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The Values of the Periodic Zeta-Function at the Nontrivial Zeros of Riemann's Zeta-Function

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Abstract: In this paper, we prove an asymptotic formula for the sum of the values of the periodic zeta-function at the nontrivial zeros of the Riemann zeta-function (up to some height) which are symmetrical on the real line and the critical line. This is an extension of the previous results due to Garunkštis, Kalpokas, and, more recently, Sowa. Whereas Sowa's approach was assuming the yet unproved Riemann hypothesis, our result holds unconditionally.

Keywords: zeta-functions; Riemann hypothesis

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1. Introduction

The periodic zeta-function, introduced by Berndt and Schorenfeld [1] in 1975, is defined by

$$F(s;\alpha) = \sum_{n\geq 1} e(n\alpha)n^{-s},$$

for $\Re(s) > 1$ and a real parameter α where the abbreviation $e(\alpha) = \exp(2\pi i\alpha)$ is used; the naming reflects the periodicity $F(s;\alpha+1) = F(s;\alpha)$ for which we may assume $\alpha \in (0,1]$ in the sequel. The periodic zeta-function admits a meromorphic continuation to the whole complex plane (details about this in the following section). Of particular interest are special values of the parameter,

$$F(s;1) = \zeta(s) \quad \text{and} \quad F\left(s;\tfrac{1}{2}\right) = \left(2^{1-s}-1\right)\zeta(s), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function. We shall prove that for no other values of α than 0 or $1/2 \bmod 1$ the quotient $F(s;\alpha)/\zeta(s)$ is an entire function.

As usual, we denote the nontrivial (non-real) zeros of $\zeta(s)$ as $\rho = \beta + i\gamma$. The number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfying $0 < \gamma < T$ is by the Riemann–von Mangoldt formula asymptotically given by

$$N(T) := \#\{\rho = \beta + i\gamma : 0 < \gamma \leq T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$$

(here, every multiple zero would be counted according to its multiplicity; however, there is no multiple ζ-zero known so far). The Riemann hypothesis states that all nontrivial zeros ρ lie on the critical line $\Re(s) = 1/2$. Recently, Sowa [2] obtained under assumption of the truth of this unproven conjecture the limit

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\gamma<T} F(\xi + i\gamma;\alpha) = \exp(2\pi i\alpha) \tag{2}$$

where the convergence type is (i) distributional whenever $\xi > 0$, (ii) in L_2-norm whenever $\xi > 1$, and (iii) uniform whenever $\xi > 3/2$. His approach relies on Fourier analysis, and his motivation came from theoretical physics. There have been notable investigations in this direction before Sowa's work. The first result of this type is probably due to Fujii [3] who proved a similar asymptotic formula with the Hurwitz zeta-function in place of the periodic zeta-function conditional to the Riemann hypothesis. Garunkštis and Kalpokas [4] succeeded with the case of the periodic zeta-function associated with a rational parameter α unconditionally. These approaches rely on approximate functional equations and contour integration, respectively. This is also the method that we shall use in our proof below. Notice that we also recover the rational case $\alpha = k/q$ with coprime integers $1 \leq k < q$ integers, namely

$$\begin{aligned}\sum_{0<\gamma\leq T} F\left(\rho;\frac{k}{q}\right) = &\left(\exp\left(2\pi i\frac{k}{q}\right) - \frac{\mu(q)}{\phi(q)}\right)\frac{T}{2\pi}\log\frac{T}{2\pi e}\\ &+ \frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \chi(k)\tau(\overline{\chi})\frac{L'}{L}(1,\chi)\frac{T}{2\pi} +\\ &+ \frac{\mu(q)}{\phi(q)}\left(\Psi\left(\frac{k}{q}\right) + \sum_{p|q}\frac{\log p}{p-1} + \gamma_0\right)\frac{T}{2\pi} +\\ &- \sum_{p|q}\log p\sum_{j=1}^{\infty}\frac{\exp\left(2\pi i\frac{p^j k}{q}\right)}{p^j}\frac{T}{2\pi} + O\left(T^{1-c(\log T)^{-\frac{3}{4}-\varepsilon}}\right).\end{aligned} \tag{3}$$

Here, μ denotes the Möbius μ-function and ϕ is Euler's totient; furthermore, Dirichlet characters $\chi \bmod q$ and their associated Gauss sums $\tau(\chi)$ and Dirichlet L-functions appear; finally, we observe the Euler–Mascheroni constant $\gamma_0 = \lim_{x\to\infty}(\sum_{n\leq x} 1/n - \log x)$ as well as the Digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$. In the special case $k/q = 1/2$, the main term vanishes since the character sum is empty, and $\Psi(1/2) = -2\log 2 - \gamma_0$ in coincidence with Formula (1).

Our main result is as follows.

