ON TESTING THE EXPONENTIAL AND GUMBEL DISTRIBUTION

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Abstract. Consider the statistical experiment $(\mathbb{R}, \mathcal{B}, \{H_\beta : \beta \in \mathbb{R}\})$, where H_β denotes the generalized Pareto distribution given by the von Mises parametrization and H_0 is the standard exponential distribution. We investigate the two-sided testing problem H_0 against H_β, $\beta \neq 0$. For that testing problem an asymptotically uniformly optimal test is established. As a main tool we show that the experiment is differentiable in quadratic mean at $\beta = 0$, which is a crucial condition in the asymptotic setting. A Monte-Carlo simulation visualizes the result. Moreover, we consider the extreme value distributions G_β, $\beta \in \mathbb{R}$, which are the most important ones in the neighborhood of the generalized Pareto distributions. We treat the testing problem Gumbel (G_0) against Fréchet (G_β, $\beta > 0$) and Weibull ($G_\beta, \beta < 0$). It turns out that the differentiability in quadratic mean carries over to the extreme value distributions.

1. Introduction and notation

Assume that a distribution function F on the real line is a member of the class of *generalized Pareto distributions* (GPDs) $H_\beta, \beta \in \mathbb{R}$, where

$$H_\beta(x) := 1 - (1 + \beta x)^{-1/\beta}, \quad x \in I_\beta \tag{1}$$

with support $I_\beta = \{x \in \mathbb{R} : 0 < (1 + \beta x)^{-1/\beta} \leq 1\}$. Interpret H_0 as $H_0(x) := \lim_{\beta \to 0} H_\beta(x) = 1 - \exp(-x)$. Explicitely we have $I_\beta = [0, \infty)$

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in the case $\beta \geq 0$ and $I_\beta = [0, -1/\beta]$ in the case $\beta < 0$. The parameter β is usually called *extreme value index.* The family of GPDs is a rather rich one. For $\beta > 0$ we obtain the usual Pareto family, H_{-1} is the uniform distribution on (0,1) and H_0 is the standard exponential distribution. We consider the two-sided testing problem

$$H_0 \text{ against } \{H_\beta : \beta \neq 0\} \tag{2}$$

and establish an asymptotically uniformly optimal unbiased test sequence for that testing problem.

Furthermore, we consider the *extreme value distributions* (EVDs)

$$G_\beta(x) := \exp(-(1+\beta x)^{-1/\beta}), \quad 1+\beta x > 0,\ \beta \in \mathbb{R}. \tag{3}$$

For $\beta = 0$ we get the Gumbel distribution $G_0(x) := \lim_{\beta\to 0} G_\beta(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, for $\beta > 0$ the class of Fréchet distributions and for $\beta < 0$ the class of Weibull distributions. Note that G_β has support $(-\infty, -1/\beta), (-1/\beta, \infty), \mathbb{R}$ if $\beta < 0, \beta > 0$, and $\beta = 0$, respectively, and that $H_\beta(x) = 1 + \log G_\beta(x)$, $x > 0$. We treat the testing problem

$$G_0 \text{ against } \{G_\beta : \beta \neq 0\} \tag{4}$$

and establish an asymptotic optimal test sequence. Recall that a distribution belongs to the domain of attraction of an EVD iff the exceedances in the peaks-over-threshold model (POT) follow approximately a GPD, which was mathematically first observed by Pickands (1975). Hence, the present paper deals with the ideal situation, where the iid exceedances follow exactly a GPD. For a review of the crucial role played by GPDs in extreme value theory we refer to the monographs by Reiss (1989, 1993), Chapters 1, 5, 9, and Chapter 5, respectively.

Though the testing problems (2) and (4) are treated in the literature (see Gomes and Montfort (1986), Gomes (1989)), a mathematical rigorous treatment of these testing problems in the framework of the theory of statistical experiments in the sense of LeCam (1986) seems to be missing. As a main tool we show that the family of GPDs is *differantiable in quadratic mean* at $\beta = 0$, which is a crucial condition in the asymptotic setting (LeCam (1986), Chapter 17, Section 3). This property, which we call DQM, implies local asymptotic normality (LAN) of the underlying sequence of statistical experiments which in turn opens the way to find asymptotic optimal procedures. It turns out that the property DQM carries over to the class of EVDs. For the background of the theory of statistical experiments we refer to LeCam (1986), LeCam and Yang (1990), Milbrodt and Strasser (1985), and Strasser (1985). Concerning tests for the domain of attraction we refer to Castillo et al. (1989), Falk (1992, 1993), and Hasofer and Wang (1992).

