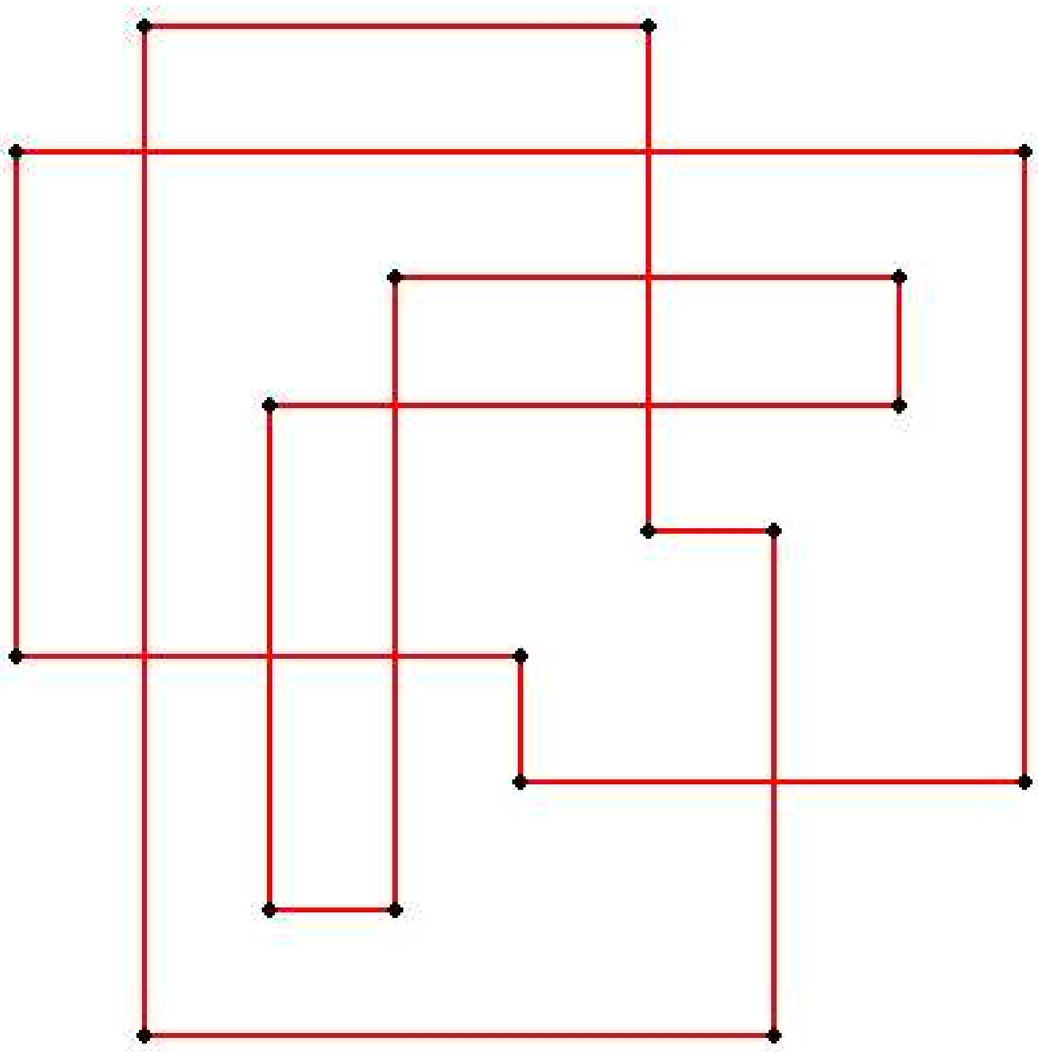

{0,1}-Matrices with Rectangular Rule

Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades der Julius–Maximilians–Universität Würzburg, vorgelegt von Thomas Gregor aus Erlenbach bei Marktheidenfeld.

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Würzburg 2008

Eingereicht am: 24.04.2008
bei der Fakultät für Mathematik und Informatik
1. Gutachter: Prof. Dr. O. Mutzbauer
2. Gutachter: Prof. Dr. P. Müller

Für meine Eltern,

Karl und Helga Gregor

Danksagung

Mein besonderer Dank gilt Prof. Otto Mutzbauer für seine freundliche und stets hilfreiche, sowie unermüdliche Unterstützung während meiner Promotion. Herzlichen Dank für den Vorschlag dieses interessanten Themas und die vielen ergiebigen Gespräche.
Weiterhin danke ich Prof. Theo Grundhöfer für die Beschäftigung als wissenschaftlicher Mitarbeiter und Prof. Peter Müller für die freundliche Aufnahme am Lehrstuhl Algebra recht herzlich.
Außerdem danke ich meinen Eltern Helga und Karl, die immer an mich glaubten. Ich danke meiner Schwester Eva und meiner lieben Freundin Stefanie, die mir immer zur Seite standen, wenn mir der Mut sank. Ich danke allen Freunden und Verwandten, die mich unterstützt haben, sei es durch aktive Beratung, durch Rücksichtnahme und Verständnis, durch aufmunternde Worte oder einfach durch gemeinsam verbrachte Zeit. Danke.

Thomas Gregor

Contents

1 Introduction

One of the famous problems in finite geometries and combinatorics is the question, whether or not a projective plane of a given order exists and how many different planes there are up to isomorphism.
It is well known, that projective planes of order n exist, if n is a prime power, cf. [9]. Moreover, there is the important result of Richard H. Bruck and Herbert J. Ryser [5] in 1949, that shows the nonexistence of projective planes of certain orders. In 1991, Galina Kolesova, Clement W. H. Lam [12] and Larry Thiel published a computational proof that there are only the known four planes of order 9. This result has been confirmed 2004 by Helmut Kramer [11]. Finally, the nonexistence of a projective plane of order 10 has been shown also by Lam, Thiel and Stanley Swiercz [13], [14] in 1989. The results of Lam and Kramer have been achieved by extensive use of numerical computations.
As finite projective planes are finite incidence structures, they can be identified with their incidence matrices. The main properties of the incidence matrices of projective planes are the constant line sum and the rectangular rule, where the rectangular rule is a translation of the geometrical condition

"two lines meet in at most one point"

into the language of $\{0, 1\}$-matrices. Those properties characterize some special combinatorial designs, the symmetric configurations.
Configurations are introduced in the paper of Harald Gropp [7], that gives a historical overview concerning the theory of configurations. In [6], Gropp defines symmetric configurations and gives an overview of existence results for symmetric configurations, as well as in [8]. In fact, the finite projective planes are the symmetric configurations where any two lines meet in exactly one point.
Another famous combinatorial challenge is the search for mutually orthogonal Latin squares, cf. Brendan D. McKay, Alison Meynert and Wendy Myrvold [16]. Ray Chandra Bose [3] has shown in 1938, that the existence of $n-1$ mutually orthogonal Latin squares of order n is equivalent to the existence of a projective plane of order n, compare also [18, Chapter 7, Theorem 4.1]. This result together with the confirmation of Eulers conjecture, that there are not even two orthogonal Latin squares of order 6, given by Gaston Tarry [19] around 1900, proves the nonexistence of a projective plane of

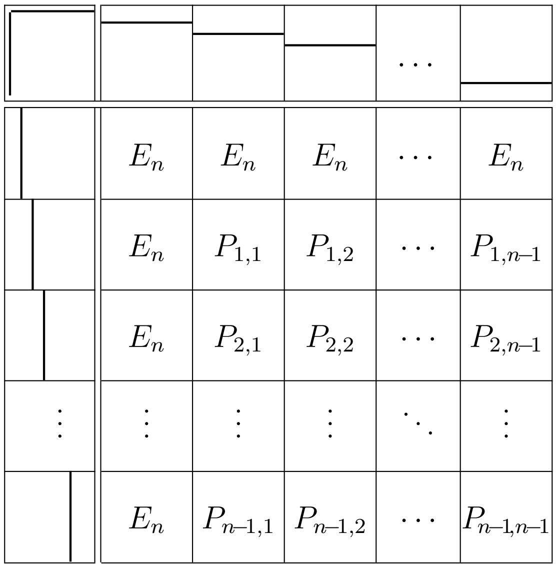

Figure 1: Doubly ordered incidence matrix of a finite projective plane

order 6. This follows also from the later result of Bruck and Ryser mentioned above. Kramer gives a translation of mutually orthogonal Latin squares into $\{0,1\}$-matrices with rectangular rule in [11, Corollary 4.4].

Summarized, there are several combinatorial objects that can be interpreted as $\{0,1\}$-matrices with rectangular rule, as there are projective planes, configuations, mutually orthogonal Latin squares and others, we did not mention yet, as for example incidence matrices of graphs.

Applying row and column permutations to incidence matrices preserve the combinatorial structure, i.e., the corresponding incidence structures are isomorphic. Therefore, we may doubly order incidence matrices of projective planes and obtain a special block structure, cf. Figure 1. The double ordering has been invented in 2001, by Adolf Mader and Otto Mutzbauer [15]. In particular, such line permutations lead to matrices with a completely determined margin, the step margin. Hence, we may omit this step margin and obtain a smaller matrix, the so called step core. This process is called peeling and has been studied by Patrick Pechmann [17] in the case of incidence matrices of finite projective planes as an application of double ordering.

Here, we study the peeling of matrices, i.e., their decomposition into step margin and step core. We will iterate the peeling of matrices and ask for the reversibility of this process.

For this, we study the rectangular rule in Section 3 and give a characterization of the rectangular rule with so called bundle structures of a matrix. They turn out to be kind of a generalization of the rectangular rule. Moreover, we determine the properties of matrices that allow peeling. Finally, Theorem 3.10 implies, that iterated peeling is possible for matrices with rectangular rule.

Section 4 is dedicated to the iterated peeling of incidence matrices of projective planes that leads to a series of special cores. Those cores of incidence matrices of projective planes have some significant properties that are collected in the term regular block matrices, cf. Section 5.

In Section 6, we introduce homogeneous configurations and Latin configurations as a specialization of symmetric configurations. Moreover, we will see, that regular block matrices are in particular (r, n)-matrices. It turns out, that (r, n)-matrices are those matrices, that are permutation equivalent to the so called step hull of a smaller regular block matrix. Finally, we introduce a block switch, that changes the block structure of block matrices. Using the block switch, we obtain a deep connection between regular block matrices and Latin configurations in Theorem 6.15. In the end, Theorem 6.20 gives a criterion, which (r, n)-matrices are permutation equivalent to regular block matrices.

Then, we ask for the reversibility of peeling regular block matrices, in Section 7. Regular block matrices, that allow such a reverse peeling are called regular envelopable. Theorem 7.9 characterizes those regular envelopable block matrices. After a short excursion to the theory of bipartite graphs and the definition of complementary indication matrices, we are able to study the iteration of the reverse peeling of regular block matrices and obtain a characterization of k-times regular envelopable block matrices in Theorem 7.20. Finally, Theorem 7.21 denotes the interesting fact, that the complementary indication matrices of k-regular envelopable block matrices are also Latin configurations.

A few further results concerning the theory of reverse peeling are given in Section 8, including the existence of some series of configurations, cf. Section 8.2.

2 Preliminaries

Before we start with some basic definitions, we have to mention a few general agreements.
The unit matrix of size n is always denoted by E_n or briefly E. In the figures with block matrices a permutation matrix is always denoted by $P_{i,j}$ and a pseudo permutation matrix is always denoted by $S_{i,j}$.
Moreover, we will sometimes speak of a "dual statement" in the context of matrices. It is the statement where the words "row" and "column" are interchanged.

2.1 Matrices and margins

For positive integers $a < b$ we denote $[a, b] := \{a, a+1, a+2, \ldots, b\}$ and $[a, a] = \{a\}$. Let $m, n \in \mathbb{N}$ and let R, C be totally ordered sets, where $|R| = m$ and $|C| = n$ and let S be any non-empty set. Usually, a *matrix* M *of size* $m \times n$ is defined as a mapping

$$\begin{array}{rccc} M: & R \times C & \longrightarrow & S \\ & (r, c) & \mapsto & M(r, c) \end{array}$$

In particular, a matrix is a $\{0,1\}$-*matrix*, if $S = \{0,1\}$. For $R' \subseteq R$ and $C' \subseteq C$, the matrix

$$\begin{array}{rccc} M': & R' \times C' & \longrightarrow & S \\ & (r', c') & \mapsto & M(r', c') \end{array}$$

or equivalently $M' := M|_{R' \times C'}$ is a *submatrix* of M and we briefly write $M' = M(R', C')$. Clearly, a *row of* M is a submatrix of M where $|R'| = 1$ and $C' = C$ and a *column of* M is a submatrix of M where $|C'| = 1$ and $R' = R$.
Let $R_1, R_2 \subseteq R$, $C_1, C_2 \subseteq C$ and $M_1 = M(R_1, C_1), M_2 = M(R_2, C_2)$ be submatrices of M. The matrix $M(R_1 \cap R_2, C_1 \cap C_2)$ is called the *intersection of* M_1 *and* M_2.
By the cardinality of R, there is an order isomorphism $\phi : R \rightarrow [1, m]$ and similarly there is an order isomorphism $\psi : C \rightarrow [1, n]$. Thus, we tacitly assume for a matrix $M : R \times C \rightarrow S$, that $R = [1, m], C = [1, n]$ for some $m, n \in \mathbb{N}$.
The element $M(r, c) \in S$ is called *entry* of M at *position* (r, c).

Let $\mathcal{M}$ be the set of $m \times n$ matrices with entries from S. Then, the symmetric group $\mathcal{S}_m$ acts on $\mathcal{M}$, by the mapping

$$\begin{array}{ccc} \mathcal{S}_m \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ (\pi, (M(i,j))_{i,j}) & \mapsto & (M(\pi(i), j))_{i,j} \end{array}$$

and for fixed $\pi \in \mathcal{S}_m$ we call the mapping

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M} \\ (M(i,j))_{i,j} & \mapsto & (M(\pi(i), j))_{i,j} \end{array}$$

a *row permutation.* We often identify π with this mapping by saying briefly π is a row permutation. A permutation π has a so called *representing matrix*, which is denoted by $P(i, \pi(i)) = 1$ and all other entries are equal to 0. Then, a matrix description of a row permutation π is left multiplication with the representing matrix P^{-1} of π^{-1}. Analogously, $\mathcal{S}_n$ acts on $\mathcal{M}$, by the mapping

$$\begin{array}{ccc} \mathcal{S}_n \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ (\psi, (M(i,j))_{i,j}) & \mapsto & (M(i, \psi(j)))_{i,j} \end{array}$$

and for fixed $\psi \in \mathcal{S}_n$ we call the mapping

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M} \\ (M(i,j))_{i,j} & \mapsto & (M(i, \psi(j))_{i,j} \end{array}$$

a *column permutation.* A matrix description of a column permutation ψ is right multiplication with the representing matrix of ψ. Finally, the group $\mathcal{S}_m \times \mathcal{S}_n$ acts on $\mathcal{M}$ by combined row and column permutations and we say $M_1, M_2 \in \mathcal{M}$ are *permutation equivalent*, if they are in the same orbit of this operation, i.e., there is $(\pi, \psi) \in \mathcal{S}_m \times \mathcal{S}_n$, where $M_2(i,j) = M_1(\pi(i), \psi(j))$ for all $(i,j) \in [1,m] \times [1,n]$. The matrix representation $M_2 = P^{-1} M_1 Q$ is not the usual one, but this deviation will be justified by Lemma 3.4.
Let M be a matrix of size $m \times n$ and let $m' > m,\ n' > n$. The matrix M can be extended to a matrix M' of size $m' \times n'$ in a natural way, by the following definition:

$$M'(i,j) = \begin{cases} M(i,j) & i \in [1,m] \wedge j \in [1,n] \\ 0 & \text{otherwise} \end{cases}$$

If we add matrices of different formats, we tacitly assume that both matrices are extended to a minimal common format. Thus, for a matrix M of size $m' \times n'$ and $m < m',\ n < n'$, the difference $\widetilde{M} = M - M([1,m],[1,n])$ is of size $m' \times n'$ and the submatrix $\widetilde{M}([1,m],[1,n])$ is the 0-matrix of size $m \times n$. We call $\widetilde{M}$ the $(m'-m) \times (n'-n)$ *margin of* M and $M([1,m],[1,n])$ the $m \times n$ *core of* M. Moreover, the submatrix $\widetilde{M}([m+1,m'],[n+1,n']) = M([m+1,m'],[n+1,n'])$ is called the $(m'-m) \times (n'-n)$-*head* or briefly the *head* of $\widetilde{M}$ or M.

2.2 Block matrices

Now let M be a matrix of size $m \times n$ and let a and b be *subdivisions* of the rows and columns of M, respectively, i.e., $a = (a_1, a_2, \dots, a_k)$ and $b = (b_1, b_2, \dots, b_l)$ where all a_i and b_j are positive integers with $1 \leq a_1 < a_2 < \dots < a_k = m$ and $1 \leq b_1 < b_2 < \dots < b_l = n$. By convention, let $a_0 = b_0 = 0$. We call (M, a, b) a *matrix with block structure* and say briefly M is a *block matrix* if the corresponding subdivisions are fixed. Moreover, we will briefly write $(M, a) := M(a, a)$ for square block matrices with equal subdivisions of the rows and of the columns. The submatrix $M([a_i+1, a_{i+1}],[1,n])$ is called a *block row* for any $i \in [0, k-1]$, the submatrix $M([1,n],[b_j+1, b_{j+1}])$ is called a *block column* for any $j \in [0, l-1]$ and the intersection of a block row and a block column is called a *block*. For $r \in [a_i+1, a_{i+1}]$, the row $g = M(r,[1,n])$ is a submatrix of the block row $M([a_i+1, a_{i+1}],[1,n])$. We call r the *absolute index of* g. Moreover, we call $r - a_i$ the *internal index of* g and $i+1$ the *block row index of* g. Analogously, we define the absolute index, the internal index and the block column index of a column. Since the block row indices of all rows in a block row are the same, we will sometimes use the formulation *block row index of a block row.* Furthermore, if a row g is a submatrix of a block row B, we say g *is in* B.

We specialize the term margin for block matrices. The margin of a block matrix (M, a, b) is the matrix $M - M([1, a_{k-1}],[1, b_{l-1}])$. A margin is always interpreted to have the same block structure as the original block matrix, i.e., $(M - M([1, a_{k-1}],[1, b_{l-1}]), a, b)$. The head of M is the matrix $M([a_{k-1}+1, a_k],[b_{l-1}+1, b_l])$. The block row and the block column containing the head are called *margin row* and *margin column*, respectively.

2.3 Partitions and incidence structures

Let $K = \{R_i \mid i \in [1,k]\}$ be a partition of a set X into k parts and $\pi \in \mathcal{S}(X)$ a permutation of X. We define $R_i^\pi := \{\pi(r) \mid r \in R_i\}$ for all $i \in [1,k]$ and denote the whole partition by $K^\pi := \{R_i^\pi \mid i \in [1,k]\}$. If the permutation π is fixed we use the term *corresponding partitions*. Moreover, let $S \subset X$. The restricted partition $\{R_i \cap (X \setminus S) \mid i \in [1,k]\}$ is denoted by $K \setminus S$.

Lemma 2.1. *Let $K = \{R_i \mid i \in [1,k]\}$ be a partition of $[1,n]$. There is a permutation $\pi \in \mathcal{S}_n$, such that*

$$K^\pi = \left\{ \Big[1, |R_1|\Big], \Big[|R_1|+1, |R_1|+|R_2|\Big], \ldots, \left[(\sum_{i=1}^{k-1} |R_i|)+1, \sum_{i=1}^{k} |R_i|\right] \right\}$$

and $\pi|_{R_i}$ is an order isomorphism.

Proof. We denote the elements of a part of K in the following way: $R_i = \{r_{i,j} | j \in [1, |R_i|]\}$, where $r_{i,1} < r_{i,2} < \cdots < r_{i,|R_i|}$. Moreover, let by convention be $|R_0| = 0$. The permutation $r_{i,j} \mapsto (\sum_{m=0}^{i-1} |R_m|) + j$ does the job. $\square$

Note, that K^π induces a subdivision $(|R_1|, |R_1|+|R_2|, \ldots, \sum_{i=1}^{k} |R_i|)$ of the set $[1,n]$.

Two partitions $R_1 = \{R_{1,i} \mid i \in [1,m]\}$ and $R_2 = \{R_{2,j} \mid j \in [1,n]\}$ of a set are called *orthogonal*, if $|R_{1,i} \cap R_{2,j}| \leq 1$ for all $(i,j) \in [1,m] \times [1,n]$, i.e., if there is no pair of elements, which is in a part of R_1 and in a part of R_2.

A (finite) *incidence structure* is a triplet $(P, \mathcal{G}, I)$, where P and $\mathcal{G}$ are disjoint finite non-empty sets and $I \subseteq P \times \mathcal{G}$ is a relation. We say $p \in P$ and $G \in \mathcal{G}$ are *connected*, if $(p, G) \in I$, and otherwise they are *not connected*.

Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ and $P = \{p_1, \ldots, p_n\}$. The matrix $I = (a_{i,j})_{i \in [1,m], j \in [1,n]}$, where

$$a_{i,j} = \begin{cases} 1 & (p_i, G_j) \in I \\ 0 & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of the incidence structure $(P, \mathcal{G}, I)$[1]. Every $\{0,1\}$-matrix is the incidence matrix of a suitable incidence structure, cf. [18], pp.53-54. We will often identify an incidence structure with its incidence matrix.

[1] This is a slightly changed definition of the one given in [18], p.53

$$\begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix}$$

Figure 2: Forbidden pattern for the rectangular rule

3 Rectangular rule

We deal with $\{0,1\}$-matrices only. Rows and columns are *lines* and row- and column permutations are briefly called *line permutations*. A *point* on a row is a column c where the intersection of this row with c has the entry 1 and vice versa a point on a column is a row r where the intersection of this column with r has the entry 1. A matrix M satisfies the *rectangular rule*, if $M(i,j)M(i,k)M(l,j)M(l,k) = 0$ for all quadruples (i,j,k,l), $i \neq l$, $j \neq k$, or equivalently, if the scalar product of different rows or different columns is 1 or 0. We use the geometric language and express this shortly by saying that they *meet in some point*, if the scalar product is 1. We give another equivalent definition: A matrix satisfies the rectangular rule, if every submatrix of size 2×2 has at least one entry 0, i.e., the matrix does not contain a pattern corresponding to Figure 2.

Line permutations maintain the rectangular rule. Let P be a set of rows of a matrix and let Q be a set of columns. P and Q are said to be *not connected* if the submatrix formed by the intersection of P and Q is the 0-matrix, otherwise P and Q are called *connected*. In particular, a single row r and a single column c are connected, if the intersection of r and c has the entry 1. If r and c are connected, we say (r,c) is a *connected pair*.