Theorem 1. *For irrational* $\alpha \in (0,1)$,

$$\sum_{0<\gamma<T} F(\rho+\xi;\alpha) = e(\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{T}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + \mathcal{E}(T;\xi,\alpha),$$

where the error term is

$$\mathcal{E}(T;\xi,\alpha) \ll \begin{cases} (\log T)^2 & \text{if } \xi > 1, \\ T^{1-\xi}\log T & \text{if } 0 < \xi \leq 1, \\ T^{4/5}(\log T)^4 & \text{if } \xi = 0. \end{cases}$$

Here and in the sequel, the abbreviation $e(\alpha) = \exp(2\pi i\alpha)$ and Λ is the von Mangoldt-function, defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, p \text{ prime}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the mean-value of $F(\rho+\xi;\alpha)$ exists and is equal to $e(\alpha)$ for any *irrational* parameter α.

The case of *rational* α, however, is rather different. The case of $\alpha = 1$ is trivial (see Formula (1)). Then, the proof of the above theorem will show that the same asymptotics hold whenever $\xi > 0$; however, for $\xi = 0$, the main term is significantly different and has been treated by different means.

In the following section, we recall several useful facts about zeta-functions in number theory as well as some classical results on exponential integrals. In Section 3, we provide the proof of the main theorem. The so far unsettled case of irrational parameters relies on an estimate of exponential sums due to Vinogradov in the context of his famous work on the Goldbach conjecture.

2. Preliminaries

We begin with the formula

$$F(s;\alpha) = \sum_{n\geq 1} e(n\alpha)n^{-s} = \frac{1_{\mathbb{Z}}(\alpha)}{s-1} + O(1),$$

valid in a neighbourhood of $s = 1$ with

$$1_{\mathbb{Z}}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

This follows for rational $\alpha = k/q$ with w.l.o.g. $\gcd(k,q) = 1$ from the representation

$$F\left(s;\frac{k}{q}\right) = q^{-s} \sum_{1\leq b\leq q} e\left(\frac{bk}{q}\right)\zeta\left(s;\frac{b}{q}\right),$$

valid for $\sigma > 1$, where the Hurwitz zeta-function to a real parameter $\beta \in (0,1]$ is defined by

$$\begin{aligned} \zeta(s;\beta) &= \sum_{m\geq 0} (m+\beta)^{-s} && \text{for } \sigma > 1, \\ &= \frac{1}{s-1} + O(1) && \text{in a neighbourhood of } s = 1, \end{aligned}$$

respectively. The formula

$$F(1;\alpha) = -\frac{1}{2}(\log 2 + \log(1 - \cos(2\pi\alpha)) + i\pi(1+2\alpha))$$

is valid for $0 < \alpha < 1$. For details, we refer to [5].

The functional equation for the Riemann zeta-function plays an important role, i.e.,

$$\zeta(s) = \Delta(s)\zeta(1-s) \qquad \text{with} \quad \Delta(s) := 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right).$$

Applying the logarithmic derivative, we deduce

$$\frac{\zeta'}{\zeta}(s) = \frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s),$$

and from Stirling's formula, we find

$$\frac{\Delta'}{\Delta}(\sigma+it) = -\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right).$$

In addition,

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= -\sum_{m\geq 2} \Lambda(m)m^{-s} && \text{for } \sigma > 1 \\ &= \frac{-1}{s-1} + O(1) && \text{in a neighbourhood of } s = 1 \end{aligned}$$

with von Mangoldt's Λ-function. Moreover, we shall use

$$\frac{\zeta'}{\zeta}(s) = O\Big((\log t)^2\Big) \qquad \text{for } -1 \leq \sigma \leq 2, \qquad |t-\gamma| \geq \frac{c}{\log t},$$

where c is a positive constant, as follows from partial fraction decomposition:

$$F(s;\alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{ e\left(\frac{1-s}{4}\right)\zeta(1-s;\alpha) + e\left(\frac{s-1}{4}\right)\zeta(1-s;1-\alpha)\right\}.$$

This functional equation and Stirling's formula plus the Phragmén-Lindelöf principle lead to

$$F(s;\alpha) \ll \begin{cases} 1 & \text{if } \sigma > 1, \\ t^{\frac{1-\sigma}{2}} & \text{if } \dfrac{-1}{\log T} \leq \sigma \leq 1 + \dfrac{1}{\log T}, 1 \leq t \leq T, \\ t^{\frac{1}{2}-\sigma} & \text{if } \sigma < 0, \end{cases}$$

as $t \to \infty$.

Furthermore, we shall make use of the so-called first derivative test:

Lemma 1. *Let $F(x)$ and $G(x)$ be real functions, $\dfrac{G(x)}{F'(x)}$ monotonic, and $\dfrac{F'(x)}{G(x)} \geq m > 0$, or $\leq -m < 0$; then,*

$$\left| \int_a^b G(x)\exp(iF(x)) \right| \leq \frac{4}{m}.$$

A proof of this statement can be found, for example, in [6].

