Isoparametric hypersurfaces with a homogeneous focal manifold

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MARTIN WOLFROM

aus

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Preface

The classification of isoparametric hypersurfaces in spheres with a homogeneous focal manifold is a project that has been started by Linus Kramer [Kra98]. It extends results by É. Cartan [Car39] and Hsiang and Lawson [HL71]. Kramer does most part of this classification in his *Habilitationsschrift* [Kra98]. In particular he obtains a classification for the cases where the homogeneous focal manifold is at least 2-connected. Results of É. Cartan [Car39], Dorfmeister and Neher [DN85], and Takagi [Tak76] also solve parts of the classification problem. This thesis completes the classification. We classify all closed isoparametric hypersurfaces in spheres with $g \geq 3$ distinct principal curvatures one of whose multiplicities is 2 such that the lower dimensional focal manifold is homogeneous.

The methods are essentially the same as in [Kra98]. The cohomology of the focal manifolds in question is known. This leads to two topological classification problems, which are also solved in this thesis. We classify simply connected homogeneous spaces of compact Lie groups with the same integral cohomology ring as a product of spheres $\mathrm{S}^2 \times \mathrm{S}^m$ and m odd on the one hand and a truncated polynomial ring $\mathbb{Q}[a] \,/\, (a^m)$ with one generator of even degree and $m \geq 2$ as its rational cohomology ring on the other hand.

Hardly anybody can do mathematical research completely on their own. There are several people that supported me during the years of preparation for this thesis. At first I would like to thank Linus Kramer for this beautiful subject. He teached me with patience and always had an open ear for my mathematical problems. I also would like to thank Prof. Horst Ibisch. With his help I could spend a year at the Université de Nantes where he introduced me to the vast field of algebraic topology. For this year in France I was supported by the DAAD (*Deutscher Akademischer Austauschdienst*). In the last two years I was supported by the *Cusanuswerk*. Nils Rosehr helped me with various problems that arose naturally, because English is not my native language. Moreover he always readily discussed TEXnical details. For our daily discussions about mathematics and other topics I would like to thank

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Contents

Introduction

Isoparametric submanifolds have been studied for a long time. They are a type of submanifolds which is easily defined: A closed submanifold is called *isoparametric* if its normal bundle is flat, and if its principal curvatures, i.e. the eigenvalues of the shape operator, are constant along any parallel normal field, cf. [PT88]. Usually, one studies isoparametric submanifolds of Euclidean and spherical spaces. An isoparametric submanifold is called *irreducible* if it is not isomorphic to a product of isoparametric submanifolds under the isometry group of the ambient space. Non-compact irreducible isoparametric submanifolds are completely classified, see [Tho91]. Compact isoparametric submanifolds always can be embedded into a sphere. Also (compact) full irreducible isoparametric submanifolds in spheres are classified except for those that are hypersurfaces.

For isoparametric hypersurfaces in spheres the following is known. (For a survey on isoparametric hypersurfaces see [Tho00].) The number of distinct principal curvatures is $g \in \{1, 2, 3, 4, 6\}$ by a theorem of Münzner in [Mün81]. Those isoparametric hypersurfaces in spheres with $g = \{1, 2, 3\}$ were classified by É. Cartan in the thirties in [Car39]. An isoparametric hypersurface in a sphere with $g = 6$ distinct principal curvatures has dimension 6 or 12 by Münzner's theorem. Those with dimension 6 have been classified by Dormeister and Neher in [DN85]. For those of dimension 12 only one example is known, and one might conjecture that this example is the only one, see [Kra98].

For $g = 4$ many examples are known. There are the classical examples, which arise from isotropy representations of symmetric spaces. Ferus, Karcher and Münzner constructed more examples using Clifford algebras in [FKM81]. These examples are called *Clifford hypersurfaces* in the sequel. Some but not all of them are classical. The classical and the Clifford examples include infinite series of arbitrary high dimension. The following steps towards a classification have been done up to now. Münzner found strong restrictions on the possible multiplicities $m_1, \dots, m_4$ of the 4 principal curvatures, see [Mün81]: One has (up to reordering the indices) $m_1 = m_3$ and $m_2 = m_4$; moreover, $m_1, m_2 \geq 2$ implies $m_1 = m_2$ or that $m_1 + m_2$ is odd. Stolz even sharpened these restrictions in [Sto99].

Immervoll proved that the point–line geometry associated to an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures is a compact connected Tits building of rank 2, i.e. a compact connected generalized polygon, see [Imm01]. This important result relates two difficult geometric problems. Note that isoparametric submanifolds in spheres that cannot be embedded as hypersurfaces also lead to Tits buildings by Thorbergsson. The same turns out to be true for isoparametric hypersurfaces in spheres with $g = 3$, see [KK95]. Yet Immervoll's result seems not to help for the classification, because compact connected generalized polygons are far from being classified.

Hsiang and Lawson proved in [HL71] that all *homogeneous* isoparametric hypersurfaces in spheres arise from isotropy representations of symmetric spaces and thus are classical in the above sense. An isoparametric hypersurface $\mathcal{F}$ is called *homogeneous* if its isometry group acts transitively on $\mathcal{F}$. To each isoparametric hypersurface $\mathcal{F}$ there belongs a so-called *isoparametric foliation*. The leaves of this foliation consist of parallel hypersurfaces, which are isometric to $\mathcal{F}$, and the two focal manifolds of $\mathcal{F}$. If the isometry group of $\mathcal{F}$ acts transitively on $\mathcal{F}$, then it also acts transitively on each leaf of the foliation. Some of the Clifford hypersurfaces are not homogeneous but have a homogeneous focal manifold. In [Kra98] Kramer generalizes the theorem of Hsiang and Lawson, i.e. the classification of all homogeneous isoparametric hypersurfaces in spheres, and asks for a classification of all isoparametric hypersurfaces in spheres that have a homogeneous focal manifold.

Kramer gives a complete classification of all isoparametric hypersurfaces in spheres with $g = 4$ distinct principal curvatures that have a 2-connected homogeneous focal manifold in [Kra98]. With an appropriate ordering on the multiplicities m_i of the principal curvatures this means $m_1 \geq 3$. The case $g = 4$, $m_1 = 1$ has been settled by Takagi [Tak76], see also [Ble02]. The cases $g \in \{1, 2, 3\}$ and $g = 6$, $m_1 = 1$ are clear from the general classifications of É. Cartan in [Car39] and of Dorfmeister and Neher in [DN85]. We complete the classification in this thesis by solving the case $m_1 = 2$. The result is as follows.

Theorem *Let $\mathcal{F}$ be a closed isoparametric hypersurface in a sphere. Assume that the isometry group of $\mathcal{F}$ acts transitively on one of the focal manifolds of $\mathcal{F}$. Then either $\mathcal{F}$ itself is homogeneous and arises from the isotropy representation of an irreducible Riemannian symmetric space* (*of* *non-compact type), or $\mathcal{F}$ is of Clifford type with multiplicities (m_1, m_2) equal to $(8, 7)$ or $(3, 4k)$.*

Note that in the higher rank case, i.e. for irreducible isoparametric submanifolds in Euclidean spaces that cannot be embedded as isoparametric hypersurfaces in spheres, Thorbergsson proves the existence of a transitive isometry group. In fact, he shows that each such submanifold arises from the isotropy representation of a symmetric space. We thus may reformulate the above theorem in a more general setting.

Theorem *Let $\mathcal{F}$ be a closed irreducible isoparametric submanifold (of rank at least 2) in an Euclidean space. Assume that the isometry group of $\mathcal{F}$ acts transitively on one of the focal manifolds of $\mathcal{F}$. Then either $\mathcal{F}$ itself is homogeneous and arises from the isotropy representation of an irreducible Riemannian symmetric space (of non-compact type), or $\mathcal{F}$ is of Clifford type with multiplicities (m_1, m_2) equal to $(8, 7)$ or $(3, 4k)$.*

The strategy for proving the case $m_1 = 2$ of the above results is essentially the same as in [Kra98] for $m_1 \geq 3$. By a theorem of Münzner in [Mün81] we know the cohomology ring of the focal manifolds. The homogeneous focal manifold $\mathcal{P}$ in question is in addition simply connected. Moreover, the isometry group of the isoparametric hypersurface acts transitively on $\mathcal{P}$. It turns out that there is a compact subgroup still acting transitively on $\mathcal{P}$. So we first classify all simply connected homogeneous spaces of compact Lie groups that have this cohomology ring and thus obtain a complete list of candidates for the homogeneous focal manifold $\mathcal{P}$. Then we use geometrical properties of the isoparametric foliation to solve the classification problem.

This strategy for the proof divides the text into two parts. The chapters 1–3 form the topological part where we classify homogeneous spaces of compact Lie groups with a specific cohomology. The last two chapters 4 and 5 form the geometrical part. There we use the results of the first part for the classification of isoparametric hypersurfaces in spheres that have a homogeneous focal manifold. The main theorems are stated in section 4.2.

The topological part may be of independent interest. We classify simply connected homogeneous spaces of compact Lie groups with a specific cohomology ring. In chapter 3 this cohomology ring, with coefficients in the field of rational numbers, is supposed to be a truncated polynomial ring $\mathbb{Q}[a] \,/\, (a^m)$ with one generator a of even degree with $m \geq 2$. We obtain a complete classification, which simultaneously generalizes results of Montgomery and Samelson [MS43], Borel [Bor49], and Uchida [Uch77]. The resulting homogeneous spaces are the even dimensional spheres, the complex projective spaces, the quaternionic projective spaces, the Grassmann manifolds of oriented planes in odd dimensional Euclidean spaces and three exceptional spaces. These exceptional spaces are $\mathrm{G}_2 / \mathrm{SO}(4)$, the octonion projective plane $\mathbb{O}\mathrm{P}^2$ and the point space of the split Cayley hexagon $\mathrm{H}(\mathbb{C})$. The result is stated in section 3.3. For the proof we use the fact that the Euler-Poincaré characteristic of the homogeneous space is $m \geq 2$. Using a result of Wang in [Wan49] we can show that the group that acts transitively is almost simple. Moreover, the point stabilizer of this action has the same torus rank, and we may use the classification of the maximal compact subgroups of the simple compact connected Lie group of maximal rank due to Borel and De Siebenthal, see [BDS49].

In chapter 2 we classify simply connected homogeneous spaces of compact Lie groups with the integral cohomology ring of a product of two spheres $\mathrm{S}^2 \times \mathrm{S}^m$ and $m \geq 3$ odd. The classification result is stated in section 2.5. Here the situation is more involved, because the Euler-Poincaré characteristic of the homogeneous space vanishes. But we can apply the methods developed in [Kra98]. There, homogeneous spaces are classified that have the same cohomology ring as a product of spheres $\mathrm{S}^{n_1} \times \mathrm{S}^{n_2}$ with $3 \leq n_1 \leq n_2$ and n_2 odd. Fortunately, many proofs carry over to the case $n_1 = 2$, and we may use Kramer's methods. The classification with rational coefficients, i.e. without attention to torsion, can be done in a similar way as the classification in [Kra98]. The integral classification, though, is more complicated, because the point stabilizer of the transitive group contains a central torus factor if the homogeneous space is not itself a product of spheres. We adapt a method developed by Kreck and Stolz to determine the exact embedding of this torus factor in the transitive group.

Chapter 1

Lie groups and topology

This first chapter collects some facts and easily obtained results that are certainly well known to the specialist. For all non-specialists, this chapter should provide all notation, notions and methods we need from Lie theory. Moreover there is a section about graded modules and spectral sequences.

All Lie groups in this text will be compact, and most of them connected. In the first two sections we introduce two invariants of compact connected Lie groups, the (torus) rank and the degree multiset. The former is a well known integral invariant, the dimension of a maximal torus. The latter may be regarded as a generalization of the rank. It is the multiset consisting of the degrees of a canonical set of generators for the rational cohomology of the group. The rank of the group turns out to be the number of elements in this multiset.

A basic problem in the later chapters is to examine embeddings between compact connected Lie groups: whether one given compact connected Lie group can be embedded into another one, or given two isomorphic closed subgroups, whether they are conjugate in the group. Section 1.3 explains how representation theory may help.

In the chapters 2 and 3 we classify homogeneous spaces with specific topological properties. In section 1.4 we therefore recall some facts about homogeneous spaces. Moreover we give several well known examples of topological spaces, written as homogeneous spaces of compact Lie groups.

Section 1.5 gives a very brief introduction to graded modules and spectral sequences with a short description of the Leray-Serre spectral sequence. The latter is a powerful tool to relate the cohomologies of the total space of a fibration (with certain properties) to the cohomologies of fiber and base space. It is used in [Kra98] for the proof of some basic results from which our classification starts. We repeat these results in the theorems 1.12 and 3.1.

The last, very short section consists of a table of all compact connected simple Lie groups and their maximal connected closed subgroups of maximal rank, which is due to Borel and De Siebenthal [BDS49].

1.1 The rank

We now recall some notions from Lie theory, in particular that of a torus, a maximal torus and the (torus) rank of a compact connected Lie group. The material presented in this section can e.g. be found in [MT91, section V.3] or [HM98, chapter 6].

Recall that an (abstract) subgroup of a Lie group G is again a Lie group if it is closed in G. A compact connected non-Abelian Lie group is called *almost simple* if all proper normal closed subgroups have dimension zero. Note that a subgroup of a compact connected Lie group of dimension zero is automatically finite. An epimorphism between Lie groups that is a finite covering, i.e. that has finite kernel, is a so-called local isomorphism. More generally, two Lie groups are *locally isomorphic*, if they have isomorphic universal coverings. Lie groups that are locally isomorphic have the same Lie algebra. This fact allows us to name a local isomorphism class by the corresponding Lie algebra. The Lie algebras of the almost simple compact connected Lie groups are simple. Moreover they are classified and known. We will give a complete list in table 1.1 on page 8.

We denote by $\mathrm{U}(1)$ the group of all complex numbers of norm 1. An *n-torus* is the direct product of n copies of $\mathrm{U}(1)$. It is a well known fact that each element of a compact connected Lie group is contained in a maximal closed torus subgroup, also called a *maximal torus*. Moreover each closed torus subgroup is contained in a maximal torus. All maximal tori in a compact connected Lie group are conjugate to each other and thus have equal dimensions. This gives a simple but useful invariant of compact connected Lie groups:

Definition Let G be a compact connected Lie group. The *rank* of G, denoted by $\operatorname{rk} G$, is the dimension of a maximal torus in G.

One of the most frequent problems to solve in this text is whether a given compact connected Lie group can be embedded into another one. Sometimes the rank gives an easy negative answer:

Proposition 1.1 *Let K and G be compact connected Lie groups with $K \subseteq G$. Then $\operatorname{rk} K \leq \operatorname{rk} G$.*

Proof. A maximal torus T in K is by the inclusion $K \subseteq G$ also a torus in G. As each torus in a compact connected Lie group is contained in a maximal one, the torus T is contained in a maximal torus in G. $\square$

In section 1.2 we will see that a compact connected Lie group is almost a (finite) product of almost simple compact connected Lie groups and a torus. This fact can be used to get a maximal torus of the group: Take a maximal torus in each factor. The (inner direct) product of these tori is a maximal torus of the group. In particular the rank of a product is the sum over the ranks of the factors.

Proposition 1.2 *Let G_1 and G_2 be compact connected Lie groups with maximal tori T_1 and T_2, respectively. Then $T_1 \times T_2$ is a maximal torus in $G_1 \times G_2$.*

Proof. Let $a = (a_1, a_2) \in G_1 \times G_2$ be a *generator* of a maximal torus in $G_1 \times G_2$, i.e. the set $\left\{ a^k \;\middle|\; k \in \mathbb{Z} \right\}$ is dense in the maximal torus. Recall that all maximal tori of a compact connected Lie group are conjugate to each other. Let $i \in \{1, 2\}$. We have $a_i \in G_i$. Hence, a_i lies in a maximal torus in G_i. So, there exists $g_i \in G_i$ such that $g_i a_i {g_i}^{-1} \in T_i$. Let $g = (g_1, g_2)$. Since a generates a maximal torus in $G_1 \times G_2$, so does gag^{-1}. Moreover $gag^{-1} = (g_1, g_2)(a_1, a_2)(g_1, g_2)^{-1} = (g_1 a_1 {g_1}^{-1}, g_2 a_2 {g_2}^{-1}) \in T_1 \times T_2$. Hence gag^{-1} generates $T_1 \times T_2$, and therefore $T_1 \times T_2$ is maximal in $G_1 \times G_2$. $\square$

Corollary 1.3 *Let G_1 and G_2 be compact connected Lie groups of rank r_1 and r_2, respectively. Then $G_1 \times G_2$ has rank $r_1 + r_2$.*

The following proposition explains how an almost simple Lie group can be embedded in a product. As a corollary we get conditions on the rank if all groups involved are compact and connected.

Proposition 1.4 *Let K, G_1 and G_2 be Lie groups such that K is almost simple and a closed subgroup of $G_1 \times G_2$.*

Then one of the following cases applies:

(a) *K is a subgroup of $G_1 \times 1$.*

(b) *K is a subgroup of $1 \times G_2$.*

(c) *K is locally isomorphic to subgroups K_1 of G_1 and K_2 of G_2.*

Proof. Consider the composite homomorphisms

$$g_1 \colon K \longrightarrow G_1 \times G_2 \longrightarrow G_1 \times 1$$

and

$$g_2 \colon K \longrightarrow G_1 \times G_2 \longrightarrow 1 \times G_2.$$

Their respective kernels $\ker g_1$ and $\ker g_2$ are normal subgroups of K. Since K is almost simple, $\ker g_1$ or $\ker g_2$, respectively, is either finite or equal to K. If $\ker g_1 = K$, this is case (b). If $\ker g_2 = K$, this is case (a). Otherwise both $\ker g_1$ and $\ker g_2$ are finite, and g_1 and g_2 provide local isomorphisms of K onto subgroups of G_1 and G_2, respectively. This is case (c). □

Corollary 1.5 *Let K, G_1 and G_2 be compact connected Lie groups such that K is almost simple and a closed subgroup of $G_1 \times G_2$.*

Then $\operatorname{rk} K \leq \operatorname{rk} G_1$ or $\operatorname{rk} K \leq \operatorname{rk} G_2$. If $\operatorname{rk} K > \operatorname{rk} G_1$ then K is a closed subgroup of $1 \times G_2$.

Proof. The first claim follows from proposition 1.4 using proposition 1.1. Now suppose $\operatorname{rk} K > \operatorname{rk} G_1$. By proposition 1.1, K is not locally isomorphic to a subgroup of G_1. This excludes the cases (a) and (c) in proposition 1.4. □

For later use, we examine how a 1-torus can be embedded in an n-torus as a closed subgroup.

Lemma 1.6 *Let $T \cong \mathrm{U}(1)^n$, let $S \cong \mathrm{U}(1)$ be a closed subgroup of T, and let $i\colon S \longrightarrow T$ be the inclusion map.*

Then there are integers $k_1, \ldots, k_n \in \mathbb{Z}$ such that $i(z) = (z^{k_1}, \ldots, z^{k_n})$ for all $z \in S$.

Proof. Let $p_j\colon T \longrightarrow \mathrm{U}(1)$ be the projection on the j-th factor of T. Consider the composite map

$$S \xrightarrow{\ i\ } T \xrightarrow{\ p_j\ } \mathrm{U}(1).$$

this is a morphism of Lie groups $\mathrm{U}(1) \longrightarrow \mathrm{U}(1)$. Thus there exists $k_j \in \mathbb{Z}$ such that the map is described by $z \mapsto z^{k_j}$ for all $z \in \mathrm{U}(1)$. □

Definition In the situation of lemma 1.6, we denote the inclusion map by $i = i_{k_1,\ldots,k_n}$. Let G be a compact connected Lie group. Let T be a maximal torus in G. Then we can consider inclusion maps $i_{k_1,\ldots,k_n}\colon \mathrm{U}(1) \longrightarrow T$ as above. We denote their composition with the inclusion map $T \longrightarrow G$ by the same symbol:

$$i_{k_1,\ldots,k_n}\colon \mathrm{U}(1) \xrightarrow{i_{k_1,\ldots,k_n}} T \longrightarrow G.$$

1.2 Degree multisets

An important topological invariant of compact connected Lie groups is the cohomology ring. We will only use simplicial cohomology in this text. The cohomology ring of a topological space X with coefficients in the ring R will be denoted by $H^*(X;R)$. As coefficient rings will occur the field of rational numbers $\mathbb{Q}$, the field with two elements $\mathbb{Z}/2$ and the integers $\mathbb{Z}$. In the first case, i.e. coefficients in $\mathbb{Q}$, we will speak of *rational cohomology*. In the last case we omit the coefficient ring $\mathbb{Z}$ and write $H^*(X;\mathbb{Z}) = H^*(X)$. In this case we will speak of *integral cohomology*.

As we will see in this section, the rational cohomology ring of a compact connected Lie group is always an exterior algebra with a finite number of generators. This means that the rational cohomology of a compact connected Lie group is completely determined by the degrees of the generators of this exterior algebra. The multiset of these degrees will be called the degree multiset of the group. We will give an explanation of the perhaps non-standard notion of a multiset later. Note that, whenever we use degree multisets, we could use rational cohomology instead. But we hope that it is easier for the reader to follow our calculations in the multiset notation.

We first have a closer look at the rational cohomology of a compact connected Lie group and cite the theorems that explain why it is an exterior algebra. The first theorem states that a compact connected Lie group is almost a product of almost simple ones and a torus. More precisely it is a product up to finite covering, i.e. up to local isomorphism. We denote the set of positive integers by $\mathbb{N}$, and its union with $\{0\}$, by $\mathbb{N}_0$.

Theorem 1.7 *Let G be a compact connected Lie group. Then there are compact connected almost simple Lie groups $G_1, \ldots, G_s$, a number $k \in \mathbb{N}_0$ and a surjection with finite kernel*

$$f\colon G_1 \times \cdots \times G_s \times \mathrm{U}(1)^k \longrightarrow G.$$

Proof. See [HM98, theorem 6.18]. □

The next theorem tells us that locally isomorphic compact connected Lie groups have the same rational cohomology.

Theorem 1.8 *If G and $\widetilde{G}$ are compact connected Lie groups, and if $f : \widetilde{G} \longrightarrow G$ is a covering map, then the induced morphism in rational cohomology*

$$H^*(\widetilde{G};\mathbb{Q}) \xleftarrow{f^*} H^*(G;\mathbb{Q})$$

is an isomorphism.

Proof. See [MT91, lemma VI.5.2]. □

So, if we know the rational cohomology of the almost simple compact connected Lie groups, we know it of all compact connected Lie groups. We denote by $\Lambda_R(x_1, \ldots, x_n)$ the exterior algebra over the ring R with (homogeneous) generators $x_1, \ldots, x_n$. If the ring R is the ring of integers $\mathbb{Z}$, we simply write $\Lambda(x_1, \ldots, x_n)$.

Theorem 1.9 *Let G be a compact connected almost simple Lie group of rank n. Then $H^*(G;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, \ldots, x_n)$ with generators $x_1, \ldots, x_n$ of odd degree.*

Proof. See [MT91, theorems III.6.5 and VI.5.10]. □

The simple compact connected Lie groups are classified and known, and so is their cohomology. Before we give a complete list we finish our general considerations and show that the rational cohomology ring of a compact connected Lie group is an exterior algebra.

Theorem 1.10 *Theorem 1.9 holds for all compact connected (not necessarily almost simple) Lie groups.*

Proof. By theorem 1.7, there are compact connected almost simple Lie groups $G_1, \ldots, G_s$, a number $k \in \mathbb{N}_0$ and a covering map

$$f\colon G_1 \times \cdots \times G_s \times \mathrm{U}(1)^k \longrightarrow G.$$

By theorem 1.8, the covering map f induces the isomorphism

$$H^*(G_1 \times \cdots \times G_s \times \mathrm{U}(1)^k;\mathbb{Q}) \xleftarrow{f^*} H^*(G;\mathbb{Q})$$

in rational cohomology. The Künneth formula implies

$$H^*(G;\mathbb{Q}) \cong H^*(G_1;\mathbb{Q}) \otimes \cdots \otimes H^*(G_s;\mathbb{Q}) \otimes H^*(\mathrm{U}(1);\mathbb{Q}) \otimes \cdots \otimes H^*(\mathrm{U}(1);\mathbb{Q}).$$

Now, for all i the cohomology group $H^*(G_i;\mathbb{Q})$ is an exterior algebra with generators of odd degree by theorem 1.9. Moreover, $\mathrm{U}(1)$ is homeomorphic to the 1-sphere S^1, whence $H^*(\mathrm{U}(1);\mathbb{Q}) \cong H^*(\mathrm{S}^1;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x)$ with $\deg x = 1$. Finally, the claim follows by the following multiplication rule for exterior algebras: $\Lambda_{\mathbb{Q}}(y_1, \ldots, y_s) \otimes \Lambda_{\mathbb{Q}}(y_{s+1}, \ldots, y_n) \cong \Lambda_{\mathbb{Q}}(y_1, \ldots, y_n)$. □

As announced in the beginning, the rational cohomology of a compact connected Lie group is an exterior algebra and depends only on the degrees of its generators. Since there is no canonical order for these degrees, and since some degrees may occur more than once, the appropriate structure to collect them is a so-called multiset.

A *multiset* is essentially the same as a set, except that the same element may occur more than once in a multiset. The *length* of a multiset is the number of its elements. Equal elements are counted with their multiplicities. We denote multisets of finite length with brackets, e.g. $[0, 0, 1]$. Note that for example $[0, 0, 1]$ is not the same as $[0, 1]$, but $[0, 0, 1] = [0, 1, 0] = [1, 0, 0]$. Equivalently a multiset of length n can be regarded as an n-tuple without ordering.

One can formalize the notion of a multiset as an equivalence class of maps. Let X be a set, and let I be another set. Think of I as a set of indices. Recall that a family in X indexed by I is a map $I \longrightarrow X$. So, a sequence in X is a family in X indexed by $I = \mathbb{N}$, and a n-tuple in X is a family in X indexed by $\{1, \ldots, n\}$. We call two families with the same index set equivalent if we can get one from the other by a permutation of the index set. The equivalence classes are the multisets in X.

Definition Let G be a compact connected Lie group. By theorem 1.10, there exist generators $x_1, \ldots, x_n$ with $H^*(G; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, \ldots, x_n)$. The multiset $[\deg x_1, \ldots, \deg x_n]$ of the degrees of the generators is called the *degree multiset* of G. Notation: $\deg G = [\deg x_1, \ldots, \deg x_n]$.