A set of $\{0,1\}$-matrices $(M_i)_{i\in[1,k]}$ of equal size is called *cover free* or a *cover free collection*, if their sum over $\mathbb{Z}$ is again a $\{0,1\}$-matrix, or equivalently, if the matrix

$$\begin{pmatrix} E & E & \cdots & E \\ M_1 & M_2 & \cdots & M_k \end{pmatrix} \tag{3.1}$$

satisfies the rectangular rule. In particular, a square matrix M is called *fixed point free*, if it is cover free with the unit matrix E, or equivalently, if the matrix

$$\begin{pmatrix} E & E \\ E & M \end{pmatrix} \tag{3.2}$$

satisfies the rectangular rule. The unit matrix of a certain size is always

denoted by E_n or E. A first application of these terms is the following straightforward lemma.
A margin is called 1-*margin*, if the row sums in the margin column except of the head and the column sums in the margin row except of the head are all at most 1.

Lemma 3.1. *Each* 1*-margin satisfies the rectangular rule if and only if its head satisfies the rectangular rule.*

The next lemma follows for example from [11, Theorem 3.10] and is proven only for completeness.

Lemma 3.2. *A matrix M of size $m \times n$ satisfies the rectangular rule, if and only if M^t does.*

Proof. $M(a,b) = M^t(b,a)$ for all $a \in [1,m]$ and $b \in [1,n]$, hence,

$$M(a,b)M(a,c)M(d,b)M(d,c) = M^t(b,a)M^t(c,a)M^t(b,d)M^t(c,d)$$

for all $a, d \in [1,m]$ and $b, c \in [1,n]$. □

Lemma 3.3. *Let $2 \leq m, n$ and let $M = (M_{i,j})_{i\in[1,m],j\in[1,n]}$ be a block matrix. Then, M satisfies the rectangular rule, if and only if the submatrices*

$$\left(\begin{array}{c|c} M_{i,j} & M_{i,k} \\ \hline M_{l,j} & M_{l,k} \end{array}\right)$$

satisfy the rectangular rule for all $1 \leq i < l \leq m$ and for all $1 \leq j < k \leq n$.

Proof. The rectangular rule inherits to submatrices.
Conversely any four positions that form a rectangular, are covered by a submatrix of four blocks in the constellation above. □

Clearly, if a block matrix $M = (M_{1,j})_{j\in[1,n]}$ is a block row, M satisfies the rectangular rule, if and only if the submatrices $(M_{1,j}\ M_{1,k})$ satisfy the rectangular rule for all $j, k \in [1,n]$, $j \neq k$. The dual statement holds for a block column.

Lemma 3.4. *Let $\pi_{1,1}, \pi_{1,2}, \pi_{2,1}, \pi_{2,2} \in \mathcal{S}_n$ be permutations with representing matrices $P_{1,1}, P_{1,2}, P_{2,1}, P_{2,2}$, respectively. Moreover, let*

$$M = \left(\begin{array}{c|c} P_{1,1} & P_{1,2} \\ \hline P_{2,1} & P_{2,2} \end{array}\right).$$

Then, M satisfies the rectangular rule, if and only if

$$\forall r, s \in [1, n] : \pi_{1,1}(r) = \pi_{2,1}(s) \implies \pi_{1,2}(r) \neq \pi_{2,2}(s).$$

Proof. M is square of size $2n$. Observe, that for $i, j \in [1, n]$,

$$P_{1,1}(i,j) = M(i,j),\ P_{1,2}(i,j) = M(i,j+n),\ P_{2,1}(i,j) = M(i+n,j)$$

and $P_{2,2}(i,j) = M(i+n, j+n)$.
Let M satisfy the rectangular rule. Moreover, let $\pi_{1,1}(r) = \pi_{2,1}(s)$, i.e., $P_{1,1}(r, \pi_{1,1}(r)) = P_{2,1}(s, \pi_{1,1}(r)) = 1$. Then, $P_{2,2}(s, \pi_{1,2}(r)) = 0$, as

$$M(r, \pi_{1,1}(r))M(r, \pi_{1,2}(r)+n)M(s+n, \pi_{1,1}(r))M(s+n, \pi_{1,2}(r)+n) = 0,$$

by the rectangular rule. Consequently, it is $\pi_{2,2}(s) \neq \pi_{1,2}(r)$.
Conversely, assume, there is a quadruple (i, j, k, l), $i \neq l$ and $j \neq k$ such that $M(i,j)M(i,k)M(l,j)M(l,k) = 1$. Without loss of generality, let $i \leq n < l \leq 2n$ and $j \leq n < k \leq 2n$, as columns in the same block column of M as well as rows in the same block row of M do not meet, obviously. Thus, it follows

$$P_{1,1}(i,j) = P_{1,2}(i, k-n) = P_{2,1}(l-n, j) = P_{2,2}(l-n, k-n) = 1,$$

or equivalently

$$\pi_{1,1}(i) = j = \pi_{2,1}(l-n) \quad \text{and} \quad \pi_{1,2}(i) = k-n = \pi_{2,2}(l-n).$$

□

3.1 Peeling of matrices

A *row step matrix* has a row where all entries are equal to 1 and all other entries are equal to 0. The *shape* of a row step matrix is the index of the row containing the entries 1. Analogously, we define a *column step matrix* and its shape. The 0-matrix is also considered to be a step matrix with shape 0. A Γ*-matrix* has all entries in the last row and in the last column equal to 1 and otherwise 0.
We define some special margins of block matrices (M, a, b) with subdivisions $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_l)$. A margin $(\widetilde{M}, a, b)$ of a block matrix (M, a, b) is called a *step margin* if its head is a Γ-matrix and the remaining blocks in its margin row are row step matrices of pairwise different shape in the range of $[0, a_k - a_{k-1} - 1]$ and the remaining blocks in the block column are column step matrices of pairwise different shape in the range of $[0, b_l - b_{l-1} - 1]$, cf. the step margin of Figure 4 and of Figure 7.

Remark 3.5. Omitting the last row and the last column of a step margin leads to a 0-head. Thus, the remaining rows of the margin row are not connected with the remaining columns of the margin column.

A margin of a block matrix is called *permutation margin*, if all blocks in the margin row and in the margin column are permutation matrices and therefore they are in particular square and all of the same size. Furthermore, a permutation margin is called *unit margin*, if all blocks in the margin are unit matrices. For a matrix with unit margin, we call the corresponding core the *unit core*.
There are some straightforward consequences summarized in the following remark.

Remark 3.6. Note, that step margins and permutation margins are 1-margins. For a block matrix $(M, (a_1, \dots, a_k), (b_1, \dots, b_l))$ with step margin it follows, that $k \leq a_k - a_{k-1} + 1$ and $l \leq b_l - b_{l-1} + 1$, since the step matrices have pairwise different shape and their shapes range between 0 and $a_k - a_{k-1} - 1$ or $b_l - b_{l-1} - 1$, respectively. For example in Figure 7, $k = l$ are maximal. For a block matrix (M, a, b) of size $m \times n$ with permutation margin, it follows $a_i = ia_1$ and $b_j = ja_1$ for all $i \in [1, k]$ and all $j \in [1, l]$. In particular, $m = ka_1$ and $n = la_1$.

Let $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_l)$ and let (M, a, b) be a matrix with step margin $(\widetilde{M}, a, b)$. The submatrix $M' = M([1, a_{k-1}], [1, b_{l-1}])$ is called a *step core of M*. The decomposition of a block matrix with step margin into its step margin and its step core is called *peeling*. Obviously,

$$(M', (a_1, \dots, a_{k-1}), (b_1, \dots, b_{l-1}))$$

is a block matrix.
A matrix M *allows peeling*, if it is permutation equivalent to a block matrix with step margin, i.e., if there are permutation matrices P, Q and subdivisions a, b, such that (PMQ^{-1}, a, b) is a block matrix with step margin. Thus, a step margin induces a block structure for a matrix, since it fixes the subdivisions a and b. Note, that M can be permutation equivalent to several matrices with different step margins and different induced block structures.
Conversely, for a block matrix $(M, (a_1, \dots, a_k), (b_1, \dots, b_l))$, we can build a *step hull*, adding a step margin

$$(\widetilde{M}, (a_1, \dots, a_k, a_k + c), (b_1, \dots, b_l, b_l + d)),$$

where $c \geq k$ and $d \geq l$. More generally, we can build a *hull* of a block matrix M by adding a margin $\widetilde{M}$, that is not necessary a step margin, in the same way, provided, $\widetilde{M}$ respects the block structure of M.

3.2 Bundle structures

The collection of all rows of a matrix M containing an entry 1 in the column c is called *(row) bundle* with *origin c*. Thus, c is a point on each of these rows. Note, that each column of M is the origin of a row bundle and a row r is in the bundle with origin c, if and only if the intersection of r and c has the entry 1, i.e., if (r, c) is a connected pair. Dually, we speak of column bundles with a row as origin.

A collection of cover free rows (or columns) of a matrix is called a *parallel bundle*.

Note, that for a matrix M, bundles with origin and also parallel bundles are in particular submatrices of M. Conversely, a collection of rows is a bundle, if and only if there is a column c of this submatrix with exclusively entries 1 such that the column sum of c in the submatrix is the same as the column sum of c in M, i.e., all entries 1 of c are in the submatrix. The dual statement holds for a collection of columns. Moreover, a collection of rows is a parallel bundle, if and only if each column of this submatrix has at most one entry 1 and dually for a collection of columns.

Let g be a row of a matrix M and B be a bundle with g as element. We call $B \setminus \{g\}$ a *g-bundle*. Let $p_1, \ldots, p_n$ be the points on the row g and let $B'_1, \ldots, B'_n$ be the (row) bundles with origin $p_1, \ldots, p_n$, respectively. We obtain the g-bundles $B_1, \ldots, B_n$ by $B_i := B'_i \setminus \{g\}$ for $i \in [1, n]$. Moreover, let B_0 be the set of rows of M which do not meet g. Note, that possibly $B_i = \emptyset$ for some $i \in [0, n]$. Then we call $(B_0, B_1, \ldots, B_r)$ the *g-bundle structure* of M, if $S = \{B_0, B_1, \ldots, B_r, \{g\}\}$ is a partition of the rows of M. In this case and if P_i is the set of the absolute indices of the rows in B_i we call the partition $(P_0, P_1, \ldots, P_r)$ the *g-partition* of M or of $(B_0, B_1, \ldots, B_r)$. Note that a g-partition is a partition of the set $[1, |S|] \setminus \{k\}$, where k is the absolute index of g. Analogously, we define these terms for columns.

The following lemma provides some straightforward technical support for Lemma 3.8, but first, we need some more notational agreements.

Let M be a matrix of size $m \times n$ and let Q be the representing matrix of a column permutation ϕ of M. We identify the rows $M(i, [1, n])$ and $MQ(i, [1, n])$

and denote both with the same symbol. Consequently, we identify g-bundles in M and MQ, for any row g of M or MQ. Analogously, we identify columns and c-bundles of a column c in M and PM for a permutation matrix P.

Lemma 3.7. *Let g be a row of the matrix M of size $m \times n$ with a g-bundle structure and let Q be the representing matrix of a column permutation ϕ of M. The g-partitions of M and MQ are equal. Moreover, the g-bundle B has the origin p in M if and only if B has the origin $\phi(p)$ in MQ. The dual statement holds for columns.*

Lemma 3.8. *A matrix M of size $m \times n$ is permutation equivalent to a matrix with step margin, i.e., M allows peeling, if and only if there is a connected pair (r, c), such that M has an r-bundle structure and a c-bundle structure.*

Proof. A matrix with step margin has bundle structures relative to the last row and the last column, which form a connected pair.
Conversely, by permutation equivalence, we may assume r and c to be the last row and the last column of M, respectively. Let $(R_0, R_1, \ldots, R_k)$ be the r-bundle structure and $(C_0, C_1, \ldots, C_l)$ be the c-bundle structure of M. For r and c are connected, c is the origin of exactly one r-bundle and without loss of generality, we may assume R_k to be this bundle. Vice versa we assume, r is the origin of the c-bundle C_l.
By Lemma 2.1, there is a row permutation π with representing matrix P that fixes the last row, such that the matrix $M' = P^{-1}M$ has the r-bundle structure $(R_0^\pi, R_1^\pi, \ldots, R_k^\pi)$ and $M'(R_i^\pi, [1, n])$ is a block row of the block matrix $(M', a, \cdot)$ for an arbitrary subdivision of the columns indicated by the dot and where $a = (a_1, \ldots, a_{k+1})$ is defined by $a_i = \sum_{j=0}^{i-1} |R_j|$ for $i \leq k$ and $a_{k+1} = (\sum_{j=0}^{k} |R_j|) + 1 = m$.
By Lemma 3.7, the matrix M' has a c-bundle structure. Analogously, there is a column permutation ϕ with representing matrix Q that fixes the last column, such that $M'' = M'Q$ has the c-bundle structure $(C_0^\phi, C_1^\phi, \ldots, C_l^\phi)$ and $M''([1, m], C_i^\phi)$ is a block column of the block matrix (M'', a, b) where $b = (b_1, \ldots, b_{l+1})$ is defined by $b_i = \sum_{j=0}^{i-1} |C_j|$ for $i \leq l$ and $b_{l+1} = (\sum_{j=0}^{l} |C_j|) + 1 = n$.
Clearly, $M''(m, n) = 1$. Moreover, $M''(R_k^\pi, n) = 1$, since c is the origin of R_k and vice versa $M''(m, C_l^\phi) = 1$. All other entries of the head of (M'', a, b) are equal to 0, since each row except of r is in exactly one r-bundle and thus, the head of (M'', a, b) is a Γ-head. It follows, that the origins of the r-bundles R_i^π,

where $i \geq 1$, are all in the last block column and the rows in R_0^π do not have any entry 1 in the last block column. Since the r-bundles are the block rows of M'', each block of the last block column has a column, where all entries are equal to 1. Moreover, the r-bundles are disjoint and thus, the blocks in the last block column are column step matrices of pairwise different shape. The analogous statement holds for the blocks in the last block row and finally, the matrix (M'', a, b) is a block matrix with step margin. □

Remark 3.9. In the proof of Lemma 3.8, we used the row r and the column c to form the Γ-head and clearly, if r has row sum m and c has column sum n, the Γ-head is of size $n \times m$.

Theorem 3.10. *A matrix satisfies the rectangular rule, if and only if it has a bundle structure relative to each line.*

Proof. Let A be a matrix with rectangular rule and g a row of A. If g has no entries 1 we may identify the matrix A without g as bundle B_0. Now let g be a nonzero row of A. By the rectangular rule, each other row meets g at most once. Hence, each row except g is in exactly one g-bundle B_i and $\{B_0, B_1, \ldots, B_r, \{g\}\}$ is a partition of the rows of A.
Conversely, suppose that there are two rows or two columns g, h in A which meet in at least two points p_1, p_2. Hence, the matrix A has no g-bundle structure, since the line h is element of at least the two g-bundles with origins p_1 and p_2. This contradicts the requirement of a bundle structure that the bundles are disjoint. □

Pseudo permutation matrices are square and have at most one entry 1 in each line. Straightforward consequences of the rectangular rule are the following two lemmata.

Lemma 3.11. *A matrix with rectangular rule allows peeling, i.e., it is permutation equivalent to a block matrix M with step margin. The blocks of M, which are the intersection of a block row with non-zero column step matrix and a block column with non-zero row step matrix, are pseudo permutation matrices.*
Conversely, the rectangular rule inherits from a block matrix with exclusively pseudo permutation matrices as blocks to its step hulls.

A set of square matrices of size k such that each matrix has precisely k entries 1 is called a *complete cover free collection* if their sum over $\mathbb{Z}$ is the

all-1-matrix J. Naturally, such a complete collection consists of k matrices. A block line is said to be *cover free*, if its blocks form a cover free collection. A square, block matrix of size h and with all blocks square of size n, i.e., $n|h$, is called of *format* h/n.
Internal permutations of a block matrix are permutations that permute only lines belonging to the same block line, i.e., the block line index is invariant for all lines under internal permutations. In particular a matrix description of internal permutations are left and right multiplication with permutation matrices with a suitable block diagonal form.

Lemma 3.12. *A block matrix with permutation margin allows internal permutations of the block lines to a block matrix with a unit margin.*

Let I be a matrix of size $m \times n$, let $S_r \subset [1, m], S_c \subset [1, n]$ and let $C = I(S_r, S_c)$ be a submatrix of I. We say a row of I is *inside* C, if it is a submatrix of $I(S_r, [1, n])$, otherwise, we say it is *outside* C and similarly for columns.

Lemma 3.13. *Let I be a matrix with rectangular rule and with submatrix C. Let a bundle of rows of C have its origin inside of C. Let g be a row of I that meets every row of the bundle but not inside of C. Then the number of points on g outside of C is an upper bound for the number of rows of the bundle. There is a dual version for columns.*

Proof. If the number of rows in the bundle is bigger than the available points on g, outside of C, then by the pigeon hole principle there must be two rows in the bundle that meet in the same point p on g, outside of C. Thus, these two bundle rows have this point p and the origin of the bundle, which is inside of C, in common. This contradicts the rectangular rule, that holds for I. □

3.3 Iterated peeling

After peeling a matrix, we can ask, if the core allows peeling again. This leads to an iterated peeling. The peeling process terminates after finitely many steps, since the number of entries 1 in the matrix is finite. We say, a matrix allows *complete peeling*, if the last core is the 0-matrix of any size including the 0-matrix of size 0×0, that is not really a matrix. The following example demonstrates that there are $\{0, 1\}$-matrices that allow peeling but not complete peeling.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By Lemma 3.8, a matrix allows peeling, if it has a connected pair (r, c), such that the matrix has an r-bundle structure and a c-bundle structure. Since a matrix with rectangular rule has this property, provided it is not the 0-matrix, we obtain easily the following corollary.

Corollary 3.14. *A nonzero matrix with rectangular rule allows complete peeling.*

Proof. By Theorem 3.10, a matrix with rectangular rule has a bundle structure relative to each line. Hence, by Lemma 3.8, it allows peeling as long as the cores have an entry 1. □

In the following, we will establish an algorithm for peeling square matrices with rectangular rule. For this, we need some notation. Let M be a matrix of size $m \times n$ and $i, j \in [1, m]$. A row $M(i, [1, n])$ is called a *proper cover* (cf. [15]) of a row $M(j, [1, n])$ if the row sum of $M(i, [1, n])$ is larger than the row sum of $M(j, [1, n])$ and

$$\forall k \in [1, n] : \; M(j, k) = 1 \implies M(i, k) = 1.$$

For a square matrix M of size n, we call the mapping

$$(M(i, j))_{i,j} \mapsto (M(n - j + 1, n - i + 1))_{i,j}$$

the *cross-transposing* of a matrix. For example, the cross-transposing of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ leads to $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$.

Let M be a matrix with rectangular rule and let r be a row and c be a column of M, such that (r, c) are a connected pair and neither r nor c have a proper cover in M. Then, the following algorithm peels M.

Algorithm (Peeling)

(1) *Permute the intersection of r and c to the position $(1, 1)$.*

(2) *Doubly order the resulting matrix.*

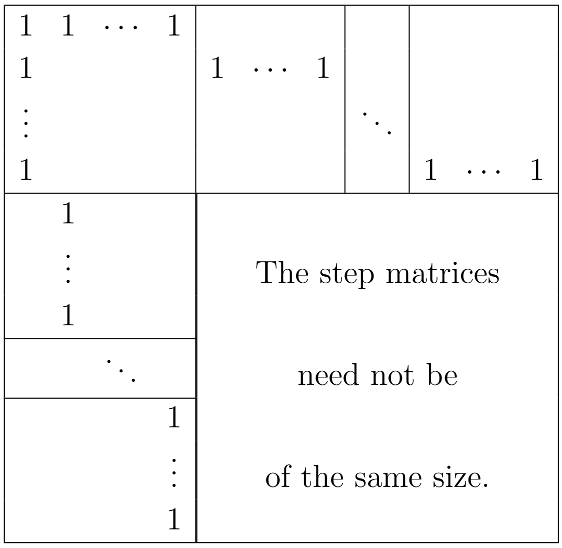

Figure 3: Partial structure of a doubly ordered matrix with rectangular rule

Since r and c do not have a proper cover, the double ordering fixes the first row and the first column, cf. [15], and produces a cross-transposed step margin in the first block row and the first block column, cf. Figure 3.

(3) *Cross-transpose the doubly ordered matrix*

(4) *Decompose the resulting matrix into step margin and step core.*

Note that lines of a matrix with rectangular rule have no proper cover, if their line sum is at least 2.

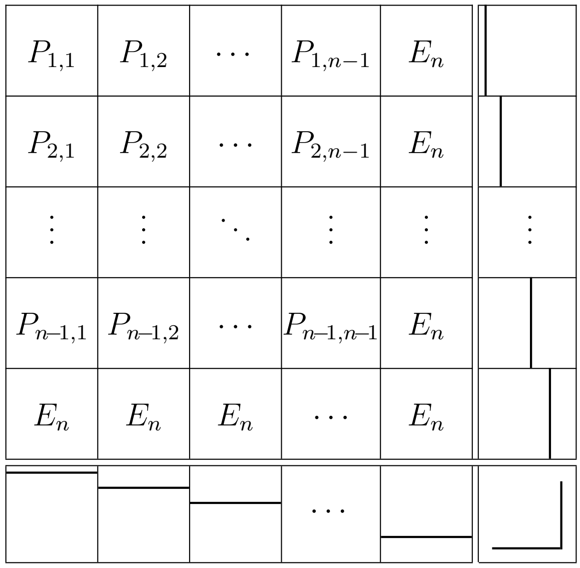

Figure 4: Peeling of an incidence matrix by double ordering and cross-transposing

4 Incidence matrices of planes

The most prominent matrices with rectangular rule are the incidence matrices of finite projective planes. There is the well known duality between columns and rows. The following lemma is well known.

Lemma 4.1. *For $n \geq 2$ a square matrix of size $n^2 + n + 1$ with constant line sum $n + 1$ is the incidence matrix of a projective plane if and only if it satisfies the rectangular rule.*

Proof. Incidence matrices of projective planes of order n clearly have the indicated property, cf. [4]. Conversely it is enough to show that two rows of such a matrix meet. For columns holds the dual result. Let g, h be two rows, let p be a point on h. If p is a point on g, we are done. Otherwise there are precisely $n + 1$ rows in the bundle with origin p. The number of points on this bundle is $n^2 + n + 1$, i.e., all points are on some row of this bundle. The row g is not in this bundle but has $n + 1$ points. Hence, g meets each row of this bundle, since it cannot meet a row twice, by rectangular rule. Thus, in particular, g meets h. □

Proposition 4.2. *A step core of an incidence matrix of some projective plane of order n is a block matrix of format n^2/n with rectangular rule and exclusively permutation matrices as blocks, cf. Figure* 4.
Conversely, a block matrix of format n^2/n with rectangular rule and exclusively permutation matrices as blocks, has a step hull, that is the incidence matrix of a projective plane of order n. In particular, the corresponding step margin increases the constant line sum by 1.

