the same position.

Remark 3.3.7. *The definition of entropy solution is done using the notion of Germs, introduced in [8]. Furthermore the entropy condition can not be distinguished from an entropy condition for a discontinuous flux problem with interfaces located at the particle positions $h_i(t)$, emphasizing the pointwise influence of the particles. In contrary to discontinuous flux problems, in the case of particles, it is not possible to use classical Kruzkov entropies because the pointwise disturbance can not easily be translated into a part of the entropy flux.*

3.3.4. Main result

The main result of this section and this work is the well-posedness of entropy solutions of the Cauchy Problem (3.13) given an arbitrary, finite number of admissible particles. The result is stated in here, while the rest of this section is devoted to the proof of this theorem.

Theorem 3.3.8. *Given initial data $u_0 \in L^\infty(\mathbb{R})$, the Cauchy problem (3.13) with a finite number $\boldsymbol{N}(T)$ of particles, admissible in the sense of (3.3.1), admits a unique entropy solution u.*
Additionally, every such entropy solution satisfies for every time $t \in [0, T]$

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \sum_{i \in N(t)} \lambda_i$$

Remark 3.3.9. *In the case of a single particle, Andreianov, Lagoutiere, Seguin and Takahashi were able to also show a bound in total variation on the constructed entropy solutions [10]. Although it is possible to recover a bound in variation in the case of splitting of a particles, as shown in section 3.3.7, it was not possible to extend their result in the presence of particle crossings or merging. In fact, it is rather unclear if such a bound even exist in those cases, as it would be theoretically possible for shockwaves to be reflected back and forth at an increasing rate between the particles approaching each other.*

3.3.5. A time-stepping construction method and L^∞ bounds

We will prove existence of entropy admissible solutions, starting with the case where in a given time interval the particle paths do not intersect. The cases of merging, splitting and crossing are discussed in the following sections and the method of constructing solutions is strongly based on the analysis done in this first case. Given initial data $u_0 \in L^\infty(\mathbb{R})$ and any finite time interval $[0, T]$, such that $h_i(t) \neq h_j(t)$ for all $i \neq j \in \mathbf{N}, t \in [0, T]$, we divide the problem into several local problems and use the existence result for the problem with a single particle from theorem 3.1.6.

Lemma 3.3.10. *Given $h \in W^{1,\infty}([0, T])$ and $u_0 \in L^\infty(\mathbb{R})$, then there exists a unique entropy admissible solution u of (3.13) with $N(t) = 1$.*

Remark 3.3.11. *Note that from $N(t) = 1$ for all $t \in [0, T]$, due to conditions (P2), (P3) and (P5), one can assume that also $\boldsymbol{N}(T) = 1$. Although it would be admissible to define two particles $(h_1, \lambda_1), (h_2, \lambda_2)$ with $t_1^2 = t_2^1$ and $h_1(t_1^2) = h_2(t_2^1)$, which means the first particles path ends at the beginning point of the second particle path, but in this case, one can simply define the two particles to be a single particle, as by (P5), $\lambda_1 = \lambda_2$.*

Several difficulties arise. Even though the behaviour of the fluid in the presence of a single particle is known, each particle generates waves interfering with the other particles, creating domains of unknown behaviour. Additionally, the possibility of crossings, merging and splitting of particles seem to destroy some of the nice properties that were holding as long as only one particle was present, e.g. the global in time bound on the total variation. One could imagine waves reflecting back and forth between two particle paths.

The proof is done using an explicit construction algorithm based on the existence result in the presence of a single particle, which we will present here for the case of two particles in case of no particle path intersections and at most two particles at the same time, i.e. $N(t) \leq 2$, in the case of merging and splitting. It is proven in a later section, that this can be easily extended to any finite number of particles by simply choosing a good timestepping, creating domains where the following analysis applies locally.

At the same time, we will prove an L^∞ bound (3.19), justifying the existence of a maximum speed of propagation, denoted L from here on, which, though a very natural property of hyperbolic equations, needs to be checked in the presence of source terms. In fact, our specific Burgers flux and the Lipschitz continuity of the particle paths (P1) is sufficient to guarantee a maximum speed of propagation, which is also physically reasonable. Both the L^∞ bound as well as the existence of solutions are constructed using a time-stepping, which ensures that the cones of influence of two particles don't intersect within the current timestep $[t^i, t^{i+1}]$.

We begin by introducing the following functions, which will bound solutions given bounded initial data. Let $c_{\min}, c_{\max}$ be defined by

$$c_{\text{min, max}}(t,x) = \begin{cases} c^1_{\text{min, max}} & \text{for } x \in \Omega_1(t), t \in [t^i, t^{i+1}] \\ c^2_{\text{min, max}} & \text{for } x \in \Omega_2(t), t \in [t^i, t^{i+1}] \\ c^3_{\text{min, max}} & \text{for } x \in \Omega_3(t), t \in [t^i, t^{i+1}] \end{cases} \tag{3.17}$$

(3.18)

with

$$\begin{aligned} \Omega_1(t) &:= (-\infty, h_1(t)) \\ \Omega_2(t) &:= (h_1(t), h_2(t)) \\ \Omega_3(t) &:= (h_2(t), \infty) \end{aligned}$$

such that for $j = 1, 2$

$$c^j_{\text{min, max}} = c^{j+1}_{\text{min, max}} + \lambda_j$$

and $c^k_{\min} = \inf\limits_{x\in\Omega_{k_1}(t^i)} u(t^i,x),\ c^l_{\max}(t,x) = \sup\limits_{x\in\Omega_{k_2}(t^i)} u(t^i,x)$ for

$$\begin{aligned} k &= \arg \min_{j=1,2,3} \big\{ \inf_{x\in\Omega_1} u(t^i,x), \inf_{x\in\Omega_2} \big(u(t^i,x) - \lambda_1\big), \inf_{x\in\Omega_3} \big(u(t^i,x) - \lambda_1 - \lambda_2\big) \big\} \\ l &= \arg \max_{j=1,2,3} \big\{ \sup_{x\in\Omega_1} u(t^i,x), \sup_{x\in\Omega_2} \big(u(t^i,x) - \lambda_1\big), \sup_{x\in\Omega_3} \big(u(t^i,x) - \lambda_1 - \lambda_2\big) \big\} \end{aligned}$$

We will now state the main result of this section, which is the existence in the case of no particle interactions and an L^∞ bound.

Lemma 3.3.12. *Given $u_0 \in L^\infty(\mathbb{R})$ and given two particles $(h_1, \lambda_1), (h_2, \lambda_2)$, admissible in the sense of (3.3.1), with $h_1(0) \neq h_2(0)$, any solution u to problem (3.13) with $|N(t)| = 2$ satisfies*

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \sum_{i=1}^{2} \lambda_i. \tag{3.19}$$

Lemma 3.3.13. *Given any time $t^i \in [0, T]$, there exists a time $t^{i+1} > t^i$, such that given problem (3.13) with two particles, admissible in the sense of (3.3.1), non-intersecting in the sense that $h_1(t) \neq h_2(t) \in [t^i, t^{i+1}]$ and initial data $u(t^i) \in L^\infty(\mathbb{R})$, there exists a solution $u(t,x) \in L^\infty([t^i, t^{i+1}] \times \mathbb{R})$, entropy admissible in the sense of (3.15).*
Additionally, if $u(t_i, x)$ satisfies for all $x \in \mathbb{R}$

$$c_{min}(t^i, x) \leq u(t^i, x) \leq c_{max}(t^i, x),$$

then $u(t,x)$ satisfies for all times $t \in [t^i, t^{i+1}], x \in \mathbb{R}$

$$c_{min}(t,x) \leq u(t,x) \leq c_{max}(t,x), \tag{3.20}$$

The last statement (3.20) is actually a stronger result than the L^∞ bound, as (3.19) follows directly from (3.20) as soon as it is established for all times $t \in [0, T]$.

Proof. To be able to make use of the existing results for the case of a single particle, we choose t^{i+1} such that the waves propagating from the two particles can not intersect in $[t^i, t^{i+1}] \times \mathbb{R}$. This is achieved by defining

$$t^{i+1} = t^i + \frac{h_2(t^i) - h_1(t^i)}{2L}.$$

We claim that such a constant L exists and assume for now that $L = \sup_{x\in\mathbb{R}}(c_{\max}(t,x), -c_{\min}(t,x))$. We define the superposition of $[t^i, t^{i+1}] \times \mathbb{R} = B_1 \cup P_1 \cup B_2 \cup P_2 \cup B_3$ such that P_1, P_2 contain the particles and all waves emanating from them, compare Figure 3.4.

$$\begin{aligned}
P_{1,2}(t) &:= [h_{1,2}(t_{i+1}) - L(t_{i+1} - t), h_{1,2}(t_{i+1}) + L(t_{i+1} - t)] \\
B_1(t) &:= (-\infty, h_1(t_{i+1}) - L(t_{i+1} - t)] \\
B_2(t) &:= [h_1(t_{i+1}) + L(t_{i+1} - t), h_2(t_{i+1}) - L(t_{i+1} - t)] \\
B_3(t) &:= [h_2(t_{i+1}) + L(t_{i+1} - t), \infty).
\end{aligned}$$

for $t \in [t^i, t^{i+1}]$.