Table 1.1 shows a complete list of all almost simple compact connected Lie groups and their degree multisets, cf. [MT91, theorems III.6.5 and VI.5.10]. By "type", we mean the local isomorphism type, represented by the corresponding Lie algebra. We use the following notation.

$$\mathfrak{a}_n = \mathfrak{su}_{n+1}$$
$$\mathfrak{b}_n = \mathfrak{so}_{2n+1}$$
$$\mathfrak{c}_n = \mathfrak{sp}_n$$
$$\mathfrak{d}_n = \mathfrak{so}_{2n}$$

Note that there are the following isomorphisms of Lie algebras.

$$\mathfrak{a}_1 = \mathfrak{b}_1 = \mathfrak{c}_1$$
$$\mathfrak{c}_2 = \mathfrak{b}_2$$
$$\mathfrak{d}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_1$$
$$\mathfrak{d}_3 = \mathfrak{a}_3$$

Type	Degree multiset	
$\mathfrak{a}_n$	$[3, 5, 7, \ldots, 2n+1]$	
$\mathfrak{b}_n$	$[3, 7, 11, \ldots, 4n-1]$	$(n \geq 2)$
$\mathfrak{c}_n$	$[3, 7, 11, \ldots, 4n-1]$	$(n \geq 3)$
$\mathfrak{d}_n$	$[3, 7, 11, \ldots, 4n-5, 2n-1]$	$(n \geq 4)$
$\mathfrak{e}_6$	$[3, 9, 11, 15, 17, 23]$	
$\mathfrak{e}_7$	$[3, 11, 15, 19, 23, 27, 35]$	
$\mathfrak{e}_8$	$[3, 15, 23, 27, 35, 39, 47, 59]$	
$\mathfrak{f}_4$	$[3, 11, 15, 23]$	
$\mathfrak{g}_2$	$[3, 11]$	

Table 1.1: The compact connected almost simple Lie groups and their degree multisets

They correspond to the following isomorphisms of Lie groups. For sake of simplicity, we take in each local isomorphism class the simply connected representative.

$$\mathrm{SU}(2) \cong \mathrm{Spin}(3) \cong \mathrm{Sp}(1)$$
$$\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$$
$$\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$
$$\mathrm{Spin}(6) \cong \mathrm{SU}(4)$$

The groups of type $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$ are classical groups. More precisely, they are locally isomorphic to the matrix groups $\mathrm{SU}(n+1) \cong \mathrm{SU}_{n+1}(\mathbb{C})$, $\mathrm{SO}(2n+1) \cong \mathrm{SO}_{2n+1}(\mathbb{R})$, $\mathrm{Sp}(n) \cong \mathrm{U}_n(\mathbb{H})$ and $\mathrm{SO}(2n) \cong \mathrm{SO}_{2n}(\mathbb{R})$ respectively. The groups of type $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$ are called *exceptional*.

Remark 1.11 Note that the length of the degree multiset of a compact connected Lie group equals its rank, and that the sum over all elements of the degree multiset is equal to the dimension of the group. The notation for the Lie algebras in table 1.1 contains the rank as index; for example, the compact connected Lie groups of type $\mathfrak{a}_n$ have rank n.

In most cases one can recognize the group by looking only at its degree multiset. For example, the number of ones in the degree multiset is the dimension of the central torus factor. The number of threes is the number of almost simple factors of (the universal covering of) the group. The number of fives is the number of factors of type $\mathfrak{a}_n$ with $n \geq 2$.

We need part of a result in [Kra98]. It is the starting point in chapter 2 for our classification of simply connected homogeneous spaces of compact Lie groups such that the rational cohomology of the homogeneous space is that of the product of a 2-sphere with an odd dimensional sphere.

Theorem 1.12 *Let $\mathcal{P}$ be a simply connected compact topological space with rational cohomology $H^*(\mathcal{P};\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m) \otimes \Lambda_{\mathbb{Q}}(v)$ with $\deg a$ even, $\deg v$ odd and $\deg a < \deg v$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then $\operatorname{rk} G = \operatorname{rk} H + 1$,

$$\deg G = [m_1, \ldots, m_{\operatorname{rk} H-1}, 3, m] \quad \textit{and}$$
$$\deg H = [m_1, \ldots, m_{\operatorname{rk} H-1}, 1]$$

for some odd $m_1, \ldots, m_{\operatorname{rk} H-1}, m \in \mathbb{N}$.

This result is proved in [Kra98, section 3.B]. Though there it is always assumed that $\deg a \geq 4$, this assumption is never used in the proof. Kramer simply does not need the result for $\deg a = 2$.

The same remark applies to the following proposition 1.13, which can be found in [Kra98, proposition 3.14].

Proposition 1.13 *Let $G\,/\,H$ be a simply connected homogeneous space of a compact Lie group G. Let the integral cohomology $H^*(G\,/\,H)$ be either of the form*

$$\Lambda_{\mathbb{Z}}(u, v)$$

with $\deg u, \deg v \geq 3$ odd or of the form

$$\mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda_{\mathbb{Z}}(u)$$

with $\deg a$ even, $\deg u$ odd and $\deg u > \deg a \geq 2$. Then there exists a normal transitive semisimple closed subgroup N of G with at most two almost simple or torus factors.

We will use this result in the following form. Recall that a G-action is called *almost effective* if the kernel of this action has dimension zero, i.e. if the kernel is finite.

Corollary 1.14 *Let $\mathcal{P}$ be a simply connected compact topological space with cohomology $H^*(\mathcal{P}) \cong \mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda_{\mathbb{Z}}(u)$ with $\deg a$ even, $\deg u$ odd and $\deg u > \deg a \geq 2$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then G has at most two (almost simple or torus) factors.

1.3 Representation theory

We briefly present the material from representation theory that we will use. See [BtD95] for details. In addition to this material we will frequently refer to the tables of low dimensional representations of compact connected Lie groups in [Kra98].

Definition Let G be a group. Let V be a vector space over the (skew) field $\mathbb{F}$. A *representation* ρ of G on V is a group morphism $\rho\colon G \longrightarrow \mathrm{End}(V)$. If ρ is a monomorphism, the representation is called *faithful*. If $\mathbb{F}$ is one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, i.e. the reals, the complex numbers or the quaternions, then the representation is called a *real*, a *complex* or a *quaternionic* representation, respectively. Two representations ρ and σ of a group G on a vector space V are called *conjugate* if there exists an element $f \in \mathrm{GL}(V)$ with $\rho(g) = f^{-1} \circ \sigma(g) \circ f$ for all $g \in G$.

The vector space V, together with the representation ρ, is also called a G-*module*, since G acts on V by $g \cdot v = \rho(g)(v)$ for $g \in G$, $v \in V$. The adjectives faithful, real, complex and quaternionic are also used for G-modules in the obvious sense. A G-module V is called *simple* if V contains no non-trivial proper G-invariant subspaces.

Let M and N be two vector spaces over the same (skew) field $\mathbb{F}$. Let M and N be G-modules. A linear map $f\colon M \longrightarrow N$ is called G-*equivariant*, if it "commutes" with the action of G, i.e. if $g \cdot f(v) = f(g \cdot v)$ for all $g \in G, v \in V$. We denote by $\mathrm{Hom}_G(M, N)$ the set of all G-equivariant linear maps $M \longrightarrow N$. Put $\mathrm{End}_G(M) = \mathrm{Hom}_G(M, M)$. The set $\mathrm{End}_G(M)$ is called the *centralizer* of the G-module M.

Let $\mathbb{F}$ be one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We denote by $\mathbb{F}(n)$ the ring of all $(n \times n)$-matrices with entries in $\mathbb{F}$. Let G be a group. We denote by $\underline{\mathbb{F}}$ the trivial (one dimensional) G-module. Let M be a G-module, let $n \in \mathbb{N}$. We denote by nM the direct sum of n copies of M, i.e. $nM = \underbrace{M \oplus \cdots \oplus M}_{n \text{ copies}}$.

Each module of a compact Lie group is the sum of simple ones, see [BtD95, proposition II.1.9]. The following lemma allows us to calculate the centralizer of a sum from the centralizers of the summands.

Lemma 1.15 *Let M and N be simple G-modules, let $l \in \mathbb{N}$, and let $\mathbb{F}$ be* *the centralizer* $\mathrm{End}_G(M)$ *of* M. *Then the following equations hold.*

$$\mathrm{Hom}_G(M,N) = \begin{cases} 0 & \text{for } M \ncong N \\ \mathbb{F} & \text{for } M \cong N \end{cases}$$

$$\mathrm{End}_G(lM) = \mathbb{F}(l)$$

$$\mathrm{End}_G(M \oplus N) = \mathrm{End}_G(M) \oplus \mathrm{End}_G(N) \text{ for } M \ncong N.$$

Proof. For the first equation see the theorems II.6.7 and II.1.10 in [BtD95] (Schur's Lemma). The other two equations are direct consequences of the first one. □

The next lemma is simple to prove but very useful.

Lemma 1.16 *Let* $\mathbb{F}$ *be one of the division algebras* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *Each subgroup of* $\mathrm{GL}_n\,\mathbb{F}$ *has a faithful representation on* $\mathbb{F}^n$.

Proof. Let H be a subgroup of $\mathrm{GL}_n\,\mathbb{F}$. The inclusion map $H \longrightarrow \mathrm{GL}_n\,\mathbb{F}$ is a monomorphism of groups and hence a representation of H on $\mathbb{F}^n$. □

Corollary 1.17 *Each subgroup of* $\mathrm{SO}(n)$ *has a faithful representation on* $\mathbb{R}^n$, *each subgroup of* $\mathrm{SU}(n)$ *has a faithful representation on* $\mathbb{C}^n$ *and each subgroup of* $\mathrm{Sp}(n)$ *has a faithful representation on* $\mathbb{H}^n$.

This corollary can be used to show that some group H cannot be a subgroup of G where G is one of the groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$. One only has to show that H has no faithful representation on $\mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{H}^n$, respectively.

Another question, which will arise in this text, is the following. Given isomorphic subgroups of one of the classical groups above, are they conjugate in this group? This question can be answered by representation theory:

Lemma 1.18 *Let* G *be one of the groups* $\mathrm{SO}(n)$, $n \geq 3$ *odd,* $\mathrm{SU}(n)$, $n \geq 2$, *or* $\mathrm{Sp}(n)$, $n \geq 1$. *Let* $\rho, \sigma\colon H \longrightarrow G$ *be monomorphisms. Via the natural inclusion of* G *into* $\mathrm{GL}_n\,\mathbb{R}$, $\mathrm{GL}_n\,\mathbb{C}$ *and* $\mathrm{GL}_n\,\mathbb{H}$, *respectively, these monomorphisms can be regarded as representations of* H. *Then* ρ *and* σ *are conjugate by an element of* G *if they are conjugate as representations on* $\mathbb{R}^n$, $\mathbb{C}^n$ *and* $\mathbb{H}^n$, *respectively.*

Proof. See [Kra98, lemmas 4.6 and 4.7]. □

1.4 Homogeneous spaces

In the chapters 2 and 3 we classify homogeneous spaces of compact connected Lie groups with specific topological properties. A homogeneous space X can always be written as $X \cong K \,/\, H$, where K is the group that acts transitively and $H = K_x$ is the stabilizer of a point $x \in X$. We now cite some facts from [Kra98, section 3.B.]. There exists always a transitive normal closed subgroup G of K that has no transitive proper normal closed subgroup. For a complete classification of the groups K acting transitively on X it is sufficient to classify all groups G acting transitively on X without transitive proper normal closed subgroup, together with the centralizer $\mathrm{Cen}_G(G_x)$ of a point stabilizer G_x in G.

In fact the group K can be recovered in the following way. Being a normal closed subgroup, G has a normal complement L in K, i.e. $K = GL$. The group L is a subgroup of the centralizer $\mathrm{Cen}_{Sym(X)}(G)$ of G in the group of all permutations of the homogeneous space X. This centralizer is isomorphic to the quotient $\mathrm{Nor}_G(H) \,/\, H$ of the normalizer of H in G and H, which in turn is locally isomorphic to the quotient $\mathrm{Cen}_G(H) \,/\, \mathrm{Cen}(H)$.

We give some examples of homogeneous spaces of compact Lie groups. All these examples are well known. We use the occasion to fix notation and to recall some transitive and almost effective group actions.

We denote by S^n the *n-sphere*, i.e. the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$. The group $\mathrm{SO}(n+1)$ acts transitively and almost effectively on S^n with point stabilizer $\mathrm{SO}(n)$, and thus $\mathrm{S}^n \cong \mathrm{SO}(n+1) \,/\, \mathrm{SO}(n)$. If the dimension of the ambient vector space $\mathbb{R}^{n+1}$ is divisible by 2 or 4, we can give it a complex or quaternionic structure, respectively, and obtain $\mathrm{S}^{2k-1} \cong \mathrm{SU}(k) \,/\, \mathrm{SU}(k-1)$ and $\mathrm{S}^{4k-1} \cong \mathrm{Sp}(k) \,/\, \mathrm{Sp}(k-1)$.

Let $\mathbb{F}$ be one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We denote by $\mathrm{V}_k(\mathbb{F}^n)$ the *Stiefel manifold* of all linearly independent k-tuples of vectors in $\mathbb{F}^n$. We can write the Stiefel manifolds over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ as homogeneous spaces: $\mathrm{V}_k(\mathbb{R}^n) \cong \mathrm{SO}(n) \,/\, \mathrm{SO}(n-k)$, $\mathrm{V}_k(\mathbb{C}^n) \cong \mathrm{SU}(n) \,/\, \mathrm{SU}(n-k)$ and $\mathrm{V}_k(\mathbb{H}^n) \cong \mathrm{Sp}(n) \,/\, \mathrm{Sp}(n-k)$. Note that in the case $k = 1$ the Stiefel manifolds are spheres with the group actions that we discussed in the preceding paragraph.

The *Grassmann manifold* of oriented subspaces of dimension k in $\mathbb{R}^n$ is denoted by $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$. It is a homogenous space of $\mathrm{SO}(n)$; more precisely we have $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n) \cong \mathrm{SO}(n) \,/\, (\mathrm{SO}(k) \cdot \mathrm{SO}(n-k))$.

Let again $\mathbb{F}$ be one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We denote by $\mathbb{F}\mathrm{P}^n$ the *projective space* over $\mathbb{F}$. It is the Grassmann manifold of (not oriented) one dimensional subspaces of $\mathbb{F}^{n+1}$. We can write the projective spaces over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ as homogeneous spaces: $\mathbb{R}\mathrm{P}^n \cong \mathrm{SO}(n+1)\,/\,\mathrm{O}(n)$, $\mathbb{C}\mathrm{P}^n \cong \mathrm{SU}(n+1)\,/\,(\mathrm{SU}(n)\cdot\mathrm{U}(1))$ and $\mathbb{H}\mathrm{P}^n \cong \mathrm{Sp}(n+1)\,/\,(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1))$. If $n = 2k-1$ is odd, the group $\mathrm{SU}(n+1) = \mathrm{SU}(2k)$ has a subgroup $\mathrm{Sp}(k)$ that still acts transitively on $\mathbb{C}\mathrm{P}^n$, and we have $\mathbb{C}\mathrm{P}^{2k-1} \cong \mathrm{Sp}(k)\,/\,(\mathrm{Sp}(k-1)\cdot\mathrm{U}(1))$.

1.5 Graded modules and spectral sequences

The theorems 1.12 and 3.1 about the cohomology of certain homogeneous spaces, which we cite from [Kra98], are proved there by means of the Leray-Serre spectral sequence. This is reason enough to give a brief introduction to the subject of spectral sequences here. For a thorough treatment see [Kra98]. We will use the Leray-Serre spectral sequence to exclude the existence of some embeddings between certain compact connected Lie groups.

Let R be a ring. Let I be an index set. An *I-graded R-module* is a direct sum of R-modules M_i, $i \in I$:

$$M = \bigoplus_{i\in I} M^i.$$

Two such index sets are of special interest for our purposes: $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$. A $\mathbb{Z}$-graded module is called a *graded module* for short:

$$M = M^* = \bigoplus_{i\in\mathbb{Z}} M^i.$$

The elements of M^i are called *homogeneous of degree i*. A $\mathbb{Z}\times\mathbb{Z}$-graded module is called a *bigraded module*:

$$M = M^{**} = \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}.$$

A *differential* of a graded module M^* is a map $d\colon M^* \longrightarrow M^*$ with $d(M^i) \subseteq M^{i+1}$ for all i and $d^2 = 0$. The pair (M, d) is called a *differential graded module*. Its cohomology is defined in the obvious way: It is the graded module $H^*(M) = \bigoplus_{i\in\mathbb{Z}} H^i(M)$ with

$$H^i(M) = \ker(M^i \xrightarrow{\ d\ } M^{i+1}) \Big/ \operatorname{im}(M^{i-1} \xrightarrow{\ d\ } M^i).$$

A *differential of bidegree $(r, s) \in \mathbb{Z}\times\mathbb{Z}$* of a bigraded module M^{**} is a map $d\colon M^{**} \longrightarrow M^{**}$ with $d(M^{i,j}) \subseteq M^{i+r,j+s}$ for all i, j and $d^2 = 0$.

The cohomology of such a bigraded differential module M^{**} is defined as $H^{**}(M) = \bigoplus_{i,j\in\mathbb{Z}} H^{i,j}(M)$ with

$$H^{i,j}(M) = \ker(M^{i,j} \xrightarrow{d} M^{i+r,j+s}) \Big/ \operatorname{im}(M^{i-r,j-s} \xrightarrow{d} M^{i,j}). \tag{1.5.1}$$

Note that, if R is a field, then $\dim_R H^{i,j}(M) \leq \dim_R M^{i,j}$.

An E_2-*spectral sequence* is a sequence of differential bigraded modules (E_r, d_r), $r = 2, 3, 4, \ldots$, whose differential d_r has bidegree $(r, 1-r)$ and with $E_{r+1} \cong H^{**}(E_r)$. If for one pair (i, j) and one $r \geq 2$ the two maps d_r in equation (1.5.1) vanish, we have $E_{r+1}^{i,j} \cong E_r^{i,j}$. If, for some $n \geq 2$, this happens for all $r \geq n$, we have $E_n^{i,j} \cong E_{n+1}^{i,j} \cong E_{n+2}^{i,j} \cong \ldots$, i.e. some sort of local convergence. In this case we write $E_\infty^{i,j} = E_n^{i,j}$. If this local convergence happens for all pairs of indices (i, j), we say that the spectral sequence *converges* to the bigraded module $E_\infty^{**} = \bigoplus_{i,j\in\mathbb{Z}} E_\infty^{i,j}$. Note that, if R is a field, we have

$$\dim_R E_2^{i,j} \geq \dim_R E_3^{i,j} \geq \dim_R E_4^{i,j} \geq \cdots \geq \dim_R E_\infty^{i,j}. \tag{1.5.2}$$

Let

$$F \longrightarrow E \longrightarrow B$$

be a fibration with path connected base space B. There is a spectral sequence relating the cohomologies of base and fiber to that of the total space. It is called the *Leray-Serre spectral sequence*. The fundamental group $\pi_1(B)$ of B acts on the fiber F. If the induced action on the cohomology $H^*(F; R)$ of F is trivial, then the fibration is called R-*simple*. This is e.g. the case if $\pi_1(B) = 0$, i.e. if B is simply connected. For R-simple fibrations the E_2-term of the corresponding Leray-Serre spectral sequence takes the following simple form.

Theorem 1.19 *Let R be a ring and let*

$$F \longrightarrow E \longrightarrow B$$

be an R-simple fibration over the path connected base space B. There exists an E_2-spectral sequence that converges to the bigraded module (associated to some bounded and convergent filtration of) $H^*(E; R)$ with $E_2^{i,j} \cong H^i(B; H^j(F; R))$.

Proof. See [Spa66, 9.4.9]. □

The filtration in theorem 1.19, which gives $H^*(E;R)$ the structure of a bigraded module, is not important for our purposes. We only need the following fact: If R is a field, we have

$$H^q(E;R) \cong \bigoplus_{i+j=q} E_\infty^{i,j}. \tag{1.5.3}$$

This leads to the following useful formula.

Corollary 1.20 *Let R be a field and let*

$$F \longrightarrow E \longrightarrow B$$

be a fibration over the simply connected base space B. Then

$$\dim_R H^q(E;R) \leq \sum_{i+j=q} \dim_R H^i(B;R) \cdot \dim_R H^j(F;R). \tag{1.5.4}$$

Proof. By theorem 1.19 there is the Leray-Serre spectral sequence with $E_2^{i,j} \cong H^i(B;H^j(F;R))$. Since R is a field, this implies $E_2^{i,j} \cong H^i(B;R) \otimes H^j(F;R)$ and hence $\dim_R E_2^{i,j} = \dim_R H^i(B;R) \cdot \dim_R H^j(F;R)$. With inequality (1.5.2) and equation (1.5.3) we calculate

$$\begin{aligned}\dim_R H^q(E;R) = \sum_{i+j=q} \dim_R E_\infty^{i,j} \leq \\ \leq \sum_{i+j=q} \dim_R E_2^{i,j} = \sum_{i+j=q} \dim_R H^i(B;R) \cdot \dim_R H^j(F;R).\end{aligned}$$

□

We will use this corollary mainly for fibrations that arise from inclusions of Lie groups.

Lemma 1.21 *Let A, B and C be Lie groups with $A \subseteq B \subseteq C$. Then*

$$B\,/\,A \longrightarrow C\,/\,A \longrightarrow C\,/\,B$$

is a fiber bundle.

Proof. See [MT91, II.2.12]. □

Recall that each fiber bundle over a paracompact space is a fibration. In the special case $A = 1$, lemma 1.21 gives the principal fiber bundle

$$B \longrightarrow C \longrightarrow C\,/\,B.$$

1.6 Maximal subgroups of the simple compact connected Lie groups

For the convenience of the reader we repeat here the table of all compact connected simple Lie groups G and their maximal compact subgroups K of maximal rank. This table is due to Borel and De Siebenthal and can e.g. be found in [BDS49, p. 200] and [Wol84, p. 281]. The Lie algebra of the 1-torus U(1) is denoted by $\mathfrak{t}^1$.

Type of G	Type of K (semisimple)	Type of K (non-semisimple)
$\mathfrak{a}_l$	—	$\mathfrak{a}_i \oplus \mathfrak{a}_{l-1-i} \oplus \mathfrak{t}^1$
$\mathfrak{b}_l$	$\mathfrak{d}_l$ $\mathfrak{b}_i \oplus \mathfrak{d}_{l-i}$ $(0 < i < l-1)$	$\mathfrak{b}_{l-1} \oplus \mathfrak{t}^1$
$\mathfrak{c}_l$	$\mathfrak{c}_i \oplus \mathfrak{c}_{l-i}$ $(0 < i < l)$	$\mathfrak{a}_{l-1} \oplus \mathfrak{t}^1$
$\mathfrak{d}_l$	$\mathfrak{d}_i \oplus \mathfrak{d}_{l-i}$ $(1 < i < l-1)$	$\mathfrak{a}_{l-1} \oplus \mathfrak{t}^1$ $\mathfrak{d}_{l-1} \oplus \mathfrak{t}^1$
$\mathfrak{e}_6$	$\mathfrak{a}_1 \oplus \mathfrak{a}_5$ $\mathfrak{a}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{d}_5 \oplus \mathfrak{t}^1$
$\mathfrak{e}_7$	$\mathfrak{a}_7$ $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ $\mathfrak{a}_2 \oplus \mathfrak{a}_5$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
$\mathfrak{e}_8$	$\mathfrak{a}_8$ $\mathfrak{d}_8$ $\mathfrak{a}_4 \oplus \mathfrak{a}_4$ $\mathfrak{e}_6 \oplus \mathfrak{a}_2$ $\mathfrak{e}_7 \oplus \mathfrak{a}_1$	—
$\mathfrak{f}_4$	$\mathfrak{b}_4$ $\mathfrak{a}_2 \oplus \mathfrak{a}_2$ $\mathfrak{a}_1 \oplus \mathfrak{c}_3$	—
$\mathfrak{g}_2$	$\mathfrak{a}_2$ $\mathfrak{a}_1 \oplus \mathfrak{a}_1$	—

Table 1.2: The maximal compact subgroups K of maximal rank in the simple compact connected Lie groups G. For the meaning of the symbols $\mathfrak{a}_\mathfrak{l}$, $\mathfrak{b}_\mathfrak{l}$, $\mathfrak{c}_\mathfrak{l}$ and $\mathfrak{d}_\mathfrak{l}$ see page 7.

Chapter 2

Product cohomology

The aim of this chapter is the classification of all simply connected homogeneous spaces G/H of compact Lie groups G with $H^*(G/H) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^m)$, $m \geq 3$ odd. The classification result can be found at the end of the chapter.

The classification is done in several steps marked as sections. In the first step we classify homogeneous spaces G/H with the right *rational* cohomology; that means we forget about torsion for the moment. A result of Kramer gives us conditions on the degree multisets of the groups G and H, which are sharp enough to obtain a complete classification.

The cohomology in question is that of a product of spheres. It is natural to ask in which cases the homogeneous space itself is a product. For our application to homogeneous focal manifolds of isoparametric hypersurfaces in chapter 4 this question will also be of importance, because such a focal manifold is in general not a product. The answer to this question is given in the second section.

The next two sections are devoted to ruling out all homogeneous spaces among those obtained in the first section that have torsion and thus do not have the right integral cohomology. For this we mainly use two different methods to detect torsion. The two sections correspond to the two methods. One method leads from the group inclusions $H \subseteq G$ to fibrations and via the Leray-Serre spectral sequence to conditions on the $\mathbb{Z}/2$-cohomology of certain spaces.

The other section deals with cases in which H has a central 1-torus. The integral cohomology of the homogeneous space G/H strongly depends on how the 1-torus is embedded in G. We adapt a method of Kreck and Stolz to calculate the integral cohomology in these cases. Here again the Leray-Serre spectral sequence, in particular the transgression, is the main tool. It allows to determine the Euler class of a certain sphere bundle. The desired integral cohomology is finally obtained from the Gysin exact sequence, in which one of the morphisms is multiplication by the Euler class.

2.1 Rational cohomology

Let $\mathcal{P}$ be a simply connected compact topological space with cohomology $H^*(\mathcal{P}) \cong \mathbb{Z}[a] \,/\, (a^2) \otimes \Lambda_{\mathbb{Z}}(u) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^m)$, $\deg a = 2$ and $m = \deg u \geq 3$ odd. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. In addition we may assume that G has no proper normal closed subgroup that is still transitive on $\mathcal{P}$, see section 1.4. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$. By [GKK00, lemma 1.3] the commutator subgroup G' of G still acts transitively on $\mathcal{P}$. Since G is supposed not to have a transitive proper normal closed subgroup, this means that G has no (central) torus factor. By corollary 1.14 we obtain that the universal covering of G has at most two simple factors. Hence G is either almost simple or an almost direct product of two almost simple closed subgroups.

Let $s = \operatorname{rk} H$. We have $\operatorname{rk} G = s + 1$, $\deg G = [m_1, \ldots, m_{s-1}, 3, m]$ and $\deg H = [m_1, \ldots, m_{s-1}, 1]$ for some odd $m_1, \ldots, m_{s-1}, m \in \mathbb{N}$ by theorem 1.12. First consider the case $s = 1$, i.e. $\deg H = [1]$ and $\deg G = [3, m]$. Remark 1.11, together with table 1.1 on page 8, shows that $H \cong \mathrm{U}(1)$, and that G is of one of the following types:

Type of G	m
$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	3
$\mathfrak{a}_2$	5
$\mathfrak{b}_2$	7
$\mathfrak{g}_2$	11

Now suppose $s \geq 2$. Since G has no torus factor, there is not the number 1 among the numbers $m_1, \ldots, m_{s-1}$. Recall that the degree multiset of H is $[m_1, \ldots, m_{s-1}, 1]$. This means that H has a central torus factor of dimension 1. By remark 1.11, there is an element 3 in the degree multiset of H, $m_1 = 3$, say. Now, the degree multiset of G is $\deg G = [3, m_2, \ldots, m_{s-1}, 3, m]$. As seen above, G is the almost direct product of at most two simple factors. The two 3's in $\deg G$ show that G has exactly two factors G_1 and G_2, say. In particular, there is no further element 3 in $[m_2, \ldots, m_{s-1}]$. This means that H is an almost direct product of $\mathrm{U}(1)$ and an almost simple closed subgroup H' with degree multiset $\deg H' = [3, m_2, \ldots, m_{s-1}]$. Note that in fact H' is the commutator subgroup of H. We may—up to reordering—assume that $\operatorname{rk} G_1 \geq \operatorname{rk} G_2$. By corollary 1.5 we have $\operatorname{rk} H' \leq \operatorname{rk} G_1$. From $\deg H' =$

$[3, m_2, \ldots, m_{s-1}]$ we see $\operatorname{rk} H' = s - 1$ and thus $\operatorname{rk} G_2 = \operatorname{rk} G - \operatorname{rk} G_1 \leq \operatorname{rk} G - \operatorname{rk} H = (s+1) - (s-1) = 2$.