Another crucial tool in our approach is Gonek's lemma [7]:

Lemma 2. *For large $A \leq r \leq B \leq 2A$,*

$$\int_A^B \exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt = (2\pi)^{1-a} r^a \exp\Big(-i\Big(r - \frac{\pi}{4}\Big)\Big) 1_{[A,B)}(r) + E(r;A,B)$$

where the real number a is fixed,

$$E(r;A,B) = O\left(A^{a-\frac{1}{2}} + \frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}} + \frac{B^{a+\frac{1}{2}}}{|B-r| + B^{\frac{1}{2}}} \right)$$

and

$$1_{[A,B)}(r) := \begin{cases} 1 & \text{if } A \leq r < B, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we need a simple form of an approximate functional equation:

Lemma 3. *For $\sigma \geq \sigma_0 > 0, |t| \leq \pi\alpha x$ and $0 < \alpha < 1$,*

$$F(\sigma + it;\alpha) = \sum_{1 \leq n \leq x} \frac{e(n\alpha)}{n^{\sigma+it}} + O(x^{-\sigma}).$$

This last lemma is a special case of an approximate functional equation for the Lerch zeta-function (see [5], page 32), and it is easily proved by partial summation.

3. Proof of the Main Theorem

Recall that the logarithmic derivative $\frac{\zeta'}{\zeta}$ of ζ has simple poles at the zeros of ζ and is analytic elsewhere except for a simple pole at $s = 1$. Hence, by Cauchy's theorem,

$$\sum_{0<\gamma<T} F(\rho+\xi;\alpha) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\zeta'}{\zeta}(s)F(s+\xi;\alpha)ds,$$

where $\mathcal{C}$ is a counter-clockwise oriented contour with vertices $a+i, a+iT, 1-a+iT$, $1-a+i$ with $a := 1+\frac{1}{\log T}$. Moreover, we assume—for the first—that $T \geq 3$ satisfies

$$|T-\gamma| \geq \frac{c}{\log T},$$

where γ is any ordinate of a ζ-zero $\rho = \beta + i\gamma$, and c is a positive constant; note that every interval $[T_0, T_0+1)$ contains such a real number T as follows from the Riemann–von Mangoldt formula.

Notice that the least ordinate γ of a ζ-zero in the upper half-plane is a little larger than 14. We rewrite the contour integral as

$$\frac{1}{2\pi i}\left\{\int_{a+i}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a+i}^{a+i}\right\}\frac{\zeta'}{\zeta}(s)F(s+\xi;\alpha)ds = \sum_{j=1}^{4} I_j,$$

for instance. The main term comes from the integral I_3 over the vertical line segment on the left; all other integrals contribute to the error term.

We begin with the integral I_4 over the lower horizontal line segment, since this is independent of T. We have $I_4 = O(1)$.

Since the vertical line segment on the right lies inside the half-plane of absolute convergence for the defining Dirichlet series, we find

$$I_1 = \frac{1}{2\pi i}\int_{a+i}^{a+iT}\left(-\sum_{m\geq 2}\Lambda(m)m^{-s}\right)\sum_{n\geq 1} e(n\alpha)n^{-s-\xi}ds$$

or, after interchanging integration and summation,

$$I_1 = -\sum_{m\geq 2}\Lambda(m)m^{-a}\sum_{n\geq 1} e(n\alpha)n^{-a-\xi}\frac{1}{2\pi}\int_1^T (mn)^{-it}dt.$$

Since $mn \geq 2$, the latter integral is bounded, and it follows from the Laurent expansions of $\frac{\zeta'}{\zeta}(s)$ and of $F(s+\xi;\alpha)$ at $s = 1$ that

$$I_1 = O\left(\left|\frac{\zeta'}{\zeta}(a)F(a+\xi;\alpha)\right|\right) = O\left((\log T)^2\right).$$

Note that

$$\frac{\zeta'}{\zeta}(a) = \frac{-1}{a-1} + O(1) = \frac{1}{1+\frac{1}{\log T}-1} + O(1) = O(\log T).$$

Observe that $I_1 = O(\log T)$ if α are irrational or $\xi > 0$; however, there will be bigger error terms.

In view of the estimates for $\frac{\zeta'}{\zeta}$ and F, we obtain for the upper horizontal integral

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT}\frac{\zeta'}{\zeta}(s)F(s+\xi;\alpha)ds \\
&= -\frac{1}{2\pi i}\int_{1-a}^{a}\frac{\zeta'}{\zeta}(\sigma+iT)F(\sigma+\xi+iT;\alpha)d\sigma \\
&= O\Big((\log T)^2\max\Big\{1,T^{\frac{1-\xi}{2}}\Big\}\Big).
\end{aligned}$$

It remains to consider the vertical integral on the left:

$$I_3 = \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i}\frac{\zeta'}{\zeta}(s)F(s+\xi;\alpha)ds.$$

Here we have to distinguish cases.