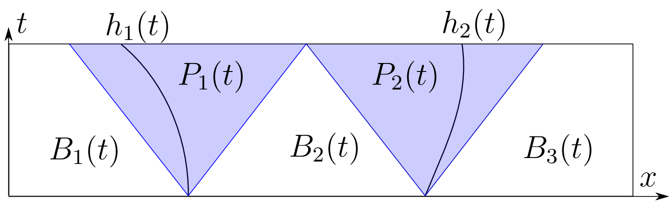

Figure 3.4.: Partition into regions influenced by the particles P_1, P_2 and regions dominated by the Burgers equation B_1, B_2, B_3. The slope of the cones is given by L.

From the analysis done for a single particle, we know that given $u(t_i, \cdot) \in L^\infty(P_1)$ and given that the solution $u(t,x)$ with $x \in \mathbb{R} \setminus P_1$ in the adjacent regions to P_1 satisfies $c_{\min}(t,x) \leq u(t,x) \leq c_{\max}(t,x)$, the bounds are also true in P_1 [1], namely

$$c_{\min}(t,x) \leq u(t,x) \leq c_{\max}(t,x) \quad \text{for } x \in P_1$$

and the same holds equivalently for P_2. Also, we know for the regions B_j, $j = 1, 2, 3$, given $u(t^i, \cdot) \in L^\infty(B_j)$, $u(t,x)$ on the boundaries of B_j and given that the solution $u(t,x)$

[1] This was a byproduct of constructing the L^∞ bound in [10] and can be found in the proof of the corresponding Lemma.

with $x \in \mathbb{R} \setminus B_j$ in the adjacent region to B_j satisfies $c_{\min}(t^i, x) \le u(t,x) \le c_{\max}(t^i, x)$, the bounds are also true in B_j

$$c_{\min}(t,x) \le u(t,x) \le c_{\max}(t,x). \quad \text{for } x \in B_j.$$

as the Burgers equation with L^∞ boundary data satisfies an L^∞ bound for any finite time. Piecing together the different regions, given $c_{\min}(t^i, x) \le u(t^i, x) \le c_{\max}(t^i, x)$, we obtain (3.20).
Therefore, defining the new superposition of $[t^i, t^{i+1}] \times \mathbb{R} = \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_1(t) = (-\infty, h_2(t^i) - L(t - t^i)]$$
$$\Sigma_2(t) = (h_1(t^i) + L(t - t^i), \infty)$$

Each of those regions contains only one particle, and therefore, applying Lemma 2 twice, we obtain existence of an entropy solution in $[t_i, t_{i+1}] \times \mathbb{R}$, compare Figure 3.5. □

Iterating this by using $t^i = t^{i+1}$ as new starting time for Lemma (3.3.13) until reaching time T gives the existence result on the whole domain $[0, T] \times \mathbb{R}$ and the L^∞ bound follows directly from property (3.20) as long as the particle paths do not intersect.

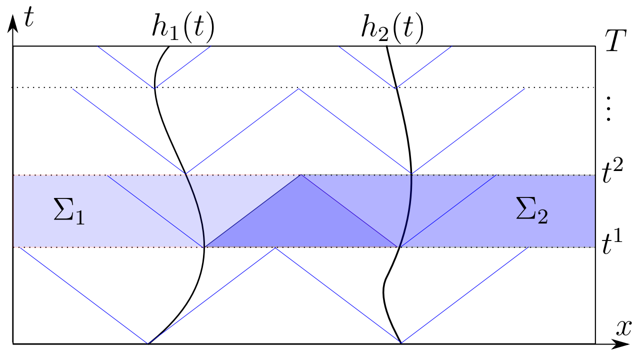

Figure 3.5.: Iteration of the construction method. The regions Σ_1, Σ_2 in time-step $[t_1, t_2]$ contain only waves emanating from one of the particles.

3.3.6. Existence in the case of two merging particles

Having proven existence of entropy solutions if there are no particle path intersections, there remain two cases which are investigated in this work. Merging, which is the case when for some time $t_{\text{merge}} \in [0, T]$, the ending points of two particle paths intersect. For the sake of simplicity, it is assumed in this section, that this event is isolated, meaning there are exactly three admissible particles in the domain, such that $h_1(t_1^2) = h_2(t_2^2) = h_3(t_3^1)$ and $t_1^2 = t_2^2 = t_3^1 = t_{\text{merge}}$. The first thing to note, is, that by the method of construction used to obtain a solution, given initial data $u(t_{\text{merge}}, \cdot) \in L^\infty$, we have

existence in the domain $[t_{\text{merge}}, T] \times \mathbb{R}$. Therefore it is sufficient if we can prove existence in $[0, t_{\text{merge}}] \times \mathbb{R}$. For simplicity of notation, we redefine $T = t_{\text{merge}}$.
We recall the specific problem, which is the Burgers equation influenced by two particles in a finite time interval $[0, T = t_{\text{merge}}]$

$$\begin{aligned} \partial_t u(t,x) + \partial_x \left(\frac{u^2(t,x)}{2} \right) &= \sum_{i \in \{1,2\}} \lambda_i (h_i' - u(t,x)) \delta_{h_i(t)} & (t,x) &\in \Omega \\ u(0,x) &= u_0(x) & x &\in \mathbb{R} \end{aligned} \tag{3.21}$$

for given particle related friction constants λ_1, λ_2 and admissible particles $(h_1(t), \lambda_1), (h_2(t), \lambda_2)$, with $h_1(T) = h_2(T)$, $h_1(t) \neq h_2(t)$ $\forall t \in [0, T)$ and $\Omega := [0, T] \times \mathbb{R}$.

The result we prove now extends the existence result of the Section 3.3.5 (Lemma 3.3.13), further working towards the proof of Theorem 3.3.8.

We state the main result for the two merging particles case.

Lemma 3.3.14. *Given initial data $u_0 \in L^\infty(\mathbb{R})$ for the Cauchy problem for two merging particles (3.21), admissible in the sense of (3.3.1), then there exists an entropy admissible solution u.*
Additionally, every such solution u satisfies

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \sum_{i=1}^{2} \lambda_i \tag{3.22}$$

Proof.
As the same result is true in the case of no particle interactions, we start by applying the same time-stepping construction method as in section 3.3.5.
The first thing to realise is that given any time $t < T$, we obtain the existence of upper and lower bounds $c_{\text{min, max}}(t,x)$ on any solution such that

$$c_{\min}(t,x) \leq u(t,x) \leq c_{\max}(t,x) \tag{3.23}$$

holds true in $[0,t] \times \mathbb{R}$, by iterating Lemma 3.3.13.
As before, combining this with the finite speed of the particles given by the Lipschitz continuity of the particle paths (P1), we obtain the desired L^∞ bound (3.22).
To explicitly construct solutions on the whole domain, it remains to check that the merging of particles does not hinder the iteration process to reach time T.
Recall that every new timestep is created by computing

$$t^{i+1} = t^i + \frac{h_2(t^i) - h_1(t^i)}{2 \sup\limits_{x \in \mathbb{R}} \left(c_{\max}(t,x), -c_{\min}(t,x) \right)}. \tag{3.24}$$

Directly, we also have naturally $t^{i+1} > t^i$, as the distance of the particles is positive (absolute values are not needed due to the particles being ordered, thus $h_2(t) \geq h_1(t) \, \forall \, t \in [0, T]$).

Remark 3.3.15. *Equation (3.24) might seem unfitting at a first glimpse, as we did not specify the time at which to evaluate the supremum of the bounds on the solution in the denominator. However, as mentioned before, the time dependence of the functions c_{min}, c_{max} is only due to the particles movement, shifting the constant states, and interactions of particles like merging, splitting or crossing. Due to the isolation of a single event, which is the merging of particles at the ending time of the observed domain in this particular case, the number and order of constant states does not change here. Therefore the denominator of the right side of (3.24) is not time dependent in $[0, T)$.*

By the method of constructing solutions, the number of timesteps goes to infinity whenever the distance between two particles goes to zero, which is the case here as $h_1(T) = h_2(T)$. Therefore, it remains to prove that $\lim_{n\to\infty} t^n = T$.
The exact formula for t_n is easily computed as

$$t^n = \sum_{i=1}^{n} \frac{h_2(t^i) - h_1(t^i)}{2\sup\limits_{x\in\mathbb{R}}(c_{\max}(t,x), -c_{\min}(t,x))}$$

Assuming there exists a time $t^{\max} < T$ where the iteration stops, i.e.

$$\exists N \in \mathbb{N} : t^{N+1} = t^N = t^{\max},$$

this obviously leads to a contradiction, as

$$t^{N+1} = t^N + \underbrace{\frac{h_2(t^N) - h_1(t^N)}{2\sup\limits_{x\in\mathbb{R}}(c_{\max}(t,x), -c_{\min}(t,x))}}_{>0,\text{ as } h_1(t^N)\neq h_2(t^N)}.$$

Actually, the progression of the timestepping behaves like a quasi-Newton method, found in most textbooks on analysis or numerics, where the slope of the timestepping remains fixed, compare Figure 3.6.
Therefore, we have $t^n \to T$ as $n \to \infty$ and we recover the existence result of Lemma 3.3.14. □

Remark 3.3.16. *The analysis for the merging of particles can be extended to three or more particles merging at time T, meaning there could be a set of particles N_{merge}, and we could consider the problem (3.21) with $h_i(T) = h_j(T)\ \forall\ i, j \in N_{merge}$. The difference in constructing the solutions would be, and this will be discussed further in section 3.3.8, that the two neighboring particles with the lowest distance define the timestepping.*

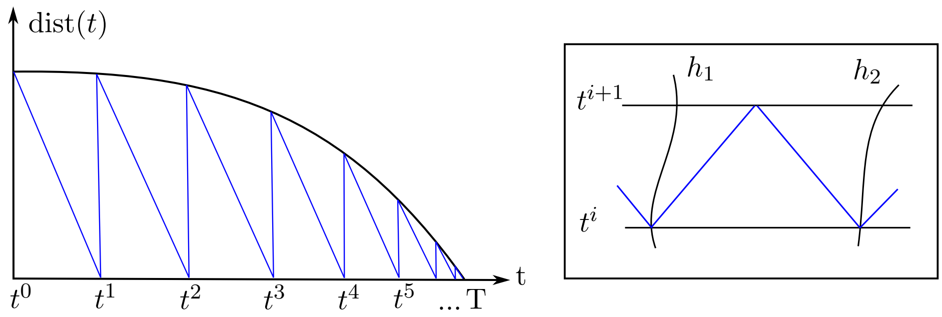

Figure 3.6.: On the left a visualization of the convergence for the construction method and $t^i \to T$. The method behaves like a quasi-Newton method, always reaching time T. On the right the construction of the length of a single timestep, given by the longest possible time, s.t. waves propagating from the two particles don't intersect.