In order to show that the case $\operatorname{rk} G_2 = 2$ cannot occur, suppose $\operatorname{rk} G_2 = 2$. We have $\operatorname{rk} G_1 = s - 1 = \operatorname{rk} H'$. This implies $H' \subseteq G_1$; for $\operatorname{rk} H' \geq 3$ this follows directly from corollary 1.5. For $\operatorname{rk} H' = 2$ we have $\operatorname{rk} G_1 = \operatorname{rk} G_2 = \operatorname{rk} H' = 2$. By proposition 1.4, H is either a subgroup of one of the groups G_1 and G_2, a subgroup of G_1, say, or isomorphic to subgroups of both G_1 and G_2. In the latter case, the centralizer of H' in G would be trivial while it should contain $\mathrm{U}(1)$. (Recall that $H = \mathrm{U}(1) \cdot H' \subseteq G$.) In the other case, i.e. in the case $H' \subseteq G_1$, the centralizer of H' in G_1 has rank 0, and thus the torus factor of H is contained in G_2. Thus $G / H \cong (G_1 / H') \times (G_2 / \mathrm{U}(1))$. Since G acts on $\mathcal{P} \cong G / H$ without transitive proper normal closed subgroup, the groups G_1 and H' cannot be locally isomorphic. By table 1.2 on page 16, G_1 is of type $\mathfrak{g}_2$, and H' is of type $\mathfrak{a}_2$. This implies $G_1 / H' \cong \mathrm{S}^6$, $\deg G_1 = [3, 11]$ and $\deg H' = [3, 5]$, whence $\deg H = [3, 5, 1]$ and $\deg G = [3, 5, 3, 11]$. Now we have $m = 11$, and by the Künneth formula we obtain on the one hand $H^*(\mathcal{P}; \mathbb{Q}) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^{11}; \mathbb{Q}) \cong H^*(\mathrm{S}^2; \mathbb{Q}) \otimes H^*(\mathrm{S}^{11}; \mathbb{Q})$. That means in particular $H^6(\mathcal{P}; \mathbb{Q}) = 0$. On the other hand we obtain $H^*(\mathcal{P}; \mathbb{Q}) \cong H^*(\mathrm{S}^6 \times (G_2 / \mathrm{U}(1)); \mathbb{Q}) \cong H^*(\mathrm{S}^6; \mathbb{Q}) \otimes H^*(G_2 / \mathrm{U}(1); \mathbb{Q})$, and in particular $H^6(\mathrm{S}^6; \mathbb{Q}) \cong \mathbb{Q}$ and thus $H^6(\mathcal{P}; \mathbb{Q}) \neq 0$. This is a contradiction.

This contradiction shows $\operatorname{rk} G_2 = 1$, and hence G_2 is of type $\mathfrak{a}_1$. Moreover, we have $\deg G_2 = [3]$, see table 1.1 on page 8. Recall from above that $\deg G = [3, m_2, \ldots, m_{s-1}, 3, m]$. This implies $\deg G_1 = [3, m_2, \ldots, m_{s-1}, m]$. Also recall from above that $\deg H' = [3, m_2, \ldots, m_{s-1}]$ and that H' is almost simple. With careful book-keeping one gets the following list of all pairs (G_1, H') with the desired degree multisets.

Type of G_1	Type of H'	m	n
$\mathfrak{a}_n$	$\mathfrak{a}_{n-1}$	$2n+1$	$n \geq 2$
$\mathfrak{a}_3$	$\mathfrak{b}_2$	5	
$\mathfrak{b}_2$	$\mathfrak{a}_1$	7	
$\mathfrak{b}_n$	$\mathfrak{b}_{n-1}$	$4n-1$	$n \geq 3$
$\mathfrak{b}_n$	$\mathfrak{c}_{n-1}$	$4n-1$	$n \geq 4$
$\mathfrak{b}_3$	$\mathfrak{g}_2$	7	
$\mathfrak{c}_n$	$\mathfrak{b}_{n-1}$	$4n-1$	$n \geq 3$
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}$	$4n-1$	$n \geq 4$
$\mathfrak{c}_3$	$\mathfrak{g}_2$	7	
$\mathfrak{d}_n$	$\mathfrak{b}_{n-1}$	$2n-1$	$n \geq 4$
$\mathfrak{d}_n$	$\mathfrak{c}_{n-1}$	$2n-1$	$n \geq 4$
$\mathfrak{g}_2$	$\mathfrak{a}_1$	11	

Corollary 1.5 shows $H' \subseteq G_1$ for $\operatorname{rk} H' \geq 2$. We now have to check for $\operatorname{rk} H' \geq 2$ if there are corresponding groups G_1 and H' with $H' \subseteq G_1$. This is not the case for the pairs $(\mathfrak{b}_n, \mathfrak{c}_{n-1})$ for $n \geq 4$, $(\mathfrak{c}_n, \mathfrak{b}_{n-1})$ for $n \geq 4$, $(\mathfrak{c}_3, \mathfrak{g}_2)$ and $(\mathfrak{d}_n, \mathfrak{c}_{n-1})$ for $n \geq 4$, cf. [Kra98].

The above discussion shows the following theorem 2.1.

Theorem 2.1 *Let $\mathcal{P}$ be a simply connected compact topological space with rational cohomology $H^*(\mathcal{P};\mathbb{Q}) \cong \mathbb{Q}[a] \,/\, (a^2) \otimes \Lambda_{\mathbb{Q}}(u)$ with $\deg a = 2$ and $m = \deg u \geq 3$ odd. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then either $H = \mathrm{U}(1)$ and G is in the list

Type of G	$m = \deg u$
$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	3
$\mathfrak{a}_2$	5
$\mathfrak{b}_2$	7
$\mathfrak{g}_2$	11

or G is an almost direct product of a closed subgroup G_1 with $\mathrm{Sp}(1)$, the point stabilizer H is an almost direct product of a closed subgroup H' with $\mathrm{U}(1)$, and the pair (G_1, H') is in the following list.

Type of G_1	*Type of H'*	$m = \deg u$	n
$\mathfrak{a}_n$	$\mathfrak{a}_{n-1}$	$2n+1$	$n \geq 2$
$\mathfrak{a}_3$	$\mathfrak{b}_2$	5	
$\mathfrak{b}_2$	$\mathfrak{a}_1$	7	
$\mathfrak{b}_n$	$\mathfrak{b}_{n-1}$	$4n-1$	$n \geq 3$
$\mathfrak{b}_3$	$\mathfrak{g}_2$	7	
$\mathfrak{c}_3$	$\mathfrak{b}_2$	11	
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}$	$4n-1$	$n \geq 4$
$\mathfrak{d}_n$	$\mathfrak{b}_{n-1}$	$2n-1$	$n \geq 4$
$\mathfrak{g}_2$	$\mathfrak{a}_1$	11	

Remark 2.2 All pairs (G, H) in theorem 2.1 belong to inclusions of groups $H \subseteq G$. Furthermore the quotients all have the right rational cohomology. Examples will occur in the following sections.

2.2 Products

The aim of the rest of the chapter is to improve on theorem 2.1. We aim at a classification with integral instead of rational cohomology. More precisely, we aim at a classification of the simply connected homogeneous spaces $G\,/\,H$ of compact Lie groups G with $H^*(G\,/\,H) \cong \mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda_{\mathbb{Z}}(u)$, $\deg a = 2$ and $\deg u \geq 3$ odd, where G does not have a transitive proper normal closed subgroup. By the universal coefficient theorem, a pair (G, H) that yields a homogeneous space $G\,/\,H$ with the above cohomology is contained in the list of theorem 2.1. We calculate the integral cohomology of the corresponding homogeneous spaces.

If in the situation of theorem 2.1 the group G is not almost simple, then G is an almost direct product $G = G_1{\cdot}\mathrm{Sp}(1)$, and so is H, namely $H = H'{\cdot}\mathrm{U}(1)$. Suppose that the subgroups of G and H satisfy $H' \subseteq G_1$ and $\mathrm{U}(1) \subseteq \mathrm{Sp}(1)$. Then $\mathcal{P} \cong G\,/\,H \cong (G_1\,/\,H') \times (\mathrm{Sp}(1)\,/\,\mathrm{U}(1)) \cong (G_1\,/\,H') \times \mathrm{S}^2$. By the Künneth formula, the homogeneous space $G_1\,/\,H'$ is a (simply connected) cohomology m-sphere, $m = \deg u \geq 3$ odd. These spaces are classified, see e.g. [MS43], [Bor49], [Bre61], [Oni94] or [Kra98]. All of these spaces are ordinary spheres.

Type of G_1	Type of H'	$G_1\,/\,H'$
$\mathfrak{a}_1$	trivial	$\mathrm{S}^3 \cong \mathrm{Sp}(1)$
$\mathfrak{a}_n$	$\mathfrak{a}_{n-1}$	$\mathrm{S}^{2n+1} \cong \mathrm{SU}(n+1)\,/\,\mathrm{SU}(n)$
$\mathfrak{a}_3$	$\mathfrak{b}_2$	$\mathrm{S}^5 \cong \mathrm{SO}(6)\,/\,\mathrm{SO}(5)$
$\mathfrak{b}_2$	$\mathfrak{a}_1$	$\mathrm{S}^7 \cong \mathrm{Sp}(2)\,/\,\mathrm{Sp}(1)$
$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathrm{S}^7 \cong \mathrm{Spin}(7)\,/\,\mathrm{G}_2$
$\mathfrak{b}_4$	$\mathfrak{b}_3$	$\mathrm{S}^{15} \cong \mathrm{Spin}(9)\,/\,\mathrm{Spin}(7)$
$\mathfrak{c}_3$	$\mathfrak{b}_2$	$\mathrm{S}^{11} \cong \mathrm{Sp}(3)\,/\,\mathrm{Sp}(2)$
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}$	$\mathrm{S}^{4n-1} \cong \mathrm{Sp}(n)\,/\,\mathrm{Sp}(n-1)$
$\mathfrak{d}_n$	$\mathfrak{b}_{n-1}$	$\mathrm{S}^{2n-1} \cong \mathrm{SO}(2n)\,/\,\mathrm{SO}(2n-1)$

Table 2.1: The odd dimensional simply connected cohomology spheres.

Let (G, H) be as above, but let $G\,/\,H$ not be a product. If $\mathrm{rk}\, H \geq 2$, we have $H' \subseteq G_1$, and the projection $G \longrightarrow G_1$ maps the torus factor of H onto an isomorphic image. In other words, there is a closed subgroup $\mathrm{U}(1)$ in the centralizer of H' in G_1. This is not the case for (G_1, H') of type $(\mathfrak{a}_3, \mathfrak{b}_2), (\mathfrak{b}_3, \mathfrak{g}_2)$ and $(\mathfrak{d}_n, \mathfrak{b}_{n-1})$ by table 1.2 on page 16. So, for these pairs $G\,/\,H$ always is a product.

2.3 Cases excluded by spectral sequences

We continue our classification of simply connected homogeneous spaces $G\,/\,H$ of compact Lie groups G with $H^*(G\,/\,H) \cong \mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda_{\mathbb{Z}}(u)$, $\deg a = 2$

and $\deg u \geq 3$ odd, where G does not have a transitive proper normal closed subgroup. We call these conditions for the homogeneous space the *setting of integral coefficients* for short. Of course we can omit from now on the case where $G \,/\, H$ is a product, as this case has been treated in the previous section 2.2.

Proposition 2.3 *Consider the case in theorem 2.1 where G and H are of type $\mathfrak{b}_n \oplus \mathfrak{a}_1$ and $\mathfrak{b}_{n-1} \oplus \mathfrak{t}^1$ with $n \geq 3$, respectively. In the setting of integral coefficients this case only occurs, if the resulting homogeneous space $G \,/\, H$ is a product.*

Proof. First consider the inclusion of subgroups $H' \subseteq G_1$ where G_1 is of type $\mathfrak{b}_n$, and H' of type $\mathfrak{b}_{n-1}$. The subgroup G_1 has a natural representation as $\mathrm{SO}(2n+1)$ on $\mathbb{R}^{2n+1}$. For $n = 3$ or $n \geq 5$, the only at most $(2n+1)$-dimensional representation of H' is the action of $\mathrm{SO}(2n-1)$ on $\mathbb{R}^{2n-1}$, see [Kra98, 4.12]. So the inclusion $H' \subseteq G_1$ is one of the natural inclusions $\mathrm{SO}(2n-1) \subseteq \mathrm{SO}(2n+1)$ or $\mathrm{Spin}(2n-1) \subseteq \mathrm{Spin}(2n+1)$. In addition to the natural representation on $\mathbb{R}^7$ there is an 8-dimensional representation of $\mathrm{Spin}(7)$ for $n = 4$, see [Kra98, 4.12]. But for the corresponding inclusion $\mathrm{Spin}(7) \subseteq \mathrm{Spin}(9)$ the centralizer of $\mathrm{Spin}(7)$ in $\mathrm{Spin}(9)$ is trivial, see [Kra98, 6.2]. So, if $(G_1, H') = (\mathrm{Spin}(9), \mathrm{Spin}(7))$, then the quotient $G \,/\, H$ is a product. So suppose that $H' \subseteq G_1$ is one of the natural inclusions. We have $G_1 \,/\, H' \cong \mathrm{SO}(2n+1) \,/\, \mathrm{SO}(2n-1) \cong \mathrm{V}_2(\mathbb{R}^{2n+1})$, see section 1.4. Consider the inclusions $H' \subseteq H \subseteq G$. By lemma 1.21 there is a fibration

$$H \,/\, H' \longrightarrow G \,/\, H' \longrightarrow G \,/\, H.$$

This is the fibration

$$\mathrm{U}(1) \longrightarrow \mathrm{V}_2(\mathbb{R}^{2n+1}) \times \mathrm{SU}(2) \longrightarrow G \,/\, H.$$

Since the homogeneous space $G \,/\, H$ is simply connected, corollary 1.20 applies. By [Bor53b, proposition 10.1] we have

$$H^*(\mathrm{V}_2(\mathbb{R}^{2n+1}); \mathbb{Z}/2) \cong H^*(\mathrm{S}^{2n} \times \mathrm{S}^{2n-1}; \mathbb{Z}/2),$$

and hence by the Künneth formula

$$H^{2n-1}(\mathrm{V}_2(\mathbb{R}^{2n+1}) \times \mathrm{SU}(2); \mathbb{Z}/2) \neq 0.$$

Thus by corollary 1.20 there are integers i, j with $i + j = 2n - 1$,

$$H^i(\mathrm{S}^2 \times \mathrm{S}^{4n-1}; \mathbb{Z}/2) \cong H^i(G \,/\, H; \mathbb{Z}/2) \neq 0$$

and

$$H^j(\mathrm{S}^1;\mathbb{Z}/2) \cong H^i(\mathrm{U}(1);\mathbb{Z}/2) \neq 0,$$

which implies $2n-1 \leq 3$ or $2n-1 \geq 4n-1$. This is not possible for $n \geq 3$. Thus in the setting of integral coefficients, the case $(\mathfrak{b}_n \oplus \mathfrak{a}_1, \mathfrak{b}_{n-1} \oplus \mathfrak{t}^1)$ only occurs as a product. □

Proposition 2.4 *Consider the case in theorem 2.1 where G and H are of type $\mathfrak{g}_2$ and $\mathfrak{t}^1$, respectively. In the setting of integral coefficients this case does not occur.*

Proof. Suppose that the case $(\mathfrak{g}_2, \mathfrak{t}^1)$ does occur, i.e.

$$H^*(\mathrm{G}_2 \,/\, \mathrm{U}(1)) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^{11}).$$

The universal coefficient theorem yields

$$H^*(\mathrm{G}_2 \,/\, \mathrm{U}(1);\mathbb{Z}/2) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^{11};\mathbb{Z}/2).$$

Consider the principal fiber bundle

$$\mathrm{U}(1) \longrightarrow \mathrm{G}_2 \longrightarrow \mathrm{G}_2 \,/\, \mathrm{U}(1).$$

Since $\mathrm{G}_2 / \mathrm{U}(1) \cong G / H$ is simply connected, corollary 1.20 applies. By [MT91, theorem VII.6.2], we have

$$H^*(\mathrm{G}_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3] \,/\, ({x_3}^4) \otimes \Lambda_{\mathbb{Z}/2}(x_5),$$

and in particular

$$H^5(\mathrm{G}_2;\mathbb{Z}/2) \cong \mathbb{Z}/2 \neq 0.$$

Thus by corollary 1.20, there exist integers i, j with $i + j = 5$,

$$H^i(\mathrm{S}^2 \times \mathrm{S}^{11};\mathbb{Z}/2) \cong H^i(\mathrm{G}_2 \,/\, \mathrm{U}(1);\mathbb{Z}/2) \neq 0$$

and

$$H^j(\mathrm{S}^1;\mathbb{Z}/2) \cong H^j(U(1);\mathbb{Z}/2) \neq 0.$$

Obviously there are no such integers. This is the desired contradiction. □

Proposition 2.5 *Consider the case in theorem 2.1 where G and H are of type $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ and $\mathfrak{a}_1 \oplus \mathfrak{t}^1$, respectively. In the setting of integral coefficients this case does not occur.*

Proof. Suppose that the case $(\mathfrak{g}_2 \oplus \mathfrak{a}_1, \mathfrak{a}_1 \oplus \mathfrak{t}^1)$ does occur, i.e.

$$H^*(\mathrm{G}_2 \cdot \mathrm{SU}(2) \,/\, \mathrm{SU}(2) \cdot \mathrm{U}(1)) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^{11}).$$

The universal coefficient theorem yields

$$H^*(\mathrm{G}_2 \cdot \mathrm{SU}(2) \,/\, \mathrm{SU}(2) \cdot \mathrm{U}(1); \mathbb{Z}/2) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^{11}; \mathbb{Z}/2).$$

Consider the principal fiber bundle

$$\mathrm{SU}(2) \cdot \mathrm{U}(1) \longrightarrow \mathrm{G}_2 \cdot \mathrm{SU}(2) \longrightarrow \mathrm{G}_2 \cdot \mathrm{SU}(2) \,/\, \mathrm{SU}(2) \cdot \mathrm{U}(1).$$

Since $\mathrm{G}_2 \cdot \mathrm{SU}(2) \,/\, \mathrm{SU}(2) \cdot \mathrm{U}(1) \cong G \,/\, H$ is simply connected, corollary 1.20 applies. By [MT91, theorem VII.6.2], we have

$$H^*(\mathrm{G}_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_3] \,/\, ({x_3}^4) \otimes \Lambda_{\mathbb{Z}/2}(x_5),$$

and thus by the Künneth formula, we have

$$\begin{aligned} H^*(\mathrm{G}_2 \times \mathrm{SU}(2); \mathbb{Z}/2) &\cong H^*(\mathrm{G}_2; \mathbb{Z}/2) \otimes H^*(\mathrm{SU}(2); \mathbb{Z}/2) \cong \\ &\cong \mathbb{Z}/2[x_3] \,/\, ({x_3}^4) \otimes \Lambda_{\mathbb{Z}/2}(x_5) \otimes \Lambda_{\mathbb{Z}/2}(y_3). \end{aligned}$$

In particular we find $H^8(\mathrm{G}_2 \cdot \mathrm{SU}(2); \mathbb{Z}/2) \neq 0$. Thus by corollary 1.20, there exist integers i, j with $i + j = 8$,

$$H^i(\mathrm{S}^2 \times \mathrm{S}^{11}; \mathbb{Z}/2) \cong H^i(\mathrm{G}_2 \cdot \mathrm{SU}(2) \,/\, \mathrm{SU}(2) \cdot \mathrm{U}(1); \mathbb{Z}/2) \neq 0$$

and

$$H^j(\mathrm{S}^3 \times \mathrm{S}^1; \mathbb{Z}/2) \cong H^j(\mathrm{SU}(2)U(1); \mathbb{Z}/2) \neq 0.$$

Obviously there are no such integers. This is the desired contradiction. □

We summarize the preceding considerations in the following proposition.

Proposition 2.6 *Let $\mathcal{P}$ be a simply connected compact topological space with cohomology $H^*(\mathcal{P}) \cong \mathbb{Z}[a] \,/\, (a^2) \otimes \Lambda(u)$ with $\deg a = 2$ and $\deg u \geq 3$ odd. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

If $G \,/\, H$ is not a product of spheres, then the pair (G, H) is in the following list.

Type of G	*Type of H*	m	n
$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{t}^1$	3	
$\mathfrak{a}_2$	$\mathfrak{t}^1$	5	
$\mathfrak{a}_n \oplus \mathfrak{a}_1$	$\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$	$2n+1$	$n \geq 2$
$\mathfrak{b}_2$	$\mathfrak{t}^1$	7	
$\mathfrak{b}_2 \oplus \mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{t}^1$	7	
$\mathfrak{c}_3 \oplus \mathfrak{a}_1$	$\mathfrak{b}_2 \oplus \mathfrak{t}^1$	11	
$\mathfrak{c}_n \oplus \mathfrak{a}_1$	$\mathfrak{c}_{n-1} \oplus \mathfrak{t}^1$	$4n-1$	$n \geq 4$

Note that we do not claim that all homogeneous spaces that arise from the list of proposition 2.6 have this integral cohomology. In fact it will turn out that this is wrong in some cases. The aim of the rest of this chapter is to rule out the spaces with the wrong integral cohomology, i.e. those whose cohomology ring has torsion.

2.4 Cases excluded by the Gysin exact sequence

In this section we calculate the integral cohomology groups of some homogeneous spaces by means of the Gysin exact sequence in order to rule out the homogeneous spaces that have an integral cohomology ring with torsion. The point stabilizers that occur in this section have a central 1-torus. The cohomology ring of the homogeneous space G/H strongly depends on the embedding of this torus in the group G.

We first examine the central 1-torus in the point stabilizer H. We show that in all cases of proposition 2.6 with $\operatorname{rk} H \geq 2$ the torus $\mathrm{U}(1)$ in H is contained in the centralizer of H' in G, and that this centralizer has rank 2. Suppose that $G = G_1 \cdot \mathrm{SU}(2)$ and $H = H' \cdot U(1)$. If $\operatorname{rk} H' \geq 2$, then $H' \subseteq G_1$ by corollary 1.5. This applies to $G_1 = \mathrm{Sp}(n)$, $n \geq 3$, and to $G_1 = \mathrm{SU}(n)$, $n \geq 4$. In both cases the inclusion of H' into G_1 is unique up to inner automorphisms of G_1, and the centralizer of H' in G has rank 2. For $G_1 = \mathrm{Sp}(2)$ and for $G_1 = \mathrm{SU}(3)$ this argument does not apply. We give alternative arguments in the following paragraph.

Consider the cases $G_1 = \mathrm{Sp}(2)$, $H' = \mathrm{Sp}(1)$ and $G_1 = \mathrm{SU}(3)$, $H' = \mathrm{SU}(2)$. If $H' \nsubseteq G_1 \times 1$, then $H' \cap G_1 \times 1$ is finite, because H' is almost simple. Moreover we have $\mathrm{U}(1) \subseteq G_1 \times 1$, because the projection of H' on the second factor $1 \times \mathrm{Sp}(1)$ has full rank. Consider the action of G_1 on G/H. The point stabilizer of this action is $G_1 \cap H$ and thus locally isomorphic to $\mathrm{U}(1)$. The orbits have dimension $\dim G_1 - \dim \mathrm{U}(1) = \dim(G_1 \cdot \mathrm{Sp}(1)) - \dim(H' \cdot U(1)) =$

dim $G\,/\,H$. By [SBG+95, corollary 96.11] this shows that G_1 acts transitively on $G\,/\,H$. This implies that G has a proper normal closed subgroup that acts transitive on $G\,/\,H$. So, also in the cases $G_1 = \mathrm{Sp}(2)$, $H' = \mathrm{Sp}(1)$ and $G_1 = \mathrm{SU}(3)$, $H' = \mathrm{SU}(2)$ we obtain $H' \subseteq G_1$. This is the natural inclusion, because otherwise the centralizer of H' in G_1 were trivial and $G\,/\,H$ were a product.

We have shown that in all cases of proposition 2.6 with $\operatorname{rk} H \geq 2$ the torus $\mathrm{U}(1)$ in H is contained in the centralizer of H' in G, and that this centralizer has rank 2. Again the inclusion map $\mathrm{U}(1) \longrightarrow G$ depends on two integral parameters $k, l \in \mathbb{Z}$ and will be denoted by $i_{k,l}$ in analogy to the definition on page 4. We calculate the first five cohomology groups of the resulting homogeneous spaces $G\,/\,H$.

Proposition 2.7 *Let $n \geq 2$, let $G = \mathrm{Sp}(n)\times\mathrm{Sp}(1)$ or $G = \mathrm{SU}(n+1)\times\mathrm{Sp}(1)$, and let $H \cong \mathrm{Sp}(n-1) \times \mathrm{U}(1)$ or $H \cong \mathrm{SU}(n) \times \mathrm{U}(1)$, respectively, be the subgroup $H = H' \times i_{k,l}(\mathrm{U}(1))$ of G, $k, l \neq 0$. Then*

$$H^q(G\,/\,H) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0,2\}, \\ 0 & \text{for } q \in \{1,3\}, \\ \mathbb{Z}\,/\,(l^2) & \text{for } q = 4. \end{cases}$$

Remark 2.8 Note that in the case $k = 0$ the homogeneous spaces as in proposition 2.7 are products of spheres but of the wrong dimensions. If $l = 0$ then $G\,/\,H$ is the product of a complex projective space of dimension at least 4 with a 3-sphere. So they cannot have the same cohomology as the product of two spheres.

Proof of proposition 2.7. The calculations of Kreck and Stolz in [KS88] lead to the cohomology of $\mathrm{SU}(3) \times \mathrm{SU}(2)\,/\,\mathrm{SU}(2) \cdot i_{k,l}(\mathrm{U}(1))$. They consider this space as a $\mathrm{U}(1)$-bundle over $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1$. We reproduce their arguments in a more explicit form and apply them to our more general situation.

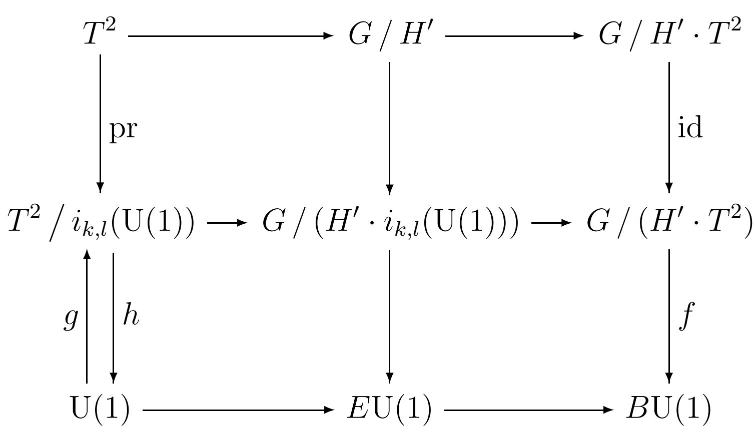
Let $G = G_1 \times \mathrm{Sp}(1)$ with $G_1 = \mathrm{Sp}(n)$ or $G_1 = \mathrm{SU}(n+1)$, respectively. Note that $G\,/\,H$ is simply connected. This can be seen from the principal fiber bundle

$$H \longrightarrow G \longrightarrow G\,/\,H$$

and the following part of its long exact sequence of homotopy groups.

$$0 = \pi_1(G) \longrightarrow \pi_1(G\,/\,H) \longrightarrow \pi_0(H) = 0$$

DIAGRAM 2.4.1: Some principal fiber bundles.