By $E_P(X)$ we denote the expectation of a random variable X with distribution P. The normal distribution with expectation μ and variance σ^2 is designated by $N(\mu, \sigma^2)$. By $\langle \cdot, \cdot \rangle_a$ we denote the inner product $\langle s, t \rangle_a = ast$, $a > 0$, $s, t \in \mathbb{R}$. The corresponding norm is denoted by $|| \cdot ||_a$. The weak convergence of a sequence of statistical experiments $(E_n)_n$ to a limit experiment E is denoted by $E_n \to E$.

2. Testing the exponential distribution

In this section we test the hypothesis H_0 within the family of GPDs (1). The following lemma is crucial for our investigations. Its proof is postponed to Section 4.

Lemma 2.1. *The statistical experiment $(\mathbb{R}, \mathcal{B}, \{H_\beta : \beta \in \mathbb{R}\})$ satisfies condition DQM at $\beta = 0$ with derivative*

$$v(x) = \begin{cases} -x + x^2/2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

We remark that v is just the score function $x \to (\partial/\partial\beta) \log h_0(x) := (\partial/\partial\beta) \log h_\beta(x)|_{\beta=0}$. As a consequence of Lemma 2.1 we get the following result. Note that $\int v^2 \, dH_0 = 2$.

Theorem 2.2 (LAN). *The log-likelihood ratio with base 0 admits the expansion*

$$\log \frac{dH^n_{\beta/\sqrt{n}}}{dH^n_0} = \langle \beta, Z_n \rangle_2 - \frac{1}{2} ||\beta||^2_2 + o_{H^n_0}(1)$$

with central sequence

$$Z_n(x_1, \ldots, x_n) := \frac{1}{2\sqrt{n}} \sum_{i=1}^{n} v(x_i).$$

That means

$$(\mathbb{R}^n, \mathcal{B}^n, \{H^n_{\beta/\sqrt{n}} : \beta \in \mathbb{R}\}) \longrightarrow (\mathbb{R}, \mathcal{B}, \{N(\beta 2, 2) : \beta \in \mathbb{R}\}).$$

Recall that the limit experiment is the Gaussian shift on $(\mathbb{R}, \langle\cdot,\cdot\rangle_2)$. It is well known that the statistician has only to solve the corresponding testing problem in the limit experiment, in our case $N(0,2)$ against $N(\beta 2, 2)$, $\beta \neq 0$, and to carry its solution over to the finite sample case, see Strasser (1985), Chapter 13, in order to obtain asymptotically optimal testing and estimating procedures. Denote by Φ the distribution function of $N(0,1)$ and by $u_\alpha = \Phi^{-1}(\alpha)$ the α–quantile of Φ.

Corollary 2.3. *Consider the asymptotic testing problem*

$$H_0^n \text{ against } \{H^n_{\beta/\sqrt{n}} : \beta \neq 0\}.$$

(i) Let $(\varphi_n)_n$ be an asymptotically unbiased test sequence of level α for this testing problem, that means,

$$\limsup_{n\to\infty} E_{H_0^n}\varphi_n \leq \alpha \text{ and } \liminf_{n\to\infty} E_{H^n_{\beta/\sqrt{n}}}\varphi_n \geq \alpha, \ \beta \neq 0.$$

Then

$$\limsup_{n\to\infty} E_{H^n_{\beta/\sqrt{n}}}\varphi_n \leq \Phi(u_{\alpha/2} + \sqrt{2}\beta) + \Phi(u_{\alpha/2} - \sqrt{2}\beta).$$

(ii) Let $X_1, \ldots, X_n$ be independent random variables with common distribution H_β. Then

$$\varphi_n^*(X_1, \ldots, X_n) = \begin{cases} 1 & > \\ & |Z_n(X_1, \ldots, X_n)| \qquad \dfrac{u_{1-\alpha/2}}{\sqrt{2}} \\ 0 & \leq \end{cases}$$

is an asymptotically uniformly optimal unbiased test sequence of asymptotic level α, that means, $(\varphi_n^*)_n$ attains the upper bound for the power function

$$\lim_{n\to\infty} E_{H^n_{\beta/\sqrt{n}}}\varphi_n^* = \Phi(u_{\alpha/2} + \sqrt{2}\beta) + \Phi(u_{\alpha/2} - \sqrt{2}\beta).$$