1st case: $\xi > 1$. Then, $F(s+\xi;\alpha)$ can be replaced by its absolutely convergent Dirichlet series. For the logarithmic derivative $\frac{\zeta'}{\zeta}$, we apply the functional equation and obtain

$$\begin{aligned}
I_3 &= \frac{1}{2\pi i}\int_{1-a+i}^{1-a+iT}\left(\frac{\zeta'}{\zeta}(1-s)-\frac{\Delta'}{\Delta}(s)\right)F(s+\xi;\alpha)ds \\
&= \frac{1}{2\pi}\int_1^T\left(-\sum_{m\geq 2}\frac{\Lambda(m)}{m^{a-it}}\right)\sum_{n\geq 1}\frac{e(n\alpha)}{n^{1+\xi-a+it}}dt \\
&\quad+\frac{1}{2\pi}\int_1^T\left(\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)\right)\sum_{n\geq 1}\frac{e(n\alpha)}{n^{1+\xi-a+it}}dt \\
&= J_1+J_2,\ \text{say}.
\end{aligned}$$

Since $\xi > 1$, we may interchange integration and summation which yields

$$J_1 = -\frac{1}{2\pi}\sum_{m\geq 2}\frac{\Lambda(m)}{m^a}\sum_{n\geq 1}\frac{e(n\alpha)}{n^{1+\xi-a}}\int_1^T\left(\frac{m}{n}\right)^{it}dt.$$

The latter integral is equal to $T+O(1)$ if $m=n$, respectively,

$$\int_1^T\exp\Big(it\log\frac{m}{n}\Big)dt = O(1)\qquad\text{if } m\neq n.$$

Hence,

$$J_1 = -\frac{T}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}}+O(\log T)$$

(by taking into account the Laurent expansion of $\frac{\zeta'}{\zeta}(s)$ at $s=1$). Obviously, the infinite series is absolutely convergent (since $\xi > 1$). For the other integral, we similarly find

$$\begin{aligned}
J_2 &= \frac{1}{2\pi}\int_1^T\left(\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)\right)dt\cdot e(\alpha) \\
&\quad+\frac{1}{2\pi}\sum_{n>1}\frac{e(n\alpha)}{n^{1+\xi-a}}\int_1^T\log\frac{t}{2\pi}\cdot\exp(-it\log n)dt+O(\log t).
\end{aligned}$$

Applying the first derivative test shows that

$$\int_u^T\log\frac{t}{2\pi}\exp(-it\log n)dt = O\left(\frac{\log T}{\log n}\right)$$

for every sufficiently large constant u. Hence, we obtain

$$J_2 = e(\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Collecting all results so far thus leads, for the case $\xi > 1$, to

$$\sum_{0<\gamma<T} F(\rho+\xi;\alpha) = e(\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{T}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + O\Big((\log T)^2\Big).$$

2nd case: $0 < \xi \leq 1$. Then, $F(s+\xi;\alpha)$ is for $s \in [1-a+i, 1-a+iT]$ no longer represented by an absolutely convergent Dirichlet series. In this case, we use the simple approximate functional equation in I_3 and arrive at

$$I_3 = \frac{1}{2\pi i}\int_{1-a+i}^{1-a+iT}\left(\frac{\zeta'}{\zeta}(1-s) - \frac{\Delta'}{\Delta}(s)\right)\left(\sum_{n\leq x}\frac{e(n\alpha)}{n^{1+\xi-a+it}} + O\Big(x^{-(1+\xi-a)}\Big)\right)ds,$$

where $\sigma = 1+\xi-a$ is positive for sufficiently large T and $T \leq \pi\alpha x$. Integrating the error term leads to an error term of size

$$O\Big(T\log T x^{-(1+\xi-a)}\Big) = O\Big(T^{1-\xi}\log T\Big) = o(T).$$

For the remaining terms, we find, similar to the first case,

$$\begin{aligned}
I_3 &= \frac{1}{2\pi}\int_1^T\left(-\sum_{m\geq 2}\frac{\Lambda(m)}{m^{a-iT}} + \log\frac{t}{2\pi} + O\left(\frac{1}{t}\right)\right)\left(e(\alpha) + \sum_{1<n\leq x}\frac{e(n\alpha)}{n^{1+\xi-a+it}}\right)dt \\
&= \sum_{k=1}^{6} J_k, \text{ say.}
\end{aligned}$$

By the same methods as above,

$$\begin{aligned}
J_1 &= \frac{1}{2\pi}\int_1^T\left(-\sum_{m\geq 2}\frac{\Lambda(m)}{m^{a-it}}\right)dt\cdot e(\alpha) \\
&= \frac{-1}{2\pi}\sum_{m\geq 2}\frac{\Lambda(m)}{m^a}\int_1^T m^{it}dt\cdot e(\alpha) = O(\log T),
\end{aligned}$$

$$\begin{aligned}
J_2 &= \frac{1}{2\pi}\int_1^T\left(-\sum_{m\geq 2}\frac{\Lambda(m)}{m^{a-it}}\right)\sum_{1\leq n\leq x}\frac{e(n\alpha)}{n^{1+\xi-a+it}}dt \\
&= \frac{-1}{2\pi}\sum_{m\geq 2}\frac{\Lambda(m)}{m^a}\sum_{1\leq n\leq x}\frac{e(n\alpha)}{n^{1+\xi-a}}\int_1^T\left(\frac{m}{n}\right)^{it}dt \\
&= -\frac{T}{2\pi}\sum_{1<n\leq x}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + O\Big((\log T)^2\Big),
\end{aligned}$$

where the latter series now is finite; however, in view of its convergence as $x \to \infty$ (since $\xi > 0$), we have

$$\sum_{1<n\leq x}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} = \sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + O\left(\frac{\log x}{x^{\xi}}\right)$$