Let T^2 be a maximal torus in the centralizer of H' in G, which contains $i_{k,l}(\mathrm{U}(1))$. From the inclusions $H' \cdot i_{k,l}(\mathrm{U}(1)) \subseteq H' \cdot T^2 \subseteq G$ we get the following fibration by lemma 1.21.

$$T^2\,/\,i_{k,l}(\mathrm{U}(1)) \longrightarrow G\,/\,(H' \cdot i_{k,l}(\mathrm{U}(1))) \longrightarrow G\,/\,(H' \cdot T^2) \tag{2.4.1}$$

We may assume that k and l do not have a common divisor. Let $r, s \in \mathbb{Z}$ with $rk + sl = 1$. Then T^2 is the (inner) direct product of $i_{k,l}(\mathrm{U}(1))$ and $i_{-s,r}(\mathrm{U}(1))$. With the action of $i_{-s,r}(\mathrm{U}(1))$ the fibration (2.4.1) becomes a principal U(1)-bundle. We will use its Gysin exact sequence to compute the cohomology of $G\,/\,H$:

$$\ldots \longrightarrow H^{q+1}(G\,/\,H) \longrightarrow H^{q}(G\,/\,(H' \cdot T^2)) \xrightarrow{\cdot\chi}$$
$$\xrightarrow{\cdot\chi} H^{q+2}(G\,/\,(H' \cdot T^2)) \longrightarrow H^{q+2}(G\,/\,H) \longrightarrow \ldots$$

For this computation we have to know the Euler class χ, which equals the first Chern class $c_1(L)$ of the corresponding complex line bundle L.

Consider the commutative diagram 2.4.1. The middle row is the principal U(1)-bundle (2.4.1) discussed above. The bottom row is the corresponding universal U(1)-bundle with classifying map f. The first row is the fibration that arises from the inclusions $H' \subseteq H' \cdot T^2 \subseteq G$, see lemma 1.21. The isomorphism g is the composite of $i_{-s,r}$ and the canonical projection $\mathrm{pr}\colon T^2 \longrightarrow T^2\,/\,i_{k,l}(\mathrm{U}(1))$. Its inverse is

$$\begin{aligned} h\colon T^2\,/\,i_{k,l}(\mathrm{U}(1)) &\longrightarrow \mathrm{U}(1), \\ [z_1, z_2] &\mapsto {z_1}^{-l}{z_2}^{k}. \end{aligned} \tag{2.4.2}$$

DIAGRAM 2.4.2: TRANSGRESSION.

$$\begin{array}{ccc}
H^1(T^2) & \xrightarrow[\cong]{\tau} & H^2(G\,/\,H'\cdot T^2) \\
\uparrow{\scriptstyle \mathrm{pr}^*} & & \uparrow{\scriptstyle \mathrm{id}} \\
H^1(T^2\,/\,i_{k,l}(\mathrm{U}(1))) & \xrightarrow{\tau} & H^2(G\,/\,H'\cdot T^2) \\
\uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle f^*} \\
H^1(\mathrm{U}(1)) & \xrightarrow[\cong]{\tau} & H^2(B\mathrm{U}(1))
\end{array}$$

Each of these fibrations has a simply connected base space. Note that $G\,/\,(H'\cdot T^2) \cong (G_1\,/\,(H'\cdot i_{1,0}(\mathrm{U}(1)))) \times (\mathrm{Sp}(1)\,/\,i_{0,1}(\mathrm{U}(1))) \cong \mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1$ with $t = 2n-1$, if $G_1 = \mathrm{Sp}(n)$, and with $t = n$, if $G_1 = \mathrm{SU}(n+1)$.

From the differential in the E_2-term of the Leray-Serre spectral sequence arises a morphism in cohomology relating the first cohomology group of the fiber F to the second one of the base space B:

$$\tau\colon H^1(F) \cong E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \cong H^2(B).$$

It is called the *transgression*. Because of the naturality of the Leray-Serre spectral sequence the transgression morphism is natural. Hence we get the commuting diagram 2.4.2. The transgression in the bottom row is an isomorphism, see [MT91, III.(4.13')]. The transgression in the first row can be embedded in the Serre exact sequence as follows:

$$H^1(G\,/\,H') \longrightarrow H^1(T^2) \xrightarrow{\tau} H^2(G\,/\,H'\cdot T^2) \longrightarrow H^2(G\,/\,H'). \tag{2.4.3}$$

The Serre exact sequence is described in [MT91, theorem III.2.14]. The exactness of sequence (2.4.3) is not stated in this theorem but it follows from its proof. We have

$$G\,/\,H' \cong (G_1\,/\,H') \times \mathrm{Sp}(1) \cong \begin{cases} \mathrm{S}^{4n-1} \times \mathrm{S}^3 & \text{for } G_1 = \mathrm{Sp}(n), \\ \mathrm{S}^{2n+1} \times \mathrm{S}^3 & \text{for } G_1 = \mathrm{SU}(n+1). \end{cases}$$

In both cases we have $H^1(G\,/\,H') = 0 = H^2(G\,/\,H')$, and hence τ in the first row of diagram 2.4.2 is an isomorphism.

We now calculate $c_1(L)$ by a chase in diagram 2.4.2. The Chern class $c_1(L) \in H^2(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1)$ is the image of the universal Chern class $c_1 \in H^2(B\mathrm{U}(1))$ under the morphism f^* induced by the classifying map f. Since the transgression in the bottom row of diagram 2.4.2 is an isomorphism, there is $\iota \in H^1(\mathrm{U}(1))$ with $\tau(\iota) = c_1$, whence $f^*\tau(\iota) = c_1(L)$. Starting with ι we now go another way through diagram 2.4.2 in direction $c_1(L)$.

But first consider again the transgression isomorphism

$$\tau\colon H^1(T^2) \longrightarrow H^2(G\,/\,(H' \cdot T^2)) \cong \\ \cong H^2(G_1\,/\,(H' \cdot i_{1,0}(\mathrm{U}(1))) \times \mathrm{Sp}(1)\,/\,i_{0,1}(\mathrm{U}(1))) \cong H^2(\mathbb{C}\mathrm{P}^t) \oplus H^2(\mathbb{C}\mathrm{P}^1). \tag{2.4.4}$$

The cohomology group $H^1(T^2) \cong H^1(\mathrm{S}^1 \times \mathrm{S}^1)$ is a free Abelian group in two generators corresponding to the two (inner direct) factors $i_{1,0}(\mathrm{U}(1))$ and $i_{0,1}(\mathrm{U}(1))$ of T^2. Let x and y be the images of these generators under τ. By the naturality of τ, the factor $H^2(\mathbb{C}\mathrm{P}^t)$ is generated by x, and the factor $H^2(\mathbb{C}\mathrm{P}^1)$ is generated by y. This implies $H^*(G\,/\,(H' \cdot T^2)) = \mathbb{Z}[x, y]\,/\,(x^{t+1}, y^2)$. Now, from (2.4.2) we conclude $\tau\mathrm{pr}^*h^*(\iota) = -lx + ky$. The commutativity of diagram 2.4.2 finally gives $c_1(L) = -lx + ky$.

Consider the Gysin exact sequence for the fiber bundle (2.4.1). For the construction of the Gysin exact sequence see [MT91, theorem III.3.5]. Since the odd dimensional cohomology groups of $\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1$ vanish, we get the two exact sequences

$$0 \longrightarrow H^1(G\,/\,H) \longrightarrow H^0(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \xrightarrow{\cdot c_1(L)} \\ \xrightarrow{\cdot c_1(L)} H^2(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \longrightarrow H^2(G\,/\,H) \longrightarrow 0$$

and

$$0 \longrightarrow H^3(G\,/\,H) \longrightarrow H^2(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \xrightarrow{\cdot c_1(L)} \\ \xrightarrow{\cdot c_1(L)} H^4(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \longrightarrow H^4(G\,/\,H) \longrightarrow 0.$$

Since in the first exact sequence the multiplication by $c_1(L) = -lx + ky$ is injective, we have $H^1(G\,/\,H) = 0$. Hence we have

$$\begin{aligned} H^2(G\,/\,H) &\cong H^2(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1)\,/\,(H^0(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \cdot c_1(L)) \\ &\cong (\mathbb{Z}x + \mathbb{Z}y)\,/\,\mathbb{Z} \cdot (-lx + ky) \\ &\cong \mathbb{Z}(rx + sy). \end{aligned}$$

For the last equality recall that $rk + sl = 1$ whence $x = -s(-lx + ky) + k(rx + sy)$ and $y = r(-lx + ky) + l(rx + sy)$.

From the second exact sequence we see $H^3(G/H) = \ker(\cdot c_1(L)) = 0$; in order to see this let $a, b \in \mathbb{Z}$, and let $ax + by \in \ker(c_1(L))$. Then $0 = (ax+by)(-lx+ky) = -alx^2 + (ak-bl)xy$. Since x^2 and xy are algebraically independent, this shows $al = 0$ and $ak - bl = 0$. The first equation shows $a = 0$ (since $l \neq 0$), and the second then shows $b = 0$ (since $k \neq 0$). The fact $H^3(G/H) = 0$ finally implies

$$\begin{aligned}
H^4(G/H) &\cong H^4(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \,/\, (H^2(\mathbb{C}\mathrm{P}^t \times \mathbb{C}\mathrm{P}^1) \cdot c_1(L)) \\
&\cong (\mathbb{Z}x^2 + \mathbb{Z}xy) \,/\, (\mathbb{Z}x + \mathbb{Z}y) \cdot (-lx + ky) \\
&\cong (\mathbb{Z}x^2 + \mathbb{Z}xy) \,/\, (-lx^2 + kxy, -lxy) \\
&\overset{(rk+sl=1)}{\cong} (\mathbb{Z}x^2 + \mathbb{Z}xy) \,/\, (-lx^2 + kxy, -lxy, -lrx^2 + xy) \\
&\cong \mathbb{Z}x^2 \,/\, (-lx^2 - klrx^2, -l^2rx^2) \\
&\overset{(rk+sl=1)}{\cong} \mathbb{Z}x^2 \,/\, (-l^2sx^2, -l^2rx^2) \\
&\overset{(\gcd(r,s)=1)}{\cong} \mathbb{Z}x^2 \,/\, (-l^2x^2).
\end{aligned}$$

This shows $H^4(G/H) \cong \mathbb{Z}/(l^2)$. □

Corollary 2.9 *Let $n \geq 2$. Let $G = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ or $G = \mathrm{SU}(n+1) \times \mathrm{Sp}(1)$, and let $H \cong \mathrm{Sp}(n-1) \times \mathrm{U}(1)$ or $H \cong \mathrm{SU}(n) \times \mathrm{U}(1)$, respectively, be the subgroup $H = H' \times i_{k,l}(\mathrm{U}(1))$ of G. Let $t = 4n-1$ or $t = 2n+1$, respectively. If $H^*(G/H) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^t)$ then $k \neq 0$ and $l \in \{\pm 1\}$.*

Proof. We have $t > 4$ whence $0 = H^4(G/H) \cong \mathbb{Z}/(l^2)$ and thus $l \in \{\pm 1\}$. □

Now consider the cases of proposition 2.6 with $H \cong \mathrm{U}(1)$. In these cases the group G is of type $\mathfrak{a}_2$ or $\mathfrak{b}_2$ and has rank 2. Up to inner automorphisms of G, the torus H is contained in a fixed (maximal) 2-torus in G. By lemma 1.6 this embedding depends on two integral parameters $k, l \in \mathbb{Z}$, and we have $H = i_{k,l}(\mathrm{U}(1))$.

We use a result by Kreck and Stolz for the case in which G and H are of type $\mathfrak{a}_2$ and $\mathfrak{t}^1$, respectively.

Proposition 2.10 *Let $G = \mathrm{SU}(3)$, and let $H \cong \mathrm{U}(1)$ be the subgroup $H = i_{k,l}(\mathrm{U}(1))$ of G. Then $H^4(G/H) \cong \mathbb{Z}/(k^2 + kl + l^2)$.*

Proof. See [KS91, p. 474]. □

Corollary 2.11 *Let $G = \mathrm{SU}(3)$, and let $H \cong \mathrm{U}(1)$ be the subgroup $H = i_{k,l}(\mathrm{U}(1))$ of G. If $H^*(G\,/\,H) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^5)$ then (up to exchanging k and l) we have $k \in \{\pm 1\}$ and $l \in \{0, -k\}$.*

Proof. By proposition 2.10, we have $0 = H^4(G\,/\,H) \cong \mathbb{Z}\,/\,(k^2 + kl + l^2)$. This implies $k^2 + kl + l^2 \in \{\pm 1\}$. The equation $k^2 + kl + l^2 = -1$ does not have a solution in $\mathbb{Z} \times \mathbb{Z}$. Consider the equation $k^2 + kl + l^2 = 1$. By a well known formula, this equation (in l) has solutions only if $k^2 - 4(k^2 - 1) \geq 0$, i.e. if $3k^2 \leq 4$, i.e. if $k \in \{-1, 0, 1\}$. The solutions are

$$l = \frac{1}{2}(-k \pm \sqrt{k^2 - 4(k^2-1)}) = \begin{cases} 0 \text{ and } -1 & \text{for } k = 1 \\ 1 \text{ and } -1 & \text{for } k = 0 \\ 1 \text{ and } 0 & \text{for } k = -1. \end{cases}$$

□

Note that multiplication of the parameters k and l by the same integer $m \in \mathbb{Z}$ does not have an effect on the resulting homogeneous space: $G\,/\,i_{k,l}(\mathrm{U}(1)) \cong G\,/\,i_{mk,ml}(\mathrm{U}(1))$. Therefore we may assume $k = 1$ in the above corollary.

Moreover the subgroups $i_{1,0}(\mathrm{U}(1))$ and $i_{1,-1}(\mathrm{U}(1))$ are conjugate to each other in $\mathrm{SU}(3)$. To see this take the standard 2-torus T^2 in $\mathrm{SU}(3)$, namely

$$T^2 = \left\{ \begin{pmatrix} s & & \\ & t & \\ & & (st)^{-1} \end{pmatrix} \middle|\ s, t \in \mathrm{U}(1) \right\}.$$

Then we get

$$i_{1,-1}(\mathrm{U}(1)) = \left\{ \begin{pmatrix} z & & \\ & z^{-1} & \\ & & 1 \end{pmatrix} \middle|\ z \in \mathrm{U}(1) \right\}$$

and

$$i_{1,0}(\mathrm{U}(1)) = \left\{ \begin{pmatrix} z & & \\ & 1 & \\ & & z^{-1} \end{pmatrix} \middle|\ z \in \mathrm{U}(1) \right\},$$

and thus

$$i_{1,-1}(z) = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \cdot i_{1,0}(z) \cdot \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

for all $z \in \mathrm{U}(1)$.

Proposition 2.12 *Let $G = \mathrm{Sp}(2)$, and let $H \cong \mathrm{U}(1)$ be the subgroup $H = i_{k,l}(\mathrm{U}(1))$ of G. Then*

$$H^q(G\,/\,H) \cong \begin{cases} \mathbb{Z} & \textit{for } q \in \{0, 2\}, \\ 0 & \textit{for } q \in \{1, 3\}, \\ \mathbb{Z}\,/\,(k^2 + l^2) & \textit{for } q = 4. \end{cases}$$

Proof. Let T^2 be a maximal torus in Sp(2). We know $H^*(BT^2) \cong \mathbb{Z}[x,y]$ with generators x, y of degree 2. The Weyl group $W(\mathrm{Sp}(2))$ of Sp(2) acts on $H^*(BT^2)$. A complete set of generators for $W(\mathrm{Sp}(2))$ is provided by the following two maps. The first one interchanges x and y and the second one maps x to $-x$ and y to $-y$. Hence $H^*(\mathrm{Sp}(2)\,/\,T^2) \cong \mathbb{Z}[x,y]\,/\,(x^2+y^2, x^2y^2) \cong \mathbb{Z}[x,y]\,/\,(x^4, y^4, x^2+y^2)$. The cohomology of $\mathrm{Sp}(2)\,/\,T^2$ can also be found in [Bor53b, p. 200, example (2)].

Without loss of generality we may assume that k and l do not have a common divisor. Let $m, n \in \mathbb{Z}$ be such that $km + ln = 1$. With the method of Kreck and Stolz described in [KS91, section 4] and in the proof of proposition 2.7, we get as the first Chern class $c_1(L) = -lx + ky$ and exact sequences

$$0 \longrightarrow H^1(G\,/\,H) \longrightarrow H^0(G\,/\,T^2) \xrightarrow{\cdot c_1(L)} H^2(G\,/\,T^2) \longrightarrow H^2(G\,/\,H) \longrightarrow 0$$

and

$$0 \longrightarrow H^3(G\,/\,H) \longrightarrow H^2(G\,/\,T^2) \xrightarrow{\cdot c_1(L)} H^4(G\,/\,T^2) \longrightarrow H^4(G\,/\,H) \longrightarrow 0.$$

Note that these are two pieces of the Gysin exact sequence of the principal fiber bundle

$$T^2 \longrightarrow G \longrightarrow G\,/\,T^2.$$

Because of $H^1(G\,/\,H) = 0$, the first exact sequence implies

$$H^2(G\,/\,H) \cong (\mathbb{Z}x + \mathbb{Z}y)\,/\,(-lx + ky) = \mathbb{Z}(mx + ny).$$

From the second exact sequence we see $H^3(G\,/\,H) = \ker(\cdot c_1(L)) = 0$; in order to show this let $a, b \in \mathbb{Z}$ and let $ax + by \in \ker(\cdot c_1)(L)$. Then $0 = (ax+by)(-lx+ky) = (ak-bl)xy + (bk+al)y^2$. Since xy and y^2 are algebraically independent, this shows $ak = bl$ and $bk = -al$. Since $\mathrm{U}(1) \cong H = i_{k,l}(\mathrm{U}(1))$, the integers k and l cannot both be zero. For, $k = 0$ or $l = 0$ would immediately imply $a = b = 0$. So, suppose $k, l \neq 0$. Now $0 \leq (ak)^2 = (ak)(bl) = (bk)(al) = -b^2k^2$, and thus $b = 0$ and $a = 0$, because $k, l \neq 0$. This shows $\ker(\cdot c_1(L)) = 0$.

Finally $H^3(G\,/\,H) = 0$ implies

$$\begin{aligned} H^4(G\,/\,H) &\cong (\mathbb{Z}x^2 + \mathbb{Z}xy)\,/\,(-lx^2 + kxy, -lxy - kx^2) \\ &= (\mathbb{Z}x^2 + \mathbb{Z}xy)\,/\,(-lx^2 + kxy, lxy + kx^2, (lm - kn)x^2 - xy) \\ &= \mathbb{Z}x^2\,/\,((-l + lmk - k^2n)x^2, (l^2m - kln + k)x^2) \end{aligned}$$

$$\overset{(km+ln=1)}{=} \mathbb{Z}x^2 \,/\, ((-l+l-l^2n-k^2n)x^2, (l^2m+k^2m-k+k)x^2)$$
$$\overset{(\gcd(m,n)=1)}{=} \mathbb{Z}x^2 \,/\, (l^2+k^2)x^2.$$

□

Corollary 2.13 *Let $G = \mathrm{Sp}(2)$, and let $H \cong \mathrm{U}(1)$ be the subgroup $H = i_{k,l}(\mathrm{U}(1))$ of G. If $H^*(G\,/\,H) \cong H^*(\mathrm{S}^2 \times \mathrm{S}^7)$ then (up to exchanging k and l) we have $k \in \{\pm 1\}$ and $l = 0$.*

Proof. By proposition 2.12, we have $0 = H^4(G\,/\,H) \cong \mathbb{Z}\,/\,(k^2+l^2)$, i.e. $k^2 + l^2 = 1$. As k and l are integers, one of them is zero and the other one equals 1 or -1. □

2.5 Summary: the classification result

In the preceding sections we assumed for technical reasons that the considered groups G do not have a proper normal closed subgroup still acting transitively on $\mathcal{P} \cong G\,/\,H$. We obtain *all* compact connected Lie groups acting transitively and almost effectively on $\mathcal{P}$ with the method described in section 1.4, if we know the centralizer $\mathrm{Cen}_G(H)$ of H in G. Note therefore that in each case of the following theorem a torus factor $\mathrm{U}(1)$ in the connected component $\mathrm{Cen}_G(H)^\circ$ of the centralizer of H in G is contained in the center $\mathrm{Cen}(H)$ of H.

We can summarize the results of this chapter in the following classification theorem:

Theorem 2.14 *Let $\mathcal{P}$ be a simply connected compact topological space with cohomology $H^*(\mathcal{P}) \cong \mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda(u)$, $\deg a = 2$ and $\deg u \geq 3$ odd. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

If $G\,/\,H$ is homeomorphic to $\mathrm{S}^2 \times \mathrm{S}^m$, then $G = G_1 \times \mathrm{SU}(2)$ and $H = H' \times \mathrm{U}(1)$, and the pair (G_1, H') is one of the following. Moreover the embedding of H into G is unique up to conjugation in G.

G_1	H'	$\mathrm{Cen}_{G_1}(H')^\circ$	m	n
$\mathrm{SU}(2)$	1	$\mathrm{SU}(2)$	3	
$\mathrm{SU}(n)$	$\mathrm{SU}(n-1)$	$\mathrm{U}(1)$	$2n-1$	$n \geq 3$
$\mathrm{Sp}(n)$	$\mathrm{Sp}(n-1)$	$\mathrm{Sp}(1)$	$4n-1$	$n \geq 4$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n-1)$	1	$2n-1$	$n \geq 3$
$\mathrm{Spin}(7)$	G_2	1	7	

If $G\,/\,H$ is not a product of spheres, then G and H are contained in the following list.

G	H	$\mathrm{Cen}_G(H)^\circ$	m	k, l, n
$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$i_{k,l}(\mathrm{U}(1))$	$(\mathrm{U}(1))^2$	3	$k, l \neq 0$, $gcd(k,l) = 1$
$\mathrm{SU}(n) \times \mathrm{SU}(2)$	$\mathrm{SU}(n-1) \times i_{k,1}(\mathrm{U}(1))$	$(\mathrm{U}(1))^2$	$2n-1$	$k \neq 0$, $n \geq 3$
$\mathrm{Sp}(n) \times \mathrm{SU}(2)$	$\mathrm{Sp}(n-1) \times i_{k,1}(\mathrm{U}(1))$	$(\mathrm{U}(1))^2$	$4n-1$	$k \neq 0$, $n \geq 2$
$\mathrm{SU}(3)$	$i_{1,0}(\mathrm{U}(1))$	$\mathrm{SU}(2) \cdot \mathrm{U}(1)$	5	
$\mathrm{Sp}(2)$	$i_{1,0}(\mathrm{U}(1))$	$\mathrm{Sp}(1) \cdot \mathrm{U}(1)$	7	

In all cases, except perhaps for the case $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, the integral cohomology of the homogeneous space $\mathcal{P} \cong G\,/\,H$ actually is $H^*(G\,/\,H) \cong \mathbb{Z}[a]\,/\,(a^2) \otimes \Lambda(u)$.

Chapter 3

Truncated polynomial cohomology

The aim of this chapter is the classification of all simply connected homogeneous spaces $G\,/\,H$ of compact Lie groups G with $H^*(G\,/\,H;\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m)$, $\deg a$ even and $m \geq 2$.

In the first section we deduce conditions on the degree multisets of the groups G and H from the structure of the cohomology rings of the corresponding homogeneous spaces. We do this in two different ways. The first approach uses strong results about rational homotopy groups. The second approach is more elementary but in turn more technical.

In the second section we use the information about the degree multisets to get a list of all possibilities for the pairs (G, H). In section 3.3 we state and discuss the classification result. It remains to prove in the last section that each pair of groups determines a unique homogeneous space with the given rational cohomology.

3.1 Calculation of the degree multisets

The specific structure of the rational cohomology rings of $G\,/\,H$, G and H as truncated polynomial ring or exterior algebras, respectively, allows us to calculate the ranks of their homotopy groups. Note that the homotopy groups—in particular the fundamental group—of a Lie group are Abelian, see [MT91, theorem V.5.19]. Recall that the *rank* of a finitely generated Abelian group, denoted by rk, is its $\mathbb{Q}$-dimension when tensored with $\mathbb{Q}$. Via the principal fiber bundle of the homogeneous space $G\,/\,H$, these homotopy groups are embedded in a long exact sequence. Tensored with $\mathbb{Q}$, this sequence splits into many short exact sequences, because many of the homotopy groups involved turn out to have rank zero. The information we get from these short exact sequences about the ranks of the homotopy groups of G and H can finally be retranslated into information about the degree multisets of G and H.

The tool to go from the rational cohomology rings to the ranks of the homotopy groups and back is the following theorem. We need some special cases of this theorem and formulate them as corollaries. We denote the homotopy groups of a topological space X by $\pi_q(X)$.

Theorem 3.1 *Let X be a simply connected topological space whose rational cohomology is the product of a polynomial ring with one generator a of even degree and an exterior algebra with homogeneous generators $u_1, \ldots, u_r$ of odd degree: $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m) \otimes \Lambda_{\mathbb{Q}}(u_1, \ldots, u_r)$. Let $r_i = |\{ j \mid \deg u_j = i\}|$ be the number of generators u_i that have degree i. Then*

$$\operatorname{rk}(\pi_k(X)) = \begin{cases} 1 & \text{for } k = \deg a, \\ r_k + 1 & \text{for } k = m \deg a - 1 \text{ and} \\ r_k & \text{otherwise,} \end{cases}$$

Proof. See [Kra98, theorem 2.4] □

Corollary 3.2 *Let X be as in theorem 3.1 with $r = 0$, i.e. $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m)$. Then*

$$\operatorname{rk}(\pi_k(X)) = \begin{cases} 1 & \text{for } k = \deg a \text{ or } k = m \deg a - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.3 *Let X be as in theorem 3.1 with $m = 1$, i.e. $H^*(X;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_1, \ldots, u_r)$. Then $\operatorname{rk}(\pi_k(X)) = r_k$.*

Corollary 3.4 *Let G be a compact connected Lie group. Then $H^*(G;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_1, \ldots, u_r)$ for some u_i of odd degree and $\operatorname{rk}(\pi_k(G)) = r_k$ for all k.*

Proof. Consider the universal covering of G. It has the form $G' \times \mathrm{U}(1)^\tau$ for some semisimple Lie group G' and some $\tau \in \mathbb{N}_0$. So, by the Künneth formula, the rational cohomology of G is an exterior algebra as claimed, with exactly τ generators of degree 1. Since $G' \times \mathrm{U}(1)^\tau \longrightarrow G$ is a finite covering, one has

$$\pi_k(G) \otimes \mathbb{Q} = \pi_k(G' \times \mathrm{U}(1)^\tau) \otimes \mathbb{Q} = \begin{cases} \pi_k(G') \otimes \mathbb{Q} & \text{for } k \geq 2, \\ \pi_k(\mathrm{U}(1)^\tau) \otimes \mathbb{Q} & \text{for } k = 1. \end{cases}$$

Now $\operatorname{rk}(\pi_k(G)) = r_k$ for $k \geq 2$ by corollary 3.3 and $\operatorname{rk}(\pi_1(G)) = \tau = r_1$. □

Theorem 3.5 *Let $\mathcal{P}$ be a simply connected compact topological space with rational cohomology $H^*(\mathcal{P};\mathbb{Q}) \cong \mathbb{Q}[a] \,/\, (a^m)$, $d = \deg a$ even and $m \geq 2$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then the degree multisets of G and H have the form

$$\begin{aligned} \deg G &= [md-1, m_2, \ldots, m_n] \text{ and} \\ \deg H &= [d-1, m_2, \ldots, m_n] \end{aligned}$$

for suitable $n, m_2, \ldots, m_n \in \mathbb{N}$. That means that G and H have equal rank, and $H^*(G;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y_1, \ldots, y_n)$ and $H^*(H;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, \ldots, x_n)$ with $\deg x_1 = d-1$, $\deg y_1 = md-1$ and $\deg x_i = \deg y_i$ for $i \geq 2$.