We briefly report some Monte–Carlo simulations for the asymptotic testing problem H_0^n against $H^n_{\beta\sqrt{n}}$, $\beta \neq 0$. We generate a normal probability plot for N independent replicates $\sqrt{2}Z_n(1), \ldots, \sqrt{2}Z_n(N)$ of $\sqrt{2}Z_n$. The plot shows the points

$$\left(\Phi^{-1}\left(\frac{i}{N+1}\right), \sqrt{2}Z_n(i:N)\right), \quad i = 1, \ldots, N$$

with $N = 400, n = 400$, where $Z_n(i : n) \leq \cdots \leq Z_n(N : N)$ denotes the order statistic pertaining to the N independent replicates of $Z_n(1), \ldots, Z_n(N)$ of Z_n. Deviations from the broken line, being the identity, visualize deviations of the distribution of $\sqrt{2}Z_n$ from the hypothetical standard normal one. The plots show the behaviour of Z_n under the hypothesis $\beta = 0$ and the alternatives $\beta = 0.4$, $\beta = -0.4$. They reflect the specific behaviour of Z_n which ought to be expected: an underlying alternative parameter $\beta = 0.4$ ($\beta = -0.4$) shifts the distribution of Z_n to the right (left).

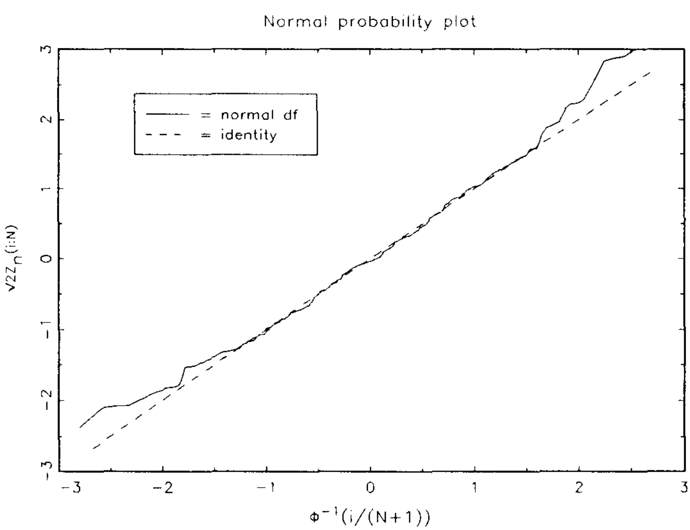

Figure 1: $\beta = 0$

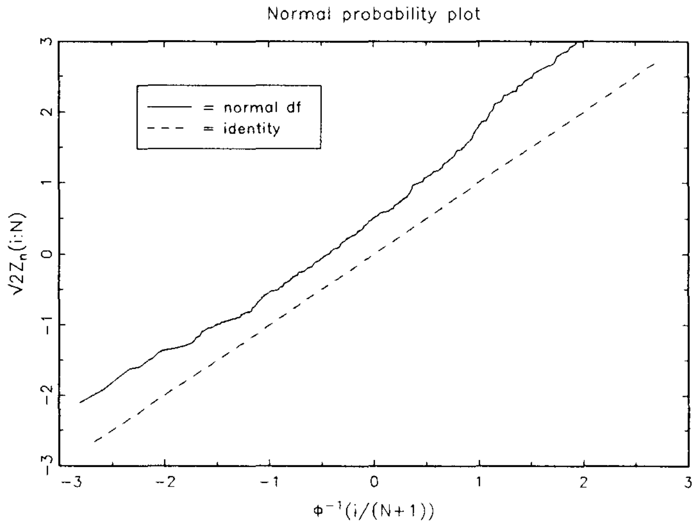

Figure 2: $\beta = 0.4$

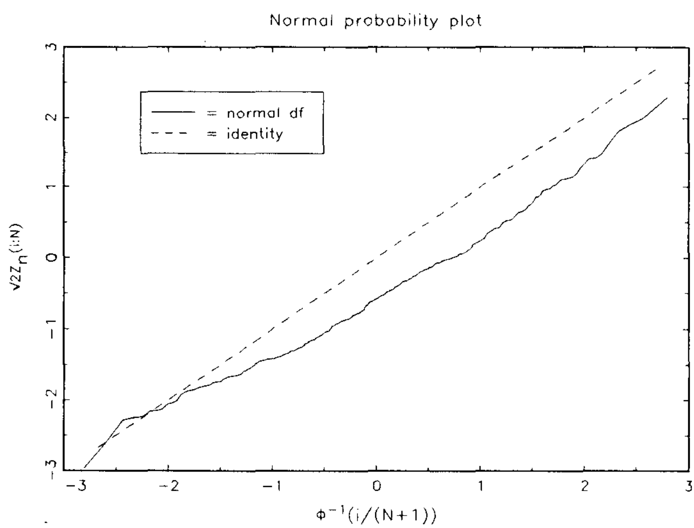

Figure 3: $\beta = -0.4$

3. Testing the Gumbel distribution

In this section, we consider the class of EVDs (3) and treat the testing problem Gumbel (G_0) against Fréchet (G_β, $\beta > 0$) and Weibull (G_β, $\beta < 0$). The proof of the following lemma is postponed to Section 4.