(since $\sum_{n>x} \frac{\log n}{n^{1+\xi}} = O\Big(\int_x^\infty \frac{\log u}{u^{1+\xi}} du\Big) = O\Big(\frac{\log x}{x^\xi}\Big)$); hence,

$$J_2 = -\frac{T}{2\pi} \sum_{n\geq 2} \frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + O\Big(T^{1-\xi}\log T\Big)$$

by choosing $T = \pi\alpha x$. The integral J_3 has already been computed in the first case and contributes to the main term. Next,

$$\begin{aligned} J_4 &= \frac{1}{2\pi}\int_1^T \log\frac{t}{2\pi} \sum_{1<n\leq x} \frac{e(n\alpha)}{n^{1+\xi-a+it}} dt \\ &= \frac{1}{2\pi} \sum_{1<n\leq x} \frac{e(n\alpha)}{n^{1+\xi-a}} \int_1^T \log\frac{t}{2\pi} \exp(-it\log n) dt \end{aligned}$$

and applying the first derivative test once more yields

$$J_4 = O(\log T).$$

Obviously, $J_5 = O(\log T)$, too, and, finally,

$$J_6 = \frac{1}{2\pi} \sum_{1<n\leq x} \frac{e(n\alpha)}{n^{1+\xi-a}} \int_1^T O\Big(\frac{1}{t}\Big) n^{-it} dt = O(\log T).$$

Collecting together, we thus obtain, for $0 < \xi \leq 1$, that

$$\sum_{0<\gamma<T} F(\rho+\xi;\alpha) = e(\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{T}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n^{1+\xi}} + O\Big(T^{1-\xi}\log T\Big).$$

It thus remains to consider the last case.

3rd case: $\xi = 0$. In view of the above formula, this is the most interesting case since the error term estimate would exceed the main terms for $\xi = 0$.

By the functional equation and substituting $s \mapsto 1-s$, we obtain

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{1-a+i}^{1-a+iT} \Big(\frac{\zeta'}{\zeta}(1-s) - \frac{\Delta'}{\Delta}(s)\Big) F(s;\alpha) ds \\ &= \frac{1}{2\pi i}\int_{a-i}^{a-iT} \Big(\frac{\Delta'}{\Delta}(1-s) - \frac{\zeta'}{\zeta}(s)\Big) F(1-s;\alpha) ds \end{aligned}$$

which is the conjugate of

$$\begin{aligned} \overline{I_3} &= -\frac{1}{2\pi i}\int_{a+i}^{a+iT} \Big(\frac{\Delta'}{\Delta}(1-s) - \frac{\zeta'}{\zeta}(s)\Big) F(1-s;1-\alpha) ds \\ &= -\frac{1}{2\pi i}\int_{a+i}^{a+iT} \Big(\frac{\Delta'}{\Delta}(1-s) - \frac{\zeta'}{\zeta}(s)\Big) \frac{\Gamma(s)}{(2\pi)^s}\Big\{e\Big(\frac{s}{4}\Big)\zeta(s;1-\alpha) + e\Big(-\frac{s}{4}\Big)\zeta(s;\alpha)\Big\} ds \\ &= \sum_{k=1}^{4} J_k, \text{ say,} \end{aligned}$$

where we also used the functional equation for F; note that conjugation here requires switching the parameter in the periodic zeta-function to $-\alpha$, respectively, $1-\alpha$. Moreover, observe that the Hurwitz zeta-functions appearing above as well as the periodic zeta-function are represented as absolutely convergent Dirichlet series. By Stirling's formula,

$$\frac{\Gamma(s)}{(2\pi)^s} e\Big(\pm\frac{s}{4}\Big)\Big|_{s=\sigma+it} = \Big(\frac{t}{2\pi}\Big)^{\sigma-\frac{1}{2}+it} \exp\Big(-i\Big(t+\frac{\pi}{4}\Big)\Big) \begin{cases} O(\exp(-\pi t)) & \text{for " + ",} \\ 1+O\Big(\frac{1}{t}\Big) & \text{for " − ".} \end{cases}$$

This implies that the integrals J_1 with $e\Big(\pm\dfrac{s}{4}\Big)$ contribute to the error term. More precisely,

$$J_1 = O\left(\sum_{m\geq 0}(m+1-\alpha)^{-a}\int_1^T\left(-\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)\right)t^{\frac{1}{2}}\exp(-\pi t)dt\right) = O(\log T)$$

where the integral is bounded, and $O(\log T)$ arises from

$$\sum_{m\geq 0}(m+1-\alpha)^{-a} = \zeta(a;1-\alpha) = O\left(\frac{1}{a-1}\right).$$

Furthermore, in a similar way as above,

$$\begin{aligned} J_3 &= O\left(\sum_{n\geq 2}\frac{\Lambda(n)}{n^a}\sum_{m\geq 0}\frac{1}{(m+1-\alpha)^a}\int_1^T t^{\frac{1}{2}}\exp(-\pi t)dt\right) \\ &= O\Big((\log T)^2\Big) \end{aligned}$$

since both Dirichlet series have a simple pole at $s=1$ (and $a = 1+\dfrac{1}{\log T}$).