Proof. The stabilizer H of a point is a closed subgroup of the compact Lie group G and thus also compact. By corollary 3.4, $H^*(G;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y_1, \ldots, y_n)$ and $H^*(H;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, \ldots, x_l)$ with generators x_i, y_i of odd degree,

$$\begin{aligned} \operatorname{rk}(\pi_k(G)) &= |\{ j \mid \deg y_j = k \}| \text{ and} \\ \operatorname{rk}(\pi_k(H)) &= |\{ j \mid \deg x_j = k \}| \,. \end{aligned} \tag{3.1.1}$$

In particular, $\pi_k(G) \otimes \mathbb{Q} = 0 = \pi_k(H) \otimes \mathbb{Q}$ for k even. By corollary 3.2,

$$\operatorname{rk}(\pi_k(\mathcal{P})) = \begin{cases} 1 & \text{for } k = d \text{ or } k = md-1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence of homotopy groups for the fibration

$$H \longrightarrow G \longrightarrow G \,/\, H \cong \mathcal{P},$$

tensored with $\mathbb{Q}$, therefore leads to short exact sequences

$$0 \longrightarrow \mathbb{Q} \longrightarrow \pi_{d-1}(H) \otimes \mathbb{Q} \longrightarrow \pi_{d-1}(G) \otimes \mathbb{Q} \longrightarrow 0,$$

$$0 \longrightarrow \pi_{md-1}(H) \otimes \mathbb{Q} \longrightarrow \pi_{md-1}(G) \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \pi_k(H) \otimes \mathbb{Q} \longrightarrow \pi_k(G) \otimes \mathbb{Q} \longrightarrow 0$$

for all $k \neq d-1, md-1$. With (3.1.1), the claim is now obvious. □

This proof uses strong results about rational homotopy groups. We will give another proof, which is more elementary. For this we need an algebraic result, which seems obvious but necessitates careful and technical argumentation to prove it.

Lemma 3.6 *Let* $X = \mathbb{Q}[x_1, \ldots, x_n]$ *and* $Y = \mathbb{Q}[y_1, \ldots, y_n]$ *be polynomial graded* $\mathbb{Q}$*-algebras and* $\eta\colon X \longrightarrow Y$ *a monomorphism of graded* $\mathbb{Q}$*-algebras. Let* $I := (\eta(x_1), \ldots, \eta(x_n)) \subseteq Y$ *be the ideal in* Y *generated by the images* $\eta(x_1), \ldots, \eta(x_n)$.

If $Y \,/\, I \cong \mathbb{Q}[a] \,/\, (a^m)$ *for some* $m \geq 2$*, then* (*after reordering*) $\deg y_1 = \deg a$*,* $\deg x_1 = \deg a^m$ *and* $\deg x_i = \deg y_i$ *for* $i \geq 2$*.*

Proof. We need some notation and some simple observations. Let $d = \deg a$. We denote by I^k the vector subspace consisting of zero and of all homogeneous elements in I of degree k. Similarly, we define Y^k and $(\mathbb{Q}[a] \,/\, (a^m))^k$. Note that $I^k = Y^k \cap I$. Let $\iota\colon I \longrightarrow Y$ be the inclusion map. It restricts to a monomorphism of vector spaces $\iota_| : I^k \longrightarrow Y^k$, since ι preserves degrees. There are isomorphisms of vector spaces

$$Y^k \,\big/\, I^k \cong (Y \,/\, I)^k \cong (\mathbb{Q}[a] \,/\, (a^m))^k \cong \begin{cases} \mathbb{Q} & \text{for } k \in d\mathbb{Z},\ 0 \leq k < md \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The proof will proceed by induction on the number n of indeterminates. For $n = 1$, i.e. for $X = \mathbb{Q}[x_1]$, $Y = \mathbb{Q}[y_1]$, the claim is obvious. So, let $n \geq 2$. We have to distinguish between two cases depending on whether there are generators with a degree that is divisible by d or not.

Let us begin with the easier case: First assume that one of the generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ has a degree that is not divisible by d. Take the smallest such degree k, say. The minimality of k implies that neither I^k nor Y^k contains powers of elements of lower degree. (We will see later that in the other case the arguments are not more difficult. In fact, we will use the same arguments. But there will always be powers of elements of lower degree, which make things more complicated.) As shown above, we have $Y^k \,\big/\, I^k = 0$, thus $I^k = Y^k$. Since there are no powers of elements of lower degree in I^k and Y^k, the sets $\{\, y_i \mid \deg y_i = k \}$ and $\{\, \eta(x_i) \mid \deg x_i = k \}$ each form a basis of the same vector space $Y^k = I^k$ and thus have the same number of elements r, say. We may (up to reordering) assume that $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are the generators of degree k. Moreover, $\eta(x_1), \ldots, \eta(x_r)$ and $y_1, \ldots, y_r$ generate the same ideal in Y. Call this ideal J. We obtain the equality $(Y \,/\, J) \,/\, (I \,/\, J) \cong Y \,/\, I \cong \mathbb{Q}[a] \,/\, (a^m)$. Hence we can apply induction to $\widetilde{X} = \mathbb{Q}[x_{r+1}, \ldots, x_n]$, $\widehat{I} = I \,/\, J \cong (\eta(x_{r+1}), \ldots, \eta(x_n))$ and $\widetilde{Y} = Y \,/\, J \cong \mathbb{Q}[y_{r+1}, \ldots, y_n]$.

Now assume that each of the generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ has a degree that is divisible by d. We have to make the right choice for y_1. Consider the canonical projection $\pi\colon Y \longrightarrow Y \,/\, I \cong \mathbb{Q}[a] \,/\, (a^m)$. Choose $y \in Y^d \setminus I^d$ such that $\pi(y) = a$. As d is the smallest degree occurring among the generators, we may without loss of generality assume that y is one of them, $y = y_1$ say, and thus $\pi(y_1) = a$.

Let ld be the smallest degree among the degrees of $y_2, \dots, y_n$.

First assume $l < m$. This is the case in which $\dim(Y^{ld} / I^{ld}) = 1$. The minimality property of l assures that there are no powers of elements of lower degree in I^{ld} and Y^{ld} except for powers of y_1. We know $\deg {y_1}^l = ld$, thus $Y^{ld} = {y_1}^l\mathbb{Q} \oplus I^{ld}$. Note that ${y_1}^l \in Y^{ld}$. So, the sets $\{{y_1}^l\} \cup \{\, y_i \mid \deg y_i = ld\}$ and $\{{y_1}^l\} \cup \{\, \eta(x_i) \mid \deg x_i = ld\}$ each form a basis of the same vector space Y^{ld} and thus have the same number of elements r, say. We may (up to reordering) assume that $x_2, \dots, x_r$ and $y_2, \dots, y_r$ are the generators of degree ld, and in addition y_1, if $l = 1$. As the elements of either of the two sets above are linear combinations of elements of the other set, we may make a change of variables such that afterwards $y_i = \eta(x_i)$ for $i \in \{2, \dots, r\}$. Define J to be the ideal generated by $y_2, \dots, y_r$. We are now in the position to continue in complete analogy to the first case: We obtain the equality $(Y / J) / (I / J) \cong Y / I \cong \mathbb{Q}[a] / (a^m)$ and can apply induction to $\widetilde{X} = \mathbb{Q}[x_1, x_{r+1}, \dots, x_n]$, $\widehat{I} = I / J \cong (\eta(x_1), \eta(x_{r+1}), \dots, \eta(x_n))$ and $\widetilde{Y} = Y / J \cong \mathbb{Q}[y_1, y_{r+1}, \dots, y_n]$.

Finally assume $l \geq m$. In this case, we have $Y^{ld} / I^{ld} = 0$ and thus $Y^{ld} = I^{ld}$. We now have to make the right choice for x_1. Let $x = {y_1}^m \in I^{md}$. Note that all generators x_i of X have degree at least md. Otherwise there would arise a contradiction to the dimensions calculated in the beginning of the proof. So, as seen for y, we may without loss of generality assume that x is the image of one of the generators x_i in I, $x = \eta(x_1)$, say. Note that ${y_1}^l \in Y^{ld}$ and $\eta(x_1) \cdot {y_1}^{l-m} \in I^{ld}$ are equal. But, by the minimality property of l, the only powers of elements of lower degrees in I^{ld} and Y^{ld} are scalar multiples of these two elements. We now can write $Y^{ld} = {y_1}^l\mathbb{Q} \oplus M$ and copy the argumentation of the case $l < m$ above word by word with M replaced by I^{ld}. This completes the induction and hence the proof. □

Proof of theorem 3.5. By assumption, the Euler-Poincaré characteristic of $\mathcal{P}$ is $m > 0$. By [Wol84, p. 285 (beginning of the proof)] or [Wan49], the groups G and H have equal rank n, say. The stabilizer H is connected because G is connected and $\mathcal{P} \cong G / H$ is simply connected. So, G and H are compact connected Lie groups. This implies $H^*(G; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y_1, \dots, y_n)$ and $H^*(H; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, \dots, x_n)$ for suitable x_i, y_i of odd degree. By [Bor53b, théorème 19.1], the corresponding classifying spaces BG and BH have the rational cohomology $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[\tilde{y}_1, \dots, \tilde{y}_n]$ and $H^*(BH; \mathbb{Q}) \cong \mathbb{Q}[\tilde{x}_1, \dots, \tilde{x}_n]$ respectively with $\deg \tilde{x}_i = 1 + \deg x_i$ and $\deg \tilde{y}_i = 1 + \deg y_i$ for all i. By [Bor53b, théorème 26.1], there is a monomorphism $\eta\colon H^*(BG; \mathbb{Q}) \longrightarrow H^*(BH; \mathbb{Q})$, and $H^*(\mathcal{P}; \mathbb{Q}) \cong \mathbb{Q}[a] / (a^m)$ is isomorphic to the quotient of $H^*(BH;\mathbb{Q}) \cong \mathbb{Q}[\tilde{x}_1,\dots,\tilde{x}_n]$ and the ideal generated by $\eta(\tilde{y}_1),\dots,\eta(\tilde{y}_n)$. Lemma 3.6 shows that (after reordering) $\deg x_1 = \deg \tilde{x}_1 - 1 = d-1$, $\deg y_1 = \deg \tilde{y}_1 - 1 = md-1$ and $\deg x_i = \deg \tilde{x}_i - 1 = \deg \tilde{y}_i - 1 = \deg y_i$ for $i \geq 2$. □

3.2 The groups

We know so far that in the situation of theorem 3.5 the following holds:

The groups G and H have equal rank n, say. Their degree multisets have the form $\deg G = [md-1, m_2, \dots, m_n]$ and $\deg H = [d-1, m_2, \dots, m_n]$, respectively, for suitable $m_2,\dots,m_n \in \mathbb{N}$. The rational cohomology ring of $G\,/\,H$ is a truncated polynomial ring. Thus it is not a tensor product. Hence by the Künneth formula the homogeneous space $G\,/\,H$ also is not a product. By [Wol84, p. 285] or [Wan49], the group G is almost simple. Thus the degree multiset of G contains exactly one 3 and no 1. The degree multisets of G and of H differ only in one entry. So, H has at most two simple or torus factors.

We show that, if H has two factors, one of them has rank 1. Suppose the contrary, i.e. suppose that H is an almost direct product of two almost simple factors of rank at least 2. Then $\operatorname{rk} G = \operatorname{rk} H \geq 4$. The multisets $\deg G$ and $\deg H$ contain a 3 for each almost simple factor, whence

$$\deg G = [3, m_2, \dots, m_n] \text{ and } \deg H = [3, 3, m_3, \dots, m_n]\,. \tag{3.2.1}$$

By table 1.2 on page 16, G is of type $\mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{d}_n, \mathfrak{e}_7, \mathfrak{e}_8$ or $\mathfrak{f}_4$ with $n \geq 4$. Groups of these types do not contain closed subgroups of exceptional type and equal rank, again by table 1.2. Moreover, as type $\mathfrak{a}_n$ is not allowed for G, the degree multiset of G contains no 5. By equation (3.2.1), the degree multiset of H also contains no 5. Hence each of the factors of H has a 7 in its degree multiset, and thus $\deg H$ contains at least two 7's. By equation (3.2.1), the degree multiset of G also contains at least two 7's. This is only possible for G of type $\mathfrak{d}_4$ with $\deg G = [3, 7, 11, 7]$. The only possibility for H is $\deg H = [3, 7, 3, 7]$. Thus H is of type $\mathfrak{b}_2 \oplus \mathfrak{b}_2$. This is not possible by table 1.2. So, if H has two factors, one of them has rank 1.

The collected information in combination with table 1.2 on page 16 shows that in the situation of theorem 3.5 the pair (G, H) is in the following list. Note that table 1.2 on page 16 contains only the *maximal* compact subgroups of equal rank. To obtain *all* compact subgroups of equal rank one has to reread the table several times: We find all such subgroups as maximal subgroups in maximal subgroups, and so on.

Type of G	Type of H	d	m	n
$\mathfrak{a}_1$	$\mathfrak{t}^1$	2	2	
$\mathfrak{b}_n$	$\mathfrak{d}_n$	$2n$	2	$n \geq 4$
$\mathfrak{b}_3$	$\mathfrak{a}_3$	6	2	
$\mathfrak{g}_2$	$\mathfrak{a}_2$	6	2	
$\mathfrak{f}_4$	$\mathfrak{b}_4$	8	3	
$\mathfrak{a}_n$	$\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$	2	$n+1$	$n \geq 2$
$\mathfrak{b}_2$	$\mathfrak{a}_1 \oplus \mathfrak{t}^1$	2	4	
$\mathfrak{b}_n$	$\mathfrak{b}_{n-1} \oplus \mathfrak{t}^1$	2	$2n$	$n \geq 3$
$\mathfrak{c}_3$	$\mathfrak{b}_2 \oplus \mathfrak{t}^1$	2	6	
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1} \oplus \mathfrak{t}^1$	2	$2n$	$n \geq 4$
$\mathfrak{g}_2$	$\mathfrak{a}_1 \oplus \mathfrak{t}^1$	2	6	$n \geq 2$
$\mathfrak{b}_2$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	4	2	
$\mathfrak{c}_3$	$\mathfrak{b}_2 \oplus \mathfrak{a}_1$	4	3	
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1} \oplus \mathfrak{a}_1$	4	n	$n \geq 4$
$\mathfrak{g}_2$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	4	3	

3.3 The classification result

It turns out that each entry in the above list belongs to a unique homogeneous space with the desired rational cohomology. We use the following notation for the occurring groups and homogeneous spaces. The exceptional simple Lie groups of type $\mathfrak{f}_4$ and $\mathfrak{g}_2$ are denoted by F_4 and G_2, respectively. Let V be a vector space. The Grassmann manifold of oriented subspaces of dimension k in V is denoted by $\widetilde{\mathrm{Gr}}_k(V)$. The projective space over the division algebra $\mathbb{F}$ is denoted by $\mathbb{F}\mathrm{P}^n$. It is the Grassmann manifold of (not oriented) one dimensional subspaces of $V = \mathbb{F}^{n+1}$. A projective space of geometric dimension 2 is called a *projective plane*. The symbol $\mathbb{O}\mathrm{P}^2$ represents the octonion projective plane. By $\mathrm{H}(\mathbb{C})$ we mean the point space of the split Cayley hexagon.

Theorem 3.7 *Let $\mathcal{P}$ be a simply connected compact topological space with rational cohomology $H^*(\mathcal{P};\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m)$, $d = \deg a$ even and $m \geq 2$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then the pair (G, H) is (up to local isomorphisms) in the following list. In each of the cases the subgroup H is uniquely embedded in G up to conjugation in G. Moreover, all pairs of groups in the list have the claimed properties.

$\mathcal{P} \cong G / H$	G	H	d	m
S^d	$\mathrm{SO}(d+1)$	$\mathrm{SO}(d)$	$d \geq 2$ *even*	2
S^6	G_2	$\mathrm{SU}(3)$	6	2
$\mathbb{C}\mathrm{P}^{m-1}$	$\mathrm{SU}(m)$	$\mathrm{SU}(m-1) \cdot \mathrm{U}(1)$	2	$m \geq 3$
$\mathbb{C}\mathrm{P}^{m-1}$	$\mathrm{Sp}(\frac{m}{2})$	$\mathrm{Sp}(\frac{m}{2}-1) \cdot \mathrm{U}(1)$	2	$m \geq 4$ *even*
$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})$	$\mathrm{SO}(m+1)$	$\mathrm{SO}(m-1) \cdot \mathrm{SO}(2)$	2	$m \geq 4$ *even*
$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$	G_2	$\mathrm{Sp}(1) \cdot \mathrm{U}(1)$	2	6
$\mathrm{H}(\mathbb{C})$	G_2	$\mathrm{Sp}(1) \cdot \mathrm{U}(1)$	2	6
$\mathbb{H}\mathrm{P}^{m-1}$	$\mathrm{Sp}(m)$	$\mathrm{Sp}(m-1) \cdot \mathrm{Sp}(1)$	4	$m \geq 3$
$\mathrm{G}_2 / \mathrm{SO}(4)$	G_2	$\mathrm{SO}(4)$	4	3
$\mathbb{O}\mathrm{P}^2$	F_4	$\mathrm{Spin}(9)$	8	3

The list of possible pairs (G, H) has been determined in the preceding sections. The uniqueness up to conjugation of the embedding $H \subseteq G$ will be proved in section 3.4. Homogeneous spaces in the list of theorem 3.7 denoted by different symbols are pairwise not homotopy equivalent. This will be shown below in this section.

Recall that in the classification result of the last chapter, theorem 2.14, all listed transitive groups G are supposed not to have a transitive proper normal closed subgroup. Note that in the above theorem 3.7 the list of transitive and almost effective group actions is complete without this additional assumption about transitive proper normal closed subgroups: A group G acting transitively and effectively on a space with this rational cohomology is automatically simple by [Wol84, p. 285] or [Wan49]. (Note that the rational cohomology ring is a truncated polynomial ring, and thus it is not a tensor product. Hence by the Künneth formula the corresponding space is not a product.)

Note that some special cases of this classification result have been proved by Montgomery and Samelson ([MS43], $m = 2$), Borel ([Bor49], m a prime number, see also [Wol84, theorem 8.10.16]) and Uchida ([Uch77], $d = 2$). The proof we give, in particular the part of the proof already developed in the first two sections 3.1 and 3.2, is not a generalization of one of the proofs of these partial results. I do not think that these proofs can be generalized. The list of the Lie algebras involved is established using different techniques. However, the question how the subgroup H is embedded in the group G does not require new methods, and we feel free to refer simply to Borel's solution for some of the low rank cases.

As a direct consequence of theorem 3.7, we get a classification of all simply connected homogeneous spaces of compact Lie groups such that the *integral* cohomology of the homogeneous space is a truncated polynomial ring.

Theorem 3.8 *Let $\mathcal{P}$ be a simply connected compact topological space with integral cohomology $H^*(\mathcal{P}) \cong \mathbb{Z}[a]\,/\,(a^m)$, $d = \deg a$ even and $m \geq 2$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then the pair (G, H) is (up to local isomorphisms) in the following list. In each of the cases the subgroup H is uniquely embedded in G up to conjugation in G. Moreover, all pairs of groups in the list have the claimed properties.

$\mathcal{P} \cong G\,/\,H$	G	H	d	m
S^d	$\mathrm{SO}(d+1)$	$\mathrm{SO}(d)$	$d \geq 2$	2
S^6	G_2	$\mathrm{SU}(3)$	6	2
$\mathbb{C}\mathrm{P}^{m-1}$	$\mathrm{SU}(m)$	$\mathrm{U}(m-1)$	2	$m \geq 3$
$\mathbb{C}\mathrm{P}^{m-1}$	$\mathrm{Sp}(\frac{m}{2})$	$\mathrm{Sp}(\frac{m}{2}-1)\cdot\mathrm{U}(1)$	2	$m \geq 4$ *even*
$\mathbb{H}\mathrm{P}^{m-1}$	$\mathrm{Sp}(m)$	$\mathrm{Sp}(m-1)\cdot\mathrm{Sp}(1)$	4	$m \geq 3$
$\mathbb{O}\mathrm{P}^2$	F_4	$\mathrm{Spin}(9)$	8	3

Proof. By the universal coefficient theorem, the homogeneous space $\mathcal{P}$ has the rational cohomology $H^*(\mathcal{P}; \mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^m)$. Thus it is in the list of theorem 3.7. The integral cohomology of the spaces in the list is well known, and we obtain the list in the announcement. □

We now show that homogeneous spaces in the list of theorem 3.7 denoted by different symbols are pairwise not homotopy equivalent. For the spheres and the projective planes and projective spaces, this is clear: Topological spaces with different rational cohomology cannot be homotopy equivalent. But there are spaces in the list whose rational cohomologies coincide. This is the case for $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})$ and $\mathbb{C}\mathrm{P}^{m-1}$, $m \geq 4$ even, for $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$, $\mathbb{C}\mathrm{P}^5$ and $\mathrm{H}(\mathbb{C})$, and for $\mathbb{H}\mathrm{P}^2$ and $\mathrm{G}_2\,/\,\mathrm{SO}(4)$. We now will have a closer look at these spaces.

For the Grassmann manifolds $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})$, $m \geq 4$ even, look at the inclusions of compact connected Lie groups

$$\mathrm{SO}(m-1) \longrightarrow \mathrm{SO}(m-1)\cdot\mathrm{SO}(2) \longrightarrow \mathrm{SO}(m+1).$$

By Lemma 1.21 they give rise to the following fiber bundle.

$$\mathrm{SO}(2) \longrightarrow \frac{\mathrm{SO}(m+1)}{\mathrm{SO}(m-1)} \longrightarrow \frac{\mathrm{SO}(m+1)}{\mathrm{SO}(m-1)\cdot\mathrm{SO}(2)}$$

Its base space is $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})$, and its total space is the Stiefel manifold $\mathrm{V}_2(\mathbb{R}^{m+1})$, cf. section 1.4. Its typical fiber is a circle S^1. This leads to the following exact sequence of homotopy groups.

$$\pi_{m-1}(\mathrm{S}^1) \longrightarrow \pi_{m-1}(\mathrm{V}_2(\mathbb{R}^{m+1})) \longrightarrow \pi_{m-1}(\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})) \longrightarrow \pi_{m-2}(\mathrm{S}^1)$$

By [Ste51, 25.6] we have $\pi_{m-1}(\mathrm{V}_2(\mathbb{R}^{m+1})) \cong \mathbb{Z}/2$. Since $m \geq 4$ and thus $\pi_{m-1}(\mathrm{S}^1) = 0 = \pi_{m-2}(\mathrm{S}^1)$, the exact sequence shows $\pi_{m-1}(\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})) \cong \pi_{m-1}(\mathrm{V}_2(\mathbb{R}^{m+1})) \cong \mathbb{Z}/2$.

For the complex projective spaces $\mathbb{C}\mathrm{P}^{m-1}$, $m \geq 4$, recall that one can construct them as the set of all 1-dimensional subspaces of $\mathbb{C}^m$. This leads to a complex line bundle. The corresponding sphere bundle is

$$\mathrm{S}^1 \longrightarrow \mathrm{S}^{2m-1} \longrightarrow \mathbb{C}\mathrm{P}^{m-1}.$$

The associated long exact sequence of homotopy groups contains the following part.

$$\pi_{m-1}(\mathrm{S}^1) \longrightarrow \pi_{m-1}(\mathrm{S}^{2m-1}) \longrightarrow \pi_{m-1}(\mathbb{C}\mathrm{P}^{m-1}) \longrightarrow \pi_{m-2}(\mathrm{S}^1)$$

Because of $m \geq 4$ and thus $\pi_{m-1}(\mathrm{S}^1) = 0 = \pi_{m-2}(\mathrm{S}^1)$, this exact sequence shows that $\pi_{m-1}(\mathbb{C}\mathrm{P}^{m-1}) \cong \pi_{m-1}(\mathrm{S}^{2m-1}) = 0$, which directly implies that $\mathbb{C}\mathrm{P}^{m-1}$ and $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+1})$ are not homotopy equivalent.

For $m = 6$ there is in addition to $\mathbb{C}\mathrm{P}^5$ and $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$ one more space in the list with rational cohomology $\mathbb{Q}[a] \,/\, (a^5)$, $\deg a = 2$. It is named $\mathrm{H}(\mathbb{C})$ in theorem 3.7. All three spaces have different integral cohomology: The integral cohomology ring of $\mathbb{C}\mathrm{P}^5$ is torsion free; those of $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$ and $\mathrm{H}(\mathbb{C})$ have torsion, but are different, see [Kra94, theorem 6.4].

Finally, there are two spaces in the list with rational cohomology ring $\mathbb{Q}[a] \,/\, (a^3)$ and $\deg a = 4$: the quaternionic projective space $\mathbb{H}\mathrm{P}^2$ and the homogeneous space $\mathrm{G}_2 \,/\, \mathrm{SO}(4)$. The former has the $\mathbb{Z}/2$-cohomology ring $H^*(\mathbb{H}\mathrm{P}^2; \mathbb{Z}/2) = (\mathbb{Z}/2)[a] \,/\, (a^3)$ by the universal coefficient theorem. The latter has as $\mathbb{Z}/2$-Poincaré polynomial $1+t^2+t^3+t^4+t^5+t^6+t^8$, cf. [Bor53a, 13.1]. This shows in particular that $H^2(\mathbb{H}\mathrm{P}^2; \mathbb{Z}/2) \ncong H^2(\mathrm{G}_2 \,/\, \mathrm{SO}(4); \mathbb{Z}/2)$, and thus $\mathbb{H}\mathrm{P}^2$ and $\mathrm{G}_2 \,/\, \mathrm{SO}(4)$ are not homotopy equivalent.

3.4 Group embeddings

We now give a proof of theorem 3.7. Recall the assumptions of this theorem:

Let $\mathcal{P}$ be a simply connected compact topological space with rational cohomology $H^*(\mathcal{P}; \mathbb{Q}) \cong \mathbb{Q}[a] \,/\, (a^m)$, $d = \deg a$ even and $m \geq 2$. Let G be a compact connected Lie group acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.

The claim in theorem 3.7 that the list of pairs (G, H) with the considered rational cohomology is complete has been proved in the sections 3.1 and 3.2. It remains to prove the claim in theorem 3.7 about the resulting homogeneous spaces. That means we have to examine all possible embeddings of H into G for the pairs (G, H) in the list of theorem 3.7.