Proof. By Lemma 3.11 an incidence matrix allows peeling. Moreover, by the constant line sum and the rectangular rule, the step core is a block matrix with exclusively permutation matrices as blocks.
Conversely, let M be a block matrix with rectangular rule and exclusively permutation matrices as blocks. By Lemma 3.11, each step hull of M also satisfies the rectangular rule. Moreover, there is a step hull that has the correct line sum and suitable size, cf. Figure 4. Hence, by Lemma 4.1, it is the incidence matrix of a projective plane. □

By double ordering, cf. [15], and cross-transposing of an incidence matrix of a projective plane of finite order we obtain a step margin. We call the corresponding step core C_0. This step core C_0 has a unit margin, as in Figure 4, and is a block matrix of format n^2/n with exclusively permutation matrices as blocks. By Corollary 3.14, incidence matrices of finite projective planes allow complete peeling. We will peel those incidence matrices iteratively in a special way, cf. Remark 5.1 and get higher cores $C_1, C_2, \ldots$, cf. Corollary 5.5 If we start with the incidence matrix of a projective plane then the cores get smaller and smaller matrices. The matrix C_0 still carries the complete geometrical information, cf. Proposition 4.2, not so the matrices C_i for $i \geq 1$. There are additional bundle properties for matrices describing the cores $C_1, C_2, \ldots$ carrying the geometrical information, which got lost by matrix peeling.

$S_{1,1}$	$S_{1,2}$	$\cdots$	$S_{1,n-1}$	E_r
$S_{2,1}$	$S_{2,2}$	$\cdots$	$S_{2,n-1}$	E_r
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$S_{n-1,1}$	$S_{n-1,2}$	$\cdots$	$S_{n-1,n-1}$	E_r
E_r	E_r	$\cdots$	E_r	E_r

Figure 5: Regular block matrix of format nr/r

5 Regular block matrices

Let r, n be non-negative integers with $1 \leq r \leq n$. A square block matrix M of format nr/r satisfying the rectangular rule, is called a *regular block matrix of format* nr/r, cf. Figure 5, if it has a unit margin and the unit core $M([1,(n-1)r],[1,(n-1)r])$ is a block matrix with constant line sum $r-1$ and exclusively pseudo permutation matrices as blocks.

There are n square blocks of size r in each block line of a regular block matrix of format nr/r. Thus, the rows of the unit margin have row sum n and the columns of the unit margin have column sum n. The rows in a block row as well as the columns in a block column do not meet, i.e., they form a parallel bundle, since the blocks are pseudo permutation matrices.

Recall, that by (3.1) and (3.2) the unit core of a regular block matrix has cover free block lines and fixed point free blocks.

In particular, the core C_0 of a plane of order n is a regular block matrix of format n^2/n, cf. Figure 4, where all pseudo permutation matrices are proper permutation matrices.

For $r \leq n$, a step margin is called of *structure* n/r if the Γ-head is square of size n and there are n row step matrices and n column step matrices of format $n \times r$ and $r \times n$, respectively, where precisely one row step matrix is 0 and precisely one column step matrix is 0. In particular, a matrix with step margin of structure n/r is a square block matrix with equal subdivision $(r, 2r, \ldots, nr, nr+n)$ of the rows and of the columns. Thus, its step core is a block matrix of format nr/r.

Remark 5.1. Let $r \leq n$. For every regular block matrix of format nr/r there is a step margin of structure n/r such that the step hull is a matrix with line sums either n or $r+1$. Note that for $r < n$, the step matrices 0 are forced to continue the rows and columns of the unit margin. If a step hull of a regular block matrix of format nr/r is formed, we tacitly assume that the step margin has structure n/r and increases the line sums as indicated above. Conversely, for $r \leq n$, every regular block matrix of format nr/r is permutation equivalent to a block matrix with step margin of structure $n/(r-1)$, by line sums and Lemma 3.8, since there are a row with row sum n and a column with column sum n which are connected.

Permuting the lines of a regular block matrix of format nr/r to obtain a matrix M with step margin of structure $n/(r-1)$ and peeling M leads to a *step core of a regular block matrix.* Briefly, we call this the *peeling of a regular block matrix.* Thus, each step core of a regular block matrix of format nr/r is a block matrix with the same subdivision $((r-1), 2(r-1), \ldots, n(r-1))$ of the rows and of the columns.

Lemma 5.2. *Each step core of a regular block matrix of format nr/r, where $r > 1$, is permutation equivalent to a regular block matrix of format $n(r-1)/(r-1)$.*
Conversely, the step hull of a regular block matrix of format $n(r-1)/(r-1)$ is permutation equivalent to a block matrix of format nr/r, with unit margin and with rectangular rule.

Proof. A regular block matrix of format nr/r is permutation equivalent to a matrix with step margin of structure $n/(r-1)$, cf. Figure 6. The block column C where the intersection with the margin row is the 0-matrix, contains the columns with column sum n and the block row R where the intersection with the margin column is the 0-matrix, contains the rows with row sum n, by Remark 5.1. Since the rows in R are parallel, the blocks in R have at most one entry 1 in each column and vice versa the blocks in C have at most one entry 1 in each row. Thus, the intersection of R and C is a pseudo permutation matrix. All other blocks in R have at most one entry 1 in each row, by the row step matrices in the margin row and the rectangular rule. Analogously, all other blocks in C have at least one entry 1 in each column.

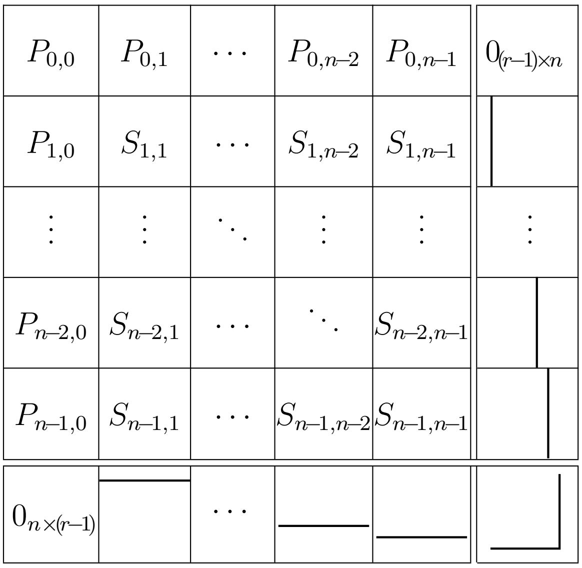

Figure 6: Double ordering and cross-transposing of a regular block matrix of format nr/r

Hence, all blocks in R and all blocks in C are pseudo permutation matrices. There are n blocks in R and C, respectively. Thus, by the line sum n, they are even permutation matrices. By Lemma 3.12 we may assume these blocks to be unit matrices applying only internal permutations, i.e., the step margin is maintained. Hence, by Lemma 3.11, all other blocks in the step core are pseudo permutation matrices. By permutations of the block lines of the step core, cf. Figure 7, we obtain a matrix with unit margin and pseudo permutation matrices in the unit core. By the rectangular rule, they are fixed point free, cf. (3.2) and the block lines form cover free collections, cf. (3.1). Thus, we obtain a regular block matrix of format $n(r-1)/(r-1)$.
Conversely, by Lemma 3.11, the step hull of a regular block matrix satisfies the rectangular rule. Moreover, by Remark 5.1, the hull has precisely r rows and r columns with line sum n and the format is as desired. These rows do not meet, and also the columns do not meet. This enables us to get a unit margin by line permutations, i.e., a block matrix with unit margin. □

Remark 5.3. The converse direction in Lemma 5.2 tells us, that the step hull of a regular block matrix satisfies the rectangular rule and allows a unit margin, but this does not mean that this step hull is permutation equivalent to a regular block matrix. Moreover, if two regular block matrices are per-

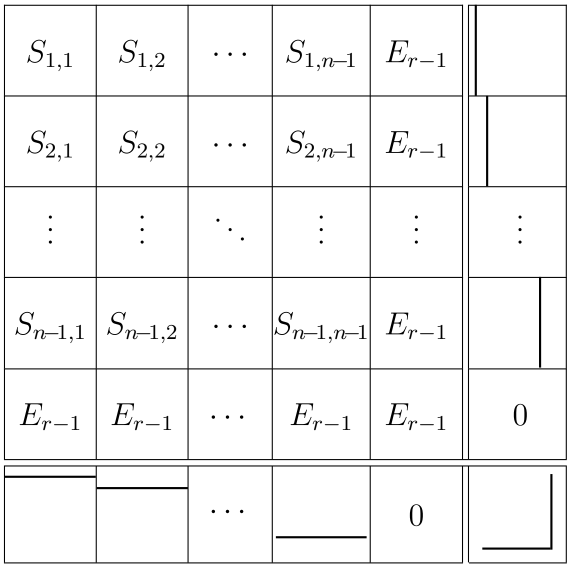

Figure 7: Peeling of a regular block matrix of format nr/r

mutation equivalent, then the respective step hulls need not be permutation equivalent, cf. 5.4.

Example 5.4. The regular block matrices

$$B_1 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \qquad B_2 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

of format $3 \cdot 2/2$ are seen to be permutation equivalent by transposition of the 4th and the 6th column. But the step hull H_1 of B_1 is not permutation equivalent to the step hull H_2 of B_2.

$$H_1 = \left(\begin{array}{ccc|cc|cc|cc} 1&1&1&0&0&0&0&0&0\\ 1&0&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&1&1\\ \hline 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&1&0&1&0&1\\ \hline 0&1&0&1&0&0&1&0&0\\ 0&1&0&0&1&1&0&0&0\\ \hline 0&0&1&1&0&0&0&0&1\\ 0&0&1&0&1&0&0&1&0 \end{array}\right) \quad H_2 = \left(\begin{array}{ccc|cc|cc|cc} 1&1&1&0&0&0&0&0&0\\ 1&0&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&1&1\\ \hline 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&1&0&1&0&1\\ \hline 0&1&0&1&0&0&0&0&1\\ 0&1&0&0&1&1&0&0&0\\ \hline 0&0&1&1&0&0&1&0&0\\ 0&0&1&0&1&0&0&1&0 \end{array}\right)$$

This follows since the columns of H_2 allow a partition into parallel bundles each of cardinality 3, while the columns of H_1 have no such partition. Clearly, this is a property that is unchanged by line permutations.

We may iterate peeling of regular block matrices by Lemmata 3.11 and 5.2. If we start with a regular block matrix of format nr/r, then a step core is permutation equivalent to a regular block matrix A_1 of format $n(r-1)/(r-1)$. This way we obtain a sequence $(A_i)_{i=1}^{r-1}$ of regular block matrices of format $n(r-i)/(r-i)$, respectively. Note that $A_{r-1} = \Gamma_n$. We specialize this procedure in the following corollary for planes.

Corollary 5.5. *Peeling n times the incidence matrix of a projective plane of order n gives sequences $C_0, \ldots, C_{n-1}$ of regular block matrices, where C_r is of format $n(n-r)/(n-r)$.*
In particular, C_0 is a regular block matrix of format n^2/n with exclusively permutation matrices as blocks and the last core is $C_{n-1} = \Gamma_n$.

Such regular block matrices C_r, for $0 \le r \le n-1$ are called *cores of the plane* or *cores of the incidence matrix*. Those sequences of cores are not unique and it might be interesting to ask for common properties of different core sequences.

6 Configurations and (r, n)-matrices

Configurations and in particular symmetric configurations are introduced for example in [6]. Since the term “connected” is used differently in [6], we give an equivalent definition of symmetric configurations using the dual axiom of the axiom (3) in [6].
A *symmetric configuration* v_k is an incidence structure of v points and v lines, such that

(1) each line contains k points,

(2) each point lies on k lines,

(3) two different lines meet in at most one point.

We identify symmetric configurations with their incidence matrices denoted by $I_{v,k}$. The following lemma is a straightforward consequence of the definition above.

Lemma 6.1. *The incidence matrix $I_{v,k}$ of a symmetric configuration v_k is square of size v, has line sum k and satisfies the rectangular rule.*
Conversely, a square matrix with constant line sum that satisfies the rectangular rule is the incidence matrix of a symmetric configuration.

The following lemma is well known and provides a necessary condition of the existence of a symmetric configuration.

Lemma 6.2. *Let $I_{v,k}$ be a symmetric configuration. Then, $v \geq k(k-1)+1$.*

Proof. Let c be a column of $I_{v,k}$. Then, c is the origin of a row bundle $\{r_1, \ldots, r_k\}$ of cardinality k, by the line sum. Each of these rows is connected with c and $k-1$ other columns. As $r_1, \ldots, r_k$ meet in c, they do not meet in any other column, by the rectangular rule. Hence, there are at least $k(k-1)+1$ columns. □

Let M be a $\{0,1\}$-matrix of size $rn \times m$. A partition $\{P_1, \ldots, P_n\}$ of the rows of M into n parallel bundles all of cardinality r is called an (r, n)-*parallel partition* of the rows. Similarly, we speak of (r, n)-parallel partitions of the columns of a $\{0,1\}$-matrix of size $m \times rn$.

Lemma 6.3. *Let B be a regular block matrix of format nr/r and let R be an (r, n)-parallel partition of the rows of B. Then, each bundle of R contains exactly one row of each internal index. The dual statement holds for the columns.*

Proof. The blocks of B are square of size r. Thus, the internal indices range in $[1, r]$. By the rectangular rule and the unit margin of B, the rows with the same internal index meet. Hence, the rows in a parallel bundle must have pairwise different internal indices. Since the bundles of R have cardinality r, every internal index appears precisely once in each bundle. □

Furthermore, let $R = \{R_i \mid i \in [1, n]\}$ be an (r, n)-parallel partition of the rows and $C = \{C_i \mid i \in [1, n]\}$ be an (r, n)-parallel partition of the columns of an incidence matrix of a symmetric configuration. We call R and C *complementary* if there is a permutation π of the set $[1, n]$ such that R_i and $C_{\pi(i)}$ are not connected for all $i \in [1, n]$. Then, the permutation π is called a *complementary indication* of those partitions.
We give another, equivalent definition of the term complementary, understanding (R, C, I) as an incidence structure with the following incidence relation I:

$$R_i I C_j :\Longleftrightarrow R_i \text{ is not connected with } C_j$$

We define for all $j \in [1, n]$ the subset $S_j := \{R_i | R_i I C_j, i \in [1, n]\}$ of R. Then, R and C are complementary, if there is a system of distinct representatives[2] for the sample $(S_1, \ldots, S_n)$.

6.1 Homogeneous and Latin configurations

Let $k + 1$ be a divisor of v, i.e., $v = (k + 1)m$. We call a symmetric configuration $I_{v,k}$ a *homogeneous configuration*, if it has an $(m, k + 1)$-parallel partition R of the rows and an $(m, k+1)$-parallel partition C of the columns, such that R and C are complementary.
The set of homogeneous configurations of a given size and line sum is in general a proper subset of the symmetric configurations of this size and line sum. An example for a symmetric configuration that is not a homogeneous configuration is given in Appendix A.
Let $v = (k + 1)m$ and $I_{v,k}$ be a symmetric configuration. We call $I_{v,k}$ a $(k+1, m)$-*Latin configuration*, if it has $(m, k+1)$-parallel partitions R, C of the

[2] Systems of distinct representatives are introduced for example in [18].

rows and columns that are complementary and $(k+1, m)$-parallel partitions of the rows and columns that are orthogonal to R and C, respectively.
Note, that the bundles in R are of cardinality m as opposed to the bundles with cardinality $k+1$ of the $(k+1, m)$-parallel partition of the rows that is orthogonal to R. In particular, the orthogonality of those partitions is strict, i.e., the intersection of any bundle of R with any bundle of the orthogonal $(k+1, m)$-parallel partition is exactly of cardinality 1.
By definition, each $(k+1, m)$-Latin configuration is a homogeneous configuration $I_{(k+1)m,k}$.

Lemma 6.4. *Let $v = (k+1)m$.*

(1) *If $I_{v,k}$ is a symmetric configuration, $m \geq k-1$.*

(2) *If $I_{v,k}$ is a $(k+1, m)$-Latin configuration, $m \geq k$.*

Proof. For (1), assume $m \leq k-2$. Then,

$$(k+1)m \leq (k+1)(k-2) < k(k-1)+1.$$

This contradicts the necessary condition given in Lemma 6.2.
We show (2). $I_{v,k}$ has a $(k+1, m)$-parallel partition R of the rows. By parallelism, there is at most one entry 1 in the intersection of a bundle of R and a column. By the column sum, R must have at least k row bundles. □

Later, as a particular consequence of Corollary 6.18, we will see that for a prime power q, there is a $(q, q-1)$-Latin configuration and a homogeneous configuration $I_{(q+1)(q-1),q}$, as there is a projective plane of order q.

Lemma 6.5. *A homogeneous configuration $I_{(k+1)m,k}$ is permutation equivalent to a block matrix $(M, (m, 2m, \ldots, (k+1)m))$ satisfying the rectangular rule, where all blocks on the main diagonal are* 0 *and all other blocks are permutation matrices.*
Conversely, a block matrix $(M, (m, 2m, \ldots, (k+1)m))$ satisfying the rectangular rule, where all blocks on the main diagonal are 0 *and all other blocks are permutation matrices is the incidence matrix of a homogeneous configuration $I_{(k+1)m,k}$.*

Proof. By Lemma 2.1, we can permute the lines of each parallel bundle of the $(m, k+1)$-parallel partitions of the rows and columns, to a block line. The resulting matrix is a block matrix with the subsections $(m, 2m, \ldots, (k+1)m)$

	m	m	m	$\cdots$	m
m	0	$P_{1,2}$	$P_{1,3}$	$\cdots$	$P_{1,k+1}$
m	$P_{2,1}$	0	$P_{2,3}$	$\cdots$	$P_{2,k+1}$
m	$P_{3,1}$	$P_{3,2}$	0	$\cdots$	$P_{3,k+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
m	$P_{k+1,1}$	$P_{k+1,2}$	$P_{k+1,3}$	$\cdots$	0

Figure 8: Homogeneous configuration $I_{(k+1)m,k}$ with standardized structure.

for the rows and columns. Each block has at most one entry 1 in a column, since the rows in a block row are parallel and vice versa each block has at most one entry 1 in a row, since the columns in a block column are parallel. Thus, all blocks are pseudo permutation matrices. Moreover, by the line sum, each row has an entry 1 in all blocks of its block row, except of exactly one block and since the $(m, k+1)$-parallel partitions are complementary, there is a 0-block in each block row and in each block column. Hence, all other blocks in a block row must be permutation matrices. The same holds for the block columns. Using block line permutations, we can place the 0-blocks at the main diagonal.
Conversely, since all blocks are pseudo permutation matrices, all block lines are parallel bundles of cardinality m. Hence, the block rows form an $(m, k+1)$-parallel partition of the rows and the block columns form an $(m, k+1)$-parallel partition of the columns. Moreover, those partitions are complementary, since the ith block row is not connected with the ith block column, by the 0-blocks on the main diagonal. Furthermore, M satisfies the rectangular rule by assumption and has constant line sum k, as there are k permutation matrices in each block line. Thus, by Lemma 6.1 we obtain a homogeneous configuration. □

Lemma 6.6. *A $(k+1, m)$-Latin configuration is permutation equivalent to a block matrix $(M, (m, 2m, \dots, (k+1)m))$ satisfying the rectangular rule, where all blocks on the main diagonal are 0 and all other blocks are permutation* *matrices, such that all block lines of M form cover free collections.*

Conversely, a block matrix $(M, (m, 2m, \ldots, (k+1)m))$ satisfying the rectangular rule, where all blocks on the main diagonal are 0 *and all other blocks are permutation matrices, such that all block lines of M form cover free collections, is the incidence matrix of a $(k+1, m)$-Latin configuration.*

Proof. A $(k+1, m)$-Latin configuration is permutation equivalent to a block matrix $(M, (m, 2m, \ldots, (k+1)m))$ where all blocks on the main diagonal are 0 and all other blocks are permutation matrices, by Lemma 6.5. Moreover, there is a $(k+1, n)$-parallel partition $R = \{R_1, \ldots, R_n\}$ of the rows which is orthogonal to the $(m, k+1)$-parallel partition of the rows, whose bundles form the block rows of M. Thus, each bundle R_i contains exactly one row of each of the $k+1$ block rows. We can permute the rows using only internal permutations of the block rows, such that the rows, that belong to the same bundle of R have the same internal index. Hence, by parallelism of these rows, the block columns form cover free collections. The blocks stay permutation matrices or the 0-block, respectively, since we applied only internal permutations. Analogously we can apply internal permutations of the block columns, such that the block rows form cover free collections.

Conversely, using Lemma 6.5 it remains to show the existence of $(k+1, m)$-parallel partitions of the rows and of the columns of M that are orthogonal to the partitions given by the block rows and block columns, respectively.

Since the block rows of M are cover free, the columns with the same internal index form a parallel bundle of cardinality $k+1$ and we obtain a $(k+1, m)$-parallel partition P of the columns of M. Moreover, the intersection of a bundle of P and a block column is obviously of cardinality 1 and thus, P is orthogonal to the partition given by the block columns. Moreover, the block columns of M are cover free, too. Analogously, we find a $(k+1, m)$-parallel partition of the rows of M that is orthogonal to the partition given by the block rows. □

We will tacitly assume, that homogeneous configurations and Latin configurations have the standardized structure as in Figure 8, provided by the Lemmata 6.5 and 6.6, respectively.

6.2 (r, n)-matrices and their block structures

Let M be a square matrix of size rn, where $r \leq n+1$. The matrix M is called an (r, n)*-matrix*, if it has r rows with row sum n, forming a parallel bundle, the so called *controlling row bundle*, r columns with column sum n forming a parallel bundle, the *controlling column bundle*, the remaining lines have line sum r and the rectangular rule is satisfied in M. Clearly, line permutations maintain these properties.
If $r \neq n$, the lines of the controlling bundles are uniquely determined by the line sum n. Obviously, a regular block matrix of format nr/r is a (r, n)-matrix where the margin row forms the controlling row bundle and the margin column forms the controlling column bundle.