It thus remains to evaluate J_2 and J_4. For this purpose, we shall use "Gonek's lemma". We begin with J_2, and as an application of the fundamental theorem of calculus,

$$\begin{aligned} J_2 &= -\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\Delta'}{\Delta}(1-s)\frac{\Gamma(s)}{(2\pi)^s}e\Big(-\frac{s}{4}\Big)\zeta(s;\alpha)ds \\ &= \int_1^T\left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\frac{\Gamma(s)}{(2\pi)^s}e\Big(-\frac{s}{4}\Big)\zeta(s;\alpha)ds\right). \end{aligned}$$

Next, we rewrite the integrator as follows:

$$\begin{aligned} J &:= \frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\frac{\Gamma(s)}{(2\pi)^s}e\Big(-\frac{s}{4}\Big)\sum_{m\geq 0}(m+\alpha)^{-s}ds \\ &= \sum_{m\geq 0}(m+\alpha)^{-a}\sum_{1\leq j\leq \mathcal{J}}\frac{1}{2\pi}\int_{2^{-j}\tau}^{2^{1-j}\tau}\frac{\Gamma(a+it)}{(2\pi)^{a+it}}e\left(-\frac{a+it}{4}\right)(m+\alpha)^{-it}dt + O(\log T) \end{aligned}$$

where $\mathcal{J}\in\mathbb{N}$ is uniquely determined by

$$2^{-\mathcal{J}}\tau < 1 \leq 2^{1-\mathcal{J}}\tau,$$

and the error term arises from

$$\int_{2^{-\mathcal{J}}\tau}^{1}\frac{\Gamma(a+it)}{(2\pi)^{a+it}}e\left(-\frac{a+it}{4}\right)(m+\alpha)^{-it}dt = O(1)$$

and summation over $m\geq 0$. Applying Gonek's lemma, we find

$$\begin{aligned} &\frac{1}{2\pi}\int_{2^{-j}\tau}^{2^{1-j}\tau}\frac{\Gamma(a+it)}{(2\pi)^{a+it}}e\left(-\frac{a+it}{4}\right)(m+\alpha)^{-it}dt \\ &= \frac{1}{2\pi}\int_A^B\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{t}{re}\right)\left(\exp\left(-\frac{\pi i}{4}\right)+O\left(\frac{1}{t}\right)\right)dt \\ &= \left(\frac{r}{2\pi}\right)^a\exp(-ir)1_{[A,B)}(r) + E(r;A,B) \end{aligned}$$

where $B = 2A = 2^{1-j}\tau, r = 2\pi(m+\alpha)$, and $E(r; A, B)$ is the corresponding error term. This leads to

$$\begin{aligned} J &= \sum_{m\geq 0}(m+\alpha)^{-a}\sum_{1\leq j\leq \mathcal{J}}\{(m+\alpha)^{a}\exp(-2\pi i(m+\alpha))1_{[2^{-j}\tau,2^{1-j}\tau)}(2\pi(m+\alpha))+ \\ &\quad + E\Big(2\pi(m+\alpha);2^{-j}\tau,2^{1-j}\tau\Big)\Big\} \\ &= \sum_{0\leq m\leq \frac{\tau}{2\pi}} e(-\alpha)+O\Big(\tau^{\frac{1}{2}}\Big) \end{aligned}$$

(since $\exp(-2\pi i(m+\alpha)) = e(-\alpha)$ and $r = 2\pi(m+\alpha) \leq \tau$ is equivalent to $m \leq \frac{\tau}{2\pi} - \alpha$, but the difference to summing up according to $m \leq \frac{\tau}{2\pi}$ is negligible); the error term estimate follows as in the case of Gonek's original work (see his Lemma 4).

Hence, $J = e(-\alpha)\frac{\tau}{2\pi} + O\Big(\tau^{\frac{1}{2}}\Big)$; plugging this into the formula for J_2 and applying partial integration, we arrive at

$$J_2 = e(-\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\Big(T^{\frac{1}{2}}\Big).$$

It remains to evaluate

$$\begin{aligned} J_4 &= \frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)\frac{\Gamma(s)}{(2\pi)^s}e\Big(-\frac{s}{4}\Big)\zeta(s;\alpha)ds \\ &= -\sum_{n\geq 2}\frac{\Lambda(n)}{n^a}\sum_{m\geq 0}\frac{1}{(m+\alpha)^a}\frac{1}{2\pi}\int_1^T\frac{\Gamma(a+it)}{(2\pi)^{a+it}}e\Big(-\frac{a+it}{4}\Big)(n(m+\alpha))^{-it}dt. \end{aligned}$$