Proof of theorem 3.7. By a theorem of Borel ([Wol84, theorem 8.10.16]), there is up to conjugation exactly one embedding $H \subseteq G$ for m a prime number, in particular for $m \in \{2, 3, 5\}$. This settles most of the low rank cases. The cases $(\mathfrak{b}_2, \mathfrak{a}_1 \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}_2, \mathfrak{a}_1 \oplus \mathfrak{t}^1)$ are treated with help of table 1.2 on page 16. For the remaining cases, we use representation theory. For the methods see section 1.3.

Let (G, H) be of type $(\mathfrak{b}_2, \mathfrak{a}_1 \oplus \mathfrak{t}^1)$. Then G and H both have rank 2, and H is contained in a maximal closed subgroup of G of rank 2. By table 1.2 on page 16, the only such maximal closed subgroups are up to conjugation $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$ and $\mathrm{SO}(3) \cdot \mathrm{SO}(2) \subseteq \mathrm{SO}(5)$. The latter inclusion is one of the desired ones. In the former inclusion, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$ contains subgroups H isomorphic to $\mathrm{Sp}(1) \times \mathrm{U}(1)$. Here, the $\mathrm{Sp}(1)$-factor of H is $\mathrm{Sp}(1) \times 1$ or $1 \times \mathrm{Sp}(1)$, because otherwise the centralizer of the $\mathrm{Sp}(1)$-factor would be trivial and thus could not contain a 1-torus $\mathrm{U}(1)$. As a consequence, the torus factor $\mathrm{U}(1)$ is contained in the other direct factor. As a maximal torus in this direct factor of rank 1, it is embedded uniquely up to conjugation. So there are (up to conjugation) only three ways to embed H in G: as $H = \mathrm{SO}(3) \cdot \mathrm{SO}(2)$ in $G = \mathrm{SO}(5)$, as $H = \mathrm{Sp}(1) \cdot \mathrm{U}(1)$ in $G = \mathrm{Sp}(2)$ or as $H = \mathrm{U}(1) \cdot \mathrm{Sp}(1)$ in $G = \mathrm{Sp}(2)$. The last two subgroups H are obviously conjugate in $G = \mathrm{Sp}(2)$, and hence there remain the two embeddings listed in the statement of theorem 3.7.

Now let (G, H) be of type $(\mathfrak{g}_2, \mathfrak{a}_1 \oplus \mathfrak{t}^1)$. Then G and H again both have rank 2, and H is contained in a maximal closed subgroup of G of rank 2. By table 1.2 on page 16, the only such maximal closed subgroups are up to conjugation $\mathrm{Sp}(2)$ and $\mathrm{Sp}(1) \cdot \mathrm{U}(1)$. The latter subgroup is one of the desired groups. The former contains by table 1.2 up to conjugation only one maximal closed subgroup of rank 2, and this subgroup is of type $\mathfrak{a}_1 \oplus \mathfrak{t}^1$.

Now consider the pairs (G, H) of type $(\mathfrak{a}_n, \mathfrak{a}_{n-1} \oplus \mathfrak{t}^1)$, $n = 3$ or $n \geq 5$. We may suppose $G = \mathrm{SU}(n+1)$. (If not, then G has $\mathrm{SU}(n+1)$ as a finite covering and we have to choose a corresponding covering of H. This does not change the quotient $G\,/\,H$, and the "new" G and H are locally isomorphic to the "old" ones.) There is a natural representation of the group G on $\mathbb{C}^{n+1}$.

So, we have to look for all at most $(2n+2)$-dimensional faithful representations of groups of type $\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$. They are calculated in [Kra98, 4.10]. For $n \geq 5$ the only at most $(2n+2)$-dimensional representation of groups of type $\mathfrak{a}_{n-1}$ is the natural representation of $\mathrm{SU}(n)$ on $\mathbb{C}^n$. For $n = 3$ there is also an 8-dimensional representation as $\mathrm{PSU}(3)$ on $\mathfrak{su}_3\mathbb{C}$. Its centralizer is $\mathbb{C}$, more precisely, scalar multiplication with complex numbers. The natural module of $\mathrm{SU}(n+1)$ does not contain scalar multiplication. So, we cannot complete $\mathfrak{su}_3\mathbb{C}$ to a faithful representation of H on $\mathbb{R}^8$ such that it becomes a subrepresentation of the natural representation of G. Thus, the only possibility for $\mathfrak{a}_{n-1}$ is the natural representation as $\mathrm{SU}(n)$ on $\mathbb{C}^n \oplus 2\underline{\mathbb{R}}$. The connected component of the centralizer of $\mathrm{SU}(n)$ in $\mathrm{SU}(n+1)$ is exactly $\mathrm{U}(1)$. This shows that (up to conjugation) the only possibility for (G, H) is $(\mathrm{SU}(n+1), \mathrm{U}(n))$ with the natural inclusion.

Next consider the pairs (G, H) of type $(\mathfrak{b}_n, \mathfrak{b}_{n-1} \oplus \mathfrak{t}^1)$, $n \geq 3$. Then either $G = \mathrm{SO}(2n+1)$ or $G = \mathrm{Spin}(2n+1)$. In both cases there is a representation of G on $\mathbb{R}^{2n+1}$. For $n = 3$ or $n \geq 5$ the only at most $(2n+1)$-dimensional representation of groups of type $\mathfrak{b}_{n-1}$ is the natural representation of $\mathrm{SO}(2n-1)$ on $\mathbb{R}^{2n-1}$. See [Kra98, 4.11] for the low dimensional representations of groups of type $\mathfrak{b}_{n-1}$. For $n = 4$ there is also an 8-dimensional representation as $\mathrm{Spin}(7)$ on $\mathbb{R}^8$. We get a 9-dimensional representation if we add a trivial representation $\underline{\mathbb{R}}$. The centralizer of this representation is $\mathbb{R} \oplus \mathbb{R}$ and thus does not contain a torus $\mathrm{SO}(2)$. Thus, the only possibility for $\mathfrak{b}_{n-1}$ is the natural representation of $\mathrm{SO}(2n-1)$ on $\mathbb{R}^{2n-1} \oplus 2\underline{\mathbb{R}}$. The centralizer of this $\mathrm{SO}(2n-1)$-module is $\mathbb{R} \oplus \mathbb{R}(2)$, and there is only one possibility to embed a torus $\mathrm{SO}(2)$ in it. This shows that (up to conjugation) the only possibility for (G, H) is $(\mathrm{SO}(2n+1), \mathrm{SO}(2n-1) \cdot \mathrm{SO}(2))$ with the natural inclusion.

Finally consider the pairs (G, H) of type $(\mathfrak{c}_n, \mathfrak{c}_{n-1} \oplus \mathfrak{t}^1)$, $n \geq 4$, or $(\mathfrak{c}_3, \mathfrak{b}_2 \oplus \mathfrak{t}^1)$, $n = 3$, or $(\mathfrak{c}_n, \mathfrak{c}_{n-1} \oplus \mathfrak{a}_1)$, $n \geq 4$. In each of these cases, G is locally isomorphic to $\mathrm{Sp}(n)$. As above for $\mathrm{SU}(n+1)$ we may suppose $G = \mathrm{Sp}(n)$. There is a natural quaternionic representation of G on $\mathbb{H}^n$. The only faithful at most n-dimensional quaternionic representation of groups of respective type $\mathfrak{c}_{n-1}$, $n \geq 4$, and $\mathfrak{b}_2$ is as $\mathrm{Sp}(n-1)$ on $\mathbb{H}^{n-1}$ or as $\mathrm{Sp}(2)$ on $\mathbb{H}^2$, respectively, see [Kra98, 4.13]. So, the only corresponding exactly n-dimensional quaternionic module is $\mathbb{H}^{n-1} \oplus \underline{\mathbb{H}}$ or $\mathbb{H}^2 \oplus \underline{\mathbb{H}}$, respectively. The connected component of the centralizer of $\mathrm{Sp}(n-1)$ in $\mathrm{Sp}(n)$, $n \geq 3$, is $\mathrm{Sp}(1)$. That shows directly that (up to conjugation) the only possibility for (G, H) of type $(\mathfrak{c}_n, \mathfrak{c}_{n-1} \oplus \mathfrak{a}_1)$, $n \geq 4$, is $(\mathrm{Sp}(n), \mathrm{Sp}(n-1) \cdot \mathrm{Sp}(1))$ with the natural inclusion. For the other two cases, note that the centralizer of $\mathrm{Sp}(n-1)$ in $\mathrm{Sp}(n)$ has torus rank 1. Thus, all 1-tori in the connected component of the centralizer are conjugate. That shows that (up to conjugation) the only possibility for (G, H) of type $(\mathfrak{c}_n, \mathfrak{c}_{n-1} \oplus \mathfrak{t}^1)$, $n \geq 4$, or of type $(\mathfrak{c}_3, \mathfrak{b}_2 \oplus \mathfrak{t}^1)$ is $(\mathrm{Sp}(n), \mathrm{Sp}(n-1) \cdot \mathrm{U}(1))$ with the natural inclusion. □

Chapter 4

Isoparametric hypersurfaces

Even though the topological results about simply connected homogeneous spaces of compact Lie groups in the preceding chapters are of independent interest, we are mainly interested in the application of these results to the classification of isoparametric hypersurfaces with a homogeneous focal manifold. The present chapter gives the geometrical background and the results of the classification. The next chapter contains the proofs of these results.

In the first section of this chapter we explain what isoparametric hypersurfaces are and discuss some of their geometric and topological properties. Most of the material in this section, including the corresponding proofs, can be found in Münzner's papers [Mün80] and [Mün81]. See also the survey [Tho00].

The second section is devoted to the presentation of our classification results about isoparametric hypersurfaces in spheres with a homogeneous focal manifold. Although we will have collected all necessary tools and properties to prove these results, we postpone the proofs in favour of a discussion of the results. The proofs will be given at the end of the text in a separate chapter.

Section 4.3 is about buildings in the sense of Tits. To each isoparametric hypersurface in our classification belongs a point-line geometry, which turns out to be a rank 2 building, i.e. a generalized polygon. We briefly describe the connection between rank 2 buildings and isoparametric hypersurfaces. Finally we show how our topological results of the chapters 2 and 3 can also be applied to the classification of certain point homogeneous compact connected generalized polygons.

Section 4.4 finally describes classical examples of isoparametric hypersurfaces. Some of them occur in the classification. This description will be useful in chapter 5 for the proofs of the classification results which we state in section 4.2.

4.1 Geometric and topological properties

A closed connected hypersurface in a sphere is called an *isoparametric hypersurface* if its principal curvatures are constant.

Recall that the principal curvatures are the eigenvalues of the shape operator. We denote by g the number of distinct principal curvatures and by $m_1, \dots, m_g$ their multiplicities. By a theorem of Münzner in [Mün80] we can reorder these multiplicities such that $m_{i+2} = m_i$ (indices modulo g); in particular all multiplicities are equal if g is odd. Now, all multiplicities are determined by m_1 and m_2. Call the pair (m_1, m_2) the *multiplicities* of the isoparametric hypersurface. Moreover, Münzner's theorem gives $g \in \{1, 2, 3, 4, 6\}$. The isoparametric hypersurfaces with $g \leq 3$ have been classified by É. Cartan in [Car39]. Those with $g = 1$ or $g = 2$ are easy to classify. They are nicely embedded spheres and Clifford tori, respectively.

So, the remaining cases are $g = 4$ and $g = 6$. By work of Münzner [Mün81] and Grove and Halperin [GH87], we know the following. If $g = 6$ then $m_1 = m_2 \in \{1, 2\}$. If $g = 4$ and $m_1, m_2 \geq 2$ then $m_1 = m_2$, or $m_1 + m_2$ is odd. Even stronger restrictions on the multiplicities for the case $g = 4$ are achieved in the paper [Sto99]; cf. [Wol99] for a discussion of its topological core.

Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere S^k. Consider submanifolds parallel to $\mathcal{F}$. As long as they are close enough to $\mathcal{F}$, they are diffeomorphic to $\mathcal{F}$ and also isoparametric. To be more explicit, choose a unit normal field N on $\mathcal{F}$ in the ambient sphere S^k and consider the one parameter groups $\{\exp_x(tN) \mid t \in \mathbb{R}\}$. The endpoint map, defined on $\mathcal{F}$ by $\eta_t \colon x \mapsto \exp_x(tN)$ gives a foliation of the sphere in the following way: There are real numbers $\theta_1 < 0 < \theta_2$ such that the sphere is the disjoint union of the sets $M_t := \eta_t(\mathcal{F})$, $t \in [\theta_1, \theta_2]$. Moreover, all M_t with t in the interior of the interval are isoparametric hypersurfaces of the sphere diffeomorphic to $\mathcal{F}$ via η_t. The sets $\mathcal{P} := M_{\theta_1}$ and $\mathcal{L} := M_{\theta_2}$ are submanifolds of lower dimension and are called the *focal manifolds* of $\mathcal{F}$. Note that the union $\mathcal{P} \cup \mathcal{L}$ consists precisely of the focal points of $\mathcal{F}$.

The foliation described above will be called an *isoparametric foliation* in the sequel. Obviously each hypersurface of the foliation determines the foliation. More surprisingly the foliation is even completely determined by one of its focal manifolds, e.g. by $\mathcal{P}$: By [Mün80, section 6] each geodesic circle in the sphere that meets $\mathcal{P}$ orthogonally is divided by its intersection points with $\mathcal{P}$ and $\mathcal{L}$ into a regular geodesic (ordinary) $2g$-gon. In particular the edges of this polygon have (spherical) length π/g. The vertices of this polygon belong alternately to $\mathcal{P}$ and to $\mathcal{L}$. More precisely, $\mathcal{L}$ consists exactly of the points on the sphere with distance π/g to $\mathcal{P}$. The other leaves of the foliation can be recovered from $\mathcal{P}$ as the sets of points in the sphere at distance θ for a fixed θ with $0 < \theta < \pi/g$. Let $\mathcal{F}$ be one of these leaves. Then each minimal geodesic arc from $\mathcal{P}$ to $\mathcal{L}$ meets $\mathcal{F}$ orthogonally and in exactly one point.

Consider the maps $\eta_{\theta_1}\colon \mathcal{F} \longrightarrow \mathcal{P}$ and $\eta_{\theta_2}\colon \mathcal{F} \longrightarrow \mathcal{L}$. They are fiber bundles. The fibers $\eta_{\theta_i}^{-1}(x)$ are curvature spheres of $\mathcal{F}$ belonging to the focal points x. The dimension of the fibers is m_2 and m_1, respectively. So $\mathcal{F}$ can be identified with the normal sphere bundle of $\mathcal{P}$ and of $\mathcal{L}$. Consider the corresponding disk bundles. Glueing together these two disk bundles along their common boundary $\mathcal{F}$ leads to the double mapping cylinder for the two maps

$$\mathcal{P} \xleftarrow{\;\eta_{\theta_1}\;} \mathcal{F} \xrightarrow{\;\eta_{\theta_2}\;} \mathcal{L},$$

and we get a decomposition of the ambient sphere S^k as the union of two disk bundles. This decomposition of the sphere is the starting point for Münzner's work [Mün81], in which he calculated the cohomology groups of the isoparametric hypersurfaces and their focal manifolds. For the ring structure see [Kra94] or [Str96]:

Theorem 4.1 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures of multiplicities (m_1, m_2), $m_1, m_2 \geq 2$. Let $\mathcal{P}$ and $\mathcal{L}$ be its focal manifolds.*

If $m_1 \neq m_2$ then $m_1 + m_2$ is odd and

$$\begin{aligned} H^*(\mathcal{P}) &\cong H^*(\mathrm{S}^{m_1} \times \mathrm{S}^{m_1+m_2}),\\ H^*(\mathcal{L}) &\cong H^*(\mathrm{S}^{m_2} \times \mathrm{S}^{m_1+m_2}),\\ H^*(\mathcal{F}) &\cong H^*(\mathrm{S}^{m_1} \times \mathrm{S}^{m_2} \times \mathrm{S}^{m_1+m_2}). \end{aligned}$$

The spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are simply connected. The ambient sphere S^k has dimension $k = 2(m_1 + m_2) + 1$.

Proof. See [Mün80], [Mün81] and [Str96]. □

Theorem 4.2 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g \geq 3$ distinct principal curvatures all of the same multiplicity $d \geq 2$. Then $H^*(\mathcal{P};\mathbb{Q}) \cong H^*(\mathcal{L};\mathbb{Q}) \cong \mathbb{Q}[a]\,/\,(a^g)$ with $\deg a = d$. Moreover, one has*

$$\begin{aligned} &\textit{either } g = 3 \textit{ and } d \in \{2,4,8\},\\ &\textit{or } g = 4 \textit{ and } d \in \{2,4\},\\ &\textit{or } g = 6 \textit{ and } d \in \{2,4\}. \end{aligned}$$

The spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are simply connected. The ambient sphere S^k has dimension $k = gd + 1$.

Proof. See [Mün80], [Mün81] and [Str96]. □

Remark 4.3 Using the differential structure, it can be shown that there is no isoparametric hypersurface in a sphere with multiplicities $(4, 4)$ and $g = 4$ or $g = 6$, cf. [GH87].

Note that the Euler-Poincaré characteristic of the focal manifolds in the above theorems is zero in the first and equal to g in the second theorem. (Recall that the Euler-Poincaré characteristic of a topological space is the alternating sum over the ranks of its homology groups.)

From our classification of homogeneous spaces with a specific cohomology ring in chapters 2 and 3 we get the following direct consequences of the above theorems. These corollaries allow us to classify all isoparametric hypersurfaces that have a homogeneous focal manifold with certain multiplicities.

Corollary 4.4 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures of multiplicities $(2, m)$, $m \geq 3$ odd. Let $\mathcal{P}$ be its lower dimensional focal manifold. Let G be a compact connected Lie subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then (G, H) is one of the pairs in theorem 2.14.

Corollary 4.5 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g \geq 3$ distinct principal curvatures all of the same multiplicity $d \geq 2$. Let $\mathcal{P}$ be one of its focal manifolds. Let G be a compact connected Lie subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then (G, H) is one of the pairs in theorem 3.7. More precisely, the pair (G, H) is (up to local isomorphisms) in the following list.

$\mathcal{P} \cong G/H$	G	H	d	g
$\mathbb{C}\mathrm{P}^{g-1}$	$\mathrm{SU}(g)$	$\mathrm{U}(g-1)$	2	$3, 4, 6$
$\mathbb{C}\mathrm{P}^{g-1}$	$\mathrm{Sp}(\frac{g}{2})$	$\mathrm{Sp}(\frac{g}{2}-1)\cdot\mathrm{U}(1)$	2	$4, 6$
$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{g+1})$	$\mathrm{SO}(g+1)$	$\mathrm{SO}(g-1)\cdot\mathrm{SO}(2)$	2	$4, 6$
$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$	G_2	$\mathrm{Sp}(1)\cdot\mathrm{U}(1)$	2	6
$\mathrm{H}(\mathbb{C})$	G_2	$\mathrm{Sp}(1)\cdot\mathrm{U}(1)$	2	6
$\mathbb{H}\mathrm{P}^{g-1}$	$\mathrm{Sp}(g)$	$\mathrm{Sp}(g-1)\cdot\mathrm{Sp}(1)$	4	$3, 4, 6$
$\mathrm{G}_2/\mathrm{SO}(4)$	G_2	$\mathrm{SO}(4)$	4	3
$\mathbb{O}\mathrm{P}^2$	F_4	$\mathrm{Spin}(9)$	8	3

By remark 4.3 we could furthermore exclude the entries $\mathbb{H}P^3$ and $\mathbb{H}P^5$ in the above list. In the two corollaries 4.4 and 4.5 there are some more pairs of groups that do not lead to focal manifolds of isoparametric hypersurfaces. The complete classification results are stated in the theorems 4.8 and 4.9 in the next section.

For the classification we will need two lemmas that allow us to apply representation theory.

Lemma 4.6 *Every isometry of $\mathcal{F}$ is induced by an isometry of the ambient sphere $S^{2(m_1+m_2)+1}$ and is thus linear, i.e. an element of $O(2(m_1+m_2)+2)$. That means that every group action on $\mathcal{F}$ by isometries gives rise to a (real) $(2(m_1+m_2)+2)$-dimensional representation of the group, and $\mathcal{P}$ is one of the orbits.*

Proof. See [CR85, p. 303]. □

Note that, since each focal manifold $\mathcal{P}$ or $\mathcal{L}$ completely determines the isoparametric foliation via its normal bundle in the ambient sphere, every isometry of $\mathcal{P}$ or $\mathcal{L}$ also is induced by an isometry of the ambient sphere.

Lemma 4.7 *Let $\mathcal{P}$ be a focal manifold of an isoparametric hypersurface $\mathcal{F}$ with $g \geq 3$ distinct principal curvatures. Let G be a closed subgroup of the isometry group of $\mathcal{F}$ acting transitively on $\mathcal{P}$.*

Then there is no non-zero invariant subspace of the corresponding representation with trivial G-action. In other words, the corresponding G-module does not have a trivial factor. Moreover, all G-orbits of dimension smaller than that of $\mathcal{P}$ are contained in the other focal manifold.

Proof. See [Kra98, lemmas 8.6 and 8.7]. □

4.2 The classification results

We classify all isoparametric hypersurfaces with g distinct principal curvatures of multiplicities (m_1, m_2) with either $g = 4$, $m_1 = 2$ and $m_2 \geq 3$ odd, or $g \geq 3$ and $m_1 = m_2 \geq 2$ that have a homogeneous focal manifold. The results for these two settings are as follows. We prove these results in the next chapter.

Theorem 4.8 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures of multiplicities $(2, m)$, $m \geq 3$ odd. Let $\mathcal{P}$ be its lower dimensional focal manifold. Let G be a compact connected Lie* *subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal subgroup.*

Let $n = \frac{m+3}{2}$. Then $\mathcal{F}$ is the classical Clifford $(2, m)$-hypersurface with the standard embedding in the sphere S^{4n-1}. The point-line geometry $(\mathcal{P}, \mathcal{L}, \mathcal{F})$, where $\mathcal{L}$ is the other focal manifold of $\mathcal{F}$, is a generalized polygon. Moreover, G is one of the groups $\mathrm{SU}(n) \times \mathrm{SU}(2)$ or $\mathrm{Sp}(\frac{n}{2}) \times \mathrm{SU}(2)$ with the natural action on $\mathcal{F}$. In the former case, the group G acts even transitively on $\mathcal{F}$. The latter case occurs if and only if n is even.

We give an explicit description of these so-called classical Clifford hypersurfaces in section 4.4 at the end of this chapter.

Theorem 4.9 *Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g \geq 3$ distinct principal curvatures all of the same multiplicity $d \geq 2$. Let $\mathcal{P}$ and $\mathcal{L}$ be its focal manifolds. Let G be a compact connected subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then G acts even transitively on $\mathcal{F}$. The point-line geometry $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a generalized polygon. More precisely, it is one of the classical planes $\mathrm{PG}_2\,\mathbb{C}$, $\mathrm{PG}_2\,\mathbb{H}$ or $\mathrm{PG}_2\,\mathbb{O}$, the complex symplectic quadrangle $\mathrm{W}(\mathbb{C})$ or its dual, or the split Cayley hexagon $\mathrm{H}(\mathbb{C})$ or its dual. The embedding in the sphere is in each case the well known Veronese embedding. The following list contains all possibilities for $\mathcal{P}$, $\mathcal{L}$, G and H up to local group isomorphisms and up to exchanging $\mathcal{P}$ and $\mathcal{L}$.

$\mathcal{P}$	$\mathcal{L}$	G	H	d	g
$\mathbb{C}\mathrm{P}^2$	$\mathbb{C}\mathrm{P}^2$	$\mathrm{SU}(3)$	$\mathrm{U}(2)$	2	3
$\mathbb{H}\mathrm{P}^2$	$\mathbb{H}\mathrm{P}^2$	$\mathrm{Sp}(2)$	$\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$	4	3
$\mathbb{O}\mathrm{P}^2$	$\mathbb{O}\mathrm{P}^2$	F_4	$\mathrm{Spin}(9)$	8	3
$\mathbb{C}\mathrm{P}^3$	$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5)$	$\mathrm{Sp}(2)$	$\mathrm{Sp}(1) \cdot \mathrm{U}(1)$	2	4
$\mathrm{H}(\mathbb{C})$	$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$	G_2	$\mathrm{Sp}(1) \cdot \mathrm{U}(1)$	2	6

The geometries that occur in the above theorems are described in [VM98, chapter 2]. For the embeddings into the spheres see [KK95] and [Kra00b].

The above theorems show that to each considered isoparametric hypersurface with a homogeneous focal manifold there is a closed subgroup of the isometry group acting transitively on the hypersurface. This fact gives the following corollaries.

Corollary 4.10 *An isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures and multiplicities $(2, m)$, $m \geq 3$ odd, whose lower dimensional focal manifold is homogeneous is also homogeneous.*

Corollary 4.11 *An isoparametric hypersurface in a sphere with $g \geq 3$ distinct principal curvatures all of the same multiplicity $d \geq 2$ and with a homogeneous focal manifold is homogeneous.*

4.3 The corresponding buildings

Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with focal manifolds $\mathcal{P}$ and $\mathcal{L}$. Recall from section 4.1 that each geodesic circle of the ambient sphere that meets $\mathcal{P}$ orthogonally is divided by its intersection points with $\mathcal{P}$ and $\mathcal{L}$ into a regular geodesic $2g$-gon. The union of all these $2g$-gons is a (topological) bipartite graph with $\mathcal{P} \cup \mathcal{L}$ as vertex set and the edges of the $2g$-gons as edges of the graph. Recall also from section 4.1 that each such edge meets $\mathcal{F}$ in a unique point. In this way $\mathcal{F}$ may be identified with the set of edges and thus with a subset of $\mathcal{P} \times \mathcal{L}$. The graph obtained in this fashion turns out to be a rank 2 building for the examples of isoparametric hypersurfaces that occur in our classification.

More generally, the isoparametric hypersurfaces in spheres with $g = 3$ or $g = 4$ distinct principal curvatures automatically lead in this fashion to graphs that are rank 2 buildings. This is clear for $g = 3$ from Cartan's classification or from [KK95]. For $g = 4$ this is clear from a recent result of Immervoll, see [Imm01]. For $g = 6$ no such result is known. But the only two known examples of isoparametric hypersurfaces in spheres with $g = 6$ distinct principal curvatures actually lead to rank 2 buildings.

Rank 2 buildings are also called *generalized polygons* or *polygons* for short. Those described above are, more precisely, *generalized g-gons*. Generalized polygons are often described as point-line geometries, i.e. as a set of *points* $\mathcal{P}$ and a set of *lines* $\mathcal{L}$ together with a relation $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$, called the *flag set*, that tells whether a line goes through a point or not. A "nice" topology on $\mathcal{P}$ and on $\mathcal{L}$ with the property that intersecting lines and joining points are continuous maps makes it into a *topological polygon*. For a definition of generalized polygons and more information, e.g. what we mean by "nice" topologies, the interested reader is referred to [GK90], [Kra94] or [VM98].

A topological polygon is called *compact* or *connected* if its point and line space is compact or connected, respectively. Note that for a compact connected polygon with finite dimensional point and line space we get a similar topological situation as for an isoparametric hypersurface in a sphere. Consider the projection maps $\mathcal{F} \longrightarrow \mathcal{P}$ and $\mathcal{F} \longrightarrow \mathcal{L}$ that arise from the inclusion $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$. They turn out to be fiber bundles with homotopy spheres as fibers. The fibers are the point rows and the line pencils of the geometry. Their dimensions (m_1, m_2) are called the *topological parameters* of the polygon. The corresponding double mapping cylinder is a homotopy sphere and thus provides a decomposition of the homotopy sphere similar to the decomposition of the ambient sphere of an isoparametric hypersurface. Münzner's theorem, which allows to calculate the cohomology of an isoparametric hypersurface in a sphere and of its focal manifolds, also applies to this more general situation. We get the following theorems in analogy to theorems 4.1 and 4.2.