Lemma 6.7. *In an (r, n)-matrix, each column is connected with exactly one row of the controlling row bundle and dually, each row is connected with exactly one column of the controlling column bundle.*

Proof. Let M be an (r, n)-matrix. The r rows in the controlling row bundle of M form a parallel bundle. By line sum and parallelism, each column of M has exactly one entry 1 in the intersection with the controlling row bundle, since M is of size nr. Hence, each column is connected with exactly one row of this bundle. The dual statement holds analogously. □

Lemma 6.8. *Let M be an (r, n)-matrix and let g be a row of M that is not in the controlling row bundle. Then g meets each row of the controlling row bundle of M. The dual statement holds for columns.*

Proof. By line sum and parallelism, there is no 0-column in the controlling row bundle of M. The row g meets each row of this bundle at most once, by the rectangular rule. Thus, by its row sum r, the row g meets r different rows of the controlling row bundle. The same holds for the columns. □

Lemma 6.9. *Let M be a square matrix of size nr, where $r \leq n + 1$. The following are equivalent:*

(1) *The matrix M is permutation equivalent to the step hull of a regular block matrix of format $n(r - 1)/(r - 1)$.*

(2) *The matrix M is an (r, n)-matrix.*

Proof. (1) $\Rightarrow$ (2): Due to Lemma 5.2, the step hull of a regular block matrix of format $n(r-1)/(r-1)$ satisfies the rectangular rule and is permutation equivalent to a block matrix of format nr/r with a unit margin. Thus, the margin row forms a parallel bundle of r rows with row sum n as well as the margin column forms a parallel bundle of r columns with column sum n. Furthermore, the line sum of the remaining lines is r.

(2) $\Rightarrow$ (1): Now assume that M is an (r, n)-matrix. Since M satisfies the rectangular rule, it has a bundle structure relative to each line, by Theorem 3.10. Moreover, by Lemma 6.7, there is a connected pair of a row from the controlling row bundle and a column from the controlling column bundle. By Lemma 3.8 and Remark 3.9, M is permutation equivalent to a matrix S with step margin, such that the last row and the last column are in the controlling bundles.

First we determine the structure of the step margin. Since the last row of the step margin row is in the controlling row bundle, all other $n-1$ rows of the step margin row are not in the controlling row bundle, as they meet in the last column. Thus, they have line sum r. Hence, all non zero row step matrices of the step margin row have size $n \times (r-1)$. The number of non zero columns in the step margin row is $n + (n-1)(r-1) = nr - (r-1)$. Thus, the remaining $r-1$ columns of the step margin row are 0-columns and form a 0-matrix of size $n \times (r-1)$. This is the remaining row step matrix of the step margin row. The same statement holds for the step margin column. Therefore, the step margin is of structure $n/(r-1)$ and hence, the step core of S is a block matrix of format $n(r-1)/(r-1)$.

Let R denote the block row containing the 0-step matrix of the margin column and let C denote the block column containing the 0-step matrix of the margin row. Beside the last row, there are $r-1$ rows in the controlling row bundle. Observe, that all rows meet the last row in the margin column, except of the rows of R. Thus, by parallelism, R contains the remaining rows of the controlling row bundle. Dually, C contains the remaining columns of the controlling column bundle.

By parallelism, there is at most one entry 1 in each column of R and at most one entry 1 in each row of C. Thus, the block, that is the intersection of R and C, must be a pseudo permutation matrix. Furthermore, the remaining blocks of R have at most one entry 1 in each row, by the step matrices in the margin row and dually, the remaining blocks of C have at most one entry 1 in each column, by the step matrices in the margin column. Thus, the blocks of R and C are pseudo permutation matrices. As the line sum in R is n, all the blocks are even permutation matrices. The same holds for the block column C. By Lemma 3.11, all other blocks in the step core are pseudo permutation matrices, too. Hence, the step core B' of S has a permutation margin. By Lemma 3.12, internal permutations of the block lines of B' lead to a block matrix B with unit margin. As we applied only internal permutations, the blocks in the unit core of B are pseudo permutation matrices, as they were pseudo permutation matrices in B'. Obviously, the unit core of B has constant line sum $r-2$. Hence, B is a regular block matrix of format $n(r-1)/(r-1)$. □

Corollary 6.10. *An (r, n)-matrix is permutation equivalent to a matrix with a unit margin and the corresponding unit core is a symmetric configuration $I_{(n-1)r,r-1}$.*

Proof. By the Lemmata 6.9 and 5.2, an (r, n)-matrix M is permutation equivalent to a matrix with unit margin. Clearly, the unit core has constant line sum $r-1$ and hence, by Lemma 6.1, the unit core is a symmetric configuration $I_{(n-1)r,r-1}$. □

Let $\widetilde{M}$ be a margin with a $k \times k$ unit-head and let all other blocks in the margin row be $k \times m$ row step matrices of pairwise different shape and all other blocks in the margin column be $m \times k$ column step matrices of pairwise different shape. Moreover, let all those shapes be greater than zero. Then we call $\widetilde{M}$ a *homogeneous margin*, cf. Figure 9. A homogeneous margin $\widetilde{M}$ is a square 1-margin of size $km + k$ and the rows in the margin row have row sum $m + 1$ and the columns in the margin column have column sum $m + 1$.

Lemma 6.11. *An (r, n)-matrix is permutation equivalent to a matrix with homogeneous margin.*

Proof. We permute the controlling row bundle to the last r rows and the controlling column bundle to the last r columns of the matrix, i.e., the new margin row and margin column. The intersection of them is a permutation matrix, by Lemma 6.7. Using internal permutations of the margin row, we can form a unit-head. The resulting matrix is denoted by M. The rows of the margin row of M have $n-1$ entries 1 in the intersection with the submatrix $M' = M([1, rn], [1, r(n-1)])$, i.e., the row sum in $M([r(n-1) + 1, rn], [1, r(n-1)])$ is constant $n-1$. Hence, each of those rows is the origin

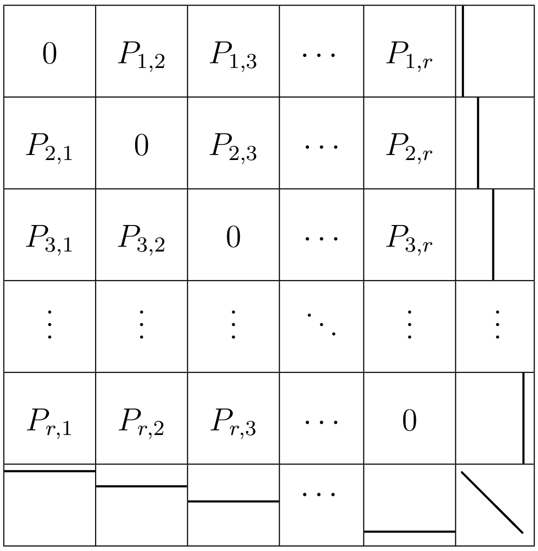

Figure 9: (r, n)-matrix with a homogeneous margin

of a bundle of $n-1$ columns of M'. By parallelism of the margin row there is exactly one entry 1 in each column of the margin row. Hence, those column bundles form a partition of M'. By Lemma 2.1, we can permute the bundles of this partition to block columns and the blocks in the margin row except of the head are row step matrices of size $r \times (n-1)$. Analogously, there is a permutation, such that the blocks in the margin column except of the head are column step matrices of size $(n-1) \times r$. Thus, we obtain a matrix with homogeneous margin. □

Lemma 6.12. *Let M be an (r, n)-matrix with a homogeneous margin. Then, $M([1,(n-1)r],[1,(n-1)r])$ is a homogeneous configuration $I_{(n-1)r,r-1}$. Conversely, each homogeneous configuration $I_{kr,r-1}$ can be extended to an $(r, k+1)$-matrix.*

Proof. All blocks in the margin except of the head are step matrices. Thus, all blocks in $M([1,(n-1)r],[1,(n-1)r])$ must be pseudo permutation matrices. By Lemma 6.7, a column c in the margin is connected with exactly one row g in the margin and vice versa. Clearly, there is a block row R_c such, that the intersection of R_c and c has entries 1 at every position and vice versa there is a block column C_g such, that the intersection of C_g and g has entries 1 at every position. By the rectangular rule, the intersection of R_c and C_g is the 0-block. Hence, there is a 0-block in each block line. By the line sum, the remaining

blocks are permutation matrices. Thus, $M([1,(n-1)r],[1,(n-1)r])$ is a homogeneous configuration.
Conversely, let $I_{kr,r-1}$ be a (standardized) homogeneous configuration (cf. Figure 8) and let H be a homogeneous margin that respects the block structure of $I_{kr,r-1}$, i.e., with row step matrices of size $r \times k$ and column step matrices of size $k \times r$, such that the step matrix in the ith block line has shape i. Obviously, H is square of size $kr + r$. We build the hull $M = I_{rk,r-1} + H$, cf. Figure 9.
We show that M satisfies the rectangular rule. For this, let g be the row of the margin row of M with internal index i and c be the column of the margin column of M with internal index i. Then g is the origin of the ith block column C_i and c is the origin of the ith block row R_i. Since the rectangular rule holds for $I_{rk,r-1}$ and also for H, we have to prove that it holds for $I_{rk,r-1}$ together with each line of H. Observe that the entries 1 of g are exactly in the columns $C_i \cup \{c\}$ and dually the entries 1 of c are exactly in the rows $R_i \cup \{g\}$, cf. Figure 9. Therefore, it is enough to show, that the intersection $M(R_i \cup \{g\}, C_i \cup \{c\})$ satisfies the rectangular rule. But the intersection of C_g and R_c is a block on the main diagonal and thus, a 0-block. Hence, $M(R_i \cup \{g\}, C_i \cup \{c\})$ is a Γ-matrix. Thus, the rectangular rule holds for $I_{rk,r-1}$ together with g and c, and inductively for M, too.
By definition, the margin row and the margin column are parallel bundles of cardinality r and line sum $k+1$. All other lines have line sum r, since their line sum in $I_{rk,r-1}$ is $r-1$ and the homogeneous margin increases the line sum by 1. □

6.3 Block switch

Using the rows and columns from the controlling bundles, we can induce two different block structures for (r,n)-matrices.

(S1) By Corollary 6.10, an (r,n)-matrix is permutation equivalent to a block matrix $(M,(r,2r,\dots,nr))$ with a unit margin.

(S2) By Lemma 6.11, an (r,n)-matrix is permutation equivalent to a block matrix $(M,((n-1),2(n-1),\dots,r(n-1),r(n-1)+r))$ with a homogeneous margin.

In particular, regular block matrices of format nr/r are (r,n)-matrices with the block structure (S1). We can switch between the block structures (S1)

and (S2).
In a square block matrix $(B, (k, 2k, \dots, mk))$ of size mk, a row r is uniquely identified by its internal index $i \in [1, k]$ and its block line index $b \in [1, m]$. The same holds for a column c. The row r has the absolute index $a \in [1, mk]$, defined by

$$a = (b-1)k + i. \tag{6.1}$$

The interchange of the block row index and the internal index of each row of B is a row permutation, i.e., the row with absolute index $(b-1)k + i$ is permuted to the absolute index $(i-1)m + b$. For the following more precise definition of this permutation $\sigma_{k,m}$, note that $b = \lceil \frac{a}{k} \rceil$, i.e., the smallest integer greater than or equal to $\frac{a}{k}$ and furthermore, $i = a - (\lceil \frac{a}{k} \rceil - 1)k$ by equation (6.1).

$$\begin{array}{rccc} \sigma_{k,m}: & [1, km] & \longrightarrow & [1, km] \\ & a & \mapsto & (a - (\lceil \frac{a}{k} \rceil - 1)k - 1)m + \lceil \frac{a}{k} \rceil \end{array}.$$

To see, that $\sigma_{k,m}$ is really a permutation, observe that

$$\{(b-1)k + i \mid i \in [1, k], b \in [1, m]\} = [1, km]$$

and also

$$\{(i-1)m + b \mid i \in [1, k], b \in [1, m]\} = [1, km].$$

We can interpret $\sigma_{k,m}$ as row permutation or column permutation of the block matrix B and hence, a matrix description of $\sigma_{k,m}$ is left multiplication with the representing matrices $\Sigma_{k,m}^t$ and right multiplication with $\Sigma_{k,m}$, respectively. We call the application of $\sigma_{k,m}$ simultaneously as a row permutation and as a column permutation the (k, m)-*block switch* or briefly the *block switch*, since the resulting matrix has a different block structure, if $k \neq m$.

$$(B, (k, 2k, \dots, mk)) \qquad \longrightarrow \qquad (\Sigma_{k,m}^t B \Sigma_{k,m}, (m, 2m, \dots, km))$$

Clearly, $\sigma_{k,m}^{-1} = \sigma_{m,k}$ and therefore, $\Sigma_{k,m}^t = \Sigma_{m,k}$.

Lemma 6.13. *For the block matrices* $(B, (k, 2k, \dots, mk))$ *and*

$$B' := (\Sigma_{k,m}^t B \Sigma_{k,m}, (m, 2m, \dots, km)),$$

the following equivalences hold:

(1) *The blocks of B are fixed point free $\iff$ B' has 0-blocks on the main diagonal.*

(2) *B has 0-blocks on the main diagonal $\iff$ the blocks of B' are fixed point free.*

(3) *The blocks of B are cover free $\iff$ the blocks of B' are pseudo permutation matrices.*

(4) *The blocks of B are pseudo permutation matrices $\iff$ the blocks of B' are cover free.*

Proof. For $i, j \in [1, m]$, let $P_{i,j}$ be the blocks of B and for $r, s \in [1, k]$, let $Q_{r,s}$ be the blocks of B'. In particular, the blocks $P_{i,j}$ are square of size k and the blocks $Q_{r,s}$ are square of size m. We show the equation

$$P_{i,j}(r, s) = Q_{r,s}(i, j). \tag{6.2}$$

By the definition of internal indices and block indices, it is

$$P_{i,j}(r, s) = B((i-1)k + r, (j-1)k + s)$$

and

$$Q_{r,s}(i, j) = B'((r-1)m + i, (s-1)m + j).$$

Moreover, by the definition of $\sigma_{k,m}$, it is $\sigma_{k,m}((i-1)k + r) = (r-1)m + i$ and $\sigma_{k,m}((j-1)k + s) = (s-1)m + j$ and thus, equation (6.2) holds.
For (1), let $P_{i,j}$ be fixed point free, or equivalently $P_{i,j}(r, r) = 0$ for all $i, j \in [1, m]$ and all $r \in [1, k]$. By equation (6.2), this is equivalent to $Q_{r,r}(i, j) = 0$ for all i, j, r, i.e., all blocks on the main diagonal of B' are 0-matrices. The proof of (2) is analogous.
For (3), let the block lines of B be cover free, or equivalently

$$\forall i, j \in [1, m] \forall r, s \in [1, k] : \left(\sum_{i=1}^{m} P_{i,j}(r, s) \;\leq 1 \right) \wedge \left(\sum_{j=1}^{m} P_{i,j}(r, s) \;\leq 1 \right).$$

By equation (6.2), this is equivalent to

$$\forall i, j \in [1, m] \forall r, s \in [1, k] : \left(\sum_{i=1}^{m} Q_{r,s}(i, j) \;\leq 1 \right) \wedge \left(\sum_{j=1}^{m} Q_{r,s}(i, j) \;\leq 1 \right),$$

i.e., there is at most one entry 1 in each row and each column of a block $Q_{r,s}$.
The proof of (4) is analogous. □

Lemma 6.14. *Let M be an (r, n)-matrix with a unit margin. Then, the matrix* $\operatorname{diag}(\Sigma^t_{r,n-1}, E_r)M \operatorname{diag}(\Sigma_{r,n-1}, E_r)$ *is an (r, n)-matrix with a homogeneous margin.*

Proof. Let $M' := \operatorname{diag}(\Sigma^t_{r,n-1}, E_r)M \operatorname{diag}(\Sigma_{r,n-1}, E_r)$. Since all entries 1 in a row of the margin row of M have the same internal index, the entries 1 of this row in $M'([1,(n-1)r],[1,(n-1)r])$ have the same block column index. The same holds for the columns in the margin column. The head is not affected by the permutation and thus, M' has a unit-head. Hence, M' is a matrix with a homogeneous margin. □

Theorem 6.15. *The following are equivalent:*

(1) *M is the unit core of a regular block matrix of format nr/r.*

(2) *$\Sigma^t_{r,n-1}M\Sigma_{r,n-1}$ is an $(r, n-1)$-Latin configuration (with standardized structure), cf.* Figure 9.

Proof. Line permutations maintain the rectangular rule. Let

$$M' := \Sigma^t_{r,n-1}M\Sigma_{r,n-1}.$$

(1) ⇒ (2): In particular, the unit core of a regular block matrix of format nr/r is a block matrix $(M,(r, 2r, \dots,(n-1)r))$ where all blocks are fixed point free pseudo permutation matrices and all block lines are cover free collections. Thus, by Lemma 6.13, the blocks of M' are pseudo permutation matrices, the blocks on the main diagonal are 0-blocks and all block lines of M' are cover free. The lines of M and thus, the lines of M' have constant line sum $r - 1$. Moreover, each block line of M' contains r blocks, where exactly one of those is a 0-block. Hence, all other blocks of M' must be proper permutation matrices. Thus, by Lemma 6.6, M' is an $(r, n-1)$-Latin configuration.
(2) ⇒ (1): Let M' be an $(r, n-1)$-Latin configuration. Again by Lemmata 6.6 and 6.13, the blocks of M are fixed point free pseudo permutation matrices and all block lines of M are cover free collections. □

The following corollary is an immediate consequence of Theorem 6.15.

Corollary 6.16. *The existence of a regular block matrix of format nr/r is equivalent to the existence of an $(r, n-1)$-Latin configuration.*

Corollary 6.17. *The existence of a (k, m)-Latin configuration is equivalent to the existence of a homogeneous configuration $I_{(k+1)m,k}$.*

Proof. By Corollary 6.16, the existence of a (k, m)-Latin configuration is equivalent to the existence of a regular block matrix B of format $(m+1)k/k$. By Remark 5.1 and Lemma 6.9, the step hull of B is a $(k + 1, m + 1)$-matrix. By Lemmata 6.11 and 6.12, the existence of a $(k+1, m+1)$-matrix is equivalent to the existence of a homogeneous configuration $I_{(k+1)m,k}$. □

Corollary 6.18. *The following are equivalent:*

(1) *There is a homogeneous configuration $I_{(k+2)k,k+1}$.*

(2) *There is a $(k + 1, k)$-Latin configuration.*

(3) *There is a projective plane of order $k + 1$.*

Proof. (1) ⇒ (2) is shown by Corollary 6.17.
(2) ⇒ (3): By Corollary 6.16, there is a regular block matrix of format $(k + 1)^2/(k+1)$. Thus, by Proposition 4.2, there is a projective plane of order $k+1$.
(3) ⇒ (1): By Corollary 5.5, the core C_0 of a projective plane of order $k + 1$ is a regular block matrix of format $(k+1)^2/(k+1)$. Thus, by Corollary 6.16, there is a $(k + 1, k)$-Latin configuration and finally a homogeneous configuration $I_{(k+2)k,k+1}$, by Corollary 6.17. □

Corollary 6.19. *All regular block matrices of format $n{\cdot}2/2$ are permutation equivalent.*

Proof. Let R_1, R_2 be regular block matrices of format $n2/2$. By Lemma 6.14, the matrices

$$R_1' = \operatorname{diag}(\Sigma_{r,n-1}^t, E_r)R_1 \operatorname{diag}(\Sigma_{r,n-1}, E_r)$$

and

$$R_2' = \operatorname{diag}(\Sigma_{r,n-1}^t, E_r)R_2 \operatorname{diag}(\Sigma_{r,n-1}, E_r)$$

are matrices with homogeneous margin and by Theorem 6.15, the cores of R_1' and R_2' are $(2, n-1)$-Latin configurations, i.e., of the form $\begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$, where P, Q are permutation matrices of size $n - 1$. Since internal permutations do not affect the homogeneous margin, the statement holds. □

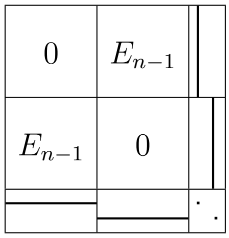

Figure 10: $(2, n)$-matrix with a homogeneous margin

By Corollary 6.19, all regular block matrices of format $n{\cdot}2/2$ are permutation equivalent to a matrix as in Figure 10.
Let $P = \{P_1, \dots, P_n\}$ be an (r, n)-parallel partition of the rows of an (r, n)-matrix. There is a parallel bundle in P that contains all rows of the controlling row bundle, by Lemma 6.8. Without loss of generality, we will always denote this bundle by P_n and analogously for the columns.

Theorem 6.20. *Let $r \leq n$. The matrix M is permutation equivalent to a regular block matrix of format nr/r if and only if M is an (r, n)-matrix and there are (r, n)-parallel partitions simultaneously for the rows and for the columns of M.*

Proof. First, let M be permutation equivalent to a regular block matrix of format nr/r. The block lines of this regular block matrix form the claimed parallel bundles and M is also permutation equivalent to a step hull of a regular block matrix of format $n(r-1)/(r-1)$ by Lemma 5.2. Hence, M is an (r, n)-matrix, by Lemma 6.9.
Conversely suppose that M is an (r, n)-matrix and there exist suitable parallel partitions of the rows and columns of M. By Lemma 2.1, we can permute the lines of each parallel bundle to a block line. Then, all single blocks are pseudo permutation matrices since each block row and each block column forms a parallel bundle. The blocks in the block row R and block column C containing the rows and columns of the controlling bundles, respectively, are permutation matrices by the line sum n. We permute R and C to the last block row and the last block column, respectively. With internal permutations as provided by Lemma 3.12 we can form a unit margin for the obtained block matrix. As we used only internal permutations, the block lines stay parallel and the blocks stay pseudo permutation matrices. The block matrix has format nr/r and satisfies the rectangular rule, by assumption. Clearly,

the unit core has constant line sum $r - 1$. Thus, we obtain a regular block matrix of format nr/r. $\square$

7 Reverse peeling of regular block matrices

In this section, we ask for properties of regular block matrices of format nr/r, to be a step core of a regular block matrix of format $n(r+1)/(r+1)$.
Let $r < n$ and let M be an (r,n)-matrix. Moreover, let $P = \{P_i \mid i \in [1,n]\}$ and $Q = \{Q_i \mid i \in [1,n]\}$ be (r,n)-parallel partitions of the rows of M where $P_n = Q_n$ is the controlling row bundle. We call P and Q *orthogonal*, if the partitions without the controlling bundles are orthogonal, i.e., $|P_i \cap Q_j| \leq 1$ for all $i,j \in [1,n-1]$. The block rows of a regular block matrix form an (r,n)-parallel partition of its rows, the so called *natural partition* of the rows. An (r,n)-parallel partition of the rows of the regular block matrix, which is orthogonal to the natural partition is called an *alternative partition* of the rows. In the same way, we define the terms orthogonal, natural and alternative for (r,n)-parallel partitions of columns.

Remark 7.1. Let B be a regular block matrix with an alternative partition of the rows. By Lemma 2.1, there is a row permutation, that leads to a matrix B', such that the natural partition of the rows of B' is formed by the rows of the alternative partition of the rows of B. Clearly, the natural partition of the rows of B then forms an alternative partition of the rows of B'. We call this procedure *to interchange the natural partition with the alternative partition of the rows.* The same holds for the columns.

The following lemma is straightforward.