Lemma 3.1. *The statistical experiment* $(\mathbb{R}, \mathcal{B}, \{G_\beta : \beta \in \mathbb{R}\})$ *satisfies condition DQM with derivative*

$$w(x) = -x + \frac{x^2}{2}(1 - \exp(-x)), \quad x \in \mathbb{R}.$$

Theorem 3.2 (LAN). *The log-likelihood ratio with base 0 admits the expansion*

$$\log \frac{dG^n_{\beta/\sqrt{n}}}{dG^n_0} = \langle \beta, Z_n \rangle_{\sigma_0^2} - \frac{1}{2}||\beta||^2_{\sigma_0^2} + o_{G_0^n}(1)$$

with central sequence

$$Z_n(x_1, \ldots, x_n) := \frac{1}{\sigma_0^2 \sqrt{n}} \sum_{i=1}^{n} w(x_i),$$

where

$$\sigma_0^2 := \int \left(\frac{1}{4}x^4 - x^3 + x^2\right) dG_0(x).$$

That means,

$$(\mathbb{R}^n, \mathcal{B}^n, \{G^n_{\beta/\sqrt{n}} : \beta \in \mathbb{R}\}) \longrightarrow (\mathbb{R}, \mathcal{B}, \{N(\beta\sigma_0^2, \sigma_0^2) : \beta \in \mathbb{R}\}).$$

PROOF. In view of Lemma 3.1 it remains to show that

$$\int w^2(x)\, dG_0(x) = \int \left(\frac{1}{4}x^4 - x^3 + x^2\right) dG_0(x).$$

First, we have

$$w^2(x) = \frac{x^4}{4} - x^3 + x^2 - \left(\frac{x^4}{2} - x^3\right) \exp(-x) + \frac{x^4}{4} \exp(-2x).$$

Partial integration yields

$$\int \frac{x^4}{4} \exp(-2x)\, dG_0(x) = \int \left(\frac{x^4}{2} - x^3\right) \exp(-x)\, dG_0(x)$$

which implies the assertion. □

Note that condition DQM implies $\int w\, dG_0 = 0$ which can also be verified directly by partial intergration. An explicit representation of the variance σ_0^2 can be derived from the equalities (see Gumbel (1958))

$$\begin{aligned} E_{G_0} X &= \gamma \\ E_{G_0} X^2 &= \gamma^2 + \pi^2/6 \\ E_{G_0} X^3 &= \gamma^3 + \gamma\pi^2/2 + 2S_3 \\ E_{G_0} X^4 &= \gamma^4 + \gamma^2\pi^2 + 8\gamma S_3 + 3\pi^4/20, \end{aligned}$$

where $\gamma = 0.5772\ldots$ is Euler's constant and $S_3 := \sum_{j=1}^{\infty} j^{-3}$. Approximately, we have $E_{G_0} X^4 \approx 23.5615$, $E_{G_0} X^3 \approx 5.4449$, $E_{G_0} X^2 \approx 1.9781$, and $\sigma_0^2 \approx 2.4236$.

Corollary 3.3. *Let $Y_1, \ldots, Y_n$ be independent random variables with common distribution G_β. Then*

$$\varphi_n^*(Y_1, \ldots, Y_n) = \begin{cases} 1 & |Z_n(Y_1, \ldots, Y_n)| \quad > \quad \dfrac{u_{1-\alpha/2}}{\sigma_0} \\ 0 & \qquad\qquad\qquad\quad \leq \end{cases}$$

is an asymptotically uniformly optimal unbiased test sequence of asymptotic level α for the testing problem

$$G_0^n \text{ against } \{G^n_{\beta/\sqrt{n}} : \beta \neq 0\}$$

with asymptotic power function

$$\lim_{n\to\infty} E_{G^n_{\beta/\sqrt{n}}} \varphi_n^* = \Phi(u_{\alpha/2} + \sigma_0\beta) + \Phi(u_{\alpha/2} - \sigma_0\beta).$$

Concluding Remark. It can be shown that LAN holds also for the extended models $H_{\beta,\sigma}$ and $G_{\beta,\sigma}$ by adding a scale parameter σ. Again, the score function is the derivative w.r.t. DQM. The details are omitted in order not to overload the present paper.