Theorem 4.12 *Let $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ be the point set, line set and flag set, respectively, of a compact connected generalized quadrangle with (finite) topological parameters (m_1, m_2), $m_1, m_2 \geq 2$.*

If $m_1 \neq m_2$ then $m_1 + m_2$ is odd and

$$\begin{aligned} H^*(\mathcal{P}) &\cong H^*(\mathrm{S}^{m_1} \times \mathrm{S}^{m_1+m_2}), \\ H^*(\mathcal{L}) &\cong H^*(\mathrm{S}^{m_2} \times \mathrm{S}^{m_1+m_2}), \\ H^*(\mathcal{F}) &\cong H^*(\mathrm{S}^{m_1} \times \mathrm{S}^{m_2} \times \mathrm{S}^{m_1+m_2}). \end{aligned}$$

The spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are simply connected.

Proof. See [Mün80], [Mün81], [Kra94] and [Str96]. □

Theorem 4.13 *Let $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ be the point set, line set and flag set, respectively, of a compact connected generalized g-gon, $g \geq 3$, with (finite) topological parameters (d, d), $d \geq 2$. Then $H^*(\mathcal{P}; \mathbb{Q}) \cong H^*(\mathcal{L}; \mathbb{Q}) \cong \mathbb{Q}[a] \,/\, (a^g)$ with* $\deg a = d$. *Moreover, one has*

$$\begin{aligned} &\textit{either } g = 3 \textit{ and } d \in \{2, 4, 8\}, \\ &\textit{or } g = 4 \textit{ and } d \in \{2, 4\}, \\ &\textit{or } g = 6 \textit{ and } d \in \{2, 4\}. \end{aligned}$$

The spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are simply connected.

Proof. See [Mün80], [Mün81], [Kra94] and [Str96]. □

The (topological) *automorphism group* of a topological polygon is the group of all collineations of the polygon that are homeomorphisms. A topological polygon is called *flag homogeneous* if its automorphism group is transitive on the flag set $\mathcal{F}$. Analogously, a topological polygon is called *point homogeneous* if its automorphism group is transitive on the point set $\mathcal{P}$. Flag homogeneous compact connected polygons have been classified in [GKK95] and [GKK00]. As a natural generalization arises the problem to classify all point homogeneous compact connected polygons.

Theorem 7.18 in [Kra98] allows to apply the results of chapters 2 and 3 to this situation: It assures that the automorphism group of the point homogeneous compact connected polygon is a Lie group and that there exists a compact connected Lie subgroup of this automorphism group acting transitively on the point space, provided the point space is simply connected. Moreover, the theorem gives that such polygons have finite dimensional point spaces. Then also the line spaces have finite dimension by [Kra94, proposition 3.1.3]. We get the following corollaries of the above theorems 4.12 and 4.13.

Corollary 4.14 *Let $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ be the point set, line set and flag set, respectively, of a point homogeneous compact connected generalized quadrangle with (finite) topological parameters $(2, m)$, $m \geq 3$ odd. Assume that* $\dim \mathcal{P} \leq \dim \mathcal{L}$. *Let G be a compact connected Lie subgroup of the automorphism group of the geometry acting transitively on $\mathcal{P}$ without transitive proper normal closed subgroup. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then (G, H) is one of the pairs in theorem 2.14.

Corollary 4.15 *Let $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ be the point set, line set and flag set, respectively, of a point homogeneous compact connected generalized g-gon, $g \geq 3$, with (finite) topological parameters (d, d), $d \geq 2$. Let G be a compact connected Lie subgroup of the automorphism group of the geometry acting transitively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.*

Then (G, H) is one of the pairs in theorem 3.7.

The point homogeneous compact connected polygons with equal topological parameters as in corollary 4.15 have been classified in [Kra94]. In order to obtain a list of homogeneous spaces as candidates for the point space from the knowledge of the cohomology ring of the polygon Kramer uses a different technique.

For point homogeneous polygons as in corollary 4.14 no classification is yet known. But at least this corollary provides a concrete list of all candidates for the point space and the corresponding possible group actions. The recovery of the geometry, i.e. of $\mathcal{F}$ and of $\mathcal{L}$, from the point space is much more difficult than in the case of a focal manifold of an isoparametric hypersurface. A priori there is no ambient sphere for the polygon, i.e. representation theory does not help. Moreover the point space alone does not at all determine the geometrical structure of the polygon.

4.4 Classical examples

The isoparametric hypersurfaces that occur in our classification are all well known and deserve to be called classical. Those with $g = 3$ distinct principal curvatures are the classical projective planes over the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. The corresponding point-line geometries and their Veronese embeddings into spheres as isoparametric hypersurfaces are described in [KK95].

The classical examples of isoparametric hypersurfaces with $g = 4$ distinct principal curvatures can be constructed from Clifford algebras and are therefore sometimes called *Clifford hypersurfaces*, cf. [FKM81]. Those with multiplicities $(2, m)$ can also be described as follows. Consider the complex vector space $V = \mathbb{C}^{n\times 2}$ of all $(n \times 2)$-matrices with entries in $\mathbb{C}$. The group $G = \mathrm{SU}(n)\times\mathrm{SU}(2)$, $n \geq 2$, acts on V by $(A, B)\cdot X = AXB^{-1}$ for $A \in \mathrm{SU}(n)$, $B \in \mathrm{SU}(2)$ and $X \in V$. The G-orbits on the unit sphere $\mathrm{S}^{4n-1} \subseteq V$ form an isoparametric foliation. The principal orbits are $(2, 2n-3)$-hypersurfaces, i.e. $m = 2n - 3$. The orbits

$$\mathcal{P} = G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \text{ and } \mathcal{L} = G \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

are their focal manifolds.

Our aim in the next chapter is to show that these are the only $(2, m)$-hypersurfaces (up to isometries of the ambient sphere S^{4n-1}) whose lower dimensional focal manifold is homogeneous, and that for n odd the described group action is the only almost effective one without transitive proper normal closed subgroup. For n even, the subgroup $\mathrm{Sp}(\frac{n}{2})\times\mathrm{SU}(2)$ still acts transitively on $\mathcal{P}$. This is the only other possible almost effective group action without transitive proper normal closed subgroup.

Similarly, one constructs isoparametric hypersurfaces with $g = 4$ distinct principal curvatures and multiplicities $(4, m)$. Consider the quaternionic vector space $V = \mathbb{H}^{n\times 2}$ of all $(n \times 2)$-matrices with entries in $\mathbb{H}$, the group $\mathrm{Sp}(n) \times \mathrm{SU}(2)$ and the G-action $(A, B) \cdot X = AXB^{-1}$ for $A \in \mathrm{Sp}(n)$, $B \in \mathrm{SU}(2)$ and $X \in V$. Again the G-orbits on the unit sphere $\mathrm{S}^{8n-1} \subseteq V$ form an isoparametric foliation. This time, the principal orbits are $(4, 4n-5)$-hypersurfaces with focal manifolds

$$\widehat{\mathcal{P}} = G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \text{ and } \widehat{\mathcal{L}} = G \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

The point-line geometries corresponding to the above examples of isoparametric hypersurfaces can be described algebraically, as polar spaces in $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively, see e.g. [Wol98]. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$, and denote by $\overline{\cdot}$ the canonical field (anti-) automorphism of $\mathbb{F}$. Consider the vector space $V = \mathbb{F}^{n+2}$ together with the hermitian form f, defined by

$$f(x, y) = -x_1\overline{y_1} - x_2\overline{y_2} + \sum_{k=3}^{n+2} x_k\overline{y_k}.$$

Call those subspaces of V on which f vanishes *totally isotropic*. As f has Witt index 2, the non-trivial totally isotropic subspaces of V have dimension 1 or 2. Let V_1 be the set of those of dimension 1 and V_2 the set of those of dimension 2. Let $V_{1,2} \subseteq V_1 \times V_2$ be the set of flags (p, l) with $p \subseteq l$. Then the geometry $(V_1, V_2, V_{1,2})$ is isomorphic to the generalized quadrangle belonging to the isoparametric hypersurface described above with multiplicities $(2, 2n-3)$ for $\mathbb{F} = \mathbb{C}$, or $(4, 4n-5)$ for $\mathbb{F} = \mathbb{H}$; an isomorphism is given by the following maps: An element of $\mathcal{P}$ or $\widehat{\mathcal{P}}$, respectively, can be written as (wc, ws) with $w \in \mathbb{F}^n$, $c \in \mathbb{R}$, $s \in \mathbb{F}$ and $|c|^2 + |s|^2 = \|w\|^2 = 1$, and is mapped onto $(c, \overline{s}, w)\mathbb{F} \in V_1$; an element (u, v) of $\mathcal{L}$ or $\widehat{\mathcal{L}}$, respectively, is mapped onto $(1, 0, \sqrt{2} \cdot u)\mathbb{F} + (0, 1, \sqrt{2} \cdot v)\mathbb{F} \in V_2$. These two maps are homeomorphisms and provide an isomorphism between the geometries.

For a more detailed description of these isoparametric hypersurfaces and the corresponding generalized quadrangles see the series of papers [KVM99], [Kra00a] and [Kra00b].

Chapter 5

Homogeneous focal manifolds

In this chapter we prove the classification results stated in theorems 4.8 and 4.9 about homogeneous focal manifolds of certain isoparametric hypersurfaces in spheres. We use their topological and geometric properties explained in the preceding chapter, in particular their cohomology. It is the starting point in both cases. The classification of simply connected homogeneous spaces with this cohomology in chapter 2 and 3 gives in each case a complete list of the possibilities for the homogeneous focal manifold and the group acting on it.

In the case in which the homogeneous focal manifold has Euler-Poincaré characteristic zero, we divide the proof (of theorem 4.8) into three sections. The transitive group G turns out to be a product of an almost simple compact connected Lie group with SU(2). In the first section we determine this first factor and its action on the focal manifold. The next section is devoted to the SU(2) factor. Since this group has many representations of small dimensions, representation theory alone cannot determine the right action of this factor. We have to calculate explicitly these representations and to use some geometry.

Having determined the group actions on the sphere that lead to isoparametric foliations of the desired form we are left with two such group actions. One of these group actions has isoparametric hypersurfaces as principal orbits and thus determines uniquely the induced isoparametric foliation. The other group action has many orbits of small dimensions as candidates for the focal manifold of an isoparametric foliation. In the third section we show that only one of these orbits actually is the focal manifold of an isoparametric foliation, and this completes the proof of theorem 4.8.

The last section deals with homogeneous focal manifolds of positive Euler-Poincaré characteristic. This case is less complicated for two reasons: The transitive group is simple, and in all cases in which it has an orbit of the right dimension on the sphere, the orbits form an isoparametric foliation, i.e. the group action uniquely determines the geometry.

5.1 Group actions I: the first factor

We start the proof of theorem 4.8. Recall the assumptions of this theorem:

Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures of multiplicities $(2, m)$, $m \geq 3$ odd. Let $\mathcal{P}$ be its lower dimensional focal manifold. Let G be a compact connected Lie subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup.

By corollary 4.4, the pair (G, H) is one of the pairs in theorem 2.14. We parse through the list given in theorem 2.14 and exclude all group actions up to the classical one described in section 4.4. In this section 5.1 we exclude the cases where G is of type $\mathfrak{a}_2$ or $\mathfrak{b}_2$, i.e. the cases where G is almost simple. This shows that G is a product $G_1 \times \mathrm{SU}(2)$. We determine the first factor G_1 to be either $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$.

First we rule out the almost simple groups G in the list of theorem 2.14.

Proposition 5.1 *The homogeneous space* $\mathrm{SU}(3) \,/\, \mathrm{U}(1)$ *does not occur as a focal manifold of a* $(2, 3)$*-hypersurface.*

Proof. Suppose that $\mathcal{P} = \mathrm{SU}(3) \,/\, \mathrm{U}(1)$ occurs as a focal manifold of a $(2, 3)$-hypersurface. By lemma 4.6, the space $\mathcal{P}$ would be an orbit of a 12-dimensional representation of $\mathrm{SU}(3)$. By lemma 4.7 this $\mathrm{SU}(3)$-module does not have a trivial factor. By [Kra98, 4.10] this module is either $\mathbb{C}^3 \oplus \mathbb{C}^3$ or $S^2\mathbb{C}^3$.

In the former case, the $\mathrm{SU}(3)$-orbit of an element (x, y) in the unit sphere $\mathrm{S}^{11} \subseteq \mathbb{C}^3 \oplus \mathbb{C}^3$ is either a 5-sphere or an (8-dimensional) Stiefel manifold $\mathrm{V}_2(\mathbb{C}^3)$ depending on whether x, y are linearly dependent or not. But the space $\mathcal{P}$ has dimension 7. This excludes the case $\mathbb{C}^3 \oplus \mathbb{C}^3$.

In the latter case $S^2\mathbb{C}^3$, the group $\mathrm{SU}(3)$ acts on the space $S^2\mathbb{C}^3 = \{\, A \in \mathbb{C}(3) \mid A^{tr} = A\}$ by $\gamma \cdot A = \gamma A \gamma^{tr}$ for $\gamma \in \mathrm{SU}(3)$. By theorem 2.14 we may suppose that the subgroup $\mathrm{U}(1)$ is the subgroup

$$i_{1,0}(\mathrm{U}(1)) = \left\{ \begin{pmatrix} z & & \\ & z^{-1} & \\ & & 1 \end{pmatrix} \,\middle|\, z \in \mathrm{U}(1) \right\}.$$

By assumption this subgroup is the stabilizer of some element in $S^2\mathbb{C}^3$. An element A in $S^2\mathbb{C}^3$ that is fixed by each element of $i_{1,0}(\mathrm{U}(1))$ is of the form

$A = \left(\begin{smallmatrix} 0 & x & \\ x & 0 & \\ & & y \end{smallmatrix}\right)$ for suitable $x, y \in \mathbb{C}$. But then the matrix $\left(\begin{smallmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{smallmatrix}\right) \notin i_{1,0}(\mathrm{U}(1))$ is also in the stabilizer of A, a contradiction. □

Proposition 5.2 *The homogeneous space* $\mathrm{Sp}(2) / \mathrm{U}(1)$ *does not occur as a focal manifold of a* $(2, 5)$*-hypersurface.*

Proof. Suppose that $\mathcal{P} = \mathrm{Sp}(2) / \mathrm{U}(1)$ occurs as a focal manifold of a $(2, 5)$-hypersurface. By lemma 4.6 the space $\mathcal{P}$ would be an orbit of a 16-dimensional representation of $\mathrm{Sp}(2)$. By lemma 4.7 this $\mathrm{Sp}(2)$-module does not have a trivial factor. By [Kra98, 4.12] this module is $\mathbb{H}^2 \oplus \mathbb{H}^2$.

The $\mathrm{Sp}(2)$-orbit of an element (x, y) in the unit sphere $\mathrm{S}^{15} \subseteq \mathbb{H}^2 \oplus \mathbb{H}^2$ is either a 7-sphere or a (10-dimensional) Stiefel manifold $\mathrm{V}_2(\mathbb{H}^2)$ depending on whether x, y are linearly dependent or not. But the space $\mathcal{P} = \mathrm{Sp}(2) / \mathrm{U}(1)$ has dimension 9. Hence it cannot be an orbit of this 16-dimensional representation of $\mathrm{Sp}(2)$. □

In the remaining G-actions in the list of theorem 2.14 the group G is an almost direct product $G_1 \cdot \mathrm{SU}(2)$ with G_1 of type $\mathfrak{a}_n$, $n \geq 2$, $\mathfrak{b}_2$ or $\mathfrak{c}_n$, $n \geq 3$. Recall that $\mathcal{P} = G / H$ is a focal manifold of an isoparametric hypersurface in a sphere with $g = 4$ distinct principal curvatures of multiplicities $(2, m)$ with $m = 2n + 1$, $m = 7$ or $m = 4n - 1$ respectively, and that G acts transitively and almost effectively on $\mathcal{P}$ without transitive proper normal closed subgroup. By lemma 4.6 and lemma 4.7 the space $\mathcal{P}$ is an orbit of a $(2m + 6)$-dimensional G-module M without trivial factors. In particular, this is a non-trivial G_1-module. Since $1 \times \mathrm{SU}(2)$ centralizes G_1 in G, we get a map $1 \times \mathrm{SU}(2) \longrightarrow \mathrm{End}_{G_1}(M)$. This map cannot be trivial, because G does not have a proper normal closed subgroup that acts transitive on $\mathcal{P}$, i.e. on one of the orbits in the unit sphere in M. Since $1 \times \mathrm{SU}(2)$ is almost simple, the map has finite kernel. We now search for the G-modules with all these properties.

Every G_1-Module is a sum of simple G_1-modules, see [BtD95, proposition II.1.9]. So, we first list all simple G_1-modules whose dimension is small enough to fit in as a summand. There is some notation to explain: For a vector space V we denote by $S^k V$ its k-th symmetric power, and by $\Lambda^k V$, its k-th exterior power. Moreover, the symbol $(S^2\mathbb{R}^3)^{\mathrm{trl.}}$ denotes the symmetric, traceless real (3×3)-matrices. Finally, some simple modules are denoted by ${}^{\mathbb{R}}\rho_{r\lambda_s}$ with $r, s \in \mathbb{N}$. For a definition of these symbols see [Kra98, chapter 4]. Only one of these modules will be discussed in detail in the next section, namely the $\mathrm{SU}(2)$-module ${}^{\mathbb{R}}\rho_{3\lambda_1}$. We denote by this same symbol the module and the corresponding representation homomorphism.

Proposition 5.3 *Let K be a compact connected Lie group of type $\mathfrak{a}_1$. The simple K-modules M of dimension at most 8 are exactly the following.*

K	M	$\dim_{\mathbb{R}} M$	$\mathrm{End}_K(M)$
SO(3)	$\mathbb{R}^3$	3	$\mathbb{R}$
SO(3)	$(S^2\mathbb{R}^3)^{trl.}$	5	$\mathbb{R}$
SO(3)	${}^{\mathbb{R}}\rho_{6\lambda_1}$	7	$\mathbb{R}$
Sp(1)	$\mathbb{H}^1$	4	$\mathbb{H}$
Sp(1)	${}^{\mathbb{R}}\rho_{3\lambda_1}$	8	$\mathbb{H}$

Proof. See [Kra98, 4.8]. □

Proposition 5.4 *Let K be a compact connected Lie group of type $\mathfrak{a}_n$, $n \geq 2$. The simple K-modules M of dimension at most $4n + 4$ are exactly the following.*

K	M	$\dim_{\mathbb{R}} M$	$\mathrm{End}_K(M)$	n
SU($n+1$)	$\mathbb{C}^{n+1}$	$2n+2$	$\mathbb{C}$	$n \geq 2$
SU(3)	$S^2\mathbb{C}^3$	12	$\mathbb{C}$	2
PSU(3)	$\mathfrak{su}_2\mathbb{C}$	8	$\mathbb{C}$	2
SO(6)	$\mathbb{R}^6$	6	$\mathbb{R}$	3
PSU(4)	$\mathfrak{su}_4\mathbb{C}$	15	$\mathbb{C}$	3
SU(5)	$\Lambda^2\mathbb{C}^6$	20	$\mathbb{C}$	4

Proof. See [Kra98, 4.10]. □

Proposition 5.5 *Let K be a compact connected Lie group of type $\mathfrak{b}_2$ or $\mathfrak{c}_n$, $n \geq 3$. The simple K-modules M of dimension at most 16 and $8n$, respectively, are exactly the following.*

K	M	$\dim_{\mathbb{R}} M$	$\mathrm{End}_K(M)$	n
Sp(n)	$\mathbb{H}^n$	$4n$	$\mathbb{H}$	$n \geq 2$
SO(5)	$\mathfrak{so}_5$	10	$\mathbb{R}$	2
SO(5)	${}^{\mathbb{R}}\rho_{2\lambda_1}$	14	$\mathbb{R}$	2
PSp(3)	${}^{\mathbb{R}}\rho_{\lambda_2}$	14	$\mathbb{R}$	3
PSp(3)	$\mathfrak{sp}_3$	21	$\mathbb{R}$	3
PSp(4)	${}^{\mathbb{R}}\rho_{\lambda_2}$	27	$\mathbb{R}$	4

Proof. See [Kra98, 4.12]. □

Now we sum up simple G_1-modules and build all G_1-modules of the right dimension.

Proposition 5.6 *Let G_1 be a compact connected Lie group of type $\mathfrak{a}_n$, $n \geq 2$. The G_1-modules M of dimension equal to $4n+4$ are exactly the following.*

M	$\mathrm{End}_{G_1}(M)$	n	*Remarks*
$\mathbb{C}^{n+1} \oplus (2n+2)\underline{\mathbb{R}}$	$\mathbb{C} \oplus \mathbb{R}(2n+2)$	$n \geq 2$	P
$2\mathbb{C}^{n+1}$	$\mathbb{C}(2)$	$n \geq 2$	—
$S^2\mathbb{C}^3$	$\mathbb{C}$	2	T
$\mathfrak{sl}_3\mathbb{C} \oplus 4\underline{\mathbb{R}}$	$\mathbb{C} \oplus \mathbb{R}(4)$	2	P
$\mathbb{C}^4 \oplus \mathbb{R}^6 \oplus 2\underline{\mathbb{R}}$	$\mathbb{C} \oplus \mathbb{R} \oplus \mathbb{R}(2)$	3	T
$\mathbb{R}^6 \oplus 10\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(10)$	3	P
$2\mathbb{R}^6 \oplus 4\underline{\mathbb{R}}$	$\mathbb{R}(2) \oplus \mathbb{R}(4)$	3	P
$\mathfrak{su}_4\mathbb{C} \oplus \underline{\mathbb{R}}$	$\mathbb{C} \oplus \mathbb{R}$	3	T
$\Lambda^2\mathbb{C}^6$	$\mathbb{C}$	4	T

Proposition 5.7 *Let G_1 be a compact connected Lie group of type $\mathfrak{b}_2$ or $\mathfrak{c}_n$, $n \geq 3$. The G_1-modules M of dimension equal to 16 and $8n$, respectively, are exactly the following.*

M	$\mathrm{End}_{G_1}(M)$	n	*Remarks*
$\mathbb{H}^n \oplus 4n\underline{\mathbb{R}}$	$\mathbb{H} \oplus \mathbb{R}(4n)$	$n \geq 2$	—
$2\mathbb{H}^n$	$\mathbb{H}(2)$	$n \geq 2$	—
$\mathfrak{so}_5 \oplus 6\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(6)$	2	P
${}^{\mathbb{R}}\rho_{2\lambda_1} \oplus 2\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(2)$	2	T
${}^{\mathbb{R}}\rho_{\lambda_2} \oplus 10\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(10)$	3	P
$\mathfrak{sp}_3 \oplus 3\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(3)$	3	T
${}^{\mathbb{R}}\rho_{\lambda_2} \oplus 5\underline{\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}(5)$	4	P

The G_1-modules that are marked with "P" only admit a non-trivial $(1 \times \mathrm{SU}(2))$-action on a trivial G_1-submodule. The corresponding G-orbits with an G-action without transitive proper normal closed subgroup are thus products of G_1-orbits and $(1 \times \mathrm{SU}(2))$-orbits. The G_1-modules marked with "T" do not admit a $(1 \times \mathrm{SU}(2))$-action at all. Hence we can exclude all G_1-modules that are marked with "T". To exclude also those marked with "P" we need the following lemma 5.8.

Lemma 5.8 *If $\mathcal{P}$ is a product, then one of the factors is S^3 with the regular* $\mathrm{SU}(2)$*-action.*

Proof. See [Kra98, lemma 8.11]. □

With a glance at the cohomology of $\mathcal{P}$, we can therefore also exclude all G_1-modules in propositions 5.6 and 5.7 that are marked with "P". The following cases are left.

G	G_1-module M	$\mathrm{End}_{G_1}(M)$	n	m
$\mathrm{SU}(n)\times\mathrm{SU}(2)$	$2\mathbb{C}^n$	$\mathbb{C}(2)$	$n\geq 2$	$2n-3$
$\mathrm{Sp}(n)\times\mathrm{SU}(2)$	$\mathbb{H}^n\oplus 4n\underline{\mathbb{R}}$	$\mathbb{H}\oplus\mathbb{R}(4n)$	$n\geq 1$	$4n-3$
$\mathrm{Sp}(n)\times\mathrm{SU}(2)$	$2\mathbb{H}^n$	$\mathbb{H}(2)$	$n\geq 1$	$4n-3$

We now show that the case $G=\mathrm{Sp}(n)\times\mathrm{SU}(2)$, $M=\mathbb{H}^n\oplus 4n\underline{\mathbb{R}}$ also leads to G-orbits that are products. This is clear if the second factor $1\times\mathrm{SU}(2)$ of G acts trivially on the G_1-submodule $\mathbb{H}^n$. Otherwise, it acts on this submodule as $\mathrm{Sp}(1)$ by scalar multiplication from the right. The G_1-orbits are spheres S^{4n-1} in $\mathbb{H}^n$. Since scalar multiplication with elements of $\mathrm{Sp}(1)$, i.e. with quaternions of norm 1, permutes these spheres, the G-orbits on which G acts without transitive proper normal closed subgroup are products with S^{4n-1} as a factor. Now, the cohomology of one of these orbits should be that of $\mathrm{S}^2\times\mathrm{S}^{4n-1}$, and lemma 5.8 yields $n=1$. But then the only G-orbits in the unit sphere of $\mathbb{H}^2$ are homeomorphic to S^3 or to $\mathrm{S}^3\times\mathrm{S}^3$. So, we can exclude the possibility that $G=\mathrm{Sp}(n)\times\mathrm{SU}(2)$ and $M=\mathbb{H}^n\oplus 4n\underline{\mathbb{R}}$.

5.2 Group actions II: the spherical factor

The two remaining cases of the above table include the classical examples, cf. section 4.4: Consider complex $(n\times 2)$-matrices and complex $(2n\times 2)$-matrices as pairs of vectors. Then the matrix multiplication with $G_1=\mathrm{SU}(n)$ and $G_1=\mathrm{Sp}(n)$, respectively, from the left corresponds to the given G_1-representation M. The matrix multiplication with $\mathrm{SU}(2)$ from the right corresponds to the standard embedding of $\mathrm{SU}(2)$ in $\mathrm{End}_{G_1}(M)=\mathbb{C}(2)$ and $\mathrm{End}_{G_1}(M)=\mathbb{H}(2)$, respectively, as (2×2)-matrices.

For $G=\mathrm{SU}(n)\times\mathrm{SU}(2)$, there is (up to conjugation) no other possibility to embed $\mathrm{SU}(2)$ in $\mathrm{End}_{G_1}(M)=\mathbb{C}(2)$, since $\mathrm{SU}(2)$ has only one non-trivial 2-dimensional complex representation by proposition 5.3. The G-orbits in the unit sphere $\mathrm{S}^{8n-1}\subseteq M$ form an isoparametric foliation. Thus there is only one possibility for a $(2,2n-3)$-hypersurface with an $(\mathrm{SU}(n)\times\mathrm{SU}(2))$-action without transitive proper normal closed subgroup on its lower dimensional focal manifold (up to isometries of the ambient sphere).