Lemma 7.2. *Let P, Q be orthogonal (r,n)-parallel partitions of the rows of an (r,n)-matrix M and let S be a subset of the rows of M. Then $P \setminus S$ and $Q \setminus S$ are orthogonal. The dual statement holds for columns.*

Furthermore, let $R = \{R_i \mid i \in [1,n]\}$ be an (r,n)-parallel partition of the rows and let $C = \{C_i \mid i \in [1,n]\}$ be an (r,n)-parallel partition of the columns of an (r,n)-matrix M. We call R and C *complementary* if the partitions without the controlling bundles are complementary, i.e., there is a permutation π of the set $[1,n-1]$ such that R_i and $C_{\pi(i)}$ are not connected for all $i \in [1,n-1]$. Then, π is a complementary indication of R and C. For example, in the matrix shown in Fig 11, the natural partition of the rows is complementary with the natural partition of the columns with the complementary indication $\pi = \mathrm{id}$. Note, that a complementary indication is not unique in general.

0	$S_{1,2}$	$\cdots$	$S_{1,n-1}$	E_r
$S_{2,1}$	0	$\cdots$	$S_{2,n-1}$	E_r
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$S_{n-1,1}$	$S_{n-1,2}$	$\cdots$	0	E_r
E_r	E_r	$\cdots$	E_r	E_r

Figure 11: Regular block matrix of format nr/r with 0-blocks on the main diagonal

Lemma 7.3. *Let M be an (r,n)-matrix, let R_1, R_2 be (r,n)-parallel partitions of the rows of M and let C be an (r,n)-parallel partition of the columns of M. Moreover, let $\pi \in \mathcal{S}_{rn}$ be a row permutation and $\phi \in \mathcal{S}_{rn}$ a column permutation of M with representing matrices P, Q respectively and $M' = P^{-1}MQ$. Then,*

(1) *R_1, R_2 are orthogonal, if and only if R_1^π and R_2^π are orthogonal.*

(2) *R_1 and C are complementary in M by the complementary indication α, if and only if R_1^π and C^ϕ are complementary in M', by the complementary indication α.*

Proof. For (1), let B_1, B_2 be any bundles of R_1, R_2. It is $r \in B_1 \cap B_2$, if and only if $\pi(r) \in B_1^\pi \cap B_2^\pi$, by definition. Furthermore, as π is a permutation and therefore injective, it is $|B_1 \cap B_2| = |B_1^\pi \cap B_2^\pi|$.
We prove (2). For this, let $R_{1,i}$ and $C_{\alpha(i)}$ be bundles of R_1 and C, respectively and $r \in R_{1,i}$ as well as $c \in C_{\alpha(i)}$. Equivalently, $\pi(r) \in R_{1,i}^\pi$ and $\phi(c) \in C_{\alpha(i)}^\phi$. By definition, the intersection of r and c in M has the same entry as the intersection of $\pi(r)$ and $\phi(c)$ in M'. Therefore, (r,c) is a connected pair in M, if and only if $(\pi(r), \phi(c))$ is a connected pair in M' and the statement holds. □

For the convenience of the reader, we illustrate the terms orthogonal, complementary and complementary indication with the following example that will accompany us throughout this section.

Example 7.4. Consider the $(2,4)$-matrix M with homogeneous margin, cf. Figure 10.

$$M = \left(\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \\ \hline 4 \\ 5 \\ 6 \\ \hline 7 \\ 8 \end{array}$$

$$\begin{array}{ccc|ccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

We denote the rows and the columns with their absolute indices as indicated above. First, we determine the $(2,4)$-parallel partitions of the rows of M. By Lemma 6.8, the rows 7 and 8 form a bundle in each $(2,4)$-parallel partition of the rows of M. Moreover, by parallelism, each of the remaining three bundles can contain at most one row of $\{1,2,3\}$ and one row of $\{4,5,6\}$. Since all bundles are of cardinality 2, each bundle must contain exactly one row of $\{1,2,3\}$ and exactly one row of $\{4,5,6\}$. Therefore, there are six $(2,4)$-parallel partitions of the rows of M. The same holds for the columns.

$$\begin{aligned} R_1 = \quad & \{\{1,4\},\{2,5\},\{3,6\},\{7,8\}\} & = C_1 \\ R_2 = \quad & \{\{1,5\},\{2,6\},\{3,4\},\{7,8\}\} & = C_2 \\ R_3 = \quad & \{\{1,6\},\{2,4\},\{3,5\},\{7,8\}\} & = C_3 \\ R_4 = \quad & \{\{1,4\},\{2,6\},\{3,5\},\{7,8\}\} & = C_4 \\ R_5 = \quad & \{\{1,5\},\{2,4\},\{3,6\},\{7,8\}\} & = C_5 \\ R_6 = \quad & \{\{1,6\},\{2,5\},\{3,4\},\{7,8\}\} & = C_6 \end{aligned}$$

Obviously, R_1, R_2, R_3 are pairwise orthogonal and R_4, R_5, R_6 are pairwise orthogonal as well as C_1, C_2, C_3 and C_4, C_5, C_6. We denote the bundles of R_i and C_i in the following way: $R_{i,4} = C_{i,4} = \{7,8\}$ and for $j \in [1,3]$, $R_{i,j} = C_{i,j}$ is the bundle that contains the row or column j, respectively. Then, all complementary indications must be fixed point free. Note, that there are only two fixed point free permutations in $\mathcal{S}_3$, namely the 3-cycles $(1\ 3\ 2)$ and $(1\ 2\ 3)$.

$$
\begin{array}{|ccccc|ccccc|}
\hline
 & & & & 1 & 1 & & & & \\
 & & & & 1 & & 1 & & & \\
 & & & & \vdots & & & \ddots & & \\
 & & & & 1 & & & & 1 & \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
1 & & & & 0 & & & & & 1 \\
 & 1 & & & 0 & & & & & 1 \\
 & & \ddots & & \vdots & & & & & \vdots \\
 & & & 1 & 0 & & & & & 1 \\
 & & & & 0 & 1 & 1 & \cdots & 1 & 1 \\
\hline
\end{array}
$$

Figure 12: Step hull of the Γ-head Γ_n

To verify, whether R_i is complementary with C_j or not (by lack of space, we restrict to $i, j \in [1, 3]$), we observe, that $C_{1,1}$ is not connected with the rows $\{2, 3, 5, 6\}$ and hence,
$C_{1,1}$ is not connected with the bundles $R_{1,2}, R_{1,3};\ R_{2,2};\ R_{3,3}$.
$C_{1,2}$ is not connected with the bundles $R_{1,1}, R_{1,3};\ R_{2,3};\ R_{3,1}$.
$C_{1,3}$ is not connected with the bundles $R_{1,1}, R_{1,2};\ R_{2,1};\ R_{3,2}$.
Thus, R_2 is complementary with C_1 by the complementary indication $\phi_{2,1} = (1\ 3\ 2)$, and R_3 is complementary with C_1 by the complementary indication $\phi_{3,1} = (1\ 2\ 3)$. Clearly, both 3-cycles are complementary indications of R_i and C_i for all $i \in [1, 3]$, denoted by $\phi_{i,i,a} = (1\ 3\ 2)$ and $\phi_{i,i,b} = (1\ 2\ 3)$. In fact, R_i and C_j are complementary for any $i, j \in [1, 3]$ and the remaining complementary indications are

$$\phi_{1,2} = (1\ 2\ 3),\ \phi_{3,2} = (1\ 3\ 2),\ \phi_{1,3} = (1\ 3\ 2),\ \phi_{2,3} = (1\ 2\ 3).$$

7.1 Regular envelopable block matrices

A *regular envelopable block matrix* is a regular block matrix whose step hull is again permutation equivalent to a regular block matrix.
A regular block matrix of format $n \cdot 1/1$ is equal to the Γ-matrix Γ_n, which is obviously regular envelopable, cf. Figure 12, where the ith row together with the $(n+i)$th row form a parallel bundle and analogously the ith column together with the $(n + i)$th column form a parallel bundle. Therefore, there is a $(2, n)$-parallel partition of the rows and a $(2, n)$-parallel partition of the

columns. By Theorem 6.20, the matrix is permutation equivalent to a regular block matrix of format $n{\cdot}2/2$.
From now on, let $r \geq 2$. The following remarks denote arguments, that are often used in later proofs.

Remark 7.5. Let S be an (r, n)-matrix with a step margin of structure $n/(r-1)$ and g a row from its margin row. All rows in the margin row meet each other. Thus, all other rows of a parallel bundle containing g, must be inside the core of S. Consequently, there is a parallel bundle containing r rows with g as an element if and only if there exists a parallel bundle in S containing $r-1$ rows inside the core of S, such that none of the rows meet g. The dual statement holds for columns.

Remark 7.6. Let $r \geq 2$ and let B be a regular block matrix of format nr/r with step hull S. Let $\{g_i \mid i \in [1, n]\}$ be the set of rows of the margin row, where g_n is the last row of S and let $N_C = \{N_{C,j} \mid j \in [1, n]\}$ be the natural partition of the columns of B. Moreover, let $\{Q_j \mid j \in [1, n+1]\}$ denote the block columns of S, such that $N_{C,j}$ is a submatrix of Q_j for all $j \in [1, n-1]$. For any $j \in [1, n-1]$ there is an $i \in [1, n-1]$, such that the intersection of Q_j and g_i has exclusively entries 1, as the step margin is of structure n/r, by Remark 5.1. Moreover, the origin of the rows g_i is the last column. By the line sum $r+1$, g_i has the entry 0 in all columns, except of the last column and precisely one block column Q_j. The dual statement holds.
Observe, that the last row and the last column of S are the origins of the step margin column and the step margin row, respectively and do not have an entry 1 outside the head, Figure 7, i.e., the last row of S is not connected with any column inside B and dually.

Based on Remark 7.6, we will sometimes speak of the origin of a block column of S or of the corresponding block column in B, i.e., its intersection with B.

Lemma 7.7. *Let $2 \leq r < n$, let B be a regular block matrix of format nr/r with step hull S and let $R = \{R_i \mid i \in [1, n]\}$ be an alternative partition of the rows of B that is complementary with the natural partition $N_C = \{N_{C,i} \mid i \in [1, n]\}$ of the columns of B by the complementary indication π. Moreover, for $i \in [1, n-1]$, let g_i denote the origin of $N_{C,i}$ in S and g_n the last row of S.*
Then, the bundles of R are parallel in S and

$$\{R_i \dot\cup \{g_{\pi(i)}\} \mid i \in [1, n-1]\} \dot\cup \{R_n \dot\cup \{g_n\}\}$$

is an $(r+1,n)$-parallel partition of the rows of S.
The dual statements hold.

Proof. Since R is orthogonal to the natural partition of the rows of B, no two lines of a bundle of R meet in an origin of a block row. Hence, the bundles of R are parallel in S as they are parallel in B.
Clearly, $g_{\pi(i)}$ does not meet any row of R_i, as R_i and $N_{C,\pi(i)}$ are not connected, since π is a complementary indication of R and N_C. Hence, $R_i \cup \left\{g_{\pi(i)}\right\}$ is a parallel bundle of cardinality $r+1$ of the rows of S for all $i \in [1, n-1]$. By Remark 7.6, it follows from $i \neq j$, that $g_i \neq g_j$. Thus, $\{R_i \dot\cup \{g_{\pi(i)}\} \mid i \in [1, n-1]\}$ is a partition of the rows with row sum $r+1$ in S.
Finally, the parallel bundle formed by the original unit margin row stays parallel and is completed by the last row of the step margin, as all its entries 1 are inside the head. □

Lemma 7.8. *Let $2 \leq r < n$ and B be a regular block matrix of format nr/r with step hull S and let $R' = \{R'_i \mid i \in [1, n]\}$ be an $(r+1, n)$-parallel partition of the rows of S. Moreover, for $j \in [1, n-1]$, let g_j denote the origin of the block column Q_j of S and let g_n denote the last row of S.*
There is a permutation $\pi \in \mathcal{S}_{n-1}$, such that $g_{\pi(i)} \in R'_i$. Furthermore, R' restricted to the rows of B is an alternative partition of the rows of B that is complementary with the natural partition of the columns of B, by the complementary indication π.
The dual statements hold.

Proof. The unit margin row forms the controlling row bundle R_n of B. Note that all rows g_i except of g_n meet some row of R_n in S. Thus, g_n is the only possibility to supplement R_n to a parallel bundle of cardinality $r+1$, namely the controlling row bundle R'_n of S.
Moreover, by Remark 7.5, two different rows from the step margin row of S must be in different bundles of any $(r+1,n)$-parallel partition of the rows of S. Hence, there is a permutation $\pi \in \mathcal{S}_{n-1}$, such that $g_{\pi(i)} \in R'_i$ for all $i \in [1, n-1]$.
For $j \in [1, n-1]$, the row g_j is the origin of the block column Q_j of S, cf. Figure 7. Thus, $R_i := R'_i \setminus \{g_{\pi(i)}\}$ is not connected with $Q_{\pi(i)}$ by parallelism. As R_i is not connected with $Q_{\pi(i)}$ and therefore not connected with the corresponding block column of B, that is a submatrix of $Q_{\pi(i)}$, the permutation π is a complementary indication of R and the natural partition of the columns of B. The dual statements hold analogously. □

Theorem 7.9. *Let $2 \leq r < n$ and M be an (r,n)-matrix. Then M is permutation equivalent to a regular envelopable block matrix, if and only if there are simultaneously two orthogonal (r,n)-parallel partitions R_1, R_2 of the rows and C_1, C_2 of the columns of M, such that R_1, C_2 are complementary as well as R_2, C_1 are complementary.*

Proof. Let B be a regular block matrix of format nr/r, which is permutation equivalent to M and let the step hull S of B be permutation equivalent to a regular block matrix of format $n(r+1)/(r+1)$. We prove the existence of an alternative row partition of B which is complementary with the natural column partition of B. By Theorem 6.20, there is an $(r+1,n)$-parallel partition R'_2 of the rows of S. By Lemma 7.8, R'_2 restricted to the rows of B, denoted by R_2 is an (r,n)-parallel partition of the rows of B, which is orthogonal to the natural partition R_1 of the rows of B and complementary with the natural partition C_1 of the columns of B. The dual statement holds, i.e., there is an alternative partition C_2 of the columns, that is complementary with R_1.
Conversely, we can permute one of the orthogonal (r,n)-parallel partitions of the rows to block rows of M, by Lemma 2.1. This way, we obtain a natural partition of the rows and an alternative partition of the rows. Now, we permute one of the orthogonal (r,n)-parallel partitions of the columns to block columns, such that the alternative partition of the rows is complementary with the natural partition of the columns and the natural partition of the rows is complementary with the alternative partition of the columns. This is possible, since line permutations maintain the complementarity, by Lemma 7.3. Applying internal permutations, we obtain a regular block matrix B of format nr/r.
By Theorem 6.20, it is enough to show that there exist $(r+1,n)$-parallel partitions simultaneously for the rows and for the columns in the step hull of B. By Lemma 7.7, there are such parallel partitions in the step hull of B, since B has an alternative partition of the rows and of the columns. □

The following corollary denotes the special situation of Theorem 7.9, where we need not permute the lines of the regular block matrix before forming the step hull. The proof is completely analogous to the proof of Theorem 7.9 without permuting lines.

Corollary 7.10. *Let $2 \leq r \leq n$ and B be a regular block matrix of format nr/r. Then B is regular envelopable, if and only if there are simultaneously* *for the rows and for the columns of B alternative partitions, such that the natural row partition is complementary with the alternative column partition, and the natural column partition is complementary with the alternative row partition.*

Corollary 7.11. *Let B be a regular envelopable block matrix and B' the regular block matrix obtained by interchanging the natural partition of the rows and of the columns of B with the alternative partition of the rows and of the columns, respectively. Then B' is regular envelopable.*

Proof. By Corollary 7.10, B has an alternative partition each of the rows and columns that are complementary with the natural partition of the columns and rows, respectively. Hence, B' has an alternative partition each of the rows and the columns, since the orthogonality of partitions is maintained by line permutations. Moreover, as we interchanged both the natural partition with the alternative partition of the rows and of the columns, provided by Remark 7.1 the alternative partition of the columns is again complementary with the natural partition of the rows and vice versa the alternative partition of the rows is again complementary with the natural partition of the columns. Again by Corollary 7.10, B' is regular envelopable. □

Corollary 7.12. *Each regular envelopable block matrix is permutation equivalent to a regular block matrix with at least one* 0*-block in each block line of the unit core, cf. Figure* 11*.*

Proof. By Corollary 7.10, a regular envelopable block matrix has in particular an alternative partition of the rows that is complementary with the natural partition of the columns. By Remark 7.1, we can interchange the natural partition of the rows with the alternative partition of the rows. Now, the natural partition of the rows is complementary with the natural partition of the columns. Hence, there is a 0-block in each block line. □

The converse direction of Corollary 7.12 is not true, in general, since for example the regular block matrix of format $6{\cdot}4/4$ given in Appendix B does not have two orthogonal $(4, 6)$-parallel partitions of the rows but a 0-block in each block line.

7.2 Complementary graphs

Here, we mention some basics of the theory of bipartite graphs[3] that can be applied to the complementarity of parallel partitions in (r,n)-matrices. For the convenience of the reader, we give the definitions of some well known terms.

First, let $G = (V, E)$ be a graph and let $V' \subseteq V$ as well as $E' = \{(v, w) \in E \mid \{v, w\} \subset V'\}$. Then we call (V', E') a *subgraph* of G. For an arbitrary subset $E' \subseteq E$, we call (V, E') a *partial graph* of G.

Recall, that an edge $e \in E$ of a bipartite graph $(V_1 \dot\cup V_2, E)$ is a pair of vertices $(v_1, v_2) \in V_1 \times V_2$ and consequently $E \subset V_1 \times V_2$. Two edges are *adjacent*, if they have a common vertex, i.e., (v_1, v_2) and (w_1, w_2) are adjacent if (either) $v_1 = w_1$ or $v_2 = w_2$. A *matching* of a bipartite graph $(V_1 \dot\cup V_2, E)$ is a subset $E_0 \subset E$ such that there are no adjacent edges in E_0. The matching E_0 is a *perfect matching*, if

$$\forall v \in V_1 \dot\cup V_2 \exists w \in V_1 \dot\cup V_2 : (v, w) \in E_0$$

or equivalently, $|E_0| = |V_1| = |V_2|$. These terms are defined more generally in [2].

Let $\mathrm{d}(v)$ denote the degree of the vertex v. A *regular graph* is a graph (V, E) where $\forall v, u \in V : \ \mathrm{d}(v) = \mathrm{d}(u)$. A bipartite graph $(V_1 \dot\cup V_2, E)$ is called *almost complete*, if $\forall v \in V_1 : \mathrm{d}(v) = |V_2| - 1$ and $\forall v \in V_2 : \mathrm{d}(v) = |V_1| - 1$. For example, $F_{m,m}$ is an almost complete bipartite graph, defined by:

$$F_{m,m} = ([1, m] \dot\cup [-m, -1], \{(i, -j) \mid i, j \in [1, m], \ i \neq j\})$$

Note, that $F_{m,m}$ is a regular graph.

Lemma 7.13. *Let $G = (R \dot\cup C, E)$ be an almost complete regular bipartite graph. Then, $|R| = |C|$ and G is isomorphic to $F_{|R|,|R|}$.*

Proof. Since G is almost complete, $\forall r \in R : \mathrm{d}(r) = |C| - 1$ and $\forall c \in C : \mathrm{d}(c) = |R| - 1$. Hence, since G is regular, $|C| = |R| =: m$. For the second part, we construct an isomorphism. Let $R = \{r_1, \ldots, r_m\}$ and $C = \{c_1, \ldots, c_m\}$. For every r_i there is exactly one c_j with $(r_i, c_j) \notin E$. For those elements, let $\phi_V(r_i) = i$ and $\phi_V(c_j) = -i$.

Obviously, $\phi_V : R \dot\cup C \to [1, m] \dot\cup [-m, -1]$ is a bijection. There is the natural bijection of the edges $\phi_E : E \to \{(i, -j) \mid i, j \in [1, m], \ i \neq j\}$ defined by $\phi_E : (v, w) \to (\phi_V(v), \phi_V(w))$. Hence, $G \cong (\phi_V(R \dot\cup C), \phi_E(E)) = F_{m,m}$. $\square$

[3] For the definitions of graph, bipartite, degree, graph isomorphism, etc., cf. [1]

Almost complete regular bipartite graphs are briefly called *acr-graphs*.

Lemma 7.14. *Let $G = (R\dot{\cup}C, E)$ be a bipartite graph. The following are equivalent:*

(1) *There is an acr-partial graph of G.*

(2) $|R| = |C|$ *and* $\forall v \in R\dot{\cup}C : \mathrm{d}(v) \geq |R| - 1$.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 7.13, since the degree of a vertex in $F_{|R|,|R|}$ is $|R| - 1$.
(2) $\Rightarrow$ (1): Note that for any $v \in R\dot{\cup}C : \mathrm{d}(v) \leq |R|$ and therefore, the degree of any vertex is either $|R| - 1$ or $|R|$. Let

$$R' := \{v \in R \mid \mathrm{d}(v) = |R|\} \quad \text{and} \quad C' := \{v \in C \mid \mathrm{d}(v) = |R|\}.$$

It follows $|R'| = |C'|$ by double counting the edges, i.e.,

$$|R'| = \sum_{v \in R} (\mathrm{d}(v) - |R| + 1) = \sum_{v \in C} (\mathrm{d}(v) - |R| + 1) = |C'|,$$

and the subgraph $G' = (R'\dot{\cup}C', E')$ of G is a complete bipartite graph, cf. Figure 13. Thus, there is a perfect matching E'_0 of G'. Hence, the partial graph $(R\dot{\cup}C, E \setminus E'_0)$ is an almost complete regular graph. □

Clearly, there are $|R'|!$ different ways to obtain an acr-partial graph of G, as the complete bipartite graph G' has $|R'|!$ different perfect matchings.
Let M be an (r, n)-matrix, let $\mathcal{R}$ be a set of pairwise orthogonal (r, n)-parallel partitions of the rows of M and let $\mathcal{C}$ be a set of pairwise orthogonal (r, n)-parallel partitions of its columns. For an element $x \in \mathcal{R}\dot{\cup}\mathcal{C}$, we define the *complementary-degree in* $\mathcal{R}\dot{\cup}\mathcal{C}$, briefly denoted as $\mathrm{cd}(x)$.

$$\mathrm{cd}(x) := \begin{cases} |\{C \in \mathcal{C} \mid x \text{ is complementary with } C\}| & \text{for } x \in \mathcal{R} \\ |\{R \in \mathcal{R} \mid R \text{ is complementary with } x\}| & \text{for } x \in \mathcal{C} \end{cases}$$

Hence, $G = (\mathcal{R}\dot{\cup}\mathcal{C}, \{(R, C) \in \mathcal{R} \times \mathcal{C} \mid R \text{ is complementary with } C\})$ is a bipartite graph, the *complementary graph of* $(M, \mathcal{R}, \mathcal{C})$ and by definition, for all $v \in \mathcal{R}\dot{\cup}\mathcal{C} :\ \mathrm{cd}(v) = \mathrm{d}(v)$.
An *acr-partial complementary graph of* $(M, \mathcal{R}, \mathcal{C})$ is an acr-partial graph of the complementary graph of $(M, \mathcal{R}, \mathcal{C})$. The existence of an acr-partial complementary graph of $(M, \mathcal{R}, \mathcal{C})$ implies $|\mathcal{R}| = |\mathcal{C}|$ and that the complementary degree of any $x \in \mathcal{R}\dot{\cup}\mathcal{C}$ is at least $|\mathcal{R}| - 1$ in $\mathcal{R}\dot{\cup}\mathcal{C}$, by Lemma 7.14.