4. Proofs of Lemma 2.1 and Lemma 3.1.

We need the following lemma.

Lemma 4.1. *The function* $f(z) := (1+z)^{-1/z}$, $z > -1$, *is strictly increasing on* $(-1, \infty)$.

PROOF: We have $f'(z) = f(z)l(z)$ with $l(z) := z^{-2}\log(1+z) - 1/(z+z^2)$ for $z \neq 0$ and $f'(0) = 1/(2e)$. We show $l(z)z^2 > 0$, $z \neq 0$. But this is immediate from $(l(z)z^2)' = z/(1+z)^2$. □

Denote by h_β the density of H_β with respect to the Lebesgue measure λ. Precisely, we have for $\beta > 0$

$$h_\beta(x) = \begin{cases} (1+\beta x)^{-(1+1/\beta)}, & x \geq 0 \\ 0, & x \leq 0, \end{cases}$$

for $\beta < 0$

$$h_\beta(x) = \begin{cases} (1+\beta x)^{-(1+1/\beta)}, & x \in [0, -1/\beta) \\ 0, & x \in I\!R \setminus [0, -1/\beta), \end{cases}$$

and for $\beta = 0$

$$h_\beta(x) = \begin{cases} \exp(-x), & x \geq 0 \\ 0, & x < 0. \end{cases}$$

PROOF OF LEMMA 2.1: First, note that H_β is dominated by H_0. Define the auxiliary function

$$q_\beta(x) := \beta^{-1}((h_\beta(x)/h_0(x))^{1/2} - 1), \quad x > 0.$$

We show

$$\lim_{\beta \to 0} q_\beta(x) = v(x)/2 \tag{5}$$

and

$$|q_\beta(x)| \leq \psi(x) \in L_2(H_0) \tag{6}$$

for β sufficiently small and some function ψ which implies condition DQM by the dominated convergence theorem of Lebesgue. By $L_2(H_0)$

we denote the space of square integrable functions w.r.t. H_0. Using the Taylor expansion $\log(1+z) = z - z^2/2 + o(z^2)$ as $z \to 0$, we obtain for $\beta \to 0$

$$h_\beta^{1/2}(x)/h_0^{1/2}(x) = \exp\Big\{ -\frac{\beta x}{2} + \frac{\beta x^2}{4} + o(\beta)\Big\}$$

which implies (5). To prove (6) we consider the two cases $\beta > 0$ and $\beta < 0$ separately.

The case $\beta > 0$: Elementary computations yield the bound

$$|q_\beta(x)| \le \beta^{-1}(h_\beta(x)/h_0(x))^{1/2}\Big|(1+\beta x)^{1/(2\beta)}\exp(-x/2) - 1\Big| \\ +\beta^{-1}(1-(1+\beta x)^{-1/2}).$$

Using the inequality

$$|e^{-y}(1+y/\alpha)^\alpha - 1| < 2y^2/\alpha \quad \text{for } \alpha \ge 1,\ y \ge -\alpha/2 \tag{7}$$

(for details see Reiss (1989), p. 322) and the Taylor expansion

$$\beta^{-1}((1+\beta y)^{-1/2} - 1) = -\frac{y}{2} + \frac{3}{8}\beta(1+\beta\xi)^{-5/2}y^2, \quad 1+\beta y > 0, \tag{8}$$

where ξ is between 0 and y, we get for $0 < \beta < 1/2$

$$|q_\beta(x)| \le (h_\beta(x)/h_0(x))^{1/2}x^2 + p_1(x), \quad x > 0$$

for an appropriately chosen polynomial p_1. Next we can find a number $x_0 > 0$ and a number $\beta_0 = \beta_0(x_0)$ such that for all $\beta \in (0, \beta_0]$

$$(1+\beta x)^{-(1+1/\beta)} < (1+x/2)^{-3},\ x > x_0. \tag{9}$$

To establish inequality (9), first note that (9) is equivalent to $\exp(s_\beta(x)) < 1$, $x > x_0$, where

$$s_\beta(x) := 3\log(1+x/2) - \frac{1+\beta}{\beta}\log(1+\beta x), \quad x > 0.$$

Standard calculations show that $(d/dx)s_\beta(x) < 0$ for $x > 1$. Choose $x_0 > 1$ such that $3\log(1+x_0/2) < x_0$. Taking into account $((1+\beta)/\beta)\log(1+\beta x_0) \sim x_0$ as $\beta \to 0$, we obtain $s_\beta(x_0) < 0$ for $0 < \beta \le \beta_0$ and some β_0. But now inequality (9) follows, since s_β is strictly decreasing on $(1, \infty)$. Hence, we can find a function $\psi_1 \in L_2(H_0)$ with $|q_\beta(x)| \le \psi_1(x)$

(choose, for example, $\psi_1(x) := (C_1 1_{(0,x_0]}(x) + (1+x/2)^{-3}x^2 1_{(x_0,\infty)}(x))/h_0^{1/2}(x) + p_1(x))$.