Applying Gonek's lemma once more (this time with $r = 2\pi n(m+\alpha)$) leads to

$$J_4 = -\sum_{\substack{n\geq 2 \\ n(m+\alpha)\leq \frac{T}{2\pi}}}\sum_{m\geq 2}\Lambda(n)e(-n\alpha) + O\Big(T^{\frac{1}{2}}\Big).$$

For rational $\alpha = \frac{k}{q}$, where w.l.o.g. k and q are coprime, we follow Kalpokas' reasoning and consider

$$\begin{aligned} \sum_{\substack{n\geq 2 \\ n(m+\alpha)\leq x}}\sum_{m\geq 2}\Lambda(n)e(-n\alpha) &= \left\{\sum_{\substack{1\leq b\leq q \\ \gcd(b,q)=1}} + \sum_{\substack{1\leq b\leq q \\ \gcd(b,q)>1}}\right\}e\Big(-\frac{bk}{q}\Big)\sum_{\substack{n\left(m+\frac{k}{q}\right)\leq x \\ n\equiv b \mod q}}\Lambda(n) \\ &= \Sigma_1 + \Sigma_2, \text{ say.} \end{aligned}$$

Using the orthogonality relation for characters

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(b)\chi(n) = \begin{cases} 1 & \text{if } n\equiv b \bmod q \\ 0 & \text{otherwise} \end{cases}$$

is valid if $\gcd(b,q)=1$. We find

$$\Sigma_1 = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \sum_{1\le b\le q} \overline{\chi}(b) e\left(-\frac{bk}{q}\right) \sum_{n\left(m+\frac{k}{q}\right)\le x} \Lambda(n)\chi(n)$$
$$= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \tau(-k,\overline{\chi}) W(x).$$

where $\tau(r,\chi) := \sum_{b \bmod q} \chi(b) e\left(\frac{rb}{q}\right)$ is the Gauss sum associated with χ and

$$W(x) := W\left(x; \frac{k}{q}, \chi\right) := \sum_{n\left(m+\frac{k}{q}\right)\le x} \Lambda(n)\chi(n),$$

which we may rewrite by Perron's formula as

$$W(x) = -\frac{1}{2\pi i} \int_{a-iu}^{a+iu} \frac{L'}{L}(s;\chi)\zeta\left(s;\frac{k}{q}\right)\frac{x^s}{s} ds + O\left(\frac{x(\log x)^2}{u}\right).$$

Moving the path of integration to the left and taking into account a suitable zero-free region for Dirichlet L-functions, one obtains

$$W(x) = -\operatorname{res}_{s=1} \frac{L'}{L}(s,\chi)\zeta\left(s;\frac{k}{q}\right)\frac{x^s}{s}$$
$$+ \frac{1}{2\pi i}\left\{\int_{a+iu}^{b+iu} + \int_{b+iu}^{b-iu} + \int_{b-iu}^{a-iu}\right\} \frac{L'}{L}(s;\chi)\zeta\left(s;\frac{k}{q}\right)\frac{x^s}{s} ds + O\left(\frac{x(\log x)^2}{u}\right)$$

where b can be chosen as $1 - c(\log u)^{-\frac{3}{4}-\varepsilon}$ with an appropriate $c>0$. As usual, we denote the principal character by χ_0. Taking into account the Laurent expansion of the integrand, one obtains

$$\operatorname{res}_{s=1} \frac{L'}{L}(s,\chi)\zeta\left(s;\frac{k}{q}\right)\frac{x^s}{s} = \begin{cases} -x\log\frac{x}{e} + x\left(\Psi\left(\frac{k}{q}\right) + \gamma_0 + \sum_{p|q}\frac{\log p}{p-1}\right) & \text{if } \chi=\chi_0, \\ \frac{L'}{L}(1,\chi)x & \text{otherwise.} \end{cases}$$

The integral can be estimated in a straightforward way; using this with $u = T^{1-b}$ and $x = \frac{T}{2\pi}$ leads to

$$\Sigma_1 = -\frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \tau(-k,\overline{\chi})\frac{L'}{L}(1,\chi)\frac{T}{2\pi}$$
$$- \frac{1}{\varphi(q)}\tau(-k,\chi_0)\left(-\frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{T}{2\pi}\left(\Psi\left(\frac{k}{q}\right) + \gamma_0 + \sum_{p|q}\frac{\log p}{p-1}\right)\right)$$
$$+ O\left(T^{1-c(\log T)^{-\frac{3}{4}-\varepsilon}}\right).$$