3.3.7. Existence in the case of two splitting particles

At this point we have existence for the domains where no particles paths intersect and for domains where two (or more) particles merge at time $t = t_{\text{split}}$. In this section we will prove the existence and L^∞ bound for the splitting of particles. Again, the splitting event can be isolated, looking at the following specific problem.

The Burgers equation influenced by two particles in a finite time interval $[0 = t_{\text{split}}, T]$ writes

$$\begin{aligned} \partial_t u(t,x) + \partial_x \left(\frac{u^2(t,x)}{2} \right) &= \sum_{i \in \{1,2\}} \lambda_i (h_i' - u(t,x)) \delta_{h_i(t)} && (t,x) \in \ \Omega \\ u(0,x) &= u_0(x) && x \in \ \mathbb{R} \end{aligned} \tag{3.25}$$

for given particle related friction constants λ_1, λ_2 and admissible particles $(h_1(t), \lambda_1), (h_2(t), \lambda_2)$, with $h_1(0) = h_2(0)$, $h_1(t) \neq h_2(t)$ $\forall t \in (0,T]$ and $\Omega := [0,T] \times \mathbb{R}$.

The result we prove now is the final building block for the proof of the existence claimed by theorem 3.3.8 further extending the existence results of Sections 3.3.5 (Lemma 3.3.13) and 3.3.6 (Lemma 3.3.14), further working towards the proof of Theorem 3.3.8.

However, proving an existence result for this case is more complicated than for merging of particles, as the time-stepping construction from section 3.3.5 is not directly applicable. This is due to the fact that $h_1(0) = h_2(0)$ and therefore $t^2 = t^1 = 0$. Again, the main result is stated first, while the rest of this section is devoted to its proof.

Lemma 3.3.17. *Given initial data $u_0 \in L^\infty(\mathbb{R})$ for the Cauchy problem for two splitting particles (3.25), admissible in the sense of (3.3.1), then there exists an entropy admissible solution u.*
Additionally, every such solution u satisfies

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \sum_{i=1}^{2} \lambda_i. \tag{3.26}$$

The following Lemmata are necessary for the proof of Lemma 3.3.17. Their proofs can be found at the end of this section.

Lemma 3.3.18. *Given problem (3.25) and initial data $u_0 \in L^\infty$ with*

$$c_{min}(0,x) \leq u(0,x) \leq c_{max}(0,x),$$

for any piece-wise constant functions c_{min}, c_{max} of the form (3.17) then any solution $u(t,x)$ satisfies for all times $t \in [0,T], x \in \mathbb{R}$

$$c_{min}(t,x) \leq u(t,x) \leq c_{max}(t,x). \tag{3.27}$$

Lemma 3.3.19. *Given $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R})$ and given particle paths $h_1(t) \neq h_2(t) \in Lip[0,T]$. Then any solution u to problem (3.13) satisfies*

$$\begin{aligned} TV(u(t,\cdot)) \leq{}& TV(u_0) + Kt \\ &+ 2\sum_{i=1}^{2} dist((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))) + \sum_{i=1}^{2} 4TV_{[0,t]}h_i' \end{aligned}$$

with $K \in \mathbb{R}$ a constant.

Lemma 3.3.20. *Given $u_0^1, u_0^2 \in L^\infty \cap BV_{loc}(\mathbb{R})$ and given admissible particle paths $h_1, h_2 \in Lip[0,T]$ with $h_1(0) \neq h_2(0)$. Then entropy solutions $u^{1,2}$ to problem (3.25) with $\boldsymbol{N}(T) = 2$ and initial data u_0^1, u_0^2 respectivly, satisfy the local L^1-contraction*

$$\int_{[-R,R]} |u^1(t,x) - u^2(t,x)| \, dx \leq \int_{[-R-Lt,R+Lt]} |u_0^1 - u_0^2| \, dx \tag{3.28}$$

for any $R \in \mathbb{R}$ and $L = \max_{x\in\mathbb{R}} (c_{max}(t,x), -c_{min}(t,x))$.

Remark 3.3.21. *Lemma 3.3.18 is the same result as for the previous cases and again, directly gives the L^∞ bound as a corollary. However, in contrary to the previous cases, an approximation is needed to obtain these bounds here.*
To be able to conclude convergence of the approximation to a solution of the original problem, compactness in BV_{loc} is needed, and Lemma 3.3.19 provides the necessary bounds in total variation.
Finally, Lemma 3.3.20 allows to uniquely identify the obtained limit as the entropy solution to problem (3.25), whose existence was claimed by Lemma 3.3.17.

Additionally, as an approximation is unavoidable due to the missing initial data in the region between the particles, the following approximated problem will also be considered.

$$\begin{aligned} \partial_t u^{\delta,\epsilon}(t,x) + \partial_x \left(\frac{(u^{\delta,\epsilon})^2(t,x)}{2} \right) &= \sum_{i\in\{1,2\}} \lambda_i((h^\epsilon)_i' - u^{\delta,\epsilon}(t,x))\delta_{h_i^\epsilon(t)} \qquad && (t,x) \in \Omega \\ u^\delta(0,x) &= u_0^\delta(x) && x \in \mathbb{R} \end{aligned} \tag{3.29}$$

The important novelties compared to the original splitting particles problem (3.25) are the approximation of initial data

$$u_0^\delta(x) = u_0(x) * \rho^\delta \tag{3.30}$$

with ρ^δ being a regularizing kernel such that $u_0^\delta \in L^\infty \cap BV_{loc}(\mathbb{R})$ and the choice a second approximation, this time for the particle paths

$$\begin{aligned} h_1^\epsilon(t) &= h_1(t) \\ h_2^\epsilon(t) &= h_2(t) + \epsilon \end{aligned} \tag{3.31}$$

removing the intersection point at time $t = 0$ and assuring, that the particle paths are strictly non-intersecting in the whole domain $[0,T] \times \mathbb{R}$.
The following corollary of Lemma 3.3.18 and 3.3.19 gives the necessary compactness of solutions to the approximated problem (3.29) in BV_{loc} in order to be able to pass to the limit using Helly's selection theorem.

Corollary 3.3.22. *Solutions to the approximated splitting particle problem (3.29) with initial data $u_0^\delta \in L^\infty \cap BV_{loc}(\mathbb{R})$ and their local total variation are uniformly bounded with respect to ϵ, i.e. the shift of particles.*

With these auxiliary results, we are now able to prove Lemma 3.3.17.

Proof of Lemma 3.3.17.
As mentioned, we are not able to start the time-stepping construction, as problem (3.25) does not have sufficient initial data to cover the region in between the splitting particles. Therefore we dismiss the approach of directly constructing the solution, and instead consider the approximated problem (3.30), (3.31), followed by a compactness argument to pass to original solutions.
Using Lemma 3.3.18, we obtain that any solution $u^{\delta,\epsilon}(t,x)$ to problem (3.25) with initial data u_0^δ and admissible particle paths $h_1^\epsilon, h_2^\epsilon$ admits bounds of the form (3.27) and therefore is in $L^\infty(\mathbb{R})$ for almost every time $0 \le t \le T$. Furthermore, as the particles are strictly apart by this approximation, Lemma 3.3.13 gives existence of a solution $u^{\delta,\epsilon}$ to the approximated problem in the whole domain $[0,T] \times \mathbb{R}$.
Using Lemma 3.3.19, we know that those solution $u^{\delta,\epsilon}$ are also in BV_{loc} for almost every time $0 \le t \le T$.
As for every $\epsilon > 0$ and almost every time $0 \le t \le T$, we have $u^{\delta,\epsilon} \in L^\infty(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R})$, and, as stated by corollary 3.3.22, those solutions are uniformly bounded and are locally

of bounded total variation, we obtain, using Helly's selection theorem, existence of a subsequence u^{δ,ϵ_k}, convergent in L^1 with respect to ϵ, such that

$$\lim_{\epsilon_k \to 0} \int_{\mathbb{R}} |u^{\delta,\epsilon_k} - u^{\delta}| \, dx = 0$$

with $u^{\delta} \in L^{\infty} \cap BV_{\text{loc}}(\mathbb{R})$.
Now we drop the regularization, letting $\delta \to 0$, and obtain existence of a subsequence δ_l and a limit function $\overline{u}$, such that, again converging in L^1