The case $\beta < 0$: For $-1/2 < \beta < 0$ and $x \in (0, -1/(2\beta))$ inequality (7) is applicable. For an appropriately chosen polynomial p_2 we get

$$\begin{aligned} |q_\beta(x)| &\leq |\beta^{-1}(1+\beta x)^{-1/2}((1+\beta x)^{-1/(2\beta)}\exp(x/2) - 1)| \\ &\quad + |\beta^{-1}((1+\beta x)^{-1/2} - 1)| \\ &\leq \sqrt{2}x^2 + p_2(x) \in L_2(H_0) \end{aligned}$$

by (7) and (8). Note that $\beta x \in (-1/2, 0)$. The case $x \in (-1/(2\beta), -1/\beta)$ is a little bit more complicated. An application of the mean value theorem yields

$$q_\beta(x) = \frac{1}{2} h_0^{-1/2}(x) h_\xi^{-1/2}(x) \frac{\partial}{\partial \beta} h_\xi(x)$$

with $\xi = \xi(\beta, x) \in (\beta, 0)$. Consequently, it remains to show the existence of a function $\psi_2 \in L_2(\lambda)$ such that

$$\left| h_\xi^{-1/2}(x) \frac{\partial}{\partial \beta} h_\xi(x) \right| \leq \psi_2(x), \quad x \in (-1/(2\beta),\ -1/\beta)$$

for $\xi \in (\beta, 0)$ and β small. Differating $h_\beta(x)$ w.r.t. β yields

$$\frac{\partial}{\partial \beta} h_\beta(x) = h_\beta(x)\Big(\beta^{-2}\log(1+\beta x) - (1+\beta)x/(\beta(1+\beta x))\Big).$$

Hence, we get

$$\begin{aligned} & h_\xi^{-1/2}(x) \frac{\partial}{\partial \beta} h_\xi(x) \\ = \;& \xi^{-2}(1+\xi x)^{-(3/2+1/(2\xi))}\Big((1+\xi x)\log(1+\xi x) - (1+\xi)\xi x\Big). \end{aligned}$$

As $\xi \to 0$, we have

$$\begin{aligned} & (1+\xi x)\log(1+\xi x) - (1+\xi)\xi x \\ = \;& -\frac{(\xi x)^2}{2} - \xi^2 x + \xi x \log(1+\xi x) + O(\xi^3) \end{aligned}$$

with

$$O(\xi^3) = \log(1+\xi x) - \xi x + \frac{(\xi x)^2}{2}.$$

Since $z^{-1}\log(1+z) \to 1$ and $z^{-2}(\log(1+z) - z + z^2/2) \to 1/2$ as $z \to 0$ we can find a polynomial p_3 such that

$$\left|h_\xi^{-1/2}(x)\frac{\partial}{\partial\beta}h_\xi(x)\right| \le p_3(x)(1+\xi x)^{-(3/2+1/(2\xi))}.$$

Note that $\xi x \in (-1, 0)$. Moreover, $(1+\xi x)^{-(3/2+1/(4\xi))} \le 1$ for $-1/6 < \xi < 0$ and $(1+\xi x)^{-1/(4\xi)} \le \exp(-x/4)$, $x \in (-1/(2\beta), -1/\beta)$ by Lemma 4.1. Hence, for small β we can find a function $\psi_2 \in L_2(\lambda)$ such that $|h_\xi^{-1/2}(x)(\partial/\partial\beta)h_\xi(x)| \le \psi_2(x)$, $x \in (-1/(2\beta), -1/\beta)$, $\beta < \xi < 0$.

Summarizing the above results, we conclude that (6) holds. The proof is complete. □

Denote by g_β the Lebesgue density of G_β. The explicit form of the densities is given by

$$g_0(x) = e^{-x}\exp(-e^{-x}), \quad x \in \mathbb{R}$$

if $\beta = 0$, and

$$g_\beta(x) = (1+\beta x)^{-(1+1/\beta)}\exp(-(1+\beta x)^{-1/\beta}) \text{ if } \begin{array}{l} x > -1/\beta,\ \beta > 0 \\ x < -1/\beta,\ \beta < 0, \end{array}$$

and $= 0$ otherwise.