For $G=\mathrm{Sp}(n)\times\mathrm{SU}(2)$, the situation is more complicated. By proposition 5.3, the group $\mathrm{SU}(2)$ has two quaternionic representations of dimension at most 2, namely $\mathbb{H}^1$ and ${}^{\mathbb{R}}\rho_{3\lambda_1}$. This gives three essentially different embeddings of $\mathrm{SU}(2)$ in $\mathbb{H}(2)$ corresponding to the 2-dimensional quaternionic $\mathrm{SU}(2)$-modules $\mathbb{H}^1\oplus\mathbb{H}^1$, $\mathbb{H}^1\oplus 4\underline{\mathbb{R}}$ and ${}^{\mathbb{R}}\rho_{3\lambda_1}$. The first one belongs to the classical example. We have to exclude the other two embeddings.

First consider the case corresponding to the SU(2)-module $\mathbb{H}^1 \oplus 4\underline{\mathbb{R}}$. This module belongs to the standard embedding of SU(2) as Sp(1) in $\mathbb{H}(2)$. The group $G = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ acts on $M = \mathbb{H}^n \oplus \mathbb{H}^n$ by $(A, h) \cdot (x, y) = (Axh^{-1}, Ay)$. The G-orbits of $(x, 0)$ and $(0, y)$ for $x, y \neq 0$ are spheres S^{4n-1}. The G-orbits of (x, y) are ($(8n-6)$-dimensional) Stiefel manifolds $\mathrm{V}_2(\mathbb{H}^n)$ for linearly independent x, y. The group G acts transitively on the linearly dependent pairs (x, y), $x, y \neq 0$, in the unit sphere $\mathrm{S}^{8n-1} \subseteq M$; so they form the only remaining G-orbit. This last G-orbit is the only candidate for $\mathcal{P}$, because of $\dim \mathcal{P} = m + 2 = 4n + 1$. But it does not have $\mathrm{Sp}(n-1) \cdot \mathrm{U}(1)$ as point stabilizer: The stabilizer of the point $((\frac{1}{\sqrt{2}}, 0, \dots, 0)^{tr}, (\frac{1}{\sqrt{2}}, 0, \dots, 0)^{tr})$ is $\mathrm{Sp}(n-1) \times 1$. This excludes the case $\mathbb{H}^1 \oplus 4\underline{\mathbb{R}}$.

Now consider the remaining undesired case, which corresponds to the SU(2)-representation ${}^{\mathbb{R}}\rho_{3\lambda_1}$. Excluding this case will take the rest of this section. The group $G = \mathrm{Sp}(n) \times \mathrm{SU}(2)$ is a subgroup of the group $\widehat{G} = \mathrm{Sp}(n) \times \mathrm{Sp}(2)$ with the obvious action on $\mathbb{H}^{n\times 2}$ by matrix multiplication from the left in the first factor and from the right in the second factor. The inclusion $G \longrightarrow \widehat{G}$ is given by $(A, B) \mapsto (A, {}^{\mathbb{R}}\rho_{3\lambda_1}(B))$. By [SBG+95, lemma 73.17] the map ${}^{\mathbb{R}}\rho_{3\lambda_1}$ is given by

$$ {}^{\mathbb{R}}\rho_{3\lambda_1} \colon \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mapsto \begin{pmatrix} a^3 + jb^3 & 3(a\overline{b}^2 + j\overline{a}^2 b) \\ ab^2 + ja^2 b & (|a|^2 - 2|b|^2)\overline{a} + j(|b|^2 - 2|a|^2)\overline{b} \end{pmatrix} $$

for $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. Here we write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ with $j^2 = -1$ and $ij = -ji$.

The orbits of the $\widehat{G}$-action on $\mathbb{H}^{n\times 2}$ form an isoparametric foliation of the unit sphere $\mathrm{S}^{8n-1} \subseteq \mathbb{H}^{n\times 2}$, which belongs to the classical $(4, 4n-5)$-hypersurface, cf. section 4.4. Its lower dimensional focal manifold is the orbit

$$ \widehat{\mathcal{P}} = \widehat{G} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \left\{ (wc, ws) \mid w \in \mathbb{H}^n, \|w\| = 1, c, s \in \mathbb{H}, |c|^2 + |s|^2 = 1 \right\}. $$

It contains all pairs $(x, y) \in \mathrm{S}^{8n-1}$ with linearly dependent x, y. Thus it has all $(4n+1)$-dimensional G-orbits $\mathcal{P}$ as submanifolds; for the orbits $G \cdot (x, y)$, $(x, y) \in \mathrm{S}^{8n-1}$, with linearly independent x, y contain as a subspace the $(\mathrm{Sp}(n) \times 1)$-orbit $(\mathrm{Sp}(n) \times 1) \cdot (x, y) \cong \mathrm{V}_2(\mathbb{H}^n)$, and thus have dimension at least $\dim \mathrm{V}_2(\mathbb{H}^n) = 8n - 6 > 4n - 3$.

Define a map

$$ \varphi \colon \begin{cases} \widehat{\mathcal{P}} \longrightarrow \mathrm{S}^4 \subseteq \mathbb{R} \oplus \mathbb{H}, \\ (wc, ws) \mapsto (|c|^2 - |s|^2, 2\overline{c}s). \end{cases} $$

This is a $\widehat{G}$-equivariant map and a fiber bundle with typical fiber $F = \{x \in \mathbb{H}^n \mid \|x\| = 1\} \cong \mathrm{S}^{4n-1}$, cf. [Kra98, section 7.D]. By the inclusion $G \subseteq \widehat{G}$ we find the subbundle $\varphi_{|\mathcal{P}} : \mathcal{P} \longrightarrow \varphi(\mathcal{P}) \subseteq \mathrm{S}^4$ for each $(4n+1)$-dimensional G-orbit $\mathcal{P}$. It is again G-equivariant, and hence $\varphi(\mathcal{P})$ is a G-orbit on $\mathrm{S}^4 \subseteq \mathbb{R} \oplus \mathbb{H} \cong \mathbb{R}^5$. Its dimension is $\dim \varphi(\mathcal{P}) = \dim \mathcal{P} - \dim F = (4n+1)-(4n-1) = 2$. Consider the following part of the long exact sequence of homotopy groups of the fibration $\varphi_{|\mathcal{P}}$.

$$0 = \pi_1(F) \longrightarrow \pi_1(\mathcal{P}) \longrightarrow \pi_1(\varphi(\mathcal{P})) \longrightarrow \pi_0(F) = 0$$

It shows $\pi_1(\mathcal{P}) \cong \pi_1(\varphi(\mathcal{P}))$. In particular, each $(4n+1)$-dimensional G-orbit $\mathcal{P}$ on S^{8n-1} has the same fundamental group as a 2-dimensional G-orbit on S^4.

We now compute these fundamental groups. From the explicit formula for φ, we see that G acts linearly on S^4 and that the first factor $\mathrm{Sp}(n) \times 1$ of G acts trivially. So, the G-orbits on S^4 are orbits of a 5-dimensional $\mathrm{SU}(2)$-module. By proposition 5.3 there are exactly four such modules: $5\underline{\mathbb{R}}$, $\mathbb{R}^3 \oplus 2\underline{\mathbb{R}}$, $\mathbb{H}^1 \oplus \underline{\mathbb{R}}$, and $(S^2\mathbb{R}^3)^{\mathrm{trl.}}$. We can exclude $5\underline{\mathbb{R}}$ and $\mathbb{H}^1 \oplus \underline{\mathbb{R}}$, because there are no 2-dimensional orbits. The 2-dimensional orbits on $\mathbb{R}^3 \oplus 2\underline{\mathbb{R}}$ are spheres S^2. The orbits on $(S^2\mathbb{R}^3)^{\mathrm{trl.}}$ form an isoparametric foliation, which belongs to the Veronese embedding of the real projective plane, see [KK95]. Its 2-dimensional orbits are the two focal manifolds, which are both real projective planes $\mathbb{R}\mathrm{P}^2$. Hence, all $(4n+1)$-dimensional G-orbits $\mathcal{P}$ on S^{8n-1} have the same fundamental group $\pi_1(\mathcal{P}) \cong \pi_1(\mathrm{S}^2) = 0$ or $\pi_1(\mathcal{P}) \cong \pi_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2$.

Finally we compute $\pi_1(\mathcal{P})$ for one of these orbits, $\mathcal{P} = G \cdot X$ with $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$. The stabilizer G_X of X obviously contains the subgroups

$$U_1 = \left\{ \left(\left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right), \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \,\middle|\, A \in \mathrm{Sp}(n-1) \right\} \cong \mathrm{Sp}(n-1)$$

and

$$U_2 = \left\{ \left(\begin{pmatrix} \overline{z}^3 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} z^3 & \\ & \overline{z} \end{pmatrix} \right) \,\middle|\, z \in \mathrm{U}(1) \right\} \cong \mathrm{U}(1),$$

with $\begin{pmatrix} z^3 & \\ & \overline{z} \end{pmatrix} = {}^{\mathbb{R}}\rho_{3\lambda_1}(\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix})$, and it contains the element

$$u = \left(\begin{pmatrix} j & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -j & \\ & j \end{pmatrix} \right)$$

with $\begin{pmatrix} -j & \\ & j \end{pmatrix} = {}^{\mathbb{R}}\rho_{3\lambda_1}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})$. The two subgroups U_1 and U_2 generate a subgroup of G_X isomorphic to $\mathrm{Sp}(n-1) \times \mathrm{U}(1)$. Together with the element u, they generate a subgroup U isomorphic to $\mathrm{Sp}(n-1) \times (\mathrm{U}(1) \rtimes \mathbb{Z}/2)$. In fact we can show $G_X = U$; for each element g of G_X is of the form

$$g = \left(\left(\begin{array}{c|c} z & \\ \hline & A \end{array} \right), {}^{\mathbb{R}}\rho_{3\lambda_1}\left(\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right) \right)$$

with $z \in \mathrm{U}(1)$, $A \in \mathrm{Sp}(n-1)$, $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Moreover, ${}^{\mathbb{R}}\rho_{3\lambda_1}(\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix})$ is a diagonal matrix, whence $ab^2 + ja^2b = 0$. Since 1 and j are linearly independent over $\mathbb{C}$ and we have $a, b \in \mathbb{C}$, this implies $a = 0$ or $b = 0$ and thus $g \in U$.

Consider the fibration

$$G_X \longrightarrow G \longrightarrow \mathcal{P} = G \,/\, G_X$$

and the following part of the associated long exact sequence of homotopy groups.

$$\pi_1(G) \longrightarrow \pi_1(\mathcal{P}) \longrightarrow \pi_0(G_X) \longrightarrow \pi_0(G)$$

We have $\pi_1(G) = 0 = \pi_0(G)$ because G is simply connected. Hence we have $\pi_1(\mathcal{P}) \cong \pi_0(G_X) = \pi_0(\mathrm{Sp}(n-1) \times (\mathrm{U}(1) \rtimes \mathbb{Z}/2)) \cong \mathbb{Z}/2$.

As shown above, all $(4n+1)$-dimensional G-orbits in S^{8n-1} have the same fundamental group, and this group is $\mathbb{Z}/2$. So, none of these orbits is simply connected. This excludes the case ${}^{\mathbb{R}}\rho_{3\lambda_1}$.

5.3 The right orbit

We now have identified the action of $G = \mathrm{Sp}(n) \times \mathrm{SU}(2)$ on $\mathrm{S}^{8n-1} \subseteq \mathbb{H}^{n\times 2}$ that leads to an isoparametric foliation. The only possibility is the one that belongs to the classical example. The orbit $\mathcal{P} = G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ is the lower dimensional focal manifold of the classical $(2, 4n-3)$-hypersurface described in section 4.4. There are other small G-orbits on S^{8n-1}. We will show in this section that they cannot belong to $(2, 4n-3)$-hypersurfaces, i.e. that $\mathcal{P}$ is the only possibility. This will complete the proof of theorem 4.8 at the end of this section.

Consider the space $\widehat{\mathcal{P}} \subseteq \mathrm{S}^{8n-1}$ as on page 65. As we have shown in the last section, all G-orbits on S^{8n-1} of dimension at most $4n+1$ are contained in this space $\widehat{\mathcal{P}}$. Since the first factor $\mathrm{Sp}(n) \times 1$ of G acts transitively on each $(4n-1)$-sphere around 0 in $\mathbb{H}^n$, the small G-orbits on S^{8n-1} have the form $G \cdot \begin{pmatrix} x & y \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ with $x \in \mathbb{R}$, $y \in \mathbb{H}$ and $x^2 + |y|^2 = 1$. We now examine these small G-orbits.

Consider first the orbits $\mathcal{L}_c = G \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ with $c \in \mathbb{H}$, $|c| = 1$. In the stabilizer of the point $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \in \mathcal{L}_c$ there are contained all elements g of the form

$$g = \left(\left(\begin{array}{c|c} \overline{a} - \overline{b}j & \\ \hline & A \end{array} \right), \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}^{-1} \right)$$

with $A \in \mathrm{Sp}(n-1)$, $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$; for

$$\begin{aligned} g \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} &= \left(\begin{array}{c|c} \overline{a} - \overline{b}j & \\ \hline & A \end{array} \right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \overline{a} - \overline{b}cj & \overline{a}cj - \overline{b}cjcj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \\ &\overset{|c|=1}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} \overline{a} - \overline{b}cj & \overline{a}cj - \overline{b} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} |a|^2 - \bar{b}cja + \bar{a}cjb + |b|^2 & -\bar{a}\bar{b} + \bar{b}cj\bar{b} + \bar{a}cj\bar{a} + \bar{b}\bar{a} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} |a|^2 + |b|^2 & (|a|^2 + |b|^2)cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & cj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

Hence the stabilizer contains a subgroup isomorphic to $\mathrm{Sp}(n-1)\cdot\mathrm{SU}(2)$. This implies $\dim \mathcal{L}_c \leq \dim G - \dim(\mathrm{Sp}(n-1)\cdot\mathrm{SU}(2)) = 4n-1$. By lemma 4.7 the sets $\mathcal{L}_c$ are contained in the other (higher dimensional) focal manifold.

Now consider those orbits in $\widehat{\mathcal{P}}$ that are neither $\mathcal{P}$ nor $\mathcal{L}_c$ for any c. If one of these orbits is a focal manifold of a $(2, 4n-3)$-hypersurface $\mathcal{F}'$, then all $\mathcal{L}_c$ with $c \in \mathbb{H}$ and $|c| = 1$ are contained in the other focal manifold of this hypersurface $\mathcal{F}'$. Thus all points in this orbit would have a spherical distance of at least $\pi/4$ to all points of $\mathcal{L}_c$, see section 4.1. We show now that this is not the case. In order to show this, take a point of each such orbit, $\begin{pmatrix} x & y \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$, say, with $x \in \mathbb{R}$, $y \in \mathbb{H}$ and $x^2 + |y|^2 = 1$. Let $y = y_1 + y_2 j$ with $y_1, y_2 \in \mathbb{C}$. The fact that this orbit is not $\mathcal{P}$ implies $x \neq 0$ and $y_2 \neq 0$. Let $c = \frac{\overline{y_2}^{-1}x}{|y_2^{-1}x|} \in \mathbb{C}$, $h = \frac{x - ycj}{|x-ycj|} \in \mathbb{H}$, and let $l \in \mathcal{L}_c$ be the element $l = \frac{1}{\sqrt{2}}\begin{pmatrix} h & hcj \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$. We show $\sqrt{2}\,\langle p, l\rangle_{\mathbb{R}} > 1$. This is equivalent to p and l having a distance less than $\pi/4$. We calculate

$$\begin{aligned} \sqrt{2}\,\langle p, l\rangle_{\mathbb{R}} &= \langle (x,y),(h,hcj)\rangle_{\mathbb{R}} = \mathrm{Re}(x\bar{h} + y(\overline{hcj})) = \mathrm{Re}(x\bar{h} - yj\bar{c}\bar{h}) \\ &= \mathrm{Re}((x - yj\bar{c})\bar{h}) = \langle x - yj\bar{c}, \bar{h}\rangle_{\mathbb{R}} = \frac{|x-ycj|^2}{|x-ycj|} = |x - ycj| \\ &= |x - (y_1 + y_2 j)cj| = |x + y_2\bar{c} - y_1 cj| = \sqrt{|x + y_2\bar{c}|^2 + |y_1 c|^2} \\ &\overset{|c|=1}{=} \sqrt{(x + y_2\bar{c})(\overline{x + y_2\bar{c}}) + |y_1|^2} \\ &= \sqrt{x^2 + y_2\bar{c}x + x\overline{y_2}c + |y_2|^2 + |y_1|^2} \\ &= \sqrt{1 + 2|x|\,|y_2|} > 1, \text{ because } x, y_2 \neq 0. \end{aligned}$$

This shows that the orbit $\mathcal{P}$ belonging to the classical example is the only one that leads to an isoparametric foliation as desired. This completes the proof of the classification result theorem 4.8.

5.4 Focal manifolds of positive characteristic

In this last section we prove theorem 4.9. Recall the assumptions of this theorem:

Let $\mathcal{F}$ be an isoparametric hypersurface in a sphere with $g \geq 3$ distinct principal curvatures all of the same multiplicity $d \geq 2$. Let $\mathcal{P}$ and $\mathcal{L}$ be its focal manifolds. Let G be a compact connected Lie subgroup of the isometry group of $\mathcal{F}$ acting transitively and almost effectively on $\mathcal{P}$. Let $H = G_x$ be the stabilizer of a point x in $\mathcal{P}$.

Corollary 4.5 gives a list of all possible pairs (G, H) with these properties. By lemma 4.6 the space $\mathcal{P}$ is an orbit of a $(dg+2)$-dimensional representation of G. By lemma 4.7 this G-module does not have a trivial factor. We now parse through the list of corollary 4.5 and check each entry. Recall that $\dim \mathcal{P} = \dim \mathcal{L} = d(g-1)$ and $\dim \mathcal{F} = dg$. For a description of the occurring Veronese embeddings and the corresponding representations see [KK95] and [Kra00b].

Proof of theorem 4.9.

$G = \mathrm{SU}(3)$, $d = 2$, $g = 3$: The simple G-modules of dimension at most 8 are the natural representation on $\mathbb{C}^3$ and the (8-dimensional) adjoint representation on $\mathfrak{su}_3\mathbb{C}$, see [Kra98, 4.10]. So, the only 8-dimensional G-module without trivial factors is $\mathfrak{su}_3\mathbb{C}$. The Veronese embedding of the complex projective plane $\mathrm{PG}_2\,\mathbb{C}$ is an example of such an isoparametric hypersurface with $d = 2$ and $g = 3$, cf. [KK95]. So, the representation we found belongs to this hypersurface. Moreover, the group SU(3) acts transitively on the hypersurface. Therefore the focal manifolds of this special hypersurface are the only two G-orbits in the unit sphere $\mathrm{S}^7 \subseteq \mathfrak{su}_3\mathbb{C}$ of lower dimension. As a consequence they are the only two candidates for our focal manifolds $\mathcal{P}$ and $\mathcal{L}$. This shows the uniqueness of the embedding of the hypersurface in the sphere, because one focal manifold already completely determines the corresponding isoparametric foliation, see section 4.1. Note that both focal manifolds are homeomorphic to $\mathbb{C}\mathrm{P}^2$ and have U(2) as point stabilizer.

$G = \mathrm{SU}(4)$, $d = 2$, $g = 4$: The simple G-modules of dimension at most 10 are $\mathbb{R}^6$ and $\mathbb{C}^4$ where G acts naturally as SO(6) and as SU(4), respectively, see [Kra98, 4.10]. There is no way to sum up 6- and 8-dimensional representations to a 10-dimensional representation without trivial factors. Therefore the case $G = \mathrm{SU}(4)$, $d = 2$, $g = 4$ can be excluded.

$G = \mathrm{SU}(6)$, $d = 2$, $g = 6$: The only simple G-module of dimension at most 14 is the natural representation on $\mathbb{C}^6$, see [Kra98, 4.10]. It has dimension 12. Therefore the case $G = \mathrm{SU}(6)$, $d = 2$, $g = 6$ can be excluded.

$G = \mathrm{Sp}(2) \cong \mathrm{Spin}(5)$, $d = 2$, $g = 4$: The simple G-modules of dimension at most 10 are the natural representation as $\mathrm{Sp}(2)$ on $\mathbb{H}^2$, the natural representation as $\mathrm{SO}(5)$ on $\mathbb{R}^5$, and the (10-dimensional) adjoint representation on $\mathfrak{so}_5\mathbb{R}$. So, there are exactly two 10-dimensional G-modules without trivial factors: $\mathbb{R}^5 \oplus \mathbb{R}^5$ and $\mathfrak{so}_5\mathbb{R}$. In the former case, the $\mathrm{SO}(5)$-orbit of an element (x, y) in the unit sphere $\mathrm{S}^9 \subseteq \mathbb{R}^5 \oplus \mathbb{R}^5$ is either a 4-sphere or a (7-dimensional) Stiefel manifold $\mathrm{V}_2(\mathbb{R}^5)$ depending on whether x, y are linearly dependent or not. But the space $\mathcal{P}$ has dimension 6. Thus $\mathfrak{so}_5\mathbb{R}$ remains as the only possibility. The Veronese embedding of the complex symplectic quadrangle $\mathrm{W}(\mathbb{C})$ is an example of such an isoparametric hypersurface with $d = 2$ and $g = 4$. So, the representation we found belongs to this hypersurface. Moreover, the group $\mathrm{Sp}(2)$ acts transitively on the hypersurface. In analogy to the above case of $\mathrm{PG}_2\,\mathbb{C}$, the uniqueness of the embedding of the hypersurface follows. The focal manifolds are homeomorphic to $\mathbb{C}\mathrm{P}^3$ and $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5)$, respectively, with point stabilizers $\mathrm{Sp}(1) \cdot \mathrm{U}(1)$ and $\mathrm{SO}(3) \cdot \mathrm{SO}(2)$, respectively.

$G = \mathrm{Sp}(3)$, $d = 2$, $g = 6$ or $d = 4$, $g = 3$: The simple G-modules of dimension at most 14 are the natural representation on $\mathbb{H}^3$ and a 14-dimensional representation, see [Kra98, 4.14]. So, the latter one is the only 14-dimensional G-module without trivial factors. The Veronese embedding of the quaternionic projective plane $\mathrm{PG}_2\,\mathbb{H}$ is an example of such an isoparametric hypersurface with $d = 4$ and $g = 3$, cf. [KK95]. So, the representation we found belongs to this hypersurface. This shows that the case $G = \mathrm{Sp}(3)$, $d = 2$, $g = 6$ is not possible. The group $\mathrm{Sp}(3)$ acts transitively on the hypersurface belonging to the Veronese embedding of $\mathrm{PG}_2\,\mathbb{H}$. In analogy to the above case of $\mathrm{PG}_2\,\mathbb{C}$, the uniqueness of the embedding of the hypersurface follows. The focal manifolds are homeomorphic to $\mathbb{H}\mathrm{P}^2$ and have $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as point stabilizer.

$G = \mathrm{Spin}(7)$, $d = 2$, $g = 6$: The simple G-modules of dimension at most 14 are the natural representation as $\mathrm{SO}(7)$ on $\mathbb{R}^7$ and an 8-dimensional representation, see [Kra98, 4.12]. So, the only 14-dimensional G-module without trivial factors is $\mathbb{R}^7 \oplus \mathbb{R}^7$. The $\mathrm{SO}(7)$-orbit of an element (x, y) in the unit sphere $\mathrm{S}^{13} \subseteq \mathbb{R}^7 \oplus \mathbb{R}^7$ is either a 6-sphere or an (11-dimensional) Stiefel manifold $\mathrm{V}_2(\mathbb{R}^7)$ depending on whether x, y are linearly dependent or not. But the space $\mathcal{P}$ has dimension 10, which excludes the case $G = \mathrm{Spin}(7)$,

$d = 2$, $g = 6$.

$G = \mathrm{G}_2$, $d = 2$, $g = 6$: The simple G-modules of dimension at most 14 are the (14-dimensional) adjoint representation on $\mathfrak{g}_2$ and the 7-dimensional representation on $\mathrm{Pu}\,\mathbb{O}$, see [Kra98, 4.26]. Recall that the group G_2 acts on the $\mathbb{R}$-algebra $\mathbb{O}$ as the (full) automorphism group. This action is $\mathbb{R}$-linear and trivial on $\mathbb{R} \subseteq \mathbb{O}$. Hence, G_2 acts on a 7-dimensional complement of $\mathbb{R}$ in $\mathbb{O}$. This complement is $\mathrm{Pu}\,\mathbb{O}$. Note that G_2 is transitive on the unit sphere in $\mathrm{Pu}\,\mathbb{O}$. (The natural norm in $\mathbb{O}$ induces a Euclidean norm on $\mathrm{Pu}\,\mathbb{O}$, and hence we can speak of “the” unit sphere in $\mathrm{Pu}\,\mathbb{O}$.) So, there are two 14-dimensional G-modules without trivial factors: $\mathrm{Pu}\,\mathbb{O} \oplus \mathrm{Pu}\,\mathbb{O}$ and $\mathfrak{g}_2$. In the former case, the G_2-orbit of an element (x, y) in the unit sphere $\mathrm{S}^{13} \subseteq \mathbb{R}^7 \oplus \mathbb{R}^7$ is either a 6-sphere or an (11-dimensional) Stiefel manifold $\mathrm{V}_2(\mathrm{Pu}\,\mathbb{O})$ depending on whether x, y are linearly dependent or not. But the space $\mathcal{P}$ has dimension 10. Thus $\mathfrak{g}_2$ is the only 14-dimensional G-module without trivial factors. The Veronese embedding of the complex split Cayley hexagon is an example of such an isoparametric hypersurface with $d = 2$ and $g = 6$. So, the representation we found belongs to this hypersurface. The group G_2 acts transitively on the hypersurface. In analogy to the above case of $\mathrm{PG}_2\,\mathbb{C}$, the uniqueness of the embedding of the hypersurface follows. The focal manifolds are homeomorphic to $\mathrm{H}(\mathbb{C})$ and $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$, respectively, with point stabilizers that are both isomorphic to $\mathrm{Sp}(1) \cdot \mathrm{U}(1)$ but that are not conjugate to each other in G.

$G = \mathrm{Sp}(4)$, $d = 4$, $g = 6$: The only simple G-module of dimension at most 26 is the natural representation on $\mathbb{H}^4$, see [Kra98, 4.14]. It has dimension 16. This excludes the case $G = \mathrm{Sp}(4)$, $d = 4$, $g = 6$.

$G = \mathrm{Sp}(6)$, $d = 4$, $g = 6$: The only simple G-module of dimension at most 26 is the natural representation on $\mathbb{H}^6$, see [Kra98, 4.14]. It has dimension 24. This excludes the case $G = \mathrm{Sp}(6)$, $d = 4$, $g = 6$.

$G = \mathrm{F}_4$, $d = 8$, $g = 3$: The only simple G-module of dimension at most 26 is of dimension 26, see [Kra98, 4.24]. The Veronese embedding of the octonion projective plane $\mathrm{PG}_2\,\mathbb{O}$ is an example of such an isoparametric hypersurface with $d = 8$ and $g = 3$, cf. [KK95]. So, the representation we found belongs to this hypersurface. The group F_4 acts transitively on the hypersurface. In analogy to the above case of $\mathrm{PG}_2\,\mathbb{C}$, the uniqueness of the embedding of the hypersurface follows. The focal manifolds are homeomorphic to $\mathbb{O}\mathrm{P}^2$ and have $\mathrm{Spin}(9)$ as point stabilizer. □

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Notation

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bei der Fakultät für Mathematik und Informatik

1. Gutachter: PD Dr. Linus Kramer

2. Gutachter: Prof. Dr. Theo Grundhöfer

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