$$\lim_{\delta_l \to 0} \int_{\mathbb{R}} |u^{\delta_l} - \overline{u}| \, dx = 0$$

with $\overline{u} \in L^{\infty}((0,T) \times \mathbb{R})$.
It remains to show that the obtained function $\overline{u}$ is indeed an entropy solution to (3.25).
It is easy to see, that solutions $u^{\delta,\epsilon}$ to the approximated problem satisfy for
$\phi \in C^{\infty}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, and all piecewise constant functions $c(t,x)$ of the form
$c(t,x) = c_1 \mathbf{1}_{x<h_1(t)} + c_2 \mathbf{1}_{h_1(t)<x<h_2(t)+\epsilon} + c_3 \mathbf{1}_{x>h_2+\epsilon}$, $(c_1,c_2,c_3) \in \mathbb{R}^3$,
the following entropy inequality.
There exists $M \in \mathbb{R}$, such that given $\mathcal{G}_{1,2}$ are the admissibility germs of h_1, h_2 respectivly

$$\int_0^T \int_{\mathbb{R}} |u^{\delta,\epsilon} - c(t,x)| \partial_t \phi + \Phi(u^{\delta,\epsilon}, c(t,x)) \partial_x \phi) \, dx dt + \int_{\mathbb{R}} |u_0^{\delta} - c(0,x)| \phi(0,x) dx$$
$$+ M \int_0^T \text{dist}((c_1,c_2),\mathcal{G}_1) \phi|_{x=h_1(t)} \, ds + M \int_0^T \text{dist}((c_2,c_3),\mathcal{G}_2) \phi|_{x=h_2(t)+\epsilon} \, ds \geq 0. \tag{3.32}$$

The critical terms are the comparison of the initial data with the piecewise constant entropy auxiliary function $c(t,x)$ and the interface terms. As convergence of u_0^{δ} in L^1 is already established, only the germ related interface terms remain.
However, as mentioned in Remark 3.3.15, the particular states c_i of $c(t,x)$ are time-independent constants and therefore the shift of particles only affects the position at which the testfunction ϕ is evaluated. This means, that the interface term doesn't change in value, but only shifts with respect to the position of the particle as $\epsilon \to 0$ and we conclude that the approximations (3.30), (3.31) have no effect on the interface terms.
Passing to the limit in (3.32), one obtains (3.16) and therefore the limit solution $\overline{u}$ is an entropy solution to (3.25) as claimed.
The proof is finished by using Lemma 3.3.20, justifying uniqueness of the discovered entropy solution. □

The remainder of this section is devoted to the proofs of Lemma 3.3.18 and 3.3.19. Lemma 3.3.20 is a result connected to uniqueness of entropy solutions and will be proven in section 3.3.9.

Proof of Lemma 3.3.18.
The proof is strongly based on the proof for the case of non-intersecting particle paths. As in the proof of Lemma 3.3.17, we choose the approximation (3.31)

$$\begin{aligned} h_1^\epsilon(t) &= h_1(t) \\ h_2^\epsilon(t) &= h_2(t) + \epsilon \end{aligned}$$

As the approximated particle paths are strictly apart in $[0, T]$, reproducing the proof of Lemma 3.3.13 with

$$\begin{aligned} P_{1,2}^\epsilon(t) &:= [h_{1,2}^\epsilon(t^{i+1}) - L(t^{i+1} - t), h_{1,2}^\epsilon(t^{i+1}) + L(t^{i+1} - t)] \\ B_1^\epsilon(t) &:= (-\infty, h_1^\epsilon(t^{i+1}) - L(t^{i+1} - t)] \\ B_2^\epsilon(t) &:= [h_1^\epsilon(t^{i+1}) + L(t^{i+1} - t), h_2^\epsilon(t^{i+1}) - L(t^{i+1} - t)] \\ B_3^\epsilon(t) &:= [h_2^\epsilon(t^{i+1}) + L(t^{i+1} - t), \infty), \end{aligned}$$

for $t \in [t^i, t^{i+1}]$ and where B_2^ϵ is respectivly small, gives an L^∞ bound in every region again using the result for the model with a single particle. Passing to the limit with respect to ϵ gives the desired bound. □

It remains to prove Lemma 3.3.19. This will require some additional Theorems from measure theory, first and foremost the theorem of Radon-Nikodỳm, which is just stated in the following, for the proof of the theorems, the reader is kindly referred to [72]. In addition, the result of existence of a bound in total variation for the model with a single particle, which can be found in [10] is a critical building block and repeated below.

We recall the bound in total variation in the presence of a single particle.

Theorem 3.3.23. *Given $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R})$ and a particle path $h_i(t) \in Lip[0, T]$. Then any solution of the Burgers equation with a single particle satisfies the bound*

$$TVu(t, \cdot) \leq TVu_0 + 2dist((u_0(h^-(0)), u_0(h^+(0))), \mathcal{G}_\lambda(h'(0)) + 4TV_{[0,t]}h'.$$

The following theorems are textbook measure theory theorems and can be found in [72].

Theorem 3.3.24 (Radon-Nikodỳm)**.**
Given a measurable space (X, Σ), if a σ-finite measure μ on (X, Σ) is absolutely continuous with respect to a σ-finite measure ν on (X, Σ), then there is a measurable function $f : X \to [0, \infty)$, such that for any measurable subset $A \subset X$

$$\mu(A) = \int_A f \, d\nu$$

Theorem 3.3.25 (Lebesgue's decomposition theorem).
For every two σ-finite signed measures μ and ν on a measurable space (Ω, Σ), there exist two uniquely determined σ-finite measures ν_0 and ν_1 such that

$$\begin{aligned} &\nu = \nu_0 + \nu_1 \\ &\nu_0 \ll \mu && \Leftrightarrow \nu_0 \text{ is absolutely continuous with respect to } \mu \\ &\nu_1 \perp \mu && \Leftrightarrow \nu_1 \text{ and } \mu \text{ are singular} \end{aligned}$$

And lastly, we define the total variation of a function f on any half-open or closed sets as the limits

$$\begin{aligned} TVf|_{(a,b]} &= \lim_{\epsilon \to 0} TVf|_{(a,b+\epsilon)} \\ TVf|_{[a,b]} &= \lim_{\epsilon \to 0} TVf|_{(a-\epsilon,b+\epsilon)} \end{aligned}$$

Remark 3.3.26. *This definition of the total variation of closed sets is needed because of the special case of appearance of shockwaves, which introduce a really point-wise contribution to total variation, and whenever one defines a superposition of domains and wants to consider the total variation, the intersection point of the domains can not be neglected due to the possibility of a shockwave sitting at that exact point.*

Proof of Lemma 3.3.19.
We start by defining for any time $0 < t < T$ the superposition $\Omega_1 \cup \Omega_2 \supset [0,t] \times \mathbb{R}$ with

$$\begin{aligned} \Omega_1(t) &= (-\infty, X(t) + L(t^1 - t)] \\ \Omega_2(t) &= [X(t) - L(t^1 - t)), \infty) \end{aligned} \tag{3.33}$$

For multiple particles, we use for any time $0 < t^1 \leq T$, with $h_1(0) + Lt^1 < h_2(0) - Lt^1$ the superposition of $[0, t^1] \times \mathbb{R}$ into $\Omega_1 \cup \Omega_2$ defined in (3.33) with a special choice of $X_1 := X(t^1)$. First we notice, that whenever we choose $X_1 \in (h_1(0) + Lt^1, h_2(0) - Lt^1)$, both regions Ω_1, Ω_2 are only influenced by one of the particles, thus in each reagion, we can make use of the result for a single particle locally, i.e. theorem 3.1.7, to obtain the TV bound

$$TVu(t,x)|_{\Omega_i(t^1)} \leq TVu_0|_{\Omega_i(0)} + 2\text{dist}((u_0(h_i^-(0)), u_0(h_i^+(0)), \mathcal{G}_{\lambda_i}(h_i'(0))) + 4TV_{[0,t^1]}h_i'$$

Clearly, by our choice of Ω_i as a superposition of $\mathbb{R}$ for any time $0 < t^1 < T$, such that $h_1(0) + Lt^1 < h_2(0) - Lt^1$, we can compute

$$\begin{aligned} &TVu(t^1,x) \leq TVu(t^1,x)|_{\Omega_1(t^1)} + TVu(t^1,x)|_{\Omega_2(t)}. \\ &\leq TVu_0|_{\Omega_1(0)} + TVu_0|_{\Omega_2(0)} + 2\sum_{i=1}^{2} \text{dist}((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))) + \sum_{i=1}^{2} 4TVh_i'|_{[0,t^1]} \\ &= TVu_0 + TVu_0|_{[X_1 - Lt^1, X_1 + Lt^1]} + 2\sum_{i=1}^{2} \text{dist}((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))) + \sum_{i=1}^{2} 4TVh_i'|_{[0,t]}. \end{aligned}$$