PROOF OF LEMMA 3.1: Define the auxiliary function

$$r_\beta(x) = \beta^{-1}((g_\beta(x)/g_0(x))^{1/2} - 1).$$

As in the proof of Lemma 2.1 we show

$$\lim_{\beta\to 0} r_\beta(x) = w(x)/2 \tag{10}$$

and

$$|r_\beta(x)| \le \psi(x) \in L_2(G_0). \tag{11}$$

Denote by $\tilde{h}_\beta$ the continuation of h_β from the support I_β of H_β to the support of G_β. Precisely,

$$\tilde{h}_\beta(x) = (1+\beta x)^{-(1+1/\beta)}, \quad 1+\beta x > 0.$$

Then g_β can be rewritten in the form

$$g_\beta(x) = \tilde{h}_\beta(x)\exp(-(1+\beta x)\tilde{h}_\beta(x)).$$

Assertion (10) follows by straightforward calculations. Note that $\beta \to \tilde{h}_\beta(x)$ is differentiable at $\beta = 0$ with derivative $(-x + x^2/2)\tilde{h}_0(x)$, $x \in \mathbb{R}$. To establish (11) we split $r_\beta(x)$ into two terms. Define

$$\tilde{q}_\beta(x) = \beta^{-1}\Big((\tilde{h}_\beta(x)/\tilde{h}_0(x))^{1/2} - 1\Big).$$

Then

$$\begin{aligned} r_\beta(x) &= \tilde{q}_\beta(x)\exp\Big(-\frac{1}{2}(1+\beta x)^{-1/\beta}\Big)\exp\Big(\frac{1}{2}e^{-x}\Big) \\ &\quad +\beta^{-1}\Big\{\exp\Big(-\frac{1}{2}(1+\beta x)^{-1/\beta}\Big) - \exp\Big(-\frac{1}{2}e^{-x}\Big)\Big\}\exp\Big(\frac{1}{2}e^{-x}\Big) \\ &=: A_1(\beta,x) + A_2(\beta,x). \end{aligned}$$

First, we treat the case that x is an element of the support I_β of H_β i.e., $x \in (0,\infty)$ for $\beta > 0$ and $x \in (0,-1/\beta)$ for $\beta < 0$. Then $\tilde{h}_\beta = h_\beta$ and thus $A_1(\beta,x)$ is dominated by some function $\psi_1 \in L_2(G_0)$. To treat the term $A_2(\beta,x)$ we need the inequality

$$e^t - e^s < e^t(t-s), \quad t > s \tag{12}$$

which is an easy consequence of the mean value theorem.
The case $\beta > 0$: Then we have $x > 0$ and Lemma 4.1 implies $\exp(-x) \le (1+\beta x)^{-1/\beta}$. Hence, by (7) and (12) with $s = -(1+\beta x)^{-1/\beta}/2$ and $t = -\exp(-x)/2$, we get $|A_2(\beta,x)| \le x^2 \in L_2(G_0)$.
The case $\beta < 0$: Then we have $x \in (0,-1/\beta)$. Lemma 4.1 implies $(1+\beta x)^{-1/\beta} \le \exp(-x)$ and applying inequality (12) we obtain

$$|A_2(\beta,x)| \le \frac{1}{2}|\beta^{-1}|(1 - e^x(1+\beta x)^{-1/\beta})e^{-x}\exp\Big(\frac{1}{2}e^{-x} - x\Big).$$

For $-1 < \beta < 0$ and $x \in (0,-1/(2\beta))$ we can apply inequality (7) and obtain $|A_2(\beta,x)| \le x^2\exp(e^{-x}/2 - x) \in L_2(G_0)$. Let now be $x \in (-1/(2\beta),-1/\beta)$. The mean value theorem implies

$$\begin{aligned} &\beta^{-1}((1+\beta x)^{-1/\beta} - e^{-x}) \\ &= \xi^{-2}(1+\xi x)^{-(1+1/\xi)}\Big((1+\xi x)\log(1+\xi x) - \xi x\Big), \end{aligned}$$

where ξ is between 0 and β. The same arguments used in the second part of the proof of Lemma 2.1 show that $|A_2(\beta,x)| \le p_1(x)\exp(-x/2) \in L_2(G_0)$ for some polynomial p_1. Note that $\xi x \in (-1,0)$ and that $(1+\xi x)^{-1/(2\xi)} \le \exp(-x/2)$ by Lemma 4.1.