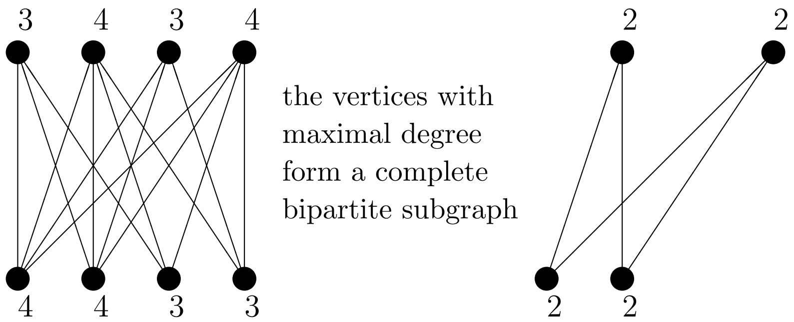

Figure 13: A bipartite graph that has an acr-partial graph and its complete bipartite subgraph

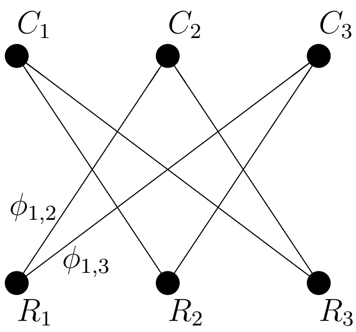

Figure 14: An acr-partial complementary graph of Example 7.4

Example 7.4 (continued). Note that the complementary graph G of $(M, \{R_1, R_2, R_3\}, \{C_1, C_2, C_3\})$ is a complete bipartite graph, i.e., each vertex has complementary degree 3. To obtain an acr-partial graph of G, we have to omit a perfect matching of the whole graph G. For example, we can omit the edges (R_i, C_i) for $i \in [1, 3]$. The resulting acr-partial graph $\widetilde{G}$ of G is equal to $F_{3,3}$, cf. Figure 14.

Remark 7.15. Let M be an (r, n)-matrix and let $\mathcal{R} = \{R_1, \ldots, R_m\}$ be a set of pairwise orthogonal (r, n)-parallel partitions of the rows of M and let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a set of pairwise orthogonal (r, n)-parallel partitions of the columns of M. If there is an acr-partial complementary graph of $(M, \mathcal{R}, \mathcal{C})$, we may assume that

$$R_i \text{ is complementary with } C_j \text{ if } i \neq j,$$

by Lemma 7.13 and Remark 7.1.

Otherwise, if there is no acr-partial complementary graph of $(M, \mathcal{R}, \mathcal{C})$, there is at least one vertex in $\mathcal{R} \dot{\cup} \mathcal{C}$ with complementary degree less than $m-1$ in $\mathcal{R} \dot{\cup} \mathcal{C}$, by Lemma 7.14. Thus, the assumption above fails.

7.3 Complementary indication matrices

A complementary indication is a permutation and has a representing matrix. Let M be an (r, n)-matrix and let $\mathcal{R} = \{R_i \mid i \in [1, k]\}$ be a set of orthogonal (r, n)-parallel partitions of the rows of M and $\mathcal{C} = \{C_i \mid i \in [1, l]\}$ be a set of orthogonal (r, n)-parallel partitions of the columns of M. Moreover, let $\widetilde{G}$ be a partial graph of the complementary graph of $(M, \mathcal{R}, \mathcal{C})$ and let $A_{i,j}$ be either the representing matrix of a complementary indication of R_i and C_j, if (R_i, C_j) is an edge of $\widetilde{G}$ or otherwise the square 0-matrix of size $n-1$. The block matrix $(A_{i,j})_{i \in [1,k], j \in [1,l]}$ is called a *complementary indication matrix* of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$.
There can be several complementary indication matrices of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$, since a partition of the rows and a partition of the columns can be complementary by several complementary indications, cf. R_1, C_1 in Example 7.4.
Note, that if $\widetilde{G}$ is an acr-partial complementary graph, the complementary indication matrices of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$ are square of size $|\mathcal{R}|(n-1)$ and there is exactly one 0-block in each block line, by Lemma 7.13.

Example 7.4 (continued). The acr-partial complementary graph given in Figure 14 has the complementary indication matrix

$$\mathcal{I} = \begin{pmatrix} 0 & P_{1,2} & P_{1,3} \\ P_{2,1} & 0 & P_{2,3} \\ P_{3,1} & P_{3,2} & 0 \end{pmatrix} \tag{7.1}$$

where $P_{i,j}$ is the representing matrix of $\phi_{i,j}$ and therefore a permutation matrix. This matrix satisfies the rectangular rule, by Lemma 3.3 and the constellation of the 0-blocks.

Let $R_1 = \{R_{1,i} \mid i \in [1, n]\}$ and $R_2 = \{R_{2,i} \mid i \in [1, n]\}$ be partitions of a finite set each into n parts and let π be a permutation of the set $[1, n]$. We say, the triplet (R_1, R_2, π) satisfies the *disjoint-condition*, if $R_{1,i} \cap R_{2,\pi(i)} = \emptyset$ for all $i \in [1, n]$. Now let π_1, π_2 be permutations of the set $[1, n]$. We say, the pairs (R_1, π_1) and (R_2, π_2) satisfy the *disjoint-condition*, if $R_{1,\pi_1^{-1}(i)} \cap R_{2,\pi_2^{-1}(i)} = \emptyset$ for all $i \in [1, n]$, i.e., if the triplet $(R_1, R_2, \pi_2^{-1}\pi_1)$ satisfies the disjoint-condition.

Let M be an (r,n)-matrix, let $R_1 = \{R_{1,i} \mid i \in [1,n]\}$ as well as $R_2 = \{R_{2,i} \mid i \in [1,n]\}$ be orthogonal (r,n)-parallel partitions of the rows of M and let C be an (r,n)-parallel partition of the columns of M where $R_{1,n}, R_{2,n}, C_n$ are the controlling bundles. Moreover, let R_1, R_2 be complementary with C by the complementary indications π_1 and π_2, respectively. Then, π_1 and π_2 are permutations of the set $[1, n-1]$. We say, (R_1, π_1) and (R_2, π_2) satisfy the *disjoint-condition*, if the bundles $R_{1,\pi_1^{-1}(i)}$ and $R_{2,\pi_2^{-1}(i)}$ are disjoint for all $i \in [1, n-1]$.

Analogously, for two orthogonal (r,n)-parallel partitions $C_1 = \{C_{1,j}\}$ and $C_2 = \{C_{2,j}\}$ of the columns of M and an (r,n)-parallel partition of the rows of M which are complementary by the complementary indications π_1 and π_2, respectively, (C_1, π_1) and (C_2, π_2) satisfy the disjoint-condition, if the bundles $C_{1,\pi_1(i)}$ and $C_{2,\pi_2(i)}$ are disjoint for all $i \in [1, n-1]$. Note the difference to the definition of the disjoint-condition for row partitions. This difference is necessary, to respect the notation we introduced for complementary indications π, namely that the ith row bundle is not connected with the $\pi(i)$th column bundle.

Finally, let $\mathcal{R}$ be a set of pairwise orthogonal (r,n)-parallel partitions of the rows of M and $\mathcal{C}$ be a set of pairwise orthogonal (r,n)-parallel partitions of the columns of M. Moreover, let $\widetilde{G}$ be an acr-partial complementary graph of $(M, \mathcal{R}, \mathcal{C})$ and let $L = (A_{i,j})_{i,j}$ be a complementary indication matrix of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$. If $A_{i,j} \neq 0_{(n-1)\times(n-1)}$, let $\alpha_{i,j}$ denote the corresponding complementary indication.

We say, $(\mathcal{R}, \mathcal{C}, L)$ satisfies the disjoint-condition, or briefly L satisfies the disjoint-condition, if

- $(R_i, \alpha_{i,j})$ and $(R_k, \alpha_{k,j})$ satisfy the disjoint-condition for all possible $i, k \in [1, |\mathcal{R}|], j \in [1, |\mathcal{C}|]$, i.e., for all i, j, k where $A_{i,j} \neq 0_{(n-1)\times(n-1)}$ as well as $A_{k,j} \neq 0_{(n-1)\times(n-1)}$ and

- $(C_i, \alpha_{i,j})$ and $(C_l, \alpha_{i,l})$ satisfy the disjoint-condition for all possible $i \in [1, |\mathcal{R}|], j, l \in [1, |\mathcal{C}|]$.

Example 7.4 (continued). Recall, that $\phi_{1,3} = (1\ 3\ 2)$ and $\phi_{2,3} = (1\ 2\ 3)$.

Since

$$\begin{aligned} R_{1,\phi_{1,3}^{-1}(1)} \cap R_{2,\phi_{2,3}^{-1}(1)} &= & R_{1,2} \cap R_{2,3} &= & \{2,5\} \cap \{3,4\} = \emptyset, \\ R_{1,\phi_{1,3}^{-1}(2)} \cap R_{2,\phi_{2,3}^{-1}(2)} &= & R_{1,3} \cap R_{2,1} &= & \{3,6\} \cap \{1,5\} = \emptyset \text{ and} \\ R_{1,\phi_{1,3}^{-1}(3)} \cap R_{2,\phi_{2,3}^{-1}(3)} &= & R_{1,1} \cap R_{2,2} &= & \{1,4\} \cap \{2,6\} = \emptyset, \end{aligned}$$

the pairs $(R_1, \phi_{1,3})$ and $(R_2, \phi_{2,3})$ satisfy the disjoint-condition. In fact, $(\{R_1, R_2, R_3\}, \{C_1, C_2, C_3\}, \mathcal{I})$ satisfies the disjoint-condition.
Not so with a complementary matrix of the originally complementary graph, since neither the pairs $(R_1, \phi_{1,1,a})$, $(R_2, \phi_{2,1})$ nor the pairs $(R_1, \phi_{1,1,b})$, $(R_2, \phi_{2,1})$ satisfy the disjoint-condition:

$$\begin{aligned} R_{1,\phi_{1,1,a}^{-1}(1)} \cap R_{2,\phi_{2,1}^{-1}(1)} &= & R_{1,2} \cap R_{2,2} &= & \{2,5\} \cap \{2,6\} = \{2\} \neq \emptyset, \\ R_{1,\phi_{1,1,b}^{-1}(1)} \cap R_{2,\phi_{2,1}^{-1}(1)} &= & R_{1,3} \cap R_{2,2} &= & \{3,6\} \cap \{2,6\} = \{6\} \neq \emptyset. \end{aligned}$$

Lemma 7.16. *Let $1 < r < n-1$. Let A, B be orthogonal (r,n)-parallel partitions of the rows of a regular block matrix of format nr/r. There is a bijection ϕ between the bundles with row sum r of A and B, such that the bundles $A' \in A$ and $\phi(A') \in B$ are disjoint, i.e., $A' \cap \phi(A') = \emptyset$. The dual statement holds for columns.*

Proof. We are not interested in the controlling row bundle and exclude it from our considerations. We label the remaining bundles of A with the numbers $1, 2, \ldots, n-1$. Since A is an (r,n)-parallel partition of a regular block matrix of format nr/r, there is precisely one row of every internal index in each bundle of A, by Lemma 6.3. The same holds for B. Thus, a row r is uniquely described by its internal index j and its membership in bundle k of A and conversely, each pair $(j,k) \in [1,r] \times [1, n-1]$ describes a row together with A. The same holds for B. Hence, for any pair (j,k), there is an l such that (j,l) together with B describes the same row as (j,k) together with A. The membership of a row with internal index j in the bundle k of A and the bundle l of B is indicated by the triplet (j,k,l). These triplets form an $r \times (n-1)$ matrix R via

$$(j,k,l) \iff R(j,k) = l.$$

Since each internal index appears exactly once in each bundle of B, it is

$$\forall j \in [1,r]: \; R(j,k_1) = R(j,k_2) \Longrightarrow k_1 = k_2,$$

i.e., the entries in a row of R are pairwise different. As all entries are elements of $[1, n-1]$ and there are $n-1$ positions in a row, each entry appears precisely once in each row of R.
By the orthogonality of A and B, it follows

$$\forall k \in [1, n-1] : \; R(j_1, k) = R(j_2, k) \Longrightarrow j_1 = j_2,$$

i.e., the entries in a column of R are pairwise different. Thus, R is an r-line Latin rectangle[4] of order $n-1$.
The bundle k of A contains the rows (j, k) for $j \in [1, r]$. Clearly, the row (j, k) is in bundle $l = R(j, k)$ of B. Thus, the bundles of B, that do not appear in the kth column of R are disjoint to the bundle k of A, i.e. if $R(j, k) \neq l_0$ for all $j \in [1, r]$, the bundle k of A and the bundle l_0 of B are disjoint. Since $r < n-1$, we can extend the Latin rectangle R to an $(r+1)$-line Latin rectangle R', cf. [9, Theorem III.2.2]. The last row of R' denotes the required bijection ϕ, by $\phi(k) = R'(r+1, k)$ for $k \in [1, n-1]$. □

7.4 k-regular envelopable block matrices

Let B_0 be a regular envelopable block matrix. There is a regular block matrix B_1 that is permutation equivalent to the step hull of B_0. We can iterate this, as long as B_i is again regular envelopable, since there is a regular block matrix B_{i+1} that is permutation equivalent to the step hull of B_i.
Let $k \in \mathbb{N}$. A regular block matrix B_0 is said to be *k-regular envelopable*, if $k-1$ is maximal such, that there is a sequence $B_0, B_1, \ldots, B_{k-1}$ of regular envelopable block matrices where B_i is permutation equivalent to the step hull of B_{i-1}, cf. Figure 15. Thus, a 1-regular envelopable block matrix is a regular envelopable matrix and a 0-regular envelopable block matrix is a regular block matrix.

Example 7.4 (continued). Theorem 7.18 will show that the matrix M is permutation equivalent to a 2-regular envelopable block matrix of format $4{\cdot}2/2$, since

- there are 3 pairwise orthogonal $(2, 4)$-parallel partitions of the rows and of the columns of M.

- those partitions have an acr-partial complementary graph such, that

[4] Latin rectangles are introduced for example in [9].

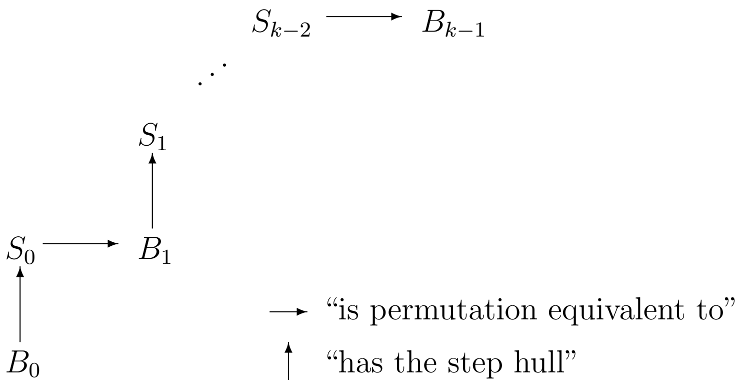

Figure 15: A sequence of regular envelopable block matrices B_i

- there is a complementary indication matrix that satisfies the disjoint-condition.

Hence, M is permutation equivalent to a matrix whose step hull is permutation equivalent to a regular envelopable block matrix of format $4{\cdot}3/3$. Thus, there is a regular block matrix of format $4{\cdot}4/4$ and consequently a projective plane of order 4, by Proposition 4.2.

Lemma 7.17. *Let R_1, R_2 be orthogonal alternative partitions of the rows of a regular block matrix B of format nr/r, that are both complementary with the natural partition N_C of the columns of B by the complementary indications π_1 and π_2, respectively. Moreover, let (R_1, π_1) and (R_2, π_2) satisfy the disjoint-condition.*
Then R_1, R_2 can be extended to orthogonal $(r+1, n)$-parallel partitions of the rows of the step hull S of B. The dual statement holds, too.

Proof. For $i \in [1, n-1]$, let g_i denote the origin of $N_{C,i}$ in S and g_n the last row of S, i.e., $g_1, \ldots, g_n$ are the rows of the margin row of S. By Lemma 7.7, $R'_1 = \{R_{1,i} \dot\cup \{g_{\pi_1(i)}\} \mid i \in [1, n-1]\}$ and $R'_2 = \{R_{2,i} \dot\cup \{g_{\pi_2(i)}\} \mid i \in [1, n-1]\}$ are $(r+1, n)$-parallel partitions of the rows of S. Therefore, $R'_{1,\pi_1^{-1}(i)} = R_{1,\pi_1^{-1}(i)} \dot\cup \{g_i\}$ and $R'_{2,\pi_2^{-1}(i)} = R_{2,\pi_2^{-1}(i)} \dot\cup \{g_i\}$. Since (R_1, π_1) and (R_2, π_2) satisfy the disjoint-condition, $R_{1,\pi_1^{-1}(i)} \cap R_{2,\pi_2^{-1}(i)} = \emptyset$. Thus, $|R'_{1,\pi_1^{-1}(i)} \cap R'_{2,\pi_2^{-1}(i)}| = 1$. All other pairs $R'_{1,i}, R'_{2,j}$ do not have a common row in the margin row of S. Hence,

$$|R'_{1,i} \cap R'_{2,j}| = |R_{1,i} \cap R_{2,j}| \leq 1,$$

if $\pi_1(i) \neq \pi_2(j)$, as R_1, R_2 are orthogonal in B. □

Theorem 7.18. *Let $2 \leq r \leq n-2$ and M be an (r,n)-matrix. Then M is permutation equivalent to a 2-regular envelopable block matrix, if and only if there is a set $\mathcal{R}$ of three pairwise orthogonal (r,n)-parallel partitions of the rows and a set $\mathcal{C}$ of three pairwise orthogonal (r,n)-parallel partitions of the columns of M with an acr-partial complementary graph $\widetilde{G}$, such that there is a complementary indication matrix of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$, that satisfies the disjoint-condition.*

Proof. Let M be permutation equivalent to a 2-regular envelopable block matrix B of format nr/r. By definition, the step hull S of B is permutation equivalent to a regular envelopable block matrix. Thus, by Theorem 7.9, the step hull S has two orthogonal $(r+1,n)$-parallel partitions R_1', R_2' of the rows and C_1', C_2' of the columns, each of complementary-degree ≥ 1 in $\{R_1', R_2'\} \dot\cup \{C_1', C_2'\}$. Moreover, by Lemma 7.3, it is enough, to show that B has the claimed properties, as they are maintained by line permutations.
By Lemmata 7.2 and 7.8, R_1' and R_2' restricted to the rows of B are orthogonal alternative (r,n)-parallel partitions of the rows of B. The same holds for the partitions C_1', C_2'. We denote the natural partitions of the rows and columns of B by N_R and N_C, respectively. The partitions R_1', R_2', C_1', C_2' restricted to the rows of B are denoted by R_1, R_2, C_1, C_2, respectively. Moreover, let $\mathcal{R} := \{N_R, R_1, R_2\}$ and $\mathcal{C} := \{N_C, C_1, C_2\}$.
Let $\widehat{S}$ denote the set of the rows of the margin row of S. Again, by Lemma 7.8, $R_1' \setminus \widehat{S}$ and $R_2' \setminus \widehat{S}$ are complementary with N_C. The dual statement holds and hence, all partitions in $\mathcal{R} \dot\cup \mathcal{C}$ have complementary-degree ≥ 2, since R_1', R_2' and C_1', C_2' have complementary-degree ≥ 1 in $\{R_1', R_2'\} \dot\cup \{C_1', C_2'\}$. Thus, $\widetilde{G} = (\mathcal{R} \dot\cup \mathcal{C}, E)$ is an acr-partial complementary graph of $(B, \mathcal{R}, \mathcal{C})$, where

$$E = \{(N_R, C_1), (N_R, C_2), (R_1, N_C), (R_1, C_2), (R_2, N_C), (R_2, C_1)\}.$$

For a block column Q_d of S, $d \in [1,n]$, let $N_{C,d}$ be the corresponding block column of B, i.e., $N_{C,d}$ is a submatrix of Q_d. We show, that $(R_1, \pi_{1,0})$ and $(R_2, \pi_{2,0})$ satisfy the disjoint-condition, whereas $\pi_{1,0}, \pi_{2,0}$ are the complementary indications provided by Lemma 7.8.
For a bundle $R_{1,a}' \in R_1'$, let $R_{1,a}$ denote the corresponding bundle of R_1, i.e., $R_{1,a}$ is a submatrix of $R_{1,a}'$. The bundles of R_2 are denoted analogously. Since R_1' and R_2' are orthogonal, any two bundles $R_{1,a}' \in R_1'$ and $R_{2,b}' \in R_2'$ have at most one common row in S. Let $\pi_{1,0}(a) = \pi_{2,0}(b) = d$. Neither $R_{1,a}$

nor $R_{2,b}$ are connected with $N_{C,d}$. The origin g_d of Q_d is in $R'_{1,a}$ as well as in $R'_{2,b}$, by Lemma 7.8. Since $g_d \in \widehat{S}$, the bundles $R_{1,a}$ and $R_{2,b}$ are disjoint in B. Hence, $(R_1, \pi_{1,0})$ and $(R_2, \pi_{2,0})$ satisfy the disjoint-condition. Dually, $(C_1, \pi_{0,1})$ and $(C_2, \pi_{0,2})$ satisfy the disjoint-condition.
Let $i \neq j$. By Remark 7.15, we may assume, that C'_j is complementary with R'_i in S. Then, C_j and N_C are complementary with R_i in B. We show, that $(N_C, \pi_{i,0})$ and $(C_j, \pi_{i,j})$ satisfy the disjoint-condition, where $\pi_{i,0}$ is the complementary indication of R_i and N_C provided by Lemma 7.8 and $\pi_{i,j}$ is a complementary indication of R'_i and C'_j. Hence, with $l = l_i := \pi_{i,0}^{-1}(d)$, the origin g_d of the block column Q_d of S is in the bundle $R'_{i,l}$. Moreover, $\{g_d\} = R'_{i,l} \cap \widehat{S}$. Thus, each column of Q_d is connected with $R'_{i,l}$. As $\pi_{i,j}$ is a complementary indication of R'_i and C'_j, the columns of the bundle $C'_{j,\pi_{i,j}(l)} \in C'_j$ are not connected with $R'_{i,l}$. Therefore, $Q_d \cap C'_{j,\pi_{i,j}(l)} = \emptyset$ and the respective bundles in the restricted partitions are disjoint, too. Dually, let $\pi_{0,j}$ be the complementary indication of N_R and C_j provided by Lemma 7.8 and let $\pi_{i,j}$ be a complementary indication of R'_i and C'_j. Then, $(N_R, \pi_{0,j})$ and $(R_i, \pi_{i,j})$ satisfy the disjoint-condition. Let $P_{i,j}$ denote the representing matrix of $\pi_{i,j}$. Hence, the matrix

$$\begin{pmatrix} 0 & P_{0,1} & P_{0,2} \\ P_{1,0} & 0 & P_{1,2} \\ P_{2,0} & P_{2,1} & 0 \end{pmatrix}$$

is a complementary indication matrix of $(\widetilde{G}, B, \mathcal{R}, \mathcal{C})$ and satisfies the disjoint-condition.
Conversely, let M have a set $\widetilde{\mathcal{R}} = \{\widetilde{R}_0, \widetilde{R}_1, \widetilde{R}_2\}$ of three pairwise orthogonal (r,n)-parallel partitions of the rows and a set $\widetilde{\mathcal{C}} = \{\widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2\}$ of three pairwise orthogonal (r,n)-parallel partitions of the columns, with an acr-partial complementary graph that has a complementary indication matrix

$$L = \begin{pmatrix} 0 & P_{0,1} & P_{0,2} \\ P_{1,0} & 0 & P_{1,2} \\ P_{2,0} & P_{2,1} & 0 \end{pmatrix}$$

which satisfies the disjoint-condition. Let $\pi_{i,j}$ be the permutation with representing matrix $P_{i,j}$ and the complementary indication of $\widetilde{R}_i$ and $\widetilde{C}_j$.
By Lemma 2.1, we can permute the bundles of $\widetilde{R}_0$ and $\widetilde{C}_0$ to block rows and block columns such, that the controlling bundles form the margin row and

the margin column. Thus, the lines in a block line are parallel. Therefore, all blocks are pseudo permutation matrices and by the line sum, the blocks in the margin row and the margin column are proper permutation matrices. Forming a unit margin with internal permutations, provided by Lemma 3.12, leads to a regular block matrix B. The set containing the partitions corresponding to $\widetilde{R}_0, \widetilde{R}_1, \widetilde{R}_2$ in B is called $\mathcal{R}$ and the set containing the partitions corresponding to $\widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2$ in B is called $\mathcal{C}$. By Theorem 7.9, it is enough to show, that the step hull S of B has two orthogonal parallel partitions of the rows and of the columns, each of complementary-degree ≥ 1. The natural partitions of the rows and columns of B are denoted by N_R, N_C, respectively. The remaining partitions of $\mathcal{R}$ are alternative row partitions in B and denoted by R_1, R_2. Analogously, the remaining partitions of $\mathcal{C}$ are alternative column partitions in B, denoted by C_1, C_2. By Remark 7.15 and Lemma 7.8, we may assume, that