Note that TVu_0 is controlled due to the fact that $u_0 \in BV_{\text{loc}}$, the distance between the particles at the initial time is finite and $TV_{[0,t]}h_i'$, $i = 1, 2$ is bounded by the Lipschitz continuity of the particle paths in time. The last remaining term is $TVu_0|_{[x-Lt^1,x+Lt^1]}$, which is also finite due to $u_0 \in BV_{loc}$.
We can not hope for this to hold up to any time T, as the condition $h_1(0)+Lt^1 < h_2(0)-Lt^1$ restricts our choice of t^1. However, we can again use a time-stepping algorithm, using $u(t^1, x) \in L^\infty \cap BV_{loc}(\mathbb{R})$ as new initial data and restart, obtaining an TV bound for any time t^2, such that $h_1(t^1) + L(t^2 - t^1) < h_2(0) - L(t^2 - t^1)$.
We obtain

$$\begin{aligned}
TVu(t^2, x) &\leq TVu(t^1, x) + TVu(t^1, x)|_{[X_2-L(t^2-t^1),X_2+L(t^2-t^1)]} \\
&\quad + \underbrace{2\sum_{i=1}^{2} \text{dist}((u(t^1, h_i^-), u(t^1, h_i^+), \mathcal{G}_{\lambda_i}(h_i'(t^1)))}_{=0} + \sum_{i=1}^{2} 4TV_{[t^1,t^2]}h_i' \\
&\leq TVu_0 + \sum_{j=1}^{2} \Big(TVu(t^{j-1}, x)|_{[X_j-L(t^j-t^{j-1}),X_j+L(t^j-t^{j-1})]} + \sum_{i=1}^{2} 4TV_{[t^{j-1},t^j]}h_i'\Big) \\
&\quad + 2\sum_{i=1}^{2} \text{dist}((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))).
\end{aligned}$$

Note that the penalization term for the distance of the solution to them Germ $\mathcal{G}_{\lambda_i}$ exists only at initial time, as the behaviour of our solution nullifies this distance for any time $t > 0$.
Reiterating finally gives for time t_{n+1}

$$\begin{aligned}
TVu(t^{n+1}) &\leq TVu_0 + \sum_{j=1}^{n} \Big(TVu(t^{j-1}, x)|_{[X_j-L(t^j-t^{j-1}),X_j+L(t^j-t^{j-1})]}\Big) \\
&\quad + 2\sum_{i=1}^{2} \text{dist}((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))) + + \sum_{i=1}^{2} 4TV_{[0,t^{n+1}]}h_i'.
\end{aligned} \tag{3.34}$$

The number of timesteps remains finite, therefore controlling the second term with respect to time. This property can be seen as the length of each timestep can be arbitrarily chosen up to the point in time where the cones of influence of the two particles intersect

$$t^k - t^{k-1} = \frac{\text{dist}(h_1(t^n), h_2(t^n))}{2L}$$

which is bounded below if the distance of the particles is bounded below, compare the time-stepping construction method in section 3.3.5. Although, for a given $\epsilon > 0$, this implies $u^{\delta,\epsilon} \in BV_{\text{loc}}$, it remains to prove the exact bound of Lemma 3.3.19 aswell as to verify corollary 3.3.22.
At first glimpse, the TV bound appears to not be uniform with respect to ϵ, as the

length of the timesteps depends linearly on the distance between the particles, therefore, if $\epsilon \to 0$, the number of timesteps goes to infinity. However, at the same time, the length of the interval $[X_j - L(t^j - t^{j-1}), X_j + L(t_j - t_{j-1})]$ goes to zero aswell. The remaining estimate to establish Lemma 3.3.19 is:
There exists a constant $K \in \mathbb{R}$, such that

$$\sum_{j=1}^{n} \left(TVu^{\epsilon,\delta}(t^{j-1}, x)|_{[X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})]} \right) \leq K \sum_{i=1}^{n} (t^{i+1} - t^i) = Kt^n. \quad (3.35)$$

The proof is based on a good selection of X_j in each timestep, using the property, that entropy solutions to the Burgers equation have Lebesgue points on a dense set on the real line, i.e. intervals of small total variation. First, we see that the sum on the left side requires us to look at the total variation in the interval $[X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})]$ for the solution at the previous timestep $u^{\epsilon,\delta}(t^{j-1}, \cdot)$. This is crucial, as for the previous timestep, we already know that $u^{\epsilon,\delta}(t^{j-1}, \cdot) \in L^\infty \cap BV_{\text{loc}}$ by the previous TV estimate for a finite number of timesteps (3.34). Additionally, $[X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})]$ lies completely in the region $B_2^\epsilon(t^j)$
Now we make use of the following well known property of functions of bounded total variation.

Given a function $f \in BV_{\text{loc}}$, there exists a Radon measure ν,
such that $f' = \nu$ in the distributional sense.

Remark 3.3.27. *This property gives the so called distributional derivative. Given the function $f \in BV$, the Radon measure is finite. This property is used to identify the weak derivative of $u^{\epsilon,\delta}$ with a measure. Note that even though u is only in BV_{loc}, as the region $[X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})]$ we are looking at is embedded in B_2^ϵ and $u \in BV_{loc}(B_2^\epsilon)$, the measure we obtain is also finite.*

The goal is to identify the total variation of $u^{\epsilon,\delta}$ with a density function using the theorem of Radon-Nikodỳm. But as the measure representing the total variation is not absolutely continuous with respect to the Lebesgue measure, we have to first use the Lebesgue decomposition theorem.

Remark 3.3.28. *The lack of absolute continuity of the total variation with respect to the Lebesgue measure can be seen as $\lim_{\epsilon\to 0} TVu|_{(x-\epsilon,x+\epsilon)} \neq 0$ if a shockwave is located at the point x. Note also that the measure representing the total variation is a signed measure.*

By the Lebesgue decomposition theorem 3.3.25, there exist unique Radon measures ν_0, ν_1 representing the total variation of $u^{\epsilon,\delta}$ in $[X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})]$ with

$\nu = \nu_0 + \nu_1$	the sum of the measures gives the total variation
$\nu_0 \ll \lambda$	ν_0 is absolutely continuous with respect to λ
$\nu_1 \perp \lambda$	ν_1 and λ are singular

It is easy to identify the measures ν_0, ν_1 with the parts of the total variation representing the variation induced by oscillations (ν_0) and the variation induced by shockwaves (ν_1), as the measures are uniquely determined.
As $u \in BV_{\text{loc}}$, the support of ν_1 can not lie dense in $[h_1(t), h_2(t)]$ and therefore we can always choose X_j such that $\nu_1([X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})] \to 0$, as $t^{j+1} - t^j \to 0$.
It remains to check ν_0. As by the Lebesgue decomposition, $\nu_0 \ll \lambda$ and ν_0 is finite, we obtain for each $K \subset [h_1(t), h_2(t)$ by the theorem of Radon-Nikodym existence of an associated positive density ρ defined by

$$\int_K d\nu_0 = \int_K \rho(x)\, dx$$

such that for almost every $x \in [h_1(t), h_2(t)]$, x is a Lebesgue point in the sense that

$$\rho(x) = \lim_{r \to 0} \frac{1}{r} \int_{B(x,r)} \rho(s)\, ds = 0. \tag{3.36}$$

The last property is due to $u^{\epsilon,\delta} \in BV_{\text{loc}}$ has already been established for the previous timestep (and is a general property of BV type solutions, and can be found in textbooks on the topic, e.g. [47]).
Therefore we have for r sufficiently small and any Lebesgue point $X \in I$

$$\begin{aligned} \frac{1}{r}\nu_0(B(X,r)) &\leq \rho(x) + 1 \qquad \forall x \in B(X,r) \\ \nu_0(B(X,r)) &\leq \rho(x) r + r \quad\ \ \forall x \in B(X,r) \end{aligned} \tag{3.37}$$

Translating (3.37) back to our setting, we have

$$\frac{1}{r} TVu(t^j, \cdot)|_{B(X_j,r)} =: \frac{1}{r} \int_{B(X_j,r)} d\nu_0 \leq \rho(X_j) + 1$$

From (3.37) and the fact that ν_0 is finite, the density $\rho(x)$ is in L^1_{loc} and positive. Therefore $\min_{X \in B(X,r)} \rho(X) = \rho(X_{\min}) < \infty$ exists and we obtain

$$\begin{aligned} &\sum_{j=1}^{n} \frac{1}{L(t^{j+1} - t^j)} \Big(TVu^{\epsilon,\delta}(t^{j-1}, x)|_{[X_{\min} - L(t^j - t^{j-1}), X_{\min} + L(t^j - t^{j-1})]} \Big) \\ &\qquad = \sum_{j=1}^{n} \frac{1}{L(t^{j+1} - t^j)} \int_{B(X_{\min}, L(t^{j+1} - t^j))} d\nu_0 \leq \sum_{j=1}^{n} C(X_j). \end{aligned}$$

with $C(X) = \rho(X_{\min}) + 1$. As the sum is finite, we take $K = L \max_{j \in [1,n]} C(X_j)$ and obtain (3.35). The bound is uniform, as due to X_j being a Lebesgue point and thus (3.36) holds, we have for $\epsilon \to 0$, that $R := [X_j - L(t^j - t^{j-1}), X_j + L(t^j - t^{j-1})] \to 0$ and

$$\rho(X_j) = \lim_{R \to 0} \frac{1}{R} \int_{B(X_j,R)} \rho(s)\, ds = 0$$

such that there exists ϵ and $K = L \max_{j\in[1,n(\epsilon)]} C(X_j)$, such that $\forall\epsilon' < \epsilon : K > L \max_{j\in[1,n(\epsilon')]} C(X_j)$, giving corollary 3.3.22. $\square$

Remark 3.3.29. *Note that the last part of the proof relies heavily on the fact that all our analysis is done in B_2^ϵ, which is governed by the Burgers equation and has compact support in $\mathbb{R}$ for any finite time. On the whole real line, the proof would lose the finiteness of ν_0 and the L^1 property of ρ.*

3.3.8. Extension to N particles

At this point, theorem 3.3.8 has been proven as long as $|N(t)| \leq 2$, for all times $t \in [0, T]$. However, even if the number of particles exceeds $|N(t)| = 2$, the possible interactions remain local, which is ensured by the finiteness of $|\mathbf{N}(T)|$ and the definition of admissible particles.