It remains to consider the two cases $x \in (-1/\beta, 0)$ for $\beta > 0$ and $x \in (-\infty, 0)$ for $\beta < 0$. We treat only the first case, since the second case is shown in a similar way. For $0 < \beta < 1$ and $x \in (-1/(2\beta), 0)$ inequality (7) is applicable. Hence we get from (7) and (8) as in the first part of the proof of Lemma 2.1

$$|\tilde{q}_\beta(x)| \le \Big(\frac{\tilde{h}_\beta^{1/2}(x)}{\tilde{h}_0^{1/2}(x)}\Big)^{1/2} x^2 + p_2(x)$$

for some polynomial p_2. Note that $\beta x \in (-1/2, 0)$. Taking into account Lemma 4.1 we have $\exp(-x) \le (1+\beta x)^{-1/\beta} \le \exp(-x\log 4)$. Hence we see that a function $\psi_2 \in L_2(G_0)$ exists such that $|A_1(\beta,x)| \le \psi_2(x)$, $x \in (-1/(2\beta), 0)$. Using (7) and (12) we get $|A_2(\beta,x)| \le x^2\exp(-x\log 4) \in L_2(G_0)$. Let now be $x \in (-1/\beta, -1/(2\beta))$. Similar to the situation in the second part of the proof of Lemma 2.1, the arguments used above do not work and we have to include the derivative of $g_\beta(x)$ w.r.t. β. The mean value theorem implies

$$r_\beta(x) = \frac{1}{2} g_0^{-1/2}(x) g_\xi^{-1/2}(x) \frac{\partial}{\partial\beta} g_\xi^{1/2}(x)$$

with $\xi = \xi(\beta, x) \in (0,\beta)$. Consequently, it remains to show the existence of a function $\psi_3 \in L_2(\lambda)$ such that

$$\Big| g_\xi^{-1/2}(x) \frac{\partial}{\partial\beta} g_\xi^{1/2}(x) \Big| \le \psi_3(x), \quad x \in (-1/\beta, -1/(2\beta)),$$

for $\xi \in (0,\beta)$ and β small. Lengthy and tedious calculations show

$$\begin{aligned} & g_\xi^{-1/2}(x)\frac{\partial}{\partial\beta} g_\xi(x) \\ &= \xi^{-2}(1+\xi x)^{-5/2-3/(2\xi)} \exp\Big(-\frac{1}{2}(1+\xi x)^{-1/\xi}\Big) \\ &\quad \times\Big\{\Big((1+\xi x)^{1+1/\xi} - (1+\xi x)\Big)\Big((1+\xi x)\log(1+\xi x) - \xi x(1+\xi)\Big) \\ &\qquad -x^2\xi^2(1+\xi x)\Big\}. \end{aligned}$$

Now, $(1+\xi x)^{-5/2-3/(2\xi)} \le (1+\xi x)^{-2/\xi}$ for $0<\xi<1/5$ and we show

$$(1+\xi x)^{-2/\xi}\exp\Big(-\frac{1}{4}(1+\xi x)^{-1/\xi}\Big)\le 1 \tag{13}$$

for small ξ and $x \in (-1/\xi, x_0)$ with $x_0 < -\log 8$. From (13) and $\exp(-x) \le (1+\xi x)^{-1/\xi}$ we deduce as in the proof of Lemma 2.1

$$\Big|g_\xi^{-1/2}(x)\frac{\partial}{\partial\beta}g_\xi(x)\Big| \le p_3(x)\exp\Big(-\frac{1}{4}e^{-x}\Big) =: \psi_3(x)$$

for some polynomial p_3. Note that $\xi x \in (-1,0)$. First, observe that (13) is equivalent to

$$s_\xi(x) := -\frac{8}{\xi}\log(1+\xi x) - (1+\xi x)^{-1/\xi} < 0$$

and

$$\frac{d}{dx}s_\xi(x) = -\frac{8}{1+\xi x} + (1+\xi x)^{-(1+1/\xi)} > 0$$

iff $(1+\xi x)^{-1/\xi} > 8$, which holds for $x < -\log 8$. Hence $s_\xi(x)$ is increasing on $(-1/\xi, -\log 8)$. Next we choose $x_0 < -\log 8$ such that $-8x_0 - \exp(-x_0) < 0$. Then $s_\xi(x_0) < 0$ for small ξ, since $\xi^{-1}\log(1+\xi x_0) \to x_0$ as $\xi \to 0$. Consequently, $s_\xi(x) < 0$ for small ξ and $x \in (-1/\xi, x_0)$.

The proof is complete. □

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