- N_R is complementary with C_1, C_2 with indications $\pi_{0,1}, \pi_{0,2}$,
- R_1 is complementary with N_C, C_2 with indications $\pi_{1,0}, \pi_{1,2}$,
- R_2 is complementary with N_C, C_1 with indications $\pi_{2,0}, \pi_{2,1}$.

By Lemmata 7.7 and 7.17, R_1 and R_2 can be extended to orthogonal $(r+1, n)$-parallel partitions R'_1 and R'_2 of the rows of S and C_1, C_2 can be extended to orthogonal $(r+1, n)$-parallel partitions C'_1 and C'_2 of the columns of S.
It remains to show that R'_1 is complementary with C'_2 in S and R'_2 is complementary with C'_1 in S.
For this, let $R_1 = \{R_{1,i} \mid i \in [1,n]\}, C_2 = \{C_{2,i} \mid i \in [1,n]\}$ and let $\{c_i \mid i \in [1,n]\}$ denote the columns of the step margin column, such that $C'_2 = \{C_{2,i} \dot{\cup} \{c_i\} \mid i \in [1,n]\}$. This is possible, since there is precisely one column c_i of the step margin column of S, in each bundle $C'_{2,i}$, by Remark 7.5.
By assumption, the (r,n)-parallel partition R_1 is complementary with C_2 by the indication $\pi_{1,2} \in \mathcal{S}_{n-1}$. First, we show that $R_{1,i}$ and $C'_{2,\pi_{1,2}(i)}$ are not connected for all $i \in [1, n-1]$. It is enough to show, that the intersection of $R_{1,i}$ and $c_{\pi_{1,2}(i)}$ is a 0-column, since $R_{1,i}$ and $C_{2,\pi_{1,2}(i)}$ are not connected, by assumption. The column $c_{\pi_{1,2}(i)}$ is the origin of a bundle $N_{R,k} \in N_R$. As the pairs $(N_R, \pi_{0,2})$ and $(R_1, \pi_{1,2})$ satisfy the disjoint-condition, the bundles $N_{R,k}$ and $R_{1,i}$ are disjoint and therefore, the intersection of $R_{1,i}$ and $c_{\pi_{1,2}(i)}$ is a 0-column.

The dual statement holds and the bundles $C_{2,\pi_{1,2}(i)}$ and $R'_{1,i}$ are not connected for all $i \in [1, n-1]$. Finally, R'_1 is complementary with C'_2, since the rows and columns with internal indices in the range of $[1, n-1]$ of the step margin row and step margin column, respectively are not connected, by Remark 3.5. The same holds for R'_2 and C'_1. $\square$

Remark 7.19. Let R_1, R_2 and C_1, C_2 be orthogonal (r, n)-parallel partitions of the rows and columns of an (r, n)-matrix M, respectively, each of complementary degree ≥ 1. Then, there exists an acr-partial complementary graph $\widetilde{G}$ of $(M, \{R_1, R_2\}, \{C_1, C_2\})$, by Lemma 7.14. Moreover, any complementary indication matrix of $(\widetilde{G}, M, \{R_1, R_2\}, \{C_1, C_2\})$ satisfies the disjoint-condition in a trivial way, since there are no triplets that have to fulfill the disjoint-condition. Moreover, the complementary indication matrix of B satisfies the rectangular rule obviously, as it is a permutation matrix:

$$\begin{pmatrix} 0 & P_{1,2} \\ P_{2,1} & 0 \end{pmatrix}$$

Now let $\mathcal{R}$ and $\mathcal{C}$ be sets of three pairwise orthogonal (r, n)-parallel partitions of the rows and the columns of an (r, n)-matrix M, respectively, with an acr-partial complementary graph $\widetilde{G}$ of $(M, \mathcal{R}, \mathcal{C})$, such that there is a complementary indication matrix of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$ that satisfies the disjoint-condition. Then, the complementary indication matrix satisfies the rectangular rule automatically, by the constellation of the 0-blocks, cf. (7.1).

Theorem 7.20. *For a positive integer k, let $2 \leq r \leq n - k$ and M be an (r, n)-matrix. Then, M is permutation equivalent to a k-regular envelopable block matrix if and only if there are sets $\mathcal{R}$ and $\mathcal{C}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the rows and of the columns of M, respectively, with an acr-partial complementary graph $\widetilde{G}$ of $(M, \mathcal{R}, \mathcal{C})$, such that there is a complementary indication matrix of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$ that satisfies the disjoint-condition and the rectangular rule.*

Proof. We use induction on k. For this, recall that the theorem is true for $k = 1$ and $k = 2$ by Theorem 7.9 and Theorem 7.18, cf. Remark 7.19. We suppose that the theorem is true for $k - 1$ and show that it is true for k.
Let M be permutation equivalent to a k-regular envelopable block matrix B. By Lemma 7.3, it is enough, to show that B has the claimed properties, as they are maintained by line permutations. By definition, the step hull S of B

is permutation equivalent to a $(k-1)$-regular envelopable block matrix. Thus, by assumption, S has a set $\mathcal{R}' = \{R'_1, \ldots, R'_k\}$ of k pairwise orthogonal $(r+1, n)$-parallel partitions of the rows and a set $\mathcal{C}' = \{C'_1, \ldots, C'_k\}$ of k pairwise orthogonal $(r+1, n)$-parallel partitions of the columns with an acr-partial complementary graph $\widetilde{G}$ that has a complementary indication matrix L that satisfies the disjoint-condition and the rectangular rule.
By Remark 7.15, we may assume, that for $i \neq j$, the partitions R'_i and C'_j are complementary and thus,

$$L = \begin{pmatrix} 0 & P_{1,2} & \cdots & P_{1,k} \\ P_{2,1} & 0 & \cdots & P_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1} & P_{k,2} & \cdots & 0 \end{pmatrix},$$

where the permutation with representing matrix $P_{i,j}$ is the complementary indication of R'_i and C'_j, denoted by $\pi_{i,j}$.
Let R'_i, C'_i restricted to the rows of B be denoted by R_i, C_i, respectively. By Lemmata 7.2 and 7.8, $R_1, \ldots, R_k$ are pairwise orthogonal alternative (r, n)-parallel partitions of the rows of B and dually for $C_1, \ldots, C_k$. Moreover, the natural (r, n)-partitions N_R, N_C of the rows and columns of B are (r, n)-parallel partitions, since S satisfies the rectangular rule.
Hence, B has a set $\mathcal{R} = \{N_R, R_1, \ldots, R_k\}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the rows and a set $\mathcal{C} = \{N_C, C_1, \ldots, C_k\}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the columns.
Let $\widehat{S}$ denote the set of the rows of the margin row of S. By Lemma 7.8, all $R'_i \setminus \widehat{S}$ are complementary with N_C. The dual statement holds and hence, all partitions in $\mathcal{R}\dot{\cup}\mathcal{C}$ have complementary-degree $\geq k$, since each partition in $\mathcal{R}'\dot{\cup}\mathcal{C}'$ has a complementary-degree $\geq k-1$. Thus, $\widetilde{G} = (\mathcal{R}\dot{\cup}\mathcal{C}, E)$ is an acr-partial complementary graph of $(B, \mathcal{R}, \mathcal{C})$, with $E = \{(R_i, C_j) \mid i, j \in [0, k] \ \wedge \ i \neq j\}$, where $R_0 = N_R$ and $C_0 = N_C$.
Let $i \neq j$. For a block column Q_d of S, $d \in [1, n]$, let $N_{C,d}$ be the corresponding block column of B, i.e., $N_{C,d}$ is a submatrix of Q_d. We show, that $(R_i, \pi_{i,0})$ and $(R_j, \pi_{j,0})$ satisfy the disjoint-condition, whereas $\pi_{i,0}, \pi_{j,0}$ are the complementary indications provided by Lemma 7.8. For a bundle $R'_{i,j} \in R'_i$, let $R_{i,j}$ denote the corresponding bundle of R_i, i.e., $R_{i,j}$ is a submatrix of $R'_{i,j}$.
Since R'_i and R'_j are orthogonal, any two bundles $R'_{i,a} \in R'_i$ and $R'_{j,b} \in R'_j$ have at most one common row in S. Let $\pi_{i,0}(a) = \pi_{j,0}(b) = d$. Neither $R_{i,a}$ nor $R_{j,b}$ are connected with $N_{C,d}$. The origin g_d of Q_d is in $R'_{i,a}$ as well as

in $R'_{j,b}$, by Lemma 7.8. Since $g_d \in \widehat{S}$, the bundles $R_{i,a}$ and $R_{j,b}$ are disjoint in B. Dually, $(C_i, \pi_{0,i})$ and $(C_j, \pi_{0,j})$ satisfy the disjoint-condition.

Again, let $i \neq j$. The partitions C_j and N_C are complementary with R_i in B, by Lemma 7.8 and since R'_i and C'_j are complementary. We show, that $(N_C, \pi_{i,0})$ and $(C_j, \pi_{i,j})$ satisfy the disjoint-condition, where $\pi_{i,0}$ is the complementary indication of R_i and N_C provided by Lemma 7.8. Hence, with $l = l_i := \pi_{i,0}^{-1}(d)$, the origin g_d of the block column Q_d of S is in the bundle $R'_{i,l}$. Moreover, $\{g_d\} = R'_{i,l} \cap \widehat{S}$. Thus, each column of Q_d is connected with $R'_{i,l}$. As $\pi_{i,j}$ is a complementary indication of R'_i and C'_j, the columns of the bundle $C'_{j,\pi_{i,j}(l)} \in C'_j$ are not connected with $R'_{i,l}$. Therefore, $Q_d \cap C'_{j,\phi(l)} = \emptyset$ and the respective bundles in the restricted partitions are disjoint, too. Dually, let $\pi_{0,j}$ be the complementary indication of N_R and C_j provided by Lemma 7.8. Then, $(N_R, \pi_{0,j})$ and $(R_i, \pi_{i,j})$ satisfy the disjoint-condition. Hence, as L, satisfies the disjoint-condition by assumption, the matrix

$$\widetilde{L} = \begin{pmatrix} 0 & P_{0,1} & P_{0,2} & \cdots & P_{0,k} \\ P_{1,0} & 0 & P_{1,2} & \cdots & P_{1,k} \\ P_{2,0} & P_{2,1} & 0 & \cdots & P_{2,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{k,0} & P_{k,1} & P_{k,2} & \cdots & 0 \end{pmatrix}$$

satisfies the disjoint-condition and is a complementary indication matrix of $(\widetilde{G}, B, \mathcal{R}, \mathcal{C})$.

Note, that L is the submatrix of $\widetilde{L}$, that contains all of the complementary indications between the alternative partitions. By assumption, the rectangular rule is satisfied in L. Thus, it remains to show that the additional block lines in $\widetilde{L}$, i.e., the block lines containing the complementary indications concerning the natural partitions of B, maintain the rectangular rule. Since the intersection of those additional block lines is the 0-block, it remains to show that L together with the additional block column satisfies the rectangular rule and L together with the additional block row satisfies the rectangular rule. We show first, that the additional block column maintains the rectangular rule, or equivalently by Lemma 3.4, that for any $a, b \in [1, n-1]$, $l, i, j \in [1, k]$ with $\pi_{i,l}(a) = \pi_{j,l}(b)$, it follows $\pi_{i,0}(a) \neq \pi_{j,0}(b)$. Observe, that $\pi_{i,l}(a) = \pi_{j,l}(b)$ implies that the bundles $R'_{i,a}, R'_{j,b}$ are disjoint in S, by assumption. Let c be such that $\pi_{i,0}(a) = \pi_{j,0}(c)$. Then the bundles $R'_{i,a}, R'_{i,c}$ are not disjoint in S, since $\pi_{i,0}, \pi_{j,0}$ are the complementary indica-

tions provided by Lemma 7.8 and it follows $c \neq b$. Hence, $\pi_{i,0}(a) \neq \pi_{j,0}(b)$ and by Lemma 3.3 the rectangular rule holds in L together with the additional block column, i.e., in $\widetilde{L}([n,(k+1)(n-1)],[1,(k+1)(n-1)])$.
Note,that the matrix L^t satisfies the rectangular rule, by Lemma 3.2. Analogously, L^t together with the additional block column satisfies the rectangular rule, or equivalently, for any $a, b \in [1, n-1]$, $l, i, j \in [1, k]$ with $\pi_{l,i}^{-1}(a) = \pi_{l,j}^{-1}(b)$, it follows $\pi_{0,i}^{-1}(a) \neq \pi_{0,j}^{-1}(b)$. Hence, the additional block row of $\widetilde{L}$ maintains the rectangular rule, too.
Conversely, let M have a set $\widetilde{\mathcal{R}} = \{\widetilde{R}_0, \ldots, \widetilde{R}_k\}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the rows and a set $\widetilde{\mathcal{C}} = \{\widetilde{C}_0, \ldots, \widetilde{C}_k\}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the columns, with an acr-partial complementary graph that has a complementary indication matrix

$$\widetilde{L} = \begin{pmatrix} 0 & P_{0,1} & P_{0,2} & \cdots & P_{0,k} \\ P_{1,0} & 0 & P_{1,2} & \cdots & P_{1,k} \\ P_{2,0} & P_{2,1} & 0 & \cdots & P_{2,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{k,0} & P_{k,1} & P_{k,2} & \cdots & 0 \end{pmatrix}$$

which satisfies the disjoint-condition and the rectangular rule. Let $\pi_{i,j}$ be the permutation with representing matrix $P_{i,j}$ and the complementary indication of $\widetilde{R}_i$ and $\widetilde{C}_j$.
By Lemma 2.1, we can permute the bundles of $\widetilde{R}_0$ and $\widetilde{C}_0$ to block rows and block columns such, that the controlling bundles form the margin row and the margin column. Thus, the lines in a block line are parallel. Therefore, all blocks are pseudo permutation matrices and by the line sum, the blocks in the margin row and the margin column are proper permutation matrices. Forming a unit margin with internal permutations, provided by Lemma 3.12, leads to a regular block matrix B. For $i \in [1, n]$, let R_i be the corresponding partition to $\widetilde{R}_i$ in B, as well as C_i is the corresponding partition to $\widetilde{C}_i$ in B. Moreover, the corresponding partitions to $\widetilde{R}_0$ and $\widetilde{C}_0$ are N_R, N_C, respectively. Let $\mathcal{R} = \{N_R, R_1, \ldots, R_k\}$ and $\mathcal{C} = \{N_C, C_1, \ldots, C_k\}$. By assumption, it is enough to show, that the step hull S of B has a set of k orthogonal parallel partitions of the rows and a set of k orthogonal parallel partitions of the columns with an acr-partial complementary graph that has a complementary indication matrix which satisfies the disjoint-condition and the rectangular rule.

By the Lemmata 7.7 and 7.17, all $R_i = \{R_{i,a} \mid a \in [1, n]\}$ can be extended to pairwise orthogonal $(r + 1, n)$-parallel partitions R'_i of the rows of S and all $C_j = \{C_{j,a} \mid a \in [1, n]\}$ can be extended to pairwise orthogonal $(r+1, n)$-parallel partitions C'_i of the columns of S.
We show that $R'_i = \{R'_{i,a} \mid a \in [1, n]\}$ is complementary with $C'_j = \{C'_{j,a} \mid a \in [1, n]\}$ in S for $i \neq j$. By Remark 7.5, we may denote the columns $\{c_a \mid a \in [1, n]\}$ of the step margin column, such that $C'_{j,a} = C_{j,a} \dot\cup \{c_a\}$ for all $a \in [1, n]$.
By assumption, the (r, n)-parallel partition R_i is complementary with C_j by the indication $\pi_{i,j} \in \mathcal{S}_{n-1}$. First, we show that $R_{i,a}$ and $C'_{j,\pi_{i,j}(a)}$ are not connected for all $a \in [1, n-1]$. It is enough to show, that the intersection of $R_{i,a}$ and $c_{\pi_{i,j}(a)}$ is a 0-column, since $R_{i,a}$ and $C_{j,\pi(a)}$ are not connected, by assumption. The column $c_{\pi_{i,j}(a)}$ is the origin of a bundle $N_{R,d} \in N_R$. As the disjoint-condition is satisfied in B, the bundles $N_{R,d}$ and $R_{i,a}$ are disjoint and therefore, the intersection of $R_{i,a}$ and $c_{\pi_{i,j}(a)}$ is a 0-column.
The dual statement holds and the bundles $C_{j,\pi_{i,j}(a)}$ and $R'_{i,a}$ are not connected for all $a \in [1, n-1]$. Finally, R'_i is complementary with C'_j, since the rows and columns with internal indices in the range of $[1, n-1]$ of the step margin row and step margin column, respectively, are not connected, by Remark 3.5. Thus, all partitions of $\mathcal{R}' := \{R'_1, \ldots, R'_k\}$ and of $\mathcal{C}' := \{C'_1, \ldots, C'_k\}$ have complementary-degree $\geq k-1$ in $\mathcal{R}' \dot\cup \mathcal{C}'$. Hence, $\widetilde{G}' = (\mathcal{R}' \dot\cup \mathcal{C}', E)$ is an acr-partial complementary graph of $(S, \mathcal{R}', \mathcal{C}')$ and the submatrix

$$L = \begin{pmatrix} 0 & P_{1,2} & \cdots & P_{1,k} \\ P_{2,1} & 0 & \cdots & P_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1} & P_{k,2} & \cdots & 0 \end{pmatrix}$$

of $\widetilde{L}$ is a complementary indication matrix of $(\widetilde{G}', S, \mathcal{R}', \mathcal{C}')$.
Assume for contradiction, that L does not satisfy the disjoint-condition. Hence, there are for instance pairs $(R'_i, \pi_{i,l})$ and $(R'_j, \pi_{j,l})$ that do not satisfy the disjoint-condition. Thus, there are integers $a, b \in [1, n-1]$, such that $\pi_{i,l}(a) = \pi_{j,l}(b)$ and $|R'_{i,a} \cap R'_{j,b}| \geq 1$. Observe, that $R_{i,a} \cap R_{j,b} = \emptyset$, since $\widetilde{L}$ satisfies the disjoint condition. Hence, the common row of $R'_{i,a}$ and $R'_{j,b}$ is a row from the step margin row. As $\pi_{i,0}, \pi_{j,0}$ are the complementary indications of R_i, R_j and N_C, respectively, it follows $\pi_{i,0}(a) = \pi_{j,0}(b)$, by Lemma 7.7. This contradicts the fact, that $\widetilde{L}$ satisfies the rectangular rule, by Lemma 3.4. Sim-

ilarly we argue for the columns. Therefore, the disjoint-condition is fulfilled in L.
Finally, L satisfies the rectangular rule, since it is a submatrix of $\widetilde{L}$, that satisfies the rectangular rule, by assumption. □

Let M be an (r, n)-matrix that is permutation equivalent to a k-regular envelopable block matrix. Then there are sets $\mathcal{R}$ and $\mathcal{C}$ of $k+1$ pairwise orthogonal (r, n)-parallel partitions of the rows and of the columns of M, respectively, with an acr-partial complementary graph $\widetilde{G}$ and a complementary indication matrix L of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$ that satisfies the disjoint-condition and the rectangular rule. We will briefly call L a *complementary indication matrix of* M, if it is a complementary indication matrix of $(\widetilde{G}, M, \mathcal{R}, \mathcal{C})$ satisfying the disjoint-condition and the rectangular rule.
Theorem 6.15, provides a deep connection between Latin configurations and regular block matrices. The following theorem will show, that this connection is even closer for k-regular envelopable block matrices.