Since

$$\tau(-k,\chi_0)=\sum_{\substack{b \bmod q\\ \gcd(b,q)=1}} e\left(-\frac{kb}{q}\right)=\sum_{\substack{c \bmod q\\ \gcd(c,q)=1}} e\left(\frac{c}{q}\right)=\mu(q)$$

as well as

$$\begin{aligned}\tau(-k,\overline{\chi})&=\sum_{b \bmod q}\overline{\chi}(b)e\left(-\frac{bk}{q}\right)=\chi(k)\sum_{c \bmod q}\overline{\chi}(c)e\left(-\frac{c}{q}\right)\\&=\chi(k)\tau(-1;\overline{\chi}),\end{aligned}$$

we finally arrive at

$$\begin{aligned}\Sigma_1=&-\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(k)\tau(-1;\overline{\chi})\frac{L'}{L}(1,\chi)\frac{T}{2\pi}\\&-\frac{\mu(q)}{\phi(q)}\left(-\frac{T}{2\pi}\log\frac{T}{2\pi e}+\frac{T}{2\pi}\left(\Psi\left(\frac{k}{q}\right)+\gamma_0+\sum_{p|q}\frac{\log p}{p-1}\right)\right)\\&+O\left(T^{1-c(\log T)^{-\frac{3}{4}-\varepsilon}}\right).\end{aligned}$$

It remains to consider

$$\begin{aligned}\Sigma_2&=\sum_{\substack{b \bmod q\\ \gcd(b,q)>1}} e\left(-\frac{kb}{q}\right)\sum_{\substack{n\geq 2\\ n\equiv b \bmod q}}\Lambda(n)\sum_{\substack{m\geq 0\\ n\left(m+\frac{k}{q}\right)\leq x}}1\\&=\frac{T}{2\pi}\sum_{p|q}\log p\sum_{j\geq 1}\frac{e\left(-\frac{kp^j}{q}\right)}{p^j}+O(\log T)\end{aligned}$$

since $\Lambda(m)\neq 0$ only if $n=p^j$ and $n\equiv b \bmod q$, where $\gcd(b,q)>1$. This gives an asymptotic formula for I_4, and, finally, it reproduces Formula (3).

If the parameter α, however, is not rational, we cannot apply the theory of Dirichlet L-functions, but we may apply a deep result of Vinogradov [8]. We have

$$\begin{aligned}\sum_{\substack{n\geq 2\\ n(m+\alpha)\leq\frac{T}{2\pi}}}\sum_{m\geq 2}\Lambda(n)e(-n\alpha)&=\sum_{2\leq n\leq\frac{T}{2\pi\alpha}}\Lambda(n)e(-n\alpha)\sum_{2\leq m\leq\frac{T}{2\pi n}-\alpha}1\\&=\sum_{n\leq\frac{T}{2\pi\alpha}}\Lambda(n)e(-n\alpha)\left(\left\lfloor\frac{T}{2\pi n}-\alpha\right\rfloor-1\right).\end{aligned}$$

Vinogradov proved

$$V(N):=\sum_{n\leq N}\Lambda(n)e(n\alpha)=O\left(\left(\frac{N}{q^{\frac{1}{2}}}+N^{\frac{4}{5}}+N^{\frac{1}{2}}q^{\frac{1}{2}}\right)(\log N)^4\right),$$

where q is from a sequence of integers tending to ∞ and satisfying $\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{q^2}$ with $\gcd(a,q)=1$. Hence,

$$V(N)=O\left(N^{\frac{4}{5}}(\log N)^4\right)$$

and thus, by partial summation,

$$\begin{aligned}\sum_{N<n\leq M}\frac{\Lambda(n)e(n\alpha)}{n} &= \left\{\sum_{n\leq M}-\sum_{n\leq N}\right\}\frac{\Lambda(n)e(n\alpha)}{n}\\ &= \frac{V(M)}{M}-\frac{V(N)}{N}+\int_N^M V(u)\frac{du}{u^2}\\ &= O\left(N^{-\frac{1}{5}}(\log N)^4+M^{-\frac{1}{5}}(\log M)^4+\int_N^M\frac{(\log u)^4}{u^{\frac{6}{5}}}du\right)\end{aligned}$$

or, after letting $M\to\infty$,

$$\sum_{n>N}\frac{\Lambda(n)e(n\alpha)}{n}=O\left(\frac{(\log N)^4}{N^{\frac{1}{5}}}\right);$$

This shows that

$$\Sigma := \sum_{n\geq 2}\frac{\Lambda(n)e(n\alpha)}{n}$$

is a constant. This gives

$$\sum_{n\geq 2}\sum_{\substack{2\\ n(m+\alpha)\leq\frac{T}{2\pi}}}\Lambda(n)e(-n\alpha)=\frac{T}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)e(-n\alpha)}{n}+O\left(T^{1-\frac{1}{5}}(\log T)^4\right).$$

Plugging this in yields

$$\sum_{0<\gamma<T}F(\rho;\alpha)=e(\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e}-\frac{T}{2\pi}\Sigma+O\left(T^{\frac{4}{5}}(\log T)^4\right).$$

4. Conclusions

Formula (2) of the Sowa's result was derived based on the Riemann hypothesis, in which the nontrivial zeros lie on the critical line. However, the nontrivial zeros are symmetrical on the real line and the critical line ($\sigma = 1/2$). In our study, we applied the contour integration over this assumption and obtained an extended formula of the former results (including Sowa's result) without conditions.

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