Remark 3.3.30. *The case of multiple distinct events, e.g. merging of particles, dictating a timestepping that scales faster than the one considered in section 3.3.5, is not possible in this setting. One could imagine for example two particles circling around each other, where the time between the crossings of the particle paths goes to zero. It is an open problem, if it is possible to prove a BV-estimate in this case. Here, this case is ruled out by conditions (P2), (P3), which would cause $|N(T)|$ to exceed any bound as the particles interact infinitely often.*

The algorithm designed in section 3.3.5 can be extended to give existence of entropy solutions to the main model (3.13) with finitely many admissible particles in the following way.
The time-stepping is chosen according to the shortest distance between two particles, such that in the i-th timestep

$$t^{i+1} = t^i + \min_{j_1,j_2\in N(t)} \frac{h_{j_1}(t^i) - h_{j_2}(t^i)}{2L}. \tag{3.38}$$

It is easy to see, that this time-stepping doesn't cause any problem for obtaining the previous results in the case of no particle interactions in $[t^i, t^{i+1}]$. It remains to check the two new possible events of multi-particle merging or splitting.

Case of multi-particle merging
Let now $N(t) > 2$ and $h_j^2(t_{merge}) = h_k^2(t_{merge})$, for all $j, k \in N(t)$, i.e. the case of a number of $|N(t)| > 2$ particles meeting at time t_{merge}. Choosing the new time-stepping (3.38), it is not as easy as before to see the convergence towards t_{merge}, as the domains defining the timestep vary between different particles. However, one can project the

convergence problem onto the following two-particles problem.

$$\begin{aligned} \partial_t u(t,x) + \partial_x \left(\frac{u^2(t,x)}{2} \right) &= \sum_{l \in \{1,2\}} \lambda_i (h'_{\text{new},l} - u(t,x)) \delta_{h_{\text{new},l}(t)}, \\ \text{with} \quad \text{dist}(h_{\text{new},1}, h_{\text{new},2})(t) &= \min_{j_1,j_2 \in N(t)} \text{dist}(h_{j_1}, h_{j_2})(t) \end{aligned} \tag{3.39}$$

Clearly, the convergence result of Lemma 3.3.14 holds and the case of multi-particle merging can be dealt with the same way as the merging of two particles in section 3.3.6.

Remark 3.3.31. *Note that dist$(h_{new,1}, h_{new,2})(t)$ is still continuous, condition (3.39) does not disturb the admissibility of the particles $h_{new,1}, h_{new,2}$.*

Case of multi-particle splitting

Let now $N(t) > 2$ and $h_j^1(t_{split}) = h_k^1(t_{split})$, for all $j,k \in N(t)$, i.e. the case of a number of $|N(t)| > 2$ particles originating from the same position at time t_{split}. As before, the time-stepping is not a problem in the case of particle splitting, but in order to obtain existence, an approximation and compactness of the approximate solutions is needed. The new approximation is straight-forward considering the two-particle splitting from section 3.3.7. The smoothing of the initial data stays untouched

$$u_0^\delta(x) = u_0(x) * \rho^\delta$$

with ρ^δ being a regularizing kernel such that $u_0^\delta \in L^\infty \cap BV_{loc}(\mathbb{R})$.
The particle paths need now all shifting, where one is free to choose the same shift at every particle position

$$\begin{aligned} h_1^\epsilon(t) &= h_1(t) \\ h_2^\epsilon(t) &= h_2(t) + \epsilon \\ h_3^\epsilon(t) &= h_3(t) + 2\epsilon \\ &\vdots \\ h_{|N(t)|}^\epsilon(t) &= h_{|N(t)|} + (|N(t)| - 1)\epsilon. \end{aligned}$$

From this point on, one recovers the L^∞-bound (and a direct extension of Lemma 3.3.12)

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \sum_{i \in N(t)} \lambda_i$$

mimicking the proof of Lemma 3.3.13 using domains $P_1, \dots, P_{|N(t)|-1}$ and $B_1, \dots, B_{|N(t)|+1}$ respective to the new number of particles. Additionally, existence of weak entropy solutions to the approximated problem follows as in the proof of Lemma 3.3.13.

The bound in total variation can be computed and proven the same way as in the two particles case of Lemma 3.3.19 and one obtains

$$
\begin{aligned}
TV(u(t,\cdot)) \leq\ & TV(u_0) + Kt \\
& + 2 \sum_{i \in N(t)} \operatorname{dist}((u_0(h_i^-), u_0(h_i^+), \mathcal{G}_{\lambda_i}(h_i'(0))) + 4 \sum_{i \in N(t)} TV_{[0,t]} h_i'.
\end{aligned}
$$

Therefore, using Helly's selection theorem, one can pass to the limit in approximate solutions and obtain existence of a weak entropy solution to the original problem, satisfying the entropy inequality (3.16).

Summarizing, a time-stepping method to construct weak entropy solutions in the presence of a finite number of particles has been shown. The time-stepping depends on the behaviour of the particles at the present time and is able to deal with merging, splitting and crossing of particles.

3.3.9. Uniqueness of entropy solutions

In this section, uniqueness of entropy solutions, admissible in the sense of (3.15), is proven using almost classical Kruzkov type analysis. It will be shown that using the dissipativity of the particle interfaces, one can obtain the Kato inequality and thus uniqueness of entropy solutions and the local L^1 contraction of Lemma 3.3.20.

Let $\Omega = [0,T] \times \mathbb{R}$, $\phi \in C_c^\infty(\Omega)$ be a classical, compactly supported testfunction and

$$w_\epsilon(x) = \begin{cases} 0, & \text{when } |x| \leq \frac{\epsilon}{2}, \\ 1, & \text{when } |x| \geq \epsilon. \end{cases}$$

with $w'_\epsilon(x) = \frac{2}{\epsilon}$ for $\frac{\epsilon}{2} \leq |x| \leq \epsilon$.

Given two entropy solutions u, v, we apply the method of doubling of variables, cf. [56], and choosing as a testfunction $\psi(x,t) = \phi(x,t) \times w_\epsilon(t, x - h_1(t)) \times \ldots \times w_\epsilon(t, x - h_N(t))$, we obtain

$$\int_\Omega |u-v| \partial_t \big(\phi \times \prod_{1 \leq i \leq N} w_\epsilon(x - h_i(t))\big) + \int_{\mathbb{R}} |u_0 - v_0| \big(\phi(0,x) \times \prod_{1 \leq i \leq N} w_\epsilon(x - h_i(0)\big) dx$$
$$+ \int_\Omega \Phi(u,v) \partial_x \big(\phi \times \prod_{1 \leq i \leq N} w_\epsilon(x - h_i(t))\big) \leq 0.$$

Note that due to the choice of testfunction, we can not see the interfaces and the method of Kruzkov works classically. In order to generalize our testfunction again, we have to consider the limit $\epsilon \to 0$. Using chain rule, we obtain

$$\Leftrightarrow \int_{\mathbb{R}^+ \times \mathbb{R}} |u-v| \left(\partial_t \phi \sum_{i=1}^N w_\epsilon(x - h_i(t)) + \phi \sum_{i=1}^N \prod_{1 \leq j \neq i \leq N} \partial_t w_\epsilon(x - h_i(t)) w_\epsilon(x - h_j(t)) \right)$$
$$+ \int_{\mathbb{R}^+ \times \mathbb{R}} \Phi(u,v) \left(\partial_x \phi \sum_{i=1}^N w_\epsilon(x - h_i(t)) + \phi \sum_{i=1}^N \prod_{1 \leq j \neq i \leq N} \partial_x w_\epsilon(x - h_i(t)) w_\epsilon(x - h_j(t)) \right)$$
$$+ \int_{\mathbb{R}} |u_0 - v_0| \big(\phi(0,x) \times \prod_{1 \leq i \leq N} w_\epsilon(x - h_i(0)\big) dx \quad \leq 0.$$

Recognizing that

$$\partial_t w_\epsilon(x - h_i(t)) = w'_\epsilon(x - h_i(t))(-h'_i(t))$$
$$\partial_x w_\epsilon(x - h_i(t)) = w'_\epsilon(x - h_i(t))$$

gives

$$\int_\Omega |u-v| \left(\partial_t\phi \prod_{1\le i\le N} w_\epsilon(x-h_i(t)) + \phi \sum_{i=1}^N \prod_{1\le j\ne i\le N} (-h_i'(t)) w_\epsilon'(x-h_i(t)) w_\epsilon(x-h_j(t)) \right)$$
$$+ \int_\Omega \Phi(u,v) \left(\partial_x\phi \prod_{1\le i\le N} w_\epsilon(x-h_i(t)) + \phi \sum_{i=1}^N \prod_{1\le j\ne i\le N} w_\epsilon'(x-h_i(t)) w_\epsilon(x-h_j(t)) \right)$$
$$+ \int_{\mathbb{R}} |u_0-v_0| \big(\phi(0,x) \times \prod_{1\le i\le N} w_\epsilon(x-h_i(0)\big) dx \quad \le \; 0.$$