Theorem 7.21. *For a positive integer k, let $2 \leq r \leq n-k$ and let M be an (r, n)-matrix that is permutation equivalent to a k-regular envelopable block matrix of format nr/r. A complementary indication matrix L of M is a $(k+1, n-1)$-Latin configuration.*

Proof. By Theorem 7.20, L satisfies the rectangular rule and by definition, all blocks are either permutation matrices or 0-matrices of size $n-1$. Moreover, there is exactly one 0-block in each block line as L is a complementary indication matrix of an acr-partial complementary graph. We may assume that the 0-blocks are on the main diagonal, by Remark 7.15. By Lemma 6.6, it remains to show, that all blocks in a block line are cover free.
M is permutation equivalent to a matrix H with homogeneous margin, by Lemma 6.11. Let $\{R_i \mid i \in [1, k+1]\}$ be the $k+1$ pairwise orthogonal (r, n)-parallel partitions of the rows of H and $\{C_l \mid l \in [1, k+1]\}$ be the $k+1$ pairwise orthogonal (r, n)-parallel partitions of the columns of H. The rows in the first block row of H, i.e., the first $n-1$ rows, must be in different bundles of any (r, n)-parallel partition of H. Therefore it is possible to label the bundles of R_i in the following way. For $j \in [1, n-1]$, let $R_{i,j} \in R_i$ be the bundle containing the row with internal index j from the first block row of H, i.e., the row with absolute index j in H. Note, that renaming the bundles of the partitions R_i corresponds to applying internal row permutations to L.

Moreover, let $P_{i,l}$ be the representing matrix of the complementary indication $\pi_{i,l}$ of R_i and C_l. Now, let $P_{i_1,l}(j,a) = 1$ and $P_{i_2,l}(m,a) = 1$ or equivalently $\pi_{i_1,l}(j) = a$ and $\pi_{i_2,l}(m) = a$. By the disjoint-condition, $R_{i_1,j} \cap R_{i_2,m} = \emptyset$ and thus, $j \neq m$, since $|R_{i_1,j} \cap R_{i_2,j}| = 1$. Therefore, the block columns are cover free. The dual statement holds and the block rows are cover free, too. □

8 Applications and open questions

8.1 Regular block matrices, Latin configurations and finite projective planes

First, we mention the relevance of Theorem 7.20 for the existence of projective planes.

Theorem 8.1. *A projective plane of order n exists, if and only if there exists a k-regular envelopable block matrix of format $n(n-k)/(n-k)$ for any $k \in [0, n-2]$.*

Proof. Peeling k-times the core C_0 of a projective plane of order n, leads to a k-regular envelopable block matrix C_k of format $n(n-k)/(n-k)$ for $k \in [0, n-2]$.
Conversely, the existence of a k-regular envelopable block matrix of format $n(n-k)/(n-k)$ implies the existence of a regular block matrix of format n^2/n. By Proposition 4.2, this is equivalent to the existence of a projective plane of order n. □

A correspondence between $(2, n)$-parallel partitions of the rows or columns of a regular block matrix of format $n{\cdot}2/2$ and permutations of the set $[1, n-1]$ will lead us to Theorem 8.5, that provides the existence of $n-1$ pairwise orthogonal $(2, n)$-parallel partitions of the rows and of the columns of a regular block matrix of format $n{\cdot}2/2$.

Remark 8.2. Let $\mathcal{M}_{n,2}$ be the standardized matrix given in Figure 10, i.e., $\mathcal{M}_{n,2}$ has a homogeneous margin and each block line of the core has two blocks, namely a unit matrix and a 0-matrix, where the 0-matrices are the blocks on the main diagonal.
By Corollary 6.19, a regular block matrix B of format $n{\cdot}2/2$ is permutation equivalent to $\mathcal{M}_{n,2}$, i.e., $B = P^{-1}\mathcal{M}_{n,2}Q$, where P is the representing matrix of a permutation π.
Certainly, R is a $(2, n)$-parallel partition of the rows of $\mathcal{M}_{n,2}$, if and only if R^π is a $(2, n)$-parallel partition of the rows of B. By Lemma 7.3, the orthogonality and the complementarity of $(2, n)$-parallel partitions of the rows and columns of B are maintained by line permutations. In the same way, the disjoint-condition is maintained by line permutations.

Therefore we may reduce our considerations to the standardized matrix $\mathcal{M}_{n,2}$, if we ask for the regular envelopability of a regular block matrix of format $n{\cdot}2/2$.

Remark 8.3. We label each row r inside the core of $\mathcal{M}_{n,2}$ by $(i,j) \in [1,2] \times [1,n-1]$, where i is the block row index and j the internal index of r. Moreover, we label each column c by $(i,j) \in [1,2] \times [1,n-1]$, where i is the block column index of the block column **not** containing c and j is the internal index of c. Thus, the row (i,j) is connected with the column (i,j) and not connected with all other columns in the core. Moreover, the rows (i,j) and (l,m) meet in $\mathcal{M}_{n,2}$, if and only if $i = l$. The same holds for the columns.

Lemma 8.4. *A $(2,n)$-parallel partition of the rows of $\mathcal{M}_{n,2}$ can be identified uniquely with a permutation $\pi \in \mathcal{S}_{n-1}$. The dual statement holds for the columns.*

Proof. We denote the rows and columns of $\mathcal{M}_{n,2}$ according to Remark 8.3. Let π be a permutation of $[1,n-1]$. Note, that the correspondence

$$\pi \longrightarrow S^{\pi} = \{\{(1,i),(2,\pi(i))\} \mid i \in [1,n-1]\} \tag{8.1}$$

is unique. The set S^{π} together with the controlling bundle is a $(2,n)$-partition of the rows, as π is a permutation. Moreover, this partition is a parallel partition, by Remark 8.3.
Conversely, let $R = \{R_i \mid i \in [1,n]\}$ be a $(2,n)$-parallel partition of the rows of $\mathcal{M}_{n,2}$. By convention, $R_{1,n}$ is the margin row and for $i \in [1,n-1]$, the bundles $R_{1,i}$ contain precisely one row of each block row of the core of $\mathcal{M}_{n,2}$, by parallelism, cf. Remark 8.3. For $i \in [1,n-1]$, let $R_i = \{(1,i),(2,m_i\}$. Then, π defined by $\pi(i) = m_i$ for all $i \in [1,n-1]$ is a permutation of $[1,n-1]$, since R is a partition. □

Theorem 8.5. *Two $(2,n)$-parallel partitions of the rows are orthogonal if and only if the corresponding permutations provided by* (8.1) *are cover free. The dual statement holds for the columns.*
In particular, there are $n-1$ pairwise orthogonal $(2,n)$-parallel partitions of the rows and of the columns of a regular block matrix of format $n{\cdot}2/2$.

Proof. We denote the rows and columns of $\mathcal{M}_{n,2}$ according to Remark 8.3. First, assume the $(2,n)$-parallel partitions R_1, R_2 of the rows of $\mathcal{M}_{n,2}$ are orthogonal. Let π_1, π_2 be the corresponding permutations, respectively. The orthogonality implies $\forall i \in [1,n-1]: \ |\{(1,i),(2,\pi_1(i))\} \cap \{(1,i),(2,\pi_2(i))\}| \leq$

1. Hence, the corresponding partitions are cover free. The reverse implication holds analogously.
We show the existence of $n-1$ pairwise orthogonal $(2, n)$-parallel partitions. By Remark 8.2, we may equivalently consider $\mathcal{M}_{n,2}$ instead of a regular block matrix of format $n2/2$. There is a Latin square of order $n-1$, cf. [9]. The rows of a Latin square denote cover free permutations of $[1, n-1]$. Thus, there are $n-1$ pairwise orthogonal $(2, n)$-parallel partitions of the rows of $\mathcal{M}_{n,2}$. The same holds for the columns. □

We know already about the relationship between regular block matrices and Latin configurations, by the Theorems 6.15 and 7.21. In the special situation of the following lemma, we can even identify a Latin configuration with the unit core of a regular block matrix.

Lemma 8.6. *An (n, n)-Latin configuration with fixed point free blocks is the unit core of a regular block matrix of format $(n + 1)n/n$.*

Proof. By Lemma 6.6, the blocks of an (n, n)-Latin configuration L are pseudo permutation matrices of size n and the block lines are cover free. Moreover, L has constant line sum $n-1$ and in each block line are n blocks. By (3.1) and (3.2), the rectangular rule is satisfied in the unit hull of L, since L satisfies the rectangular rule. □

Theorem 8.7. *The existence of a projective plane of order n implies the existence of an $(n-1, n)$-Latin configuration with fixed point free blocks.*

Proof. The step core C_0 of an incidence matrix of a finite projective plane of order n is a regular block matrix of format n^2/n, with exclusively permutation matrices as blocks, by Corollary 5.5. By (3.2), the blocks in the unit core are fixed point free and of size n. Clearly, the blocks in the unit core are also cover free, by (3.1). Moreover, there are $n-1$ blocks in each block line of the unit core of C_0. Replacing the permutation blocks on the main diagonal with 0-blocks leads to a $(n-1, n)$-Latin configuration, as the rectangular rule holds for the obtained matrix since it is satisfied in C_0 and removing entries 1 obviously maintains the rectangular rule. □

For example, the construction of the regular block matrix in Appendix B is based on Theorem 8.7.

8.2 The existence of some series of configurations

In terms of incidence matrices, a configuration (v_r, b_k), cf. [7], is a $\{0,1\}$-matrix of size $v \times b$ with constant row sum r and constant column sum b that satisfies the rectangular rule. Symmetric configurations are those configurations with $v = b$ and thus, $r = k$. Without loss of generality, one usually assumes that $v \leq b$.

Remark 8.8. A projective plane of order n is a symmetric configuration $(n^2 + n + 1)_{n+1}$, by Lemma 4.1 and Lemma 6.1. Moreover, by Proposition 4.2 and Lemma 6.1, the step core C_0 of an incidence matrix of a projective plane of order n is a symmetric configuration n^2_n.

The following well known Lemma is a slightly changed version of a theorem from Dénes König, cf. [10, Theorem F]. It can also be found in [4, Corollary 1.2.5].

Lemma 8.9. (König 1916). *A square $\{0,1\}$-matrix with constant line sum k is the sum (over $\mathbb{Z}$) of k cover free permutation matrices.*

Corollary 8.10. *The existence of a symmetric configuration v_k implies the existence of all symmetric configurations v_i with $i < k$.*

Proof. By Lemma 8.9, the incidence matrix $I_{v,k}$ of v_k is the sum of k cover free permutation matrices $P_1, \ldots, P_k$. Let $S \subset [1, k]$ with $|S| = i$. Then,

$$I_{v,i} := \sum_{j \in S} P_j$$

is the incidence matrix of a symmetric configuration v_i for all $i < k$. □

Theorem 8.11. *Given a finite projective plane of order n, the following configurations exist:*

(1) *The configurations (nk_m, nm_k) for $1 \leq k \leq m \leq n$.*

(2) *The Latin configurations $(n-1)k_{k-1}$ for $1 \leq k \leq n$.*

Proof. For (1), note that by Remark 8.8, the step core C_0 of the plane is a symmetric configuration n^2_n. Moreover, the rows in a block row of C_0 are parallel and have row sum n. Hence, there is exactly one entry 1 in the intersection of any column of C_0 with any block row of C_0. The dual

statement holds. Thus, a submatrix of C_0 that is the intersection of k block rows and m block columns has constant column sum k and constant row sum m and therefore, it is a configuration (nk_m, nm_k).
We show (2). Peeling of an incidence matrix of a projective plane leads to a series of regular block matrices, by Corollary 5.5. Moreover, the existence of a regular block matrix of format nk/k is equivalent to the existence of a Latin configuration $(n-1)k_{k-1}$, by Corollary 6.16. □

Remark 8.12. Given a finite projective plane of order n, the first statement of Theorem 8.11 implies the existence of the symmetric configurations nk_k for $1 \leq k \leq n$.

Theorem 8.13. *For a positive integer k, let $2 \leq r \leq n-k$. Given a k-regular envelopable block matrix of format nr/r, the following configurations exist:*

(1) *The Latin configurations $(n-1)i_{i-1}$ for $0 < i < r$.*

(2) *The Latin configurations $(n-1)(r+i)_{r+i-1}$ for $0 \leq i \leq k$.*

(3) *The homogeneous configuration $(n-1)(r+k+1)_{r+k}$, and if $r+k=n$, the projective plane of order n.*

Proof. Given a k-regular envelopable block matrix of format nr/r, there is a regular block matrix of format $n(r+k)/(r+k)$, by Theorem 7.20. Thus, (1) and (2) follow from Lemma 5.2 and Corollary 6.16.
We show (3). Note, that (2) implies the existence of a $(r+k, n-1)$-Latin configuration $(n-1)(r+k)_{r+k-1}$. Thus, there is a homogeneous configuration $(n-1)(r+k+1)_{r+k}$, by Corollary 6.17. Moreover, if $r+k=n$, there is a projective plane of order n, by Corollary 6.18. □

8.3 Open questions

Peeling of incidence matrices of finite projective planes leads to regular block matrices. Since regular block matrices are a combinatorial structure, that has not been investigated in detail yet, there remain a lot of natural questions.
For example, since a regular block matrix of format $n \cdot 1/1$ is equal to a Γ-matrix Γ_n and therefore unique, cf. Section 7.1, and all regular block matrices of format $n \cdot 2/2$ are permutation equivalent, by Corollary 6.19, a natural question is:

$$\begin{pmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,n-1} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1} & P_{k,2} & \cdots & P_{k,n-1} \end{pmatrix} \xrightarrow{\substack{\text{block}\\ \text{switch}}} \begin{pmatrix} 0_{k\times n-1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,1} & 0_{k\times n-1} & \cdots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & 0_{k\times n-1} \end{pmatrix}$$

Figure 16: Applying the block switch to k-MOLS, i.e., $S' \to \widetilde{S}$.

- How many permutation equivalence classes of regular block matrices of a given format exist?

Of course, this is a hard question, as it includes in particular the question, how many projective planes there are of a given order. But of course, partial results would be interesting.

The block switch, introduced in Section 6.3, provides a translation between Latin configurations and regular block matrices. Each cover free block row of a regular block matrix of format n^2/n, cf. Figure 4, can be considered as a Latin square of order n. The cover free block rows are all block rows, except of the margin row. The Latin squares corresponding to different cover free block rows are orthogonal, cf. [11]. Hence, those block rows form a set of $n-1$ mutually orthogonal Latin squares (MOLS) of order n. A submatrix S consisting of k cover free block rows form k-MOLS of order n. We may also apply the block switch to k-MOLS, cf. Figure 16.

For this, note that the intersection of S and the margin column is a block column with k unit matrices as blocks. Omitting this block column, leads to a matrix S' with fixed point free permutation matrices $P_{i,j}$ of size n as blocks. Moreover, all block lines of S' are cover free. The block switch results in the matrix

$$\widetilde{S} = \left(\Sigma_{n,k}^t S' \Sigma_{n,n-1}, (k, 2k, \ldots, nk), (n-1, 2n-2, \ldots, n^2-n)\right),$$

cf. Figure 16.

All blocks in $\widetilde{S}$ are of size $k \times n-1$. The blocks on the main diagonal are 0-matrices, and all other blocks $B_{i,j}$ have exactly one entry 1 in each row and at most one entry 1 in each column[5]. Thus, the block rows of $\widetilde{S}$ can be

[5] This statements follow from Lemma 6.13, if we add $n-1-k$ block rows with exclusively 0-blocks to S', apply the block switch and then delete the 0-rows in the resulting matrix again, i.e., delete the rows with internal indices $> k$ in $\widetilde{S}$.

considered as k-line Latin rectangles of order $n-1$, interpreting the blocks as representing matrices of partial permutations. This leads to the question:

- Does the block switch simplify the search for MOLS, for example backtracking search?

Those examples are far from being a complete list of directions, that future research can follow.

A An example of a symmetric configuration 16_3 that is not a homogeneous configuration

$$M = \left(\begin{array}{ccccccc|cc|ccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right)$$

The matrix M is square of size 16 with constant line sum 3 and satisfies the rectangular rule. Therefore, it is the incidence matrix of a symmetric configuration 16_3, by Lemma 6.1. Moreover, $k+1=4$ is a divisor of $v=16$. But there is no $(4,4)$-parallel partition of the rows of M. To a parallel bundle containing the first row, we can add either the 8th or the 9th row, since they meet and one of the last seven rows, since they meet all. Thus, the first row cannot be contained in a parallel bundle with cardinality greater than 3. The dual statement holds for the first column. The block in the bottom right corner is the doubly ordered and cross-transposed incidence matrix of the Fano-plane, i.e., of the projective plane of order 2.

B An example of a regular block matrix of format $6{\cdot}4/4$ with a 0-block in every block line, that is not regular envelopable

$$
\left(\begin{array}{cccc|cccc|cccc|cccc|cccc||cccc}
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1\\
\hline
0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\
\hline
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
\hline
0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
\hline
0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\
\hline\hline
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1
\end{array}\right)
$$

We show, that this matrix does not have two orthogonal $(4,6)$-parallel partitions of the rows. For this, label the rows with their absolute indices and observe, that the parallel rows 9 and 10 both meet the rows $\{1,2,3,4,7,8,17,18\}$. Moreover, row 9 meets the rows $\{5,13\}$ and row 10 meets the rows $\{6,14\}$. Therefore, the row 9 and 10 are both parallel to the rows $\{11,12,15,16,19,20\}$. The row 9 has internal index 1. It is parallel to three rows with internal index 2, namely $\{6,10,14\}$. The row 10 has internal index 2 and is parallel to the three rows $\{5,9,13\}$ with internal index 1. By

Lemma 6.3, in each parallel bundle is a row of each internal indices. Suppose, there are two orthogonal $(4,6)$-parallel partitions R_1, R_2 of the rows the matrix above. The rows 9 and 10 can be in a common bundle of at most one of them. Thus, we may assume, that the rows 9 and 10 are in different bundles of R_1. Let $9 \in R_{1,1}$ and $10 \in R_{1,2}$. Observe that

- row 5 meets the rows $\{15, 16, 19, 20\}$,
- row 6 meets the rows $\{15, 16, 19, 20\}$,
- row 13 meets the rows $\{11, 12, 19, 20\}$ and
- row 14 meets the rows $\{11, 12, 19, 20\}$.

Finally, there are two possibilities to complete those bundles. We have to choose either

$$R_{1,1} = \{6, 9, 11, 12\}; R_{1,2} = \{10, 13, 15, 16\}$$

or

$$R_{1,1} = \{9, 14, 15, 16\}; R_{1,2} = \{5, 10, 11, 12\}.$$

By orthogonality reasons, there must be a bundle $R_{2,1} \in R_2$, that contains both the rows 9 and 10. Again, by orthogonality and parallelism, it follows that $R_{2,1} = \{9, 10, 19, 20\}$. Thus, the rows 19 and 20 are in different bundles of R_1, i.e., $19 \in R_{1,3}$ and $20 \in R_{1,4}$. Again by Lemma 6.3, exactly one of the rows $\{4, 8\}$ is in bundle $R_{1,3}$ and exactly one of the rows $\{3, 7\}$ is in bundle $R_{1,4}$ and by parallelism and orthogonality, the possible rows with internal indices 1 or 2 for the bundles $R_{1,3}, R_{1,4}$ are $\{1, 2, 17, 18\}$. Since row 4 meets the rows $\{17, 18\}$ as well as row 3 and the row 8 meets $\{1, 2\}$ as well as row 7, there are two possibilities to complete the bundles $R_{1,3}$ and $R_{1,4}$. Either,

$$R_{1,3} = \{1, 2, 4, 19\}; R_{1,4} = \{7, 17, 18, 20\}$$

or

$$R_{1,3} = \{8, 17, 18, 19\}; R_{1,4} = \{1, 2, 3, 20\}.$$

But then, either $\{3, 8\} \subset R_{1,5}$ or $\{4, 7\} \subset R_{1,5}$. In both cases, $R_{1,5}$ is not a parallel bundle. This contradicts the assumption, that there are two orthogonal $(4,6)$-parallel partitions of the rows.

References

[1] M. Aigner, *Graphentheorie.*
Stuttgart : Teubner 1984

[2] C. Berge, *Graphs and Hypergraphs.*
Amsterdam : North-Holland Publishing Company 1973

[3] R. C. Bose, *On the application of the properties of galois fields to the problem of construction of hyper-graeco-latin squares.*
Sankhyã Vol. 3 (1938), pp. 323-338

[4] R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory.*
Cambridge; New York : Cambridge University Press 1991

[5] R. H. Bruck, H. J. Ryser, *The nonexistence of certain projective planes.*
Canadian Journal of Mathematics Vol. 1 (1949), pp. 187-195

[6] H. Gropp, *Configurations.*
in: C. J. Colbourn, J. H. Dinitz (Eds.), *Handbook of combinatorial Designs.* (2nd Edition)
Boca Raton, FL : CRC Press 2006, pp. 353-355

[7] H. Gropp, *Configurations between geometry and combinatorics.*
Discrete Applied Mathematics Vol. 138 (2004), pp. 79-88

[8] H. Gropp, *Nonisomorphic configurations* n_k.
Electronic Notes in Discrete Mathematics Vol. 27 (2006), pp. 43-44

[9] K. Jacobs, D. Jungnickel, *Einführung in die Kombinatorik.* (2. Auflage)
Berlin; New York : de Gruyter, 2004

[10] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre.*
Mathematische Annalen Vol. 77 (1916), pp. 453-465

[11] H. Kramer, *Inzidenzmatrizen endlicher projektiver Ebenen.*
Dissertation (Würzburg), 2004

[12] C. W. H. Lam, G. Kolesova, L. Thiel, *A computer search for finite projective planes of order* 9.
Discrete Mathematics Vol. 92 (1991), pp. 187-195

[13] C. W. H. Lam, *The search for a finite projective plane of order* 10.
American Mathematical Monthly Vol. 98 (1991), pp. 305-318

[14] C. W. H. Lam, L. Thiel, S. Swiercz, *The non-existence of finite projective planes of order* 10.
Canadian Journal of Mathematics Vol. 41 (1989), pp. 1117-1123

[15] A. Mader, O. Mutzbauer, *Double ordering of* $(0, 1)$*-matrices.*
Ars Combinatoria Vol. 61 (2001), pp. 81-95

[16] B. D. McKay, A. Meynert, W. Myrvold, *Small Latin Squares, Quasi-groups, and Loops.*
Journal of Combinatorial Designs Vol. 15 (2007), pp. 98-119

[17] P. R. Pechmann, *Doppelordnung und Anwendungen.*
Diplomarbeit (Würzburg), 2004

[18] H. J. Ryser, *Combinatorial Mathematics.*
New York : Wiley, 1963

[19] G. Tarry, *Le problème des* 36 *officiers.*
Compte Rendu de l'Assoc. Français Avanc. Sci. Naturel
Vol. 1 (1900), pp. 122-123; Vol. 2 (1901), pp. 170-203

List of Figures

Index