Now we use the specific form of the derivative of w_ϵ, namely $w_\epsilon'(x-h_i(t)) = -\frac{2}{\epsilon}\mathbf{1}_{[h_i-\epsilon, h_i-\frac{\epsilon}{2}]} + \frac{2}{\epsilon}\mathbf{1}_{[h_i+\frac{\epsilon}{2}, h_i+\epsilon]}$ and obtain

$$\begin{aligned}
\Leftrightarrow \; & \int_{\mathbb{R}^+\times\mathbb{R}} |u-v| \partial_t\phi \prod_{1\le i\le N} w_\epsilon(x-h_i(t)) \\
& + \int_{\mathbb{R}^+} \frac{2}{\epsilon}(-h_i'(t)) \sum_{i=1}^N \int_{h_i-\epsilon}^{h_i-\frac{\epsilon}{2}} |u-v| \left(\phi \prod_{1\le j\ne i\le N} w_\epsilon(x-h_j(t)) \right) \\
& - \int_{\mathbb{R}^+} \frac{2}{\epsilon}(-h_i'(t)) \sum_{i=1}^N \int_{h_i+\frac{\epsilon}{2}}^{h_i+\epsilon} |u-v| \left(\phi \prod_{1\le j\ne i\le N} w_\epsilon(x-h_j(t)) \right) \\
& + \int_{\mathbb{R}^+\times\mathbb{R}} \Phi(u,v) \partial_x\phi \prod_{1\le i\le N} w_\epsilon(x-h_i(t)) \\
& + \int_{\mathbb{R}^+} \frac{2}{\epsilon} \sum_{i=1}^N \int_{h_i-\epsilon}^{h_i-\frac{\epsilon}{2}} \Phi(u,v) \left(\phi \prod_{1\le j\ne i\le N} w_\epsilon(x-h_j(t)) \right) \\
& - \int_{\mathbb{R}^+} \frac{2}{\epsilon} \sum_{i=1}^N \int_{h_i+\frac{\epsilon}{2}}^{h_i+\epsilon} \Phi(u,v) \left(\phi \prod_{1\le j\ne i\le N} w_\epsilon(x-h_j(t)) \right) \\
& + \int_{\mathbb{R}} |u_0-v_0| \big(\phi(0,x) \times \prod_{1\le i\le N} w_\epsilon(x-h_i(0)\big) dx \qquad \le \; 0.
\end{aligned}$$

Letting $\epsilon \to 0$, recognizing

$$\lim_{\epsilon\to 0} w_\epsilon(x-h_i(t)) = \mathbf{1}_{\mathbb{R}}$$

reincorporates the interfaces created by the particles and the related terms. Making use of the traces $\gamma_i^{\pm}(u), \gamma_i^{\pm}(v)$ respectivly at the position of the interfaces $h_i(t)$, we obtain

$$
\begin{aligned}
&\Leftrightarrow \int_{\mathbb{R}^+\times\mathbb{R}} |u-v|\partial_t\phi + \int_{\mathbb{R}^+\times\mathbb{R}} \Phi(u,v)\partial_x\phi \\
&+\sum_{i=1}^{N}\int_{\mathbb{R}^+} h_i'|\gamma^-(u(h_i,t)) - \gamma^-(v(h_i,t))|\phi(h_i,t) - h_i'|\gamma^+(u(h_i,t)) - \gamma^+(v(h_i,t))|\phi(h_i,t) \\
&-\sum_{i=1}^{N}\int_{\mathbb{R}^+} \Phi(\gamma_i^-(u),\gamma_i^-(v))\phi(h_i,t) + \sum_{i=1}^{N}\int_{\mathbb{R}^+} \Phi(\gamma_i^+(u),\gamma_i^+(v))\phi(h_i,t) \quad \leq 0
\end{aligned}
$$

and therefore, using the definition of the entropy flux $\overline{\Phi}$

$$
\begin{aligned}
&\int_\Omega |u-v|\partial_t\phi + \Phi(u,v)\partial_x\phi\,dx\,dt + \int_{\mathbb{R}} |u_0-v_0|\phi(0,x)\,dx \\
&\quad\leq \quad \sum_{i=1}^{N}\int_0^T \left(\overline{\Phi}(h_i',\gamma_i^-(u),\gamma_i^-(v))\phi(h_i(s),s) - \overline{\Phi}(h_i',\gamma_i^+(u),\gamma_i^+(v))\phi(h_i(s),s)\right)\,ds.
\end{aligned}
$$

Using the dissipativity of the germs for each particle, given by Proposition 3.1.4, we get the good signs of the right-side terms of the last inequality, which we then can drop to obtain the Kato inequality, which classically gives uniqueness of entropy solutions. Furthermore, integrating along the cone $C := \{(x,t), |x| = R + L(T-t), t \in [0,T]\}$ gives the L^1-contraction property.

4. Discussion and Conclusions

This thesis considered a model for interaction between a fluid, represented by the Burgers equation, and an arbitrary, finite number of particles, which act as point-wise drag forces on the fluid and manifest in multiple singular source terms.
The model featured the new aspects of particle interactions, where the interfaces created by the particles were allowed to interfere, merge or split, the consideration of entropy admissible weak solutions and the interaction of particles and shockwaves. To this goal, the theory of interface admissibility for conservation laws, with examples like discontinuous or restrained flux function problems and the conservative or non-conservative coupling of regions dominated by fluid equations has been revisited. The theory on entropy admissibility, which has developed multiple approaches over the last decade, like extending the Kruzkov formulation and corresponding entropy inequality, deriving a kinetic formulation or studying (A, B) connections, summarized by the theory of germs, is studied and compared for their use in the analysis of the particle model.
Well-posedness for the Riemann problem is proven, including the computation of exact solutions for the case of multiple particles drifting away from the origin.
A definition of entropy solutions was derived, using and extending the notion of germs to generalized germs, which give dissipative behavior across a domain influenced by multiple particles.
For the Cauchy problem, existence was proven using an explicit, time-stepping construction method and the existence results from the model with a single particle. An L^∞ bound on weak entropy solutions was proven and the method was adapted to prove existence in the case of merging and splitting of particles. The latter needed an approximation on the particle paths and therefore compactness arguments, which were derived using tools from functional analysis and the BV_{loc} control property of the Burgers equation with inital data with bounded total variation.
Uniqueness was proven using the dissipativity of the germs derived for the behavior of the fluid at the position of the traveling particle interfaces.
A couple of conclusions can be drawn besides the results of well-posedness of the extended fluid-particle interaction model. The treatment of basically regularizing the particles to handle the non-conservative product in the source term might be a first idea to extend the model even further to the case of particles with a given volume.
Although it was not possible to prove a global bound in total variation, as the property was lost for the case of merging particles, and it is not clear if it is even possible at all, as waves might be reflected back and forth between the particles, the nature of the Burgers equation used to derive the compactness for the case of splitting particles nourishes the hope that such a bound might indeed exist. Even though it was not the topic of this work, there is almost no numerical result for the model with multiple particles, but simulations for two particles using a wave-front tracking algorithm have so far not brought up any case of divergent total variation.

The model presented another example of the excellent properties of the theory of germs, and demonstrated the impressive possibility to extend the theory to the interesting cases of non-conservative coupling and even coupling across domains.
The existence result might be possible to alternatively be obtained by using the numerical approach of wave-front tracking, considering the method has proven to be very useful for the study of shocks and moving interfaces.
There remain some interesting directions to go further. For the theory with a single particle, it was possible to do numerics on a coupling with an ordinary differential equation for the particle, which was not yet done for the case of multiple particles, and is probably not an easy task, as the analysis there did not yet consider traveling interfaces and is therefore probably hard to adapt.
In the spirit of extensive analysis on the particles, it might be interesting to consider the limit $N(T) \to \infty$, using a Vlasov-type equation for the particles. There exist some considerations of coupling the Vlasov equation to a fluid equation, see [16, 27], but one would probably need to be very careful with the limit to not loose the well-posedness results for entropy solutions in this model.
Finally it would of course be of great interest to consider systems of balance laws to represent the fluid, like the Euler equations, however, as mentioned before, the notion of entropy admissibility is an unsolved problem for those systems in higher dimensions. As recent developments by De Lellis, Székelyhidi and Isett [35, 51] raise the question if it is even possible to find a suitable condition to obtain uniqueness or if all the preceding analysis was maybe done in the wrong function space, it seems unlikely that it will be possible to extend the model in that direction.

A. Appendix

A.1. Notation

Dealing with partial differential equations, and in particular conservation laws, this thesis uses the customary notation in the field and additionally

u, v, w	unknowns of the (hyperbolic) differential equation at hand
F, f	flux functions of the (hyperbolic) differential equation at hand
∂_t, ∂_x	partial differential operator with respect to time, space
ϕ, ψ, ω	testfunctions, i.e. functions from C_0^∞
C, c, K, k, κ	constants
$\eta, \Phi, \bar{\Phi}$	entropy, entropy fluxes
γ^-, γ^+	left and right trace towards an interface
χ	the special χ-function related to kinetic formulation
m, q	kinetic entropy defect measures
ρ, ν	measures
$h_i(t)$	the given position of the i-th particle at time t
λ_i	the friction constant corresponding to the i-th particle
$N(t)$	arbitrary but finite number of particles at time t
supp g	support of a given function g
sgn g	sign function of a given function g
TV g	total variation a given function g
$(\,\cdot\,)^\pm$	positive/negative part of the argument

The notion of cut-off function is also used several times throughout this work. Whenever it is referred to that notion, a function of the following properties is meant.

Definition A.1.1 (Cut-off function).
ζ is called a cut-off function, if

1. $\zeta \in C_0^\infty(\mathbb{R}^d)$,
2. *ζ is nonnegative and its support is contained in the unit ball,*
3. $\int_{\mathbb{R}^d} \zeta(x)dx = 1$,
4. $\zeta(-x) = \zeta(x)$.

It will often be used to recover the interface condition for a given equation, in which case the limit of

$$\zeta_\epsilon(x) = \frac{1}{\epsilon^d}\zeta(\frac{x}{\epsilon})$$

is used.

A.2. Study of the interface term for discontinuous flux Kruzkov entropies

This section contains the case-by-case study with respect to $\kappa, \gamma^{\pm}(u)$ and $\gamma^{\pm}(v)$. It is done in a similar manner as the study of the interface term for a slightly different model equation in [13].
Recall that the inequality to prove is (2.11)

$$0 \;\le\; k_L\Phi\big(\gamma^-(u), \gamma^-(v)\big) - k_R\Phi\big(\gamma^+(u), \gamma^+(v)\big) \;=:\; I \tag{2.11}$$

Proof. Studying the original entropy inequality already gave

$$k_L\Phi\big(\gamma^-(u), \gamma^-(v)\big) - k_R\Phi\big(\gamma^+(u), \gamma^+(v)\big) - |k_L - k_R| f(\kappa) \;\le\; 0. \tag{2.12}$$

The case-by-case study is partitioned by the sign inside the entropy flux terms in (2.11).

1. Assume $\operatorname{sgn}(\gamma^-(u)(t) - \gamma^-(v)(t)) = \operatorname{sgn}(\gamma^+(u)(t) - \gamma^+(v)(t)) = c$, such that

$$\begin{aligned} &k_L\Phi(\gamma^-(u)(t), \gamma^-(v)(t)) - k_r q(\gamma^+(u)(t), \gamma^+(v)(t)) \\ &= c[k_L f(\gamma^-(u)(t) - k_L f(\gamma^-(v)(t)] - c[k_R f(\gamma^+(u)(t) - k_R f(\gamma^+(v)(t)] \\ &= 0 = I \;\text{ because of the Rankine-Hugoniot condition (1.4).} \end{aligned}$$

2. Assume $\gamma^-(u) \ge \gamma^-(v)$ and $\gamma^+(u) < \gamma^+(v)$.

   By the Rankine-Hugoniot condition (1.4) for u and v respectivly, the following holds true in this case

$$2k_L(f(\gamma^-(u)) - f(\gamma^-(v))) = 2k_R(f(\gamma^+(u)) - f(\gamma^+(v))).$$

   such that

   - $\gamma^+(v) \le \gamma^-(u)$
     means $\gamma^+(u) < \gamma^+(v) \le \gamma^-(u)$, and by choosing $\kappa = \gamma^+(u)$ in (2.12) we get

$$2k_R f(\gamma^+(u)) - 2k_R f(\gamma^+(u)) \ge 0 \to \; I \;\ge\; 0.$$

- $\gamma^+(v) > \gamma^-(u)$
  means $\gamma^-(v) \leq \gamma^-(u) < \gamma^+(v)$ and choosing $\kappa = \gamma^-(u)$ in (2.12) gives

$$2k_L f(\gamma^-(u)) - 2k_L f(\gamma^-(v)) \geq 0 \rightarrow \ I \ \geq \ 0.$$

3. Assume $\gamma^-(u) < \gamma^-(v)$ and $\gamma^+(u) \geq \gamma^+(v)$

   By the Rankine-Hugoniot condition (1.4) for u and v respectivly, the following holds true in this case

$$2k_L(f(\gamma^-(v)) - f(\gamma^-(u))) = 2k_R(f(\gamma^+(u)) - f(\gamma^+(v))).$$

   such that

   - $\gamma^-(v) \leq \gamma^+(u)$
     means $\gamma^-(u) < \gamma^-(v) \leq \gamma^+(u)$, and by choosing $\kappa = \gamma^-(v)$ in (2.12) we get

$$2k_L f(\gamma^-(v)) - 2k_L f(\gamma^-(u)) \geq 0 \rightarrow \ I \ \geq \ 0.$$

   - $\gamma^-(v) > \gamma^+(u)$
     means $\gamma^+(v) \leq \gamma^+(u) < \gamma^-(v)$ and choosing $\kappa = \gamma^+(u)$ in (2.12) gives

$$2k_R f(\gamma^+(v)) - 2k_R f(\gamma^+(u)) \geq 0 \rightarrow \ I \ \geq \ 0.$$

This completes the proof of (2.11). □

A.3. Kinetic indicator function and defect measure properties

The function χ used as an indicator function in the kinetic formulation operates on the new variable ξ and uses the original solution $u(t,x)$ as a parameter.For any $\xi \in \mathbb{R}, u \in \mathbb{R}$ it is defined as

$$\chi(\xi; u) = \begin{cases} 1, & \text{for } 0 < \xi < u, \\ -1, & \text{for } u < \xi < 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following useful properties of χ are proven in this section. In fact, there are a lot more interesting properties of χ and the function is originally found as a description for the weak limit of a family $S(u_n)$ of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in L^∞. But as the focus of this work is on uniqueness of weak entropy solutions to conservation laws, more details on the function χ would not serve the purpose of this work and I want to recommend the interested reader the book of Perthame [68] for further information and details.

Theorem A.3.1 **(Main properties of χ).**
Let $S' \in L^\infty_{loc}$, then the following properties related to χ hold true.

$$\partial_\xi \chi(\xi; u) = \delta(\xi) - \delta(\xi - u) \tag{A.1}$$

$$\int_{\mathbb{R}} S'(\xi)\chi(\xi; u)\ d\xi = S(u) - S(0) \tag{A.2}$$

$$\int_{\mathbb{R}} |\chi(\xi; u) - \chi(\xi; v)|\ d\xi = |u - v| \tag{A.3}$$

Proof. In view of the fact that χ takes value 1 in $0 < \xi < u$, -1 in $u < \xi < 0$ for positive, negative u respectively and is zero elsewhere, the first and third property are obviously true by simple computation. The second one also follows by investigating the taken values inside the integral. Because of the symmetry, consideration of the case $u \geq v$ is sufficient. Then $\chi(\xi; u) - \chi(\xi; v)$ takes value 1 in $v < \xi < u$ and zero otherwise, justifying the second property by integration. ☐

The kinetic entropy defect measure $m(t, x, \xi)$

The nonnegative measure $m(t, x, \xi)$ on the right hand side is the second important part of the kinetic formulation. As one can see in the derivation of the kinetic formulation in the next section, the kinetic formulation already contains the entropy condition and thus chooses the physically admissible solution without further adjustments. In fact, the entropy condition is encoded within the positivity of the kinetic entropy defect measure, which is seen by investigating the behavior of m in the presence of shock waves, compare section 3.3 of the book by Perthame [68].
We assume convexity on the flux, which is classically and reflects the physical problems represented by conservation laws such as traffic models. Therefore we already know $f'(\cdot)$ is increasing and from the Rankine-Hugoniot condition (1.4), that for shockwaves of speed s

$$s = \frac{f(u_R) - f(u_L)}{u_R - u_L} = f(\xi_0) \qquad \text{for some } \xi_0 \in [u_R, u_L].$$

The last equality being true due to Rolle's lemma. Computing the kinetic formulation near the jump of the shock, one ends up with a formula for the kinetic entropy defect measure

$$\frac{\partial}{\partial t}\chi(\xi, u(t,x)) + f'(\xi) \cdot \nabla_x \chi(\xi, u(t,x)) = [a(\xi) - s][\chi(\xi; u_r) - \chi(\xi; u_l)]\delta(x - st).$$

Integrating over the right side with respect to ξ and using the definition of the kinetic entropy defect measure, i.e. $\partial_\xi m = \partial_t \chi(\xi, u) + f'(\xi) \cdot \partial_x \chi(\xi, u)$, one obtains

$$m(t, x, \xi) = c(\xi)\delta(x - st)\text{sgn}(u_r - u_l). \tag{A.4}$$

with

$$c(\xi)\begin{cases} = 0, & \text{for } \xi \leq \min(u_r, u_l) =: u_m, \\ = 0, & \text{for } \xi \geq \max(u_r, u_l) =: u_M, \\ < 0 & \text{otherwise.} \end{cases}$$

It follows for $u_m \leq \xi \leq u_M$

$$c(\xi) = \int_{u_m}^{\xi} (f'(\zeta) - s) d\zeta = f(\xi) - f(u_m) - s(\xi - u_m).$$

Observe, that for $u_m \leq \xi \leq \xi_0$, $c(\xi)$ is decresing from zero to a negative value and increasing back to value zero for $\xi_0 \leq \xi \leq u_M$. Therefore $c(\xi)$ is nonpositive and to ensure the demanded positivity of the kinetic entropy defect measure, compare (A.4), it has to be $u_l < u_r$. This already excludes nonphysical rarefaction shocks, and therefore the kinetic formulation does not need an additional entropy condition for shockwaves rising from the evolution of the solution, in contrary to the Kruzkov formulation.

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